Required time at t is

$$\frac{10m_1}{100} = ce^{-kt} = m_1 e^{-kt} \Rightarrow \frac{1}{10} = e^{-kt}$$

$$\Rightarrow t = \frac{1}{k}\log(10)$$

$$= \frac{10\log(10)}{\log 10 - \log 7} = 64.5 \text{ days.}$$

II .LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Definition: An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n(x)y = Q(x)$

where $P_1(x), P_2(x), P_3(x)\ldots P_n(x)$ and $Q(x)$ (functions of x) are continuous is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def: An equation of the form $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = Q(x)$ where $P_1, P_2, P_3 \ldots P_n$, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ' n' with constant coefficients.

Note:

1. Operator $D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$; …………………… $D^n = \frac{d^n}{dx^n}$

$$D\,y = \frac{dy}{dx};\ D^2y = \frac{d^2y}{dx^2};\ldots\ldots\ldots\ldots\ D^n\,y = \frac{d^n y}{dx^n}$$

2. Operator $\frac{1}{D}Q = \int Q dx$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of f(D).y = 0 :

Here $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$ is a polynomial in D.

Now consider the auxiliary equation: f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$

where $P_1, P_2, P_3 \ldots P_n$ are real constants.

Let the roots of f(m) =0 be $m_1, m_2, m_3 \ldots m_n$.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E $f(m) = 0$	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + \ldots c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ and two roots are equal i.e., m_1, m_2 are equal and real (i.e repeated twice) &the rest are real and different.	$y_c = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \ldots c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real (i.e repeated thrice) &the rest are real and different.	$y_c = (c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots c_ne^{m_nx}$
4.	Two roots of A.E are complex say α+i β α -iβ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots c_ne^{m_nx}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\right] + c_5e^{m_5x} + \ldots c_ne^{m_nx}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x + c_3x^2)\cos\beta x + (c_4 + c_5x + c_6x^2)\sin\beta x\right]$ $+ c_7e^{m_7x} + \ldots c_ne^{m_nx}$
7.	If roots of A.E.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots. + c_ne^{m_nx}$

	irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	

Solved Problems

1. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol : Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 - 3D + 2)y = 0$

Now consider the auxiliary equation f (m) = 0

$f(m) = (m^3 - 3m + 2)y = 0 \Rightarrow$ (m-1) (m-1) (m+2) = 0

$\Rightarrow$ m =1, 1, -2

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1 + c_2)e^x + c_3e^{-2x}$

2. Solve $(D^4-2D^3-3D^2+4D+4)y = 0$

Sol : Given f(D) = $(D^4 - 2D^3 - 3D^2 + 4D + 4)$ y = 0 …(1)

Auxiliary equation is f(m)=0

$\Rightarrow m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$ …(2)

By inspection m+1 is its factor.

$(m+1)(m^3 - 3m^2 + 4) = 0$ …(3)

By inspection m+1 is factor of $(m^3 - 3m^2 + 4)$.

$\therefore$ (3) is $(m+1)(m+1)(m^2 - 4m + 4) = 0$

$\Rightarrow (m+1)^2(m-2)^2 = 0$

$\Rightarrow m = -1, -1, 2, 2$

Hence general solution of (1) is

$y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^{2x}$

3.Solve $(D^4 + 8D^2 + 16)$ y = 0

Sol : Given f(D) = $(D^4 + 8D^2 + 16)$ y = 0

Auxiliary equation f(m) = $(m^4 + 8m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2\ (m+2i)^2 = 0$

⇨ $m= 2i\ ,2i\ ,\ -2i\ ,\ -2i$

Here roots are complex and repeated

Hence general solution is

$y_c = \ [(c_1+c_2x)\cos 2x + (c_3+c_4x)\ \sin 2x)]$

4.Solve $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol : Given equation is $y^{11}+6y^1+9y = 0$

Auxiliary equation $f(D)\ y = 0 \ \Rightarrow\ (D^2 +6D +9)\ y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 +6m +9) = 0$

$\Rightarrow m = -3\ ,-3$

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x $\Rightarrow$ $y^1 =(c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1\ (0) =14 \ \Rightarrow c_1 = -4$ & $c_2 =2$

