End Semester Monsoon 2022-23 | Date: 19.2.2023, Time: 2.00 PM to 5.00 PM

Mathematics I (MC1101) | Full Marks: 100

Instructions: 1. All questions are compulsory.

2. Both parts of each question must be solved together at one place in the answer sheet.

Q-1 a) Find the Taylor's series expansion (up to three non-zero terms) of $f(x) = \sin(a \sin^{-1} x)$ about $x = 0$, where a is a constant. [7]

b) Show that the radius of curvature at any point (r, θ) on the curve $r^2 = a^2 \sec 2\theta$, is proportional to r^3. [6]

Q-2 a) Express the integral $\int_{-1}^{+1} (1+x)^{p-1} (1-x)^{q-1} dx$ in terms of Gamma function. [7]

b) Find the value of p for which the integral $\int_0^1 \frac{\ln(1+x)}{x^p} dx$ converges. [6]

Q-3 a) Examine the differentiability of the following function $f: \mathbb{R}^2 \to \mathbb{R}$ at (0,0),

$$f(x,y) = \begin{cases} \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2+y^2} & \text{for } y \neq 0 \\ 0 & \text{for } y = 0. \end{cases}$$

[6]

b) If $x^x y^y z^z = k$ (a constant), find the value of $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at $(x,y,z) = (2,2,2)$. [6]

Q-4 a) Find the critical points of function $f(x,y) = e^{1+x^2-y^2}$. Use the second derivative test for $f(x,y)$ to decide whether the each of critical points is a local maximum, local minimum or neither. [6]

b) Using the Lagrange multiplier method, find the extrema of $f(x,y,z) = x - y + z$ subject to $x^2 + y^2 + z^2 = 2$. [6]

Q-5 a) Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)\, dxdy$, where R is the Rhombus with successive vertices at $(\pi, 0), (2\pi, \pi), (\pi, 2\pi)$ and $(0, \pi)$. [6]

b) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{dxdydz}{\sqrt{x^2+y^2+z^2}}$ by changing into spherical coordinate system. [6]

Q-6 a) Find the area and moment of inertia I_x and I_y of the plane region in the first quadrant which is inside the circle $r = 4a \cos\theta$ and outside the circle $r = 2a$. [6]

b) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3$ and $z = 0$. [6]

Q-7 a) Prove that div(f**v**) = f(div **v**) + grad(f).**v** where f is a scalar function and **v** is a vector function given by $\mathbf{v}(x,y,z) = v_1(x,y,z)\,\hat{\mathbf{i}} + v_2(x,y,z)\,\hat{\mathbf{j}} + v_3(x,y,z)\,\hat{\mathbf{k}}$. [6]

b) Find the equation of the tangent plane and the unit normal vector to the surface $yz - zx + xy + 5 = 0$ at (1,-1,2). [7]

Q-8 a) B[illegible] find the area of the square region bounded by the sides with vertices (2,0), (3,1), (2,2) and (1,1). [7]

b) Use Gauss' divergence theorem to evaluate $\iint_S \boldsymbol{F}.\hat{\mathbf{n}}\, ds$ where $\boldsymbol{F} = 2xz\,\hat{\mathbf{i}} + (1 - 4xy^2)\,\hat{\mathbf{j}} + (2z - z^2)\,\hat{\mathbf{k}}$ and S is the surface of the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$. [6]

1. (a) Find the Taylor's series expansion of $f(x) = sin(asin^{-1}x)$ about $x = 0$, where a is a constant. (first term [illegible] term)

(7 Marks)

(b) Show that the radius of curvature at any point (r, θ) on the curve $r^2 = a^2 sec2\theta$, is proportional to r^3.

(6 Marks)

Solution: (a) The Taylor's series expansion of $f(x)$ about $x = 0$ is given by [formula; ...]

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

(1 Mark)

Now, for $y = f(x) = sin(asin^{-1}x)$, $y' = \frac{a}{\sqrt{1-x^2}}cos(asin^{-1}x)$, or that $(1-x^2)y_1^2 = a^2cos^2(asin^{-1}x) = a^2(1-sin^2(asin^{-1}x)) = a^2(1-y^2)$. Differentiating again w.r.t. x, we have $2(1-x^2)y_1y_2 - 2xy_1^2 = -2a^2yy_1$, or that $(1-x^2)y_2 - xy_1 + a^2y = 0$ (as $y_1 \neq 0$).

