Therefore, $f(2.1, 3.2) \approx f(2, 3) + f_x(2, 3)\, \Delta x + f_y(2, 3)\, \Delta y$

$$= 8 + 12(0.1) + (2.408)\,(0.2) = 9.6816.$$

The exact value is $f(2.1, 3.2) = 10.7424$.

Example 2.16 Find the percentage error in the computed area of an ellipse when an error of 2% is made in measuring the semi major and semi minor axes.

Solution Let the major and minor axes of the ellipse be $2a$ and $2b$ respectively. The errors Δa and Δb in computing the lengths of the semi major and minor axes are

$$\Delta a = a(0.02) = 0.02\, a \quad \text{and} \quad \Delta b = b(0.02) = 0.02\, b.$$

The area of the ellipse is given by $A = \pi ab$. Therefore, we have the following:

Maximum absolute error in computing the area of ellipse is

$$|dA| = \left|\frac{\partial A}{\partial a}\right| |\Delta a| + \left|\frac{\partial A}{\partial b}\right| |\Delta b| = \pi b(0.02a) + \pi a(0.02b) = 0.04\, \pi ab.$$

Maximum relative error is

$$\left|\frac{dA}{A}\right| = (0.04\, \pi ab)\left(\frac{1}{\pi ab}\right) = 0.04.$$

$$\text{Percentage error} = \left|\frac{dA}{A}\right| \times 100 = 4\%.$$

2.3.3 Derivatives of Composite and Implicit Functions (*Chain Rule*)

Let $z = f(x, y)$ be a function of two independent variables x and y. Suppose that x and y are themselves functions of some independent variable t, say $x = \phi(t)$, $y = \psi(t)$. Then, $z = f[\phi(t), \psi(t)]$ is a composite function of the independent variable t. Now, assume that the partial derivatives f_x, f_y are continuous functions of x, y and $\phi(t)$, $\psi(t)$ are differentiable functions of t.

Let Δx, Δy and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then we have

$$\Delta z = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2 \Delta y.$$

Dividing both sides by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}. \tag{2.32}$$

Now as $\Delta t \to 0$; $\Delta x \to 0$, $\Delta y \to 0$ and $\varepsilon_1\left(\frac{\Delta x}{\Delta t}\right) \to 0$, $\varepsilon_2\left(\frac{\Delta y}{\Delta t}\right) \to 0$. Therefore, taking limits on both sides in Eq. (2.32) as $\Delta t \to 0$, we obtain

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2.33}$$

Now, let x and y be functions of two independent variables u and v, say $x = \phi(u, v)$, $y = \psi(u, v)$. Then, $z = f[\phi(u, v), \psi(u, v)]$ is a composite function of two independent variables u and v. Assume that the functions $f(x, y)$, $\phi(u, v)$, $\psi(u, v)$ have continuous partial derivatives with respect to their arguments. Now, consider v as a constant and give an increment Δu to u. Let $\Delta_u x$ and $\Delta_u y$ be the corresponding increments in x and y. Then, the increment Δz in z is given by (using Eq. (2.20))

$$\Delta z = \frac{\partial f}{\partial x}\Delta_u x + \frac{\partial f}{\partial y}\Delta_u y + \varepsilon_1 \Delta_u x + \varepsilon_2 \Delta_u y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta u \to 0$.
Dividing both sides by Δu, we get

$$\frac{\Delta z}{\Delta u} = \frac{\partial f}{\partial x}\frac{\Delta_u x}{\Delta u} + \frac{\partial f}{\partial y}\frac{\Delta_u y}{\Delta u} + \varepsilon_1 \frac{\Delta_u x}{\Delta u} + \varepsilon_2 \frac{\Delta_u y}{\Delta u}. \tag{2.34}$$

Taking limits on both sides in Eq. (2.34) as $\Delta u \to 0$, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}. \tag{2.35}$$

Similarly, keeping u as constant and varying v, we obtain

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{2.36}$$

The rules given in Eqs. (2.35) and (2.36) are called the *chain rules*. These rules can be easily extended to a function of n variables $z = f(x_1, x_2, \ldots, x_n)$. If the partial derivatives of f with respect to all its arguments are continuous and $x_1, x_2, \ldots, x_n$ are differentiable functions of some independent variable t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}. \tag{2.37}$$

Example 2.17 Find df/dt at $t = 0$, where

(i) $f(x, y) = x\cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$.

(ii) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Solution

(i) When $t = 0$, we get $x = 1$, $y = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (\cos y + e^x \sin y)(2t) + (-x\sin y + e^x \cos y)(3t^2 + 1).$$

Substituting $t = 0$, $x = 1$ and $y = 0$, we obtain $(df/dt) = e$.

(ii) When $t = 0$, we get $x = 1$, $y = 1$, $z = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (3x^2 + z^2 + yz)(e^t) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2).$$

Substituting $t = 0$, $x = 1$, $y = 1$, $z = 0$, we obtain $(df/dt) = 3$.

