MATRIX

Rank of matrix: $\rho(A)$

* $\rho(A)$ = order of largest Non-zero minor of A
* For $m\times n$ matrix $\rho \le \min(m,n)$
* for Square matrix of order n, $\rho = n$, if $|A| \neq 0$, otherwise, $\rho < n$
* $\rho(\text{Null Matrix}) = 0$

Elementary Row Operation:

(1) $R_i \leftrightarrow R_j$
(2) $R_i \leftrightarrow KR_i$
(3) $R_i \leftrightarrow R_i + KR_j$

Echelon form of Matrix:

* Leading Element of each Row must be unity (1).
* Zero Row must be at last.
* In Each Row No of zeros must be more than previous row.

\# No of Non-zero row in Row Echelon form is the Rank of Matrix.

L.I. & L.D of Vectors:

$\boxed{M-1}$: Vectors $\{V_1, V_2, V_3, \dots, V_n\}$ are L.I if $a_1V_1 + a_2V_2 + a_3V_3 + \dots + a_nV_n = 0$ implies $a_1 = a_2 = a_3 = \dots = a_n = 0$ otherwise L.D

$\boxed{M-2}$: Reduce matrix to Row Echleon form. then
if there is zero Row $\rightarrow$ L.D.
if there is No Zero Row $\rightarrow$ L.I.

$\boxed{M-3}$: If A is Square Matrix then
if $|A| = 0 \rightarrow$ L.D
if $|A| \neq 0 \rightarrow$ L.I

Gauss Elimination Method:

\# **Non-Homogenous System:**

$\Rightarrow$ Write System in $AX=B$ matrix form
$\Rightarrow$ Write augmented matrix $[A:B]$
$\Rightarrow$ Reduce $[A:B] \rightarrow$ Row Echleon form.
$\Rightarrow$ n = No of variables

$\boxed{C-1}$: $\rho(A) \neq \rho(A:B) \rightarrow$ Inconsistent $\rightarrow$ No Sol^n
$\boxed{C-2}$: $\rho(A) = \rho(A:B) = n \rightarrow$ Consistent $\rightarrow$ Unique Sol^n
$\boxed{C-3}$: $\rho(A) = \rho(A:B) < n \rightarrow$ Consistent $\rightarrow \infty$ Sol^n

Note: If there is a zero row in Reduced Row Echleon form, then we assign that variable as an arbitary value which corresponds to the column not containing the pivot element.

\# **Homogenous System:**

$\hookrightarrow$ Always Consistent b'coz there is always a zero Sol^n.

$\Rightarrow$ Write System in $AX=0$ form

$A \rightarrow$ Row reduced Echleon form

$\rho < n$	$\rho = n$
Non-Zero Sol^n	Zero Sol^n
Non-Trival Sol^n	Trival Sol^n
$\|A\| = 0$	$\|A\| \neq 0$

* We found $|A|$ only when A is square matrix.

$\Rightarrow$ If (No of Eq^n) < (No of Variable) (n) $\rightarrow \infty$ Sol^n

Eigen Values & Eigen Vectors:

$\Rightarrow$ Write A matrix as $(A-\lambda I)X = 0$
A is square matrix, X is Non-zero Vector, λ is scalar.

$\Rightarrow$ Polynomial of λ is form after solving $(A-\lambda I) = 0$ is characteristic Eq^n of A.

$\Rightarrow$ $\lambda^2 - (\text{Trace of } A)\lambda + \det(A) = 0$
$\lambda^3 - (\text{Trace of } A)\lambda^2 + (\text{Sum of minor along main diagonal})\lambda - \det(A) = 0$
This is alternate methode to find characteristic Eq^n.

$\Rightarrow$ Roots of Characteristic Eq^n is characteristic Root or Eigen Values of Matrix A.

$\Rightarrow$ Product of Eigen Values = $|A|$

$\Rightarrow$ After putting value of λ in $(A-\lambda I)X = 0$ values of variable (x,y,z) is Eigen Vectors.

