Differential Equations

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SECOND ORDER (homogeneous)

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A Tutorial Module for learning to solve 2nd order (homogeneous) differential equations

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1. Theory

In this Tutorial, we will practise solving equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

i.e. second order (the highest derivative is of second order),
linear (y and/or its derivatives are to degree one) with
constant coefficients (a, b and c are constants that may be zero).

There are no terms that are constants and no terms that are only a function of x. If such terms were present, it would be conventional to collect them together on the right-hand-side of the equation. Here, we simply have a zero on the right-hand-side of the equals sign and this type of ordinary differential equation (o.d.e.) is called "homogeneous".

Since the o.d.e. is second order, we expect the general solution to have two arbitrary constants (these will be denoted A and B).

A trial solution of the form $y = Ae^{mx}$ yields an "auxiliary equation":

$$am^2 + bm + c = 0,$$

that will have two roots (m_1 and m_2).

The general solution y of the o.d.e. is then constructed from the possible forms (y_1 and y_2) of the trial solution. The auxiliary equation may have:

i) real different roots,
m_1 and $m_2 \rightarrow y = y_1 + y_2 = Ae^{m_1 x} + Be^{m_2 x}$

or ii) real equal roots,
$m_1 = m_2 \rightarrow y = y_1 + xy_2 = (A + Bx)e^{m_1 x}$

or iii) complex roots,
$p \pm iq \rightarrow y = y_1 + y_2 \equiv e^{px}(A\cos qx + B\sin qx)$

2. Exercises

Find the general solution of the following equations. Where boundary conditions are also given, derive the appropriate particular solution too.

Click on Exercise links for full worked solutions (there are 16 exercises in total).

$\left[\text{Notation:} \quad y'' = \frac{d^2y}{dx^2}, \quad y' = \frac{dy}{dx}\right]$

Exercise 1. $2y'' + 3y' - 2y = 0$

Exercise 2. $y'' - 2y' + 2y = 0$

Exercise 3. $y'' - 2y' + y = 0$

Exercise 4. $y'' = -4y$

Exercise 5. $y'' = 4y$

Exercise 6. $36y'' - 36y' + 13y = 0$

Exercise 7. $3y'' + 2y' = 0$

Exercise 8. $16y'' - 8y' + y = 0$

Exercise 9. $y'' + 4y' + 5y = 0\,; \quad y(0) = 0 \quad \text{and} \quad y'(0) = 2$

Exercise 10. $y'' + 6y' + 13y = 0\,; \; y(0) = 2 \quad \text{and} \quad y'(0) = 1$

Exercise 11. $y'' + 6y' + 9y = 0\,; \quad y(0) = 1 \quad \text{and} \quad y'(0) = 2$

Exercise 12. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

Exercise 13. $\dfrac{d^2y}{d\tau^2} - 6\dfrac{dy}{d\tau} + 9y = 0$

Exercise 14. $\dfrac{d^2y}{d\tau^2} + 7\dfrac{dy}{d\tau} + 12y = 0$

Exercise 15. $\dfrac{d^2x}{d\tau^2} + 5\dfrac{dx}{d\tau} + 6x = 0$

Exercise 16. $4\dfrac{d^2x}{d\tau^2} + 8\dfrac{dx}{d\tau} + 3x = 0$

3. Answers

1. $y = Ae^{\frac{1}{2}x} + Be^{-2x}$,

2. $y = e^{x}(A\cos x + B\sin x)$,

3. $y = (A + Bx)e^{x}$,

4. $y = A\cos 2x + B\sin 2x$,

5. $y = Ae^{2x} + Be^{-2x}$,

6. $y = e^{\frac{1}{2}x}(A\cos\frac{x}{3} + B\sin\frac{x}{2})$,

7. $y = A + Be^{-\frac{2}{3}x}$,

8. $y = (A + Bx)e^{\frac{1}{4}x}$,

9. $y = e^{-2x}(A\cos x + B\sin x);\quad y = 2e^{-2x}\sin x$,

10. $y = e^{-3x}(A\cos 2x + B\sin 2x);\quad y = \frac{1}{2}e^{-3x}(4\cos 2x + 7\sin 2x)$,

11. $y = (A + Bx)e^{-3x};\quad y = (1 + 5x)e^{-3x}$,

12. $y = e^{-\frac{1}{2}x}(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}}{2}x)$,

13. $y = (A + B\tau)e^{3\tau}$,

14. $y = Ae^{-4\tau} + Be^{-3\tau}$,

15. $x = Ae^{-2\tau} + Be^{-3\tau}$,

16. $x = Ae^{-\frac{3}{2}\tau} + Be^{-\frac{1}{2}\tau}$.

