Improper Integrals

The idea of definite integrals cannot be used in cases like

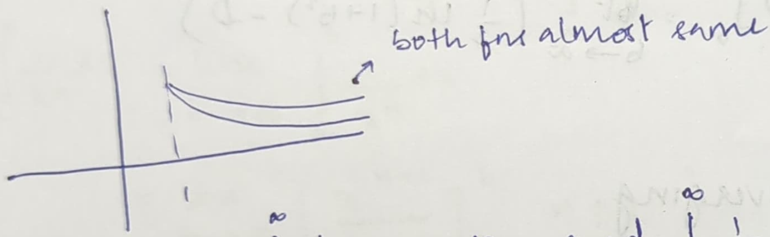
1) $\int_{-\infty}^{\infty} x\,dx \neq 0$

2) $\int_{-1}^{1} \frac{1}{x^4}\,dx$

$= 2\int_0^1 x^{-4}\,dx = \left[2\frac{x^{-3}}{-3}\right]_0^1 = \frac{-2}{3}$ here even the function is +ve, integral comes out -ve.

since at $x=0$, $f(x) \to \infty$.

3) $\int_1^{\infty} \frac{1}{x}\,dx$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$:



but still, $\int_1^{\infty} \frac{1}{x}\,dx = \infty$ and $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$

By definition of proper / definite integrals,

$\int_a^b f(x)\,dx$ where a, b are finite and $f(x)$ is finite b/w a and b.

By definition of improper integrals:

$\int_a^b f(x)\,dx$ (i) if a or b or both are $\to \pm\infty$ (first kind)

(ii) if $f(x) \to \pm\infty$ for $a \leq x \leq b$ (second kind)

Improper Integrals of First Kind

They are of the form:

$\int_a^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ or $\int_{-\infty}^{\infty} f(x)\,dx$

$\cdot \int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

If limit exists and is finite, the integral converges. Otherwise, it diverges.

$\cdot \int_{-\infty}^{b} f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx$

same.

$\cdot \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

do separately.

Ex: $\int_0^{\infty} \frac{x}{1+x^2}\,dx$

$= \lim_{b\to\infty} \int_0^b \frac{x}{1+x^2}\,dx$

$= \lim_{b\to\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_0^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(1+b^2) - 0\right)$

$= \frac{1}{2}\ln(\infty) =$ ND.

$\therefore$ The integral is diverging.

$* \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

$= \lim_{a\to -\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$

converge + converge = converge

1 converge + 1 diverge = diverge

diverge + diverge = diverge.

Ex: $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$

Method 1:

$\lim_{c\to\infty} \int_{-c}^{c} \frac{x}{1+x^2}\,dx = 0$ ✗

Method 2:

$\lim_{c\to\infty} \left[\int_{-c}^{0} \frac{x}{1+x^2}\,dx + \int_0^c \frac{x}{1+x^2}\,dx\right]$

$= \lim_{c\to\infty} \left[-\frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(1+x^2)\right] = 0$ ✗

Method 3:

$$\lim \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{x}{1+x^2}\,dx + \lim_{b \to \infty} \int_{0}^{b} \frac{x}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_a^0 + \lim_{b\to\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_0^b$$

$$= \lim_{a \to -\infty} \left[0 - \frac{1}{2}\ln(1+a^2)\right] + \lim_{b \to \infty} \left[\frac{1}{2}\ln(1+b^2) - 0\right]$$

$$= -\frac{1}{2}\ln(\infty) + \frac{1}{2}\ln(\infty)$$

= Divergent integral.

* Discuss the converge of $\int_a^\infty \frac{1}{x^n}\,dx$ $n \in$ Real, $a > 0$

$$\Rightarrow \int_a^\infty \frac{1}{x^n}\,dx = \lim_{b\to\infty} \int_a^b \frac{1}{x^n}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-n+1}}{-n+1}\right]_a^b \qquad n \neq 1$$

$$= \lim_{b\to\infty} \frac{1}{1-n}\left(b^{1-n} - a^{1-n}\right)$$

$$= \begin{cases} \infty & 1-n > 0 \quad n < 1 \\ \frac{1}{1-n}(-a^{1-n}) & 1-n < 0 \quad \text{or } n > 1 \end{cases}$$

$\therefore$ Integral is diverging for $n < 1$
converging for $n > 1$
diverging for $n = 1$

when $n = 1$,

$$= \int_a^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_a^b \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln x]_a^b$$

$$= \lim_{b\to\infty} (\ln b - \ln a)$$

$$= \infty$$

* $\boxed{\int_a^\infty \frac{1}{x^n}\,dx = \begin{cases}\text{diverging} & n \le 1 \\ \text{converging} & n > 1\end{cases} \qquad n \in \text{Real},\ a > 0}$

Requirements for Improper Integrals:

1) Consider $f(x) \geq 0$.

If some function say $g(x) \leq 0$

then $-g(x) \geq 0$

and $\phi(x) = -g(x)$

or $\phi(x) \geq 0$.

Finding $\int_a^\infty f(x)\,dx$, $f(x) \geq 0$: (Comparision Test)

Result:

If $f(x)$, $g(x)$ are two function and

$f(x) \geq 0$, $g(x) \geq 0$

or $0 \leq f(x) \leq g(x)$ $\forall$ $a \leq x < \infty$

Then:

(i) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

(ii) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

Ex: 1) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Now, $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$

$$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$$

this is converging.

So, technically, $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is also converging.

2) $\int_2^\infty \frac{1}{\ln x}\,dx$

$\ln x < x$

$\frac{1}{\ln x} > \frac{1}{x}$

$$\int_2^\infty \frac{1}{\ln x}\,dx > \int_2^\infty \frac{1}{x}\,dx$$

diverges

So, $\int \frac{1}{\ln x}$ is diverging

- Choose the function carefully.

