Mid Semester 2022-23
Mathematics 1 (MCI101)

Date: 16.12.2022, Time: 10.00 AM to 12 Noon
Full Marks: 32

Instructions: 1. All questions are compulsory and carry 4 marks each.
2. Each question should be solved only at one place in the answer sheet.

Q-1) Find the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Also, find the maximum error when $0 \le x \le 0.5$. [4]

Q-2) Find the radius of curvature at the point $(a, 0)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [4]

Q-3) Find the asymptotes of the following curve:

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$$

[4]

Q-4) Discuss the convergence of the integral:

$$\int_1^2 \frac{\sqrt{x}}{\log x} dx$$

[4]

Q-5) Using the identity of beta and gamma functions find the value of $\Gamma(2n)$ in terms of $\Gamma\left(n + \frac{1}{2}\right)$ and $\Gamma(n)$. [4]

Q-6) Evaluate the following integral:

$$\int_0^\pi \frac{\sin^{m-1} x}{(2 + \cos x)^m} dx, \qquad m > 0.$$

[4]

Q-7) If $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ $(x \ne 0, y \ne 0)$ and $f(0, y) = f(x, 0) = f(0,0) = 0$, using the definition of partial derivative find $f_{yx}(0,0)$ and $f_{xy}(0,0)$.
[Note: $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$] [4]

Q-8) Examine the continuity of $f(x, y)$ at (0,0) where

$$f(x, y) = \begin{cases} \frac{5x^2y^2}{x^2+y^2} & (x, y) \ne (0,0) \\ 0 & at\ (0,0) \end{cases}$$

[4]

$\Gamma 2n$ in terms of $\Gamma n+\frac{1}{2}$ and Γn. $\sin^{m-1} x$

1. Find the fourth degree Taylor's polynomial approximation to $f(x) = e^{2x}$ about $x = 0$. Also, find the maximum error when $0 \leq x \leq 0.5$. (4 Marks)

Solution: The fourth degree Taylor's polynomial is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0).$$

(1 Mark)

As for $f(x) = e^{2x}$, $f^{(r)}(x) = 2^r e^{2x}$. Thus $f(0) = 1, f'(0) = 2, f''(0) = 2^2, f'''(0) = 2^3, f^{(iv)}(0) = 2^4$. Therefore

$$f(x) = 2 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

(1 Mark)

Further, $R_4(x) = \frac{x^5}{5!}f^{(v)}(c) = \frac{32x^5}{5!}e^{2c}, 0 < c < x$. Thus

$$|R_4(x)| \leq \frac{32}{120}.max\{x^5 : 0 \leq x \leq 0.5\}.max\{e^{2c} : 0 \leq x \leq 0.5\} \leq \frac{e}{120}.$$

(2 Marks)

Sol^n: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$$\rho = \frac{\left(1+(y')^2\right)^{3/2}}{y''}$$

Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow y' = -\frac{b^2}{a^2}\frac{x}{y} \quad \text{--- (1)}$$

$$y'' = -\frac{b^2}{a^2}\left[\frac{y - xy'}{y^2}\right] = \frac{-b^2}{a^4 y^2}\left(a^2y^2 + b^2x^2\right)$$

$$= \frac{-b^2}{a^4y^3}a^2b^2\left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right)$$

$$= \frac{-b^4}{a^2y^3} \quad \text{--- (2)}$$

$$\rho = \frac{\left(1+(y')^2\right)^{3/2}}{y''} = -\frac{\left\{1 + \frac{b^4x^2}{a^4y^2}\right\}^{3/2}}{b^4/a^2y^3}$$

$$= \frac{-\left(a^4y^2 + b^4x^2\right)^{3/2}}{\frac{b^4}{a^2y^3} \times \left(a^4y^2\right)^{3/2}}$$

$$= \frac{-\left(a^4y^2 + b^4x^2\right)^{3/2}}{a^4b^4}$$

$$\therefore \rho \text{ at } (a,0) = -\frac{\left(b^4a^2\right)^{3/2}}{a^4b^4} = -\frac{b^6a^3}{a^4b^4} = -\frac{b^2}{a} \quad \text{--- (4)}$$

Solution Q3

Given curve is

$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$

As the coefficient of x^3 and y^3 is constant. So no Horizontal and vertical asymptotes are present.