4. Solving quadratics

To solve the quadratic equation:

$$am^2 + bm + c = 0, \quad \text{where } a, b, c \text{ are constants,}$$

one can sometimes identify simple linear factors that multiply together to give the left-hand-side of the equation.

Alternatively, one can <u>always</u> use the quadratic formula:

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

to find the values of m (denoted m_1 and m_2) that satisfy the quadratic equation.

5. Tips on using solutions

● When looking at the THEORY, ANSWERS, SOLVING QUADRATICS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1. $\boxed{2y'' + 3y' - 2y = 0}$

$$\text{Set} \quad y = Ae^{mx} \rightarrow \frac{dy}{dx} = mAe^{mx} = my$$
$$\frac{d^2y}{dx^2} = m^2Ae^{mx} = m^2y$$

i.e. $2m^2y + 3my - 2y = 0$

i.e. $2m^2 + 3m - 2 = 0$: <u>Auxiliary Equation</u> (A.E.)

i.e. $(2m-1)(m+2) = 0$

i.e. $m_1 = \frac{1}{2}$ and $m_2 = -2$: <u>Two Different Real Roots</u>

i.e. $y_1 = Ae^{\frac{1}{2}x}$ and $y_2 = Be^{-2x}$: Two Independent Solutions

<u>General solution</u> is $y = y_1 + y_2 = Ae^{\frac{1}{2}x} + Be^{-2x}$, ($A$, B are arbitrary constants).

Return to Exercise 1

Exercise 2. $\boxed{y'' - 2y' + 2y = 0}$

$$\text{Set} \quad y = Ae^{mx}, \ \frac{dy}{dx} = my, \ \frac{d^2y}{dx^2} = m^2 y$$

$\rightarrow \quad m^2 - 2m + 2 = 0$ <u>Auxiliary Equation</u> (A.E.)

i.e. $\quad am^2 + bm + c = 0$ with solutions $m = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$.

Since $\quad b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 < 0$,

expect <u>Complex (Conjugate) Solutions</u>.

In fact, $\quad m = \frac{1}{2}\left(2 \pm \sqrt{-4}\right) = \frac{1}{2}\left(2 \pm 2\sqrt{-1}\right) = 1 \pm i$.

For $m = p \pm iq$, $\quad y = e^{px}(A\cos qx + B\sin qx)$

i.e. $\quad y = e^{x}(A\cos x + B\sin x)$, since $p = 1$ and $q = 1$.

Return to Exercise 2

Exercise 3. $\boxed{y'' - 2y' + y = 0}$

$$m^2 - 2m + 1 = 0 \quad \text{(A.E.)}$$

$$\text{i.e.} \quad (m-1)^2 = 0 \qquad \text{i.e.} \quad m = 1 \ \text{(twice)}$$

$$\text{i.e.} \quad \underline{\text{EQUAL REAL ROOTS}}$$

Multiply one solution by x, to get two independent solutions

$$\text{i.e.} \quad y = y_1 + xy_2 = Ae^x + xBe^x = (A + Bx)e^x \ .$$

Return to Exercise 3

Exercise 4. $\boxed{y'' = -4y}$

$m^2 = -4$ i.e. $m = \sqrt{-1} \cdot \sqrt{4} = i \cdot (\pm 2) = \pm 2i$

(A.E.) i.e. complex conjugate solutions of form $p \pm iq$ with $p = 0,\ q = 2$.

General solution, $y = e^{px}\,(A\cos qx + B\sin qx)$

i.e. $y = e^{0}\,(A\cos 2x + B\sin 2x)$

i.e. $y = A\cos 2x + B\sin 2x$.