3) $\int_1^\infty e^{-x^2} dx$

$$x \le x^2$$
$$-x \geq -x^2$$
$$e^{-x} \geq e^{-x^2}$$
$$\int_1^\infty e^{-x} dx \geq \int_1^\infty e^{-x^2} dx$$

$$\int_1^\infty e^{-x} dx = [-e^{-x}]_1^\infty = 0 - (-\tfrac{1}{e}) = \tfrac{1}{e}.$$
↳ converging.

So, $\int_1^\infty e^{-x^2} dx$ is also converging.

* Result :

$\int_a^\infty f(x) dx$ and $f(x) \geq 0$, $g(x) \geq 0$ $\quad a \le x < \infty$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l \quad 0 < l < \infty$

then $\int_a^\infty f(x) dx$ and $\int_a^\infty g(x) dx$ converges or diverges together

Ex: $\int_1^\infty \frac{x}{x^2+1} dx$

Let $f(x) = \frac{x}{x^2+1} = \frac{x}{x^2(1+\frac{1}{x^2})}$

$$= \frac{1}{x(1+\frac{1}{x^2})}$$

let $g(x) = \frac{1}{x}$.

then $\lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}(1+\frac{1}{x^2})}$,

$$\lim_{x\to\infty} \left(\frac{1}{1+\frac{1}{x^2}}\right) \neq 1$$

Then $\int_1^\infty g(x) dx$

$= \int_1^\infty \frac{1}{x} dx$
↳ diverges.

So, $\int_1^\infty \frac{x}{x^2+1} dx$ → diverges.

2) $\int_2^\infty \frac{x\tan^{-1}x}{(1+x)^{7/2}}\,dx$

Let $f(x) = \frac{x\tan^{-1}x}{(1+x)^{7/2}} = \frac{\tan^{-1}x}{x^{5/2}\left(\frac{1}{x}+1\right)^{7/2}}$

Let $g(x) = \frac{1}{x^{5/2}}$

$\int_2^\infty \frac{1}{x^{5/2}}\,dx$ = converges.

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{x^{5/2}\left(\frac{1}{x}+1\right)^{7/2}} \cdot x^{5/2} = \frac{\tan^{-1}x}{\left(\frac{1}{x}+1\right)^{7/2}}$

$= \lim_{x\to\infty} \frac{\tan^{-1}x}{\left(1+\frac{1}{x}\right)^{7/2}}$.

$= \pi/2$.

$\therefore \int_2^\infty \frac{x\tan^{-1}x}{(1+x)^{7/2}}$ converges.

3) $\int_1^\infty \frac{1}{e^x+1}\,dx$

$f(x) = \frac{1}{e^x+1} = \frac{1}{e^x(1+e^{-x})}$

$g(x) = \frac{1}{e^x} = e^{-x}$

$\lim_{x\to\infty} = \frac{e^x}{e^x(1+e^{-x})} = \frac{1}{1+e^{-x}} = 1$

$\therefore \int_1^\infty e^{-x} = \frac{1}{e}$ converges.

x^{-2}

$\ln x\left(\frac{1}{x^2}\right)$

$\ln x\left(\frac{x^{-1}}{-1}\right) - \int \frac{1}{x}\frac{x^{-1}}{-1}\,dx$

$-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$

$+ \frac{x^{-1}}{-1}$

$-\frac{\ln x}{x} - \frac{1}{x}$

$\left[-\frac{(1+\ln x)}{x}\right]_2^\infty$

$-\frac{\frac{1}{x}}{1} + \frac{1+\ln 2}{2}$

$\frac{1+\ln 2}{2}$.

4) $\int_2^\infty \frac{\ln x}{x^2}\,dx$

$f(x) = \frac{\ln x}{x^2}$

$g(x) = \ln x$

$\lim_{x\to\infty} = \frac{\ln x}{x^2 \ln x} = \frac{1}{x^2} = 0$ (finite)

~~$\ln x < x$~~

~~$\frac{\ln x}{x^2} < \frac{x}{x^2}$~~

~~$\int_2^\infty \frac{\ln x}{x^2} < \int_2^\infty \frac{1}{x}\,dx \to$ diverges~~

~~So, it also diverges.~~

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ Then $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

g(x) has to converge

1) If $\lim \frac{f(x)}{g(x)} \to 0$

then $\frac{f(x)}{g(x)} < \epsilon$ [ϵ is any real number > 0.]

or $f(x) < \epsilon g(x)$

2) If $\lim \frac{f(x)}{g(x)} \to \infty$

then $\frac{f(x)}{g(x)} > k$

$f(x) > k\,g(x)$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ Then $\int_a^\infty f(x)\,dx$ diverges if $\int_a^\infty g(x)\,dx$ diverges.

g(x) has to diverge.

So, $\int_2^\infty \frac{\ln x}{x^2}\,dx$

$f(x) = \frac{\ln x}{x^2}$ and $g(x) = \frac{1}{x^{3/2}}$

$\lim_{x\to\infty} = \frac{\ln x}{x^2 \cdot \frac{1}{x^{3/2}}} = \frac{\ln x}{x^{1/2}} = 0.$

and $\int_2^\infty \frac{1}{x^{3/2}}\,dx$ converging ✓.

$\therefore \int_2^\infty \frac{\ln x}{x^2}\,dx$ is converging.

3) $\int_2^\infty \frac{1}{\ln x}\,dx$

$f(x) = \frac{1}{\ln x}$

$g(x) = \frac{1}{x}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{x}{\ln x} = \infty$

and $\int_2 \frac{1}{x} dx$ = diverging.

So, this satisfies and $\int_2^\infty \frac{1}{\ln x} dx$ is diverging.

Improper Integrals of Second Kind

$f(x) \longrightarrow \pm\infty$ in $\int_a^b f(x)\,dx$

Cases:

point of infinite discontinuity

1) $f(x) \to \infty$ at $x = a$ Ex: $\int_0^b \frac{1}{x^2} dx$

2) $f(x) \to \infty$ at $x = b$ Ex: $\int_0^2 \frac{1}{x-2} dx$

3) $f(x) \to \infty$ at $x = c$ $a < c < b$ Ex: $\int_{-1}^1 \frac{1}{x^3} dx$

4) $f(x) \to \infty$ at $x = a$ and $x = b$.   Ex: $\int_0^3 \frac{1}{3x - x^2} dx$

I. $f(x) \to \infty$ at $x = a$

$$\int_a^b f(x)\,dx$$

$$= \lim_{h \to 0} \int_{a+h}^b f(x)\,dx$$ if limit exists and is finite, integral converges.