Properties of Eigen Values & Eigen Vectors:

(1) A & $A^T \rightarrow$ have same Eigen Values.
(2) Eign Values of Triangular matrix is just the diagonal elements.
(3) αA has Eigen Values $\alpha\lambda$ and the Eigen Vector will remain X.
(4) $(A-KI)$ has Eigen Values $(\lambda - K)$ and Eigen vector will remain X.
(5) Sum of Eigen Values = Sum of elements of Principle diagonal.
(6) Eigen Values of Idempotent Matrix $(A^2 = A)$ is either zero or unity.
(8) λ is Eigen Value of A then $\frac{1}{\lambda}$ is Eigen value of A^{-1} and the Eigen Vector will remain X.
(9) $(A-KI)^{-1}$ has Eigen Value $\frac{1}{(\lambda - K)}$ and Eigen Vector will remain X.
(10) λ is Eigen Value of Orthogonal Matrix $(A^{-1} = A^T)$ then $\frac{1}{\lambda}$ is also its Eigen Value.
(11) $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n \rightarrow$ Eigen Value of A, then $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m \rightarrow$ Eigen value of A^m $(m \rightarrow +ve \text{ integer})$
(12) For a real matrix A $(\alpha + i\beta)$ is an Eigen value then $(\alpha - i\beta)$ is also an Eigen Value.
(13) λ is Eigen Value of Non-Singular Matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of Matrix adj(A).

Gauss Jordan Method:

$\hookrightarrow$ used to find A^{-1}

$$[A|I] \xrightarrow[\text{operation}]{\text{Elementary Row}} [I|A^{-1}]$$

Trick: Proceed like this

Cayley-Hamilton Theorem:

$\Rightarrow$ Every Square Matrix satisfies its characteristic Eq^n.
$\Rightarrow$ Also used to find A^{-1}

Eg: $A^2 - 8A + I = 0$
$A^{-1}(A^2 - 8A + I) = A^{-1}0$
$A - 8I + A^{-1} = 0$
$A^{-1} = 8I - A$

LINEAR DIFFERENTIAL EQUATION

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x)$$

* $y \rightarrow$ dependent variable $x \rightarrow$ independent
* $a_0(x) \neq 0$
* If $r(x) = 0 \rightarrow$ Homogenous Eq^n otherwise Non-Homogenous Eq^n

Theorem: If $a_0(x), a_1(x), \dots, a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$, then there exist a unique Sol^n to the initial value problem
$a_0(x)y^n(x) + a_1(x)y^{n-1}(x) \dots + a_n(x)y(x) = r(x)$
$y(x_0) = C_1,\ y'(x_0) = C_2 \dots,\ y^{n-1}(x_0) = C_n,\ x_0 \in I$

$\Rightarrow$ A point $x_0 \in I$, for which $a_0(x) \neq 0$ is called ordinary point or regular point.
$\Rightarrow$ If the condit^n of theorem are satisfied the D.E is Normal on I

Order & Degree:

Ex: $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = K^2\left(\frac{d^3y}{dx^3}\right)^2$ Order = 3, Degree = 2

Linear Combination of Sol^n:

$\Rightarrow$ If every Sol^n of Homogenous Eq^n satisfies its Eq^n then the Sol^n are linear Combination
$\Rightarrow$ Ex: e^{-x}, e^{x} and their linear combination $c_1e^{-x} + c_2e^{x}$ are Sol^n of $y'' - y = 0$

Linear Independence & Dependence:

$\boxed{M-1}$: $f_1(x), f_2(x) \dots f_n(x)$ be n function if $c_1f_1 + c_2f_2 + \dots + c_nf_n = 0$ implies $c_1 = 0, c_2 = 0, \dots, c_n = 0$, then these functions are L.I on some Interval I otherwise L.D.