(2 Marks)

Differentiating above n-times, we get

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - (xy_{n+1} + ny_n) + a^2y_n = 0,$$

or

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0.$$

Now, for $x = 0$,

$$y_{n+2}(0) = (n^2 - a^2)y_n(0), n = 0, 1, 2, ...$$

(2 Marks)

Thus

$$f(0) = y(0) = 0,$$
$$f'(0) = y_1(0) = a,$$
$$f''(0) = y_2(0) = 0$$
$$f'''(0) = y_3(0) = a(1^2 - a^2),$$
$$f^{iv}(0) = y_4(0) = 0,$$
$$f^{v}(0) = y_5(0) = a(1^2 - a^2)(3^2 - a^2),$$
$$....$$

(1 Mark)

Therefore, the required Taylor's series is

$$f(x) = ax + \frac{a(1^2 - a^2)}{3!}x^3 + \frac{a(1^2 - a^2)(3^2 - a^2)}{5!}x^5 + ...$$

(1 Mark)

(b) $r^2 = a^2sec2\theta \Rightarrow 2lnr = lna^2 + lnsec2\theta \Rightarrow 2\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{sec2\theta}.sec2\theta.tan2\theta.2$. Thus $\frac{dr}{d\theta} = r' = r.tan2\theta$.

(2 Marks)

Also, $r'' = \frac{d^2r}{d\theta^2} = tan2\theta\frac{dr}{d\theta} + r.2.sec^2 2\theta$, or that $r'' = \frac{d^2r}{d\theta^2} = r.tan^2 2\theta + 2r.sec^2\theta$.

(1 Mark)

Now, $k = \frac{r^2+2r'^2-rr''}{[r^2+r'^2]^{3/2}} = \frac{r^2+2r^2tan^2 2\theta - r(r.tan^2 2\theta + 2r.sec^2\theta)}{[r^2+r^2tan^2 2\theta]^{3/2}} = \frac{-r^2sec^2 2\theta}{[r^2sec^2 2\theta]^{3/2}} = -\frac{1}{rsec2\theta} = -\frac{1}{r.\frac{r^2}{a^2}} = -\frac{a^2}{r^3}$.

Thus $\rho = \frac{r^3}{a^2}$, which shows that the radius of curvature is proportional to r^3.

(3 Marks)

Q1: Express the integral $\int_{-1}^{+1} (1+x)^{p-1} (1-x)^{q-1} dx$ in
(a)

Gamma function. [7 Marks]

Sol^n: Given $\int_{-1}^{+1} (1+x)^{p-1} (1-x)^{q-1} dx$

Put $x = \cos 2\theta$ then $dx = -2\sin 2\theta \, d\theta$ 2

$$\therefore \int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1} dx = \int_{\pi/2}^{0} (1+\cos 2\theta)^{p-1} (1-\cos 2\theta)^{q-1} (-2\sin 2\theta \, d\theta)$$

$$= \int_{\pi/2}^{0} (1+2\cos^2\theta - 1)^{p-1} (1-1+2\sin^2\theta)^{q-1} (-4\sin\theta\cos\theta \, d\theta)$$

$$= 4\int_{0}^{\pi/2} 2^{p-1} \cos^{2p-2}\theta \cdot 2^{q-1} \sin^{2q-2}\theta \sin\theta \cos\theta \, d\theta)$$

$$= 2^{p+q} \int_{0}^{\pi/2} \sin^{2q-1}\theta \cos^{2p-1}\theta \, d\theta \quad 3$$

$$= 2^{p+q} \frac{\Gamma\frac{2q}{2} \cdot \Gamma\frac{2p}{2}}{2 \cdot \Gamma\frac{2q+2p}{2}}$$

$$= 2^{p+q-1} \frac{\Gamma p \, \Gamma q}{\Gamma p+q} \quad 2$$

Q2: Find the value of p for which the integral $\int_0^1 \frac{\ln(1+x)}{x^p}\,dx$
(b) converges.