Hence we get $y =(-4 + 2x)\ (e^{-3x})$

5.Solve $4y^{111} + 4y^{11} +y^1 = 0$

Sol : Given equation is $4y^{111} + 4y^{11} +y^1 = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation $f(m) = 0$

$4m^3 +4m^2 + m = 0$

$m(4m^2 +4m + 1) = 0$

$m\,(2m+1)^2 = 0$

$m = 0\ ,\ -1/2\ ,-1/2$

$y =c_1 + (c_2 + c_3x)\ e^{-x/2}$

6.Solve $(D^2 - 3D +4)\ y = 0$

Sol : Given equation $(D^2 - 3D +4)\ y = 0$
A.E. $f(m) = 0$

$m^2-3m + 4 = 0$

$$m = \frac{3\pm\sqrt{9-16}}{2} = \frac{3\pm i\sqrt{7}}{2}$$

$$\alpha\pm i\beta \ = \frac{3}{2}\pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}}\ (c_1\cos\frac{\sqrt{7}}{2}x + c_2\sin\frac{\sqrt{7}}{2}x)$$

<u>To Find General solution of $f(D)\ y = Q(x)$</u>

It is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of $f(D)\,y = Q(x)$

Is evaluated as $\quad P.I = \dfrac{1}{f(D)}Q(x)$

Depending on the type of function of $Q(x)$, P.I is evaluated .

1. Find $\dfrac{1}{D}(x^2)$

Sol : $\dfrac{1}{D}(x^2) = \int x^2 dx = \dfrac{x^3}{3}$

2. **Find Particular value of** $\dfrac{1}{D+1}(x)$

Sol : $\dfrac{1}{D+1}(x) = e^{-x}\int xe^x dx$ (By definition)

$$= e^{-x}(xe^x - e^x)$$

$$= x-1$$

General methods of finding Particular integral :

P.I of $f(D)y=Q(x)$, when $\dfrac{1}{f(D)}$ is expressed as partial fractions**.**

Q. Solve $(D^2 + a^2)y = \sec ax$

Sol : Given equation is …(1)

Let $f(D) = D^2 + a^2$

The AE is $f(m) = 0$ i.e $m^2 + a^2 = 0$ …(2)

The roots are m= -ai , -ai

$y_c = c_1\cos ax + c_2\sin ax$

$$y_p = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right]\sec ax \quad \ldots(3)$$

$$\frac{1}{D - ai}\sec ax = e^{iax}\int \sec ax\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx$$

$$= e^{iax}\int (1 - i\tan ax)dx = e^{iax}\left[x + \frac{i}{a}\log\cos ax\right] \ldots(4)$$

Similarly we get $\frac{1}{D+ai}\sec ax = e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]$ ...(5)

From (3),(4) and (5), we get

$$y_p = \frac{1}{2ai}\left[e^{iax}\left\{x+\frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x-\frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{x(e^{iax}-e^{-iax})}{2ai} + \frac{1}{a^2}(\log\cos ax)\frac{(e^{iax}+e^{-iax})}{2}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

RULES FOR FINDING P.I IN SOME SPECIAL CASES;

Type 1. P.I of f (D)y=Q(x) where Q(x) $=e^{ax}$, where 'a' is constant.

Case1.$P.I = \frac{1}{f(D)}.Q(x) = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

provided $f(a) \neq 0$

i.e In f(D), put D=a and Particular integral will be calculated.

Case 2: If f(a)=0 then the above method fails. Then if $f(D) = (D-a)^k \phi(D)$ (i.e 'a' is repeated root k times).

Then $P.I = \frac{1}{\phi(a)}e^{ax}.\frac{1}{k!}x^k$ provided $\phi(a) \neq 0$

Type 2. P.I of $f(D)y = Q(x)$ where Q(x) = sinax or Q(x) = cosax where 'a' is constant then P.I=$\frac{1}{f(D)}Q(x)$.

