... function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ is bounded, if there exists a real finite positive number M such that $|f(x, y)| \le M$ for all $(x, y) \in D$.

Remark 1

(a) The domain of a function of n variables $z = f(x_1, x_2, \ldots, x_n)$ is the set of all n-tuples in $\mathbb{R}^n$ for which f is defined.

(b) For functions of three variables, we need a four-dimensional space and an $(n+1)$ - dimensional space for a function of n variables, for its graphical representation. Therefore, it is not possible to represent a function of three or more variables by means of a graph in space.

(c) For a function of n variables, we define the distance between two points $P(x_{10}, x_{20}, \ldots, x_{n0})$ and $Q(x_{11}, x_{21}, \ldots, x_{n1})$ in $\mathbb{R}^n$ as

$$|PQ| = \sqrt{(x_{11} - x_{10})^2 + (x_{21} - x_{20})^2 + \ldots + (x_{n1} - x_{n0})^2}$$

and the neighborhood $N_\delta(P)$ of the point $P(x_{10}, x_{20}, \ldots, x_{n0})$ is the set of all points $(x_1, x_2, \ldots, x_n)$ inside an open ball

$$\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta.$$

2.2.1 Limits

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.8}$$

then the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$. Symbolically, we write it as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Note that for the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If $f(x, y)$ is not defined at $P(x_0, y_0)$, then we write

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This definition is called the δ-ε approach to study the existence of limits.

Remark 2

(a) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$, if it exists is unique.

(b) Let $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$. Then, we can define the limit given in Eq. (2.8) as

$$\lim_{r\to 0} |f(r\cos\theta, r\sin\theta) - L| < \varepsilon, \quad \text{whenever} \quad r < \delta, \text{ independent of } \theta.$$

(c) Since $(x, y) \to (x_0, y_0)$ in the two-dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit is unique, the limit is same along all the paths, that is the limit is independent of the path. Thus, the limit of a function cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

Let $u = f(x, y)$ and $v = g(x, y)$ be two real valued functions defined in a domain D. Let

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = L_2.$$

Then, the following results can be easily established.

(i) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [kf(x, y)] = kL_1$ for any real constant k.

(ii) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x, y) \pm g(x, y)] = L_1 \pm L_2.$

(iii) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x, y)\, g(x, y)] = L_1 L_2.$

(iv) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x, y)/g(x, y)] = L_1/L_2,\; L_2 \neq 0.$

Remark 3

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Then, for any fixed point $P_0(x_{10}, x_{20}, \ldots, x_{n0})$ in D

$$\lim_{P\to P_0} f(x_1, x_2, \ldots, x_n) = L.$$

if $|f(x_1, x_2, \ldots, x_n) - L| < \varepsilon$, whenever $\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta$ where $P(x_1, x_2, \ldots, x_n)$ is a point in the neighborhood or the deleted neighborhood of P_0.

Example 2.1 Using the δ-ε approach, show that

(i) $\displaystyle\lim_{(x,y)\to(2,1)} (3x + 4y) = 10,$ (ii) $\displaystyle\lim_{(x,y)\to(1,1)} (x^2 + 2y) = 3.$

Solution

(i) Here $f(x, y) = 3x + 4y$ is defined at (2, 1). We have

$|f(x, y) - 10| = |3x + 4y - 10| = |3(x - 2) + 4(y - 1)| \le 3|x - 2| + 4|y - 1|.$

If we take $|x - 2| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 10| < 7\delta < \varepsilon$, which is satisfied when $\delta < \varepsilon/7$.

Hence, $\displaystyle\lim_{(x,y)\to(2,1)} f(x, y) = 10.$

Note that the value of δ is not unique.

(ii) Here $f(x, y) = x^2 + 2y$ is defined at (1, 1). We have

$$\begin{aligned} |f(x, y) - 3| &= |x^2 + 2y - 3| = |(x - 1 + 1)^2 + 2(y - 1 + 1) - 3| \\ &= |(x - 1)^2 + 2(x - 1) + 2(y - 1)| \le |x - 1|^2 + 2|x - 1| + 2|y - 1| \end{aligned}$$

If we take $|x - 1| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 3| < \delta^2 + 4\delta < \varepsilon$ which is satisfied when

$$(\delta + 2)^2 < \varepsilon + 4 \text{ or } \delta < \sqrt{\varepsilon + 4} - 2.$$

Hence, $$\lim_{(x,y)\to(1,1)} f(x, y) = 3.$$

We can also write $$|f(x, y) - 3| < \delta^2 + 4\delta < 5\delta < \varepsilon$$

which is satisfied when $\delta < \varepsilon/5$.

Example 2.2 Using δ-ε approach, show that

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0,$ (ii) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \left(\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right) = 0$

Solution

(i) Here $f(x, y) = xy/(\sqrt{x^2+y^2})$ is not defined at (0, 0). We have

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{1}{2}\frac{(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2} < \varepsilon,\ (x, y) \ne (0, 0)$$

since $|xy| \le (x^2 + y^2)/2$. If we choose $\delta < 2\varepsilon$, then we get

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

Hence, $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

Alternative Writing $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \lim_{r\to 0}\left|\frac{r^2\sin\theta\cos\theta}{r}\right| = 0$$

which is independent of θ.

