6 - notes of unit-5 multivariate Calculus of MTH174

Engineering Mathematics (Lovely Professional University)

Extrema of functions of two variables

(Maximum and minimum values of a function)

Let $f(x,y)$ be a continuous function and possesses continuous first and second order partial derivatives.

Critical Points

Let $z = f(x,y)$ be a function of two variables. The point (a,b) is said to be critical point of $f(x,y)$ if one of the following conditions hold:

(1) $f_x(a,b) = f_y(a,b) = 0$

(2) Either $f_x(a,b)$ or $f_y(a,b)$ do not exist.

How to find critical points

Find f_x and f_y

Put $f_x = 0$ and $f_y = 0$

Find the values of x and y solving $f_x = 0$ and $f_y = 0$.

Ex $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2$, $f_y = x - 2y - 2$

$f_x = 0$ and $f_y = 0$

$$\Rightarrow \left.\begin{array}{l} y - 2x - 2 = 0 \Rightarrow 2x - y = -2 \\ x - 2y - 2 = 0 \Rightarrow x - 2y = 2 \end{array}\right\} \Rightarrow (x,y) = (-2,-2)$$

$\therefore (-2,-2)$ is the critical point.

Local (relative) and global (absolute) Extrema

Let $z = f(x,y)$ be a function of two variables.
Then, f has a <u>local maximum</u> at (a,b) if
$f(x,y) \le f(a,b)$ for all (x,y) in some open disc centered at (a,b).

If $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f, then f has <u>global maximum</u> at (a,b).

f has a <u>local minimum</u> at (a,b) if
$f(x,y) \ge f(a,b)$ for all (x,y) in some open disc centered at (a,b).

If $f(x,y) \ge f(a,b)$ for all (x,y) in the domain of f, then f has <u>glocal minimum</u> at (a,b).

How to find maxima/minima at critical point (a,b)

Calculate f_{xx}, f_{yy}, f_{xy}

Let $r = f_{xx}(a,b)$, $s = f_{xy}(a,b)$, $t = f_{yy}(a,b)$.

$$\begin{bmatrix} r & s \\ s & t \end{bmatrix}$$

1. <u>Relative Minima</u> : $r > 0$ and $rt - s^2 > 0$.
2. <u>Relative Maxima</u> : $r < 0$ and $rt - s^2 > 0$.
3. <u>Saddle Point (No maxima or minima)</u> : ~~rt~~

   $rt - s^2 < 0$
4. If $rt - s^2 = 0$, test is inconclusive.

Calculate $rt-s^2$

- $rt-s^2 \neq 0$
  - $rt-s^2 > 0$
    - $r > 0$ (Local Minima)
    - $r < 0$ (Local Maxima)
  - $rt-s^2 < 0$ (Saddle Point) No maxima or minima
- $rt-s^2 = 0$ $\downarrow$ No result.

Q: Test the function for relative maxima or minima.

$$f(x,y) = xy + \frac{9}{x} + \frac{3}{y}.$$

Sol.

$$f_x = y - \frac{9}{x^2}, \quad f_y = x - \frac{3}{y^2}$$

$f_x = 0$ and $f_y = 0$

$$\Rightarrow y - \frac{9}{x^2} = 0 \Rightarrow y = \frac{9}{x^2}$$

$$x - \frac{3}{y^2} = 0 \Rightarrow x - \frac{3}{\frac{81}{x^4}} = 0$$

$$\Rightarrow x - \frac{3}{81}x^4 = 0$$

$$\Rightarrow x\left(1 - \frac{1}{27}x^3\right) = 0$$

$$\Rightarrow 27 - x^3 = 0$$

$$\Rightarrow x^3 = 27 \Rightarrow x = 3$$

$$\Rightarrow y = \frac{9}{9} = 1$$

$\therefore$ (3,1) is the critical point.

$r = f_{xx} = \frac{18}{x^3}$, $s = f_{xy} = 1$, $t = f_{yy} = \frac{6}{y^3}$

At $(3,1)$, $r = \frac{18}{3.9} = \frac{2}{3}$, $s = 1$, $t = \frac{6}{1} = 6$

$rt - s^2 = \frac{12}{3} - 1 = 3 > 0$.

and $r > 0$

So, $(3,1)$ is the point of relative minima.

$f_{min}(3,1) = (3)(1) + \frac{9}{3} + \frac{3}{1}$

$= 3 + 3 + 3$

$= 9$.