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then $P.I = \frac{\sin ax}{f(-a)^2}$

Case 2: If $f(-a^2)$=0 then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of f(D). Then let f(D)=$(D^2 + a^2)$ $f(D) = (D^2+a^2)\phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\phi(-a^2)}\frac{-x\cos ax}{2a}$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\phi(-a^2)}\frac{x\sin ax}{2a}$$

Type 3.P.I for f(D)y=Q(x) where Q(x)=x^k where k is a positive integer .f(D) can be expressed as $f(D)=[1\pm\phi(D)]$

Express $\frac{1}{f(D)}=\frac{1}{[1\pm\phi(D)]}=[1\pm\phi(D)]^{-1}$

Hence $P.I=\frac{1}{[1\pm\phi(D)]}Q(x)$

$=[1\pm\phi(D)]^{-1}x^k$

Type 4.P.I of f(D)y=Q(x) when Q(x)=e^{ax} V where 'a' is a constant and V is function of x. where V =sinax or cosax or x^k

Then $P.I=\frac{1}{f(D)}Q(x)$

$=\frac{1}{f(D)}e^{ax}V$

$= e^{ax}\left[\frac{1}{f(D+a)}V\right]$ & $\frac{1}{f(D+a)}V$ is evaluated depending on V.

Type 5. P.I of f(D)y=Q(x) when Q(x)=xV where V is a function of x.

Then $P.I=\frac{1}{f(D)}Q(x)$

$=\frac{1}{f(D)}V$

$=\left[x-\frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V$

Type 6. P.I. of f(D)y = Q(x) where Q(x)= x^m v where v is a function of x.

When $P.I.=\frac{1}{f(D)}\times Q(x)=\frac{1}{f(D)}x^m v$, where v = cosax or sinax

i. $P.I.=\frac{1}{f(D)}x^m\sin ax=I.P.\text{ of }\frac{1}{f(D)}x^m e^{iax}$

ii. $P.I.=\frac{1}{f(D)}x^m\cos ax=R.P.\text{ of }\frac{1}{f(D)}x^m e^{iax}$

Formulae

1. $\frac{1}{1-D}=(1-D)^{-1}=1+D+D^2+D^3+$ ------------------
2. $\frac{1}{1+D}=(1+D)^{-1}=1-D+D^2-D^3+$ ------------------

3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \text{------------------}$

4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \text{------------------}$

5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \text{------------------}$

6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \text{------------------}$

Solved Problems

1. Solve $(4D^2 - 4D + 1)y = 100$

Sol : A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{-1}{2}$

$$C.F = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{Now } P.I = \frac{100}{4D^2 - 4D + 1} = \frac{100e^{0x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100 \ \{ \text{since } 100e^{0x} = 100 \}$$

Hence the general solution is $y = C.F + P.F = (c_1 + c_2 x)e^{\frac{x}{2}} + 100$

2. Solve the differential equation $(D^2 + 4)y = \sinh 2x + 7$.

Sol : Auxillary equation is $m^2 + 4 = 0$

$$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$$

$\therefore$ C.F is $y_c = c_1\cos 2x + c_2 \sin 2x \ \ldots(1)$

To find P.I :

$$y_p = \frac{1}{D^2+4}(\sinh 2x + 7)$$

$$= \frac{1}{D^2+4}\left(\frac{e^{2x}+e^{-2x}}{2} + 7e^0\right)$$

$$= \frac{1}{2}\cdot\frac{e^{2x}}{D^2+4} + \frac{1}{2}\frac{e^{-2x}}{D^2+4} + 7\frac{e^0}{(D^2+4)}$$

$$= \frac{e^{2x}}{2(4+4)} + \frac{e^{-2x}}{2(4+4)} + \frac{7}{(0+4)}$$

$$= \frac{e^{2x}+e^{-2x}}{16} + \frac{7}{4} = \frac{1}{8}\sinh 2x + \frac{7}{4} \qquad \ldots(2)$$

$$y = y_c + y_p$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}\sinh 2x + \frac{7}{4}$$

3. Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol : The given equation is

$$(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x \qquad \ldots(1)$$

This is of the form $f(D)y = e^{-2x} + 2\sinh x$

A.E is $f(m) = 0 \Rightarrow (m+2)(m-1)^2 = 0 \therefore m = -2,1,1$

The roots are real and one root is repeated twice.