(ii) Here $f(x, y, z) = (xy + xz + yz)/\sqrt{x^2+y^2+z^2}$ is not defined at (0, 0, 0).

Since $|xy| \le (x^2 + y^2)/2$, $|xz| \le (x^2 + z^2)/2$, $|yz| \le (y^2 + z^2)/2$, we get

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} - 0\right| \le \frac{1}{2}\left[\frac{x^2+y^2+x^2+z^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}\right] = \left|\sqrt{x^2+y^2+z^2}\right| < \varepsilon.$$

If we choose $\delta < \varepsilon$, we obtain

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} - 0\right| < \varepsilon \quad \text{whenver} \quad 0 < \sqrt{x^2+y^2+z^2} < \delta.$$

Hence, $$\lim_{(x,y,z)\to(0,0,0)} \left[\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right] = 0.$$

Example 2.3 Show that the following limits

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+\sqrt{y}}{x^2+y^2}$,

(iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$. (iv) $\lim\limits_{(x,y)\to(0,1)} \tan^{-1}\left(\dfrac{y}{x}\right)$.

do not exist.

Solution The limit does not exist if it is not finite, or if it depends on a particular path.

(i) Consider the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

Alternative Setting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$$

which depends on θ. Hence, the limit is dependent on different radial paths θ = constant. Hence, the limit does not exist.

(ii) Choose the path $y = mx^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y} = \lim_{x\to 0} \frac{1+\sqrt{m}}{(1+m)x} = \infty.$$

Since the limit is not finite, the limit does not exist.

(iii) Choose the path $y = mx^3$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{mx^6}{(1+m^2)x^6} = \frac{m}{1+m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

(iv) We have

$$\lim_{(x,y)\to(0,1)} \tan^{-1}\frac{y}{x} = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$$

depending on whether the point (0, 1) is approached from left or from right along the line $y = 1$. If we approach from left, we obtain the limit as $-\pi/2$ and if we approach from right, we obtain the limit as $\pi/2$. Since the limit is not unique, the limit does not exist as $(x, y) \to (0, 1)$.

2.2.2 Continuity

A function $z = f(x, y)$ is said to be *continuous* at a point (x_0, y_0), if

(i) $f(x, y)$ is defined at the point (x_0, y_0), (ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

If any one of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0, y_0).

Therefore, a function $f(x, y)$ is continuous at (x_0, y_0) if

$$|f(x,y)-f(x_0,y_0)|<\varepsilon, \quad \text{whenever } \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta.$$

If $f(x_0, y_0)$ is defined and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$ exists, but $f(x_0, y_0) \neq L$, then the point (x_0, y_0) is called a point of *removable discontinuity*. We can redefine the function at the point (x_0, y_0) as $f(x_0, y_0) = L$ so that the new function becomes continuous at the point (x_0, y_0).

If the function $f(x, y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

In the definition of continuity, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$ holds for all paths going to the point (x_0, y_0).

Hence, if the continuity of a function is to be proved, we cannot choose a path and find the limit. However, to show that a function is discontinuous, it is sufficient to choose a path and show that the limit does not exist.

A continuous function has the following properties:

P1 A continuous function in a closed and bounded domain D attains atleast once its maximum value M and its minimum value m at some point inside or on the boundary of D.

P2 For any number μ that satisfies $m < \mu < M$, there exists a point (x_0, y_0) in D such that $f(x_0, y_0)=\mu$.

P3 A continuous function, in a closed and bounded domain D. that attains both positive and negative values will have the value zero at some point in D.

Example 2.4 Show that the following functions are continuous at the point (0, 0).

(i) $f(x,y)=\begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$ (ii) $f(x,y)=\begin{cases} \dfrac{2x(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0), \end{cases}$

(iii) $f(x,y)=\begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0) \\ 1/2, & (x,y)=(0,0). \end{cases}$

Solution

(i) Let $x=r\cos\theta, y=r\sin\theta$. Then $r=\sqrt{x^2+y^2}\neq 0$. We have

$$|f(x,y)-f(0,0)|=\left|\frac{2x^4+3y^4}{x^2+y^2}\right|=\left|\frac{r^4(2\cos^4\theta+3\sin^4\theta)}{r^2(\cos^2\theta+2\sin^2\theta)}\right|$$

$$<r^2[2|\cos^4\theta|+3|\sin^4\theta]<5r^2<\varepsilon$$

or
$$r = \sqrt{x^2 + y^2} < \sqrt{\varepsilon/5}.$$

If we choose $\delta < \sqrt{\varepsilon/5}$, we find that $|f(x, y) - f(0, 0)| < \varepsilon$, whenever $0 < \sqrt{x^2 + y^2} < \delta$. Therefore, $\lim\limits_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at (0, 0).

(ii) Let $x = r\cos\theta$, $y = r\sin\theta$. Then, $r = \sqrt{x^2 + y^2} \neq 0$. We have

$$|f(x, y) - f(0, 0)| = \left|\frac{2x(x^2 - y^2)}{x^2 + y^2}\right| = \left|\frac{2r^3(\cos^2\theta - \sin^2\theta)\cos\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$
$$= |2r\cos 2\theta\cos\theta| \le 2r < \varepsilon$$

or
$$r = \sqrt{x^2 + y^2} < \varepsilon/2.$$

If we choose $\delta < \varepsilon/2$, we find that

$$|f(x, y) - f(0, 0)| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at (0, 0).

