Monsoon, 2022-23

Set #1 (MC-101)

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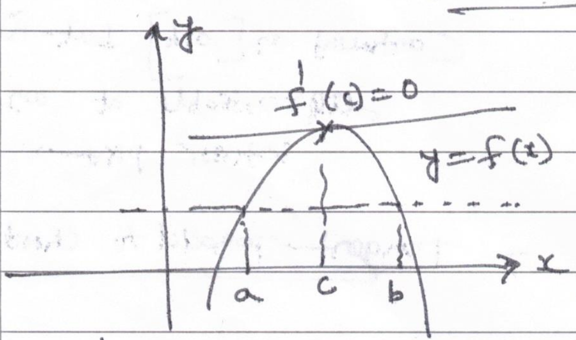
Taylor's Theorem: If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number $c \in (a,b) \ni$

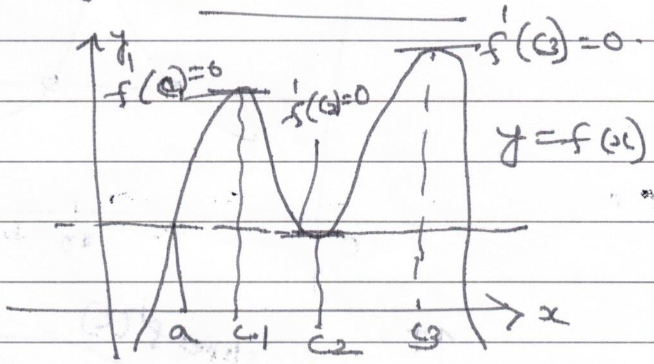
$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{n}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}}{(n+1)!}(b-a)^{n+1}.$$

Replace by x and a by 0,

Maclaurin series

It is a generalization of the Mean value Theorem



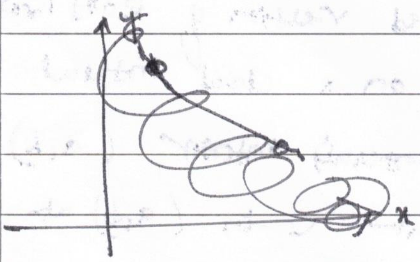


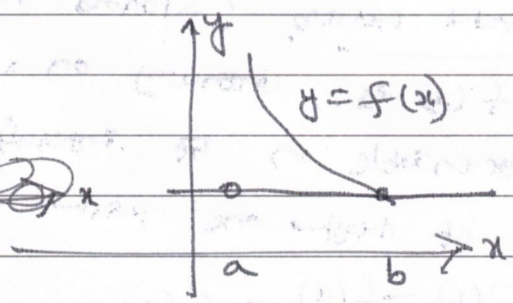
Rolle's theorem:

Suppose that $y=f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of its interior (a,b) if $f(a)=f(b)$, then there is at least one number $c \in (a,b) \ni f'(c)=0$.

Rolle's theorem says that a differentiable curve has atleast one horizontal tangent between any two points where it crosses a horizontal line.

No Horizontal tangent





$\longrightarrow$ (a).

Discontinuity at an endpoint of $[a,b]$

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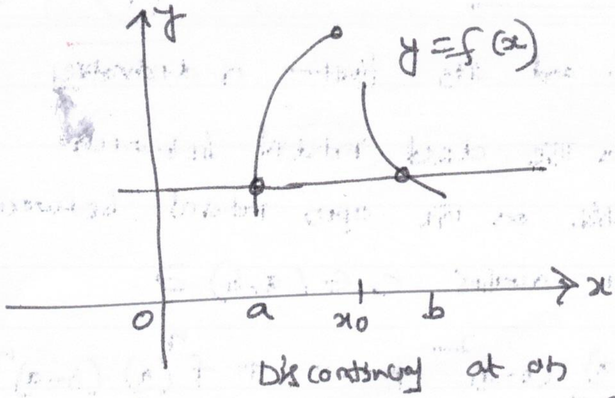
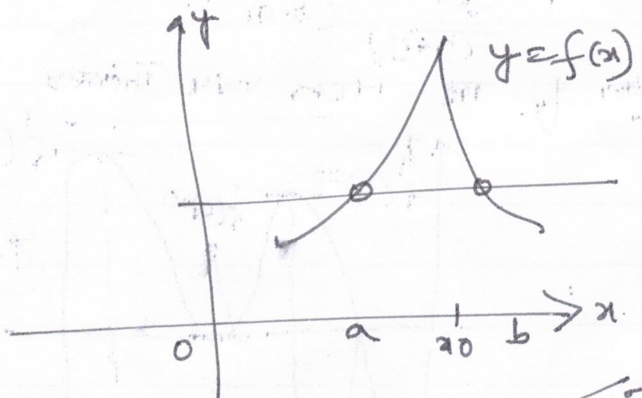


→ (B)

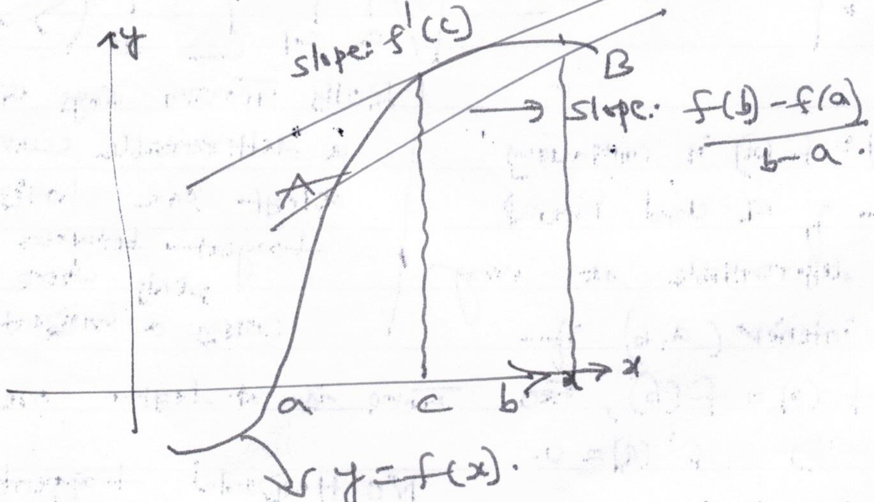
Discontinuity at an interior point of $[a,b]$



→ (C)

Continuous on $[a,b]$ but not differentiable at an interior point.

Tangent parallel to chord AB



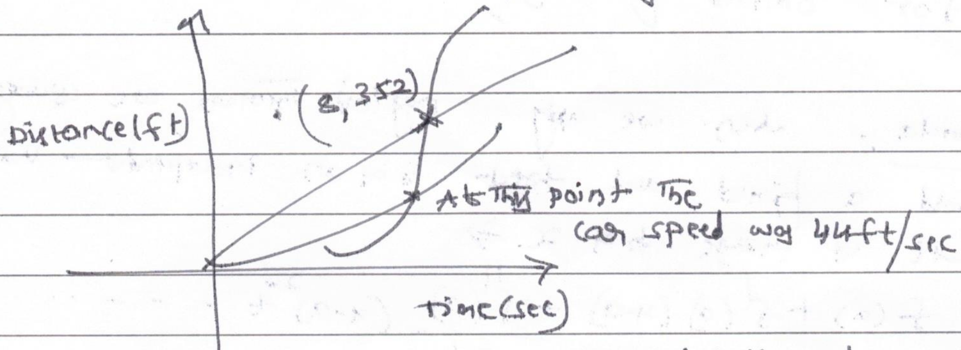
The Mean value Theorem (slanted version of Rolle's theorem).

Suppose $y = f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one point c in (a,b) at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

A physical interpretation of MVT.

If we think of number $\frac{f(b)-f(a)}{(b-a)}$ as the average change in f over $[a,b]$ and $f'(c)$ as an instantaneous change, then MVT says that at some interior point the instantaneous change must equal the average change over the entire interval.



i.e. If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec $= 30$ mph. At some point during its acceleration, the MVT says, the speedometer must read exactly 30 mph (44 ft/sec).

Corollary: Functions with the same derivative differ by a constant.

If $f'(x) = g'(x)$ at each point x in an open interval (a,b), then there exist a constant C $\ni$ $f(x) = g(x) + C$ $\forall$ $x \in (a,b)$. i.e. $f-g=$ constant.

Ex: Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0,2)$.

Sol: $f(x)$ has same derivative as $g(x) = -\cos x$ $\Rightarrow$

$f(x) = -\cos x + C$, and $f(0) = 2$ $\Rightarrow$

$f(0) = -1 + C = 2 \Rightarrow C = 3$.

$\therefore f(x) = -\cos x + 3$.

$D, D^1, D^2 \ldots D^n \longrightarrow$ nth derivatives.

— Leibnitz's theorem for the n^{th} derivative of product of two functions.

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + \cdots {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n.$$

proof by Induction.

— for rational functions, decompose into partial fractions.

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Taylor's formula. when we apply Taylor's theorem we usually want to hold a fixed and treat b as an independent variable. Let us change b as $x \Rightarrow$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x).$$

where

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}, \quad c \in (a,b).$$

$$\therefore f(x) = P_n(x) + R_n(x).$$

$P_n(x)$ — Taylor formula.

$R_n(x)$ — remainder of order n (error).

If $R_n(x) \longrightarrow 0$ as $n \to \infty$ $\forall x \in I$, we say that that the taylor series generated by f at $x=a$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Ex: $f(x) = e^x$ at $x=0$ converges $\forall$ x.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x).$$

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}, \quad c \in (0,x)$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x.

When x is negative, so is c and $e^c < 1$.

When x is zero, $e^x = 1$ and $R_n(x) = 0$.

When x is +ve, so is c, and $e^c < e^x$,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0,$$

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0.$$

Finally $\lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = 0 \ \forall x$.

$\therefore \lim_{n \to \infty} R_n(x) = 0$, the series converges $\forall x$.

$$\therefore e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots$$

The Remainder Estimation Theorem: If there is a positive constant $M \ni |f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

If this condition holds for every n and the other conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

Ex: Calculate e with an error less than 10^{-6}.

We know

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \dots$$

$R_n(x)$.

put $x=1$.

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + R_n(1)$$

where

$$R_n(1) = \frac{e^c}{(n+1)!}, \quad c \in (0,1).$$

we know $e \in (2,3)$, i.e $e < 3$.

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

By repeated experiment,

$$\frac{1}{9!} > 10^{-6} \text{ while } \frac{3}{10!} < 10^{-6}.$$

Thus, we should take $(n+1)$ to be at least 10, or n to be at least 9.

$$e = \underbrace{1 + 1 + \frac{1}{2} + \cdots \frac{1}{9!}}_{x} = 2.7182825$$

Ex 2: For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} \Big| + \frac{x^5}{5!} - \cdots$$

After $\left(x - \frac{x^3}{3!}\right)$ no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}$$

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$$\frac{|x|^5}{120} < 3 \times 10^{-4} \Rightarrow |x| < \sqrt[5]{360 \times 10^{-4}} \simeq 0.514$$

Ex 3: Compute to two decimals the integral

$$\int_0^1 \frac{\sin x}{x} \, dx.$$

$$\sin x = x - \frac{x^3}{3!} + R_5(x) \text{ and } |R_5(x)| \leq \frac{|x|^5}{5!}$$

Hence,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{R_5(x)}{x} \text{ and } \left|\frac{R_5(x)}{x}\right| \leq \frac{|x|^4}{4!}$$

~~Hence,~~

$$\int_0^1 \frac{\sin x}{x} \, dx = x - \frac{x^3}{3\,3!}\Big|_0^1 + E, \text{ where}$$

$$E = \int_0^1 \frac{R_5(x)}{x} \, dx.$$

The error term.

$$|E| \le \int_0^1 \left|\frac{R_5(x)}{x}\right| dx \le \int_0^1 \frac{x^4}{5!}\, dx = \frac{x^5}{5!\,5}\Big|_0^1 = 1/600$$

furthermore,

$$x - \frac{x^3}{3!\,3}\Big|_0^1 = 1 - \frac{1}{18} = 17/18.$$

$$\therefore \int_0^1 \frac{\sin x}{x}\, dx = \frac{17}{18} + E \text{ where } |E| \le 1/600.$$

Imp. Ex: Compute $I = \int_0^1 \sin x^2\, dx$.

Let $x^2 = u$, $\sin u = u - \frac{u^3}{3!} + R_5(u)$.

$$I = \int_0^1 \left(x^2 - \frac{x^6}{3!}\right) dx + \int_0^1 R_5(x^2)\, dx = \frac{1}{3} - \frac{1}{42} + E.$$

where $E = \int_0^1 R_5(x^2)\, dx$.

$$|R_5(u)| \le \frac{|u|^5}{5!}$$

$$\therefore |E| \le \int_0^1 \frac{x^{10}}{5!}\, dx = \frac{1}{11 \cdot 120} < 10^{-3}.$$

$$\therefore I = \frac{1}{3} - \frac{1}{42} + E. \text{ where } E = 10^{-3}.$$

For Logarithmic function:

$$\log x = \log 0 + \log'(0)x + \cdots$$

$\log 0$ is not defined.

now put $a = 1$, ~~b=x~~. $b = x$.

$$\log b = \log 1 + \log'(1)(b-1) + \log''(1)\frac{(b-1)^2}{2!} \cdots$$

now put $b = x+1 \Rightarrow$

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4} \cdots$$

For $-1 \le x \le 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + R_{n+1}(x).$$

$$R_{n+1}(x) = (-1)^n \int_0^x \frac{t^n}{1+t}\,dt.$$

For instance,

compute $\log 2$ to 3 decimals.

we write $2 = \frac{4}{3} \times \frac{3}{2}$, $\frac{4}{3} = 1 + \frac{1}{3}$, $\frac{3}{2} = 1 + \frac{1}{2}$.

$$\log 2 = \log\left(\frac{4}{3} \times \frac{3}{2}\right) = \log\left(1+\frac{1}{3}\right) + \log\left(1+\frac{1}{2}\right).$$

to find $\log\left(1+\frac{1}{3}\right)$, we use $x = 1/3$.

$$|R_{n+1}(x)| \le \int_0^x t^n\,dt \le \frac{x^{n+1}}{n+1}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + R_6(x).$$

$$|R_6(1/3)| \le \frac{1}{6}\left(\frac{1}{3}\right)^6 \le \frac{1}{4}\times 10^{-3}$$

Case 1: $0 < x \le 1$

$$\log\frac{4}{3} = \log\left(1+\frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{(1/3)^2}{2} + \frac{(1/3)^3}{3} - \frac{(1/3)^4}{4} + \frac{(1/3)^5}{5} + E_1$$

$$|R_{n+1}(x)| \le \frac{(-x)^{n+1}}{(n+1)(1+x)} = \frac{|x|^{n+1}}{(n+1)(1+x)}$$

Case 2: $-1 < x < 0$

$$= A + E_1.$$

with $E_1 = R_6(1/3)$, $|E_1| \le \frac{1}{4}\times 10^{-3}$.

