# MONOTONOCITY

## A. DEFINITIONS

The function $f(x)$ is called strictly increasing on the open interval (a, b) if for any two points $x_1$ and $x_2$ belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds true.

The function $f(x)$ is called strictly decreasing on the open interval (a, b) if for any points $x_1$ and $x_2$ belonging to the indicated interval and satisfying the inequality $x_1 < x_2$ the inequality $f(x_1) > f(x_2)$ holds true.

A function f is said to be non-decreasing in an interval I contained in the domain of f

If $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$ for all numbers $x_1$, $x_2$ in I.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all numbers $x_1$, $x_2$ in I,

then f is said to be strictly increasing in the interval I. Non-increasing and strictly decreasing functions are defined in a similar way. If f is strictly increasing in I, then the graph of f is rising as we traverse it from left to right; if f is strictly decreasing in I, the graph of f is falling in I. Some examples are show in Figure.



Strictly decreasing in [a, b]    Non-decreasing in [a, b]    Non-increasing in [a, b]
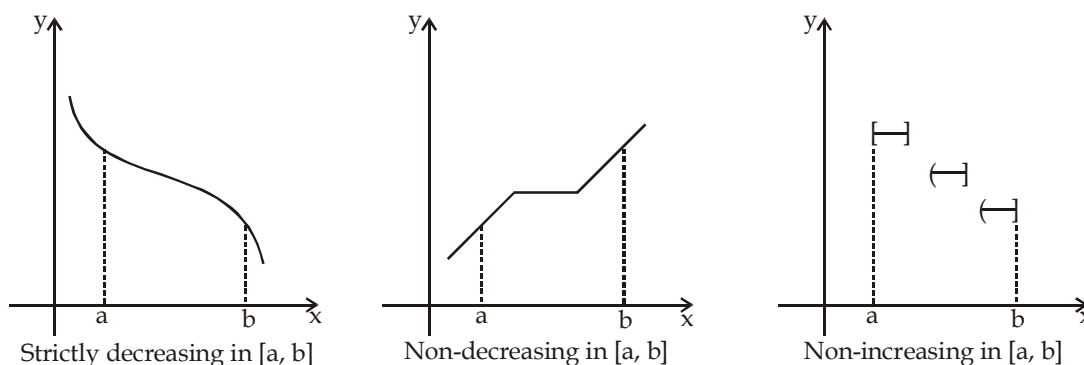
**Figure**

If a function f is either non-decreasing in an interval I or non-increasing in I, then f is said to be monotonic in I. Similarly, f is said to be strictly monotonic in I if f is either strictly increasing in I or strictly decreasing in I.

**BASIC DEFINITION TEST :**

The function $f(x)$ is said to be strictly increasing at a point $x_0$ if for a sufficiently small h > 0 the condition (Fig. 1)     $f(x_0 - h) < f(x_0) < f(x_0 + h)$     is fulfilled.

The function $f(x)$ is said to be strictly decreasing at a point $x_0$ if for a sufficiently small h > 0 the condition (Fig. 2)     $f(x_0 - h) > f(x_0) > f(x_0 + h)$     is fulfilled.
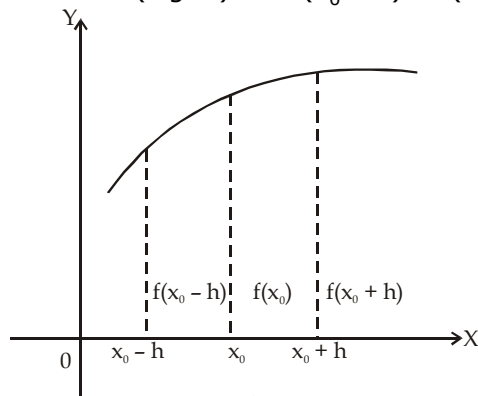


**Fig. 1**                    **Fig. 2**

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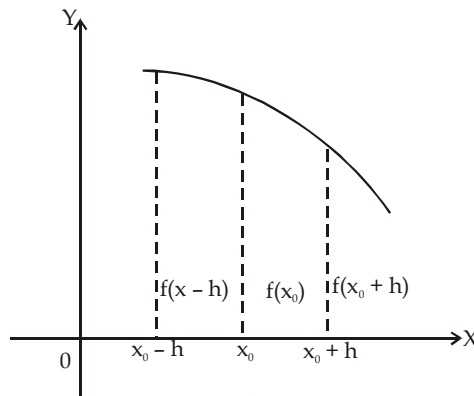MOTION IIT-JEE|AIEEE CBSE|SAT|NTSE OLYMPIADS
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A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

**SUFFICIENCY TEST :**
If the derivative function f′(x) in an interval (a , b) is every where positive, then the function f(x) in this interval is Increasing ; If f′(x) is every where negative, then f(x) is Decreasing.

**Note :** The test (criterion) also holds true when the derivative takes on zero values in the interval (a, b) so long as f(x) does not identically become zero throughout the interval (a, b) or in some interval (a′, b′) comprising a part of (a, b). The function f(x) would be a constant on such an interval.
If f′(a) = 0 then examine the sign of f′(a⁺) and f′(a⁻)
**(a)** If f′(a⁺) > 0 and f′(a⁻) > 0 then strictly increasing
**(b)** If f′(a⁺) < 0 and f′(a⁻) < 0 then strictly decreasing

**Note :** If a function is invertible it has to be either increasing or decreasing.
If a function is continuous in the intervals in which it rises and falls may be separated by points at which its derivative is zero or it fails to exist.

## B.  CRITICAL POINT

A critical point of a function f is a number c in the domain of f such the either f′(c) = 0 or f′(c) does not exist.

**Ex.1**  Find the critical points of f(x) = x^{3/5} (4 − x).

**Sol.**  $f'(x) = \dfrac{3}{5}x^{-2/5}(4-x) + x^{3/5}(-1) = \dfrac{3(4-x)}{5x^{2/5}} - x^{3/5} = \dfrac{3(4-x)-5x}{5x^{2/5}} = \dfrac{12-8x}{5x^{2/5}}$

Therefore, f′(x) = 0 if 12 − 8x = 0, that is, $x = \dfrac{3}{2}$, and f′(x) does not exist when x = 0.

Thus, the critical points are $\dfrac{3}{2}$ and 0.

**Ex.2**  Find the critical numbers for the function   $f(x) = \dfrac{e^x}{x-2}$

**Sol.**  $f'(x) = \dfrac{(x-2)e^x - e^x(1)}{(x-2)^2} = \dfrac{e^x(x-3)}{(x-2)^2}$

The derivative is not defined at x = 2, but f is not defined at 2 either, so x = 2 is not a critical number. The actual critical numbers are found by solving f′(x) = 0 :

$\dfrac{e^x(x-3)}{(x-2)^2} = 0$          x = 3 This is the only critical number since e^x > 0.

**Ex.3** Find all possible values of the parameter 'b' for which the function,

$f(x) = \sin 2x - 8(b+2)\cos x - (4b^2 + 16b + 6)x$

is monotonic decreasing throughout the number line and has no critical points.

**Sol.** $f'(x) = 2\cos 2x + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= 2(1 - 2\sin^2 x) + 8(b+2)\sin x - (4b^2 + 16b + 6)$

$= -4[\sin^2 x - 2(b+2)\sin x + (b^2 + 4b + 1)]$

for monotonic decreasing and no critical points $f'(x) > 0 \ \forall x \in R$

Now, $D = 4(b+2)^2 - 4(b^2 + 4b + 1) = 4[3] = 12$ which is always positive .

Now let $\sin x = y$ ; $y \in [-1, 1]$

$g(y) = y^2 - 2(b+2)y + (b^2 + 4b + 1)$

we have to find those values of ' b ' for which $g(y) > 0$ for all $y \in (-1, 1)$

Conditions are $g(-1) > 0$ & $-\dfrac{b}{2a} < -1$ or $g(1) > 0$ and $-\dfrac{b}{2a} > 1$

First condition gives $1 + 2(b+2) + b^2 + 4b + 1 > 0$

$b^2 + 6b + 6 > 0$ .....(1) & $\dfrac{2(b+2)}{2} < -1$ or $b < -3$ .....(2)

(1) & (2) $\Rightarrow$ $b < -\left(3 + \sqrt{3}\right)$  Similarly second condition gives $b > \sqrt{3} - 1$

Hence $b \in \left(-\infty, -\left(3 + \sqrt{3}\right)\right) \cup \left(\sqrt{3} - 1, \infty\right)$

**Ex.4** If $f(x) = \dfrac{x^2}{2 - 2\cos x}$ ; $g(x) = \dfrac{x^2}{6x - 6\sin x}$ where $0 < x < 1$, then

**Sol.** Put $x = \pi/6$ & $\pi/3$ and observe the behavior of $f(x)$ & $g(x)$ . Alternatively

$f'(x) = \dfrac{1}{2}\left[\dfrac{(1 - \cos x)2x - x^2 \sin x}{(1 - \cos x)^2}\right]$  consider $= 2(1 - \cos x) - x\sin x$

$= 4\sin^2 - 2x\sin\dfrac{x}{2}\cos\dfrac{x}{2} = 2x\sin\dfrac{x}{2}\cos\dfrac{x}{2}\left[\dfrac{\tan\frac{x}{2}}{\frac{x}{2}} - 1\right] \Rightarrow$ f is increasing.