Ex: 1) $\int_0^1 \frac{1}{x^2} dx = \lim_{h \to 0} \int_{0+h}^1 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_h^1$

$$= \lim_{h \to 0} -1 + \frac{1}{h}$$

$= \infty$ diverging.

2) $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{h \to 0} \int_h^1 x^{-1/2} dx = \left[\frac{x^{1/2}}{1/2}\right]_h^1$

$$= \lim_{h \to 0} 2(1 - \sqrt{h})$$

$= 2$ converging

II. $f(x) \to \infty$ at $x = b$

$$\int_a^b f(x)\,dx$$

$= \lim_{h \to 0} \int_a^{b-h} f(x)\,dx$ if limit exists and is finite - converges

Ex: 1) $\int_1^3 \frac{1}{\sqrt{3-x}}\,dx$

$= \lim_{h \to 0} \int_1^{3-h} (3-x)^{-1/2}\,dx$

$3-x = t$

$dx = -dt$

$-t^{-1/2}dt$

$-\frac{t^{1/2}}{1/2} \quad -2t^{1/2}$

$-2\sqrt{3-x}$

$= \left[-2\sqrt{3-x}\right]_1^{3-h}$

$= -2\sqrt{3-3+h} + 2\sqrt{3-1}$

$= 0 + 2\sqrt{2}$

$= 2\sqrt{2}$ converges.

III. $f(x) \to \infty$ at $x = a$ and $x = b$.

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad a < c < b$

$= \lim_{h \to 0} \int_{a+h}^{c} f(x)\,dx + \lim_{h \to 0} \int_c^{b-h} f(x)\,dx$

both converges - converges

any 1 is diverges - diverges

both diverges - diverges.

Ex: 1) $\int_0^2 \frac{dx}{2x-x^2}$

$\int \frac{dx}{2x-x^2}$

$\frac{dx}{x(2-x)}$

$\frac{1}{2}\left\{\frac{1}{x} + \frac{1}{2-x}\right\}$

$= \int_0^1 \frac{dx}{2x-x^2} + \int_1^2 \frac{dx}{2x-x^2}.$

$= \lim_{h \to 0}\left[\int_{0+h}^{1} \frac{dx}{2x-x^2} + \int_1^{2-h} \frac{dx}{2x-x^2}\right].$

$= \frac{1}{2}\left[\frac{1}{x} + \frac{1}{2-x}\right]_{0+h}^{1} + \frac{1}{2}\left[\frac{1}{x} + \frac{1}{2-x}\right]_1^{2-h}.$

$= \lim_{h \to 0} \frac{1}{2}\left(1 + \frac{1}{2-1} - \frac{1}{h} - \frac{1}{2-h}\right) + \frac{1}{2}\left(\frac{1}{2-h} + \frac{1}{2-2+h} - \frac{1}{1} - \frac{1}{1}\right)$

$= \lim_{h \to 0} \frac{1}{2}\left(2 - \frac{1}{h} - \frac{1}{2-h}\right) + \frac{1}{2}\left(\frac{1}{2-h} + \frac{1}{h} - 2\right)$

= diverging.

iv. $f(x) \to \infty$ at $x = c$ $\quad a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{h\to 0}\left[\int_a^{c-h} f(x)\,dx + \int_{c+h}^b f(x)\,dx\right]$$

same method. both converge - converge
otherwise diverge.

Ex: $\int_{-1}^{1} \frac{1}{x^3}\,dx$

$$= \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$$

$$= \lim_{h\to 0}\left[\int_{-1}^{0-h} \frac{1}{x^3}\,dx + \int_{0+h}^{1} \frac{1}{x^3}\,dx\right]$$

$$= \lim_{h\to 0}\left[\frac{x^{-2}}{-2}\right]_{-1}^{0-h} + \lim_{h\to 0}\left[\frac{x^{-2}}{-2}\right]_{0+h}^{1}$$

$$= \lim_{h\to 0}\left(\frac{-2}{h^2} - \left(\frac{-2}{1}\right)\right) + \ldots$$

$$= \lim_{h\to 0} \frac{-2}{h^2} + 2 \rightarrow \text{diverges}$$

* $\int_a^b \frac{1}{(x-a)^n}\,dx$ and $\int_a^b \frac{1}{(b-x)^n}\,dx$

i) $\int_a^b \frac{1}{(x-a)^n}\,dx$

$$= \lim_{h\to 0} \int_{a+h}^{b} (x-a)^{-n}\,dx$$

$$= \lim_{h\to 0}\left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+h}^{b} \qquad n \neq 1$$

$$= \lim_{h\to 0}\left[\frac{(b-a)^{1-n} - h^{1-n}}{1-n}\right]$$

$$= \begin{cases} \text{finite} & \text{if } 1-n > 0 \quad n < 1 \\ \infty & \text{if } 1-n < 0 \quad n > 1 \end{cases}$$

At $n = 1$,

$$\int_a^b \frac{1}{x-a}\,dx = [\ln(x-a)]_a^b \rightarrow \text{diverging.}$$

$$\therefore \boxed{\left.\begin{array}{l} \int_a^b \frac{1}{(x-a)^n}\,dx \\ \int_a^b \frac{1}{(b-x)^n}\,dx \end{array}\right] \begin{array}{l} \text{converges if } n<1 \\ \text{diverges if } n \geq 1 \end{array}}$$

• Let there be two functions $f(x)$ and $g(x)$ such that $0 \leq f(x) \leq g(x)$ for $a \leq x \leq b$ then

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)\,dx$ converges

(ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)\,dx$ ~~conv~~ diverges.