(iii) Let $x + 2y = t$. Therefore, $t \to 0$ as $(x, y) \to (0, 0)$.

We can now write

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}2t} = \lim_{t\to 0}\left[\frac{(\sin^{-1}t)/t}{(\tan^{-1}(2t))/(2t)}\right]\left[\frac{t}{2t}\right] = \frac{1}{2}.$$

Since $\lim\limits_{(x, y)\to(0, 0)} f(x, y) = f(0, 0) = \frac{1}{2}$, the given function is continuous at $(x, y) = (0, 0)$.

Example 2.5 Show that the following functions are discontinuous at the given points

(i) $f(x, y) = \begin{cases} \dfrac{x - y}{x + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at the point (0, 0).

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at the point (0, 0).

(iii) $f(x, y) = \begin{cases} \dfrac{x^2 + xy + x + y}{x + y}, & (x, y) \neq (2, 2) \\ 4, & (x, y) = (2, 2) \end{cases}$

at the point (2, 2).

Solution

(i) Choose the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y)\to(0, 0)} \frac{x - y}{x + y} = \lim_{x\to 0}\frac{(1 - m)x}{(1 + m)x} = \frac{1 - m}{1 + m}$$

which depends on m. Since, the limit does not exist, the function is not continuous at $(0, 0)$.

(ii) Choose the path $y = m^2x^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - x\sqrt{y}}{x^2 + y} = \lim_{x\to 0} \frac{(1 - m)x^2}{(1 + m^2)x^2} = \frac{1 - m}{1 + m^2}$$

which depends on m. Since the limit does not exist, the function is not continuous at $(0, 0)$.

(iii) $$\lim_{(x, y)\to(2, 2)} f(x, y) = \lim_{(x, y)\to(2, 2)} \frac{(x + y)(x + 1)}{(x + y)} = \lim_{(x, y)\to(2, 2)} (x + 1) = 3.$$

Since $\lim_{(x, y)\to(2, 2)} f(x, y) \neq f(2, 2)$, the function is not continuous at $(2, 2)$.

Note that the point $(2, 2)$ is a point of removable discontinuity.

Example 2.6 Let $f(x, y) = \begin{cases} \dfrac{x^4y - 3x^2y^3 + y^5}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Find a $\delta > 0$ such that $|f(x, y) - f(0, 0)| < 0.01$, whenever $\sqrt{x^2 + y^2} < \delta$.

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^4y - 3x^2y^3 + y^5}{(x^2 + y^2)^2} \right|.$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$|f(x, y) - f(0, 0)| = \left| \frac{r^5(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)}{r^4(\cos^2\theta + \sin^2\theta)^2} \right|$$

$$= |r(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)|$$

$$\leq r(1 + 3 + 1) = 5r = 5\sqrt{x^2 + y^2} < 0.01.$$

Therefore, $\sqrt{x^2 + y^2} \leq 0.01/5 = 0.002$. Hence, $\delta < 0.002$.

Exercise 2.1

Using the δ-ε approach, establish the following limits.

1. $\lim_{(x, y)\to(1, 1)} (x^2 + y^2 - 1) = 1.$

2. $\lim_{(x, y)\to(2, 1)} (x^2 + 2x - y^2) = 7.$

3. $\lim_{(x, y)\to(0, 0)} \dfrac{x + y}{x^2 + y^2 + 1} = 0.$

4. $\lim_{(x, y)\to(0, 0)} \dfrac{x^3 + y^3}{x^2 + y^2} = 0.$

5. $\lim_{(x, y)\to(0, 0)} \left[y + x\cos\left(\dfrac{1}{y}\right) \right] = 0.$

6. $\lim_{(x, y)\to(0, 0)} (x^2 + y^2)\sin\dfrac{1}{xy} = 0.$

Determine the following limits if they exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$.

8. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3-y^3}{x-y}$.

9. $\lim\limits_{(x,y)\to(a,0)} \left(1+\dfrac{x}{y}\right)^y$.

10. $\lim\limits_{(x,y)\to(0,0)} \cot^{-1}\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$.

11. $\lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)\tan^2 x}{x^2(y^2-1)}$.

12. $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)\sin y}{y\ln x}$.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x-y}{x^2+y^2}$.

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^3+y^3}$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{(x^4+y^2)^2}$.

17. $\lim\limits_{(x,y,z)\to(0,0,0)} \log\left(\dfrac{z}{xy}\right)$.

18. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+z}{x+y+z^2}$.

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy^2z^2}{x^4+y^4+z^8}$.

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x(x+y+z)}{x^2+y^2+z^2}$.

Discuss the continuity of the following functions at the given points.

21. $f(x,y)=\begin{cases} \dfrac{(x-y)^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

22. $f(x,y)=\begin{cases} \dfrac{1}{1+e^{1/x}}+y^2, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

23. $f(x,y)=\begin{cases} \dfrac{e^{xy}}{x^2+1}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

24. $f(x,y)=\begin{cases} \dfrac{x^2+y^2}{xy}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

25. $f(x,y)=\begin{cases} \dfrac{x^2+y^2}{\tan xy}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

26. $f(x,y)=\begin{cases} \dfrac{x^2-2xy+y^2}{x-y}, & (x,y)\neq(1,-1) \\ 0, & (x,y)=(1,-1) \end{cases}$

at (1, –1).