$g'(x) = \dfrac{1}{6}\left[\dfrac{(x - \sin x)2x - x^2(1 - \cos x)}{(x - \sin x)^2}\right]$  consider $x - 2\sin x + x\cos x$

$= 2x\cos^2 x - 4\sin x \cos x = 2x\cos^2 x \left[1 - \dfrac{\tan\frac{x}{2}}{\frac{x}{2}}\right] \Rightarrow$ g is decreasing.

**Ex.5** Find possible values of a such that $f(x) = e^{2x} - (a+1)e^x + 2x$ is strictly increasing for $x \in R$.

**Sol.** $f(x) = e^{2x} - (a+1)e^x + 2x$

$f'(x) = 2e^{2x} - (a+1)e^x + 2$

Now, $2e^{2x} - (a+1)e^x + 2 \geq 0 \ \forall x \in R \Rightarrow 2\left(e^x + \dfrac{1}{e^x}\right) - (a+1) \geq 0 \ \forall \ x \in R$

$(a+1) \leq 2\left(e^x + \dfrac{1}{e^x}\right) \ \forall x \in R \Rightarrow a + 1 \leq 4 \Rightarrow a \leq 3$ $\left(\because \ e^x + \dfrac{1}{e^x} \text{ has minimum value 2}\right)$

**Aliter :** $2e^{2x} - (a + 1) e^x + 2 \geq 0 \ \forall \ x \in R$

putting $e^x = t$ ; $\quad t \in (0, \infty)$

$\qquad 2t^2 - (a + 1) t + 2 \geq 0 \quad \forall \ t \in (0, \infty)$

Hence either

**(i)** $D \leq 3 \Rightarrow (a + 1)^2 - 4 \leq 0 \Rightarrow (a + 5) (a - 3) \leq 0 \Rightarrow a \in [-5, 3]$

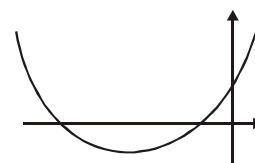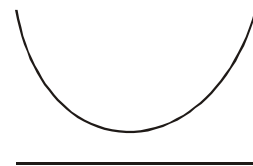**(ii)** both roots are negative

$D \geq 0 \ \& \qquad -\dfrac{b}{2a} < 0 \ \& \qquad f(0) \geq 0$

$\Rightarrow \quad a \in (-\infty, -5] \cup [3, \infty) \qquad \& \qquad \dfrac{a+1}{4} < 0 \ \& \qquad 2 \geq 0$

$\Rightarrow \quad a \in (-\infty, -5] \cup [3, \infty) \qquad \& \qquad a < -1 \qquad \& \qquad a \in R$

$\Rightarrow \quad a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

**Ex. 6** Prove that the function $f(x) = \dfrac{\ell n\ x}{x}$ is strictly decreasing in $(e, \infty)$. Hence, Prove that $303^{202} < 202^{303}$.

**Sol.** We have $f(x) = \dfrac{\ell n\ x}{x}, x > 0 \qquad$ Then $f'(x) = \dfrac{1 - \ell n\ x}{x^2} < 0 \ \forall \ x > e$

$\Rightarrow \ f(x)$ strictly decreases in $(e, \infty) \quad$ Thus, we have $\quad f(303) < f(202)$

i.e. $\dfrac{\ell n\ (303)}{303} < \dfrac{\ell n\ (202)}{202} \qquad$ i.e. $\quad 202 \ \ell n\ (303) < 303 \ \ell n\ (202)$

$\Rightarrow \ 303^{202} < 202^{303} \qquad$ which is the desired result.
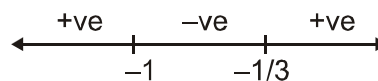
**Ex.7** Let $f(x) = x^3 + 2x^2 + x + 5$. Show that $f(x)$ has only one real root $\alpha$ such that $[\alpha] = -3$.

**Sol.** We have $f(x) = x^3 + 2x^2 + x + 5, x \in R$ and $f'(x) = 3x^2 + 4x + 1 = (x + 1) (3x + 1), x \in R$

Drawing the number line for $f'(x)$, we have $f(x)$ strictly increases in $(-\infty, -1)$

strictly decreases in $(-1, -1/3)$

strictly increases in $(-1/3, \infty)$



Also, we have $f(-1) = -1 + 2 - 1 + 5 = 5$ and $f\left(\dfrac{-1}{3}\right) = \dfrac{-1}{27} + \dfrac{2}{9} - \dfrac{1}{3} + 5 = 5 - \dfrac{4}{27} = 4.85$
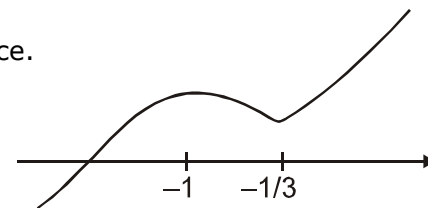
The graph of $f(x)$ (see fig.) shows that $f(x)$ cuts the X-axis only once.

Now, we have $f(-3) = -27 + 12 - 3 + 5 = -13$

and $\qquad f(-2) = -8 + 8 - 2 + 5 = 3$.

Which are of opposite signs. This proves that the curve cuts
the X-axis somewhere between $-2$ and $-3$.

$\Rightarrow \ f(x) = 0$ has a root $\alpha$ lying between $-2$ and $-3$. Hence $[\alpha] = -3$

**Ex.8** Find the number of real roots of the equation $\displaystyle\sum_{i=1}^{n} \frac{a_i^2}{x - b_i} = c$ where $b_1 < b_2 < \dots < b_n$.

**Sol.** Consider the function $f(x) = \displaystyle\sum_{i=1}^{n} \frac{a_i^2}{x - b_i} - c = \frac{a_1^2}{x - b_1} + \frac{a_2^2}{x - b_2} + \dots + \frac{a_n^2}{x - b_n} - c$

and $f'(x) = -\left[ \dfrac{a_1^2}{(x - b_1)^2} + \dfrac{a_2^2}{(x - b_2)^2} + \dots + \dfrac{a_n^2}{(x - b_n)^2} \right] < 0 \ \forall \ x \in R \sim \{b_1, b_2, \dots, b_n\}$

$\Rightarrow$ f(x) strictly decreases in $(-\infty, b_1) \cup (b_1, b_2) \cup \dots \cup (b_{n-1}, b_n)$
Now, we have
   $f(-\infty) = -c = f(\infty)$
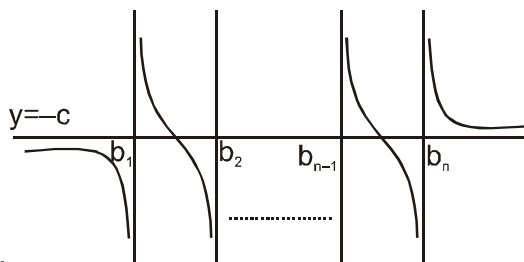   $f(b_1^-) = -\infty$ and $f(b_1^+) = \infty$
   $f(b_2^-) = -\infty$ and $f(b_2^+) = \infty$
   ............
   ............
   $f(b_n^-) = -\infty$ and $f(b_n^+) = \infty$
The plot of the curve y = f(x) is shown alongside.



**Ex.9** If $f : R \rightarrow R$ and f is a polynomial with f(x) = 0 has real and distinct roots, show that the equation, $[f'(x)]^2 - f(x) \cdot f''(x) = 0$ cannot have real roots.

**Sol.** Let $f(x) = c(x - x_1)(x - x_2) \dots (x - x_n)$

Again Let $h(x) = \dfrac{f'(x)}{f(x)} = \left( \dfrac{1}{x - x_1} + \dfrac{1}{x - x_2} + \dots + \dfrac{1}{x - x_n} \right)$

$h'(x) = \dfrac{f(x) \cdot f''(x) - [f'(x)]^2}{f^2(x)} = -\left( \dfrac{1}{(x - x_1)^2} + \dfrac{1}{(x - x_2)^2} + \dots + \dfrac{1}{(x - x_n)^2} \right)$

$\Rightarrow h'(x) < 0 \Rightarrow f(x) \cdot f''(x) - [f'(x)]^2 < 0$
Alternatively : a function f(x) satisfying the equation $[f'(x)]^2 - f(x) \cdot f''(x) = 0$ is

$f(x) = c \cdot e^{c_1 x}$ which can't have any root.