[6 Marks]

Sol^n: We know,

$$\ln(1+x) \le x \qquad \text{for } x \in [0,1]$$

$$\Rightarrow \quad \frac{\ln(1+x)}{x^p} \le \frac{x}{x^p} = x^{1-p}$$

Here, $f(x) = \frac{\ln(1+x)}{x^p}$ and choose $g(x) = x^{1-p}$ so that

3

$$0 \le f(x) \le g(x)$$

Thus,
$$\int_0^1 g(x)\,dx = \int_0^1 x^{1-p}\,dx = \int_0^1 \frac{1}{x^{p-1}}\,dx$$

$$= \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{x^{p-1}}\,dx$$

$$= \lim_{\epsilon \to 0}\left[\frac{x^{2-p}}{2-p}\right]_\epsilon^1 = \lim_{\epsilon \to 0}\left[\frac{1}{2-p} - \frac{\epsilon^{2-p}}{2-p}\right]$$

$$= \frac{1}{2-p}\lim_{\epsilon \to 0}\left[1 - \frac{1}{\epsilon^{p-2}}\right]$$

$g(x) = \frac{1}{x^p}$

if $p<1$.

$\Rightarrow \int_0^1 g(x)\,dx$ will be convergent

$\Rightarrow \int_0^1 \frac{\ln(1+x)}{x^p}\,dx$ will be also convergent.

Clearly, if $p-2<0$, then limit will exists and Integral $\int_0^1 g(x)\,dx$ converges.

Therefore, by Comparison test $\int_0^1 f(x)\,dx = \int_0^1 \frac{\ln(1+x)}{x^p}\,dx$ will converges when $p-2<0$ i.e. $p<2$.

3

Question: Examine the differentiability of the following function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$,

$$f(x,y) = \begin{cases} \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2+y^2} & \text{for } y \neq 0, \\ 0 & \text{for } y = 0. \end{cases}$$

[6]

Solution: First we find

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

....**(1M)**

In order to be differentiable at $(0,0)$, we must have

$$f(0+h, 0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + h\phi(h,k) + k\psi(h,k),$$

where $\phi \to 0, \psi \to 0$ as $(h,k) \to (0,0)$.

$$\implies f(h,k) - 0 = h\cdot 0 + k \cdot 0 + h\phi(h,k) + k\psi(h,k)$$
$$\implies \left(1 - \cos\frac{h^2}{k}\right)\sqrt{h^2+k^2} = h\phi(h,k) + k\psi(h,k)$$
$$\implies \left(1 - \cos\frac{h^2}{k}\right) = \frac{h}{\sqrt{h^2+k^2}}\phi(h,k) + \frac{k}{\sqrt{h^2+k^2}}\psi(h,k).$$

....**(2M)**

In particular, if $k = mh^2$, $m \in \mathbb{R}$ we get

$$1 - \cos\frac{1}{m} = \frac{1}{\sqrt{1+m^2h^2}}\phi(h, mh^2) + \frac{mh}{\sqrt{1+m^2h^2}}\psi(h, mh^2)$$

Taking $h \to 0$, we get

$$1 - \cos\frac{1}{m} = 0,$$

which is not true for all values of m (say $m = 1$).
$\therefore$ The given function $f(x,y)$ is not differentiable at $(0,0)$. ....**(3M)**

Question: If $x^x y^y z^z = k$ (a constant), find the value of $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at $(x,y,z) = (2,2,2)$. **[6]**

Solution: Given that $x^x y^y z^z = k$.
Taking logarithm on both sides, we get

$$x\log x + y\log y + z\log z = \log k \tag{1}$$

Differentiating (1) partially with respect to x, we get

$$x\cdot\frac{1}{x} + \log x + z\cdot\frac{1}{z}\frac{\partial z}{\partial x} + \log z\frac{\partial z}{\partial x} = 0$$
$$\implies (1+\log x) + (1+\log z)\frac{\partial z}{\partial x} = 0$$
$$\implies \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}.$$

Similarly, we get

$$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z}.$$

....(2M)

Now, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$= -\frac{1}{(1+\log z)^2}\left[(1+\log z)\cdot\frac{1}{x} - (1+\log x)\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$
$$= -\frac{1}{(1+\log z)^2}\left[(1+\log z)\cdot\frac{1}{x} + (1+\log x)^2\cdot\frac{1}{z}\cdot\frac{1}{(1+\log z)}\right].$$