Return to Exercise 4

Exercise 5. $\boxed{y'' = 4y}$

$$\begin{array}{rl} \text{A.E. is} & m^2 = 4 \\ \text{i.e.} & m = \pm 2 \\ \text{i.e.} & \text{real different roots} \\ \therefore & y = y_1 + y_2 = Ae^{2x} + Be^{-2x} \ . \end{array}$$

Return to Exercise 5

Exercise 6. $\boxed{3y'' - 36y' + 13y = 0}$

$$\begin{aligned}
\text{A.E. is} \quad & 36m^2 - 36m + 13 = 0 \\
\text{i.e.} \quad & m = \tfrac{1}{2\cdot 36}\left(36 \pm \sqrt{(-36)^2 - 4\cdot 36\cdot 13}\right) \\
& \quad = \tfrac{1}{2}\left(1 \pm \sqrt{1 - \tfrac{4\cdot 13}{36}}\right) \\
& \quad = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{-16}{36}} = \tfrac{1}{2} \pm \tfrac{i}{2}\cdot\tfrac{4}{6} \\
\text{i.e.} \quad & m = \tfrac{1}{2} \pm \tfrac{1}{3}i \\
\therefore \quad & y = e^{\frac{1}{2}x}\left[A\cos\left(\tfrac{1}{3}x\right) + B\sin\left(\tfrac{1}{3}x\right)\right] .
\end{aligned}$$

Return to Exercise 6

Exercise 7. $\boxed{3y'' + 2y' = 0}$

$$\text{A.E. is} \quad 3m^2 + 2m = 0 \quad \text{i.e.} \quad m(3m+2) = 0$$

$$\text{i.e.} \quad m_1 = 0 \quad \text{and} \quad m_2 = -\tfrac{2}{3}$$

$$\text{Real different roots} \quad : \quad y = Ae^{0\cdot x} + Be^{-\frac{2}{3}x}$$

$$\text{i.e.} \quad y = A + Be^{-\frac{2}{3}x} \ .$$

Return to Exercise 7

Exercise 8. $\boxed{16y'' - 8y' + y = 0}$

$$\text{A.E. is} \quad 16m^2 - 8m + 1 = 0 \quad \text{i.e.} \quad (4m-1)^2 = 0$$

$$\text{i.e.} \quad m = \tfrac{1}{4} \quad \text{(twice)}$$

Real equal roots: $y = (A + Bx)e^{\frac{1}{4}x}$.

Return to Exercise 8

Exercise 9. $\boxed{y'' + 4y' + 5y = 0 \quad ; \quad y(0) = 0 \quad \text{ and } \quad y'(0) = 2}$

A.E. $\quad m^2 + 4m + 5 = 0, \quad m = \frac{1}{2}\left(-4 \pm \sqrt{16 - 20}\right)$

$$= -2 \pm \tfrac{2}{2} i = -2 \pm i$$

General solution is $\quad y = e^{-2x}(A\cos x + B\sin x)\,.$

Particular solution <u>has</u> $\quad y = 0$ when $x = 0$

i.e. $\quad 0 = e^0(A\cos(0) + B\sin(0))$

$$= 1 \cdot (A + 0) = A$$

i.e. $\quad A = 0\,.$

<u>And</u> $\quad \frac{dy}{dx} = 2$ when $x = 0$

$$A = 0 \text{ gives} \qquad y = e^{-2x} B \sin x$$
$$\frac{dy}{dx} = -2e^{-2x} B \sin x + e^{-2x} B \cos x$$
$$= Be^{-2x}(\cos x - 2 \sin x)$$
$$\text{i.e.} \qquad 2 = B \cdot e^{0}[\cos(0) - 2\sin(0)]$$
$$= B \cdot 1[1 - 0]$$
$$\text{i.e.} \qquad B = 2\,.$$

$\therefore$ particular solution is $\quad y = 2e^{-2x} \sin x$.

Return to Exercise 9

Exercise 10. $\boxed{y'' + 6y' + 13y = 0 \quad ; \quad y(0) = 2 \quad \text{and} \quad y'(0) = 1}$

$$\begin{aligned} \text{A.E.} \quad m^2 + 6m + 13 = 0 \quad \text{i.e.} \quad m &= \tfrac{1}{2}\left(-6 \pm \sqrt{36 - 52}\right) \\ &= -3 \pm \tfrac{1}{2}\sqrt{-16} \\ \text{i.e.} \quad m &= -3 \pm 2i \end{aligned}$$