$|||^{ly}$ for

$$\log 3/2 = \log\left(1+\frac{1}{2}\right) = \frac{1}{2} - \frac{(1/2)^2}{2} + \cdots + \frac{(1/2)^7}{7} + E_2$$

$$= A_2 + E_2.$$

$E_2 = R_8(1/2)$. $\Rightarrow E_2 = |R_8(1/2)| \le \frac{1}{2}\times 10^{-3}$.

For $-1 < x \le 1$,

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots (-1)^{n-1}\frac{x^n}{n} + R_{n+1}(x)$.

$R_{n+1}(x) = (-1)^n \int_0^x \frac{t^n}{1+t}\,dt$.

it is essential $x > -1$. (t is dummy variable for x)

$$\log 2 = \log(1+1/2) + \log(1+1/3)$$

$$= A_1 + A_2 + E_1 + E_2.$$

$$|E| \le |E_1| + |E_2|$$

$$\le \frac{1}{4}\times 10^{-3} + \frac{1}{2}\times 10^{-3} < 10^{-3}.$$

which lies within the desired accuracy.

We needed six or eight terms in the polynomial expressions to get three decimal accuracy.

EXPT. NO. NAME: Binomial expansion: Page No.: 9 Date:

General method of Taylor's formula:

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_{n+1}(x).$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_{n+1}(x).$$

Apply this formula to the function $f(x) = (1+x)^s$.

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \cdots$$

$$+ \cdots \frac{s(s-1)\cdots(s-n+1)}{n!}x^n + R_{n+1}(x).$$

Ex: Find $\sqrt{1.2}$ to 2 decimals.

Let $x = 0.2 = 2\times10^{-1}$ and $s = 1/2 \Rightarrow$

$$\sqrt{1.2} = (1+0.2)^{1/2} = 1 + \frac{1}{2}0.2 + R_2(0.2)$$

$$= 1 + 0.1 + R_2(1/5)$$

we must estimate $R_2(1/5)$.

Since $0 \le c \le 1/5$ and $s-2 = -3/2$, we find

$$(1+c)^{-3/2} = \frac{1}{(1+c)^{3/2}} \le 1.$$

this smallest possible value of c is 0.

$$|R_2(1/5)| \le \frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{2}(0.2)^2$$

$$\le \frac{1}{8}4\times10^{-2} = \frac{1}{2}\times10^{-2}.$$

$(1+x)^s = 1 + sx + R_2(x)$

$R_2(x) = f^{(2)}(c)\frac{x^2}{2!}$.

$= s(s-1)(1+c)^{s-2}\frac{x^2}{2}$.

Teacher's Signature:

Ex. compute $\sqrt{0.8}$ to 2 decimal.

$s = 1/2, \quad x = -0.2$

$$\sqrt{0.8} = (1-0.2)^{1/2} = 1 - \frac{1}{2}0.2 + R_2(0.2)$$
$$= 0.9 + R_2(-0.2)$$

$-2 \le c \le 0,$

$$\frac{1}{(1+c)^{3/2}} \le 1/(0.8)^{3/2}.$$

$$|R_2(-0.2)| \le \frac{1}{2}\left|\frac{1}{2}-1\right| \frac{1}{2!}\frac{1}{(0.8)^{3/2}}(0.2)^2$$
$$4 \times 10^{-2}.$$
$$\le \frac{1}{8}\frac{1}{(0.8)^{3/2}}$$

Ex. find the value of $\sqrt{26}$ to two decimal.

$$26 = 25 + 1 = 25\left(1 + \frac{1}{25}\right) =$$

$$\sqrt{26} = 5\left(1 + \frac{1}{25}\right)^{1/2}.$$

By binomial taylor formula,

$$(1 + 1/25)^{1/2} = 1 + 1/50 + R_2(x).$$

$x = 1/25, \quad s = 1/2, \quad c > 0.$

$$\left|R_2\left(\frac{1}{25}\right)\right| \le \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\left(\frac{1}{25}\right)^2 \le \frac{1}{8}\,\frac{1}{625} = \frac{1}{5000}$$

$$\sqrt{26} = 5\left(1 + \frac{1}{50}\right) + 5R_2(1/25) = 5.1 + E$$

where

$$E = 5\left|R_2(1/25)\right| < 10^{-3}.$$

Ex: find $\sqrt{26}$ to four decimals.

$$\left(1+\frac{1}{25}\right)^{1/2} = 1+\frac{1}{2}\frac{1}{25}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{25}\right)^2 + R_3\left(1/25\right).$$

$$\left|R_3\left(1/25\right)\right| \le \frac{1}{2}\left|\frac{1}{2}-1\right|\left|\frac{1}{2}-2\right|\frac{1}{3!}\left(\frac{1}{25}\right)^3$$

$$\le \frac{1}{24}\times 10^{-4}.$$

$$\therefore (26)^{1/2} = 5\left(1+1/25\right)^{1/2} = 5\left(1+\frac{1}{50}-\frac{1}{4625}\right) + E.$$

$$E = 5R_3\left(1/25\right)$$

$$|E| \le \frac{5}{24}\times 10^{-4} < 10^{-4}.$$

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Monotonic functions and first derivative test:

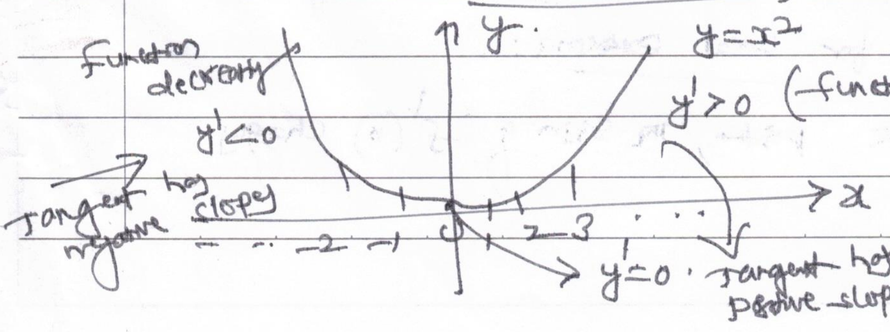
Def: Increasing, decreasing function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1). If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be increasing on I.

2). If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be decreasing on I.

A function that is increasing or decreasing on I is called monotonic on I.



The function $f(x)=x^2$ is monotonic on $(-\infty,0]$ and $[0,\infty)$ but is not monotonic on $(-\infty,\infty)$.

Corollary: First Derivative test for monotonic functions.

Suppose f is cont on $[a,b]$ and diff on (a,b)

if $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$.

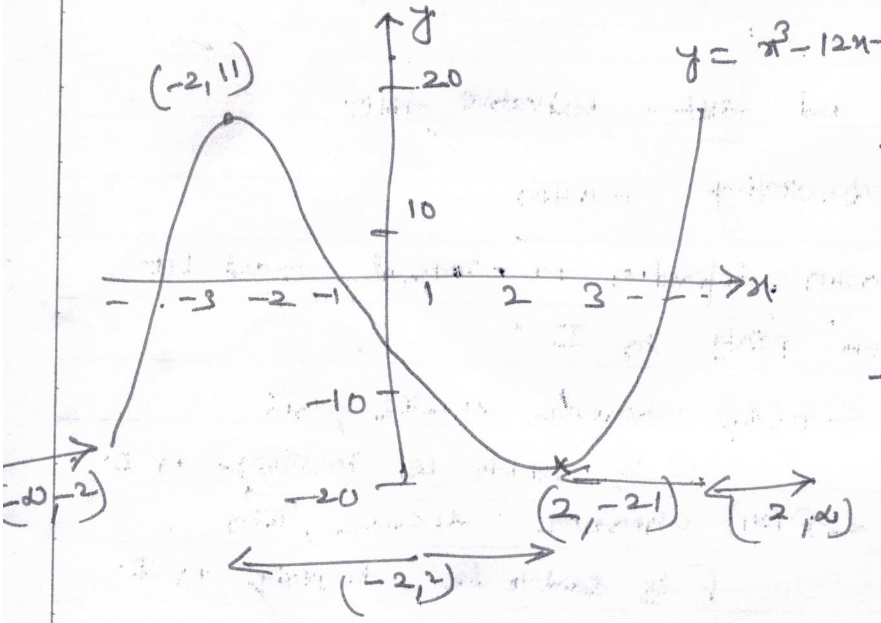
if $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.

Ex: find critical points of $f(x) = x^3 - 12x - 5$ and, identify the intervals on which f is increasing and decreasing.

Sol: $f(x) = x^3 - 12x - 5$ is a polynomial of degree 3 and is cont. and differenciable.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x+2)(x-2) = 0 \quad \text{for } x = 2, -2$$

These are critical points. The critical points subdivide the domain of f into intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.



$-\infty < x < -2$

$f'(-3) = 15$ (+ve)

f is increasing

$-2 < x < 2$

$f'(0) = -12$ (-ve)

f is decreasing

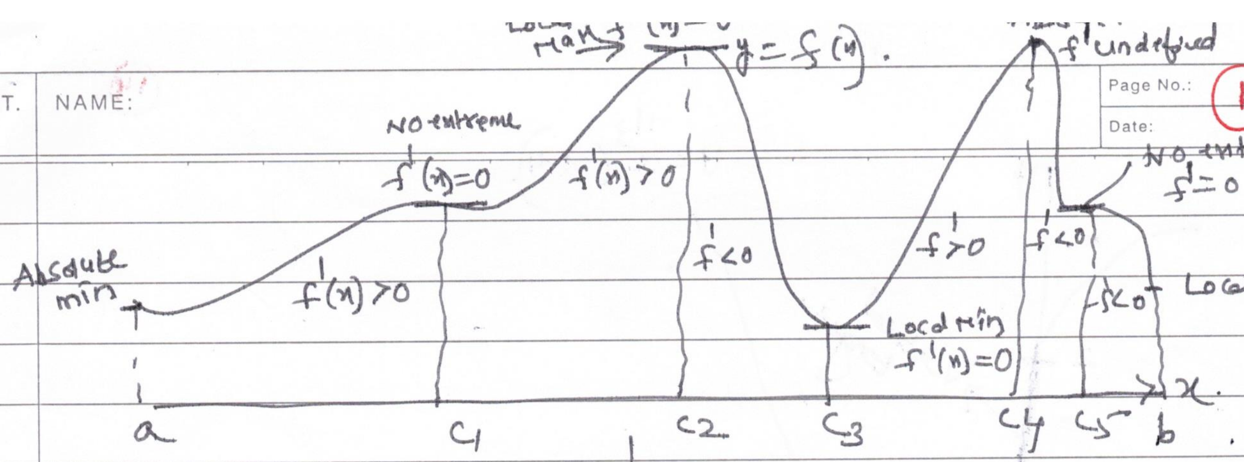
$2 < x < \infty$

$f'(3) = 15$ (+ve)

f is increasing

First derivative test for Local extrema:

At a local extreme point, the sign of $f'(x)$ changes.



f' tells how the graph rises and falls.

Suppose that C is a critical point of a cont. function f, and that f is differentiable at every point in some interval containing C ~~except~~ except possibly at C itself.

1) if f' changes from $-ve$ to $+ve$ at C, then f has a local minimum at C.

2). If f' changes from $+ve$ to $-ve$ at C, then f has a local maximum at C.

3). If f' does not change sign at C (i.e f' is $+ve$ on both sides of C or $-ve$ on both sides of C), then f has no local extremum at C.

Ex: $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$.

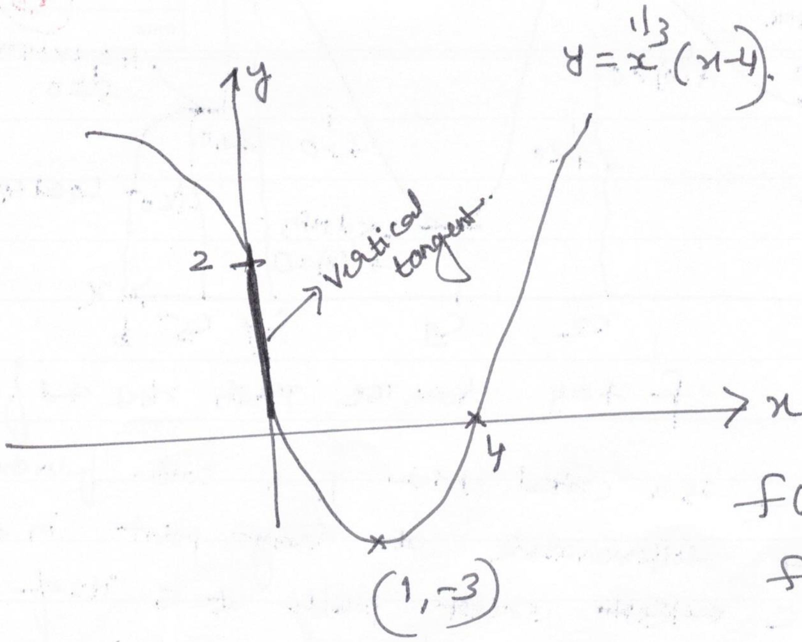
Sol: It is cont as it is product of two cont. functions.

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4(x-1)}{3x^{2/3}} = 0 \text{ for } x=1$$

and undefined at $x=0$.