27. $f(x,y)=\begin{cases} \dfrac{xy(x-y)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

28. $f(x,y)=\begin{cases} \dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

29. $f(x,y)=\begin{cases} \dfrac{\sin\sqrt{|xy|}-\sqrt{|xy|}}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

at (0, 0).

30. $f(x,y)=\begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

at (0, 0).

31. $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$

at (0, 0).

32. $f(x, y) = \begin{cases} \dfrac{x^5 - y^5}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$

at (0, 0).

33. $f(x, y) = \begin{cases} \dfrac{x^2y}{1 + x}, & x \neq -1 \\ y \quad , & (x, y) = (-1, \alpha) \end{cases}$

at $(-1, \alpha)$.

34. $f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2 + y^2 + z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad , & (x, y, z) = (0, 0, 0) \end{cases}$

at (0, 0, 0).

35. $f(x, y, z) = \begin{cases} \dfrac{2xy}{x^2 - 3z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0 \quad , & (x, y, z) = (0, 0, 0) \end{cases}$

at (0, 0, 0).

2.3 Partial Derivatives

The derivative of a function of several variables with respect to one of the independent variables keeping all the other independent variables as constant is called the *partial derivative* of the function with respect to that variable.

Consider the function of two variables $z = f(x, y)$ defined in some domain D of the x-y plane. Let y be held constant, say $y = y_0$. Then, the function $f(x, y_0)$ depends on x alone and is defined in an interval about x, that is $f(x, y_0)$ is a function of one variable x. Let the points (x, y_0) and $(x + \Delta x, y_0)$ be in D, where Δx is an increment in the independent variable x. Then

$$\Delta_x z = f(x + \Delta x, y_0) - f(x, y_0) \tag{2.10}$$

is called the *partial increment* in z with respect to x and is a function of x and Δx.

Similarly, if x is held constant, say $x = x_0$, then the function $f(x_0, y)$ depends only on y and is defined in some interval about y, that is $f(x_0, y)$ is a function of one variable y. Let the points (x_0, y) and $(x_0, y + \Delta y)$ be in D, where Δy is an increment in the independent variable y. Then

$$\Delta_y z = f(x_0, y + \Delta y) - f(x_0, y) \tag{2.11}$$

is called the partial increment in z with respect to y and is a function of y and Δy.

When both x and y are given increments Δx and Δy respectively, then the increment Δz in z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{2.12}$$

This increment is called the *total increment* in z and is a function of x, y, Δx and Δy.

In general, $\Delta z \neq \Delta_x z + \Delta_y z$. For example, consider the function $z = f(x, y) = xy$ and a point (x_0, y_0). We have

$$\Delta_x z = (x_0 + \Delta x) y_0 - x_0 y_0 = y_0 \Delta x$$

$$\Delta_y z = x_0(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y$$

$$\Delta z = (x_0 + \Delta x)\,(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y + y_0 \Delta y + \Delta x \Delta y \neq \Delta_x z + \Delta_y z.$$

Now, consider the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}. \tag{2.13}$$

If this limit exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to x at the point (x_0, y_0) and is denoted by $z_x(x_0, y_0)$ or $f_x(x_0, y_0)$ or $(\partial f/\partial x)\,(x_0, y_0)$ or $(\partial z/\partial x)\,(x_0, y_0)$.

Similarly, if the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2.14}$$

exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to y at the point (x_0, y_0) and is denoted by $z_y(x_0, y_0)$ or $f_y(x_0, y_0)$ or $(\partial z/\partial y)\,(x_0, y_0)$ or $(\partial f/\partial y)\,(x_0, y_0)$.

Remark 4

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Let $P_0(x_1, x_2, \ldots, x_n)$ be a point in D. If the limit

$$\lim_{\Delta x_i \to 0} \frac{\Delta_{x_i} z}{\Delta x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots x_n)}{\Delta x_i}$$

exists, then it is called the partial derivative of f at the point P_0 and is denoted by $(\partial f/\partial x_i)\,(P_0)$.

Remark 5

The definition of continuity, $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ can be written in alternate forms. Set $x = x_0 + \Delta x$, $y = y_0 + \Delta y$. Define $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Then, $\Delta x \to 0$, $\Delta y \to 0$ implies that $\Delta\rho \to 0$. We note that $|\Delta x| < \Delta\rho$ and $|\Delta y| < \Delta\rho$.

The above definition of continuity is equivalent to the following forms:

(i) $\lim_{\Delta x \to 0,\, \Delta y \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(ii) $\lim_{\Delta\rho \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$

(iii) $\lim_{\Delta\rho \to 0} \Delta z = 0.$

Example 2.7 Find the first order partial derivatives of the following functions

(i) $f(x, y) = x^2 + y^2 + x$, (ii) $f(x, y) = y\,e^{-x}$, (iii) $f(x, y) = \sin(2x + 3y)$

at the point (x, y) from the first principles.