Ex: 1) $\int_1^3 \frac{dx}{\sqrt{-1+x^3}}$

$$= \int_1^3 \frac{1}{\sqrt{x^3-1}} = \frac{1}{\sqrt{(x-1)(x^2+x+1)}} = \frac{1}{\sqrt{x-1}\,\sqrt{x^2+x+1}} \leq \frac{1}{\sqrt{x-1}}$$

$$= \frac{1}{(x-1)^{1/2}}$$

$\int_a^b \frac{1}{(x-a)^n}\,dx$ converges if $n<1$

$\therefore \int_1^3 \frac{1}{\sqrt{x-1}}\,dx$ converges.

~~or~~ $\int_1^3 \frac{dx}{\sqrt{x^3-1}}$ also converges.

* Discuss the convergence of $\int_0^{\pi/2} \frac{\sin^m x}{x^n}\,dx$

$$\int_0^{\pi/2} \frac{\sin^m x}{x^n} = \begin{cases} \text{converges} & \text{if } m \geq n \\ \text{diverges} & \text{if } m<n \end{cases}$$

When $m<n$,

$$\frac{\sin^m x}{x^n} = \frac{\sin^m x}{x^m \cdot x^{n-m}} = \left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}$$

$$\leq \frac{1}{x^{n-m}} \rightarrow \text{converges if } n-m<1$$

$\int_a f(x)\,dx$ where $f(x) \geq 0$
$g(x) \geq 0$

At a,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = l \quad , \; 0 < l < \infty$$

Then if limit exists, $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converges or diverges together.

At b,

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = l \quad , \; 0 < l < \infty$$

Then if limit exists, $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converges or diverges together.

Now,

$$\int_0^{\pi/2} \frac{\sin^m x}{x^n} \quad \text{where } n > m$$

$$= \int_0^{\pi/2} \left(\frac{\sin x}{x}\right)^m \frac{1}{x^{n-m}}\,dx$$

Let $g(x) = \frac{1}{x^{n-m}}$

At $x = 0$, $\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)^m = 1$

So, $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converges or diverges together.

Now, checking $\int_0^{\pi/2} \frac{1}{x^{n-m}}\,dx = \int_0^{\pi/2} \frac{1}{(x-0)^{n-m}}\,dx$ → converges if $n-m < 1$
diverges if $n-m \geq 1$

Q. $\int_2^3 \frac{3+x^2}{9-x^2}\,dx$

Sol: Problem is when $x = 3$.

$$f(x) = \frac{3+x^2}{9-x^2} = \frac{3+x^2}{(3+x)(3-x)}$$

Let $g(x) = \frac{1}{3-x}$

$$\lim_{x \to 3^-} \frac{3+x^2}{3+x} = \frac{12}{6} = 2$$

$\therefore \int_2^3 \frac{1}{(3-x)} dx \rightarrow$ so it is diverging.

$\therefore \int_2^3 \frac{3+x^2}{9-x^2} dx$ will also diverge.

Q. $\int_1^2 \frac{\sqrt{x}}{\log x} dx$

Sol: Problem when $x=1$

$f(x) = \frac{\sqrt{x}}{\log x}$

① $g(x) = \frac{1}{x \log x}$

$\lim_{x \to 1^+} = \frac{\sqrt{x}}{\log x} \cdot \frac{x \log x}{1} = x^{3/2} = 1$

$\int_1^2 \frac{1}{x \log x} dx \qquad \log x = t$

$\qquad \frac{1}{x} dx = dt$

$\int_1^2 \frac{dt}{t} = \cancel{\ln t \, 2} \cdot \ln t$

$= [\log(\log x)]_1^2 = \cancel{\log}$

$\rightarrow$ diverging.

So, $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ is diverging.

② $g(x) = \frac{1}{x-1}$ (LH)

$\lim_{x \to 1^+} \frac{\sqrt{x}(x-1)}{\log x} = \frac{\frac{3}{2} x^{1/2} - \frac{1}{2} \frac{1}{\sqrt{x}}}{1/x} = \left(\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}\right) x$

$= \frac{3}{2} x^{3/2} - \frac{1}{2} x^{1/2}$

$= 1$

$= \int_1^2 \frac{1}{x-1} dx$

$\rightarrow$ diverging.

So, $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ is diverging.

So, $g(x)$ can be more than one.

* If $l = 0$ or $l = \infty$

$\lim_{x \to a^+} \frac{f(x)}{g(x)}$ or $\lim_{x \to b^-} \frac{f(x)}{g(x)} = 0$ then $\int_a^b f(x) dx$ converges if $\int_a^b g(x) dx$ converges.

$\lim_{x \to a^+} \frac{f(x)}{g(x)}$ or $\lim_{x \to b^-} \frac{f(x)}{g(x)} = \infty$ then $\int_a^b f(x)\,dx$ diverges if $\int_a^b g(x)\,dx$ diverges.

If $l = \infty$ and $\int_a^b g(x)\,dx$ converges then it is of no use.
and vice-versa for $l = 0$.

Q. $\int_0^2 \frac{\log x}{\sqrt{x}}\,dx$

$\frac{\log x}{\sqrt{x}}$ is <0 in $\int_0^1$.

So, $f(x) = -\frac{\log x}{\sqrt{x}}$

let $g(x) = \frac{1}{x^{3/4}}$

$\lim_{x \to 0^+} \frac{-\log x \; x^{3/4}}{x^{1/2}} = -\log x \cdot x^{1/2} = 0 = l$

$\int_0^2 \frac{1}{x^{3/4}}\,dx \to$ converges.

So, $\int_0^2 \frac{\log x}{\sqrt{x}}\,dx$ converges.

Q. $\int_0^{\pi/2} \log(\sin x)\,dx$

- Properties of definite integrals can be used only when the improper integral converges.

Q. Discuss the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m, n \in$ Real

Sol: $\underbrace{\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx}_{I_1} + \underbrace{\int_{1/2}^1 x^{m-1}(1-x)^{n-1}\,dx}_{I_2}$.