There are no endpoints in the domain, so the critical points $x=0$ and $x=1$ are the only places where f might have an extreme value.

$x<0$	$0<x<1$	$x>1$
$f'(x) \to -ve$	$f'(x) \to -ve$	$f'(x) \to +ve$
$f(x)$ is decreasing	$f(x)$ is decreasing	$f(x)$ increasing



$f(x)$ decreases when $x < 1$ and $f(x)$ increases when $x > 1$.

Note that $\lim_{x \to 0} f'(x) \to -\infty$, so the graph f has a vertical tangent at the origin.

$f \to$ decreases on $(-\infty, 0)$, decreases on $(0, 1)$ and increases on $(1, \infty)$.

~~$f'(x)$ does not have change in sign and f has a local min at $x = 1$~~

By the first derivative test for local ~~max~~ extreme f does not have an extreme value at $x = 0$ (f' does not change sign) and f has a local minimum at $x = 1$ (f' changes from negative to positive).

—— x ——

Remainder,

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}, \quad R_n(x) = \frac{f^{(n)}(c)}{n!} x^n, \quad c \in (0, x)$$

It does not make any difference except amounts merely to a change of indices and convenience.

More precisely, $\frac{|f^{(n)}(c)|}{n!} |x|^n \leq$ something

So, we need to give a bound for the nth derivative of $|f^{(n)}(c)|$.

We can also see $c \in (a, b)$ arbitrary interval instead of $(0, x)$.

Theorem: In Taylor's formula, $\exists$ a number $c \in (a, b) \ni$ the remainder R_n is given by

$$R_n = \frac{f^{(n)}(c)(b-a)^n}{n!}$$

If M_n is a number $\ni$ $|f^{(n)}(x)| \leq M_n \ \forall \ x \in (a, b)$, i.e. M_n is an upper bound for $|f^{(n)}(x)|$, then

$$|R_n| \leq \frac{M_n |b-a|^n}{n!}$$

$$\therefore |R_n| = \left| \frac{f^{(n)}(c) |b-a|^n}{n!} \right| \leq M_n \frac{|b-a|^n}{n!}$$

Ex: Compute $\sin(0.1)$ to 3 decimals.

Here, $x = 0.1$, we want to find $n \ni$

$$\frac{|x|^n}{n!} \leq 10^{-3}.$$

By inspection we see for $n=3$, will work,

$$|R_3(0.1)| \leq \frac{(0.1)^3}{3!} = \frac{10^{-3}}{6}$$

$$\therefore \sin x = x + R_3(x) \Rightarrow$$

$$\sin(0.1) = 0.100 + \underbrace{E}_{R_3(0.1)} \ni$$

$$|E| \leq \frac{1}{6} \times 10^{-3}.$$

Def: In general, some expression has the value ~~$\equiv$~~ A ~~$|E| \le \frac{1}{6}10^{-3}$~~ with an accuracy of 10^{-n} or to n decimals if the expression is equal to $A+E$ with an error term $E \ni$

$$|E| \le 10^{-n}.$$

So, $\sin(0.1)$ has the value 0.1 with an accuracy of 10^{-3} or 3 decimals.

Warning: Do not write $\sin(0.1) = 0.1$

Always write $\sin(0.1) = 0.1 + E$, Then given an estimate for $|E|$.

Ex: $\sin 10^\circ$ with an accuracy of 10^{-3}

$$10^\circ = \frac{10\pi}{180} = \pi/18 \text{ rad.} \qquad \pi = 3.14159.$$

$$\pi < 3.2$$

$$\frac{\pi}{18} < 1/5.$$

Essentially $\sin(\pi/18)$ with the taylor polynomial of some degree and a remainder which has to be estimated This requires trial and error.

$$\sin(\pi/18) \simeq \pi/18 - \frac{1}{6}(\pi/18)^3 + R_5(\pi/18).$$

$$|R_5(\pi/18)| \le \frac{1}{5!}(\pi/18)^5 < \frac{1}{120}(1/5)^5 < \frac{1}{3}\times 10^{-5}$$

by easy arithmetic.

Hence the first two terms $\pi/18 - 1/6(\pi/18)^3$ gives an approximation of $\sin 10^\circ$ to an accuracy of 10^{-5}, which is better than what we wanted originally. $\therefore \sin(\pi/18) = \pi/18 + R_3(\pi/18)$.

Ex compute $\sin\left(\pi/6+0.2\right)$ to an accuracy of 10^{-4}.

Sol: Taylor's formula for $f(a+b)$, $a=\pi/6, b=0.2$.

$$\sin(a+b) = \sin a + \cos(a)\frac{b}{1} - \sin(a)\frac{b^2}{2!} - \cos(a)\frac{b^3}{3!} + R_4$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(0.2) - \frac{1}{2}\frac{(0.2)^2}{2} - \frac{\sqrt{3}}{2}\frac{(0.2)^3}{6} + R_4.$$

So, we need to estimate

$$|R_4| \le \frac{(0.2)^4}{4!} = \frac{16\times10^{-4}}{24} \le 10^{-4}.$$

which is within the required bound of accuracy.

Theorem: Let c be a number, then $\frac{c^n}{n!}$ approaches 0 as n becomes very large.

$$|R_n(x)| \le \frac{|x|^n}{n!} \longrightarrow 0$$
$$n\to\infty$$

Caution: Sometimes a definite integral cannot be evaluated from an indefinite one, but we can find simple approximation to it by using Taylor expansions.

$$\left(\left|\int_a^b f(x)\,dx\right|\right) \le \int_a^b |f(x)|\,dx \le M(b-a)$$

Let $a<b$ and let f be cont on $[a,b]$,
Let M be a number $\ni |f(x)| \le M \ \forall x$.
Then

Ex Compute $\log 1.1$ to 3 decimal.

$\log(1+0.1)$, $x=0.1$, $n=2$.

$$|R_3(x)| \le \frac{1}{3}\times10^{-3}.$$

$$\log(1.1) = 0.1 - 0.005 + E$$

with an error $E \ni$ $|E| \le \frac{1}{3}\times10^{-3}$.

Convexity and concavity of a curve :-

Theorem: If the function f and its derivative f' can be differenciated at a point a, then

f is convex if $f''(a) > 0$

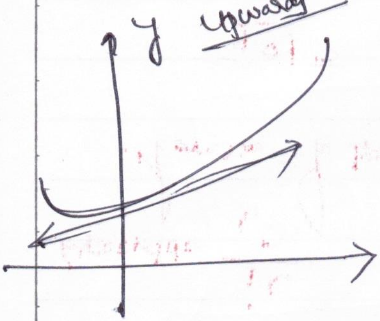
f is concave if $f''(a) < 0$.

Rate of increase of slope

(A) Ex.

$f(x) = 3x^2 + 7x + 9$.

$f'(x) = 6x + 7$, $f''(x) = 6$ ⇒ the function convex upwards

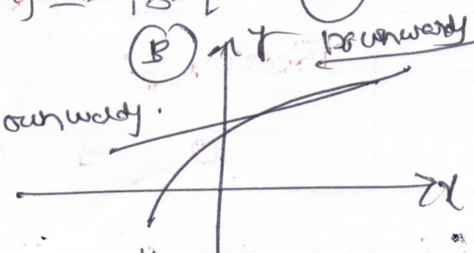


A convex function always bends upwards.

Ex: $f(x) = -9x^2 - x - 1$, $f'(x) = -18x - 1$. (B)

$f''(x) = -18 < 0$.

f is concave and bends downwards.

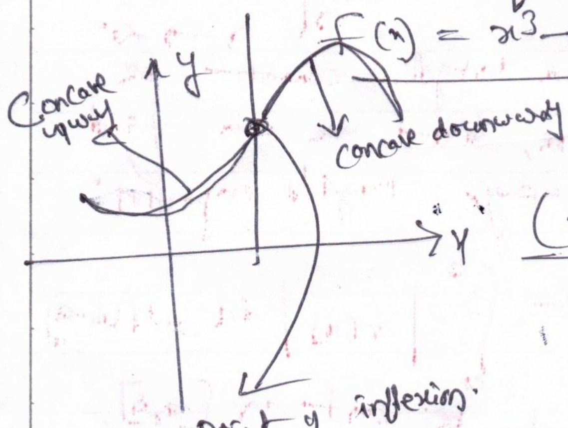


Ex1: Intervals of concavity and convexity.

$f(x) = x^3 - 3x + 2$, $f'(x) = 3x^2 - 3$, $f''(x) = 6x$.

$f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$.

root is 0.



$(-\infty, 0)$ and $(0, \infty)$.

$f''(x) = 6x$.

$f''(-1) = -6 < 0$. say

$f''(1) = 6 > 0$ say

$f''(x) = 0$, not sufficient to guarantee point of inflexion.

$f''(x)$ must change sign of x crossing a.

$f''(x) > 0$, f is convex

$f''(x) < 0$, f is concave.

Concavity : $(-\infty, 0)$

Convexity : $(0, \infty)$

Ex2 :

$f(x) = \ln(3x^2 + 12)$.

$f'(x) = \dfrac{6x}{3x^2+12}$, $f''(x) = -\dfrac{2(x^2-4)}{(x^2+4)^2}$

$f'(x) = 0 \Rightarrow -2x^2+8=0$

$\Rightarrow x = \pm 2$

we have got two roots of the function

$(-\infty, -2), (-2, 2), (2, \infty)$

$f''(x) \in (-\infty, -2)$, say $x=-3$, $f''(-3) = -10 < 0$

$f''(x) \in (-2, 2) \rightarrow$ say $x=1$, $f''(1) = 6 > 0$

$f''(x) \in (2, \infty)$, say $x=3$, $f''(3) = -10 < 0$

Concavity: $x < -2$ and $x > 2$

Convexity: $(-2, 2)$.

Ex: $f(x) = x^4$, $f''(0) = 0$ but $x=0$ is not a point of inflection.
since $f''(x)$ does not change sign as x crossing zero.
($x=0$ is a local minimum).

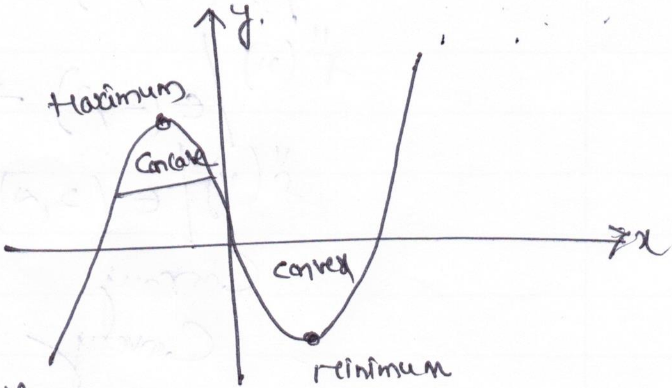
	FODT (first order derivative test)	SODT (second)	HODT (Higher)
Maxima	$f'(a)=0$ $f'(x)=0$ changes sign from +ve to -ve as x crossing a	$f'(a)=0$ $f''(a)<0$	$f'(a)=0$ $f''(a)=0$ $f^{n-1}(a)=0$ $f^{n}(a)<0$ when n is even (If n is odd, $x=a$ is not an extremum point, It is a point of ~~inflexion~~ inflection.
Minima	$f'(a)=0$ $f'(x)$ changes sign from -ve to +ve as x crossing a.	$f'(a)=0$ $f''(a)>0$	$f'(a)=0$ $f''(a)=0$ $f^{n-1}(a)=0$ $f^{n}(a)>0$ when n is even (If, n is odd, $x=a$ is not an extremum point, It is a point of inflection.

point of inflection $\longrightarrow$ $f'(a) = f''(a) = 0$

$f''(x)$ changes sign as x crosses a.

$y = x^2$, $\frac{dy}{dx} = 2x$, $\frac{d^2y}{dx^2} = 2 > 0$, always Convex.

$y = -x^2$, $\frac{d^2y}{dx^2} < 0$, always concave.



Spoon : $\longrightarrow$ front part $\longrightarrow$ Concave

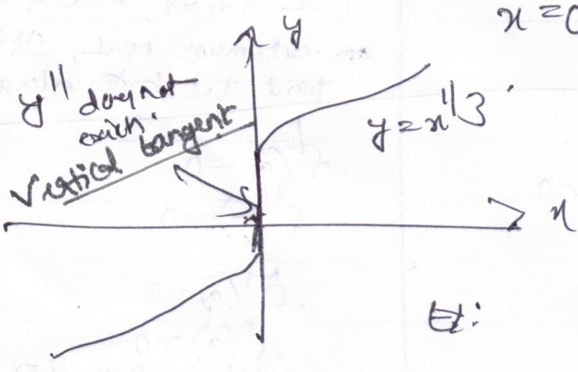
$\longrightarrow$ back side $\longrightarrow$ Convex

Ex: $y = f(x) = x^{1/3}$

$\frac{dy}{dx} = \frac{1}{3} x^{-2/3}$

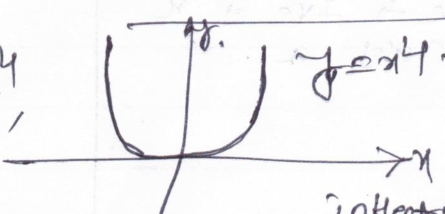
$\frac{d^2y}{dx^2} = \frac{-2}{9} x^{-5/3}$, y'' does not exist at $x = 0$

$x = 0$ is a point of inflection.