Q:- The critical points of the function $f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$ are

(a) $\left(1, \frac{3}{2}\right)$, (b) $\left(1, \frac{2}{3}\right)$ (c) $(1,1)$ (d) $(0,0)$

Sol:- $f_x = 8x - 8$, $f_y = 18y - 12$

$f_x = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = \frac{12}{18} = \frac{2}{3}$.

$\left(1, \frac{2}{3}\right)$ is the critical point.

Q:- Find the relative maximum and minimum values of the function $f(x,y) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

Sol. $f_x = 4x - 4x^3$, $f_y = -4y + 4y^3$

$f_x = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1$.

$f_y = 0 \Rightarrow -4y(1 - y^2) = 0 \Rightarrow y = 0, 1, -1$.

Critical points are $(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)$

$r = f_{xx} = 4 - 12x^2$, $s = f_{xy} = 0$, $t = f_{yy} = -4 + 12y^2$

Critical points	r	$rt - s^2$
$(0,0)$	4	-16
$(0,1)$	4	32
$(0,-1)$	4	32
$(1,0)$	-8	32
$(1,1)$	-8	$-\cancel{32}\ 64$
$(1,-1)$	-8	-64
$(-1,0)$	-8	$+32$
$(-1,1)$	-8	-64
$(-1,-1)$	-8	-64

Local minima $r > 0$, $rt - s^2 > 0$

$(0,1)$, $(0,-1)$ are points of local minima.

Minimum value $= 2(0-1) - 0 + 1 = -2+1 = -1$.

Local maxima $r < 0$, $rt - s^2 > 0$.

$(1,0)$, $(-1,0)$ are points of local maxima

Maximum value $= 1$

Saddle points $rt - s^2 < 0$.

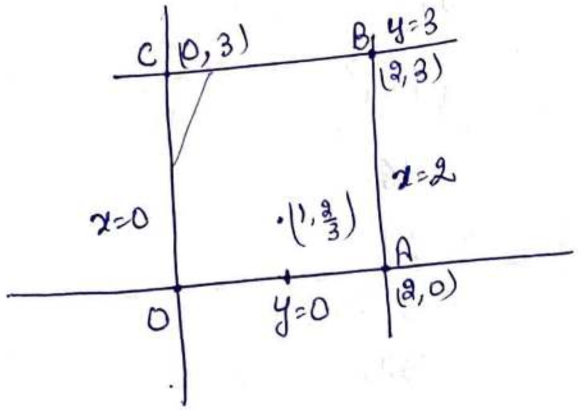
$(0,0)$, $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ are saddle points.

Ex Find the absolute maximum and minimum values of

$$f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the first quadrant bounded by the lines $x=2$, $y=3$ and the coordinate axes.

Sol: The function f can attain maximum/minimum values at the critical points or on the boundary of the rectangle formed by the lines $x=2$, $y=3$ and the coordinate axes.



$f_x = 8x-8,\ f_y = 18y-12$

$f_x = 0$ and $f_y = 0 \Rightarrow \left(1, \frac{2}{3}\right)$ is the critical point.

$r = f_{xx} = 8,\ s = f_{xy} = 0,\ t = 18$

$rt - s^2 = 144 > 0$ and $r > 0$

$\Rightarrow \left(1, \frac{2}{3}\right)$ is the point of local minimum and minimum value is $4 + 9\left(\frac{4}{9}\right) - 8 - 12\left(\frac{2}{3}\right) + 4 = -4.$

On boundary line OA $\quad y=0,\ f(x,y) = f(x,0) = 4x^2 - 8x + 4$

$\Rightarrow g(x) = 4x^2 - 8x + 4$, which is a function of one variable.

$g'(x) = 8x - 8,\ g'(x) = 0 \Rightarrow x = 1$

$g''(x) = 8 > 0.$

$\therefore$ At $x=1$, the function $g(x)$ has a minima.

The minimum value is $g(1) = 0$.

At the corners $(0,0),\ f(0,0) = 4$

$(2,0),\ f(2,0) = 4$

On boundary line AB $\quad x=2,\ f(x,y) = f(2,y) = h(y) = 9y^2 - 12y + 4$

$h'(y) = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = \frac{2}{3}$

$h''(y) = 18 > 0$

$\therefore y = \frac{2}{3}$ is a point of minimum and minimum value $= 0$

At the corner $(2,3)$, $f(2,3) = 49$.