General solution is $\quad y = e^{-3x}(A\cos 2x + B\sin 2x)\,.$

Particular solution has $\quad y = 2$ when $x = 0$

i.e. $\quad 2 = e^0(A\cos(0) + B\sin(0))$

i.e. $\quad A = 2\,.$

$$\begin{aligned} \frac{dy}{dx} &= -3e^{-3x}(A\cos 2x + B\sin 2x) + e^{-3x}(-2A\sin 2x + 2B\cos 2x) \\ &= e^{-3x}\left[(2B-3A)\cos 2x - (3B+2A)\sin 2x\right] \\ 1 &= e^{0}\left[(2B-3A)\cos(0) - (3B+2A)\sin(0)\right] \end{aligned}$$

$\left(\text{i.e. } \frac{dy}{dx} = 1 \text{ when } x = 0\right)$

i.e. $1 = 2B - 3A$

i.e. $1 = 2B - 6 \quad (\text{using } A = 2)$

i.e. $\frac{7}{2} = B.$

$\therefore$ Particular solution is $y = \frac{1}{2}e^{-3x}(4\cos 2x + 7\sin 2x)$.

Return to Exercise 10

Exercise 11. $\boxed{y'' + 6y' + 9y = 0 \quad ; \quad y(0) = 1 \quad \text{and} \quad y'(0) = 2}$

A.E. is $m^2 + 6m + 9 = 0$ i.e. $(m+3)^2 = 0$

i.e. $m = -3$ (twice)

General solution : $y = (A + Bx)e^{-3x}$.

Particular solution has $y = 1$ when $x = 0$

i.e. $1 = (A + 0)e^0$

i.e. $A = 1$.

$$\begin{aligned}\frac{dy}{dx} &= Be^{-3x} + (A+Bx)\cdot(-3)e^{-3x}\\ &= e^{-3x}\left[B - 3(A+Bx)\right]\end{aligned}$$

$$\begin{array}{lll} \text{i.e.} & 2 = e^0[B-3(A+0)], & \text{since } \frac{dy}{dx} = 2 \text{ when } x = 0 \\ \text{i.e.} & 2 = B - 3A & \\ \text{i.e.} & B = 5 & (\text{using } A = 1)\,. \end{array}$$

$\therefore$ Particular solution is $\quad y = (1+5x)e^{-3x}$.

Return to Exercise 11

Exercise 12. $\boxed{\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0}$

$$\text{Set} \quad y = Ae^{mx}\,, \quad \frac{dy}{dx} = my\,, \quad \frac{d^2y}{dx^2} = m^2 y$$

A.E. $\quad m^2 + m + 1 = 0$

i.e. $\quad m = \frac{1}{2}[-1 \pm \sqrt{1-4}] = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad$ i.e. $p \pm iq$ with $\quad p = -\frac{1}{2}$

$q = \frac{\sqrt{3}}{2}$

General solution: $\quad y = e^{px}(A\cos qx + B\sin qx)$

$$\text{i.e.} \quad y = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right)\,.$$

Return to Exercise 12

Exercise 13. $\boxed{\frac{d^2y}{d\tau^2} - 6\frac{dy}{d\tau} + 9y = 0}$

$$\begin{array}{ll} \text{A.E.} & m^2 - 6m + 9 = 0 \\ \text{i.e.} & (m-3)^2 = 0 \\ \text{i.e.} & m = 3 \quad \text{(twice)} \\ \text{and} & y = Ae^{3\tau} \quad \text{(twice)} \end{array}$$

To get two independent solutions, multiply one by τ

$$\text{i.e.} \qquad y = (A + B\tau)e^{3\tau}\ .$$

Return to Exercise 13

Exercise 14. $\boxed{\frac{d^2y}{d\tau^2} + 7\frac{dy}{d\tau} + 12y = 0}$

$$\begin{array}{ll} \text{A.E.} & m^2 + 7m + 12 = 0 \\ \text{i.e.} & (m+4)(m+3) = 0 \\ \text{i.e.} & m_1 = -4 \quad \text{and} \quad m_2 = -3 \end{array}$$

i.e. two different real roots giving two independent solutions

$$\text{i.e.} \qquad y_1 = Ae^{-4\tau} \quad \text{and} \quad y_2 = Be^{-3\tau}$$

general solution is $y = y_1 + y_2 = Ae^{-4\tau} + Be^{-3\tau}$.