$\therefore$ C.F is $y_c = c_1e^{-2x} + (c_2 + c_3x)e^x$.

$$P.I = \frac{e^{-2x} + 2\sinh x}{(D+2)(D-1)^2} = \frac{e^{-2x} + e^x - e^{-x}}{(D+2)(D-1)^2} = y_{p_1} + y_{p_2} + y_{p_3}$$

Now $y_{p_1} = \dfrac{e^{-2x}}{(D+2)(D-1)^2}$

Hence $f(-2) = 0$. Let $f(D) = (D-1)^2$. Then $\phi(2) \neq 0$ and $m = 1$

$$\therefore y_{p_1} = \frac{e^{-2x}x}{9} = \frac{xe^{-2x}}{9}$$

and $y_{p_2} = \dfrac{e^x}{(D+2)(D-1)^2}$. Here f(1)=0

$$= \frac{e^x x^2}{(3)2!} = \frac{x^2e^x}{6}$$

and $y_{p_3} = \dfrac{e^{-x}}{(D+2)(D-1)^2}$

Putting D=-1, we get $y_{p_3} = \dfrac{e^{-x}}{(1)(-2)^2} = \dfrac{e^{-x}}{4}$

$\therefore$ The general solution is $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

i.e $y = c_1e^{-2x} + (c_c + c_3x)e^x + \dfrac{xe^{-2x}}{9} + \dfrac{x^2e^x}{6} - \dfrac{e^{-x}}{4}$

4. Solve the differential equation $(D^2 + D + 1)y = \sin 2x$.

Sol : A.E is $m^2 + m + 1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore y_c = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) \qquad \ldots(1)$$

To find P.I :

$$y_p = \frac{\sin 2x}{D^2 + D + 1} = \frac{\sin 2x}{-4 + D + 1}$$

$$= \frac{\sin 2x}{D-3} = \frac{(D+3)\sin 2x}{D^2 - 9} = \frac{(D+3)\sin 2x}{-4-9}$$

$$= \frac{D\sin 2x + 3\sin 2x}{-13} = \frac{2\cos 2x + 3\sin 2x}{-13}$$

$$\therefore y = y_c + y_p = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

5. Solve $(D^2-4)y=2\cos^2x$

Sol : Given equation is $(D^2-4)y=2\cos^2x$ …(1)

Let $f(D)=D^2-4$ A.E is $f(m)=0$ i.e $m^2-4=0$

The roots are m=2,-2. The roots are real and different.

$\therefore C.F=y_c=c_1e^{2x}+c_2e^{-2x}$

$$P.I=y_p=\frac{1}{D^2-4}(2\cos^2x)=\frac{1}{D^2-4}(1+\cos2x)$$

$$=\frac{e^{0x}}{D^2-4}+\frac{\cos2x}{D^2-4}=P.I_1+P.I_2$$

$$P.I_1=y_{p_1}=\frac{e^{0x}}{D^2-4}\ \text{[Put D=0]}\ =\frac{e^{0x}}{-4}=-\frac{1}{4}$$

$$P.I_2=y_{p_2}=\frac{\cos2x}{D^2-4}=\frac{\cos2x}{-8}\ [\text{Put } D^2=-2^2=-4\,]$$

$\therefore$ The general solution of (1) is $y=y_c+y_{p_1}+y_{p_2}$

$$y=c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}-\frac{\cos2x}{8}$$

6. Solve $(D^2+1)y=\sin x\sin2x$

Sol : Given D.E is $(D^2+1)y=\sin x\sin2x$

A.E is $m^2+1=0\Rightarrow m=\pm i$

The roots are complex conjugate numbers.

C.F is $y_c=c_1\cos x+c_2\sin x$

w.k.t 2sinAsinB=cos(A-B)–cos(A+B)

$$P.I=\frac{\sin x\sin2x}{(D^2+1)}=\frac{1}{2}\frac{\cos x-\cos3x}{(D^2+1)}=P.I_1+P.I_2$$

Now $P.I_1=\frac{1}{2}\frac{\cos x}{D^2+1}$

Put $D^2=-1$ we get $D^2+1=0$

$$\therefore P.I_1=\frac{1}{2}\frac{x\sin x}{2}=\frac{x\sin x}{4}\qquad\left[\because \text{Case of failure}: \frac{\cos ax}{D^2+a}=\frac{x}{2a}\sin ax\right]$$

and $P.I_2=-\frac{1}{2}\frac{\cos3x}{D^2+1}$

Put $D^2=-9$, we get

$$P.I_2=-\frac{1}{2}\frac{\cos3x}{-9+1}=\frac{\cos3x}{16}$$

General solution is

$$y=y_c+y_{p_1}+y_{p_2}=c_1\cos x+c_2\sin x+\frac{x\sin x}{4}+\frac{\cos3x}{16}$$