## C.    INTERVALS OF MONOTONOCITY

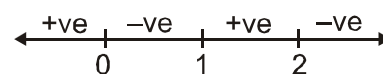**Ex.10** Find the intervals of monotonocity of the following functions :

**(a)** $f(x) = \dfrac{|x - 1|}{x^2}$          **(b)** $f(x) = 2x^2 - \ln |x|$     **(c)** $f(x) = \dfrac{x^3}{x^4 + 27}$

**Sol.** **(a)** We have $f(x) = \dfrac{1 - x}{x^2}, x < 1 ; f(x) = \dfrac{x - 1}{x^2}, x \geq 1$

and $f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x - 2}{x^3}, x < 1 ; f'(x) = \dfrac{2 - x}{x^3}, x > 1$



Now, from the sign scheme for f'(x), we have $\Rightarrow$ f(x) strictly increases i $(-\infty, 0)$
strictly decreases in (0, 1) ; strictly increases in (1, 2) ; strictly decreases in $(2, \infty)$
Ans. : Increases in $(-\infty, 0)$, (1, 2); Decreases in (0, 1), $(2, \infty)$

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**(b)** We have $f(x)=2x^2 - \ln|x|$ and $f'(x) = 4x - \dfrac{1}{x} = \dfrac{4\left(x+\dfrac{1}{x}\right)\left(x-\dfrac{1}{2}\right)}{x}$



Now, from the sign scheme for f'(x), we have　　　⇒　　f(x) strictly decreases in (−∞, −1/2)
strictly increases in (−1/2, 0) ; strictly decreases in (0, 1/2) ; strictly increases in (−1/2, ∞)

Ans. : Increases in $\left(-\dfrac{1}{2},0\right),\left(\dfrac{1}{2},\infty\right)$ ; Decreases in $\left(-\infty,-\dfrac{1}{2}\right),\left(0,\dfrac{1}{2}\right)$

**(c)** We have $f(x)=\dfrac{x^3}{x^4+27}$ and $f'(x)=\dfrac{(x^4+27)(3x^2)-x^3(4x^3)}{(x^4+27)^2}=\dfrac{-x^2(x^4-81)}{(x^4+27)^2}=\dfrac{x^2(x^2+9)(x+3)(x-3)}{(x^4+27)^2}$

Now, from the sign scheme for f'(x), we have



⇒ f(x) strictly decreases in (−∞, −3) ; strictly increases in (−3, 3) ; strictly decreases in (3, ∞).
Ans : Increases in (−3, 3) ; Decreases in (−∞, −3), (3, ∞)

**Ex.11** If $f(x) = x^2\, e^{-x^2/a^2}$ is an increasing function then (Here a > 0)

**Sol.**　$f'(x) = 2x\, e^{-x^2/a^2} + x^2 e^{-x^2/a^2} \times \dfrac{-2x}{a^2} = 2x\, e^{-x^2/a^2}\left(1-\dfrac{x^2}{a^2}\right) = \dfrac{2e^{-x^2/a^2}}{a^2}\,x(a^2-x^2)$

　　　= positive x [−1 (x + a) (x − 0) (x − a)]
f(x) is increasing when f'(x) ≥ 0 i.e.　x ∈ (−∞, −a] ∪ [0, a]



**Ex.12** A function f (x) is given by the equation, x² f' (x) + 2x f (x) − x + 1 = 0　(x ≠ 0). If f (1) = 0,
then find the intervals of monotonocity of f.

**Sol.**　$\dfrac{d}{dx}$ [x² y] = x − 1 ⇒ x²y = ∫ (x − 1) dx　　　where　y = f (x)

This gives　$y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{2x^2}$ . Find $\dfrac{dy}{dx}$ and solve $\dfrac{dy}{dx} > 0$ & $\dfrac{dy}{dx} < 0$
Ans. : I in (− ∞, 0) ∪ (1, ∞) ;　D in (0, 1)

# D.　OPERATIONS ON MONOTONOUS FUNCTIONS

**I. (a) Negative :** If f is an increasing function then its negative
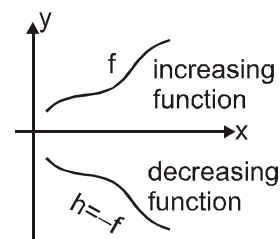i.e. h = −f is a decreasing function.
**By derivative**　　　　　　　　　　　　　　　　　　　　**By Graph**
h'(x) = −f'(x)　　∵　f'(x) > 0　　∴　　h'(x) < 0
⇒　　h is a decreasing function
In short　　− (an increasing function) = a decreasing function
i.e.　−I = D　　　Similarly −D = I



**(b) Reciprocal :** Reciprocal of an increasing function is a decreasing function

In short $\dfrac{1}{\text{an increasing function}}$ = a decreasing function i.e.　(i) $\dfrac{1}{I}$ = D　&　(ii) $\dfrac{1}{D}$ = I

**II.(a) Sum :** If f is an increasing function and g is also an increasing function their h = f + g is an increasing function.

**By derivative**                                    **By Graph**

$h'(x) = f'(x) + g'(x)$ f & g are increasing function,

$\Rightarrow$ f'(x) & g'(x) are positive $\Rightarrow$ f'(x) + g'(x) is positive

$\Rightarrow$ f(x) + g(x) increases

In short, An increasing function + An increasing function = An increasing function

i.e.        (i)    I + I = I            (ii)   I + D = can't say            (iii)   D + D = D

**(b) Difference :** Monotonocity of the difference of two function can be predicted using I(a) and II(a)
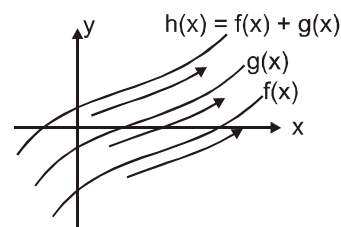
I – I = I + (–I) = I + D = can't say

I – D = I + (–D) = I + I = increasing

D – I = D + (–I) = D + D = decreasing

D – D = D + (–D) = D + I = can't say

**III. (a) Product :** Consider h = f × g

**Case I :** Both the function involved in the product i.e. f & g are positive

If f & g both are increasing function then h = f × g is also an increasing function.

In short I × I = I, I × D = can't say, D × D = D.

**Case II :** If any of the function takes negative values then we can predict the monotonocity by using I(a) & case I of III(a). If a function f is increasing & takes negative values & another function g is decreasing & takes positive values.

then $h(x) = f(x) \times g(x) = (-\phi(x)) \times g(x) = -\left[\underset{\text{Decrreasing}}{\phi(x)} \times \underset{\text{Decrreasing}}{g(x)}\right]$ = increasing

**(b) Division :** Monotonocity of division of two functions can be predicted by using I(b) & III(a).

$\frac{I}{D} = I \times \frac{I}{D} = I \times I = I$  (assuming that both the functions I & D take positive values).

**IV. Composition :**

(I)  I (I)= I            (II)   I(D) = D      (III)  D (I) = D    (IV) D(D) = I

Let  h(x) = D(D(x))    x increases  $\Rightarrow$  D(x) decreases  $\Rightarrow$  D(D(x)) increases

# E.    INEQUALITIES

**GENERAL APPROACH TO PROVE INEQUALITIES :**

To prove $f(x) \geq g(x) \forall x \geq a$, we Assume h(x) = f(x) – g(x)

Find  $h'(x) = f'(x) – g'(x)$

If    $h'(x) \geq 0$    Apply increasing function h on $x \geq a$ to get $h(x) \geq h(a)$.

If    $h(a) \geq 0$    then  $h(x) \geq 0 \forall x \geq a$   i.e. then given inequality is true.

If $h'(x) \leq 0$        Apply decreasing function h on $x \geq a$ to get $h(x) \leq h(a)$.

If    $h(a) \leq 0$    then  $h(x) \leq 0 \forall x \geq a$   i.e. the given inequality is false

**Note :** If the sign of h'(x) is not obvious then to determine its sign assume g(x) = h'(x) & apply the above procedure on g(x).

**Ex.13** Prove that, $2x \sec x + x > 3 \tan x$ for $0 < x < \pi/2$ .

**Sol.** $f(x) = 2x \sec x + x - 3 \tan x$

$f'(x) = 2 \sec x + 2x \sec x \ \tan x + 1 - 3 \sec^2 x = \sec^2 x \ [2 \cos x + 2x \sin x + \cos^2 x - 3]$

Consider $g(x) = 2 \cos x + 2x \sin x + \cos^2 x - 3$

$g'(x) = -2 \sin x + 2x \cos x + 2 \sin x - 2 \sin x \ \cos x = 2 \cos x \ (x - \sin x) > 0$ for $x \in (0, \pi/2)$

**Ex.14** Prove that $\tan x > x + \dfrac{x^3}{3}$ for all $x \in \left(0, \dfrac{\pi}{2}\right)$.

**Sol.** Let $f(x) \tan x - x - \dfrac{x^3}{3}$          ...(1)

Clearly, $f(x)$ is defined at all $x \in (0, \pi/2)$.

Now,      $f'(x) = \sec^2 x - 1 \ x - x^2$          ...(2)

$f''(x) = 2 \sec^2 x \ . \tan x - 2x$          ...(3)

$f'''(x) = 2 \sec^4 x + 4 \sec^2 x. \tan^2 x - 2 = 2 (1 + \tan^2 x)^2 + 4 \sec^2 x . \tan^2 x - 2$

$= 2\tan^4 x + 4 \tan^2 x + 4 \sec^2 x . \tan^2 x > 0$      for all $x \in (0, \pi/2)$

$\Rightarrow$ $f'''(x) > 0$ in the interval $(0, \pi/2)$ $\Rightarrow$ $f''(x)$ is monotonic increasing in $(0, \pi/2)$

$\therefore$ $f''(x) > f''(0)$ when $x \in (0, \pi/2)$.

But from (3), $f''(0) = 0$. Thus, $f''(x) > 0$ for all $x \in (0, \pi/2)$

$\therefore$ $f'(x)$ is monotonic increasing in $(0, \pi/2)$          $\therefore$ $f'(x) > f'(0)$ when $x \in (0, \pi/2)$

But from (2), $f'(0) = 1 - 1 - 0 = 0$.