Step 1 Let $x = 1$, $y = m$.

so $\phi_3(m) = m^3 - m^2 - m + 1$

$\phi_2(m) = 2 - 4m^2 + 2m$

$\phi_3(m) = 1 + m$ — (1)

Step 2: $\phi_3(m) = 0$,

or $m^3 - m^2 - m + 1 = 0$

$(m^2 - 1)(m - 1) = 0$

$m = 1, 1, -1$

Step 3 for $m = -1$

$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{4m^2 - 2m + 2}{3m^2 - 2m - 1} = \frac{4}{4} = 1$

So Asymptote is $y = -x + 1$

or $x + y = 1$

(2)

for $m = 1$

$\frac{c^2}{2}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$

$\frac{c^2}{2}$

$\frac{c^2}{2}(6m-2) + c(-8m+2) + 1+m = 0$

As $m = 1$

$\frac{c^2}{2}(4) + c(-6) + 2 = 0.$

$c^2 - 3c + 1 = 0$

$c = \frac{3 \pm \sqrt{5}}{2}$

Asymptotes are $y = x + \frac{3+\sqrt{5}}{2}$

$2y - 2x = (3+\sqrt{5})$

and $y = x + \frac{3-\sqrt{5}}{2}$

$2y - 2x = (3-\sqrt{5})$

So Asymptotes are $x + y = 1$

$2(y-x) = 3+\sqrt{5}$

$2(y-x) = 3-\sqrt{5}$

4

4th-Solution :— We have $f(x) = (\sqrt{x}/\ln x) \geq 0$, $1 < x \leq 2$.
The point $x=1$ is the only point of infinite discontinuity.

Let $g(x) = 1/(x \ln x)$. Then, we have

$$\lim_{x \to 1^+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(1+h)}{g(1+h)} = \lim_{h \to 0} \left[\frac{\sqrt{1+h}}{\ln(1+h)}\right] [(1+h)\ln(1+h)]$$

$$= \lim_{h \to 0} (1+h)^{3/2} = 1. \qquad (1)$$

Therefore, both the integrals $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

Now, $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{x \ln x} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^2 \frac{dx}{x \ln x}$

$$= \lim_{\varepsilon \to 0} \left[\ln(\ln x)\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0} \left[\ln(\ln 2) - \ln(\ln(1+\varepsilon))\right] \to \infty. \qquad (1)$$

Since $\int_1^2 g(x)\,dx$ is divergent, the given integer $\int_1^2 f(x)\,dx$ is also divergent by Comparison Test 4. (1)+(1)

(In proper way)

Q. Using the identity of Beta and Gamma functions, find the value of $\Gamma(2n)$ in terms of $\Gamma(n+1/2)$ and $\Gamma(n)$. (4 marks)

1

we know $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{(2n-1)}\theta \, d\theta. \longrightarrow$ 1

set, $m=n$, $\dfrac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{(2n-1)}(\theta)\, d\theta.$

$= \dfrac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{(2n-1)}(2\theta)\, d\theta.$

substitute $2\theta = \pi/2 - \phi$, $d\theta = \frac{-1}{2}d\phi \Rightarrow$

$$\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{-1}{2^{2n-1}}\int_{\phi=+\pi/2}^{-\pi/2} \cos^{2n-1}(\phi)\,d\phi = \frac{1}{2^{2n-1}}\int_{-\pi/2}^{\pi/2} \cos^{2n-1}\phi\, d\phi.$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2} \cos^{2n-1}(\theta)\,d\theta \longrightarrow (*)$$ 1

Again put $m = 1/2 = n$, $\dfrac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} = 2\int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta.$ (**) 1

from $\longrightarrow$ (*) & (**) $\Rightarrow$

$$\frac{(\Gamma(n))^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\left[\frac{\Gamma(n)\,\Gamma(1/2)}{\Gamma(n+1/2)}\right]. \longrightarrow 1$$

$$\Rightarrow \Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\,\Gamma(n+1/2) \quad \because \Gamma(1/2) = \sqrt{\pi}.$$

X

Teacher's Signature:

Question 6: Evaluate the following integral:

$$\int_0^{\pi} \frac{\sin^{m-1} x}{(2+\cos x)^m}\, dx, \quad m > 0. \qquad [4]$$

Solution:

$$\begin{aligned}
&\int_0^{\pi} \frac{\sin^{m-1} x}{(2+\cos x)^m}\, dx \\
&[\text{put} \quad 1+\cos x = y \implies -\sin x\, dx = dy] \\
=&\int_0^{2} \frac{(\sin^2 x)^{\frac{m-2}{2}}}{(1+y)^m}\, dy \\
=&\int_0^{2} \frac{\left(1-(y-1)^2\right)^{m/2-1}}{(1+y)^m}\, dy \\
=&\int_0^{2} \frac{y^{m/2-1}(2-y)^{m/2-1}}{(1+y)^m}\, dy \ldots\ldots\ldots\ldots \textbf{(2 Marks)} \\
&\left[\text{put} \quad z = y/(2-y) \implies dz = \frac{2}{(2-y)^2}\, dy, \quad y = \frac{2z}{z+1}\right] \\
=&\int_0^{\infty} \frac{y^{m/2-1}(2-y)^{m/2-1}}{(1+y)^m} \frac{(2-y)^2}{2}\, dz \\
=&\frac{1}{2}\int_0^{\infty} \frac{\left(\frac{2z}{z+1}\right)^{m/2-1}(2-\frac{2z}{z+1})^{m/2+1}}{(1+\frac{2z}{z+1})^m}\, dz \\
=&\frac{1}{2}\int_0^{\infty} \frac{(2z)^{m/2-1}2^{m/2+1}}{(1+3z)^m}\, dz \\
=&2^{m-1}\int_0^{\infty} \frac{z^{m/2-1}}{(1+3z)^m}\, dz \\
=&\frac{2^{m-1}}{3^{m/2}}\int_0^{\infty} \frac{(3z)^{m/2-1}}{(1+3z)^m}\, d(3z) \\
=&\frac{2^{m-1}}{3^{m/2}} \quad \beta(m/2, m/2) \ldots\ldots\ldots\ldots \textbf{(2 Marks)}
\end{aligned}$$

(8) $f(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} \quad (x \neq 0, y \neq 0)$

$f(0,y) = f(x,0) = f(0,0) = 0.$

Sol. $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$\left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right|_{(0,0)} = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

We now find $f_x(0,k)$ and $f_x(0,0)$

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{\left(h^2 \tan^{-1}\frac{k}{h} - k^2 \tan^{-1}\frac{h}{k}\right)}{h}$$

$$= \lim_{h \to 0} h \tan^{-1}\frac{k}{h} + \lim_{h \to 0} (-k) \frac{\tan^{-1}(h/k)}{(h/k)}$$

$$= 0 - k = -k$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

Thus $f_{yx}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1$

(2 Marks)

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Bigg|_{(0,0)} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

we find $f_y(h,0)$ and $f_y(0,0)$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{\left(h^2 \tan^{-1}\frac{k}{h} - k^2 \tan^{-1}\frac{h}{k}\right)}{k}$$

$$= \lim_{k\to 0} h \frac{(\tan^{-1} k/h)}{(k/h)} - \lim_{k\to 0} k \tan^{-1} h/k$$

$$= h - 0 = h$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Bigg|_{(0,0)} = f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

(2 Marks)

Qn. Examine the continuity of $f(x,y)$ at $(0,0)$

where $f(x,y) = \begin{cases} \dfrac{5x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$

Sol.*

given $f(0,0) = 0$

for continuity of $f(x,y)$ at $(0,0)$ we have to show

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

or for $\epsilon > 0 \ \exists\, \delta > 0$ st $|f(x,y) - 0| < \epsilon$ for (x,y) in δ-nbd of $(0,0)$

$$|f(x,y) - 0| = \left| \frac{5x^2y^2}{x^2+y^2} - 0 \right|$$

$$= 5|x||y| \frac{|xy|}{x^2+y^2}$$

$$\leq 5|x||y| \qquad \left\{ \frac{|xy|}{x^2+y^2} \leq 1 \right.$$

$$\leq 5\sqrt{x^2+y^2}\sqrt{x^2+y^2} \qquad \left\{ |x| \leq \sqrt{x^2+y^2} \right.$$

$$< 5\delta^2 \qquad \left\{ \sqrt{x^2+y^2} < \delta \right.$$

$$= \epsilon$$

So, $|f(x,y) - 0| < \epsilon$ for all (x,y) in δ-nbd of $(0,0)$ where $\delta = \sqrt{\epsilon/5}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$ and hence f is continuous at $(0,0)$

* One of the possible solutions.