At $(x, y, z) = (2, 2, 2)$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(1+\log 2)^2}\left[(1+\log 2)\cdot\frac{1}{2} + (1+\log 2)^2\cdot\frac{1}{2}\cdot\frac{1}{(1+\log 2)}\right]$$
$$= -\frac{1}{(1+\log 2)}\left(\frac{1}{2}+\frac{1}{2}\right) = -\frac{1}{(1+\log 2)} = -\frac{1}{\log(2e)}.$$

....(2M)

Again, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1+\log y}{1+\log z}\right)$$
$$= (1+\log y)\cdot\frac{1}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$
$$= \frac{(1+\log y)}{(1+\log z)^2}\cdot\frac{1}{z}\left(-\frac{1+\log x}{1+\log z}\right)$$
$$= -\frac{(1+\log y)(1+\log x)}{z(1+\log z)^3}$$

At $(x, y, z) = (2, 2, 2)$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log 2)^2}{2(1+\log 2)^3} = -\frac{1}{2(1+\log 2)} = -\frac{1}{2\log(2e)}$$

....(2M)

Q4(a) Second derivative Test:

$$f(x,y) = e^{1+x^2-y^2}$$

$$f_x = \frac{\partial f}{\partial x} = (2x)e^{1+x^2-y^2}$$

$$f_y = \frac{\partial f}{\partial y} = (-2y)e^{1+x^2-y^2}$$

Solving $f_x = 0$ and $f_y = 0$; we get: $(x_0, y_0) = (0,0)$. → (1 Mark)

(0,0) is a critical point.

$$f_{xx} = 2e^{1+x^2-y^2} + (2x)^2 e^{1+x^2-y^2}$$

Let $A = f_{xx}(0,0) = 2e$

$$f_{xy} = (2x)(-2y)e^{1+x^2-y^2} = -4xy\,e^{1+x^2-y^2}$$

Let $B = f_{xy}(0,0) = 0$

$$f_{yy} = -2e^{1+x^2-y^2} + (2y)(2y)e^{1+x^2-y^2} = -2e^{1+x^2-y^2} + 4y^2 e^{1+x^2-y^2}$$

Let $C = f_{yy}(0,0) = -2e$ → (2 Marks)

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = \begin{vmatrix} 2e & 0 \\ 0 & -2e \end{vmatrix}$$

$$D = (2e)(-2e) = -4e^2 < 0$$ → (2 Marks)

Hence, the critical point (0,0) is a saddle point. At this point, $f(x,y)$ has neither maximum nor minimum. → (1 Mark)

Q.4 b)

$$f(x,y,z) = x-y+z$$

$$g(x,y,z) = x^2+y^2+z^2 = 2$$

Lagrange Multiplier method

$\nabla f = \lambda \nabla g$ where λ is a constant.

This yields (2 Marks)
$$\begin{cases} 1 = 2x\lambda \Rightarrow \lambda = 1/2x \text{ or } x = \frac{1}{2\lambda} \\ -1 = 2y\lambda \Rightarrow \lambda = -1/2y \text{ or } y = \frac{-1}{2\lambda} \\ 1 = 2z\lambda \Rightarrow \lambda = 1/2z \text{ or } z = \frac{1}{2\lambda} \end{cases}$$

we have $x^2+y^2+z^2=2$

Substituting x, y & z values in $x^2+y^2+z^2=2$, we get

$$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2} = 2$$

$$\frac{3}{4\lambda^2} = 2 \Rightarrow \lambda^2 = \frac{3}{8}$$

Thus $\lambda = \pm\sqrt{\frac{3}{8}}$ or $\lambda = \pm\frac{1}{2}\sqrt{\frac{3}{2}}$ (1 Marks)

For $\lambda = \frac{1}{2}\sqrt{\frac{3}{2}}$, $x = \frac{1}{2\lambda} = \sqrt{\frac{2}{3}}$

Similarly $y = -\sqrt{\frac{2}{3}}$, $z = \sqrt{\frac{2}{3}}$

Hence, For $\lambda = \frac{1}{2}\sqrt{\frac{3}{2}}$, we get $\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ (1 Mark)