7. Solve the differential equation $(D^3-3D^2-10D+24)y=x+3$.

Sol : The given D.E is $(D^3-3D^2-10D+24)y=x+3$

A.E is $m^3-3m^2-10m+24=0$

$\Rightarrow$m=2 is a root.

The other two roots are given by $m^2-m-2=0$
$\Rightarrow (m-2)(m+1)=0$

$\Rightarrow$m=2 (or) m = -1
One root is real and repeated, other root is real.
C.F is $y_c = e^{2x}(c_1+c_2x)+c_3e^{-x}$

$$y_p = \frac{x+3}{(D^3-3D^2-10D+24)} = \frac{1}{24}\frac{x^3+3}{1+\left(\frac{D^3-3D^2-10D}{24}\right)}$$

$$= \frac{1}{24}\left[\frac{1+D^3-3D^2-10D}{24}\right]^{-1}(x+3)$$

$$= \frac{1}{24}\left[1-\left(\frac{D^3-3D^2-10D}{24}\right)\right](x+3)$$

$$= \frac{1}{24}\left[x+3+\frac{10}{24}\right] = \frac{24x+82}{576}$$

General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{2x}(c_1+c_2x)+c_3e^{-x}+\frac{24x+82}{576}$$

8. Solve the differential equation $(D^2-4D+4)y = e^{2x}+x^2+\sin 3x$.

Sol : The A.E is $(m^2-4m+4)=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

$\therefore y_c = (c_1+c_2x)e^{2x}$ …(1)

To find y_p : $y_p = \frac{1}{D^2-4D+4}(e^{2x}+x^2+\sin 3x)$

$$= \frac{e^{2x}}{(D-2)^2}+\frac{x^2}{(D-2)^2}+\frac{\sin 3x}{D^2-4D+4}$$

$$= \frac{x^2}{2!}e^{2x}+\frac{x^2}{4\left(1-\frac{D}{2}\right)^2}+\frac{\sin 3x}{-9-4D+4}$$

$$= \frac{x^2}{2}e^{2x}+\frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2-\frac{(4D-5)\sin 3x}{(5+4D)}$$

$$= \frac{x^2}{2}e^{2x}+\frac{1}{4}\left(1+\frac{2D}{2}+\frac{3D^2}{4}\right)x^2-\frac{(4D-5)\sin 3x}{16D^2-25}$$

$$= \frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}-\frac{(12\cos 3x-5\sin 3x)}{-144-25}$$

$$= \frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}+\frac{(12\cos 3x-5\sin 3x)}{169} \quad \text{…(2)}$$

$$y = y_c+y_p = (c_1+c_2x)e^{2x}+=\frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}+\frac{(12\cos 3x-5\sin 3x)}{169}$$

9. Solve the differential equation $(D^2+4)y = x\sin x$.

Sol : Auxiliary equation is $m^2+4-0 \Rightarrow m^2=(2i)^2$

$\therefore m=\pm 2i.$ The roots are complex and conjugate.

Hence Complementary Function, $y_c=c_1\cos 2x+c_2\sin 2x$

Particular integral, $y_p=\dfrac{1}{D^2+4}x\sin x$

$$=\text{I.P of }\frac{1}{D^2+4}xe^{ix}$$

$$=\text{I.P of}e^{ix}\frac{1}{(D+i)^2+4}x=\text{I.P of }e^{ix}\frac{1}{D^2+2Di+3}x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1}x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1-\frac{D^2+2Di}{3}+\ldots\right)x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1-\frac{2}{3}Di\right)x\left[D^2(x)=0,\text{etc}\right]$$

$$=\text{I.P of }\frac{1}{3}(\cos x+i\sin x)\left(x-i\frac{2}{3}\right)$$

$$=\frac{1}{3}\left(-\frac{2}{3}\cos x+x\sin x\right)$$

Hence the general solution is

$$y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}\left(x\sin x-\frac{2}{3}\cos x\right)$$

where c_1 and c_2 are constants.