Acceleration changes sign

Ex: $y = x^4$,

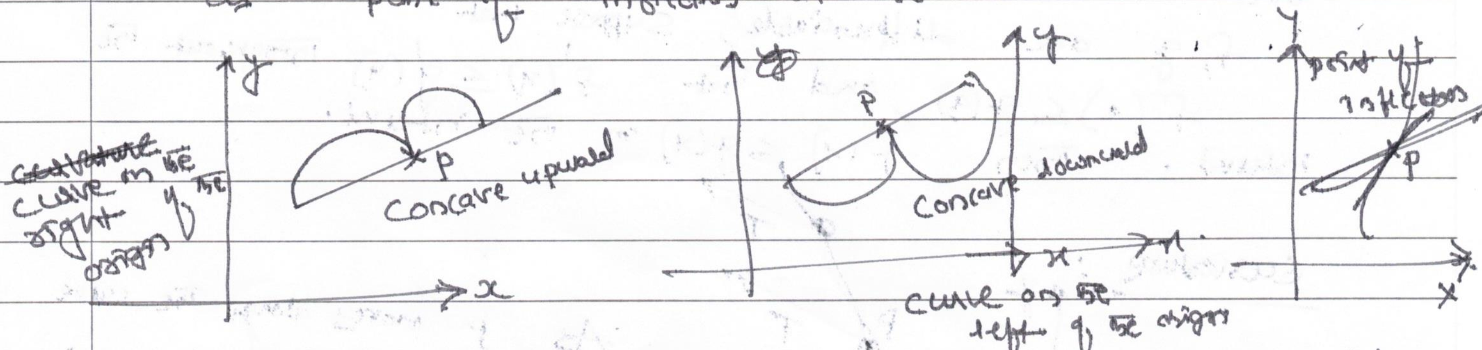


$x = 0$ is not a point of inflection no change in sign

→ Consider the curve that crosses the tangent.

→ For example, x-axis is the tangent to the curve $y = x^3$ at $(0,0)$.

– A point where a curve crosses the tangent is known as point of inflection on the curve.



– In case a curve does not cross the tangent at a point, we have two possibility.

a). A portion of the curve on both sides of p, however small it may be, which lies above the tangent at p (i.e towards the positive direction of y-axis).

in this case, the curve is concave upwards or convex downwards.

b). There is a portion of the curve on both sides of p, however small which lies below the tangent at p. (i.e towards negative direction of y-axis).

in this case, the curve is concave downwards or convex upwards at p.

Remark: At a point of inflection, the curve changes from concave upwards to concave downwards or vice-versa.

Ex: P.T $\sin x < x \ \forall \ x > 0$.

Let $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x$.

Take, $0 < x < \pi/2$, $f'(x) > 0$ since $\cos x < 1$

$\therefore f(x)$ is strictly increasing for $0 \le x \le \pi/2$.

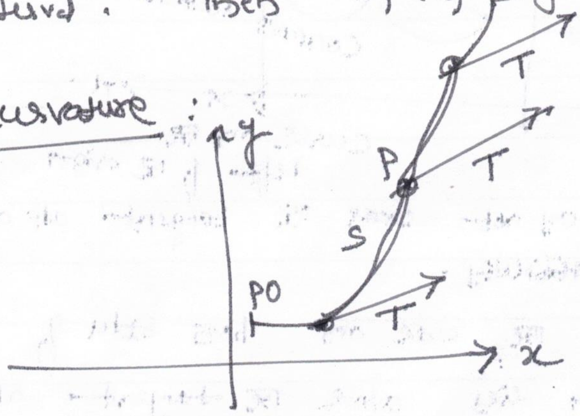
But $f(0) = 0 - \sin 0 = 0$. Hence we must have $f(x) > 0$ for $0 < x \le \pi/2$.

If $x \geq \pi/2$, Then $x > 1$ (Because $\pi \approx 3.14$)

and so $\sin x < x$ whenever $x > \pi/2$.

Theorem: Suppose we have two functions f and g and we assume that over a certain interval $[a, b]$ and f, g are differentiable, suppose that $f(a) \leq g(a)$, and that $f'(x) \leq g'(x)$ throughout the interval. Then $f(x) \leq g(x)$ in the interval.

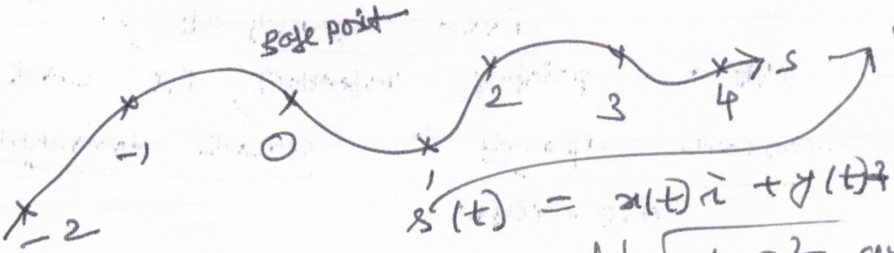
Curvature:



As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The rate of $|dT/ds|$ at P is called curvature of curve at P.

T length remains constant, only direction changes.

Essentially, the rate at which T turns per unit of length along the curve is called the curvature.



arc length parameter.

$$r(t) = x(t)i + y(t)j + z(t)k, \quad a \leq t \leq b.$$

Length of a smooth curve $\longrightarrow L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$|v| \longrightarrow$ Length of velocity vector dr/dt.

Arc length $L = \int_a^b |v|\, dt$.

$$\frac{dr}{ds} = \frac{dr}{dt} \times \frac{dt}{ds} = v\frac{1}{|v|} = \frac{v}{|v|} = T$$

unit tangent vector of a smooth curve.

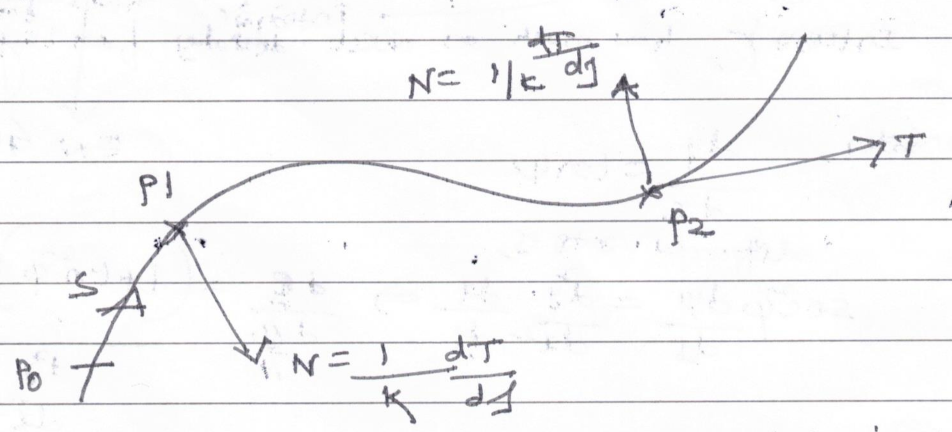
Curvature $\kappa = \left|\frac{dT}{ds}\right|$

If $r(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|,$$ where $T = v/|v|$ is the unit tangent vector.

<u>Remark</u>: At a point where $\kappa \neq 0$, the principal unit normal vector for a smooth curve in the plane is

$$N = \frac{1}{\kappa}\frac{dT}{ds}.$$



<u>Remarks</u>: — The curvature of a circle is constant.

— The curvature of a straight line is zero.

— Intuitively curvature is the amount by which a geometric object such as surface deviates from being a flat plane or curve from being straight as in the case of a line.

— The rate of change of direction of a curve with respect to distance along the curve.

$T \longrightarrow$ tangent

$s \longrightarrow$ arc length.

Its unit is $1/m$.

~~for~~ circle, It is the reciprocal of radius of curvature.

$$2\pi / 2\pi R = 1/R.$$

(Turn 2π over the length $2\pi R$).

— A curved line gradually changes direction from one point to the next.

The rate of this change in direction per unit length along the curve is called the curvature.

Curvature — extrinsic (Independent of how the surface embedded) — intrinsic (Depends on how surface embedded) (Gaussian curvature)

— Intrinsically, how much an object deviates from being flat.

Ex: Geodesic curve

Consider, $\frac{dy}{dx} = \tan\psi$

∴ diff. w.r.to s,

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds} \Rightarrow \frac{ds}{d\psi} = \frac{\left(1+\tan^2\psi\right)\frac{ds}{dx}}{\frac{d^2y}{dx^2}}.$$

Length of a curve.

$$L = \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

$\rho > 0$ if $\frac{d^2y}{dx^2} > 0$,

$\rho < 0$ if $\frac{d^2y}{dx^2} < 0$.

~~At~~ a point of inflection $\rho = 0$.

Ex 2 : For cycloid, $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

find ρ ?

we have $\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a \sin t$..

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{a}\sin t}{\cancel{a}(1+\cos t)} = \frac{\cancel{2}\sin t/_2 \cancel{\cos t/_2}}{\cancel{2}\cos^2 t/_2} = \tan t/_2 .$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \sec^2 t/_2 \frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{t}{2} \frac{1}{a(1+\cos t)} = \frac{1}{4a} \frac{1}{\cos^4 t/2}$$

$$\therefore \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1 + \tan^2 \frac{1}{2}t\right)^{3/2}}{\frac{1}{4a} \frac{1}{\cos^4(t/2)}} = 4a \cos t/_2 .$$

Ex 3 : find curvature, $r^m = a^m \cos m\theta$,

$r^m = a^m \cos m\theta \Rightarrow m \log r = m \log a + \log \cos m\theta$.

$\Rightarrow \frac{dr}{d\theta} = r_1 = -r \tan m\theta$.

$$r_2 = \frac{d^2r}{d\theta^2} = -r\, m \sec^2 m\theta - \tan m\theta \frac{dr}{d\theta}$$

$$= -r m \sec^2 m\theta + r \tan^2 m\theta$$

$$\therefore \boxed{\rho = \frac{1}{m+1} \frac{a^m}{r^{m-1}}}$$

Ex 3 : In the ellipse $x^2/a^2 + y^2/b^2 = 1$. find the radius of curvature ~~at~~ one end of major axis.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, diff. w.r.to x.

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2 x}{a^2 y},$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2} \left[y - x \left(\frac{-b^2 x}{a^2 y} \right) \right] \Big/ y^2 .$$

$$= \frac{-b^4}{a^2 y^3}.$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = -\frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}.$$

at one end of major axis, $x=a, y=0$.

$$= \frac{(b^4a^2)^{3/2}}{a^4b^4} = b^2/a.$$

———— x ————

Critical point ⟶ Refers to ~~first~~ derivative at $x=a$ if either $f'(a)=0$ or $f'(a)$ not defined.

Inflection point ⟶ Refers to Second derivative

$f''(a)=0$ or $f''(a)$ not defined | $f''(x)>0,\ a>0$
$f''(x)<0,\ a<0.$

———— x ————

(27) Unit # 1, 25/09/2019

Semester: Monsoon AY: 2021-22 Dt: 20/01/2022

18/01/2022

20/1/2021

Expt. No. ______ Page No. Monsoon, 2019 - 2020 Date ______

MATHEMATICS - I

Extreme value of function (one variable).

Def: Let f be a function with domain D, then f has an absolute maximum value on D at a point c if

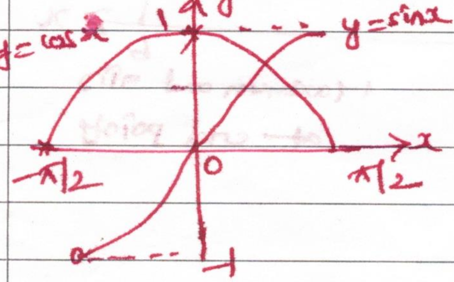
$$f(x) \le f(c) \quad \forall \; x \text{ in } D.$$

and an absolute minimum value on D at c if

$$f(x) \ge f(c) \; \forall \; x \text{ in } D.$$

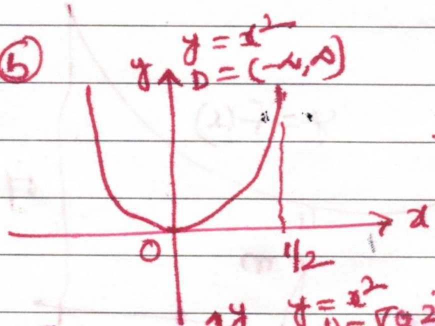
Remark: Absolute maximum and minimum values are called absolute extrema. Absolute extrema are also called global extrema.

(a) for instance, $f(x) = \cos x$, $x \in [-\pi/2, \pi/2]$



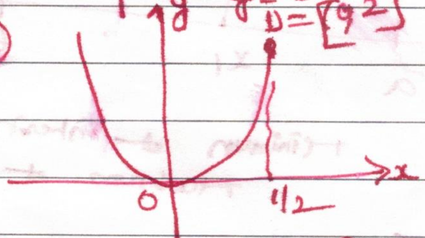
$y = \cos x$ takes absolute maximum value of 1 (once) and absolute minimum value of 0 (twice). whereas $y = \sin x$ takes absolute maximum value of 1 (once) and a minimum value of -1 (once) (depends on domain).

(b)



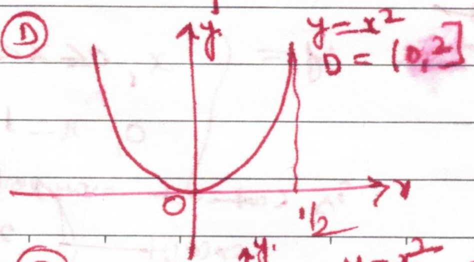
$\longrightarrow$ abs min only.

(c)



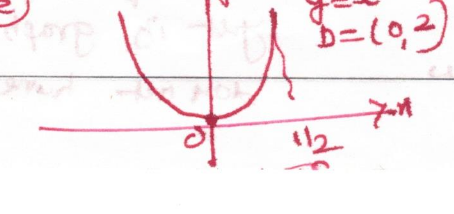
$\longrightarrow$ abs max and min

(D)



$\longrightarrow$ abs max only.