For $\lambda = -\frac{1}{2}\sqrt{\frac{3}{2}}$, $(x,y,z) = \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$ (1 Mark)

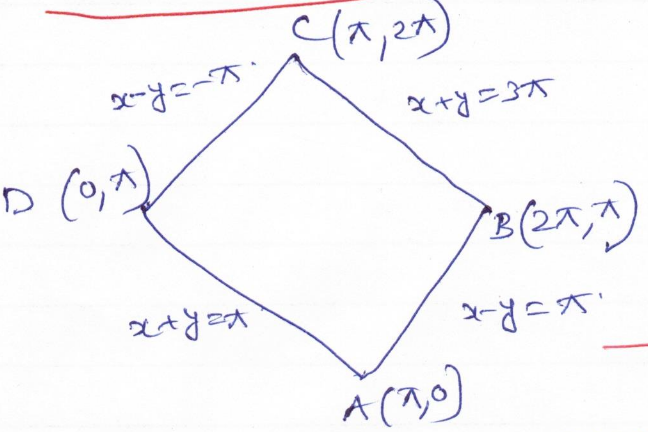
$$f_{max} = \sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}} = 3\sqrt{\frac{2}{3}} = \sqrt{\frac{18}{3}} = \sqrt{6}$$

$$f_{min} = -\sqrt{\frac{2}{3}}-\sqrt{\frac{2}{3}}-\sqrt{\frac{2}{3}} = -3\sqrt{\frac{2}{3}} = -\sqrt{\frac{18}{3}} = -\sqrt{6}$$

maximum of $f(x,y,z) = \sqrt{6}$
minimum of $f(x,y,z) = -\sqrt{6}$ } (1 Marks)

UNIT # 4 (first part), 12 Marks, 25 minute

1) a Evaluate the integral $\iint\limits_R (x-y)^2\cos^2(x+y)\,dx\,dy$, where R is the rhombus with successive vertex at $(\pi,0),(2\pi,\pi),(\pi,2\pi),(0,\pi)$.



$\longrightarrow$ (2)

AB: $\dfrac{x-\pi}{2\pi-\pi}=\dfrac{y-0}{\pi-0}=t$

$x-\pi=\pi t$

$x=(\pi+1)t$

$y=\pi t$

$y-0=\dfrac{\pi}{\pi}(x-\pi)$

$y=x-\pi$

$x-y=\pi$

Now substitute $y-x=u$, $y+x=v$ then

$-\pi\le u\le\pi,\quad \pi\le v\le 3\pi$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{vmatrix}=\frac{1}{2}$$ $\longrightarrow$ (2)

$$\therefore I=\iint\limits_R (x-y)^2\cos^2(x+y)\,dx\,dy=\frac{1}{2}\int_{\pi}^{3\pi}\int_{-\pi}^{\pi}u^2\cos^2 v\,du\,dv$$

$$=\frac{\pi^3}{3}\int_{\pi}^{3\pi}\cos^2 v\,dv=\frac{\pi^3}{6}\int_{\pi}^{3\pi}(1+\cos 2v)\,dv=\pi^4/3$$ $\longrightarrow$ (2)

1(b). Evaluate $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$$

By changing into polar coord system

$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z$

$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$ (2)

ϕ: 0 to $\pi/4$

$r=0$ to $\sec\phi$

θ: 0 to $\pi/2$

$|J|=r^2\sin\phi$ $\longrightarrow$ (2)

$\rho^2=x^2+y^2+z^2$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\phi}\frac{1}{\cancel{r}}\,r^2\sin\phi\,dr\,d\phi\,d\theta=(\sqrt{2}-1)\pi/4$$ $\longrightarrow$ (2)

6/(a) Find the area and moment of inertia I_x and I_y of the plane region in the first quadrant which is inside the circle $r = 4a\cos\theta$ and outside the circle $r = 2a$.

