Euler Cauchy Equation :- The differential equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = b(x) \quad \text{—— (1)}$$

(where $a_0, a_1, \ldots, a_n$ are constants) is called Euler - Cauchy equation

Transformation of Euler Cauchy equation :-

Consider $x = e^t \Rightarrow t = \log x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$.

Similarly —— (2)

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dt}\right) \quad (\text{by } (2))$$

$$= -\frac{1}{x^2} \times \frac{d}{dt} + \frac{1}{x} \times \frac{d}{dx}\left(\frac{d}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x} \times \frac{d}{dt}\left(\frac{d}{dt}\right) \cdot \frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x}\frac{d^2}{dt^2} \times \frac{1}{x} = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2}$$

$$\Rightarrow \boxed{x^2 \frac{d^2}{dx^2} = \frac{d^2}{dt^2} + \frac{d}{dt} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)} \quad \text{—— (3)}$$

Similarly we can consider

$$x^n \frac{d^n}{dx^n} = \frac{d}{dt}\left(\frac{d}{dt}-1\right) \cdots \left(\frac{d}{dt}-(n-1)\right) \quad \text{--- (4)}$$

After substituting $D = \frac{d}{dx}$ and $\theta = \frac{d}{dt}$ in equation (2), (3) and (4) we have

$$\boxed{\begin{array}{l} xD = \theta, \quad x^2D^2 = \theta^2 - \theta = \theta(\theta-1) \\ \& \quad x^n D^n = \theta(\theta-1) \cdots (\theta-(n-1)) \end{array}}$$

After substituting these value in equation (1), we obtain non-homogeneous L.D. equation with constant coefficients.

P.T.O

Ex ① Find the general solution of the differential equation

$$2x^2y'' + 3xy' - 3y = x^3 . \qquad \text{——— ①}$$

Solution: The given differential equation is of Euler - Cauchy type so.

we consider $x = e^t \Rightarrow t = \log x$

The equation ① can be written as

$$(2x^2D^2 + 3xD - 3)y = x^3$$

$\Rightarrow$ As $x^2D^2 = \theta(\theta - 1)$, $xD = \theta$

Therefore,

$$[2\theta(\theta-1) + 3\theta - 3]y = e^{3t}$$

$$[2\theta^2 - 2\theta + 3\theta - 3]y = e^{3t}$$

(or) $[2\theta^3 + \theta - 3]y = e^{3t}$ ——— ②

The above equation is ~~sed~~ second order differential equation with constant coefficients.

The equation ② can be written as

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 3y = e^{3t}$$

P.T.O

$$2\frac{d^2y}{dt^2}+\frac{dy}{dt}-3y=e^{3t}$$

The A.E. $2m^2+m-3=0$

$$2m^2+3m-2m-3=0$$

$$m(2m+3)-1(2m+3)=0$$

$$\Rightarrow (m-1)(2m+3)=0 \Rightarrow m_1=1,\ m_2=-3/2$$

S.

$$y_c(t)=c_1e^{t}+c_2e^{-3t/2}$$

$$\Rightarrow y_c(x)=c_1x+c_2(x)^{-3/2} \qquad (\text{As } x=e^{t})$$

$$\boxed{y_c(x)=c_1x+\frac{c_2}{x\sqrt{x}}} \quad \text{—— (3)}$$

$$x^{-3/2}=\frac{1}{(x)^{3/2}}=\frac{1}{\sqrt{x^3}}=\frac{1}{x\sqrt{x}}$$

Next

$$y_p(t)=\frac{1}{(2\theta^2+\theta-3)}e^{3t}, \quad \text{Here } a=3,$$

$$f(\theta)=2\theta^2+\theta-3; \quad f(3)=18+3-3=18\neq 0$$

$$\Rightarrow y_p(t)=\frac{1}{f(3)}e^{3t}=\frac{1}{18}e^{3t} \Rightarrow y_p(t)=\frac{e^{3t}}{18}$$

$$\Rightarrow \boxed{y_p(x)=\frac{x^3}{18}} \quad \text{—— (4)}$$

Thus General solution is

$$\boxed{y(x)=c_1x+\frac{c_2}{x\sqrt{x}}+\frac{x^3}{18}}$$

7) (2) $x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+3y=\log x$ —— (1)

Solution: The given diff equation is Euler-Cauchy form so we consider $x=e^t$ then $\Rightarrow t=\log x$,

The equation (1) can be written as

$$\cancel{(2x^2D^2+3xD-3)y=x}$$

$$(x^2D^2+5xD+3)y=\log x$$

As $x^2D^2=\theta(\theta-1)$, $xD=\theta$

Therefore, $$[\theta(\theta-1)+5\theta+3]y=\log e^t$$

$$[\theta^2-\theta+5\theta+3]y=t$$

$$[\theta^2+4\theta+3]y=t$$

$$\Rightarrow \frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=t$$

A.E. $m_1=-1$, $m_2=-3$,

$$y_c(t)=c_1e^{-t}+c_2e^{-3t} \Rightarrow y_c(x)=c_1\frac{1}{x}+c_2\frac{1}{x^3}$$

$$\rightarrow \boxed{y_c(x)=\frac{c_1}{x}+\frac{c_2}{x^3}} \quad \text{—— (2)}$$

$$P.I.=\frac{1}{\theta^2+4\theta+3}t=\frac{1}{3\left[1+\left(\frac{\theta^2+4\theta}{3}\right)\right]}t$$

P.T.O

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$\Rightarrow$ P.I. $= \dfrac{1}{3\left[1+\left(\dfrac{\theta^2+4\theta}{3}\right)\right]} t$

$= \dfrac{1}{3}\left[1+\left(\dfrac{\theta^2+4\theta}{3}\right)\right]^{-1} t$

$= \dfrac{1}{3}\left[1-\dfrac{(\theta^2+4\theta)}{3}+\left(\dfrac{\theta^2+4\theta}{3}\right)^2+\cdots\right] t$

$= \dfrac{1}{3}\left[t-\dfrac{(\theta^2+4\theta)\,t}{3}+\left(\dfrac{\theta^2+4\theta}{3}\right)^2 t-\cdots\right]$

$= \dfrac{1}{3}\left[t-\dfrac{(0+4)}{3}\right] = \dfrac{t}{3}-\dfrac{4}{9}$

$\Rightarrow \quad y_p(t) = \dfrac{t}{3}-\dfrac{4}{9}$

$\Rightarrow y_p(x) = \dfrac{\log x}{3}-\dfrac{4}{9}$

Therefore general solution is

$$\boxed{y(x) = \frac{c_1}{x}+\frac{c_2}{x^3}+\frac{1}{3}\log x-\frac{4}{9}}$$