Example 2.18 If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Solution Using the chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}.$$

Therefore,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

Change of variables

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We want to determine $\partial f/\partial x$, $\partial f/\partial y$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\dfrac{\partial f}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\dfrac{\partial f}{\partial v}\dfrac{\partial x}{\partial u} - \dfrac{\partial f}{\partial u}\dfrac{\partial x}{\partial v}} = \frac{1}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}}.$$

The determinant

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

is called the *Jacobian* of the variables of transformation. Similarly, we write

$$\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(f, y)}{\partial(u, v)} = \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

and

$$\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x, f)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial f/\partial u & \partial f/\partial v \end{vmatrix} = -\frac{\partial(f, x)}{\partial(u, v)}.$$

Hence, we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(u, v)}\right] \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(u, v)}\right]. \tag{2.38}$$

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables u, v, w given by $x = F(u, v, w)$, $y = G(u, v, w)$, $z = H(u, v, w)$, then by chain rule, we have

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}.$$

Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f}{dx}=\frac{1}{J}\left[\frac{\partial(f,y,z)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{dy}=\frac{1}{J}\left[\frac{\partial(x,f,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\left[\frac{\partial(f,x,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{dz}=\frac{1}{J}\left[\frac{\partial(x,y,f)}{\partial(u,v,w)}\right]=\frac{1}{J}\left[\frac{\partial(f,x,y)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\end{vmatrix} \tag{2.39}$$

where

$$J=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

is the Jacobian of the variables of transformation.

Example 2.19 If $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Solution The variables of transformation are r and θ. We have

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\frac{\partial(f,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos\theta\frac{\partial f}{\partial r}-\sin\theta\frac{\partial f}{\partial\theta}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial x/\partial r & \partial x/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \cos\theta & -r\sin\theta\end{vmatrix}=-r\sin\theta\frac{\partial f}{\partial r}-\cos\theta\frac{\partial f}{\partial\theta}.$$

Hence, using Eq. (2.38), we obtain

$$\frac{\partial f}{\partial x}=\frac{1}{J}\left[\frac{\partial(f,y)}{\partial(r,\theta)}\right]=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{J}\left[\frac{\partial(f,x)}{\partial(r,\theta)}\right]=\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}.$$

Squaring and adding, we obtain the required result.

Example 2.20(a) If $u=f(x,y,z)$ and $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\phi}\right)^2.$$

Solution The variables of transformation are r, θ and ϕ. We have

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}=r^2\sin\theta$$

$$\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$=r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r}+r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial\theta}-r\sin\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f,x,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi\\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$= -r^2 \sin^2\theta \sin\phi \frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi \frac{\partial f}{\partial \theta} - r\cos\phi \frac{\partial f}{\partial \phi}.$$

$$\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial \theta & \partial f/\partial \phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\theta & r\sin\theta\cos\phi \end{vmatrix}$$

$$= r^2 \sin\theta\cos\theta \frac{\partial f}{\partial r} - r\sin^2\theta \frac{\partial f}{\partial \theta}.$$

Using Eq. (2.39), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\cos\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\sin\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)}\right] = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}.$$

Squaring and adding, we obtain the required result.

Remark 7

The variables of transformation $u = f(x, y, z)$, $v = g(x, y, z)$, $w = h(x, y, z)$ are functionally related if

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0,$$

that is, there exists a relationship between the variables u, v, w and the transformation is not independent.

Example 2.20(b) Show that the variables $u = x - y + z$, $v = x + y - z$, $w = x^2 + xz - xy$, are functionally related. Find the relationship between them.

Solution The Jacobian of transformation is given by

$$J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2x + z - y & -x & x \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 2x + z - y & -x & x \end{vmatrix} = 0.$$

Hence, the variables are related.

Now, $w = x(x - y + z) = xu$, and $u + v = 2x$. Therefore, $2w = u(u + v)$.

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}.$$

Therefore,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x-y}{2y+x} = \frac{y-2x}{2y+x},\quad y \neq -\frac{x}{2}.$$

Exercises 2.2

1. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

has partial derivatives $f_x(0, 0)$, $f_y(0, 0)$, but the partial derivatives are not continuous at (0, 0).

2. Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2+y^2}{x-y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

possesses partial derivatives at (0, 0), though it is not continuous at (0, 0).

3. For the function

$$f(x, y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

compute $f_x(0, y)$, $f_y(x, 0)$, $f_x(0, 0)$ and $f_y(0, 0)$, if they exist.

4. Show that the function $f(x, y) = \sqrt{x^2+y^2}$ is not differentiable at (0, 0).

5. Show that the function

$$f(x, y) = \begin{cases} (x^2+y^2)\cos\left[\dfrac{1}{\sqrt{x^2+y^2}}\right], & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is differentiable at (0, 0) and that f_x, f_y are not continuous at (0, 0). Does this result contradict Theorem 2.2?

Find the first order partial derivatives for the following functions at the specified point:

6. $f(x, y) = x^4 - x^2y^2 + y^4$ at $(-1, 1)$.

7. $f(x, y) = \ln(x/y)$ at (2, 3).

8. $f(x, y) = x^2 e^{y/x}$ at (4, 2).

9. $f(x, y) = x/\sqrt{x^2+y^2}$ at (6, 7).

10. $f(x, y) = \cot^{-1}(x+y)$ at (1, 2).

11. $f(x, y) = \ln\left[\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$ at (3, 4).

12. $f(x, y, z) = (x^2+y^2+z^2)^{-1/2}$ at (2, 1, 2).

13. $f(x, y, z) = e^{x/y} + e^{z/y}$ at (1, 1, 1).

14. $f(x, y, z) = (xy)^{\sin z}$ at $(3, 5, \pi/2)$.

15. $f(x, y, z) = \ln(x + \sqrt{y^2+z^2})$ at (2, 3, 4).

Find dw/dt in following problems.

16. $w = x^2 + y^2, x = (t^2 - 1)/t, y = t/(t^2 + 1)$ at $t = 1$.

17. $w = x^2 + y^2 + z^2, x = \cos t, y = \ln (t + 1), z = e^t$ at $t = 0$.

18. $w = e^x \sin (y + 2z), x = t, y = 1/t, z = t^2$.

19. $w = xy + yz + zx, x = t^2, y = te^t, z = te^{-t}$.

20. $w = z \ln y + y \ln z + xyz, x = \sin t, y = t^2 + 1, z = \cos^{-1} t$ at $t = 0$.