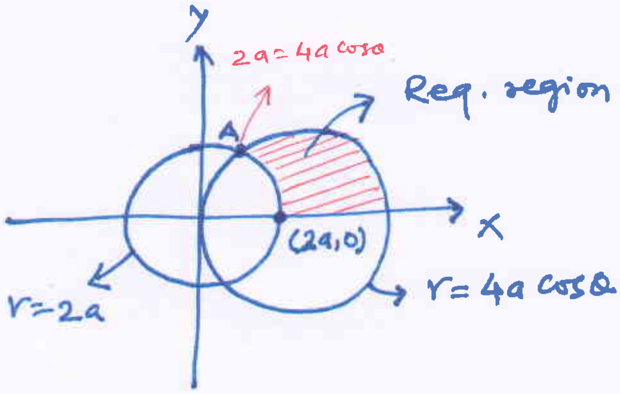
(A) Area : for limit of θ

from 0 to the point A

Point A → $2a = 4a\cos\theta$

$\cos\theta = 1/2$ or $\theta = \cos^{-1}(1/2)$

$\theta = \pi/3$.



limits: $r: 2a \longrightarrow 4a\cos\theta$

$\theta: 0 \longrightarrow \pi/3$

$$\text{Area} = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} r\,dr\,d\theta = \int_0^{\pi/3}\left[\frac{r^2}{2}\right]_{2a}^{4a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}(16a^2\cos^2\theta - 4a^2)\,d\theta$$

$$= 8a^2\int_0^{\pi/3}\cos^2\theta\cdot d\theta - 2a^2\int_0^{\pi/3} d\theta$$

$$= \frac{8a^2}{2}\int_0^{\pi/3} 2\cos^2\theta\cdot d\theta - \frac{2a^2\pi}{3}$$

$$= \frac{8a^2}{2}\int_0^{\pi/3}(1+\cos 2\theta)\,d\theta - \frac{2a^2\pi}{3}$$

$$= \frac{8a^2}{2}\left[\frac{\pi}{3} + \left(\frac{\sin 2\theta}{2}\right)_0^{\pi/3}\right] - 2a^2\pi/3$$

(2) marks.

$$= \frac{8a^2}{2}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right] - 2a^2\pi/3$$

$$= 4a^2\frac{\pi}{3} - 2a^2\pi/3 + a^2\sqrt{3}$$

$$\text{Area} = \frac{2a^2\pi}{3} + a^2\sqrt{3}$$

$$\frac{a^2(2\pi + 3\sqrt{3})}{3}$$

Moment of inertia I_x

Density function $\rho = 1$ (unit)

$$I_x = \iint_R \rho y^2 \, dA$$

$$= \int_0^{\pi/3} \int_{2a}^{4a\cos\theta} (r \sin\theta)^2 \, r \, dr \, d\theta$$

$$= \int_0^{\pi/3} \left[\frac{r^4}{4} \right]_{2a}^{4a\cos\theta} \sin^2\theta \, d\theta$$ —— 2 marks

$$= \frac{1}{4} \int_0^{\pi/3} \left[(4a\cos\theta)^4 - (2a)^4 \right] \sin^2\theta \cdot d\theta$$

$$= 4a^4 \int_0^{\pi/3} \left(16\cos^4\theta - 1\right) \sin^2\theta \, d\theta$$

$$= \frac{4\pi + 9\sqrt{3}}{6} a^4$$

using formula

$$\int_a^b \cos^n\theta \, d\theta = \left[\frac{\cos^{n-1}\theta \sin\theta}{n} \right]_a^b + \frac{n-1}{n} \int_a^b c$$

$$I_y = \iint_R \rho x^2 \, dA = \int_0^{\pi/3} \int_{2a}^{4a\cos\theta} (r\cos\theta)^2 \, r \, dr \, d\theta$$

$$= \int_0^{\pi/3} \left[\frac{r^4}{4} \right]_{2a}^{4a\cos\theta} \cos^2\theta \, d\theta$$ —— 2 marks

$$= \frac{1}{4} \int_0^{\pi/3} \left[(4a\cos\theta)^4 - (2a)^4 \right] \cos^2\theta \, d\theta$$