(e)

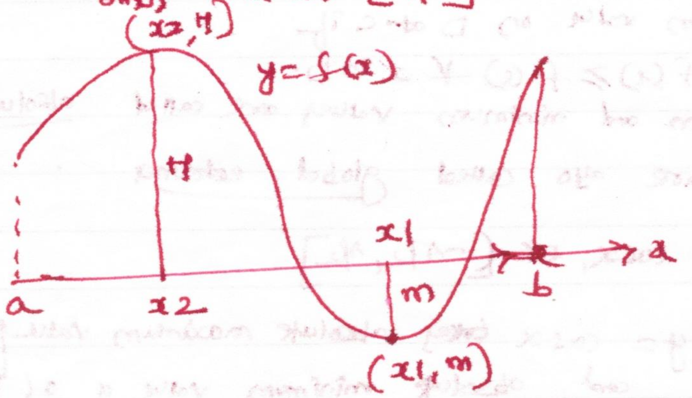


$\longrightarrow$ No max or min

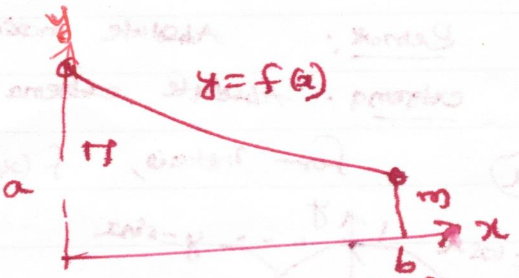
Teacher's Signature : ______

Theorem: The extreme value Theorem:

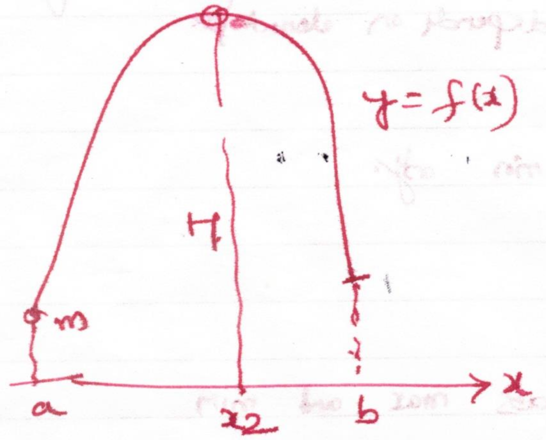
If f is cont. on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$. That is, there are numbers x_1 and x_2 in $[a,b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a,b]$.



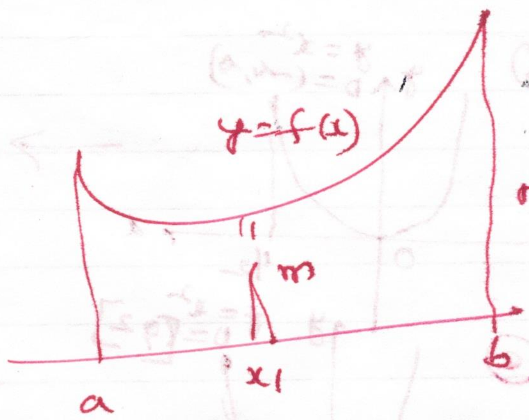
maximum and minimum at interior points



Maximum and min at end points

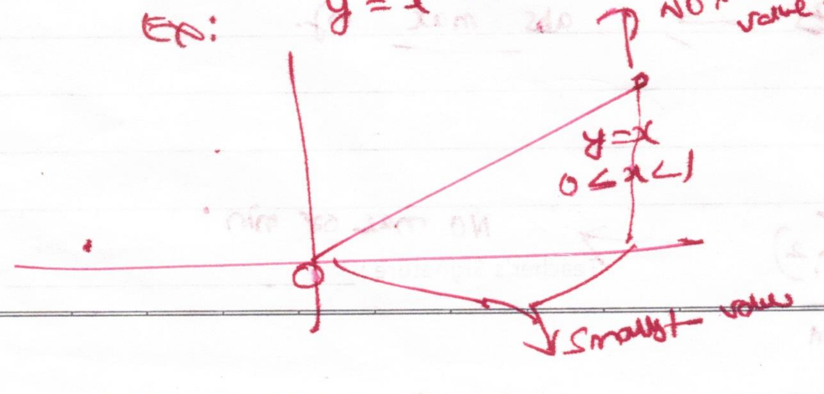


Max. at interior point, minimum at end point.



Minimum at interior point, Maximum at end point.

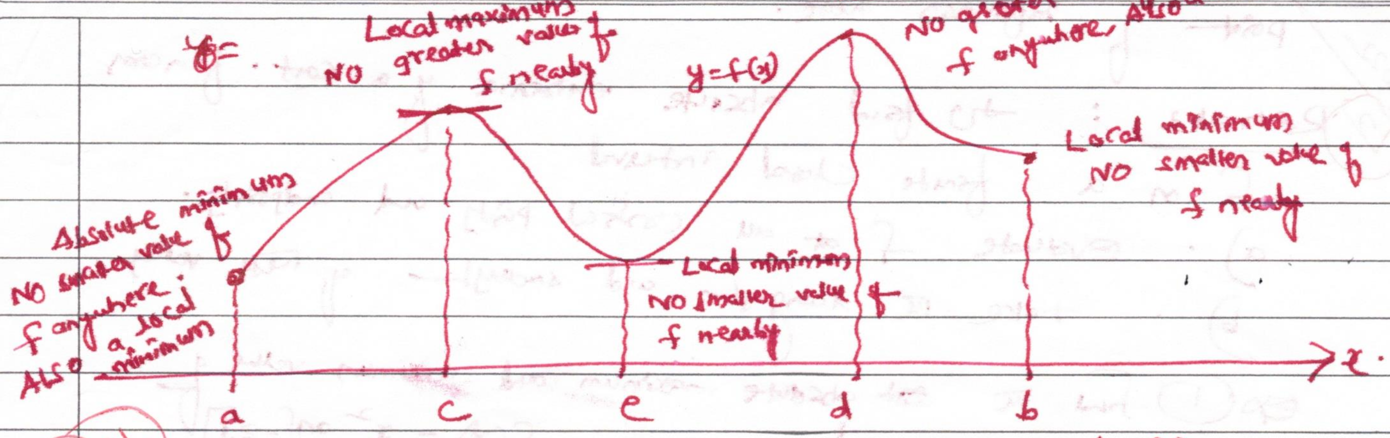
Ex: $y = x$



$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is cont at every point of $[0,1]$ except $x=1$, yet its graph over $[0,1]$ does not have a highest point

Expt. No. Local (Relative) extreme values: Page No. Date



20/1/2021 L2

How to classify maxima and minima.

Local maximum, Local minimum.

A function f has a local maximum value at an interior point c of its domain if $f(x) \le f(c)$ $\forall$ x in some open interval containing c.

~~$f(x) \le f(c)$~~

A function f has a local minimum value at an interior point c of its domain if

$$f(x) \geq f(c).$$

Theorem: If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Def: An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

Remarks A differentiable function may have a critical point at $x = c$ without having a local extreme value there.

Ex: $f(x) = x^3$ has a critical point at origin and zero value there, but is positive to the right of the origin and negative to the left. So it cannot have a local extreme value at the origin. Instead, it has a point of inflection there.

21/01/2022 L2

Remark : to find absolute extrema of a cont. function f on a finite closed interval

a) Evaluate f at all critical points and endpoints.

b) take the largest and smallest of these values.

Ex (1) find the absolute maximum and minimum value of $f(x) = x^2$ on $[-2,1]$

The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x=0$. We need to check the function's value at $x=0$ and at the endpoints $x=-2$ and $x=1$.

Critical point value : $f(0)=0$

Endpoints : $f(-2)=4$

$f(1)=1$

The function has an absolute maximum value of 4 at $x=-2$ (endpoints) and an absolute minimum value of 0 at $x=0$.

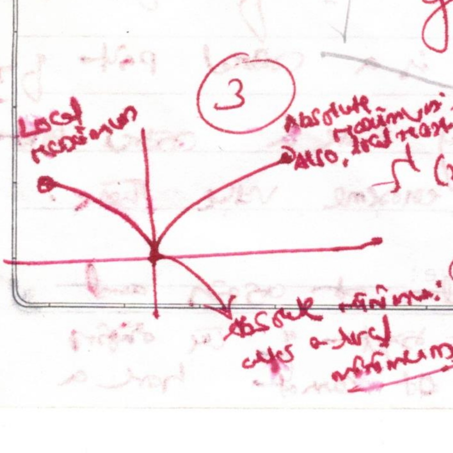
(2) $g(t) = 8t - t^4$ on $[-2,1]$.

The function is differentiable on its entire domain, so the only critical points occur where $g'(t)=0$.

$8 - 4t^3 = 0 \Rightarrow t = \sqrt[3]{2} > 1$, a point not in the given domain. The function's absolute extrema therefore occur at the endpoints,

$g(-2) = -32$ (absolute minimum) and

$g(1) = 7$ (absolute maximum).



(3) $f(x) = x^{2/3}$, $x \in [-2,3]$.

$f'(x) = \frac{2}{3\sqrt[3]{x}}$ has no zero but is undefined at interior point $x=0$.

Critical point $f(0)=0 \longrightarrow$ absolute minimum.

end points $f(-2) = \sqrt[3]{4}$

$f(3) = \sqrt[3]{9} \longrightarrow$ absolute maximum

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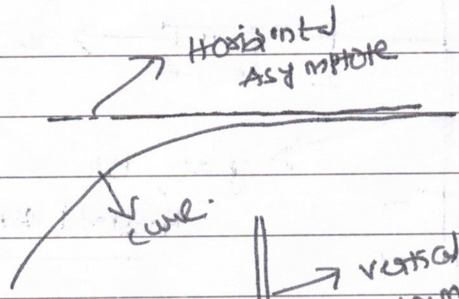
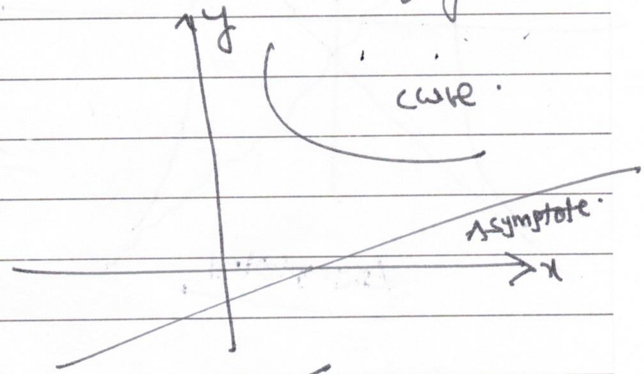
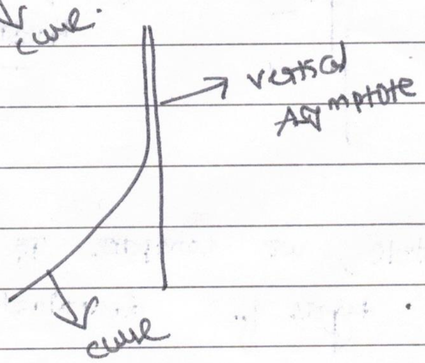
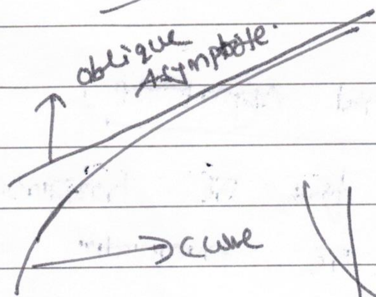
Asymptotes.

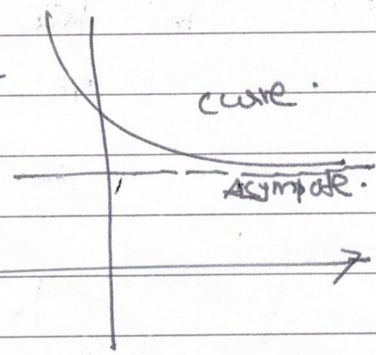
An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the x and y coordinates tend to infinity.