Verify the given results in the following problems:

21. If $z = f(ax + by)$, then $b\dfrac{\partial z}{\partial x} - a\dfrac{\partial z}{\partial y} = 0$.

22. If $z = \log [(x^2 - y^2)/(x^2 + y^2)]$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

23. If $u = f(x - y, y - z, z - x)$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

24. If $z = f(x, y)$, $x = r \cosh \theta$, $y = r \sinh \theta$, then

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

25. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then $u\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.

26. If $w = f(u, v)$, $u = \sqrt{x^2 + y^2}$, $v = \cot^{-1}(y/x)$, then

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2}\left[(x^2 + y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

27. If $z = f(x, y)$, $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$, where α is a constant, then

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

28. If $z = \ln (u^2 + v)$, $u = e^{x+y^2}$, $v = x + y^2$, then $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

29. If $w = \sqrt{x^2 + y^2 + z^2}$, $x = u \cos v$, $y = u \sin v$, $z = uv$, then

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1 + v^2}}$$

30. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2)/(x + y + z)$, then

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \tan w.$$

Check whether the variables in the following transformations are functionally related. If so, find the relationship between them.

31. $u = x^2 - y^2 - z^2$, $v = x^2 - y^2 + z^2$, $w = x^4 + y^4 + z^4 - 2x^2 y^2$.

32. $u = x + 3z$, $v = x - y - z$, $w = y^2 + 16z^2 + 8yz$.

33. $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$, $w = x^3 + y^3 + z^3 - 3xyz$.

34. $u = (x + y)/(1 - xy)$, $v = \tan^{-1} x + \tan^{-1} y$, $x > 0, y > 0, xy < 1$.

35. $u = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$, $v = \sin^{-1} x + \sin^{-1} y$, $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$.

Using implicit differentiation, obtain the following:

36. $\dfrac{dy}{dx}$, when $x^y + y^x = \alpha$, α any constant, $x > 0, y > 0$.

37. $\dfrac{dy}{dx}$, when $\cot^{-1}(x/y) + y^3 + 1 = 0,\ x > 0,\ y > 0.$

38. $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$, when $\cos xy + \cos yz + \cos zx = 1$.

39. $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$, when $x^3 + 3xy - 2y^2 + 3xz + z^2 = 0$.

40. $y\left(\dfrac{\partial x}{\partial y}\right)_y + z\left(\dfrac{\partial x}{\partial z}\right)_y$, when $f\left(\dfrac{z}{y}, \dfrac{x}{y}\right) = 0$.

Using differentials, obtain the approximate value of the following quantities:

41. $\sqrt{(298)^2 + (401)^2}$.

42. $(4.05)^{1/2}(7.97)^{1/3}$.

43. $\cos 44° \sin 32°$.

44. $\dfrac{1}{\sqrt{1.05}} + \dfrac{1}{\sqrt{3.97}} + \dfrac{1}{\sqrt{9.01}}$.

45. $\sin 26° \cos 57° \tan 48°$.

46. A certain function $z = f(x, y)$ has values $f(2, 3) = 5$, $f_x(2, 3) = 3$ and $f_y(2, 3) = 7$. Find an approximate value of $f(1.98, 3.01)$.

47. The radius r and the height h of a conical tank increases at the rate of $(dr/dt) = 0.2''/\text{hr}$ and $(dh/dt) = 0.1''/\text{hr}$. Find the rate of increase dV/dt in volume V when the radius is 5 feet and the height is 20 feet

48. The dimensions of a rectangular block of wood are 60″, 80″ and 100″ with possible absolute error of 3″ in each measurement. Find the maximum absolute error and the percentage error in the surface area.

49. Two sides of a triangle are measured as 5 cm and 3 cm and the included angle as 30°. If the possible absolute errors are 0.2 cm in measuring the sides and 1° in the angle, then find the percentage error in the computed area of the triangle.

50. The sides of a rectangular box are found to be a feet, b feet and c feet with a possible error of 1% in magnitude in each of the measurements. Find the percentage eror in the volume of the box caused by the errors in individual measurements.

51. The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 6 cm and 8 cm respectively. The maximum absolute error in each measurement is 0.2 cm. Find the maximum absolute error and the percentage error in the computed value of the volume.

52. The power consumed in an electric resistor is given by $P = E^2/R$ (in watts). If $E = 80$ volts and $R = 5$ Ohms, by how much the power consumption will change if E is increased by 3 volts and R is decreased by 0.1 Ohms.

53. If two resistors with resistences R_1 and R_2 in Ohms are connected in parallel, then the resistence of the resulting circuit is $R = [(1/R_1) + (1/R_2)]^{-1}$. Find an approximate value of the percentage change in resistence that results by changing R_1 from 2 to 1.9 Ohms and R_2 from 6 to 6.2 Ohms.

54. Suppose that $u = xze^y$ and x, y, z can be measured with maximum absolute errors 0.1, 0.2 and 0.3 respectively. Find the percentage error in the computed value of u from the measured values $x = 3$, $y = \ln 2$ and $z = 5$.

55. If the radius r and the altitude h of a cone are measured with an absolute error of 1% in each measurement, then find the approximate percentage change in the lateral area of the cone if the measured values are $r = 3$ feet and $h = 4$ feet.