Other Method (using type 5): $y_p=\dfrac{1}{D^2+4}x\sin x$

$$=\left\{x-\frac{2D}{D^2+4}\right\}\frac{\sin x}{D^2+4}$$

$$=\frac{x\sin x}{3}-\frac{2(D\sin x)}{3(D^2+4)}$$

$$=\frac{x\sin x}{3}-\frac{2\cos x}{9}$$

Hence the general solution is

$$y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}\left(x\sin x-\frac{2}{3}\cos x\right)$$

10. Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x)=e^{x}$

Auxiliary equation is $f(m)=m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

m=-2 or m=-3

The roots are real and distinct

C.F=y_c= $c_1e^{-2x} + c_2e^{-3x}$

Particular Integral=y_p= $\frac{1}{f(D)}Q(x)$

$$= \frac{1}{D^2+5D+6}e^x = \frac{1}{(D+2)(D+3)}e^x$$

Put D = 1 in f(D)

P.I= $\frac{1}{(3)(4)}e^x$

Particular Integral = y_p = $\frac{1}{12}e^x$

General solution is $y = y_c + y_p$

$$y = c_1e^{-2x} + c_2e^{-3x} + \frac{e^x}{12}$$

11. Solve $y'' - 4y' + 3y = 4e^{3x}, y(0) = -1, y'(0) = 3$

Sol : Given equation is $y'' - 4y' + 3y = 4e^{3x}$

i.e $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$ it can be expressed as

$D^2y - 4Dy + 3y = 4e^{3x}$

$(D^2 - 4D + 3)y = 4e^{3x}$

Here Q(x) = $4e^{3x}$; f(D)= D^2-4D+3

Auxiliary equation is f(m)=m^2-4m+3 = 0

m^2- 3m - m+3=0

m(m-3) -1(m-3)=0⟹m=3 or 1

The roots are real and distinct.

C.F= $y_c = c_1e^{3x} + c_2e^x$

P.I= $y_p = \frac{1}{f(D)}Q(x)$

$$= \frac{1}{D^2 - 4D + 3}4e^{3x}$$

$$= \frac{1}{(D-1)(D-3)}4e^{3x}$$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x'}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y = y_c + y_p$

$$y = c_1e^{3x} + c_2e^{x} + 2xe^{3x} \qquad \ldots(3)$$

Equation (3) differentiating with respect to 'x'

$$y' = 3c_1e^{3x} + c_2e^{x} + 2e^{3x} + 6xe^{3x} \qquad \ldots(4)$$

By data, $y(0) = -1$, $y^1(0)=3$

From (3), $-1=c_1+c_2$ …(5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ …(6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^x + (1+2x)\, e^{3x}$

12. **Solve $y'' + 4y' + 4y = 4\cos x + 3\sin x, y(0) = 0, y'(0) = 0$**

Sol : Given differential equation in operator for

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

$\therefore$ C.F is $y_c = (c_1 + c_2x)e^{-2x}$

P.I is $= y_p = \frac{4\cos x + 3\sin x}{(D^2 + 4D + 4)}$ put $D^2 = -1$

$$y_p = \frac{4\cos x + 3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x + 3\sin x)}{(4D-3)(4D+3)}$$

$$= \frac{(4D-3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

$$y_p = \frac{(4D-3)(4\cos x + 3\sin x)}{-16-9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

$\therefore$ General equation is $y=y_c+y_p$

$$y = (c_1 + c_2x)e^{-2x} + \sin x \qquad \ldots(1)$$

By given data $y(0) = 0$, $c_1 = 0$ and

Differentiating (1) w.r.t 'x', $y' = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x$ …(2)

given $y'(0) = 0$

Substitute in (2) $\Rightarrow$ $-2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$ Required solution is $y = -xe^{-2x} + \sin x$

13. Solve $(D^2+9)y = \cos 3x$

Sol : Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore m = \pm 3i$

$y_c = C.F = c_1\cos 3x + c_2\sin 3x$

$$y_p = P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)}\sin 3x = \frac{x}{6}\sin 3x$$

