Department of Mathematics and Computing
Mathematics I
Tutorial Sheet-II
(Improper Integrals, Beta and Gamma Functions)

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1. Evaluate, when possible, the following integrals:

(i) $\displaystyle\int_0^\infty \frac{x}{x^2+4}\,\mathrm{d}x$ (ii) $\displaystyle\int_1^\infty \frac{dx}{x(1+x)}$ (iii) $\displaystyle\int_{-\infty}^\infty \frac{x}{x^4+1}\,\mathrm{d}x$ (iv) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, a,b>0$
(v) $\displaystyle\int_0^\infty \frac{x}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x, a,b>0$ (vi) $\displaystyle\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}$, where n is an integer greater than 1.

Ans: (i) Diverges (ii) $-\log\left(\frac{1}{2}\right)$ (iii) 0 (iv) $\dfrac{\pi}{2ab(a+b)}$ (v) $\dfrac{1}{(a^2-b^2)}\log\left(\frac{a}{b}\right)$ (vi) $\dfrac{n}{n^2-1}$.

2. Examine the convergence of following integrals:

(i) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ (ii) $\displaystyle\int_1^\infty \frac{\log x}{x^2+1}\,\mathrm{d}x$ (iii) $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x$ (iv) $\displaystyle\int_0^\infty \frac{x^{3/2}}{3x^2+5}\,\mathrm{d}x$
(v) $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$ (vi) $\displaystyle\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (vii) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ (viii) $\displaystyle\int_1^\infty \frac{x^{m-1}}{x+1}\,\mathrm{d}x$
(ix) $\displaystyle\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,\mathrm{d}x$

Ans: (i) Converges (ii) Converges (iii) Converges (iv) Diverges (v) Diverges (vi) Converges (vii) Diverges (ix) Converges.

3. Evaluate, when possible, the following integrals:

(i) $\displaystyle\int_0^\pi \frac{dx}{1+\cos x}$ (ii) $\displaystyle\int_{-1}^1 \frac{dx}{x^3}$ (iii) $\displaystyle\int_0^\pi \frac{\sin x}{\cos^2 x}\,\mathrm{d}x$ (iv) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^3}$ (v) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^p}\,\mathrm{d}x.$

Ans: (i) Diverges (ii) Diverges (iii) Diverges (iv) Diverges (v) Converges, if $p<2$.

4. Examine the convergence of following integrals:

(i) $\displaystyle\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}\,\mathrm{d}x$ (iii) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}\,\mathrm{d}x$ (iv) $\displaystyle\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}\,\mathrm{d}x$
(v) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\,\mathrm{d}x$ (vi) $\displaystyle\int_0^1 \frac{x^{m-1}}{1+x}\,\mathrm{d}x$ (vii) $\displaystyle\int_0^\pi \frac{dx}{\sqrt{\sin x}}$ (viii) $\displaystyle\int_0^1 x^{n-1}\log x\,\mathrm{d}x$
(ix) $\displaystyle\int_1^\infty \frac{dx}{x\log x}$ x $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x.$

Ans: (i) Converges (ii) Converges (iii) Diverges (iv) Diverges (v) Converges (vi) Converges, if $m>0$ (vii) Converges (viii) Converges, if $n>0$ (ix) Diverges x Converges.

5. Discuss the convergence of $\displaystyle\int_0^1 \log(\Gamma x)dx$.

6. Show that $\displaystyle\int_0^{\pi/2} \log\sin x dx$ converges and hence evaluate it.

7. Using substitution $x=e^{-n}$, show that $\displaystyle\int_0^1 x^{m-1}(\log x)^n dx$ converges for $m>0, n>-1$.

8. Express the following integrals in terms of Gamma function:

(i) $\displaystyle\int_0^\infty e^{-k^2x^2}dx$ (ii) $\displaystyle\int_0^\infty x^{p-1}e^{-kx}dx, (k>0)$ (iii) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,\mathrm{d}x, (c>1)$ (iv) $\displaystyle\int_0^1 \left(\log\left(\frac{1}{y}\right)\right)^{n-1} dy.$

Ans: (i) $\dfrac{1}{2k}\Gamma\left(\dfrac{1}{2}\right)$ (ii) $\dfrac{\Gamma(p)}{k^p}$ (iii) $\dfrac{\Gamma(c+1)}{(\log(c))^{c+1}}$ (iv) $\Gamma(n)$.

9. Show that

(i) $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$ (ii) $\int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\sec\theta})\,d\theta = \frac{1}{2}\left[\Gamma\left(\frac{1}{4}\right)\left(\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma(3/4)}\right)\right].$

10. Show that $\int_0^1 x^m (\log x)^n dx = (-1)^n \frac{n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

11. Show that

(i) $\int_0^1 \frac{x}{\sqrt{1-x^5}}\,\mathrm{d}x = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\Gamma(1/4)}{4\Gamma(3/4)}.$

12. Show that

(i) $\int_0^1 \frac{\sin^{2m-1}\theta \cdot \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,\mathrm{d}\theta = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)}$ (ii) $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,\mathrm{d}x$

(iii) $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m,n)$ (iv) $\beta(n,n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}\Gamma(n+1/2)}.$

13. Show that for $n > -1$, $m < 1$, $\frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!(n+3)} + \frac{m(m+1)(m+2)}{3!(n+4)} + \ldots.. = \beta(n+1, 1-m).$