2.4 Higher Order Partial Derivatives

Let $z = f(x, y)$ be a function of two variables and let its first order partial derivatives exist at all the points in the domain of definition D of the function f. Then, the first order partial derivatives are also functions of x and y. We define the second order partial derivatives as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_x(x + \Delta x, y) - f_x(x, y)}{\Delta x}\right]$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{yx}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y}\right]$$

(differentiate partially first with respect to x and then with respect to y)

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{xy}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}\right]$$

(differentiate partially first with respect to y and then with respect to x)

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_y(x, y + \Delta y) - f_y(x, y)}{\Delta y}\right]$$

if the limits exist. The derivatives f_{xy} and f_{yx} are called *mixed derivatives*. If f_{xy} and f_{yx} are continuous at a point $P(x, y)$, then at this point $f_{xy} = f_{yx}$. That is, the order of differentiation is immaterial in this case. There are four partial derivatives of second order for $f(x, y)$. If all the second order partial derivatives exist at all points in D, then these derivatives are also functions of x and y and can be further differentiated.

Example 2.22 Find all the second order partial derivatives of the function

$$f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x),\ (x, y) \neq (0, 0).$$

Solution We have

$$f_x(x, y) = \frac{2x}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{2x - y}{x^2 + y^2}$$

$$f_y(x, y) = \frac{2y}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(\frac{1}{x}\right) = \frac{2y + x}{x^2 + y^2}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(-1) - (2x - y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(1) - (2y + x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2x - y)(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2 + 2xy}{(x^2 + y^2)^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2y + x)(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2}$$

We note that $f_{xy} = f_{yx}$.

Example 2.23 For the function

$$f(x, y) = \begin{cases} \dfrac{xy(2x^2 - 3y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution We obtain the required derivatives as

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0,\ f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y[2(\Delta x)^2 - 3y^2]\Delta x}{[(\Delta x)^2 + y^2]\,\Delta x} = -3y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x[2x^2 - 3(\Delta y)^2]\Delta y}{[x^2 + (\Delta y)^2]\,\Delta y} = 2x.$$

Now,

$$f_{xy}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_{(0,0)} = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x - 0}{\Delta x} = 2$$

$$f_{yx}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-3\Delta y - 0}{\Delta y} = -3.$$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Example 2.24 Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at (0, 0).

Solution We have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0,\ f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y^3 \Delta x}{[\Delta x + y^2]\,\Delta x} = y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x(\Delta y)^3}{[x + (\Delta y)^2]\,\Delta y} = 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = 0$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1.$$

Since $f_{xy}(0, 0) \neq f_{yx}(0, 0)$, f_{xy} and f_{yx} are not continuous at (0, 0).

Alternative We find that for $(x, y) \neq (0, 0)$

$$f_{yx}(x,y) = \frac{y^6 + 5xy^4}{(x+y^2)^3} = f_{xy}(x,y).$$

Along the path $x = my^2$, we obtain

$$\lim_{(x,y)\to(0,0)} f_{yx}(x, y) = \lim_{y\to 0} \frac{y^6(1+5m)}{y^6(1+m)^3} = \frac{1+5m}{(1+m)^3}.$$

Since the limit does not exist, f_{yx} is not continuous at (0, 0).

Example 2.25 For the implicit function $f(x, y) = 0$ of one independent variable x, obtain $y'' = d^2y/dx^2$. Assume that $f_{xy} = f_{yx}$.

Solution Taking the differential of $f(x, y) = 0$, we obtain

$$y' = \frac{dy}{dx} = -\left(\frac{f_x}{f_y}\right).$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = -\frac{d}{dx}\left[\frac{f_x}{f_y}\right] = -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2}$$

$$= -\frac{f_y[f_{xx} + (f_{yx})y'] - f_x[f_{xy} + (f_{yy})y']}{f_y^2}$$

$$= -\frac{(f_y f_{xx} - f_x f_{xy}) + (f_y f_{yx} - f_x f_{yy})y'}{f_y^2}.$$

Substituting $y' = -f_x/f_y$, we obtain

$$\frac{d^2y}{dx^2} = -\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}, \text{ since } f_{yx} = f_{xy}.$$

2.4.1 Homogeneous Functions

A function $f(x, y)$ is said to be *homogeneous* of degree n in x and y, if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. (2.46)

(ii) $f(x, y) = x^n g(y/x)$. (2.47)

(iii) $f(x, y) = y^n g(x/y)$. (2.48)

Similarly, a function $f(x, y, z)$ of three variables is said to be homogeneous, of degree n, if it can be written as $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$, or $f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ etc.

Some examples of homogeneous functions are the following:

f	degree of homogeneity
$x^2 + xy$	2
$\tan^{-1}(y/x)$	0
$1/(x+y)$	-1
$1/(x^4 + y^4 + z^4)$	-4
$xyz/(x^4 + y^4 + z^4)$	-1
$\sqrt{x}/\sqrt{x^2 + y^2 + z^2}$	$-1/2$

The function $f(x, y) = (x^2 + y)/(x + y^2)$ is not homogeneous.

An important result concerning homogeneous functions is the following.

Theorem 2.4 **(Euler's theorem)** If $f(x, y)$ is a homogeneous function of degree n in x and y and has continuous first and second order partial derivatives, then

(i)
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{2.49}$$

(ii)
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f. \tag{2.50}$$

Proof Since $f(x, y)$ is a homogeneous function of degree n in x and y, we can write $f(x, y) = x^n g(y/x)$.