Return to Exercise 14

Exercise 15. $\boxed{\frac{d^2x}{d\tau^2} + 5\frac{dx}{d\tau} + 6x = 0}$

$$\text{Set} \quad x = Ae^{m\tau}, \quad \text{i.e.} \quad \frac{dx}{d\tau} = mx, \quad \text{and} \quad \frac{d^2x}{d\tau^2} = m^2x$$

A.E. $\quad m^2 + 5m + 6 = 0$

i.e. $\quad (m+2)(m+3) = 0$

i.e. $\quad m_1 = -2 \quad$ and $\quad m_2 = -3 \quad$ (two different real roots)

These give two independent solutions $x(\tau)$

i.e. $\quad x_1 = Ae^{-2\tau} \quad$ and $\quad x_2 = Be^{-3\tau}$

general solution is $\quad x(\tau) = Ae^{-2\tau} + Be^{-3\tau}$.

Return to Exercise 15

Exercise 16. $\boxed{4\frac{d^2x}{d\tau^2} + 8\frac{dx}{d\tau} + 3x = 0}$

$$
\begin{array}{ll}
\text{A.E.} & 4m^2 + 8m + 3 = 0 \\
\text{i.e.} & (2m+3)(2m+1) = 0 \\
\text{i.e.} & m_1 = -\frac{3}{2} \quad \text{and} \quad m_2 = -\frac{1}{2} \quad \text{(different real roots)} \\
\text{i.e.} & x_1(\tau) = Ae^{-\frac{3}{2}\tau} \quad \text{and} \quad x_2(\tau) = Be^{-\frac{1}{2}\tau} \quad \text{(independent solutions)}
\end{array}
$$

$\therefore$ general solution is $x(\tau) = Ae^{-\frac{3}{2}\tau} + Be^{-\frac{1}{2}\tau}$.

Return to Exercise 16

Second-Order Linear Differential Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions. We saw in Section 7.1 that equations of this type arise in the study of the motion of a spring. In *Additional Topics: Applications of Second-Order Differential Equations* we will further pursue this application as well as the application to electric circuits.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1. Such equations are called **homogeneous** linear equations. Thus, the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous** and is discussed in *Additional Topics: Nonhomogeneous Linear Equations*.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

Proof Since y_1 and y_2 are solutions of Equation 2, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and $$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y \\
&\qquad = P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&\qquad = P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&\qquad = c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\
&\qquad = c_1(0) + c_2(0) = 0
\end{aligned}$$

Thus, $y = c_1y_1 + c_2y_2$ is a solution of Equation 2. ■■

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2. This means that neither y_1 nor y_2 is a constant multiple of the other. For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions P, Q, and R are constant functions, that is, if the differential equation has the form

5
$$\boxed{ay'' + by' + cy = 0}$$

where a, b, and c are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus, $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

6
$$\boxed{ar^2 + br + c = 0}$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

7
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

CASE I ▫ $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions of Equation 5. (Note that e^{r_2x} is not a constant multiple of e^{r_1x}.) Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

■■ In Figure 1 the graphs of the basic solutions $f(x) = e^{2x}$ and $g(x) = e^{-3x}$ of the differential equation in Example 1 are shown in black and red, respectively. Some of the other solutions, linear combinations of f and g, are shown in blue.

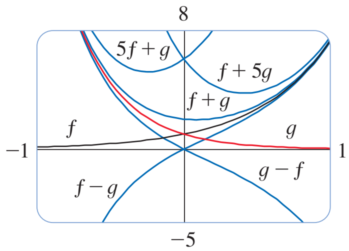


FIGURE 1

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

SOLUTION The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$. Therefore, by (8) the general solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

We could verify that this is indeed a solution by differentiating and substituting into the differential equation. ■■

EXAMPLE 2 Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

SOLUTION To solve the auxiliary equation $3r^2 + r - 1 = 0$ we use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{13}}{6}$$

Since the roots are real and distinct, the general solution is

$$y = c_1 e^{(-1+\sqrt{13})x/6} + c_2 e^{(-1-\sqrt{13})x/6}$$ ■■

CASE II □ $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

9 $$r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

The first term is 0 by Equations 9; the second term is 0 because r is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 xe^{rx}$$

EXAMPLE 3 Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

Figure 2 shows the basic solutions $f(x) = e^{-3x/2}$ and $g(x) = xe^{-3x/2}$ in Example 3 and some other members of the family of solutions. Notice that all of them approach 0 as $x \to \infty$.