Thus, $f'(x) > 0$ for all $x \in (0, \pi/2)$

$\therefore$ $f(x)$ is monotonic increasing in $(0, \pi/2)$          $\therefore$ $f(x) > f(0)$ when $x \in (0, \pi/2)$

But from (1), $f(0) = 0$.

Thus, $f(x) > 0$ for all $x \in (0, \pi/2)$          $\therefore$ $\tan x - x = \dfrac{x^3}{3} > 0$ for all $x \in (0, \pi/2)$

or $\tan x > x + \dfrac{x^3}{3}$ for all $x \in (0, \pi/2)$

**Ex.15** Show that     $1 + x \log (x + \sqrt{x^2 + 1}) \ge \sqrt{1 + x^2}$ for all $x \ge 0$.

**Sol.** Let $f(x) = 1 + x \log (x + \sqrt{x^2 + 1}) - \sqrt{1 + x^2}$

$\therefore$ $f'(x) = \dfrac{x}{[x + \sqrt{x^2 + 1}]} \cdot \left[1 + \dfrac{1.(2x)}{2\sqrt{x^2 + 1}}\right] + \log(x + \sqrt{x^2 + 1}).1 - \dfrac{1(2x)}{2\sqrt{1 + x^2}}$

or $f'(x) = \dfrac{x}{(x + \sqrt{x^2 + 1})}\left[\dfrac{\sqrt{x^2 + 1} + x}{2\sqrt{x^2 + 1}}\right] + \log(x + \sqrt{x^2 + 1}) - \dfrac{x}{\sqrt{1 + x^2}}$

or $f'(x) = \dfrac{x}{\sqrt{x^2 + 1}} + \log(x + \sqrt{x^2 + 1}) - \dfrac{x}{\sqrt{1 + x^2}}$     or     $f'(x) = = \log (x + \sqrt{x^2 + 1})$

we know $x \ge 0$     so, $\log (x + \sqrt{x^2 + 1}) \ge 0$     $\Rightarrow$     $f'(x) \ge 0$     or $f(x)$ is increasing for $x \ge 0$

Since, $f(x)$ is increasing for,     $x \ge 0 \Rightarrow$     $f(x) \ge f(0)$

$\Rightarrow$ $1 + x \log (x + \sqrt{x^2 + 1}) - \sqrt{1 + x^2} \ge 1 + 0 - \sqrt{1}$ $\Rightarrow$ $1 + x \log (x + \sqrt{x^2 + 1} \ge \sqrt{1 + x^2}$ , for $x \ge 0$.

**Ex.16** Examine which is greater sin x tan x or $x^2$. Hence evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \dfrac{\pi}{2}\right)$

**Sol.** Let $f(x) = \sin x . \tan x - x^2$

$f'(x) = \cos x . \tan x + \sin x . \sec^2 x - 2x = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow$ $f''(x) = \cos x + \cos x \sec^2 x + 2 \sec^2 x \sin x \tan x - 2$

$\Rightarrow$ $f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = (\sqrt{\cos x} - \sqrt{\sec x})^2$ and $2 \sec^2 x \tan x . \sin x > 0$ because $x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow$ $f''(x) > 0$ $\Rightarrow$ $f'(x)$ is M.I. Hence $f'(x) > f'(0)$

$\Rightarrow$ $f'(x) > 0$ $\Rightarrow$ $f(x)$ is M.I. $\Rightarrow$ $f(x) > 0$ $\Rightarrow$ $\sin x \tan x - x^2 > 0$

Hence $\sin x \tan x - x^2$ $\Rightarrow$ $\dfrac{\sin x \tan x}{x^2} > 1$ $\Rightarrow$ $\lim\limits_{x\to 0}\left[\dfrac{\sin x \tan x}{x^2}\right] = 1$

**Ex.17** Prove : $1 + \cot x \le \cot \dfrac{x}{2}$ $\forall x \in (0, \pi)$

**Sol.** Consider the function $f(x) = \cot\left(\dfrac{x}{2}\right) - 1 - \cot x$, $x \in (0, \pi)$

Then $f'(x) = \dfrac{-1}{2} \csc^2\left(\dfrac{x}{2}\right) + \csc^2 x = \dfrac{1}{\sin^2 x} - \dfrac{1}{2\sin^2(x/2)} = \dfrac{1}{2\sin^2(x/2)}\left[\dfrac{1}{2\cos^2(x/2)} - 1\right]$

$= \dfrac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)} = \dfrac{-\cos x}{\sin^2 x} < 0 \ \forall \ x \in (0, \pi/2)$ $> 0 \ \forall \ x \in (\pi/2, \pi)$

$\Rightarrow$ $f(x)$ strictly decreases in $(0, \pi/2)$ strictly increases in $(\pi/2, \pi)$

$\Rightarrow$ $f(x)$ has least value at $x = \pi/2$ $\Rightarrow$ $f(x) \ge f(\pi/2) = 0$

i.e. $\cot\left(\dfrac{x}{2}\right) \ge 1 + \cot x$ which proves the desired result.

**Ex.18** Prove that $\ell n\left(1+\dfrac{1}{x}\right) > \dfrac{1}{1-x^2}$, x>0. Hence, show that the function $f(x)=\left(1+\dfrac{1}{x}\right)^2$ strictly increases in $(0, \infty)$.

**Sol.** Consider the fucntion $g(x) = \ell n\left(1+\dfrac{1}{x}\right) > \dfrac{1}{1-x}$ $\forall x > 0$.

Then $g'(x) = \dfrac{-1/x^2}{1+\dfrac{1}{x}} + \dfrac{1}{(1+x)^2} = \dfrac{-1}{x(1+x)} + \dfrac{1}{(1+x)^2} = \dfrac{-1}{x(1+x)} < 0 \ \forall \ x > 0$

$\Rightarrow$ $g(x)$ strictly decreases in $(0, \infty)$ $\Rightarrow$ $g(x) > \lim\limits_{x\to\infty} g(x) = 0$ i.e. $\ell n\left(1+\dfrac{1}{x}\right) > \dfrac{1}{x+1}$ ...(1)

which gives the desired result.

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Now, we have $f(x) = \left(1 + \dfrac{1}{x}\right)^x$, $x > 0$ and $f'(x) = \left(1 + \dfrac{1}{x}\right)^x \ell n \left(1 + \dfrac{1}{x}\right) + x \left(1 + \dfrac{1}{x}\right)^{x-1} \left(\dfrac{-1}{x^2}\right)$

$= \left(1 + \dfrac{1}{x}\right)^x \left[\ell n \left(1 + \dfrac{1}{x}\right) - \dfrac{1}{1+x}\right] > 0 \ \forall \ x > 0$ [using result (1)] $\Rightarrow$ f(x) strictly increases in (0, ∞)

**Ex.19** Prove that sin x tan x > x² $\forall \ x \in \left(0, \dfrac{\pi}{2}\right)$.

**Sol.** Let f(x) = sin x tan x – x² $\Rightarrow$ f'(x) = sin x sec²x + sin x – 2x

$\Rightarrow$ f''(x) = 2 sin x sec² x tan x + cos x – 2 + sec x

$= 2$ sin x tan x sec² x + (cos x + sec x – 2) > 0 $\forall \ x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow$ f'(x) is an increasing function.  $\Rightarrow$  f'(x) > f'(0) $\Rightarrow$ sin x sec² x + sin x – 2x > 0

$\Rightarrow$ f(x) is an increasing function $\Rightarrow$ f(x) > f(0) sin x tan x – x² > 0 $\Rightarrow$ sin x tan x > x².

**Ex.20** Prove that sin 1 > cos (sin 1). Also show that the equation sin (cos (sin x)) = cos (sin (cos x)) has only

one solution in $\left[0, \dfrac{\pi}{2}\right]$.

**Sol.** sin 1 > cos (sin 1) if cos $\left(\dfrac{\pi}{2} - 1\right)$ > cos (sin 1)  $\Rightarrow$  if $\dfrac{\pi}{2}$ – 1 < sin 1 $\Rightarrow$ if sin 1 > $\left(\dfrac{\pi - 2}{2}\right)$  ...(1)

and sin 1 > sin $\dfrac{\pi}{4} > \dfrac{1}{\sqrt{2}}$.  Hence (1) is true  $\Rightarrow$  sin 1 > cos (sin 1).

Now let f(x) = sin (cos (sin x)) – cos (sin (cos x))

$\Rightarrow$ f'(x) < 0 $\forall \ x \in \left[0, \dfrac{\pi}{2}\right]$  $\Rightarrow$ f(x) is decreasing in $\left[0, \dfrac{\pi}{2}\right]$

and f(0) = sin 1 – cos (sin 1) > 0  $f\left(\dfrac{\pi}{2}\right)$ = sin (cos (1)) – 1 > 0

Since f(0) is positive and $f\left(\dfrac{\pi}{2}\right)$ is negative.  f(x) = 0 has one solution in $\left[0, \dfrac{\pi}{2}\right]$.

**Ex.21** Using calculus establish the inequality, $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$, where $x > 0$, $y > 0$ and $b > a > 0$.