Differentiating partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1}g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = nx^{n-1}g\left(\frac{y}{x}\right) - yx^{n-2}g'\left(\frac{y}{x}\right).$$

$$\frac{\partial f}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1}g'\left(\frac{y}{x}\right).$$

Hence, we obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1}g'\left(\frac{y}{x}\right) + yx^{n-1}g'\left(\frac{y}{x}\right) = nx^n g\left(\frac{y}{x}\right) = nf.$$

Differentiating Eq. (2.49) partially with respect to x and y, we get

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x\,\partial y} = n\frac{\partial f}{\partial x} \tag{2.51}$$

and
$$x\frac{\partial^2 f}{\partial y\,\partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y}. \tag{2.52}$$

Multiplying Eq. (2.51) by x and Eq. (2.52) by y and adding, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2} + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + xy\left(\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y\,\partial x}\right) + y^2\frac{\partial^2 f}{\partial y^2} = n\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

or
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Example 2.26 If $u(x, y) = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0 < x, y < 1$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Solution For all x, y, $0 < x, y < 1$, $(x + y)/[\sqrt{x} + \sqrt{y}] < 1$, so that $u(x, y)$ is defined. The given function can be written as

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x[1+y/x]}{\sqrt{x}\,[1+\sqrt{y/x}]} = \sqrt{x}\left[\frac{1+(y/x)}{1+\sqrt{y/x}}\right]$$

Therefore, $\cos u$ is a homogeneous function of degree 1/2. Using the Euler's theorem for $f = \cos u$ and $n = 1/2$, we obtain

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

or $$-x(\sin u)\frac{\partial u}{\partial x} - y(\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u, \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Example 2.27 If $u(x, y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, $x > 0$, $y > 0$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

Solution We have $u(\lambda x, \lambda y) = \lambda^2 u(x, y)$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Theorem 2.4 (ii) for $f = u$ and $n = 2$, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u.$$

Example 2.28 Let $u(x, y) = [x^3 + y^3]/[x + y]$, $(x, y) \neq (0, 0)$. Then evaluate

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x}.$$

Solution We have $u(x, y) = \dfrac{x^2[1+(y/x)^3]}{[1+(y/x)]}$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

Differentiating partially with respect to x, we obtain

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x}, \text{ or } x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x} = 0.$$

Example 2.29 Let $f(x, y)$ and $g(x, y)$ be two homogeneous functions of degree m and n respectively where $m \neq 0$. Let $h = f + g$. If $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$, then show that $f = \alpha g$ for some scalar α.

Solution Since f and g are homogeneous functions of degrees m and n respectively, we obtain on using Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf \text{ and } x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng.$$

Adding the two results, we get

$$x\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + y\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) = mf + ng$$

or
$$x\frac{\partial h}{\partial x}+y\frac{\partial h}{\partial y}=mf+ng=0,\text{ where } h=f+g.$$

Therefore, $f=-\dfrac{n}{m}g=\alpha g$, where $\alpha=-\dfrac{n}{m}$ is a scalar.

2.4.2 Taylor's Theorem

In section 1.3.6 we have derived the Taylor's theorem in one variable. If $f(x)$ has continuous derivatives upto $(n+1)$th order in some interval containing $x=a$, then

$$f(x)=f(a)+(x-a)f'(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x) \tag{2.53}$$

where $R_n(x)$ is the remainder term given by

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],\ a<\xi<x,\ 0<\theta<1. \tag{2.54}$$

We now extend this theorem to functions of two variables.

Theorem 2.5 **(Taylor's theorem)** Let a function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ have continuous partial derivatives upto $(n+1)$th order in some neighborhood of a point $P(x_0, y_0)$ in D. Then, for some point (x_0+h, y_0+k) in this neighborhood, we have

$$f(x_0+h, y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$
$$+\ldots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n \tag{2.55}$$

where R_n is the remainder term given by

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,\ y_0+\theta k),\ 0<\theta<1. \tag{2.56}$$

Proof Let $x=x_0+th$, $y=y_0+tk$, where the parameter t takes values in the interval $[0, 1]$. Define a function $\phi(t)$ as $\phi(t)=f(x, y)=f(x_0+th, y_0+tk)$.

Using the chain rule, we get

$$\phi'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f$$

$$\phi''(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial f}{\partial y}\right)^2 f,\ldots,\ \phi^{(n+1)}(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f.$$

Using the Taylor's theorem for a function of one variable (see Eq. (2.53)) with $t=1$ and $a=0$, we obtain

$$\phi(1)=\phi(0)+\phi'(0)+\frac{1}{2!}\phi''(0)+\ldots+\frac{1}{n!}\phi^{(n)}(0)+\frac{1}{(n+1)!}\phi^{(n+1)}(\theta) \tag{2.57}$$

$$f(1.1, 0.8) \approx 0.7354 + \frac{1}{2}\left\{(0.01)\left(-\frac{1}{2}\right) + 2(0.1)(-0.2)(0) + (0.04)\left(-\frac{1}{2}\right)\right\}$$

$$= 0.7354 - 0.0125 = 0.7229.$$

The exact value of $f(1.1, 0.8)$ to four decimal places is 0.7217. Thus, the accuracy increases as the order of approximation increases.

Exercises 2.3

Find all the partial derivatives of the specified order for the following functions at the given point:

1. $f(x, y) = [x - y]/[x + y]$, second order at (1, 1).
2. $f(x, y) = x \ln y$, third order at (2, 3).
3. $f(x, y) = \ln [(1/x) - (1/y)]$, second order at (1, 2).
4. $f(x, y) = e^x \ln y + (\cos y) \ln x$, third order at $(1, \pi/2)$.
5. $f(x, y) = e^{\sin(x/y)}$, second order at $(\pi/2, 1)$.
6. $f(x, y, z) = [x + y]/[x + z]$, second order at (1, –1, 1).
7. $f(x, y, z) = e^{x^2 + y^2 + z^2}$, second order at (–1, 1, –1).
8. $f(x, y, z) = \sin xy + \sin xz + \sin yz$, second order at $(1, \pi/2, \pi/2)$.
9. $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$, second order at (1, 2, 3).
10. $f(x, y, z) = x^x y^y z^z$, $\dfrac{\partial^2 f}{\partial x\, \partial y}$ at any point $(x, y, z) \neq (0, 0, 0)$.
11. For the function $f(x, y) = \begin{cases} \dfrac{x^2 y(x - y)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