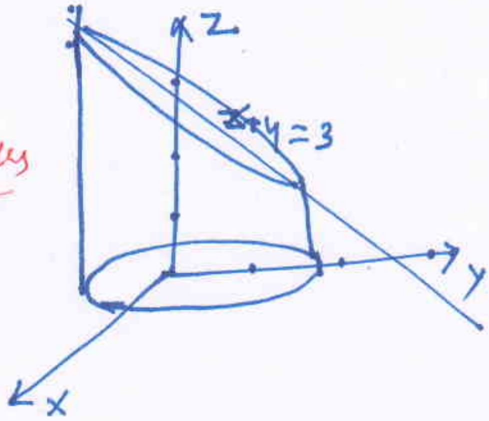
$$= 4a^4 \int_0^{\pi/3} \left(16\cos^6\theta - \cos^2\theta\right) d\theta$$

$$= \frac{12\pi + 11\sqrt{3}}{2} a^4$$

using formula

$$\int_a^b \cos^n\theta \, d\theta = \left[\frac{\cos^{n-1}\theta \sin\theta}{n} \right]_a^b + \frac{n-1}{n} \int_a^b \cos^{n-2}\theta$$

6/(b) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=3$ and $z=0$



Sol.

$$\left.\begin{aligned} z &: 0 \longrightarrow 3-y \\ y &: -\sqrt{4-x^2} \text{ to} \rightarrow \sqrt{4-x^2} \\ x &: -2 \text{ to} \rightarrow 2 \end{aligned}\right\}$$

(2) marks

Appropriate integral is:

$$I = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3-y} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[3y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\cdot dx = \int_{-2}^{2}\left(3\sqrt{4-x^2}-\frac{(4-x^2)}{2}+3\sqrt{4-x^2}+\frac{4-x^2}{2}\right)$$

(3) marks

$$= \int_{-2}^{2} 6\sqrt{4-x^2}\,dx = \frac{6}{2}\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right.$$

$$\boxed{\int_c^d \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]_c^d}$$

Formula

$$\left.\begin{aligned} I &= 3\cdot 4\left[\sin^{-1}(1)-\sin^{-1}(-1)\right] \\ &= 12\left(\pi/2-(-\pi/2)\right) \\ &= 12\pi \quad \text{Ans.} \end{aligned}\right\}$$

(01) mark

7 a) LHS $= \operatorname{div}(f\vec{v})$

$$= \nabla \cdot (f\vec{v})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(f v_1 \hat{i} + f v_2 \hat{j} + f v_3 \hat{k}\right)$$

$$= \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3) \quad — 1$$

$$= \left(v_1 \frac{\partial f}{\partial x} + f \frac{\partial v_1}{\partial x}\right) + \left(v_2 \frac{\partial f}{\partial y} + f \frac{\partial v_2}{\partial y}\right) + \left(v_3 \frac{\partial f}{\partial z} + f \frac{\partial v_3}{\partial z}\right) \quad — 3$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \left(v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y} + v_3 \frac{\partial f}{\partial z}\right)$$

$$= f\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}\right) +$$

$$\left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}\right)\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \quad — 5$$

$$= f(\nabla \cdot \vec{v}) + \vec{v} \cdot (\nabla f)$$

$$= f(\operatorname{div}(\vec{v})) + \operatorname{grad}(f) \cdot \vec{v} \quad — 6$$

$= \text{RHS}.$

Hence Proved.

7.

b) Let $f(x,y,z) = yz - 3x + xy + 5 = 0$ be the surface.

Then the normal vector is given by

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz - 3x + xy + 5).$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

$$= \hat{i}(-3+y) + \hat{j}(z+x) + \hat{k}(y-x)$$

At $(1,-1,2)$, the normal vector is given as.

$$\nabla f(1,-1,2) = -3\hat{i} + 3\hat{j} + (-2)\hat{k} \quad \text{— } 2$$

Any vector lying in the plane containing $(1,-1,2)$ is of form:

$$(x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k} \quad \text{— } 3$$

$\therefore$ The tangent plane which consists of vectors perpendicular to the normal vector $\nabla f(1,-1,2)$ is given by:

$$(\nabla f(1,-1,2)) \cdot ((x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k}) = 0$$

$$\Rightarrow (-3\hat{i} + 3\hat{j} - 2\hat{k}) \cdot ((x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k}) = 0$$

$$\Rightarrow -3(x-1) + 3(y+1) - 2(z-2) = 0.$$

$$\Rightarrow -3x + 3y - 2z + 10 = 0 \quad \text{— } 5 \text{ Ans.}$$

Equation of tangent plane.