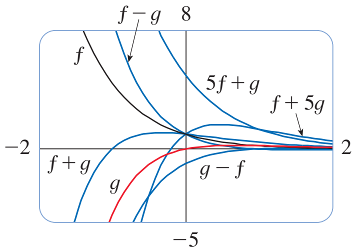


FIGURE 2

so the only root is $r = -\frac{3}{2}$. By (10) the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

CASE III $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. (See Appendix I for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix I, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real. We summarize the discussion as follows.

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

Figure 3 shows the graphs of the solutions in Example 4, $f(x) = e^{3x}\cos 2x$ and $g(x) = e^{3x}\sin 2x$, together with some linear combinations. All solutions approach 0 as $x \to -\infty$.

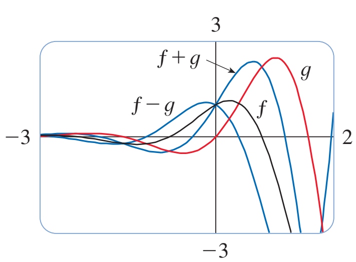


FIGURE 3

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

By (11) the general solution of the differential equation is

$$y = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where y_0 and y_1 are given constants. If P, Q, R, and G are continuous on an interval and $P(x) \neq 0$ there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 and 6 illustrate the technique for solving such a problem.

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

■■ Figure 4 shows the graph of the solution of the initial-value problem in Example 5. Compare with Figure 1.

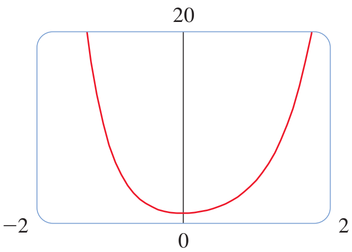


FIGURE 4

SOLUTION From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

To satisfy the initial conditions we require that

$$y(0) = c_1 + c_2 = 1 \tag{12}$$

$$y'(0) = 2c_1 - 3c_2 = 0 \tag{13}$$

From (13) we have $c_2 = \frac{2}{3}c_1$ and so (12) gives

$$c_1 + \tfrac{2}{3}c_1 = 1 \qquad c_1 = \tfrac{3}{5} \qquad c_2 = \tfrac{2}{5}$$

Thus, the required solution of the initial-value problem is

$$y = \tfrac{3}{5}e^{2x} + \tfrac{2}{5}e^{-3x}$$

■■

EXAMPLE 6 Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

■■ The solution to Example 6 is graphed in Figure 5. It appears to be a shifted sine curve and, indeed, you can verify that another way of writing the solution is

$$y = \sqrt{13}\sin(x + \phi) \quad \text{where } \tan\phi = \tfrac{2}{3}$$

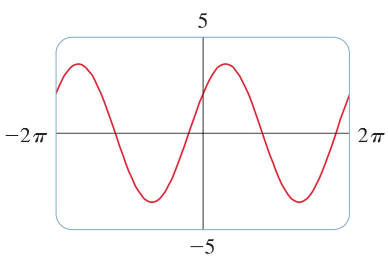


FIGURE 5

SOLUTION The auxiliary equation is $r^2 + 1 = 0$, or $r^2 = -1$, whose roots are $\pm i$. Thus $\alpha = 0$, $\beta = 1$, and since $e^{0x} = 1$, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Since $$y'(x) = -c_1 \sin x + c_2 \cos x$$

the initial conditions become

$$y(0) = c_1 = 2 \qquad y'(0) = c_2 = 3$$

Therefore, the solution of the initial-value problem is

$$y(x) = 2\cos x + 3\sin x$$

■■

A **boundary-value problem** for Equation 1 consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution.

EXAMPLE 7 Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

SOLUTION The auxiliary equation is

$$r^2 + 2r + 1 = 0 \qquad \text{or} \qquad (r + 1)^2 = 0$$

■■ Figure 6 shows the graph of the solution of the boundary-value problem in Example 7.

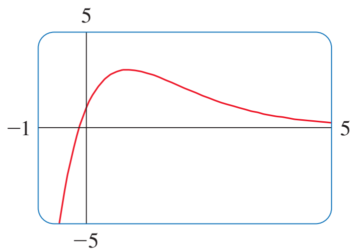


FIGURE 6

whose only root is $r = -1$. Therefore, the general solution is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

The boundary conditions are satisfied if

$$y(0) = c_1 = 1$$

$$y(1) = c_1 e^{-1} + c_2 e^{-1} = 3$$

The first condition gives $c_1 = 1$, so the second condition becomes

$$e^{-1} + c_2 e^{-1} = 3$$

Solving this equation for c_2 by first multiplying through by e, we get

$$1 + c_2 = 3e \qquad \text{so} \qquad c_2 = 3e - 1$$

Thus, the solution of the boundary-value problem is

$$y = e^{-x} + (3e - 1)xe^{-x}$$ ■■

Summary: Solutions of $ay'' + by' + c = 0$

Roots of $ar^2 + br + c = 0$	General solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Exercises

A Click here for answers.   S Click here for solutions.