    show that $f_{xy} \neq f_{yx}$ at (0, 0).
12. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = x^y$.
13. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = \log\,[x + \sqrt{y^2 + x^2}\,]$.
14. Show that $f_{xyz} = f_{yzx}$ for all (x, y, z), when $f(x, y, z) = e^{xy} \sin z$.
15. Show that $f_{xyyz} = f_{yyxz}$ for all (x, y, z), when $f(x, y, z) = z^2 e^{x + y^2}$.
16. If $z = e^x \sin y + e^y \cos x$, where x and y are implicit functions of t defined by the equations $x^3 + x + e^t + t^2 + t - 1 = 0$ and $yt^3 + y^3 t + t + y = 0$, then find dz/dt at $t = 0$.
17. If x and y are defined as functions of u, v by the implicit equations $x^2 - y^2 + 2u^2 + 3v^2 - 1 = 0$ and $2x^2 - y^2 - u^2 + 4v^2 - 2 = 0$, then find $\partial x/\partial u$, $\partial y/\partial u$, $\partial^2 x/\partial u^2$ and $\partial^2 y/\partial u^2$.
18. If u and v are defined as functions of x and y by the implicit equations $4x^2 + 3y^2 - z^2 - u^2 + v^2 = 6$, $3x^2 - 2y^2 + z^2 + u^2 + 2v^2 = 14$, then find $(\partial u/\partial x)_{y,z}$ and $(\partial v/\partial y)_{x,z}$ at $x = 1$, $y = -1$, $z = 2$. Assume that $u > 0$, $v > 0$.
19. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$, c any constant, $|x| < 1$, $|y| < 1$, then find dy/dx and d^2y/dx^2.
20. Find dy/dx and d^2y/dx^2 at the point $(x, y) = (1, 1)$, for $e^y - e^x + xy = 1$.

21. If $z = u^v$, $u = (x/y)$, $v = xy$, then find $\partial^2 z/\partial x^2$.

22. If $u = \ln(1/r)$, $r = \sqrt{(x-a)^2 + (y-b)^2}$, then show that $u_{xx} + u_{yy} = 0$.

23. If $F = f(u, v)$, $u = y + ax$, $v = -y - ax$, a any constant, then show that $F_{xx} = a^2 F_{yy}$.

24. If $f(x, y) = x \log(y/x)$, $(x, y) \neq (0, 0)$, then show that $x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = 0$.

25. If $f(x, y) = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that $f_{xx} + f_{yy} = 0$.

26. Find α and β such that $u(x, y) = e^{\alpha x + \beta y}$ satisfies the equation $u_{xx} - 7u_{xy} + 12u_{yy} = 0$.

27. If $z = f(u, v)$, $u = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that
$z_{uu} + z_{vv} = (x^2 + y^2)^2 (z_{xx} + z_{yy})$.

28. If $x = r\cos\theta$, $v = r\sin\theta$, then show that

(i) $\dfrac{\partial^2 \theta}{\partial x\, \partial y} = -\dfrac{\cos 2\theta}{r^2}$, (ii) $\dfrac{\partial^2 r}{\partial x\, \partial y} = -\dfrac{\sin 2\theta}{2r}$.

Using Euler's theorem, establish the following results.

29. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

30. If $u = \log\left[\dfrac{\sqrt{x^2 + y^2}}{x}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

31. If $u = \sqrt{y^2 - x^2}\,\sin^{-1}\left(\dfrac{x}{y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

32. If $u = \dfrac{y^3 - x^3}{y^2 + x^2}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$ and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\, \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

33. If $\tan u = \dfrac{x^3 + y^3}{x - y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ and

$x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$.

34. Obtain the Taylor's series expansion of the maximum order for the function
$f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ about the point (2, 2).

35. Obtain the Taylor's linear approximation to the function $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point (–1, 1). Find the maximum error in the region $|x + 1| < 0.1$, $|y - 1| < 0.1$.

36. Obtain the first degree Taylor's series approximation to the function $f(x, y) = e^y \ln(x + y)$ about the point (1, 0). Estimate the maximum absolute error over the rectangle $|x - 1| < 0.1$, $|y| < 0.1$.

37. Obtain the second order Taylor's series approximation to the function $f(x, y) = xy^2 + y\cos(x - y)$ about the point (1, 1). Find the maximum absolute error in the region $|x - 1| < 0.05$, $|y - 1| < 0.1$.

38. Expand $f(x, y) = \sqrt{x + y}$ in Taylor's series upto second order terms about the point (1, 3). Estimate the maximum absolute error in the region $|x - 1| < 0.2$, $|y - 3| < 0.1$.

39. Obtain the Taylor's series expansion, upto third degree terms, of the function $f(x, y) = e^{2x+y}$ about the point (0, 0). Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.2$.

40. Expand $f(x, y) = \sin(x + 2y)$ in Taylor's series upto third order terms about the point (0, 0). Find the maximum error over the rectangle $|x| < 0.1$, $|y| < 0.1$.

41. Expand $f(x, y) = \sin x \sin y$ in Taylor's series upto second order terms about the point $(\pi/4, \pi/4)$. Find the maximum error in the region $|x - \pi/4| < 0.1$, $|y - \pi/4| < 0.1$.

42. Expand $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ in Taylor series upto first order terms about the point $(2, 2, 1)$. Obtain the maximum error in the region $|x - 2| < 0.1$, $|y - 2| < 0.1$, $|z - 1| < 0.1$.