At $(1,-1,2)$, the normal vector is given as.

$$\nabla f(1,-1,2) = -3\hat{i} + 3\hat{j} + (-2)\hat{k}$$

$\therefore$ the unit normal vector is given as.

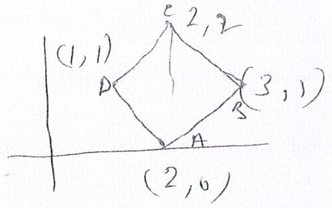
$$\frac{-3\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{3^2 + 3^2 + 2^2}} = \frac{-3}{\sqrt{22}}\hat{i} + \frac{3}{\sqrt{22}}\hat{j} + \frac{(-2)}{\sqrt{22}}\hat{k}$$

$\therefore$ The unit normal vector is $-\frac{3}{\sqrt{22}}\hat{i} + \frac{3}{\sqrt{22}}\hat{j} - \frac{2}{\sqrt{22}}\hat{k}$

— 7

1) By using ~~only line integrals~~ Green's theorem find the area of the surface bounded by the following points:

(Hint: ~~Apply Green's Theorem~~)



Eqn of AB: $y - 0 = \frac{1-0}{3-2}(x-2)$

$y = (x-2)$

Eqn of BC: $y - 1 = \frac{2-1}{2-3}(x-3)$.

$y - 1 = -(x-3)$

$y = -x+4$

Eqn of CD $y - 2 = \frac{2-1}{2-1}(x-2)$

$y = x$

Eqn of DA $y - 0 = \frac{0-1}{2-1}(x-2)$

$y = -(x-2) = -x+2$

side $= \sqrt{(3-2)^2+(1-0)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

Area $= 2$

2

Required area $= \frac{1}{2}\oint_C (x\,dy - y\,dx)$

$= \frac{1}{2}\left[\oint_A^B (x\,dy - y\,dx) + \int_B^C (x\,dy - y\,dx) + \int_C^D (x\,dy - y\,dx) + \int_D^A (x\,dy - y\,dx)\right]$

$= \frac{1}{2}(I_1 + I_2 + I_3 + I_4)$

2

Now for I_1

$y = x-2$

$x: 2 \to 3$

$\int_2^3 (x\,d(x-2) - (x-2)\,dx)$

$= \int_2^3 (x\,dx - x\,dx + 2\,dx)$

$= 2$

For I_2

$y = -x+4$

$x: 3 \to 2$

$\int_3^2 (-x\,dx - (-x+4)\,dx)$

$= \int_3^2 -4\,dx$

$= +4$

2

For I_3

$y = x$

$x: 2 \to 1$

$I_3 = 0$

For I_4

$x: 1 \to 2$

$y = -x+2$

$\int_1^2 -x\,dx - (-x+2)\,dx$

$= \int_1^2 -2\,dx$

$= -2$

Area $= \frac{1}{2}(2+4+0-2) = 2$ 1

3. Use divergence theorem to evaluate $\iint_S F \cdot ds$

where $\vec{F} = 2xz\,\hat{i} + (1-4xy^2)\hat{j} + (2z-z^2)\hat{k}$

and S is the surface of the solid bounded by

$z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$.

$0 = 6 - 2x^2 - 2y^2 \qquad x^2+y^2 = 3$

statement;

$\int F \cdot \hat{n}\, ds$

$= \iiint \operatorname{div}(F)\, dv$

1

$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{3}$

$0 \le z \le 6 - 2r^2$

2

$2z - 8xy$

$+2 - 2z$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} (2 - 8xy)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} (2 - 8r^2 \cos\theta\sin\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (6-2r^2)(2r - 8r^3\cos\theta\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (12r - 4r^3 - (48r^3 - 16r^5)\cos\theta\sin\theta)\, dr\, d\theta$$

2

$$= \int_0^{2\pi} \left(6r^2 - r^4 - (12r^4 - \tfrac{8}{3}r^6)\cos\theta\sin\theta\right)\Big|_0^{\sqrt{3}}\, d\theta$$

$$= \int_0^{2\pi} (9 - 18\sin 2\theta)\, d\theta = 9.2\pi = 18\pi$$

1