MTH174-5.Homogeneous Functions

Engineering Mathematics (Lovely Professional University)

Homogeneous Functions

A function $f(x,y)$ is said to be homogeneous function of degree n in x and y, if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

(ii) $f(x,y) = x^n g\left(\frac{y}{x}\right)$

(iii) $f(x,y) = y^n g\left(\frac{x}{y}\right)$

Q: Check if the following functions are Homogeneous:

(i) $f(x,y) = x^2 + xy$

(ii) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

(iii) $f(x,y) = \frac{1}{x+y}$

(iv) $f(x,y,z) = \frac{xyz}{x^4+y^4+z^4}$

(v) $f(x,y,z) = \frac{\sqrt{x}}{\sqrt{x^2+y^2+z^2}}$

(vi) $f(x,y) = \frac{x^2+y}{x+y^2}$

Sol: (i) $f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 xy = \lambda^2(x^2+xy)$
$= \lambda^2 f(x,y)$

$\therefore$ $f(x,y)$ is a homogeneous function of degree 2.

(ii) $f(x,y) = x^0 \tan^{-1}\left(\frac{y}{x}\right) = x^0 g\left(\frac{y}{x}\right)$

or $f(\lambda x, \lambda y) = \tan^{-1}\left(\frac{\lambda y}{\lambda x}\right) = \tan^{-1}\left(\frac{y}{x}\right) = \lambda^0 f(x,y)$

$\Rightarrow$ $f(x,y)$ is a homogeneous function of degree 0.

(iii) $f(x,y) = \dfrac{1}{x+y}$

$$f(\lambda x, \lambda y) = \frac{1}{\lambda x + \lambda y} = \frac{1}{\lambda}\frac{1}{x+y} = \lambda^{-1} f(x,y)$$

$\Rightarrow$ $f(x,y)$ is a homogeneous function of degree -1.

(iv) $f(x,y,z) = \dfrac{xyz}{x^4+y^4+z^4}$

$$f(\lambda x, \lambda y, \lambda z) = \frac{\lambda^3 (xyz)}{\lambda^4 (x^4+y^4+z^4)} = \lambda^{-1} f(x,y,z)$$

$\Rightarrow$ $f(x,y,z)$ is a homogeneous function of degree -1.

(v) $f(x,y,z) = \dfrac{\sqrt{x}}{\sqrt{x^2+y^2+z^2}}$

$$f(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{\lambda x}}{\sqrt{\lambda^2 x^2 + \lambda^2 y^2 + \lambda^2 z^2}} = \frac{\sqrt{\lambda}\sqrt{x}}{\lambda\sqrt{x^2+y^2+z^2}}$$

$$= (\lambda)^{-1/2} f(x,y,z)$$

$\Rightarrow$ $f(x,y,z)$ is a homogeneous function of degree $-\frac{1}{2}$.

(vi) $f(x,y) = \dfrac{x^2+y}{x+y^2}$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda y}{\lambda x + \lambda^2 y^2} = \frac{\cancel{\lambda}(\lambda x^2 + y)}{\cancel{\lambda}(x + \lambda y^2)} = \frac{\lambda x^2 + y}{x + \lambda y^2} \neq f(x,y)$$

$\Rightarrow$ $f(x,y)$ is not a homogeneous function.

Euler's Theorem

If $f(x,y)$ is a homogeneous function of degree n in x and y and has continuous first and second order partial derivatives, then

1. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$
2. $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$

Q-1 Using Euler's th^m, establish the following results:

(i) If $u = \frac{y^3 - x^3}{y^2 + x^2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Sol: $u = \frac{y^3 - x^3}{y^2 + x^2}$

$$u(\lambda x, \lambda y) = \frac{\lambda^3(y^3 - x^3)}{\lambda^2(y^2 + x^2)} = \lambda\frac{(y^3 - x^3)}{y^2 + x^2} = \lambda u(x,y).$$

$\therefore$ u is a homogeneous function of degree 1.

By Euler's th^m,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \quad \text{and}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 1(1-1)u = 0.$$

(ii) If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$

Sol: Here $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right) \Rightarrow \sin u = \frac{x^2 + y^2}{x + y}$

Let $f(x,y) = \sin u = \dfrac{x^2+y^2}{x+y}$

$$f(\lambda x, \lambda y) = \frac{\lambda^2(x^2+y^2)}{\lambda(x+y)} = \lambda\left(\frac{x^2+y^2}{x+y}\right) = \lambda f(x,y)$$

$\therefore f(x,y) = \sin u$ is a homogeneous function of degree 1.

$$\therefore x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

Hence proved.

(iii) If $u = \log\left(\dfrac{\sqrt{x^2+y^2}}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Sol> Here $u = \log\left(\dfrac{\sqrt{x^2+y^2}}{x}\right)$

$$\Rightarrow f(x,y) = e^u = \frac{\sqrt{x^2+y^2}}{x}$$

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda^2(x^2+y^2)}}{\lambda x} = \lambda^0\frac{\sqrt{x^2+y^2}}{x} = \lambda^0 f(x,y)$$

$\Rightarrow f(x,y) = e^u$ is a homogeneous function of degree 0.

$\therefore$ By Euler's th^m,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.f$$

$$\Rightarrow x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 0$$

$$\Rightarrow x e^u\frac{\partial u}{\partial x} + y e^u\frac{\partial u}{\partial y} = 0 \Rightarrow e^u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 0$$

Hence, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

MCQ

The function $f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ is a

(a) homogeneous function of degree 0.

(b) homogeneous function of degree 1.

(c) homogeneous function of degree 2.

(d) Not a homogeneous function.

Sol→ $f(\lambda x, \lambda y) = \lambda^2 x^2 \tan^{-1}\left(\frac{y}{x}\right) - \lambda^2 y^2 \tan^{-1}\left(\frac{x}{y}\right)$

$= \lambda^2\left[x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right]$

$= \lambda^2 f(x,y).$

(c) is correct.

Q: If $u(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, $x>0, y>0$, then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Sol: u is a homogeneous function of degree 2.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u.$$

Q: If $u(x,y) = \frac{x^3+y^3}{x+y}$, $(x,y) \neq (0,0)$. Then, evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial u}{\partial x}.$$

Sol: u is a homogeneous function of degree 2.

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Differentiating partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} = 0.$$