$I_1 = \int_0^{1/2} \frac{(1-x)^{n-1}}{x^{1-m}}\,dx = f(x)$

let $g(x) = \frac{1}{x^{1-m}}\,dx$

$\int_0^{1/2} \frac{1}{x^{1-m}}\,dx \quad \lim_{x \to 0^+} \frac{(1-x)^{n-1}}{x^{1-m}}(x^{1-m}) = 1$

= converges if $1-m < 1$

or $m > 0$

$\therefore I_1$ converges if $m > 0$ and $\forall\, n \in$ Real

$$I_2 = \int_{1/2}^{1} \frac{x^{m-1}}{(1-x)^{1-n}}\, dx.$$

$$f(x) = \frac{x^{m-1}}{(1-x)^{1-n}} \quad \text{and} \quad g(x) = \frac{1}{(1-x)^{1-n}}.$$

$$\lim_{x \to 1^-} x^{m-1} = 1$$

$$\frac{1}{x(1-x)} = \frac{1-x+x}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$\int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}\, dx$ converges if $1-n < 1$

or $n > 0$

$\therefore I_2$ converges if $n > 0$ and $\forall\, m \in$ Real

$\therefore I_1$ and I_2 together converge when $m, n > 0$ $\quad \ln x - \ln(1-x)$

$\therefore$ $\boxed{\int_0^1 x^{m-1}(1-x)^{n-1}\, dx \text{ converges if } m, n > 0}$ $\quad \left[\ln\left(\frac{x}{1-x}\right)\right]_0^1$

Beta function.

$$\therefore \boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad m, n > 0}$$

Q. Discuss the convergence of $\int_0^{\infty} \frac{x^{n-1}}{1+x}\, dx$

Sol: when $n-1 < 0$, then 0 creates a problem and ∞ creates problem at both num and denom.

$$\therefore \underbrace{\int_0^1 \frac{x^{n-1}}{1+x}\, dx}_{I_1} + \underbrace{\int_1^{\infty} \frac{x^{n-1}}{1+x}\, dx}_{I_2}$$

$$I_1 = \int_0^1 \frac{x^{n-1}}{1+x}\, dx$$

$$f(x) = \frac{x^{n-1}}{1+x} = \frac{1}{x^{1-n}(1+x)}$$

$$\text{and} \quad g(x) = \frac{1}{x^{1-n}}$$

$$\lim_{x \to 0^+} \frac{1}{1+x} = 1$$

$\int_0^1 \frac{1}{x^{1-n}}\, dx$ = Converges if $n > 0$

$\therefore I_1$ converges if $n>0$

$$I_2 = \int_1^{\infty} \frac{x^{n-1}}{1+x}\,dx$$

$$f(x) = \frac{x^{n-1}}{1+x} = \frac{1}{x^{1-n}(1+x)} = \frac{1}{x^{2-n}\left(\frac{1}{x}+1\right)}$$

$$g(x) = \frac{1}{x^{2-n}}$$

$$\lim_{x\to\infty} \frac{1}{\frac{1}{x}+1} = 1$$

$\int_1^{\infty} \frac{1}{x^{2-n}}\,dx$ converges if $2-n>1$

$n<1$

$\therefore I_2$ converges if $n<1$

$\therefore I$ converges if $n \in (0,1)$

$\therefore \int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx$ converges if $0<n<1$

* $$\boxed{\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \quad 0<n<1}$$ Proof from Residue Theorem using complex numbers

Q. Discuss the convergence of $\int e^{-x}x^{n-1}\,dx$ $n \in$ Real

Sol: $\int_0^1 e^{-x}x^{n-1}\,dx + \int_1^{\infty} e^{-x}x^{n-1}\,dx$.

$$I_1 = \int_0^1 e^{-x}x^{n-1}\,dx$$

$f(x) = \frac{e^{-x}}{x^{1-n}}$ and $g(x) = \frac{1}{x^{1-n}}$

$$\lim_{x\to 0^+} e^{-x} = 1$$

$\int \frac{1}{x^{1-n}}\,dx$ converges if $n>0$. So, I_1 converges if $n>0$

$$I_2 = \int_1^{\infty} e^{-x}x^{n-1}\,dx$$

$$f(x) = e^{-x}x^{n-1}$$

Let $g(x) = \frac{1}{x^2}$

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = e^{-x} x^{n-1} \cdot x^2 = \frac{x^{n+1}}{e^x} = 0$

$\int_1^\infty \frac{1}{x^2} dx \Rightarrow$ converges.

So, I_2 converges $\forall\, n \in$ Real

$\therefore$ I converges for $n > 0$

* $\therefore \int_0^\infty e^{-x} x^{n-1} dx$ converges for $n > 0$.

* $\boxed{\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n \quad n > 0}$
  ↳ Gamma function

- For beta function,

(i) $\beta(m,n) = \beta(n,m)$ (Taking $x = 1-t$)

(ii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ (Taking $x = \sin^2\theta$)

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ (Taking $x = \frac{1}{1+t}$)

$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ (Taking $x = \frac{t}{1+t}$)

(iv) $\beta(1-n, n) = \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$

- Gamma function:

$\Gamma 1 = \int_0^\infty e^{-x} dx = 1$

$\Gamma n+1 = \int_0^\infty e^{-x} x^n = \left[-e^{-x} x^n\right]_0^\infty + \int_0^\infty e^{-x} n x^{n-1} dx$

$= n\int_0^\infty e^{-x} x^{n-1} dx = n\Gamma n$

$\therefore \boxed{\Gamma n+1 = n\Gamma n}$

Ex: $\Gamma \frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

Ex: i) $\Gamma 5 = 4\Gamma 4 = 4\cdot 3\Gamma 3 = 4\cdot 3\cdot 2\Gamma 2 = 4\cdot 3\cdot 2\cdot 1\Gamma 1$
$= 4!$

$\therefore \boxed{\Gamma n+1 = n!}$

- $$\boxed{\int_0^\infty e^{-zx} x^{n-1} dx = \frac{\Gamma n}{z^n}}$$

Proof: $zx = t$, $dx = \frac{dt}{z}$

$$= \int_0^\infty e^{-t} \left(\frac{t}{z}\right)^{n-1} \frac{dt}{z} = \frac{1}{z^n}\int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{1}{z^n}\int_0^\infty e^{-x} x^{n-1} dt$$

$$= \frac{\Gamma n}{z^n}$$

Relation between β and γ functions:

- $$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Proof:

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma n = \int_0^\infty e^{-zx} x^{n-1} z^n dx$$

and $\Gamma n \cdot e^{-z} \cdot z^{m-1} = e^{-z} z^{m-1} \int_0^\infty e^{-zx} \cdot x^{n-1} z^n dx$

$$= \int_0^\infty e^{-z(1+x)} x^{n-1} \cdot z^{m+n-1} dx$$

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty\left(\int_0^\infty e^{-z(1+x)} \cdot x^{n-1} \cdot z^{m+n-1} dx\right)dz$$

$$\Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left(\underbrace{\int_0^\infty e^{-z(1+x)} \cdot z^{m+n-1} dz}_{\text{by } \gamma \text{ function,}}\right)dx$$ } multiple integration

$$= \int_0^\infty x^{n-1} \frac{\Gamma m+n}{(1+x)^{m+n}} dx$$

or $\dfrac{\Gamma n\,\Gamma m}{\Gamma n+m} = \displaystyle\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

$= \beta(m,n)$

and we know that, $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta.$

let $2m-1 = p$ and $2n-1 = q$.

$\therefore \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n} = 2\displaystyle\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta$

or $\boxed{\dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \displaystyle\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta}$

Ex: $\displaystyle\int_0^{\pi/2} \sin^5\theta\cos^9\theta\,d\theta.$

$= \dfrac{\Gamma\frac{5+1}{2}\,\Gamma\frac{9+1}{2}}{2\,\Gamma\frac{5+9+2}{2}} = \dfrac{\Gamma\frac{6}{2}\,\Gamma 5}{2\,\Gamma 8} = \dfrac{2!\,4!}{2!\,7!} = \dfrac{4!}{7!}$

2) $\displaystyle\int_0^{\pi/2} \sin^{10}\theta\,d\theta$

$= \displaystyle\int_0^{\pi/2} \sin^{10}\theta\cos^0\theta\,d\theta.$

$= \dfrac{\Gamma\frac{10+1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{10+0+2}{2}} = \dfrac{\Gamma\frac{11}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 6} = \dfrac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 6}$

Computing $\Gamma\frac{1}{2}$:

Put $p=0$ $q=0$.

$\dfrac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\cancel{2}\,\Gamma 1} = \displaystyle\int_0^{\pi/2} d\theta = \frac{\pi}{\cancel{2}}.$

or $\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$

Q. $\int_0^\infty e^{-x^2} dx$

Sol: $x^2 = t \qquad 2x\,dx = dt.$

$$= \int_0^\infty \frac{e^{-t} x^{-1} dt}{2} = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{1/2-1} dt \qquad n = \frac{1}{2}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Q. $\int_0^\infty a^{-bx^2} dx$

Sol: let $a^{-bx^2} = e^{-t}.$

$$\cancel{-}bx^2 \ln a = \cancel{-}t$$

$$\text{or } x^2 = \frac{t}{b\ln a.} \qquad x = \frac{\sqrt{t}}{\sqrt{b\ln a}.}$$

$$2x\,dx = \frac{dt}{b\ln a.}$$

$$dx = \frac{dt}{2b\ln a\sqrt{t}}\sqrt{b\ln a} = \frac{dt}{2\sqrt{tb\ln a}.}$$

$$= \int_0^\infty \frac{e^{-t}\,dt\,t^{-1/2}}{2\sqrt{b\ln a}} = \frac{1}{2\sqrt{b\ln a}}\int_0^\infty e^{-t} t^{1/2-1} dt$$

$$= \frac{1}{2\sqrt{b\ln a}}(\sqrt{\pi}) = \frac{1}{2}\sqrt{\frac{\pi}{\cancel{2}b\ln a}}$$

- If $m+n=1$,

$$\frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\frac{\Gamma n\,\Gamma 1-n}{1} = \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi} \qquad 0 < n < 1$$

Ex: $\int_0^{\pi/2} \sqrt{\tan x}\, dx$

$$= \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \cancel{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}} = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2}$$

$$= \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{\frac{\pi}{\sin\frac{\pi}{4}}}{2}$$

$$= \frac{\pi}{2\sin\frac{\pi}{4}} = \frac{\pi}{2\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\pi}{\sqrt{2}}$$

- $$\int_0^\infty e^{-zx}x^{n-1}\,dx = \frac{\Gamma n}{z^n}$$

let $z = a+ib$

$$\text{or } \frac{\Gamma n}{(a+ib)^n} = \int_0^\infty e^{-ax}\cdot e^{-ibx}\cdot x^{n-1}\,dx$$

$$= \int_0^\infty e^{-ax}(\cos bx - i\sin bx)\,x^{n-1}\,dx$$

$$= \int_0^\infty e^{-ax}\cos bx\cdot x^{n-1}\,dx - i\int_0^\infty e^{-ax}\sin bx\cdot x^{n-1}\,dx \quad -①$$

$a = r\cos\theta$, $b = r\sin\theta$

$$\text{or } \frac{\Gamma n}{(r\cos\theta + ir\sin\theta)^n} = \frac{\Gamma n}{r^n(\cos\theta + i\sin\theta)^n}$$

$$= \frac{\Gamma n}{r^n(\cos n\theta + i\sin n\theta)}$$

$$= \frac{\Gamma n(\cos n\theta - i\sin n\theta)}{r^n}$$

$$= \frac{\Gamma n}{r^n}\cos n\theta - i\frac{\Gamma n}{r^n}\sin n\theta \quad -②$$

Comparing ① and ②,

$$\int_0^\infty e^{-ax}\cos bx\cdot x^{n-1}\,dx = \frac{\Gamma n}{r^n}\cos n\theta \quad -\mathrm{I}$$

and $$\int_0^\infty e^{-ax}\sin bx\cdot x^{n-1}\,dx = \frac{\Gamma n}{r^n}\sin n\theta \quad -\mathrm{II}.$$