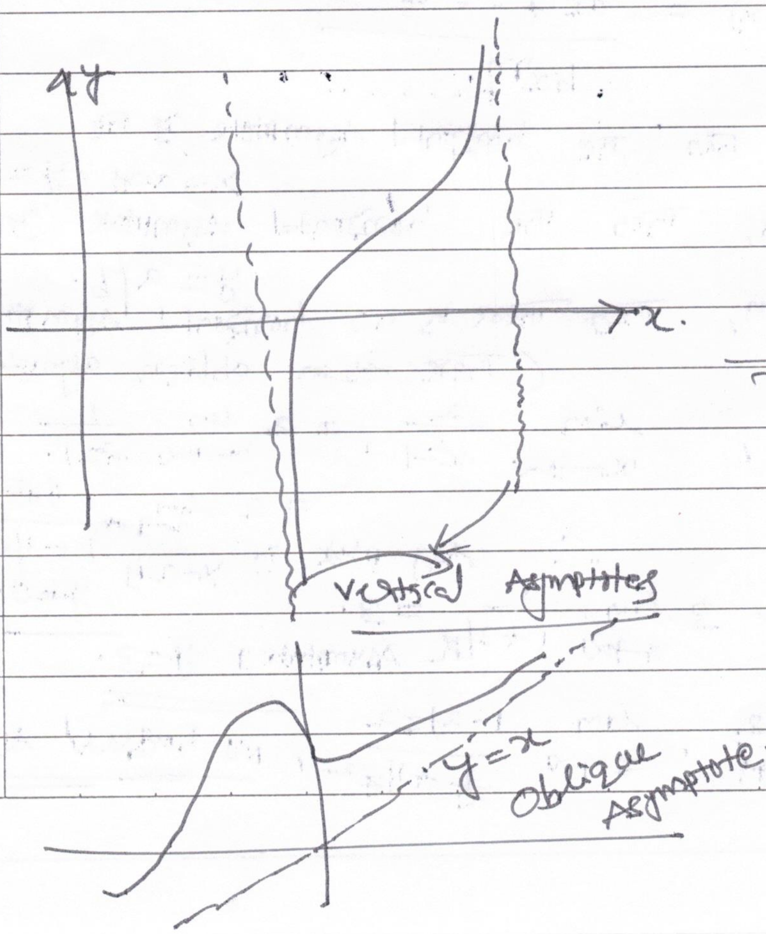


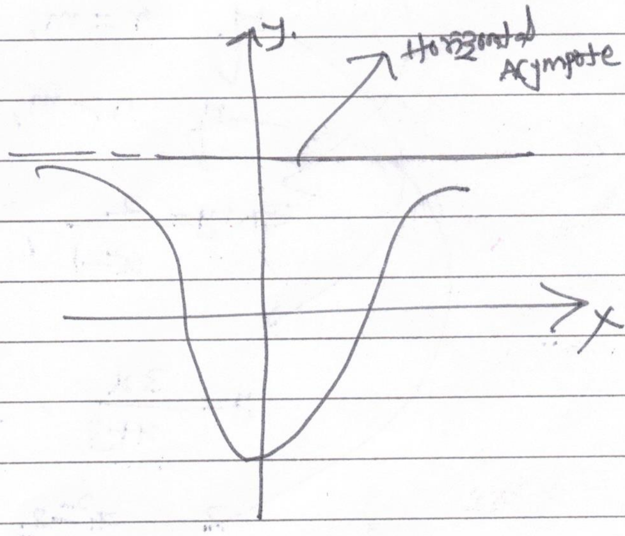




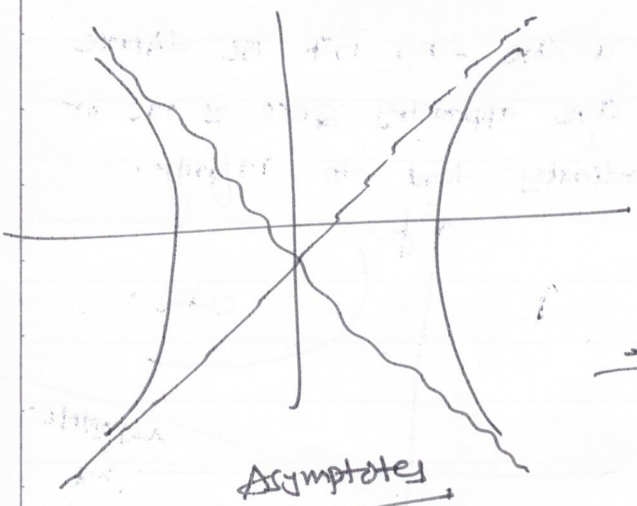




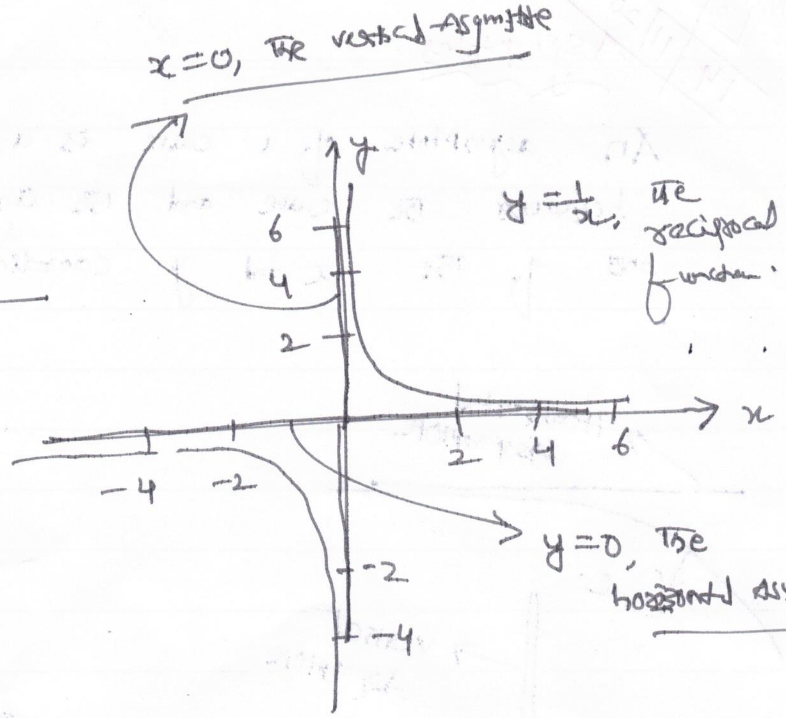




18. Hyperbola :



Asymptotes



Horizontal Asymptote :

to find the horizontal Asymptote, we compare the degree of the numerator with the degree of denominator.

$$f(x) = \frac{ax^n + \cdots}{bx^m + \cdots}$$

If $n < m$, then the horizontal Asymptote is the x-axis ($y=0$).

If $n = m$, then the horizontal Asymptote is $y = a/b$.

If $n > m$, Then there is no horizontal Asymptote (There is an oblique asymptote).

Ex: $y = \dfrac{2}{x^2-1}$, $\displaystyle\lim_{x\to\infty} \frac{2}{x^2-1} = 2 \lim_{x\to\infty} \frac{1}{x^2-1}$

$$= 2 \lim_{x\to\infty} \frac{1/x^2}{1-1/x^2} = 0$$

$\therefore$ Asymptote is x-axis $y=0$.

$y = \dfrac{3x}{x+3}$, $\displaystyle 3\lim_{x\to\infty} \frac{1}{1+3/x} = 3$ Asymptote is $y=3$.

2. $\dfrac{z^2-2}{z+1}$, $\displaystyle\lim_{z\to\infty} \frac{1-2/z^2}{\frac{1}{z}+1/z^2}$, No horizontal Asymptotes

Vertical Asymptote

A rational function $g(x) = p(x)/q(x)$ may have a vertical asymptote $x=a$ only if the denominator is zero at $x=a$ and the numerator is different from 0 at $x=a$, then $g(x)$ does have a vertical asymptote at $x=a$.

Ex: $f(x) = \dfrac{2(x+1)(x-3)}{x(x-5)}$, then

$f(x)$ has two vertical asymptotes $x=0$, $x=5$.

Ex: Find horizontal and vertical asymptotes of a curve $f(x) = 4 - \dfrac{3}{x}$.

$$\lim_{x\to\infty}\left(4 - 3/x\right) = 4 - \lim_{x\to\infty} 3/x = 4.$$

$\therefore$ $y = 4$ is a horizontal asymptote of $f(x)$.

$$\lim_{x\to-\infty}\left(4-3/x\right) = 4 - 3\lim_{x\to-\infty} 1/x = 4$$

vertical asymptote is, $x=0$.

Ex: $f(x) = \dfrac{2x^2+x}{x^2+1} = \dfrac{2+1/x}{1+1/x^2}$, $\lim_{x\to\infty} \dfrac{2+1/x}{1+1/x^2} = 2$

$\therefore$ Horizontal Asymptote $y=2$

$f(x) = \dfrac{3x+2}{4x^2}$, $\rightarrow$ num degree < bottom.
$\therefore$ Horizontal Asymptote, $y=0$

$f(x) = \dfrac{2x^3+4}{x+3}$, $\Rightarrow$ num degree > bottom.
No horizontal Asymptote.
(There is a slant Asymptote / oblique)

Ex: $y = \dfrac{x-3}{x^2-6x+8} = \dfrac{x-3}{(x-4)(x-2)}$ $\rightarrow$ set denominator to 0

$\therefore$ $x=4$, $x=2$
two vertical Asymptotes.

Oblique/slant Asymptote.

If $m = n+1$, then, $f(x) = p(x)/q(x)$ can be written as

$f(x) = mx + c + \dfrac{\text{Remainder}}{Q(x)}$ $\rightarrow 0$, $Q(x) \rightarrow$ lower degree term

$\boxed{y = mx+c}$ is slant Asymptote.

Say $R(x) = \dfrac{x^3 + 2x^2 - 3x + 5}{x^2 - 3x + 4}$ → degree : 3 / → degree : 2

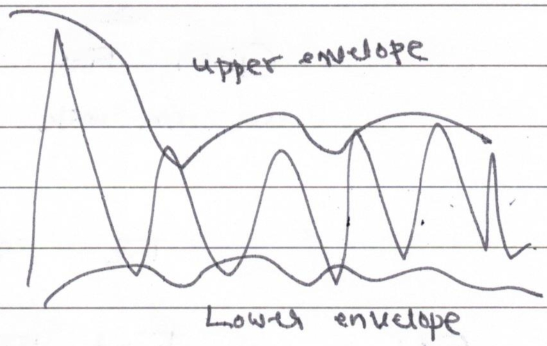
$$x^2 - 3x + 4 \overline{\left) \begin{array}{l} x^3 + 2x^2 - 3x + 5 \\ \cancel{x^3} - 3x^2 + 4x \\ - \quad + \quad - \\ \hline \quad 5x^2 - 7x + 5 \\ \quad 5x^2 - 15x + 20 \\ \quad - \quad + \quad - \\ \hline \qquad 8x - 15 \end{array} \right(} x + 5$$

$\therefore$ $x + 5$ + a remainder

$\therefore$ Oblique asymptote at $y = x + 5$.

Envelope :

It is a smooth curve outlining its extremes.
The envelope thus generalizes the concept of a constant amplitude into an instantaneous amplitude.



— It is defined as the locus of the ultimate intersections of consecutive curves.

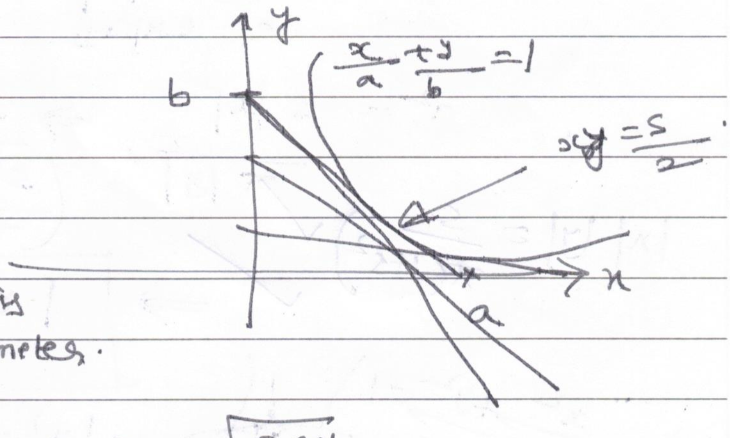
Ex: Envelope of the family of straight lines with the line segments intercepted by the coordinate axes form triangles of equal area.

write the eqn of line in intercept form

$$\frac{x}{a}+\frac{y}{b}=1,$$

where a, b are intercepts of x and y respectively.
By symmetry, it suffices to consider the lines in the first quadrant. i.e we assume $a>0$, $b>0$,

Area of right triangle $S=\frac{ab}{2}$.



$$\Rightarrow b=\frac{2S}{a}.$$

$$\frac{x}{a}+\frac{y}{2S/a}=1$$

line segment a is parameter.

Diff. w.r.to a,

$$\frac{-x}{a^2}+\frac{y}{2S}=0 \Rightarrow a=\sqrt{\frac{2Sx}{y}}$$

$$\therefore \frac{x}{a}+\frac{ay}{2S}=1 \Rightarrow \frac{x}{\sqrt{2Sx/y}}+\frac{\sqrt{\frac{2Sx}{y}}\,y}{2S}=1 \Rightarrow$$

$$\frac{\sqrt{xy}}{\sqrt{2s}} + \frac{\sqrt{xy}}{\sqrt{2s}} = 1 \Rightarrow 2\sqrt{xy} = \sqrt{2s}$$

$$\Rightarrow \sqrt{xy} = \sqrt{\frac{s}{2}}$$

Given that $x>0,\ y>0$ in the first quadrant, we can write $\boxed{xy = \frac{s}{2}}$

The eqn of envelope is hyperbola.

Ex: Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having the same area.

Sol: The area of an ellipse is

$$S = \pi ab \Rightarrow b = S/\pi a.$$

$$\frac{x^2}{a^2} + \frac{y^2}{(S/\pi a)^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{\pi^2 y^2 a^2}{S^2} = 1.$$

where semi-axis a is a parameter

Diff. w.r to $a \Rightarrow$

$$-\frac{2x^2}{a^3} + \frac{2\pi^2 y^2 a}{S^2} = 0 \Rightarrow \frac{x^2}{a^3} = \frac{\pi^2 y^2 a}{S^2} \Rightarrow S^2 x^2 = \pi^2 y^2 a^4 \Rightarrow$$

$$S|x| = \pi |y| a^2$$

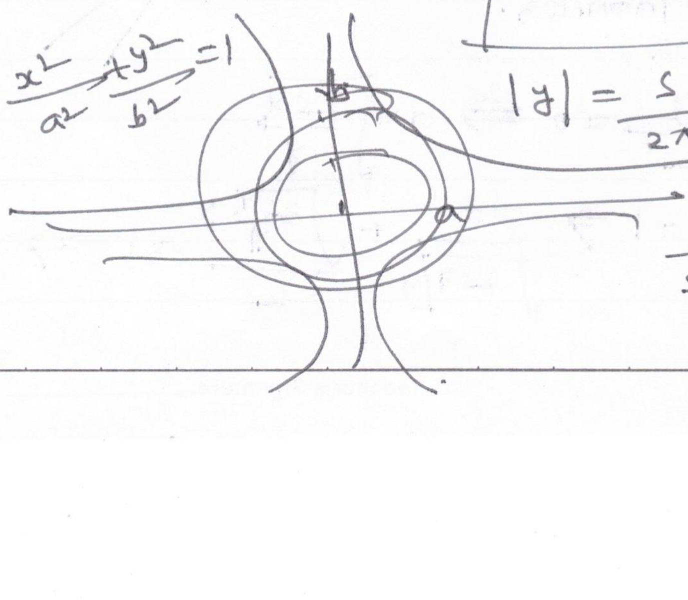
we express a^2 and substitute into the first eqn.

$$a^2 = \frac{S|x|}{\pi|y|}, \Rightarrow \frac{x^2}{\left(\frac{S|x|}{\pi|y|}\right)} + \frac{\frac{\pi^2 y^2 S|x|}{\pi|y|}}{S^2} = 1$$

$$|x|\,|y| = \frac{S}{\pi(1+s^2)}$$

$$\Rightarrow \boxed{|y| = \frac{S}{2\pi|x|}} \Rightarrow |x|\,|y| = \frac{S}{2\pi}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$|y| = \frac{S}{2\pi|x|} \quad \Rightarrow \frac{x^2\pi|y|}{S|x|} + \frac{\pi^2 y^2 \cancel{S}|x| S^2}{\pi|y|} = 1$$

$$\frac{x^2}{S^2 a^2}, \quad \frac{x^2}{S x}\pi y + \frac{y^2}{S^2}\pi^2 \frac{\cancel{S}\sqrt{x}}{\sqrt{\pi y}} = 1$$

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1). consider the curve $y = x^3 + y^3 - 3axy = 0$.

find all Asymptoty.