**Sol.** $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a} \Rightarrow \left(\left(\dfrac{x}{y}\right)^b + 1\right)^{1/b} < \left(\left(\dfrac{x}{y}\right)^a + 1\right)^{1/a}$          [ $\because$ $y > 0$ ]

or  T P T      $(t^b + 1)^{a/b} < t^a + 1$                    [ $\dfrac{x}{y} = t > 0$ ]

Let  $f(t) = (t^b + 1)^{a/b} - t^a - 1 \Rightarrow f'(t) = \dfrac{a}{b}\left(t^b + 1\right)^{\frac{a}{b} - 1} \cdot b\, t^{b-1} - a\, t^{a-1}$

$\Rightarrow f'(t) = a\, t^{a-1}\left[t^{b-a}\left(t^b + 1\right)^{\frac{a}{b} - 1} - 1\right] = a\, t^{a-1}\left[\left(1 + \dfrac{1}{t^b}\right)^{\frac{a}{b} - 1} - 1\right]$

Now since $1 + \dfrac{1}{t^b} > 1$  &  $\dfrac{a}{b} - 1 < 0$      therefore $\left(1 + \dfrac{1}{t^b}\right)^{\frac{a}{b} - 1} < 1$

Hence  $f'(t) < 0$    i.e.  $f(t)$ is decreasing function
So  $f(t) < f(0)$    but  $f(0) = 0$    $\Rightarrow$    $(t^b + 1)^{a/b} < t^a + 1$          Hence proved

**Ex.22** If $1/6 < x < 5/6$, then $\dfrac{1}{2} < 3\left(x + \dfrac{1}{2\pi} - \dfrac{\sin \pi x}{x}\right) < \dfrac{5}{2}$. This statement is true or false.

**Sol.** Consider $f(x) = 3\left(x + \dfrac{1}{2\pi} - \dfrac{\sin \pi x}{x}\right)$.      Now, $f'(x) = 3(1 - \cos \pi x) > 0\ \forall\, x \in (1/6, 5/6)$

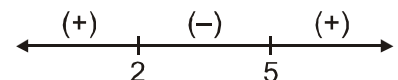Applying the increasing function f on the given inequality $\dfrac{1}{6} < x < \dfrac{5}{6}$

we get $f\left(\dfrac{1}{6}\right) < f(x) < f\left(\dfrac{5}{6}\right)$          i.e.   $\dfrac{1}{2} < f(x) < 5/2$          Hence, the statement is true.

**Ex.23** Prove that the function $f(x) = 2x^3 + 21x^2 - 60x + 41$ is strictly positive in the interval $(-\infty, 1)$.
**Sol.**   $f(x) = -2x^3 + 21x^2 - 60x + 41$
$\therefore$   $f'(x) = -6x^2 + 42x - 60 = -6(x^2 - 7x + 10) = -6(x - 5)(x - 2)$
The sign-scheme for $x^2 - 7x + 10$,     $x \in R$ is as follows :
$\therefore$   $x \in (2, 5) \Rightarrow$    $f'(x) > 0$, i.e., $f(x)$ is m.i.
and $x \notin (2, 5) \Rightarrow$    $f'(x) < 0$ i.e., $f(x)$ is m.d.    $\therefore$    $x \in (-\infty, 1) \Rightarrow$    $f(x)$ is m.d.
When     $x \in (-\infty, 1)$, $x < 1$; so, $f(x) > f(1)$.
But $f(1) = -2 + 21 - 60 + 41 = 0$.
$\therefore$   $x \in (-\infty, 1) \Rightarrow$    $f(x) > f(1) = 0$   $\therefore$    $f(x)$ is strictly positive in the interval $(-\infty, 1)$.

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## F. ROLLE'S THEOREM

Let f be a function that satisfies the following three hypotheses :

**1.** f is continuous on the closed interval [a, b].

**2.** f is differentiable on the open interval (a, b).

**3.** f(a) = f(b)

Then there is a number c in (a, b) such that $f'(c) = 0$

Before given the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graph of four such functions. In each case it appears that there is at least one point (c, f(c)) on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus, Rolle's Theorem is plausible.
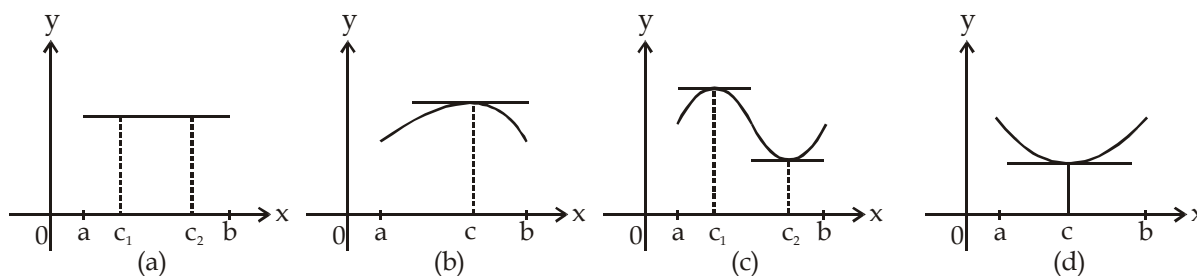


Figure 1

**Proof :** There are three cases :

**Case I :** f(x) = k, a constant. Then $f'(x) = 0$, so the number c can taken to be any number in (a, b).

**Case II :** f(x) > f(a) for some x in (a, b) [as in Figure 1(b) or (c)]

By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in [a, b]. Since f(a) = f(b), it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable at c. Therefore, $f'(c) = 0$ by Fermat's Theorem.

**Case III :**     f(x) < f(a) for some x in (a, b) [as in Figure 1(c) or (d)]

By the Extreme Value Theorem, f has minimum value in [a, b] and, since f(a) = f(b), it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem.

**Ex.24** Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

**Sol.** First we use the Intermediate Value Theorem to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that f(c) = 0. Thus, the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then f(a) = 0 = f(b) and, since f is a polynomial, it is differentiable on (a, b) and continuous on [a, b]. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But $f'(x) = 3x^2 + 1 \geq 1$   for all x

(since $x^2 \geq 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore, the equation can't have two real roots.

**Ex.25** Let f(x) & g(x) be differentiable for $0 \le x \le 1$, such that f(0) = 2, g(0) = 0, f(1) = 6 . Let there exist a real number c in [0, 1] such that f'(c) = 2 g'(c) , then the value of g(1)

**Sol.** Consider $\phi(x) = f(x) - 2g(x)$ defined on [0, 1] since f(x) and g(x) are differentiable for $0 \le x \le 1$, therefore $\phi(x)$ is differentiable on (0, 1) and continuous on [0, 1]

$\therefore \phi(0) = f(0) - 2g(0) = 2 - 0 = 2$        $\phi(1) = f(1) - 2g(1) = 6 - 2g(1)$

Now f'(x) = f'(x) - 2g'(x)    $\Rightarrow \phi'(c) = f'(c) - 2g'(x) = 0$ (given)

$\Rightarrow$ f(x) satisfies Rolle's theorem on [0, 1] $\therefore \phi(0) = \phi(1)$    $\Rightarrow 2 - 6 - 2g(1)$    $\Rightarrow g(1) = 2$

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

**Ex.26** If f(x) is continuous in [a, b] and differentiable in (a, b), prove that there is atleast one c $\in$(a, b), such that $\dfrac{f'(c)}{3c^2} = \dfrac{f(b) - f(a)}{b^3 - a^3}$ .

**Sol.** Let us consider a function, $h(x) = f(x) - f(a) + A(x^3 - a^3)$

Where A is obtained from the relation h(b) = 0.

So that, $0 = h(b) = f(b) - f(a) + A(b^3 - a^3)$            ...(i)

also,     h(a) = 0

Since,    **(1)** h(x) is continuous in [a, b] **(2)** h(x) is differentiable in (a, b) and **(3)** h(a) = 0 = h(b)

hence, all the three condition of Rolle's theorem. Then there must exists a'c' $\in$ (a, b) such that f'(c) = 0.

$\Rightarrow$ f'(c) + A (3c^2) = 0    or    $f'(c) = 3c^2 \dfrac{f(b) - f(a)}{b^3 - a^3}$ {using (i)}    $\Rightarrow$   $\dfrac{f'(c)}{3c^2} = \dfrac{f(b) - f(a)}{b^3 - a^3}$

## G. THE MEAN VALUE THEOREM

Let f be a function that satisfies the following hypotheses :

**1.** f is continuous on the closed interval [a, b].

**2.** f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

**(I)** $f'(c) = \dfrac{f(b) - f(a)}{b - a}$       or, equivalently,

**(II)** f(b) - f(a) = f'(c)(b - a)

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures (a) and (b) show that points A(a, f(a)) and B(b(b, f(b)) on the graphs of two differentiable functions. The slope of the secant line AB is

**(III)** $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$

which is the same expression as on the right side of Equation 1. Since f'(c) is the slope of the tangent line at the point (c, f(c)), the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P(c, f(c)) on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB.
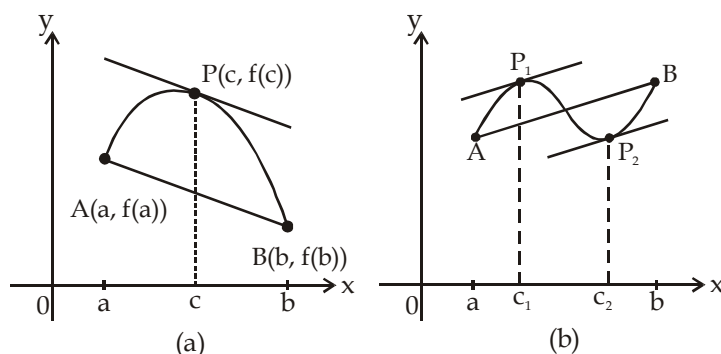
**Proof** We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using Equation 3, we see that the equation of the line AB can be written as

$y - f(x) = \dfrac{f(b) - f(a)}{b - a}(x - a)$ or as $y = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$

**(IV)** $h(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$
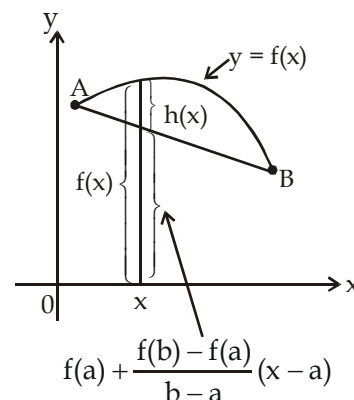
**First we must verify that h satisfies the three hypotheses of Rolle's Theorem.**

1.  The function h is continuous on [a, b] because it is the sum of f and a first-degree polynomial, both of which are continuous.