Q. $\int_0^\infty \cos x^2 \, dx$

Sol: From I,

$$\int_0^\infty e^{-ax} \cos bx \cdot x^{n-1} dx = \frac{\Gamma n}{r^n} \cos n\theta \qquad \tan\theta = \frac{b}{a}$$

$$r = \sqrt{a^2+b^2}$$

Put $a = 0$, $b = 1$ then $r = 1$ and $\theta = \frac{\pi}{2}$.

$$\therefore \int_0^\infty \cos x \, x^{n-1} dx = \Gamma n \cos n\frac{\pi}{2}$$

Now, let $x^n = t$

$$n x^{n-1} dx = dt$$

$$= \int_0^\infty \cos t^{1/n} \frac{dt}{n} = \Gamma n \cos \frac{n\pi}{2}$$

$$= \int_0^\infty \cos t^{1/n} dt = n \Gamma n \cos n\frac{\pi}{2}$$

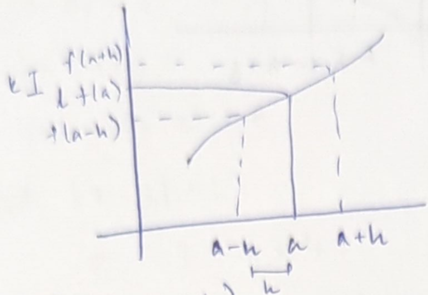
let $1/n = 2$

$$\int_0^\infty \cos t^2 dt = \frac{1}{2} \Gamma\frac{1}{2} \cos\frac{\pi}{4}$$

$$= \frac{1}{2\sqrt{2}} \sqrt{\pi}$$

Functions of Multiple Variables

Functions of Two Variables



$x \in (a-h, a+h)$

$f(x) \in (l-k, l+k)$.

or $|x-a| < h$ and $|f(x)-l| < k$

And $\lim_{x \to a} f(x) = l$,

For $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x)-l| < \epsilon$ and $0 < |x-a| < \delta$

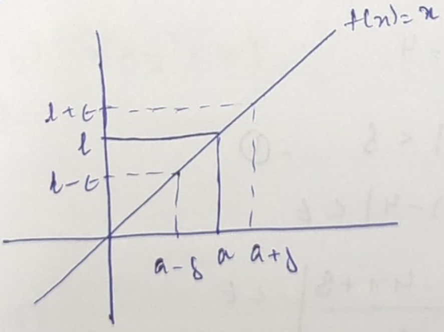
Here $h \approx \delta$ and $k \approx \epsilon$

where ϵ and δ are some constants

Also, $N_\delta(a) = (a-\delta, a+\delta)$

where $N_\delta(a) = \delta$-neighbourhood of a.

Relation between ϵ and δ



Ex: 1) Let $f(x) = x$

$\lim_{x \to a} f(x) = a = l$

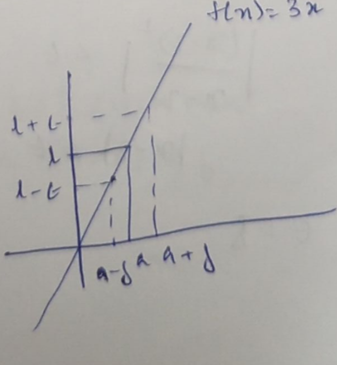
So, $l = a$

Since curve is $y = x$,

$a - \delta = l - \epsilon$

$= a - \epsilon$

or $\delta = \epsilon$ in this function.



2) Let $f(x) = 3x$

$\lim_{x \to a} f(x) = 3a = l$.

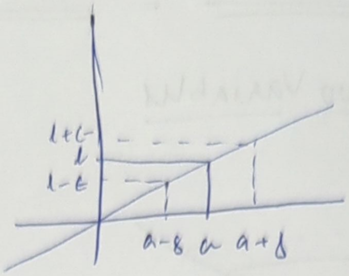
and $l - \epsilon = 3(a-\delta) = 3a - 3\delta$

$= l - 3\delta$

or $\epsilon = 3\delta$ //.

3) Let $f(x) = \frac{x}{3}$

$\lim_{x \to a} f(x) = \frac{a}{3} = l$.



$l - \epsilon = \frac{a - \delta}{3}$

$l - \epsilon = l - \frac{\delta}{3}$

or $\delta = 3\epsilon$ //

4) $f(x) = 2x + 3$

$\lim_{x \to 1} f(x) = 5$

Sol: $0 < |x - 1| < \delta$

$|f(x) - 5| < \epsilon$

$|f(x) - 5| = |2x - 2| = 2|x - 1| < \epsilon$ — ①

and $|x - 1| < \delta$ — ②

$\therefore \forall \epsilon, \exists \delta = \frac{\epsilon}{2}$ such that $|f(x) - 5| < \epsilon$ if $0 < |x - 1| < \delta$

and $\lim_{x \to 1} f(x) = 5$

Q. $f(x) = \frac{x^2 - 4}{x - 2}$

Sol: Let $a = 2$.

$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$

$0 < |x - 2| < \delta$ — ①

and $|f(x) - 4| < \epsilon$

$\left| \frac{x^2 - 4 - 4x + 8}{x - 2} \right| < \epsilon$

$\left| \frac{x^2 - 4x + 4}{x - 2} \right| < \epsilon$

$\left| \frac{(x-2)^2}{x - 2} \right| < \epsilon$

$|x - 2| < \epsilon$ — ②

So, $\epsilon = \delta$

Q. $f(x) = x^2$

Sol: Let $a = 3$.

$\lim_{x \to 3} f(x) = 9$.

$0 < |x-3| < \delta$ — ①

$|x^2 - 9| < \epsilon$

$|x+3||x-3| < \epsilon$

Let $|x-3| < 1$

or $2 < x < 4$

$5 < x+3 < 7$.

$\therefore 7|x-3| < \epsilon$ — ②

From ① and ②, $\delta = \frac{\epsilon}{7}$.