43. Expand $f(x, y, z) = \sqrt{xy + yz + xz}$ in Taylor's series upto first order terms about the point $(1, 3, 3/2)$. Obtain the maximum error in the region $|x - 1| < 0.1$, $|y - 3| < 0.1$, $|z - 3/2| < 0.1$.

44. Expand $f(x, y, z) = e^z \sin(x + y)$ in Taylor's series upto second order terms about the point $(0, 0, 0)$. Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.1$, $|z| < 0.1$.

45. Expand $f(x, y, z) = e^x \sin(yz)$ in Taylor's series upto second order terms about the point $(0, 1, \pi/2)$. Obtain the maximum error in the region $|x| < 0.1$, $|y - 1| < 0.1$, $|z - \pi/2| < 0.1$.

2.5 Maximum and Minimum Values of a Function

Let a function $f(x, y)$ be defined and continuous in some closed and bounded region R. Let (a, b) be an interior point of R and $(a + h, b + k)$ be a point in its neighborhood and lies inside R. We define the following.

(i) The point (a, b) is called a point of *relative* (or *local*) *minimum*, if

$$f(a + h, b + k) \geq f(a, b) \tag{2.67a}$$

for all h, k. Then, $f(a, b)$ is called the *relative* (or *local*) *minimum* value.

(ii) The point (a, b) is called a point of *relative* (or *local*) *maximum*, if

$$f(a + h, b + k) \leq f(a, b) \tag{2.67b}$$

for all h, k. Then $f(a, b)$ is called the *relative* (or *local*) *maximum* value.

A function $f(x, y)$ may also attain its minimum or maximum values on the boundary of the region. The smallest and the largest values attained by a function over the entire region including the boundary are called the *absolute* (or *global*) *minimum* and *absolute* (or *global*) *maximum* values respectively.

The points at which minimum / maximum values of the function occur are also called *points of extrema* or the *stationary points* and the minimum and the maximum values taken together are called the *extreme values* of the function.

We now present the necessary conditions for the existence of an extremum of a functon.

Theorem 2.6 **(Necessary conditions for a function to have an extremum)** Let the function $f(x, y)$ be continuous and possess first order partial derivatives at a point $P(a, b)$. Then, the necessary conditions for the existence of an extreme value of f at the point P are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point $P(a, b)$. Then, P will be a point of maximum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \leq 0 \quad \text{for all} \quad h, k \tag{2.68}$$

and a point of minimum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \geq 0 \quad \text{for all} \quad h, k. \tag{2.69}$$

Using the Taylor's series expansion about the point (a, b), we obtain

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)_{(a,b)} + \frac{1}{2}[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{(a,b)} + \ldots \quad (2.70)$$

Neglecting the second and higher order terms, we get

$$\Delta f \approx hf_x(a, b) + kf_y(a, b). \quad (2.71)$$

The sign of Δf in Eq. (2.71) depends on the sign of $hf_x(a, b) + kf_y(a, b)$ which is a function of h and k. Letting $h \to 0$, we find that Δf changes sign with k. Therefore, the function cannot have an extremum unless $f_y = 0$. Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore, the necessary conditions for the existence of an extremum at the point (a, b) is that

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0. \quad (2.72)$$

A point $P(a, b)$, where $f_x(a, b) = 0$ and $f_y(a, b) = 0$ is called a *critical point* or a *stationary point*. A point P is also called a critical point when one or both of the first order partial derivatives do not exist at this point.

Remark 10

To find the minimum/maximum values of a function f, we first find all the critical points. We then examine each critical point to decide whether at this point the function has a minimum value or a maximum value using the sufficient conditions.

Theorem 2.7 (Sufficient conditions for a function to have a minimum/maximum) Let a function $f(x, y)$ be continuous and possess first and second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is a point of

$$\textit{relative minimum} \text{ if } rt - s^2 > 0 \text{ and } r > 0 \quad (2.73a)$$

$$\textit{relative maximum} \text{ if } rt - s^2 > 0 \text{ and } r < 0 \quad (2.73b)$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

No conclusion about an extremum can be drawn if $rt - s^2 = 0$ and further investigation is needed. If $rt - s^2 < 0$, then the function f has no minimum or maximum at this point. In this case, the point P is called a *saddle point*.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point $P(a, b)$. Since P is a critical point, we have $f_x(a, b) = 0$, and $f_y(a, b) = 0$. Neglecting the third and higher order terms in the Taylor's series expansion of $f(a + h, b + k)$ about the point (a, b), we get

$$\Delta f = f(a+h, b+k) - f(a, b) \approx \frac{1}{2}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

$$= \frac{1}{2}[h^2 r + 2hks + k^2 t] = \frac{1}{2r}[h^2 r^2 + 2hkrs + k^2 rt]$$

$$= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)]. \quad (2.74)$$

Since $(hr + ks)^2 > 0$, the sufficient condition for the expression $(hr + ks)^2 + k^2(rt - s^2)$ to be positive is that $rt - s^2 > 0$.

Hence, if $rt - s^2 > 0$, then

$$\Delta f > 0 \text{ if } r > 0 \quad \text{and} \quad \Delta f < 0 \text{ if } r < 0.$$

Therefore, a sufficient condition for the critical point $P(a, b)$ to be a point of relative minimum is $rt - s^2 > 0$ and $r > 0$

point of relative maximum is $rt - s^2 > 0$ and $r < 0$.

If $rt - s^2 < 0$, then the sign of Δf in Eq. (2.74) depends on h and k. Hence, no maximum/minimum of f can occur at $P(a, b)$ in this case.

If $rt - s^2 = 0$ or $r = t = s = 0$, no conclusion can be drawn and the terms involving higher order partial derivatives must be considered.