2.  The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact we can compute h' directly from Equation 4 :

$h'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$     (Note that f(a) and [f(b) – f(a)]/(b – a) are constants.)

3.  $h(a) = f(a) - f(a) - \dfrac{f(b) - f(a)}{b - a}(a - a) = 0$     $h(b) = f(b) - f(a) - \dfrac{f(b) - f(a)}{b - a}(b - a)$

$= f(b) - f(a) - [f(b) - f(a)] = 0$     Therefore, h(a) = h(b)

Since h satisfies the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that h'(c) = 0. Therefore

$0 = h'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a}$     and so     $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

**Ex.27** To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. since f is a polynomial, it is continuous and differentiable for x, so it is certainly continuous on [0, 2] and differentiable on (0, 2) such that $f(2) - f(0) = f'(c)(2 - 0)$

**Sol.** Now $f(2) = 6$, $f(0) = 0$, and $f'(x)$ $3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \dfrac{4}{3}$, tha is , $c = \pm 2/\sqrt{3}$ .

But c must lie in (0, 2), so $c = 2/\sqrt{3}$ .

The tangent line at this value of c is parallel to the secant line OB.
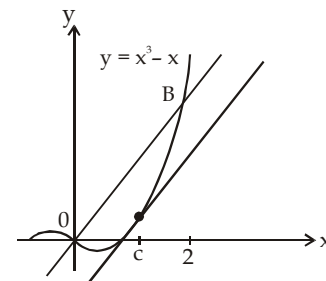
**Figure**

**Ex.28** If $f'(x) = \dfrac{1}{1+x^2}$ for all x and $f(0) = 0$, show that $0.4 < f(2) < 2$

**Sol.** Given $f'(x) = \dfrac{1}{1+x^2}$ for all x                    ...(1)

$\therefore$    $f'(x) > 0$ for all x $[\because$    $1 + x^2 > 0]$
Also given $f(0) = 0$                    ...(2)
From (1), it follows that $f(x)$ is differentiable at all x, therefore $f(x)$ is also continuous at all x
$\therefore$    by Lagrange's mean value theorem in [0, 2]

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) = \frac{1}{1+c^2}\text{, where } 0 < c < 2 \text{ or, } \frac{f(2) - 0}{2 - 0} = \frac{1}{1+c^2} \text{ or } f(2) = \frac{2}{1+c^2} \quad ...(3)$$

Now $0 < c < 2$        $\therefore$ $\dfrac{2}{1+c^2} < \dfrac{2}{1+0^2}$    or    $\dfrac{2}{1+c^2} < 2$ ...(4)

and $\dfrac{2}{1+c^2} > \dfrac{2}{1+2^2} = \dfrac{2}{5} = 0.4$    or,    $\dfrac{2}{1+c^2} > 0.4$        ...(5)

From (3), (4) and (5) it follows that $0.4 < f(2) < 2$.

# H.    CURVE SKETCHING

The following checklist is intended as a guide to sketching a curve $y = f(x)$. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

**I.**    **Domain** It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

**II.    Intercepts** The y-intercept is $f(0)$ and this tells us where the curve intersect the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

**III. Symmetry**

**(a)** If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an even function and the curve is symmetric about the y-axis. This means that out work is cut in half. If we know what the curve looks like for $x \geq 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples : $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

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**(b)** If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geq 0$. [Rotate 180º about the origin; see Figure (b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.
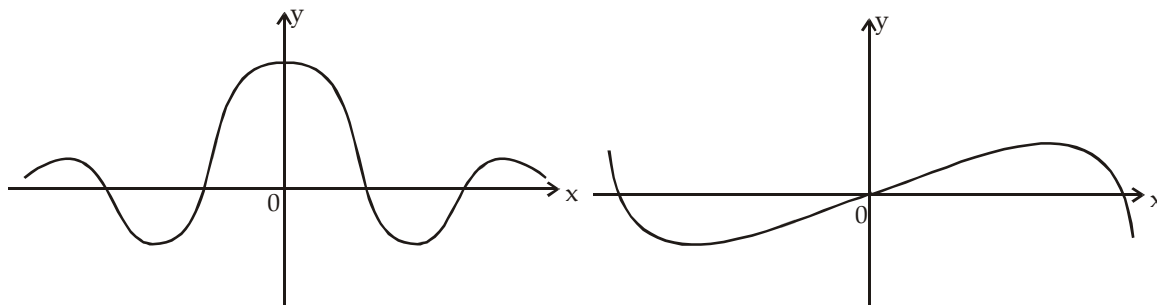


(a) Even function : reflectional symmetry      (b) Odd function  rotational symmetry

**Figure**

**(c)** If $f(x + p) = f(x)$ for all x in D, where p is positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period $2\pi$ and $y = \tan x$ has period $\pi$. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure ).
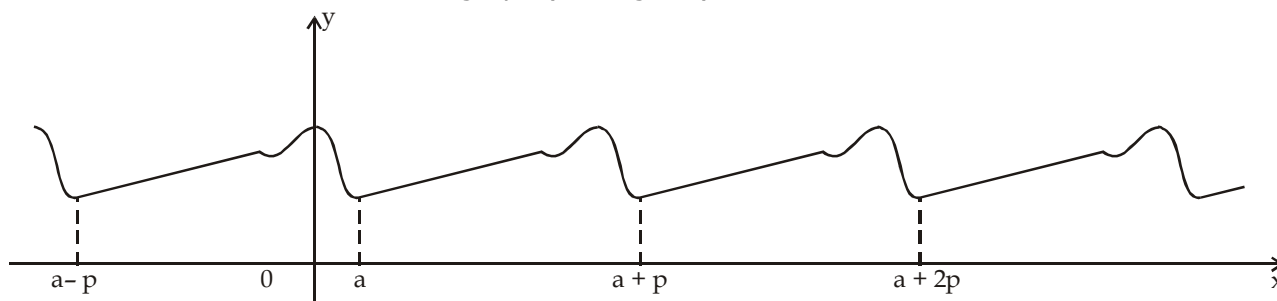


Figure      Periodic Function : translational symmetry

## IV. Asypmtotes

**(a) Horizontal Asymptotes.** If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim\limits_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

**(b) Vertical Asymptotes.** The line $x = a$ is a vertical asymptote if at least one of the following statements is true :

$$\lim\limits_{x\to a^+} f(x) = \infty \qquad\qquad \lim\limits_{x\to a^-} f(x) = \infty$$

$$\lim\limits_{x\to a^+} f(x) = -\infty \qquad\qquad \lim\limits_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (ii) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute

$\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$, whether or not this limit is infinite,

**V.   Interval of Increase / Decrease** Use the I/D Test. Compute f′(x) and find the intervals on which f′(x) is positive (f is increasing) and the intervals on which f′(x) is negative (f is decreasing).

**VI.  Local Maximum and Minimum Value** Find the critical numbers of f [the number c where f′(c) = 0 or f′(c) does not exist]. Then use the First Derivative Test. If f′ changes from positive to negative at a critical number c, then f(c) is a local maximum. If f′ changes from negative to positive at c, then f(c) is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if c is a critical number such that f″(c) ≠ 0. Then f″(c) > 0 implies that f(c) is a local minimum, whereas f″(c) < 0 implies that f(c) is a local maximum.

**VII. Concavity and Points of Inflection** Compute f′(x) and use the Concavity Test. The curve is concave upward where f″(x) > 0 and concave downward where f″(x) < 0. Inflection points occur where the direction of concavity changes.

**VIII. Sketch the Curve** Using the information in items A – G, draw the graph. Sketch the asymptotes as dashed lines, Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

**Ex.29** Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

**Sol.**   **I.**   The domain is $\{x | x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

   **II.**  The x-and y-intercepts are both 0.

   **III.** Since f(−x) = f(x), the function f is even. The curve is symmetric about the y-axis.

   **IV.**  $\displaystyle\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2.$ Therefore, the line y = 2 is a horizontal asymptote.

   Since the Denominator is 0 when x = ±1, we compute the following limits :

   $\displaystyle\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty,$    $\displaystyle\lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = \infty,$    $\displaystyle\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty,$    $\displaystyle\lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = -\infty$

   Therefore, the lines x = 1 and x = −1 are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure, showing the parts of the curve near the aymptotes.