1–13 ■ Solve the differential equation.

1. $y'' - 6y' + 8y = 0$

2. $y'' - 4y' + 8y = 0$

3. $y'' + 8y' + 41y = 0$

4. $2y'' - y' - y = 0$

5. $y'' - 2y' + y = 0$

6. $3y'' = 5y'$

7. $4y'' + y = 0$

8. $16y'' + 24y' + 9y = 0$

9. $4y'' + y' = 0$

10. $9y'' + 4y = 0$

11. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - y = 0$

12. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 4y = 0$

13. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

14–16 ■ Graph the two basic solutions of the differential equation and several other solutions. What features do the solutions have in common?

14. $6\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

15. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = 0$

16. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0$

17–24 ■ Solve the initial-value problem.

17. $2y'' + 5y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -4$

18. $y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$

19. $4y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.5$

20. $2y'' + 5y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 4$

21. $y'' + 16y = 0, \quad y(\pi/4) = -3, \quad y'(\pi/4) = 4$

22. $y'' - 2y' + 5y = 0, \quad y(\pi) = 0, \quad y'(\pi) = 2$

23. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1$

24. $y'' + 12y' + 36y = 0, \quad y(1) = 0, \quad y'(1) = 1$

25–32 ■ Solve the boundary-value problem, if possible.

25. $4y'' + y = 0, \quad y(0) = 3, \quad y(\pi) = -4$

26. $y'' + 2y' = 0, \quad y(0) = 1, \quad y(1) = 2$

27. $y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y(3) = 0$

28. $y'' + 100y = 0, \quad y(0) = 2, \quad y(\pi) = 5$

29. $y'' - 6y' + 25y = 0, \quad y(0) = 1, \quad y(\pi) = 2$

30. $y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y(1) = 0$

31. $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y(\pi/2) = 1$

32. $9y'' - 18y' + 10y = 0, \quad y(0) = 0, \quad y(\pi) = 1$

33. Let L be a nonzero real number.

(a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

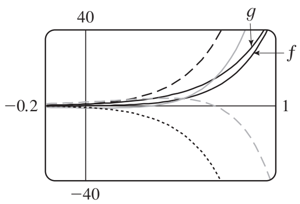
(b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

34. If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x \to \infty} y(x) = 0$.

Answers

S Click here for solutions.

1. $y = c_1 e^{4x} + c_2 e^{2x}$ **3.** $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$
5. $y = c_1 e^{x} + c_2 x e^{x}$ **7.** $y = c_1 \cos(x/2) + c_2 \sin(x/2)$
9. $y = c_1 + c_2 e^{-x/4}$ **11.** $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$
13. $y = e^{-t/2}\left[c_1 \cos\left(\sqrt{3}t/2\right) + c_2 \sin\left(\sqrt{3}t/2\right)\right]$
15.



All solutions approach 0 as $x \to -\infty$ and approach $\pm\infty$ as $x \to \infty$.
17. $y = 2e^{-3x/2} + e^{-x}$ **19.** $y = e^{x/2} - 2xe^{x/2}$
21. $y = 3 \cos 4x - \sin 4x$ **23.** $y = e^{-x}(2 \cos x + 3 \sin x)$
25. $y = 3 \cos\left(\frac{1}{2}x\right) - 4 \sin\left(\frac{1}{2}x\right)$ **27.** $y = \dfrac{e^{x+3}}{e^3 - 1} + \dfrac{e^{2x}}{1 - e^3}$
29. No solution
31. $y = e^{-2x}(2 \cos 3x - e^{\pi} \sin 3x)$
33. (b) $\lambda = n^2\pi^2/L^2$, n a positive integer; $y = C \sin(n\pi x/L)$

Solutions: Second-Order Linear Differential Equations

1. The auxiliary equation is $r^2 - 6r + 8 = 0 \quad\Rightarrow\quad (r-4)(r-2) = 0 \quad\Rightarrow\quad r = 4, r = 2$. Then by (8) the general solution is $y = c_1 e^{4x} + c_2 e^{2x}$.

