Simultaneous differential equation ~~equation~~ by Operation method :- In this section, we consider the solution of a system of two linear first order equations in two dependent variables y_1 and y_2 and one independent variable t.

$$a_1 \frac{dy_1}{dt} + a_2 \frac{dy_2}{dt} + a_3 y_1 + a_4 y_2 = a_5 \quad \text{--- ①}$$

$$b_1 \frac{dy_1}{dt} + b_2 \frac{dy_2}{dt} + b_3 y_1 + b_4 y_2 = a_6 \quad \text{--- ②}$$

(Our aim to find y_1 & y_2)

Ex :

$$\frac{dy_1}{dt} + 2\frac{dy_2}{dt} - 2y_1 - y_2 = e^{2t} \quad \text{--- ①}$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0 \quad \text{--- ②}$$

writing equation ① and ② in operator form.

$$Dy_1 + 2Dy_2 - 2y_1 - y_2 = e^{2t}$$

i.e. $$(D-2)y_1 + (2D-1)\, y_2 = e^{2t} \quad \text{--- ③}$$

and $$Dy_2 + y_1 - 2y_2 = 0$$

$$y_1 + (D-2)\, y_2 = 0 \quad \text{--- ④}$$

Operating with $(D-2)$ in equation ④ we have

$$(D-2)\, y_1 + (D-2)^2 y_2 = 0 \quad \text{--- ⑥}$$

$$\cancel{(D-2)y_1} + (2D-1)y_2 = e^{2x}$$
$$-\cancel{(D-2)y_1} \pm (D-2)^2 y_2 = 0$$

$$(2D-1)y_2 - (D-2)^2 y_2 = e^{2x}$$

$$\Rightarrow (D-2)^2 y_2 - (2D-1)y_2 = -e^{2x}$$

$$\Rightarrow \cancel{D^2 - 2D + 4} \left[(D-2)^2 - (2D-1)\right] y_2 = -e^{2x}$$

$$\Rightarrow \left[D^2 - 4D + 4 - 2D + 1\right] y_2 = e^{2x}$$

$$\left[D^2 - 6D + 5\right] y_2 = e^{2x}$$

$$\boxed{y_2(t) = c_1 e^{t} + c_2 e^{5t} + \frac{1}{3} e^{2t}} \quad \text{——— (1)}$$

Using (1) After using the value of $y_2(t)$ in equation (2)
we have,

$$y_1(t) = 2y_2(t) - \frac{dy_2}{dt}$$

$$y_1(t) = 2\left[c_1 e^{t} + c_2 e^{5t} + \frac{1}{3} e^{2t}\right] - \left[c_1 e^{t} + 5c_2 e^{5t} + \frac{2}{3} e^{2t}\right]$$

$$= 2c_1 e^{t} + 2c_2 e^{5t} + \cancel{\frac{2}{3} e^{2t}} - c_1 e^{t} - 5c_2 e^{5t} - \cancel{\frac{2}{3} e^{2t}}$$

$$= c_1 e^{t} - 3c_2 e^{5t}$$

$$\boxed{y_1(t) = c_1 e^{t} - 3c_2 e^{5t}}$$

Ex $(3D+1)y_1 + 3Dy_2 = 3t+1$ — (1) (Tutorial)

$(D-3)y_1 + Dy_2 = 2t$ — (2)

multiply in equation (2) by (3) and substraction from (1), we have.

$$(3D+1)y_1 + \cancel{3Dy_2} = 3t+1$$
$$-\ 3(D-3)y_1 + \cancel{3Dy_2} = 6t$$

$$\left[(3D+1) - 3(D-3)\right]y_1 = 3t+1-6t$$

$$\left[\cancel{3D}+1-\cancel{3D}+9\right]y_1 = 1-3t$$

$$10y_1 = 1-3t$$

$$\Rightarrow \boxed{y_1 = \frac{(1-3t)}{10}}$$

Substituting the value of y_1 in equation (2) we have

$$(D-3)\left(\frac{1-3t}{10}\right) + Dy_2 = 2t$$

$$\frac{1}{10}(D-3)(1-3t) + Dy_2 = 2t$$

$$\frac{1}{10}\left[(-3) - 3(1-3t)\right] + Dy_2 = 2t$$

$$\frac{1}{10}\left[-3-3+9t\right] + Dy_2 = 2t$$

$$\frac{1}{10}\left[-6+9t\right] + Dy_2 = 2t$$

$$Dy_2 = 2t - \frac{1}{10}[9t - 6]$$

$$Dy_2 = \frac{20t - 9t + 6}{10}$$

$$Dy_2 = \frac{11t + 6}{10}$$

$$\frac{dy_2}{dt} = \frac{(11t + 6)}{10}$$

$$dy_2 = \frac{(11t + 6)}{10} dt$$

After Integration, we have

$$\int dy_2 = \int \frac{(11t + 6)}{10} dt + C$$

$$y_2 = \frac{1}{10} \int (11t + 6) dt + C$$

$$y_2 = \frac{1}{10} \left[11\frac{t^2}{2} + 6t \right] + C$$

(or) $$\boxed{y_2 = \frac{11}{20} t^2 + \frac{6}{10} t + C}$$