Solution we have

(i) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + y^2 + (x + \Delta x)] - [x^2 + y^2 + x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x + 1)\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} [2x + 1 + \Delta x] = 2x + 1.$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 + (y + \Delta y)^2 + x] - [x^2 + y^2 + x]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\, \Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \to 0} [2y + \Delta y] = 2y.$$

(ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{ye^{-(x + \Delta x)} - ye^{-x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-ye^{-x}(1 - e^{-\Delta x})}{\Delta x} = -ye^{-x} \lim_{\Delta x \to 0} \frac{1 - e^{-\Delta x}}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)e^{-x} - ye^{-x}}{\Delta y} = e^{-x}.$$

(iii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin(2(x + \Delta x) + 3y) - \sin(2x + 3y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos(2x + 3y + \Delta x)\sin \Delta x}{\Delta x}$$

$$= 2\cos(2x + 3y).$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y + \Delta y)) - \sin(2x + 3y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\cos(2x + 3y + 3\Delta y/2)\sin(3\Delta y/2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} [3\cos(2x + 3y + 3\Delta y/2)] \frac{\sin(3\Delta y/2)}{(3\Delta y/2)} = 3\cos(2x + 3y).$$

Example 2.8 Show that the function

$$f(x, y) = \begin{cases} (x + y)\sin\left(\dfrac{1}{x + y}\right), & x + y \neq 0 \\ 0, & x + y = 0 \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left| (x + y)\sin\left(\frac{1}{x + y}\right) \right| \leq |x + y| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

If we choose $\delta < \varepsilon/2$, then

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0), the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x \sin(1/\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

does not exist. Therefore, the partial derivative f_x does not exist at (0, 0).

Similarly at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y \sin(1/\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \sin\left(\frac{1}{\Delta y}\right)$$

does not exist. Therefore, the partial derivative f_y does not exist at (0, 0).

Example 2.9 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^2 + y^2}{|x| + |y|} \right| \leq \frac{[\,|x| + |y|\,]^2}{|x| + |y|} = |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon/2$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0).$

Hence, the given function is continuous at (0, 0).

Now, at (0, 0) we have

$$\lim_{\Delta x \to 0} \frac{\Delta_x f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{|\Delta x|} = \begin{cases} 1, \text{ when } \Delta x > 0 \\ -1, \text{ when } \Delta x < 0. \end{cases}$$

Hence, the limit does not exist. Therefore, f_x does not exist at (0, 0).

Also at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{|\Delta y|} = \begin{cases} 1, \text{ when } \Delta y > 0 \\ -1, \text{ when } \Delta y < 0 \end{cases}$$

does not exist. Therefore, f_y does not exist at (0, 0).

Example 2.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

Solution Choose the path $y = mx$. Since the limit

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{mx^2}{(1 + 2m^2)x^2} = \frac{m}{1 + 2m^2}$$

depends on m, the function is not continuous at (0, 0). We now have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0.$$

Therefore, the partial derivatives f_x and f_y exist at (0, 0).

Theorem 2.1 (Sufficient condition for continuity) A sufficient condition for a function $f(x, y)$ to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives exists and is bounded in the neighborhood of (x_0, y_0) and that the other exists at (x_0, y_0).

Proof Let the partial derivative f_x exist and be bounded in the neighborhood of the point (x_0, y_0) and f_y exist at (x_0, y_0). Since f_y exists at (x_0, y_0), we have

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0).$$

Therefore, we can write

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y \tag{2.15}$$

where ε_1 depends on Δy and tends to zero as $\Delta y \to 0$. Since f_x exists in the neighborhood of (x_0, y_0), we can write using the Lagrange mean value theorem

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = \Delta x\, f_x(x_0 + \theta\, \Delta x, y_0 + \Delta y),\ 0 < \theta < 1. \tag{2.16}$$

Now, using Eqs. (2.15) and (2.16), we obtain

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \\ &= \Delta x\, f_x(x_0 + \theta \Delta x, y_0 + \Delta y) + \Delta y\, f_y(x_0, y_0) + \varepsilon_1 \Delta y. \end{aligned} \tag{2.17}$$

Since f_x is bounded in the neighborhood of the point (x_0, y_0), we obtain from Eq. (2.17)

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0).$$

Hence, the function $f(x, y)$ is continuous at the point (x_0, y_0).

Geometrical interpretation of partial derivatives

Let $z = f(x, y)$ represent a surface as shown in Fig. 2.3. Let the plane $x = x_0$ = constant intersect the surface $z = f(x, y)$ along the curve $z = f(x_0, y)$. Let $P(x_0, y, 0)$ be a particular point in the x-y plane and $R(x_0, y, z)$ be the corresponding point on the surface, where $z = f(x_0, y)$. Let $Q(x_0, y + \Delta y, 0)$ be a point in the x-y plane in the neighborhood of P and $S(x_0, y + \Delta y, z + \Delta_y z)$ be the corresponding point on the surface $z = f(x, y)$. From Fig. 2.3, we find that $\Delta y = PQ = RS'$ and the function z is increased by $SS' = (z + \Delta_y z) - z = \Delta_y z$. Now, let θ^* be the angle which the chord RS makes with the positive y-axis. Then, from $\Delta RSS'$, we have

$$\tan \theta^* = \frac{SS'}{RS'} = \frac{\Delta_y z}{\Delta y}.$$

Let $\Delta y \to 0$. Then, $\Delta_y z \to 0$. Hence,

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y} = \tan \theta$$

where in the limit, θ is the angle made by the tangent to the curve $z = f(x_0, y)$ at the point $R(x_0, y, z)$ on the surface $z = f(x, y)$ with the positive y-axis.