   **V.** f′(x) = $\dfrac{4x(x^2 - 1) - 2x^2.2x}{(x^2 - 1)^2} = \dfrac{-4x}{(x^2 - 1)^2}$

   Since f′(x) > 0 when x < 0 (x ≠ −1) and f′(x) < 0 when x > 0 when x > 0 (x ≠ 1), f is increasing on (−∞, −1) and (−1, 0) and decreasing on (0, 1) and (1, ∞).

   **VI.** The only critical number is x = 0. Since f′ changes from positive to negative at 0, f(0) = 0 is local maximum by the First Derivative Test.

**VII.** $f''(x) = \dfrac{-4(x^2-1)^2 - 4x.2(x^2-1)2x}{(x^2-1)^4} = \dfrac{12x^2+4}{(x^2-1)^3}$



Figure
Finished sketch of $y = \dfrac{2x^2}{x^2-1}$

Since $12x^2 + 4 > 0$ for all x, we have

$f''(x) > 0 \quad \Leftrightarrow \quad x^2 - 1 > 0 \Leftrightarrow \quad |x| > 1$

and $f''(x) < 0 \Leftrightarrow |x| < 1$. Thus, the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and −1 are not in the domain of f.

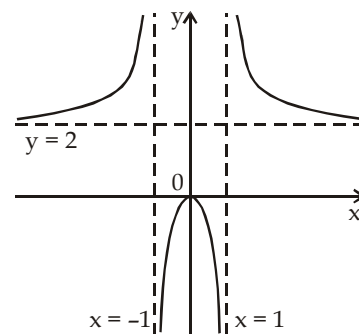**VIII.** Using the information in V – VI, we finish the sketch in Figure.

**Ex.30** Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

**Sol.** **I.** Domain = $\{x|x + 1 > 0\} = \{x|x > -1\} = (-1, \infty)$

**II.** The x- and y-intercepts are both 0.

**III.** Symmetry : None

**IV.** Since $\lim\limits_{x\to\infty}\dfrac{x^2}{\sqrt{x+1}} = \infty$ there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to 1^+$ and f(x) is

always positive, we have $\lim\limits_{x\to -1^+}\dfrac{x^2}{\sqrt{x+1}} = \infty$ and so the line x = −1 is a vertical asymptote.

**V.** $f'(x) = \dfrac{2x\sqrt{x+1} - x^2.1/(2\sqrt{x+1})}{x+1} = \dfrac{x(3x+4)}{2(x+1)^{3/2}}$

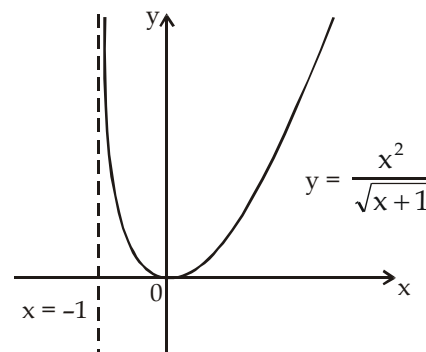We see that $f'(x) = 0$ when x = 0 (notice that $-\dfrac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when x> 0, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

**VI.** Since $f'(0) = 0$ and f′ changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

**VII.** $f''(x) = \dfrac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \dfrac{3x^2+8x+8}{4(x+1)^{5/2}}$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of $x^2$ is positive. Thus, $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.



$y = \dfrac{x^2}{\sqrt{x+1}}$

Figure

**VIII.** The curve is sketched in Figure.

**Ex.31** Sketch the graph of $f(x) = xe^x$

**Sol.**    **I.**   The domain is R.

**II.**   The x- and y-intercepts are both 0.

**III.**  Symmetry : None

**IV.**  Because both x and $e^x$ become large as $x \to \infty$, we have $\lim\limits_{x \to \infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and

so we have an indeterminate product that requires the use of L'Hospital's Rule :

$\lim\limits_{x \to -\infty} xe^x = \lim\limits_{x \to -\infty} \dfrac{x}{e^{-x}} = \lim\limits_{x \to -\infty} \dfrac{1}{-e^{-x}} = \lim\limits_{x \to -\infty} (-e^x) = 0$ . Thus, the x-axis is horizontal asymptote.

**V.**   $f'(x) = xe^x + e^x = (x + 1)e^x$. Since $e^x$ is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(X) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

**VI.**  Because $f'(-1) = 0$ and f changes from negative to positive at $x = -1$,

$f(-1) = -e^{-1}$ is a local (and absolute) minimum.

**VII.** $f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$,

f is concave upward on $(-2, \infty)$ and concave downward

on $(-\infty, -2)$. The inflection points in $(-2, -2e^{-2})$.

**VIII.** We use this information to sketch the curve in Figure.



Figure
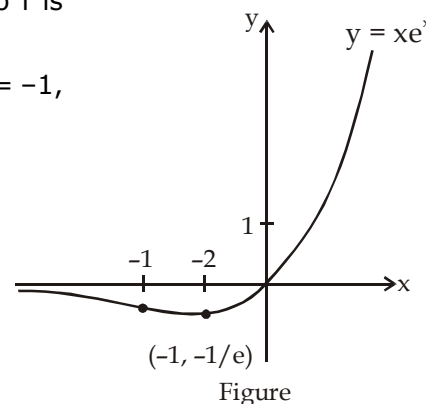
**Ex.32** Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

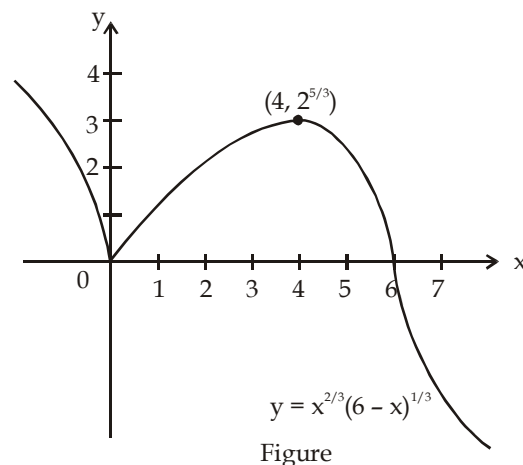**Sol.**   You can use the differentiation rules to check that the first two derivatives are

$f'(x) = \dfrac{4 - x}{x^{1/3}(6 - x)^{2/3}}$ ,   $f''(x) = \dfrac{-8}{x^{4/3}(6 - x)^{5/3}}$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not

exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4 and 6.

| Interval | $4 - x$ | $x^{1/3}$ | $(6 - x)^{2/3}$ | $f'(x)$ | f |
|----------|---------|-----------|------------------|---------|---|
| $x < 0$ | + | – | + | – | decreasing on$(-\infty,0)$ |
| $0 < x < 4$ | + | + | + | + | increasing on$(0,4)$ |
| $4 < x < 6$ | – | + | + | – | decreasing on $(4,6)$ |
| $x > 6$ | – | + | + | – | decreasing on $(6,\infty)$ |

To find the local extreme values we use the First Derivative Test. Since f′ changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f′ changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f′ does not change at 6, so there is no minimum or maximum there.

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geq 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.



Figure

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**Ex.33** Plot the following curves :

**(a)** $y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 6$     **(b)** $y = \dfrac{x}{\ell n\ x}$     **(c)** $y = x\ \ell n\ x$

**(d)** $y = \dfrac{\ell n\ x}{x}$     **(e)** $y = \dfrac{x+1}{(x-1)(x-7)}$     **(f)** $2^{|x|}\ |y| + 2^{|x|\ -\ 1} = 1.$

**Sol.**    **(a)** We have $\quad y = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + 6$ whose domain is $x \in R$, and

$y' = x^2 - 3x + 2 = (x - 1)\ (x - 2) > 0 \ \forall\ x \in (-\infty, 1) \cup (2, \infty)$

$\qquad\qquad\qquad\qquad\qquad < 0\ \forall\ x \in (1, 2)$

$\Rightarrow \quad$ y strictly increases in $(-\infty, 1)$

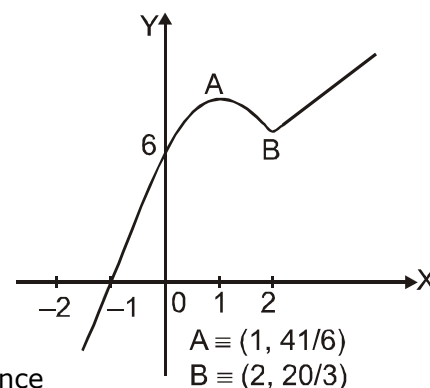$\qquad$ strictly decreases in $(1, 2)$ ; strictly increases in $(2, \infty)$

Now we have $y(1) = \dfrac{1}{3} - \dfrac{3}{2} + 2 + 6 = \dfrac{41}{6}$ , $y(2) = \dfrac{8}{3} - 6 + 4 + 6 = \dfrac{20}{3}$

$y\ (-\infty)\ -\infty,\ y(\infty) = \infty$

The curve cuts the Y-axis at $(0, 6)$.

