Department of Mathematics and Computing

Mathematics I

Tutorial Sheet-I

(Taylor's Series, Convexity and Concavity, Asymptotes, Curvature, Curve tracing)

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1. Find the first four non-zero terms of the Taylor series generated by function $f(x)$.

   (i) $f(x) = \sqrt{3+x^2}$ at $x = -1$ (ii) $f(x) = \frac{1}{1-x}$ at $x = 2$ (iii) $f(x) = \dfrac{1}{1+x}$ at $x = 3$

   (iv) $f(x) = \dfrac{1}{x}$ at $x = a > 0$ (v) $f(x) = \dfrac{1}{1+x^2}$ at $x = -2$ (vi) $f(x) = \sin x^2$ at $x = 1$

   (vii) $f(x) = \log(1-x^2)$ at $x = 2$ (viii) $f(x) = e^{-2x}$ at $x = \dfrac{1}{2}$ (ix) $f(x) = \cosh x$ at $x = 1$

   (x) $f(x) = \tan x$ at $x = 1.$

2. Find the Maclaurin series for the following functions
   (i) $f(x) = \dfrac{1}{1-2x}$ (ii) $f(x) = \dfrac{1}{1+x^3}$ (iii) $f(x) = \sin \pi x$ (iv) $f(x) = \sin \dfrac{2x}{3}$
   (v) $f(x) = \cos(x^{5/2})$ (vi) $f(x) = \cos\sqrt{5x}$ (vii) $f(x) = e^{(\pi x/2)}$ (viii) $f(x) = e^{-x^2}.$
   (ix) $f(x) = \log(1+x)$ (x) $f(x) = \dfrac{1}{1+x^2}$ (xi) $f(x) = \sinh x$

3. (i) Calculate e with an error of 10^{-6}.
   (ii) For what values of x can we replace sin x by $x - \left(\frac{x^3}{3!}\right)$ with an error of magnitude no greater than 3×10^{-4}?
   (iii) For approximately what values of x can you replace sin x by $x - \left(\frac{x^3}{6}\right)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.
   (iv) How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$?
   (v) The estimate $\sqrt{1+x} = 1 + \left(\frac{x}{2}\right)$ is used when x is small. Estimate the error when $|x| < 0.01$.
   (vi) The approximation $e^x = 1 + x + \left(\frac{x^2}{2}\right)$ when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.
   (vii) Estimate the error in the approximation $\sinh x = x + \left(\frac{x^3}{3!}\right)$ when $|x| < 0.5$. (Hint: R_4 not R_3).
   (viii) When $0 \le h \le 0.01$, show that e^h may be replaced by $1 + h$ with an error of magnitude no greater than 0.6% of h. Use $e^{0.01} = 1.01$.
   (ix) For what values of x can you replace $\ln(1+x)$ by x with an error of magnitude no greater than 1% of the value of x?

4. Find the point of inflection and the intervals in which the given curves are concave upward and concave downward.
   (i) $y = x^3 - 3x^2 + 6x + 5$ (ii) $y = \dfrac{1}{x-3}$ (iii) $y = x^4 - x^3$
   (iv) $y = \cot^{-1} x + x$ (v) $y = x^3 \ln(x), x \gtrsim 0$ (vi) $y = (1+x^2)e^x.$

5. Find the curvature and radius of curvature of the following curves at the indicated points. The constant a is positive.
   (i) $x = a\,(t - \sin t),\ y = a\,(1 - \cos t)$ at $t = \pi$ (ii) $y = a\cosh(x/a)$ at $(0, a)$
   (iii) $y = x^2 + \ln\,(x + \sqrt{1+x^2})$ at $(0,\ 0)$ (iv) $x = a\ln\,(\sec t + \tan t), y = a\sec t$ at $t = 0.$

6. Determine the curvature of the parabola $y^2 = 2px$:
   (a) at an arbitrary point $M(x, y)$;
   (b) at the point $M_1(0, 0)$;
   (c) at the point $M_2(\frac{p}{2}, p)$.

7. Find the equation of the envelope of the given family of the curves (p is a parameter).
   (i) $y = px + 3/(2p)$ (ii) $(x-p)^2 + (y-p)^2 = p^2$ (iii) $x \tan p + y \sec p = 5.$

8. Find the envelope of all the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which have a constant area $A = \pi ab$.

9. Find all the asymptotes to the given curves.
(i) $y = e^{2/x} - 1$ (ii) $y = \frac{x+1}{\sqrt{x^2-4}}$
(iii) $(y-2)(x^2-1) = 5$ (iv) The *hyperbolic spiral* $r = a/\theta$
(v) $y = \dfrac{x-4}{x^2+4x+3}$ (vi) $x^5 + y^5 = 5ax^2y^2$.

10. Sketch the graph of the following curves.
(i) $y = x/(1+x^2)$ (ii) $y^2x = a^2\,(a-y)$ (iii) $y = \frac{(x-1)(x-3)}{x^2}$
(iv) $r = a\,(1+\cos\theta)$, $a > 0$ (v) $y = x^4 - 6x^2$ (vi) $y^2 = (x-1)(x-2)^2$, (vii) $y = x^5 - 5x^4$.
(viii) $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$ as t varies from $-\pi$ to π
(ix) $r = ae^{\theta \cot \alpha}$ (x) $r = a \sin 3\theta, a > 0$ (xi) $y = \frac{5(x-2)(x+1)}{x^2+2x+4}$.