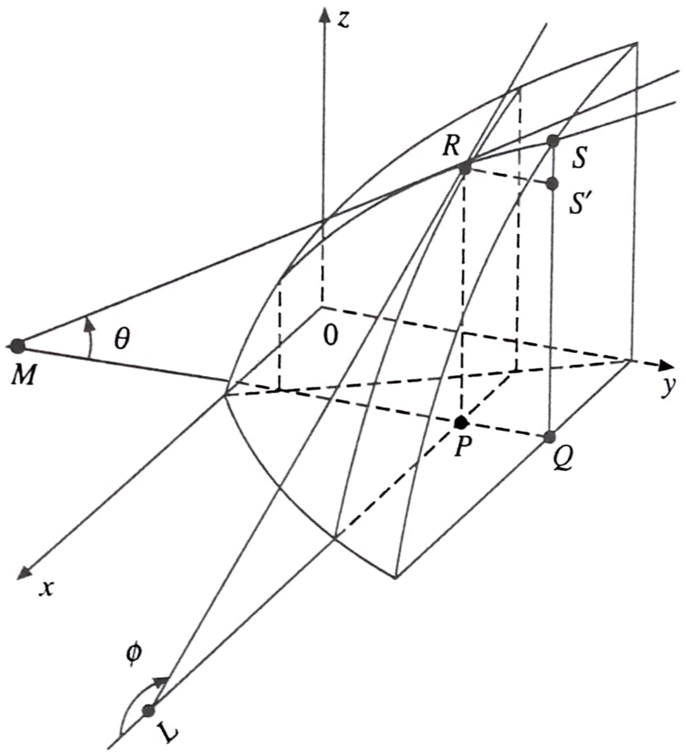

Fig. 2.3. Geometrical representation of partial derivatives.

Now, consider the intersection of the plane $y = y_0 =$ constant with the surface $z = f(x, y)$. Following the similar procedure, we obtain $\partial z/\partial x = \tan \phi$, where ϕ is the angle made by the tangent to the curve $z = f(x, y_0)$ at the point (x, y_0, z) on the surface $z = f(x, y)$ with the positive x-axis.

It can be observed that this representation of partial derivatives is a direct extension of the one dimensional case.

2.3.1 Total Differential and Differentiability

Let a function of two variables $z = f(x, y)$ be defined in some domain D in the x-y plane. Let $P(x, y)$ be any point in D and $(x + \Delta x, y + \Delta y)$ be a point in the neighborhood of (x, y), in D. Then,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

The function $z = f(x, y)$ is said to be *differentiable* at the point (x, y), if at this point Δz can be written as

$$\Delta z = (a\,\Delta x + b\,\Delta y) + (\varepsilon_1 \Delta x + \varepsilon_2 \Delta y) \tag{2.18}$$

where a, b are independent of Δx, Δy and $\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$ are infinitesimals and functions of Δx, Δy such that $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The first part $a\,\Delta x + b\,\Delta y$ in Eq. (2.18) which is linear in Δx and Δy is called the *total differential* or simply the differential of z at the point (x, y) and is denoted by dz or df. That is

$$dz = a\,\Delta x + b\Delta y \quad \text{or} \quad dz = a\,dx + b\,dy.$$

Let $\Delta y = 0$ in Eq. (2.18). Then, $\Delta z = a\,\Delta x + \varepsilon_1 \Delta x$. Dividing by Δx and taking limits as $\Delta x \to 0$, we obtain $a = \partial z/\partial x$. Similarly, letting $\Delta x = 0$ in Eq. (2.18), dividing by Δy and taking limits as $\Delta y \to 0$, we obtain $b = \partial z/\partial y$. Therefore,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x \Delta x + f_y \Delta_y \tag{2.19}$$

assuming that the partial derivatives exist at P. Hence,

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{2.20}$$

Therefore, *existence of partial derivatives f_x and f_y at a point $P(x, y)$ is a necessary condition for differentiability of $f(x, y)$ at P.*

The second part $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is the infinitesimal nonlinear part and is of higher order relative to Δx, Δy or $\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Note that $(\Delta x, \Delta y) \to (0, 0)$ implies $\Delta \rho \to 0$. Eq. (2.20) can be written as

$$\frac{\Delta z - dz}{\Delta \rho} = \varepsilon_1 \left(\frac{\Delta x}{\Delta \rho}\right) + \varepsilon_2 \left(\frac{\Delta y}{\Delta \rho}\right) \tag{2.21}$$

Now, if $f(x, y)$ is differentiable, then as $\Delta \rho \to 0$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Taking the limit as $\Delta \rho \to 0$ in Eq. (2.21), we obtain

$$\lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = \lim_{\Delta \rho \to 0} \left[\varepsilon_1 \left(\frac{\Delta x}{\Delta \rho}\right) + \varepsilon_2 \left(\frac{\Delta y}{\Delta \rho}\right)\right] = 0 \tag{2.22}$$

since $|\Delta x/\Delta \rho| \le 1$ and $|\Delta y/\Delta \rho| \le 1$.

Therefore, to test differentiability at a point $P(x, y)$, we can use either of the following two approaches.