$\boxed{M-2}$: **Wronskian:**

$$W(f_1, f_2 \dots f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \dots & f_n \\ f_1' & f_2' & f_3' & \dots & f_n' \\ f_1'' & f_2'' & f_3'' & \dots & f_n'' \\ \vdots & & & & \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} & \dots & f_n^{n-1} \end{vmatrix}$$

$\Rightarrow$ If $W(x) = 0 \rightarrow$ LD
$W(x) \neq 0 \rightarrow$ LI

$\Rightarrow$ $W(x)$ of functions exist if all functions are differentiable $(n-1)$ times on interval I. If any function not diff. then $W(x)$ not exist

Fundamental Sol^n (Basis) of D.E.

$\Rightarrow$ How to check if set of functions forms basis
① Check that functions of set satisfies D.E or not.
② If satisfies, then find $W(x)$, and if it is L.I, then set of functions forms basis.

Abel's formula:

Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \rightarrow 2^{nd}$ order D.E
then $W(x) = ce^{-\int [a_1(x)/a_0(x)]dx}$

Homogenous Linear D.E.

Process: ① Write Auxiliary Eq^n
② Find roots of Auxiliary Eq^n
③ There are 3 possibilities
(a) Roots are real & distinct
(b) Roots are real and equal
(c) Roots are complex.

case1: Roots are real & Distinct

$m_1, m_2, m_3, \dots, m_n$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

case2: Roots are real & Equal

$m_1 = m_2 = m_3 = \dots = m_n = m$

$$y = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1}) e^{mx}$$

case 3: Roots are complex

$\alpha \pm i\beta$

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Sol^n of Non Homogenous D.E:

⇒ Operator Method:

y = complementary function + Particular Integral
(CF) (PI)

⇒ CF is is finded by the process of Homogenous DE

$\boxed{\text{Type 1}}$: (a) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

(b) $f(a) = 0$, then differentiate $f(D)$ then put a, if again $f'(a) = 0$, then again diff.

$\boxed{\text{Type 2}}$: $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$, $f(-a^2) \neq 0$

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$, $f(-a^2) \neq 0$

⇒ after putting $-a^2$ at D^2, if $\frac{1}{D+c}$ some like that remain, then $\frac{D-c}{D^2-c^2}$, do like that

⇒ $D \to \frac{d}{dx}$, $\frac{1}{D} \to$ integral

⇒ Case of Failure: $f(-a^2) = 0$, then differentiate the $f(D)$, then put $-a^2$ at the place of D^2.

$\boxed{\text{Type 3}}$: $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$, V is any f^n of x.

$\boxed{\text{Type 4}}$: $\frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \left\{\frac{d}{dD} \frac{1}{f(D)}\right\} V$

$\boxed{\text{Type 5}}$: $\frac{1}{D-\alpha} Q = e^{\alpha x} \int Q e^{-\alpha x} dx$

$(1-x)^{-1} = 1 + x + x^2 + x^3 \dots$, $|x| < 1$

$(1+x)^{-1} = 1 - x + x^2 - x^3 \dots$, $|x| < 1$

Variation of Parameters:

⇒ consider 2nd order Non-Homogenous Eq^n

$$a_0(x) y'' + a_1(x) y' + a_2(x) y = r(x), \quad a_0(x) \neq 0$$

⇒ $y = CF + PI$

⇒ $CF = c_1 y_1 + c_2 y_2$

⇒ $PI = A(x) y_1 + B(x) y_2$

$$A(x) = -\int \frac{y_2 X}{W} dx, \quad B(x) = \int \frac{y_1 X}{W} dx$$

Where,

y_1 & y_2 = two linearly independent sol^n of $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$

$X = \frac{r(x)}{a_0(x)}$

W = Wronskian of y_1 and y_2

Method of Undetermined Coefficient:

⇒ Process selecting trial sol^n of P.I.

RHS $r(x)$	Trial sol^n
e^{mx}	Ce^{mx}
$\sin mx$ or $\cos mx$	$A \cos mx + B \sin mx$
x^m	$C_0 x^m + C_1 x^{m-1} + \dots + C_m$
$e^{ax} \cos bx$ or $e^{ax} \sin bx$	$e^{ax}(c_1 \cos bx + c_2 \sin bx)$

⇒ steps:

1. make P.I according to Trial sol^n
2. Then put PI in D.E and compare LHS & RHS & then find value of Arbitary constant.
3. Put Arbitary constant value in PI.