On the boundary line BC $y=3$, $f(x,y) = f(x,3) = 4x^2 + 9(9) - 8x - 36 + 4$
$= 4x^2 - 8x + 49$

$f' = 8x - 8 \Rightarrow f' = 0 \Rightarrow x = 1$
$f'' = 8 > 0$

$\therefore$ $(1,3)$ is a point of minima and minimum value is $f(1,3) = 45$.

At the corner $(0,3)$, $f(0,3) = 49$.

On the boundary line OC $x=0$, $f(x,y) = f(0,y) = 9y^2 - 12y + 4$,

which is same case as $x=2$.

Points	Value of f
$(1, \frac{2}{3})$ is point of local minima.	-4
$(1,0)$	0
$(0,0)$	4
$(2,0)$	4
$(2, \frac{2}{3})$	0
$(2,3)$	49
$(1,3)$	45
$(0,3)$	49

$\therefore$ Absolute minimum value is -4 which occurs at the point $(1, \frac{2}{3})$.
Absolute maximum value is 49 at the points $(2,3)$ and $(0,3)$.

Q: Test the function for relative maxima and minima

$$f(x,y) = \sqrt{a^2 - x^2 - y^2}, \quad a > 0.$$

Sol: $$f_x = \frac{-2x}{2\sqrt{a^2-x^2-y^2}} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$$

$$f_y = \frac{-2y}{2\sqrt{a^2-x^2-y^2}} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

For critical points, $f_x = 0$ and $f_y = 0$

$$\frac{-x}{\sqrt{a^2-x^2-y^2}} = 0 \quad \text{and} \quad \frac{-y}{\sqrt{a^2-x^2-y^2}} = 0 \Rightarrow x=0,\ y=0$$

$\Rightarrow$ $(0,0)$ is the critical point.

$$r = f_{xx} = \frac{\sqrt{a^2-x^2-y^2}\,(-1) + x\frac{1}{\cancel{2}\sqrt{a^2-x^2-y^2}}(-\cancel{2}x)}{(a^2-x^2-y^2)}$$

$$= \frac{-(a^2-x^2-y^2) - x^2}{(a^2-x^2-y^2)^{3/2}} = \frac{-a^2+\cancel{x^2}+y^2-\cancel{x^2}}{(a^2-x^2-y^2)^{3/2}} = \frac{y^2-a^2}{(a^2-x^2-y^2)^{3/2}}$$

$$t = f_{yy} = \frac{(a^2-x^2-y^2)^{1/2}(-1) + y\frac{1}{\cancel{2}\sqrt{a^2-x^2-y^2}}(-\cancel{2}y)}{(a^2-x^2-y^2)} = \frac{-a^2+x^2+y^2-y^2}{(a^2-x^2-y^2)^{3/2}} = \frac{x^2-a^2}{(a^2-x^2-y^2)^{3/2}}$$

$$s = f_{xy} = \frac{\sqrt{a^2-x^2-y^2}\,(0) + x\frac{1}{\cancel{2}\sqrt{a^2-x^2-y^2}}(-\cancel{2}y)}{(a^2-x^2-y^2)} = \frac{-xy}{(a^2-x^2-y^2)^{3/2}}$$

At $(x,y) = (0,0)$

~~$rt-s^2=$~~ $r = \dfrac{-a^2}{(a^2)^{3/2}} = \dfrac{-1}{a}$

$s = 0$

$t = \dfrac{-1}{a}$

~~rt~~ $rt - s^2 = \dfrac{1}{a^2} > 0$ and $r < 0$.

$\therefore$ $(0,0)$ is the point of relative (local) maxima.

and max value $= f(0,0) = a$.

Q:- Test the function ~~f(x)~~ $f(x,y) = 4x^2 - 4y^2 + 12x - 6y$ for relative maxima and minima.

Sol:- $f(x,y) = 4x^2 - 4y^2 + 12x - 6y$

$f_x = 8x + 12$, $f_y = -8y - 6$

$f_x = 0 \Rightarrow 8x + 12 = 0 \Rightarrow x = -\frac{3}{2}$

$f_y = 0 \Rightarrow -8y - 6 = 0 \Rightarrow y = -\frac{3}{4}$

$\therefore \left(-\frac{3}{2}, -\frac{3}{4}\right)$ is the critical point.

$r = f_{xx} = 8$, $s = f_{xy} = 0$, $t = f_{yy} = -8$

$rt - s^2 = -64 < 0$

$\Rightarrow \left(-\frac{3}{2}, -\frac{3}{4}\right)$ is neither a point of maxima nor a point of minima. It is a saddle point.