(i) Show that $\displaystyle\lim_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0$ (2.23)

(ii) Find the expressions for $\varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2(\Delta x, \Delta y)$ from Eq. (2.20) and then show that $\lim \varepsilon_1 \to 0$ and $\lim \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ or $\Delta \rho \to 0$.

Note that the function $f(x, y)$ may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x, f_y exist at P (see Example 2.12). However, if the first order partial derivatives are continuous at the point P, then the function is differentiable at P. We present this result in the following theorem.

Theorem 2.2 (Sufficient condition for differentiability) If the function $z = f(x, y)$ has continuous first order partial derivatives at a point $P(x, y)$ in D, then $f(x, y)$ is differentiable at P.

Proof Let $P(x, y)$ be a fixed point in D. By the Lagrange mean value theorem, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x\, f_x(x + \theta_1 \Delta x, y),\ 0 < \theta_1 < 1$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x + \Delta x, y + \theta_2 \Delta y),\ 0 < \theta_2 < 1.$$

Since f_x and f_y are continuous at (x, y), we can write

$$f_x(x + \theta_1 \Delta x, y) = f_x(x, y) + \varepsilon_1$$

and

$$f_y(x + \Delta x, y + \theta_2 \Delta y) = f_y(x, y) + \varepsilon_2$$

where ε_1, ε_2 are infinitesimals, are functions of Δx, Δy and tend to zero as $\Delta x \to 0$, $\Delta y \to 0$, that is, as $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$. Therefore, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x f_x(x, y) + \varepsilon_1 \Delta x \tag{2.24}$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x, y) + \varepsilon_2\, \Delta y \tag{2.25}$$

Now, the total increment is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)].\end{aligned}$$

Using Eqs. (2.24) and (2.25), we obtain

$$\Delta z = f_x\, \Delta x + f_y\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y \tag{2.26}$$

where the partial derivatives are evaluated at the point $P(x, y)$. Hence, $f(x, y)$ is differentiable at P.

Remark 6

(a) For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, we write the total differential as

$$dz = f_{x_1} dx_1 + f_{x_2}\, dx_2 + \ldots + f_{x_n} dx_n. \tag{2.27}$$

(b) Note that continuity of the first partial derivatives f_x and f_y at a point P is a sufficient condition for differentiability at P, that is, a function may be differentiable even if f_x and f_y are not continuous (Problem 5, Exercise 2.2).

(c) The conditions of Theorem 2.2 can be relaxed. It is sufficient that one of the first order partial derivatives is continuous at (x_0, y_0) and the other exists at (x_0, y_0).

Example 2.11 Find the total differential of the following functions

(i) $z = \tan^{-1}(x/y)$, $(x, y) \neq (0, 0)$, (ii) $u = \left(xz + \dfrac{x}{z}\right)^y$, $z \neq 0$.

Solution

(i)
$$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right),\ f_x = \frac{1}{1 + (x/y)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2 + y^2}$$

and
$$f_y = \frac{1}{1 + (x/y)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}.$$

Therefore, we obtain the total differential as

$$dz = f_x\, dx + f_y\, dy = \frac{1}{x^2 + y^2}(y\, dx - x\, dy).$$

(ii)
$$f(x, y, z) = \left(xz + \frac{x}{z}\right)^y,\ f_x = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$f_y = \left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right),\ f_z = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right).$$

Therefore, we obtain the total differential as

$$du = \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z + \frac{1}{z}\right)dx + xy\left(1 - \frac{1}{z^2}\right)dz\right] + \left[\left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right)\right]dy.$$

Example 2.12 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad, & (x, y) = (0, 0) \end{cases}$$

(i) is continuous at (0, 0),

(ii) possesses partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$,

(iii) is not differentiable at (0, 0).

Solution

(i) Let $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$|f(x, y) - f(0, 0)| = \left| \frac{r^3(\cos^3\theta + 2\sin^3\theta)}{r^2} \right| \leq r[\,|\cos^3\theta| + 2\,|\sin^3\theta|\,]$$

$$\leq 3r = 3\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon/3$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to f(0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, $f(x, y)$ is continuous at (0, 0).

(ii) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x - 0}{\Delta x} = 1$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{2\Delta y - 0}{\Delta y} = 2.$

Therefore, the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(iii) We have $dz = \Delta x + 2\Delta y$. Using Eq. (2.20), we get

$$\Delta z = \Delta x + 2\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Let $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Now,

$$\Delta z = f(\Delta x, \Delta y) - f(0, 0) = \frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

Hence

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \to 0} \frac{1}{\Delta\rho} \left[\frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - (\Delta x + 2\Delta y) \right]$$

$$= \lim_{\Delta\rho \to 0} - \left[\frac{\Delta x \Delta y(\Delta y + 2\Delta x)}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}} \right]$$

Let $\Delta x = r\cos\theta$ and $\Delta y = r\sin\theta$. As $(\Delta x, \Delta y) \to (0, 0)$, $\Delta\rho = r \to 0$ for arbitrary θ. Therefore,

$$\lim_{\Delta\rho\to 0}\frac{\Delta z - dz}{\Delta\rho} = -\lim_{r\to 0}[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

$$= -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)].$$

The limit depends on θ and does not tend to zero for arbitrary θ. Hence, the given function is not differentiable. Alternately, we can write

$$\frac{\Delta z - dz}{\Delta\rho} = -\frac{1}{\Delta\rho}\left[\frac{\Delta x(\Delta y)^2 + 2(\Delta x)^2\Delta y}{(\Delta x)^2 + (\Delta y)^2}\right] = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)$$

where $\qquad \varepsilon_1 = -\dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}$ and $\varepsilon_2 = -\dfrac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}.$

Substituting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$, we find that ε_1 and ε_2 depend on θ and do not tend to zero for arbitrary θ, in the limit as $r \to 0$.