⇒ If any term in the choice of PI is also sol^n of CF then multiply that term with x or x^m in PI.

Ex: $CF = c_1 e^{3x} + c_2 e^{-x}$

$PI = Axe^{-x} + Be^{x}$

Sol^n of Euler-Cauchy Equation:

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x y' + a_n y = r(x)$$

$x \neq 0$

$a_0, a_1, \dots a_n$ = constant

1. let $z = \log x$ or $x = e^z$
2. Replace $xy' = Dy$
   $x^2 y'' = D(D-1)y$
   $x^3 y''' = D(D-1)(D-2)y$
3. at last in sol^n replace z with x function

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

$a_0, a_n, b_n \to$ fourier constant

Interval → $\alpha < x < \alpha + 2c$

$$a_0 = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) dx$$

$$a_n = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) \cos \frac{n\pi x}{c} dx$$

$$b_n = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x) \sin \frac{n\pi x}{c} dx$$

Some Important integral:

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = 0$
2. $\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = 0$
3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = 0$
4. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = 0$
5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos mx \, dx = 0$
6. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = 0$
7. $\int_{\alpha}^{\alpha+2\pi} \sin^2 mx \, dx = 0$
8. $\int_{\alpha}^{\alpha+2\pi} \cos^2 mx \, dx = 0$

⇒ $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

⇒ $\int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

Dirichlet's Condition:

Any $f(x)$ can be developed as a fourier series provided:

1. $f(x)$ is periodic, single valued & finite
2. $f(x)$ has a finite no of discontinuities in any one period
3. $f(x)$ has at most a finite no of maxima & minima.

fourier Series Expansion of Even & odd function in interval $(-c, c)$:

$a_0 = \frac{1}{c} \int_{-c}^{c} f(x) dx \qquad b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} dx$

$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} dx$

$\boxed{\text{case I}}$: $f(x)$ is even function

$$a_0 = \frac{1}{c} \int_{-c}^{c} f(x) dx = \frac{2}{c} \int_{0}^{c} f(x) dx$$

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} dx = \frac{2}{c} \int_{0}^{c} f(x) \cos \frac{n\pi x}{c} dx$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} dx = 0$$

⇒ Hence if periodic f^n $f(x)$ is even, its fourier expansion contain only cosine terms

$\boxed{\text{case II}}$: $f(x)$ is odd function

$$a_0 = \frac{1}{c} \int_{-c}^{c} f(x) dx = 0$$

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} dx = 0$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} dx = \frac{2}{c} \int_{0}^{c} f(x) \sin \frac{n\pi x}{c} dx$$

⇒ Hence if periodic f^n $f(x)$ is odd, its fourier expansion contain only sine terms

Half Range Series:

1. Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

$$b_n = \frac{2}{c} \int_{0}^{c} f(x) \sin \frac{n\pi x}{c} dx$$

2. cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c} \int_{0}^{c} f(x) dx$$

$$a_n = \frac{2}{c} \int_{0}^{c} f(x) \cos \frac{n\pi x}{c} dx$$

Multivariate Calculus:

Limit:

⇒ directly put limit x, y in function and check limit exist or not

⇒ If not exist by putting directly then put $y = mx$, or $y = mx^2$ or $y = mx^3$ then put limit of x. If limit is dependent on the value of m, limit does not exist

Continuity:

If f^n $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if.

① $f(x,y)$ defined at point (x_0,y_0)

② $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

③ $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

⇒ If $f(x_0,y_0)$ defined and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists, but $f(x_0,y_0) \neq L$ then the point (x_0,y_0) is point of removal discontinuity

Partial Derivative:

Let $z = f(x,y)$, $\dfrac{\partial z}{\partial x} = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$

$$\frac{\partial z}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

⇒ Here we can use rules of diffentiation.

⇒ If we partial diff. w.r.t x then all other variable will remain constant.