$f(x,y) = x^3 + y^3 - 3axy \Rightarrow \frac{y}{x^3} = 1 + \left(\frac{y}{x}\right)^3 - 3ax\left(\frac{y}{x^3}\right)$

$\Rightarrow \frac{y}{x^3} = 1 + \left(\frac{y}{x}\right)^3 - \frac{3ay}{x^2}$

$\frac{y}{x^3} = 1 + m^3 - \frac{3ay}{x^2}$ | let $y/x = m$.

$\lim_{x \to \infty} \frac{y}{x^3} = \lim_{x \to \infty} \left(1 + m^3 - \frac{3ay}{x^2}\right) = 1 + m^3 = 0$. $y + x = 0$

$(m+1)(m^2 - m + 1) = 0$.

$m = -1$ is only real root. when $m = 1$, $\frac{y}{x} = 1$

~~diff~~

~~$3x^2 + 3y^2$~~

$x + y = p$ $\Rightarrow y + x = p$.

$y = p - x$. say.

p is a variable $\to c$

when $x \to \infty$

$y = p - x$.

$x^3 + (p-x)^3 - 3ax(p-x) = 0$.

$\cancel{x^3} + p^3 - \cancel{x^3} - 3p^2x + 3px^2 - 3apx + 3ax^2 = 0$

$3ax^2 - 3apx + p^3 = 0$.

$\Rightarrow 3a - \frac{3apx}{x^2} + \frac{p^3}{x^2} = 0$.

$\Rightarrow x \to \infty$.

~~$3a = 3apx$~~

$(p-x)^3$

$= p^3 - x^3 - 3p^2x + 3px^2$

$(3a)x^2 + (-3ap)x + p^3 = 0$.

~~$x = \frac{3ap \pm \sqrt{9a^2p^2 - 12ap^3}}{6a}$~~

~~replace p by (p+a)~~. replace a by $a+p$.

$3(p+a)x^2 - 3(a+p)px + p^3 = 0$.

$3(p+a)x^2 - 3(p^2 + ap)x + p^3 = 0$.

udir in second deal.

$3(p+a) - 3(p^2 + ap)\frac{1}{x} + \frac{p^3}{x^2} = 0$.

$x \to \infty$.

$3(p+a) = 0$ or $\Rightarrow p = -a$.

$\Rightarrow y = -a - x$.

$\Rightarrow x + y = -a$.

$x + y + a = 0$

Teacher's Signature:

$$x^3+y^3-3axy=0.$$

$$\underline{y=mx+c}$$

$$x^3+(mx+c)^3-3ax(mx+c)=0.$$

$$x^3+m^3x^3+c^3+3m^2cx^2+3c^2mx-3amx^2-3axc=0$$

$$(1+m^3)x^3+x^2(3m^2c-3am)+3cx(cm-a)+c^3=0.$$

$$\Rightarrow (1+m^3)+\frac{1}{x}(3m^2c-3am)+\frac{3c}{x^2}(cm-a)+\frac{c^3}{x^3}=0$$

$$x\to\infty.$$

$$\Rightarrow\quad m^3+1=0\Rightarrow(m+1)(m^2-m+1)=0$$

complx

$$\Rightarrow m=-1$$

$$y=-x+c\Rightarrow \underline{x+y=c}\Rightarrow \underline{y=c-x}.$$

$$x^3+(c-x)^3-3ax(c-x)=0.$$

$$\cancel{x^3}+c^3-\cancel{x^3}-3c^2x+3cx^2-3acx+3ax^2=0$$

$$3(a+c)x^2-3cx(a+c)+c^3=0.$$

quadratic $\quad a+c=p.$

divide by x^2.

$$3(a+c)-\frac{3c}{x}(a+c)+\frac{c^3}{x^2}=0.$$

$$\downarrow 0 \qquad x\to 0$$

$$a=-c.$$

$$\Rightarrow c=-a.$$

$$\therefore y=-a-x\Rightarrow x+y+a=0.$$

Ex:2 : Find Asymptotes of $x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0$.

Substitute $\underline{y=mx+c}$

$$x^3+2x^2(mx+c)-x(mx+c)^2-2(mx+c)^3+x(mx+c)-(mx+c)^2-1=0.$$

$$\Rightarrow$$

$$x^3(1+2m-m-2m^3)+x^2(2c-2mc-6m^2c+m-m^2)+\ldots=0$$

$\therefore$ $1+2m-m-2m^3=0$ and

$2c-2mc-6m^2c+m-m^2=0$.

$\Rightarrow c = m^2-m \Big/ 2(1-m-3m^3)$.

when, $m=1, c=0$

$m=-1, c=-1$

$m=-1/2, c=1/2$

$\therefore$ the required envelopes are:

$y=mx+c$, $y=x$	$y=mx+c$, $y=-x-1$, $\Rightarrow y+x+1=0$	$y=mx+c$, $=\frac{-1}{2}x-\frac{1}{2}$. $\Rightarrow 2y+x+1=0$.

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Curve Tracing

(1). A curve is symmetrical about x-axis if the powers of y which occur in its eqn are all even.

(2). A curve is symmetrical about y-axis if the powers of x which occur in its eqn are even.

(3). A curve is symmetrical about the line $y=x$ if on interchanging x and y its eqn does not change.

(4). A curve is symmetric about the line $y=-x$ if the eqn of the curve remains unchanged when x and y are replaced by $-y$ and $-x$ respectively.

Ex1: Trace the curve $y=x^3-12x-16$.

Since the powers of x and y are not even, this is not symmetric about any line.

put $x=0, y=0 \Rightarrow 0 \neq 0-0-16$

Hence origin does not lie on it.

$y=x^3-12x-16 \Rightarrow \frac{dy}{dx}=0 \Rightarrow 3x^2-12=0 \Rightarrow$

$x=\pm 2$.

when $x=-2, y=-8+24-16=0$

$x=2, y=8-24-16=-32$

$\frac{d^2y}{dx^2}=6x$.

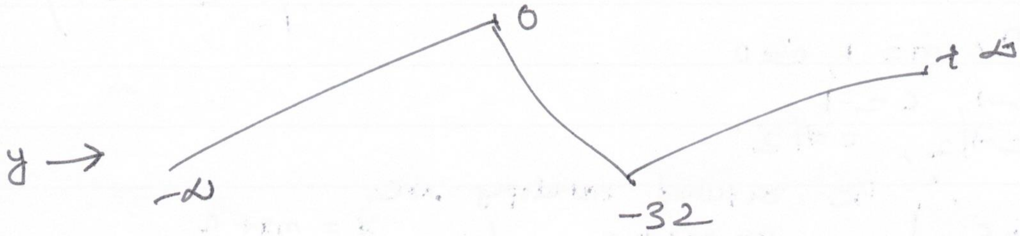
$x^3 - 12x - 16 = 0$

$x = -2$

$x = 4$,

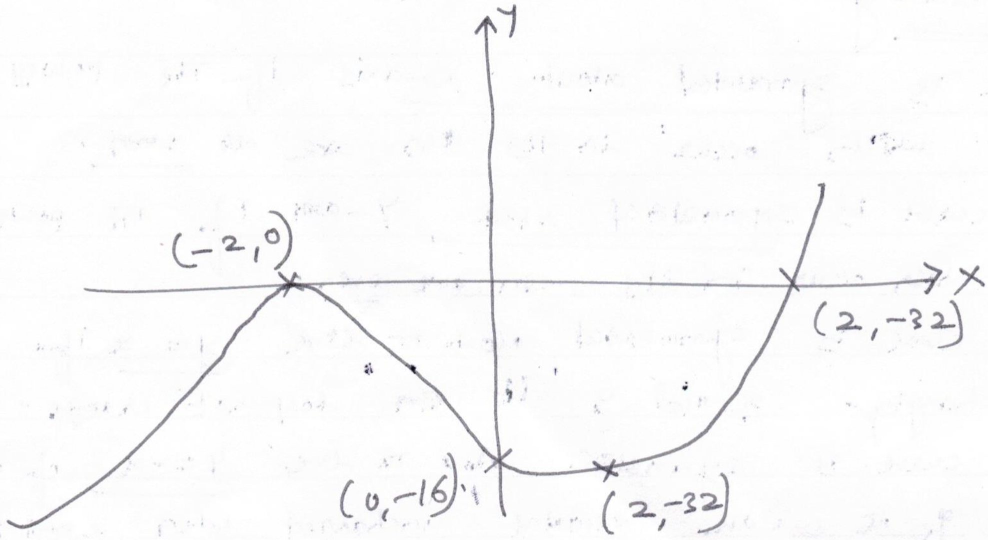
$\cancel{-8 + 24 - 16} = 0$

$64 - 48 - 16 = 0$



∴ The points of intersection are

$(-2, 0), (4, 0), (0, -16)$



Sketching of curve: Steps

1) Intersection with coordinate axes.
2) Critical points.
3) Region of increase & region of decrease
4) Maxima and minima
5) Behavior of y as x becomes → large +ve and large → −ve.
6) Values of x near which y becomes large +ve or large −ve.

Remark: Suppose, you have a function f defined for all sufficiently large numbers, we get substantial information concerning our function by investigating how it becomes as x becomes large.

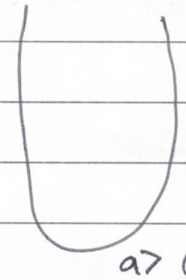
For example, $\sin x$ oscillates between -1 and $+1$ no matter how large x.

However, polynomials do not oscillate.

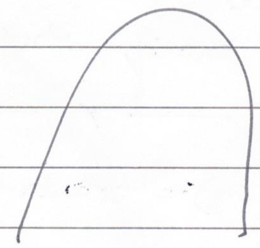
$$f(x) = x^2, \quad x \to \text{large}, \ x^2 \to \text{too large}.$$

$y = ax^2+bx+c; \ a \neq 0$.

Two essential cases,



$a > 0$



$a < 0$

Ex2: Sketch the graph $y = -3x^2+5x-1$.

$$y = f(x) = x^2\left(-3+\frac{5}{x}-\frac{1}{x^2}\right), \quad x \to \text{large +ve and −ve}$$

$$x^2 \to \text{large +ve}$$

$f'(x) = -6x+5, \ f'(x)=0 \Rightarrow -6x+5=0 \Rightarrow x = 5/6$.

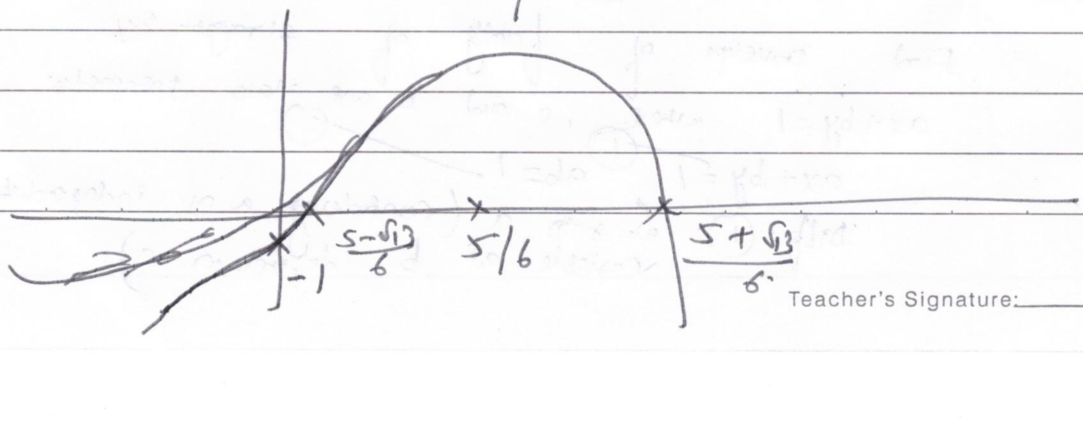
exactly one critical point.

$$f''(5/6) = -1 > 0$$

$\therefore$ The critical point is a maximum, since the parabola bends down.

The curve crosses the x-axis where $-3x^2+5x-1=0$.

$$x = (5 \pm \sqrt{13})/6$$



Remark :

For any parabola :

① . Looking at what happens when x becomes large +ve or −ve tells us whether the parabola bends up or down.

② . A quadratic function $f(x) = ax^2+bx+c$, with $a \neq 0$ has only one critical point when $f'(x) = 2ax+b = 0 \Rightarrow x = -b/2a$. Knowing whether the parabola bends up or down tells us whether the critical point is max/min and the value $x = -b/2a$ tells us exactly where this critical point lies.

———— X ————

Ex 3 : $f(x) = x^3 - 2x + 1$, Sketch the graph.

Ex : Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants.

Sol : Diff. $y = mx + am^p$ w.r.to the parameter m,

$$0 = x + apm^{p-1} \Rightarrow m = \left(\frac{-x}{pa}\right)^{1/p-1}$$

using this, eliminate $m \Rightarrow$

$$y = \left(\frac{-x}{pa}\right)^{1/p-1} + a\left(\frac{-x}{pa}\right)^{p/p-1}$$

$$\Rightarrow y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^p$$

$$\Rightarrow \underline{ap^p y^{p-1} = -x^p p^{p-1} + (-x)^p}$$ which is required envelope.

Ex 2 : Find envelope of family of straight lines $ax+by=1$, where a and b are two parameters

$ax+by=1$ ① $ab=1$ ②

Diff. ① w.r.to a (considering a as independent variable and b depends on a).