3. The auxiliary equation is $r^2 + 8r + 41 = 0 \quad\Rightarrow\quad r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.

5. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0 \quad\Rightarrow\quad r = 1$. Then by (10), the general solution is $y = c_1 e^x + c_2 x e^x$.

7. The auxiliary equation is $4r^2 + 1 = 0 \quad\Rightarrow\quad r = \pm\frac{1}{2}i$, so $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$.

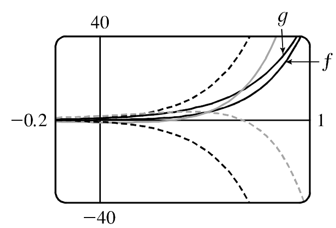
9. The auxiliary equation is $4r^2 + r = r(4r+1) = 0 \quad\Rightarrow\quad r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2 e^{-x/4}$.

11. The auxiliary equation is $r^2 - 2r - 1 = 0 \quad\Rightarrow\quad r = 1 \pm \sqrt{2}$, so $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$.

13. The auxiliary equation is $r^2 + r + 1 = 0 \quad\Rightarrow\quad r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$.

15. $r^2 - 8r + 16 = (r-4)^2 = 0$ so $y = c_1 e^{4x} + c_2 x e^{4x}$.

The graphs are all asymptotic to the x-axis as $x \to -\infty$, and as $x \to \infty$ the solutions tend to $\pm\infty$.



17. $2r^2 + 5r + 3 = (2r+3)(r+1) = 0$, so $r = -\frac{3}{2}, r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \quad\Rightarrow\quad c_1 + c_2 = 3$ and $y'(0) = -4 \quad\Rightarrow\quad -\frac{3}{2}c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

19. $4r^2 - 4r + 1 = (2r-1)^2 = 0 \quad\Rightarrow\quad r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \quad\Rightarrow\quad c_1 = 1$ and $y'(0) = -1.5 \quad\Rightarrow\quad \frac{1}{2}c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2xe^{x/2}$.

21. $r^2 + 16 = 0 \quad\Rightarrow\quad r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$. Then $y\left(\frac{\pi}{4}\right) = -3 \quad\Rightarrow\quad -c_1 = -3 \quad\Rightarrow\quad c_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 4 \quad\Rightarrow\quad -4c_2 = 4 \quad\Rightarrow\quad c_2 = -1$, so the solution to the initial-value problem is $y = 3\cos 4x - \sin 4x$.

23. $r^2 + 2r + 2 = 0 \quad\Rightarrow\quad r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \quad\Rightarrow\quad c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2\cos x + 3\sin x)$.

25. $4r^2 + 1 = 0 \quad\Rightarrow\quad r = \pm\frac{1}{2}i$ and the general solution is $y = c_1 \cos\left(\frac{1}{2}x\right) + c_2 \sin\left(\frac{1}{2}x\right)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right)$.

27. $r^2 - 3r + 2 = (r-2)(r-1) = 0 \quad\Rightarrow\quad r = 1, r = 2$ and the general solution is $y = c_1 e^x + c_2 e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1 e^3 + c_2 e^6$ so $c_2 = 1/(1 - e^3)$ and $c_1 = e^3/(e^3 - 1)$. The solution of the boundary-value problem is $y = \dfrac{e^{x+3}}{e^3 - 1} + \dfrac{e^{2x}}{1 - e^3}$.

29. $r^2 - 6r + 25 = 0 \quad\Rightarrow\quad r = 3 \pm 4i$ and the general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1 e^{3\pi} \quad\Rightarrow\quad c_1 = 2/e^{3\pi}$, so there is no solution.

31. $r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i$ and the general solution is $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$. But $2 = y(0) = c_1$ and $1 = y\left(\frac{\pi}{2}\right) = e^{-\pi}(-c_2)$, so the solution to the boundary-value problem is $y = e^{-2x}(2\cos 3x - e^{\pi} \sin 3x)$.

33. (a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow$ $y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus, $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ (distinct and real since $\lambda < 0$) $\quad \Rightarrow \quad y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1 + c_2$ $(*)$ and $0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$. Thus, $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\,L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,L = n\pi$ where n is an integer $\Rightarrow \quad \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.