Total Differential:

let $z = f(x,y)$

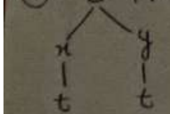
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

if $z = f(x_1, x_2, \ldots x_n)$

$$dz = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

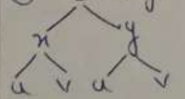
Derivative of composite & Implicit function:

① $z = f(x,y)$



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

② $z = f(x,y)$



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Higher Order Derivatives:

let $z = f(x,y)$, then its 2nd order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

⇒ If f_{xy} & f_{yx} are continuous at a point $P(x,y)$ then at this point $f_{xy} = f_{yx}$

Differentiation of Implicit function:

If $f(x,y) = c$ be an implicit relation b/w x & y then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Change of Variables (Jacobian):

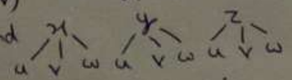
Suppose $f(x,y)$ is function of two independent variables, and $x = \phi(u,v)$, $y = \phi(u,v)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{1}{J}\cdot\frac{\partial(f,y)}{\partial(u,v)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J}\cdot\frac{\partial(x,f)}{\partial(u,v)} = \frac{-1}{J}\frac{\partial(f,x)}{\partial(u,v)}$$

⇒ If $f(x,y,z)$ and



$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y,z)}{\partial(u,v,w)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J}\frac{\partial(x,f,z)}{\partial(u,v,w)} = \frac{-1}{J}\frac{\partial(f,x,z)}{\partial(u,v,w)}$$

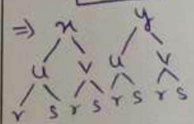
$$\frac{\partial f}{\partial z} = \frac{1}{J}\frac{\partial(x,y,f)}{\partial(u,v,w)} = \frac{-1}{J}\frac{\partial(f,x,y)}{\partial(u,v,w)}$$

⇒ If $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r, \quad J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$$\boxed{JJ' = 1}$$

⇒



$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(r,s)}$$

Functional Relation:

The variables of transformation $u = f(x,y,z)$, $v = g(x,y,z)$, $w = h(x,y,z)$ are said to be functionally related if $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$, that is there exists a relationship between variables u,v,w and the transformation is not independent.

Homogenous function:

$f(x,y)$ said to be homogenous of degree n in x and y, if it can be written any one of following form:

① $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

② $f(x,y) = x^n g\left(\frac{y}{x}\right)$

③ $f(x,y) = y^n g\left(\frac{x}{y}\right)$

⇒ similarily for 3 variable for homogenous $f(\lambda x, \lambda y, \lambda z) = \lambda^n (f(x,y,z))$

Euler's Theorem:

If $f(x,y)$ is homogenous f^n of degree n in x and y and has continuous first and second order partial derivatives then

① $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

② $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f$

⇒ If f is homogenous function of degree n in $x, y, z, t, \ldots$ then

$$x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} + t\cdot\frac{\partial f}{\partial t} + \cdots = nf$$

Maxima & Minima:

⇒ Working Rule:

① f_x and f_y → find it and equate to zero solve it and find x and y. The points we get is called critical points or stationary points.

② Calculate f_{xx}, f_{xy}, f_{yy}

③ If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$ Point of relative minimum.

④ If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0$ point of relative maximum.

⑤ If $f_{xx}f_{yy} - (f_{xy})^2 < 0$ → saddle point

⑥ If $f_{xx}f_{yy} - (f_{xy})^2 = 0$ → No conclusion

Lagrange Method of Multipliers:

- We want to find extremum of $f(x_1, x_2, \ldots x_n)$ under the condition $\phi_i(x_1, x_2, \ldots x_n) = 0$, $i = 1, 2, \ldots k$
- We construct the auxiliary function: $F = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i \phi_i(x_1, x_2, \ldots x_n)$
- λ_i are undetermined parameter and are known as Lagrange's multipliers.
- find $\dfrac{\partial F}{\partial x_1} = 0$, $\dfrac{\partial F}{\partial x_2} = 0$, ---- $\dfrac{\partial F}{\partial x_n} = 0$
- Find stationary points by solving all equations.
- further investigation is needed to determine exact nature of the points.