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$$x + \frac{db}{da}\, y = 0 \Rightarrow \frac{db}{da} = -\frac{x}{y}$$

Diff — (2) w r to a.

$$b + a\frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -b/a$$

$$\therefore \; -\frac{x}{y} = -\frac{b}{a} \Rightarrow \frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{2x},\; b = \frac{1}{2y} \;—\; *$$

$$\rightarrow \frac{1}{2x}x + \frac{1}{2y}y = 1 \qquad ab = 1$$

$$\text{using } — (*) \text{ in eqn} — (2) \Rightarrow \frac{1}{4xy} = 1 \Rightarrow$$

$$xy = 1/4.$$

Ex.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;—\; (1)$$

$$\pi ab = S \Rightarrow ab = S/\pi \;—\; (2)$$

Diff. — w. r. t. a.

$$\cancel{\frac{2x}{a^2}} \quad -\frac{x^2}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{b^2x}{a^2y}$$

Diff — (2) w. r. t. a.

$$\pi\left(b + a\frac{db}{da}\right) = 0 \Rightarrow$$

$$\cancel{\pi\left(b + a\frac{b^2x}{a^2y}\right)} = b\pi\left(1 - \frac{bx}{ay}\right) = 0.$$

$$b \neq 0 \Rightarrow bx = ay \Rightarrow \frac{x}{a} = \frac{y}{b} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\Rightarrow \frac{x/a}{1} = \frac{y/b}{1} = \cancel{\frac{x+y}{a+b}} = \frac{1}{2} \quad 1/2$$

$$= \frac{x^2/a^2 + y^2/b^2}{1+1} = \frac{1}{2} \Rightarrow \frac{x^2}{a^2} = 1/2 \quad y^2/b^2 = 1/2$$

$$\cancel{\frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \Rightarrow 2x^2 = 2a^2 \Rightarrow x = \pm a} \qquad y = \pm b$$

$$\Rightarrow \cancel{xy = S/\pi.}$$

$$\frac{x}{a} = \sqrt{1/2},\quad y/b = \sqrt{1/2}, \quad x = \frac{a}{\sqrt{2}},\; y = \frac{b}{\sqrt{2}}.$$

$$\Rightarrow ab = S/\pi \Rightarrow \sqrt{2}x \times \sqrt{2}y = S/\pi.$$

$$xy = \frac{S}{2\pi}.$$

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EX2: Sketch the graph of $f(x) = x^3 - 2x + 1$.

① If $x \to \infty$, then $f(x) \to \infty$

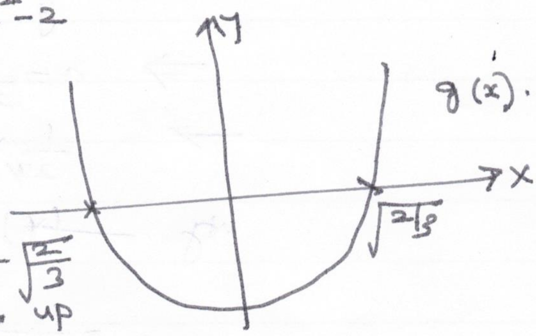
$x \to -\infty$, $f(x) \to -\infty$

② $f'(x) = 0 \Rightarrow 3x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2/3}$.

The critical points are $x = \sqrt{2/3}$, $x = -\sqrt{2/3}$

③. Let $f'(x) = g(x) = 3x^2 - 2$

The graph of g is a parabola and the intersects of the graph g are precisely the critical points of f.



g is a parabola bending up

$a = 3 > 0$.

$f'(x) = g(x) > 0 \Leftrightarrow x > \sqrt{2/3}$ and $x < -\sqrt{\frac{2}{3}}$ when $g(x) > 0$.

f is strictly increasing on

$x \geq \sqrt{2/3}$ and $x \leq \sqrt{+\frac{2}{3}}$

IIIly $f'(x) = g(x) < 0 \Leftrightarrow -\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$, when $g(x) < 0$.

f is strictly decreasing in this interval.

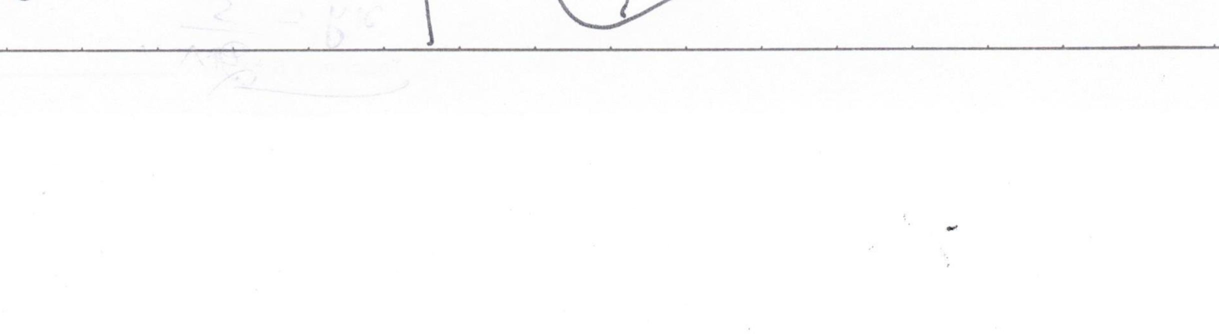
$\therefore -\sqrt{\frac{2}{3}}$ is a local maximum for f and $\sqrt{\frac{2}{3}}$ is a local minimum.

④. $f''(x) = 6x$, $f''(x) > 0$ if and only if $x > 0$.

$f''(x) < 0$ if and only if $x < 0$.

$\therefore$ f is bending up for $x > 0$ and f is bending down for $x < 0$.

$\therefore$ There is an inflection point at $x = 0$.



EXPT. NO. | NAME: Ex: Sketch the graph of the function

$$f(x) = \frac{x^2+2x-6}{x-1}$$

Sol: $f(x) = \frac{x^2+2x-6}{x-1} = x+3-\frac{3}{x-1}$.

$\therefore$ $x=1$ and $y=x+3$ are two asymptoty.

$f'(x) = 1+\frac{3}{(x-1)^2}$, $f''(x) = \frac{-6}{(x-1)^3}$.

f is increasing in $(-\infty,1)$ and $(1,\infty)$.

f is convex if $x<1$ and concave if $x>1$.

Ex: consider a quotient of polynomial like

$$Q(x) = \frac{x^3+2x-1}{2x^3-x+1} = \frac{\cancel{x^3}\left(1+\frac{2}{x^2}-1/x^3\right)}{\cancel{x^3}\left(2-1/x^2+1/x^3\right)} = \frac{1+\frac{2}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x^2}+\frac{1}{x^3}}$$

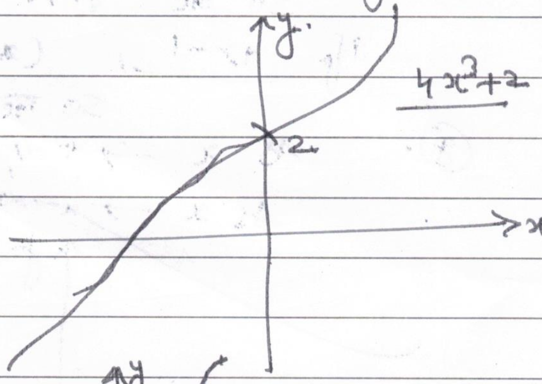
$\lim_{x\to\infty} Q(x) = 1/2$.

If $x \to \pm\infty$ then $Q(x) \longrightarrow 1/2$.

Ex: Sketch the graph of the curve $f(x) = 4x^3+2$.

$f'(x) = 12x^2 > 0 \ \forall\ x \neq 0$, There is only one critical point $x=0$, Hence f is strictly increasing $\forall x$.

$f''(x) = 24x$, $\therefore x=0$ is also a point of inflection.



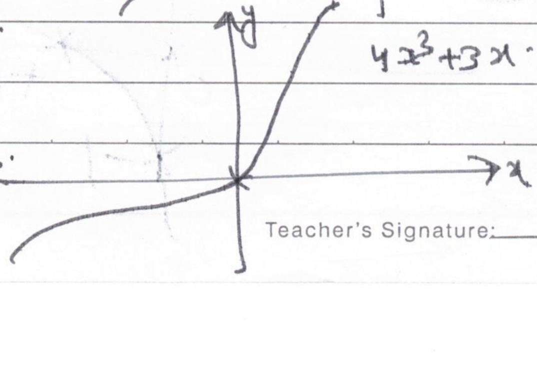
Ex: $f(x) = 4x^3+3x$.

$f''(x) = 24x$.

$x=0$ is an inflection point.

$\forall\ x<0$, bendy down +

$\forall\ x>0$, bendy up.



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CASE 1: The inflection point is a critical point, where the derivative of f is zero.

So the curve is flat at the critical point.

CASE 2: The derivative at the inflection point is $f'(0) = 3$, it is +ve.

EX: $y = f(x) = \frac{x-1}{x+1} = 1 - \frac{1}{x+1}$

(a) when $x=0$, $f(x) = -1$
$x=1$, $f(x)=0$

(b) $\frac{dy}{dx} = \frac{2}{(x+1)^2}$ never zero.
$\therefore$ NO critical points.

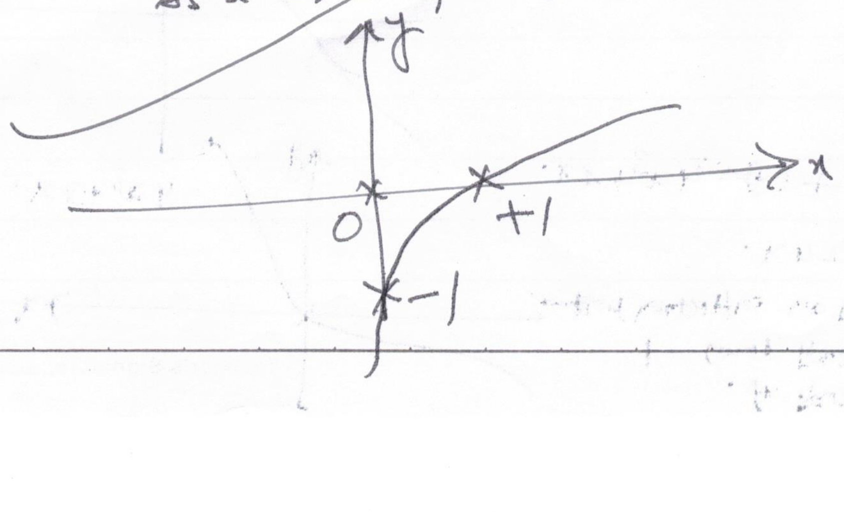
(c) $\because (x+1)^2 > 0$ $\therefore \frac{dy}{dx}$ always +ve, whenever it is defined for $x \neq -1$ and neither is the derivative.
$f'(x) > 0 \; \forall \; x \neq -1$
$\therefore$ f is increasing in the region $x < -1$ and increasing $x > 1$.

(d) There is no region of decrease.

(e) $\because \frac{dy}{dx} \neq 0$, there is no relative max or min.

(f) further, $f''(x) = -4/(x+1)^3$
There is no point of inflection since $f''(x) \neq 0 \; \forall \; x$ whenever f is defined.
if $x < -1$, then $(x+1)^3$ is -ve and $f''(x) > 0$.
So the graph is bending upward.
if $x > -1$, $(x+1)^3$ is +ve and $f''(x) < 0$.
So the graph is bending downward.

(g) As $x \to +\infty$, $f(x) \to 1$
As $x \to -\infty$, $f(x) \to 1$



EXPT. NO. | NAME: | Page No. 47 | Date:

EX1: A farmer buys a bull weighing 600 lbs, at a cost of $180. It costs 15 cents per day to feed the animal, which gains 1 lb per day. Every day that the bull is kept, the sale price per pound declines according to the formula

$$B(t) = 0.45 - 0.00025t,$$ where t is the number of days.

How long should the farmer wait to maximize profit?

Sol: total cost after time t is given by

$$f(t) = 180 + 0.15t.$$

total sales amount =

$$s(t) = (0.45 - 0.00025t)(600 + t).$$
$$= -0.00025t^2 + 0.30t + 270.$$

Hence profit, $P(t) = s(t) - f(t) = -0.00025t^2 + 0.15t + 90$

$\therefore P'(t) = -0.0005t + 0.15 = 0 \Rightarrow$

$$t = \frac{0.15}{0.0005} = 300$$

Hence 300 days for the farmer to wait before selling the bull, provided the value of t give a max.

Check point be a max.

EX2: A business makes automobile transmissions selling for $400. The total cost of marketing x units is

$$f(x) = 0.02x^2 + 160x + 400,000.$$

How many transmissions should be sold for maximum profit?

SOL: Let $P(x)$ be the profit coming from selling x units. Then $P(x)$ is the difference between the total receipts and the costs of marketing. Hence,

$$P(x) = 400x - (0.02x^2 + 160x + 400,000).$$
$$= -0.02x^2 + 240x - 400000$$

we want to know when $P(x)$ is max $\Rightarrow$

$$P'(x) = 0.$$

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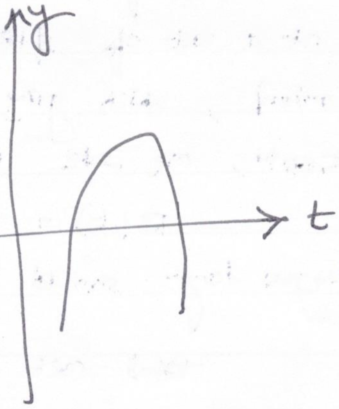
$p'(x) = -0.04x + 240 = 0$

$\Rightarrow x = 6000$

$y = px$ is a parabola which bends down because the leading coefficient is $-0.02 < 0$.

Hence, the critical point is a maximum.



x

EX: A wire 24cm long is cut in two, and one part is bent into the shape of a circle, and the other into the shape of a square. How should it be cut if the sum of the areas of a circle and the square is to be

(a) Minimum (b) Maximum.

Sol: Let x be the side of the square.

$4x$ is the perimeter of the square.

$0 \leq 4x \leq 24 \Rightarrow 0 \leq x \leq 6$

Also, length of the circle, i.e circumference: $24 - 4x = 2\pi r$

$\Rightarrow r = \frac{12 - 2x}{\pi}$

Area of a circle, πr^2.

Sum of areas: $\pi r^2 + x^2$.

$\therefore f(x) = \pi \left(\frac{12 - 2x}{\pi} \right)^2 + x^2$

for max/min, $f'(x) = 0$.

END of UNIT #1