General equation is $y=y_c+y_p$

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x$$

14. Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

Sol : Given equation can be written as

$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$

A.E is $m^3 + 2m^2 - m - 2 = 0$

$(m^2 - 1)(m+2) = 0$

$m^2 = 1$ or $m = -2$

$m = 1, -1, -2$

C.F$= c_1e^x + c_2e^{-x} + c_3e^{-2x}$

$$P.I = \frac{1}{(D^3+2D^2-D-2)}(1-4x^3) \qquad = \frac{-1}{2\left[1-\frac{(D^3+2D^2-D)}{2}\right]}(1-4x^3)$$

$$= \frac{-1}{2}\left[1-\frac{(D^3+2D^2-D)}{2}\right]^{-1}(1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+\ldots\right](1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{1}{2}\left(D^3+2D^2-D\right)+\frac{1}{4}\left(D^2-4D^3\right)+\frac{1}{8}\left(-D^3\right)\right]\left(1-4x^3\right)$$

$$=\frac{-1}{2}\left[1-\frac{5}{8}D^3+\frac{5}{4}D^2-\frac{1}{2}D\right](1-4x^3)$$

$$=\frac{-1}{2}\left[(1-4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)\right]$$

$$=\frac{-1}{2}\left[-4x^3+6x^2-30x+16\right]$$

$$=[2x^3-3x^2+15x-8]$$

The general solution is

y= C.F + P.I

$$y=c_1e^x+c_2e^{-x}+c_3e^{-2x}+[2x^3-3x^2+15x-8]$$

15. Solve $(D^3-7D^2+14D-8)y=e^x\cos 2x$

Sol : Given equation is

$$(D^3-7D^2+14D-8)y=e^x\cos 2x$$

A.E is $(m^3-7m^2+14m-8)=0$

(m-1)(m-2)(m-4)=0

Then m=1,2,4

C.F= $c_1e^x+c_2e^{2x}+c_3e^{4x}$

$$\text{P.I}=\frac{e^x\cos 2x}{(D^3-7D^2+14D-8)}$$

$$=e^x\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}\cos 2x$$

$$\left[\because \text{P.I}=\frac{1}{f(D)}e^{ax}v=e^{ax}\frac{1}{f\left(D+a\right)}v\right]$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(-4D+3D+16)}\cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$$=e^x\frac{1}{(16-D)}\cos 2x$$

$$=e^x\frac{16+D}{(16-D)(16+D)}\cos 2x$$

$$= e^{x}\frac{16+D}{256-D^{2}}\cos 2x$$

$$= e^{x}\frac{16+D}{256-(-4)^{2}}\cos 2x$$

$$= \frac{e^{x}}{260}(16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^{x}}{260}\left(8\cos 2x - \sin 2x\right)$$

$$= \frac{e^{x}}{130}\left(8\cos 2x - \sin 2x\right)$$

General solution is $y = y_c + y_p$

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{4x} + \frac{e^{x}}{130}\left(8\cos 2x - \sin 2x\right)$$

16. Solve $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

Sol : Given $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F= $(c_1 + c_2x)e^{2x}$

$$\text{P.I}= \frac{x^2\sin x + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2\sin x) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$$

$$\text{Now } \frac{1}{(D-2)^2}(x^2\sin x) = \frac{1}{(D-2)^2}(x^2) \qquad (\text{I.P of } e^{ix})$$

$$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)e^{ix}$$

$$= \text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

$$\text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

On simplification, we get

$$\frac{1}{(D+i-2)^2}(x^2\sin x) = \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right]$$

$$\text{and } \frac{1}{(D-2)^2}e^{2x} = \frac{x^2}{2}e^{2x},$$

$$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$$

$$\text{P.I}= \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

$y = y_c + y_p$

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

Linear equations of second order with variable coefficients

An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, where P(x), Q(x), R(x) are real valued functions of 'x'is called linear equation of second order with variable coefficients.

Variation of Parameters :

This method is applied when P,Q in above equation are either functions of 'x' or real constants but R is a function of 'x'.