$\therefore \forall \epsilon > 0, \exists \delta = \frac{\epsilon}{7}$ such that $|f(x) - 9| < \epsilon$ if $0 < |x-3| < \delta$

Functions of Two Variables

Domain: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

Range: $\mathbb{R}$.

$\therefore f: D \longrightarrow R = \mathbb{R}^2 \rightarrow \mathbb{R}$

Ex: i) $f(x,y) = \sqrt{4-(x^2+y^2)}$

Hemisphere

Domain: $D = \{(x,y) : x^2+y^2 \leq 4\}$

Range: $R = [0, 2]$

ii) $f(x,y) = \frac{x+y}{x-y}$

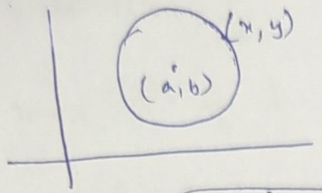
$D = \{(x,y) : x \neq y\}$

$R = \mathbb{R}$

- If $\lim_{x \to a} f(x) = l$

For all $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$



$d = \sqrt{(x-a)^2 + (y-b)^2}$

$N_\delta((a,b)) = \{(x,y) : \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$

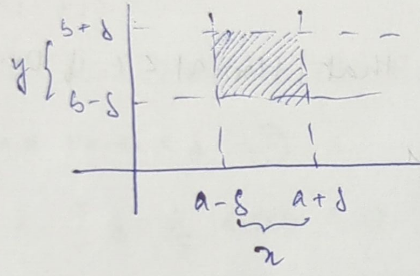
N = Neighbourhood.

Here Nbhd of (a, b) is $< \delta$

Another definition of nbhd:

$N_\delta((a,b)) = \{(x,y) : |x-a| < \delta \text{ and } |y-b| < \delta\}$

Graphically,



· For two variable,

Let there be a function $f(x,y)$

Let $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$

Then $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$|f(x,y) - l| < \epsilon$ whenever $0 < |x-a| < \delta$ and $0 < |y-b| < \delta$

OR

whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$

Ex: 1) $f(x,y) = \sqrt{1-x^2-y^2}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$

2) $\lim\limits_{(x,y)\to(1,3)} (4x+2y) = 10$

Now, $|f(x) - l| = |4x + 2y - 10|$

$= |4x - 4 + 2y - 6|$

$\leq 4|x-1| + 2|y-3|$

$\leq 4\delta + 2\delta$

$\leq 6\delta \leq \epsilon$

or $\delta \leq \frac{\epsilon}{6}$

Q. Show that $\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Sol: $l = 0$.

$\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \epsilon$

$\frac{|xy|}{\sqrt{x^2+y^2}} < \epsilon$

We know that, $(x-y)^2 \geq 0$

$x^2 + y^2 \geq 2xy \geq xy$.

or $\frac{|xy|}{x^2+y^2} \leq 1$

$\therefore \frac{|xy|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}$

$\leq \sqrt{(x-0)^2 + (y-0)^2} < \delta < \epsilon$.

or $\delta \leq \epsilon$

Q. Show that $\lim_{(x,y) \to (0,0)} (x^2+y^2) \sin\left(\frac{1}{x^2+y^2}\right) = 0$

Sol: $\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| \leq |x^2+y^2|$ coz $\sin\theta < 1$

and $\sqrt{(x-0)^2 + (y-0)^2} < \delta$

or $(x-0)^2 + (y-0)^2 < \delta^2$.

$x^2 + y^2 < \delta^2$.

$\therefore \left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq \delta^2 < \epsilon$

or $\delta^2 \leq \epsilon$

In Polar - Coordinate System,

$x = r\cos\theta, \quad y = r\sin\theta$

Ex: $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

$= \frac{r\cos\theta \cdot r\sin\theta}{\sqrt{r^2}} = \frac{r^2 \sin\theta\cos\theta}{r}$

$\lim_{r\to 0} r\sin\theta\cos\theta = 0$ ✓

as x, y come to origin, r becomes 0.

Ex: 2) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$= \frac{r\cos\theta \cdot r\sin\theta}{r^2} = \cos\theta\sin\theta$

$\lim_{r\to 0} = \cos\theta\sin\theta$

that means limit changes with values of θ.
Hence, limit does not exist.

2) $\lim_{(x,y)\to(0,0)} \frac{2x(x^2-y^2)}{x^2+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$= \lim_{r\to 0} \frac{2r\cos\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \lim_{r\to 0} \frac{2r\cos\theta\cos 2\theta \cdot \cancel{r^2}}{\cancel{r^2}}$

$= \lim_{r\to 0} 2r\cos\theta\cos 2\theta$

$= 0$

$\left|\frac{2x(x^2-y^2)}{x^2+y^2} - 0\right| \leq \left|\frac{2x^3}{x^2+y^2}\right| + \left|\frac{2xy^2}{x^2+y^2}\right|$

and $x^2 \leq x^2+y^2$
$y^2 \leq x^2+y^2$

In Polar - Coordinate System,

$x = r\cos\theta, \quad y = r\sin\theta$

Ex: $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

$= \frac{r\cos\theta . r\sin\theta}{\sqrt{r^2}} = \frac{r^{\cancel{2}}\sin\theta\cos\theta}{\cancel{r}}$

$\lim_{r\to 0} r\sin\theta\cos\theta = 0$ ✓

as x, y come to origin, r becomes 0.

Ex: 2) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$= \frac{r\cos\theta . r\sin\theta}{r^2} = \cos\theta\sin\theta$.

$\lim_{r\to 0} = \cos\theta\sin\theta$.

that means limit changes with values of θ.
Hence, limit does not exist.

2) $\lim_{(x,y)\to(0,0)} \frac{2x(x^2-y^2)}{x^2+y^2}$

$x = r\cos\theta \quad y = r\sin\theta$

$= \lim_{r\to 0} \frac{2r\cos\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$.

$= \lim_{r\to 0} \frac{2r\cos\theta\cos 2\theta . \cancel{r^2}}{\cancel{r^2}}$

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