Example 2.13 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x, y) \neq (1, -1) \\ 0, & (x, y) = (1, -1) \end{cases}$$

is continuous and differentiable at (1, –1).

Solution We have

$$\lim_{(x,y)\to(1,-1)}\frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,-1)}(x + y) = 0 = f(1, -1).$$

Therefore, the function is continuous at (1, –1).

The partial derivatives are given by

$$f_x(1,-1) = \lim_{\Delta x\to 0}\frac{f(1+\Delta x, -1) - f(1, -1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{(1+\Delta x)^2 - 1}{(1+\Delta x) + 1} - 0\right] = \lim_{\Delta x\to 0}\frac{2+\Delta x}{2+\Delta x} = 1.$$

$$f_y(1,-1) = \lim_{\Delta y\to 0}\frac{f(1, -1+\Delta y) - f(1, -1)}{\Delta y} = \lim_{\Delta y\to 0}\frac{1}{\Delta y}\left[\frac{1-(-1+\Delta y)^2}{1-(-1+\Delta y)} - 0\right] = \lim_{\Delta y\to 0}\frac{2-\Delta y}{2-\Delta y} = 1.$$

Therefore, the first order partial derivatives exist at (1, –1).

Now, we have

$$f_x(x, y) = \frac{(x-y)(2x) - (x^2 - y^2)(1)}{(x-y)^2} = \frac{x^2 - 2xy + y^2}{(x-y)^2} = \frac{(x-y)^2}{(x-y)^2}, \ (x, y) \neq (1, -1)$$

and $\qquad f_x(x, y) = 1, (x, y) = (1, -1).$

Since $\qquad \displaystyle\lim_{(x,y)\to(1,-1)} f_x(x, y) = \lim_{(x,y)\to(1,-1)}\frac{(x-y)^2}{(x-y)^2} = 1 = f_x(1, -1)$

the partial derivative f_x is continuous at $(1, -1)$. Also $f_y(1, -1)$ exists. Hence, $f(x, y)$ is differentiable at $(1, -1)$.

Alternately, we can show that $\lim_{\Delta\rho \to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

2.3.2 Approximation by Total Differentials

From Theorem 2.2, we have for a function $f(x, y)$ of two variables

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x \, \Delta x + f_y \, \Delta y$$

or

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x \, \Delta x + f_y \, \Delta y \tag{2.28}$$

where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Consider now a function of n variables $x_1, x_2, \ldots, x_n$. Let the function $z = f(x_1, x_2, \ldots, x_n)$ be differentiable at the point $P(x_1, x_2, \ldots, x_n)$. Let there be errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in measuring the values of $x_1, x_2, \ldots, x_n$ respectively. Then, the computed value of z using the inexact values of the arguments will be obtained with an error

$$\Delta z = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) - f(x_1, x_2, \ldots, x_n). \tag{2.29}$$

When the errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are small in magnitude, we obtain (using the Remark 6 (a), Eq. (2.27))

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \approx f(x_1, x_2, \ldots, x_n) + f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n \tag{2.30}$$

where the partial derivatives are evaluated at the point $(x_1, x_2, \ldots, x_n)$. This is the generalization of the result for functions of two variables given in Eq. (2.28).

Since the partial derivatives and errors in arguments can be both positive and negative, we define the *absolute error* as (using Eq. (2.29))

$$|\Delta z| \approx |dz| = |df| = |f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n|.$$

Then,

$$|df| \le |f_{x_1}| \, |\Delta x_1| + |f_{x_2}| \, |\Delta x_2| + \ldots + |f_{x_n}| \, |\Delta x_n| \tag{2.31}$$

gives the *maximum absolute error* in z. If $\max |\Delta x_i| \le \Delta x$, then we can write

$$|df| \le \Delta x \, [\, |f_{x_1}| + |f_{x_2}| + \ldots + |f_{x_n}| \,].$$

The expression $|df|/|f|$ is called the *maximum relative error* and $[\, |df|/|f| \,] \times 100$ is called the *percentage error*.

The maximum relative error can also be written as

$$\frac{|df|}{|f|} \le \left| \frac{\partial f/\partial x_1}{f} \right| |\Delta x_1| + \left| \frac{\partial f/\partial x_2}{f} \right| |\Delta x_2| + \ldots + \left| \frac{\partial f/\partial x_n}{f} \right| |\Delta x_n|$$

$$\le \left| \frac{\partial}{\partial x_1} [\ln |f|] \right| |\Delta x_1| + \left| \frac{\partial}{\partial x_2} [\ln |f|] \right| |\Delta x_2| + \ldots + \left| \frac{\partial}{\partial x_n} [\ln |f|] \right| |\Delta x_n|.$$

Example 2.14 Find the total increment and the total differential of the function $z = x + y + xy$ at the point $(1, 2)$ for $\Delta x = 0.1$ and $\Delta y = -0.2$. Find the maximum absolute error and the maximum relative error.