Working Rule :

1. Find C.F. Let C.F= $y_c = c_1u(x)+c_2u(x)$
2. Take P.I= y_p=Au+Bv where A= $-\int\frac{vRdx}{uv'-vu'}$ and $B=\int\frac{uRdx}{uv'-vu'}$
3. Write the G.S. of the given equation $y=y_c+y_p$

1. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+y=\text{cosecx}$

Sol : Given equation in the operator form is $(D^2+1)y=\text{cosecx}$ …(1)

A.E is $(m^2+1)=0$

$\therefore m=\pm i$

The roots are complex conjugate numbers.

C.F is $y_c=c_1\cos x+c_2\sin x$

Let y_p= A cosx+ B sinx be P.I. of (1)

$u\frac{dv}{dx}-v\frac{du}{dx}=\cos^2x+\sin^2x=1$

A and B are given by

$$A=-\int\frac{vRdx}{uv'-vu'}=-\int\frac{\sin x\,\text{cosec}x}{1}dx=-\int dx=-x$$

$$B=\int\frac{uRdx}{uv^1-vu^1}=\int\cos x.\text{cosec}x dx=\int\cot x dx=\log(\sin x)$$

$\therefore y_p=-x\cos x+\sin x.\log(\sin x)$

$\therefore$ General solution is y= y_c+y_p.

$y=c_1\cos x+c_2\sin x-x\cos x+\sin x.\log(\sin x)$

2. Solve $(D^2-2D+2)y=e^x\tan x$ by method of variation of parameters.

Sol : A.E is $m^2-2m+2=0$

$$\therefore m=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm i2}{2}=1\pm i$$

We have $y_c = e^x(c_1\cos x + c_2\sin x) = c_1e^x\cos x + c_2e^x\sin x$

$= c_1(u) + c_2(u)$

where $u = e^x\cos x, v = e^x\sin x$

$$\frac{du}{dx} = e^x(-\sin x) + e^x\cos x, \frac{dv}{dx} = e^x\cos x + e^x\sin x$$

$$u\frac{dv}{dx} - v\frac{du}{dx} = e^x\cos x(e^x\cos x + e^x\sin x) - e^x\sin x(e^x\cos x - e^x\sin x)$$

$$= e^{2x}(\cos^2x + \cos x\sin x - \sin x\cos x + \sin^2x) = e^{2x}$$

Using variation of parameters,

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} = -\int \frac{e^x\tan x}{e^{2x}}(e^x\sin x)dx$$

$$= -\int \tan x\sin x dx = \int\left(\frac{\sin^2x}{\cos x}dx\right) = \int \frac{(1-\cos^2x)}{\cos x}dx$$

$$= \int(\sec x - \cos x)dx = \log(\sec x + \tan x) - \sin x$$

$$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$$

$$= \int \frac{e^x\cos x.e^x\tan x}{e^{2x}}dx = \int \sin x dx = -\cos x$$

General solution is given by $y = y_c + Au + Bv$

i.e $y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - \sin x]e^x\cos x - e^x\cos x\sin x$

or $y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - 2\sin x]e^x\cos x$

3. Solve the differential equation $(D^2 + 4)\, y = \sec 2x$ by the method of variation of parameters.

Sol. Given equation is $(D^2 + 4)\, y = \sec 2x$ …..(1)

∴ A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

The roots are complex conjugate numbers.

∴ $y_c = C.F = c_1 \cos 2x + c_2 \sin 2x$

Let $y_p = P.I = A \cos 2x + B \sin 2x$

Here $u = \cos 2x$, $v = \sin 2x$ and $R = \sec 2x$.

∴ $\frac{du}{dx} = -2 \sin 2x$ and $\frac{dv}{dx} = 2 \cos 2x$

∴ $u\frac{dv}{dx} - v\frac{du}{dx} = (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2$

A and B are given by :

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx = -\int \frac{\sin 2x \sec 2x}{2}dx = -\frac{1}{2}\int \tan 2x\, dx = \frac{1}{2}\frac{\log|\cos 2x|}{2}$$

$$\Rightarrow A = \frac{\log|\cos 2x|}{4}$$

$$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx = \int \frac{\cos 2x \sec 2x}{2}dx = \frac{1}{2}\int dx = \frac{x}{2}$$

∴ $y_p = P.I = \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

∴ The general solution is given by :

$y = y_c + y_p = C.F. + P.I$

i.e., $y = c_1 \cos 2x + c_2 \sin 2x + \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$