The cure cuts the $-$ve X-axis somewhere between $-1$ and $-2$, since



$A \equiv (1, 41/6)$
$B \equiv (2, 20/3)$

$y(-1) = \dfrac{-1}{3} - \dfrac{3}{2} - 2 + 6 > 0$ and $y\ (-2) = \dfrac{-8}{3} - \dfrac{12}{2} - 4 + 6 < 0$. The plot of the curve is shown along side.

**(b)** We have $y = \dfrac{x}{\ell n\ x}$ Whose domain is $x \in (0, \infty) \sim \{1\}$, and

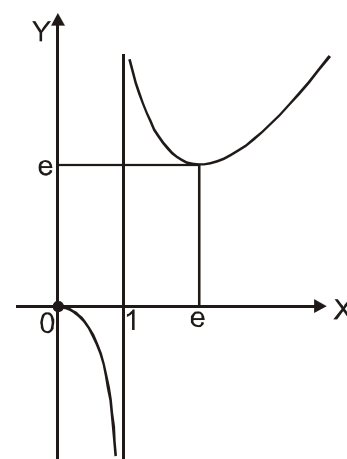$y' = \dfrac{\ell n\ x - 1}{\ell n^2\ x} < 0\ \forall\ x \in (0, 1) \cup (1, e)$

$\qquad\qquad > 0\ \forall\ x \in (e, \infty)$

$\Rightarrow$ y strictly decreases in $(0, 1) \cup (1, e)$ ; strictly increases in $(e, \infty)$

Now, we have $\displaystyle\lim_{x \to 0^+} \dfrac{x}{\ell n\ x} = 0,\ y\ (e) = e$



$\displaystyle\lim_{x \to 0^-} \dfrac{x}{\ell n\ x} = -\infty,\ \lim_{x \to 1^+} \dfrac{x}{\ell n\ x} = \infty$

$\displaystyle\lim_{x \to \infty} \dfrac{x}{\ell n\ x}\left(\dfrac{\infty}{\infty}\right) = \lim_{x \to \infty} \dfrac{1}{1/x} = \infty$   The plot of the curve is shown alongside.

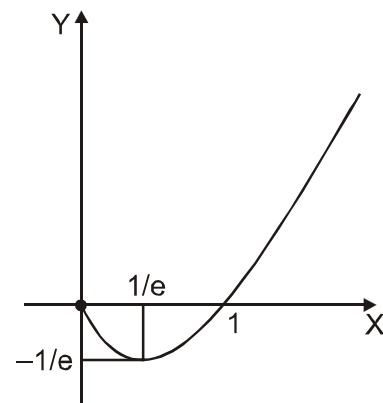**(c)** We have    $y = x \, \ell n \, x$ whose domain is $x \in (0, \infty)$, and

$y' = 1 \, \ell n \, x < 0 \, \forall \, x \in (0, e^{-1})$

$\qquad\qquad > 0 \, \forall \, x \in (e^{-1}, \infty)$

$\Rightarrow$   y strictly decreases in $(0, e^{-1})$ ; strictly increases in $(e^{-1}, \infty)$.

Now, we have  $\lim\limits_{x \to 0^+} x \, \ell n \, x = \lim\limits_{x \to 0^+} \dfrac{\ell n \, x}{1/x} \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x \to 0^+} \dfrac{1/x}{1/x^2} = 0$

$\lim\limits_{x \to \infty} x \, \ell n \, x = \infty, \, y(e^{-1}) = \dfrac{-1}{e}$

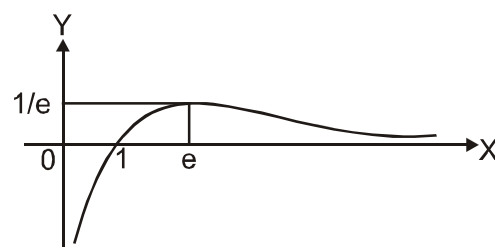The curve cuts the X-axis at $(1, 0)$. The plot of the curve is shown above.

**(d)** We have    $y = \dfrac{\ell n \, x}{x}$  whose domain is $x \in (0, \infty)$, and

$y' = \dfrac{1 - \ell n \, x}{x^2} > 0 \, \forall \, x \in (0, e)$

$\qquad\qquad < 0 \, \forall \, x \in (e, \infty)$

$\Rightarrow$ y strictly increases in $(0, e)$; strictly decreases in $(e, \infty)$.

Now, we have    $\lim\limits_{x \to \infty} \dfrac{\ell n \, x}{x} \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0, \; \lim\limits_{x \to \infty} \dfrac{\ell n \, x}{x} - \infty, y(e) = \dfrac{1}{e}$

The curve cuts the X-axis at $(1, 0)$. The plot of the curve is shown above.

**(e)** We have $y = \dfrac{x + 1}{(x - 1)(x - 7)}$  whose domain is $x \in R \sim \{1.7\}$, and

$y' = \dfrac{(x^2 - 8x + 7) - (2x - 8)(x + 1)}{(x - 1)^2(x - 7)^2} = \dfrac{-(x^2 + 2x - 15)}{(x - 1)^2(x - 7)^2} = \dfrac{-(x + 5)(x - 3)}{(x - 1)^2(x - 7)^2}$

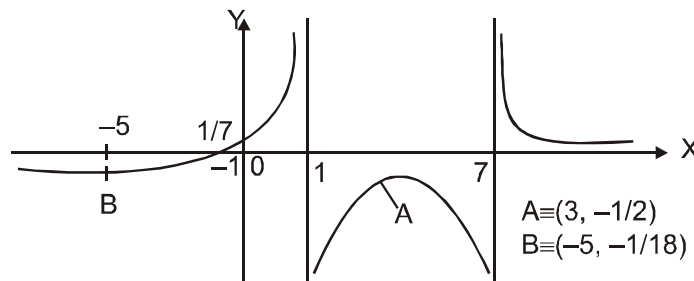$\Rightarrow$   y strictly decreases in $(-\infty, -5)$

$\qquad$ strictly increases in $(-5, 1) \cup (1, 3)$ ; strictly decreases in $(3, 7) \cup (7, \infty)$

Now, we have $y(3) = \dfrac{4}{(2)(-4)} = \dfrac{-1}{2}$ , $y(-5) = \dfrac{-4}{(-6)(-12)} = \dfrac{-1}{18}$

$\lim\limits_{x \to 1^-} \dfrac{x + 1}{(x - 1)(x - 7)} = \infty, \; \lim\limits_{x \to 1^+} \dfrac{x + 1}{(x - 1)(x - 7)} = -\infty$

$\lim\limits_{x \to 7^-} \dfrac{x + 1}{(x - 1)(x - 7)} = -\infty, \; \lim\limits_{x \to 7^+} \dfrac{x + 1}{(x - 1)(x - 7)} = \infty$

$A \equiv (3, -1/2)$
$B \equiv (-5, -1/18)$

$\lim\limits_{x \to \infty} \dfrac{x + 1}{(x - 1)(x - 7)} = 0, \; \lim\limits_{x \to \infty} \dfrac{x + 1}{(x - 1)(x - 7)} = 0$

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The curve cuts the Y-axis at (0, 1/7). The curve cuts the X-axis at (−1, 0). The plot of the curve is shown below.
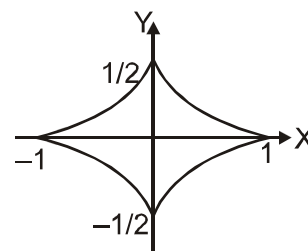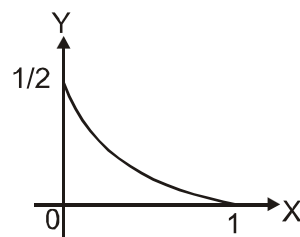


**(f)** We have $2^{|x|}\,|y| + 2^{|x|-1} = 1$ i.e. $|y| = 2^{-|x|} - \dfrac{1}{2}$

The curve is symmetrical about the X-axis as well as the Y-axis. In the first quadrant the equation of the

curve reduces to $y = 2^{-x} - \dfrac{1}{2}$

whose plot is shown above.

The complete curve is drawn by taking the mirror image of the above shown curve in the X-axis and the Y-axis as shown alongside.



**Ex.34** Sketch a possible graph of a function f that satisfies the following conditions :

**(i)** $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$n $(1, \infty)$

**(ii)** $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

**(iii)** $\lim\limits_{x \to -\infty} f(x) = -2, \lim\limits_{x \to \infty} f(x) = 0$

**Sol.** Condition (i) tells us that f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. Condition (ii) says that f is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$. From condition (iii) we know that the graph of f has two horizontal

asymptotes: $y = -2$ and $y = 0$.
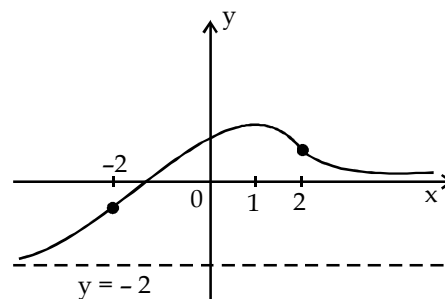


Figure

We first draw the horizontal asymptote $y = -2$ as a dashed line (see Figure). We then draw the graph of f approaching this asymptote at the far left, increasing to its maximum point at $x = 1$ and decreasing toward the x-axis at the far right. We also make sure that the graph has inflection points when $x = -2$ and 2. Notice that we made the curve bend upward for $x < -2$ and $x > 2$. and bend downward when x is between −2 and 2.

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