Remark 11

(a) We can also write Eq. (2.74) as

$$\Delta f = \frac{1}{2t}\,[k^2t^2 + 2hkst + h^2rt] = \frac{1}{2t}\,[(kt + hs)^2 + (rt - s^2)h^2].$$

Hence, a sufficient condition for a critical point $P(a, b)$ to be a

point of relative minimum is $rt - s^2 > 0$ and $t > 0$

point of relative maximum is $rt - s^2 > 0$ and $t < 0$.

From these conditions and Eqs. (2.73a, 2.73b), we find that when an extremum exists, then $rt - s^2 > 0$, and both r and t have the same sign either positive or negative.

(b) Alternate statement of Theorem 2.7

A real symmetric matrix $\mathbf{A} = (a_{ij})$ is called a positive definite matrix, if

$$\mathbf{x}^T \mathbf{A}\,\mathbf{x} > 0 \text{ for all real vectors } \mathbf{x} \neq \mathbf{0}$$

or
$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j > 0 \text{ for all } x_i, x_j \text{ (see section 3.5.3).}$$

A sufficient condition for the matrix $\mathbf{A}$ to be positive definite is that the minors of all its leading submatrices are positive. Now we state the result. Let

$$\mathbf{A} = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b) = f_{yx}(a, b)$ and $t = f_{yy}(a, b)$. Then, the function $f(x, y)$ has a relative minimum at a critical point $P(a, b)$, if the matrix $\mathbf{A}$ is positive definite. Since all the leading minors of $\mathbf{A}$ are positive, we obtain the conditions $r > 0$ and $rt - s^2 > 0$.

The function $f(x, y)$ has a relative maximum at $P(a, b)$, if the matrix $\mathbf{B} = -\mathbf{A} = \begin{bmatrix} -r & -s \\ -s & -t \end{bmatrix}$ is positive definite. Since all the leading minors of $\mathbf{B}$ are positive, we obtain the conditions $-r > 0$ and $rt - s^2 > 0$, that is $r < 0$ and $rt - s^2 > 0$.

This alternative statement of the Theorem 2.7 is useful when we consider the extreme values of the functions of three or more variables. For example, for the function $f(x, y, z)$ of three variables, we have

$$\mathbf{A} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

where $f_{yx} = f_{xy}, f_{zx} = f_{xz}, f_{zy} = f_{yz}$. The matrix **A** or the matrix **B** = – **A** can be tested whether it is positive definite, to find the points of minimum/maximum. Therefore, a critical point (a point at which $f_x = 0 = f_y = f_z$)

(i) is a point of relative minimum if **A** is positive definite and f_{xx}, f_{yy}, f_{zz} are all positive.

(ii) is a point of relative maximum if **B** = – **A** is positive definite (that is, the leading minors of **A** are alternately negative and positive) and f_{xx}, f_{yy}, f_{zz} are all negative.

Example 2.34 Find the relative maximum and minimum values of the function

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4.$$

Solution We have

$$f_x = 4x - 4x^3 = 0, \text{ or } x = 0, \pm 1$$

$$f_y = -4y + 4y^3 = 0, \text{ or } y = 0, \pm 1.$$

Hence, (0, 0), (0, ±1), (±1, 0), (±1, ±1) are the critical points. We find that

$$r = f_{xx} = 4 - 12x^2, \quad s = f_{xy} = 0, \quad t = f_{yy} = -4 + 12y^2$$

and
$$rt - s^2 = -16(1 - 3x^2)(1 - 3y^2).$$

At the points (0, 1) and (0, –1), we have $rt - s^2 = 32 > 0$ and $r = 4 > 0$. Therefore, the points (0, 1) and (0, –1) are points of relative minimum and the minimum value at each point is –1.

At the points (–1, 0) and (1, 0), we have $rt - s^2 = 32 > 0$ and $r = -8 < 0$. The points (–1, 0) , (1, 0) are points of relative maximum and the maximum value at each point is 1.

At (0, 0), we have $rt - s^2 = -16 < 0$. At (±1, ±1), we have $rt - s^2 = -64 < 0$. Hence, the points (0, 0) (±1, ±1) are neither the points of maximum nor minimum.

Example 2.35 Find the absolute maximum and minimum values of

$$f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the first quadrant bounded by the lines $x = 2$, $y = 3$ and the coordinate axes.

Solution The function f can attain maximum/minimum values at the critical points or on the boundary of the rectangle *OABC* (Fig. 2.4).

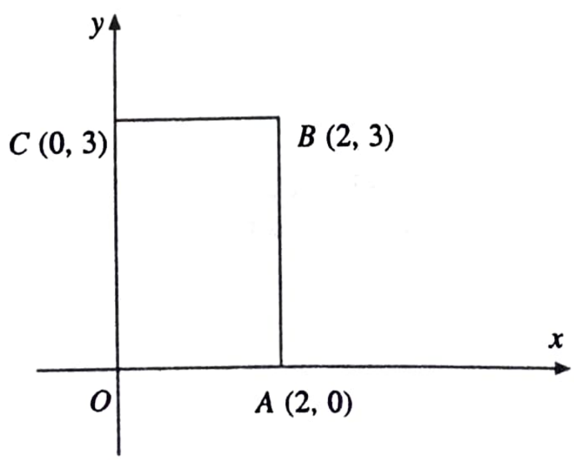

Fig. 2.4. Region in Example 2.35.

We have $f_x = 8x - 8 = 0$, $f_y = 18y - 12 = 0$. The critical point is $(x, y) = (1, 2/3)$. Now,

$$r = f_{xx} = 8, \quad s = f_{xy} = 0, \quad t = f_{yy} = 18, \quad rt - s^2 = 144.$$

Since $rt - s^2 > 0$ and $r > 0$, the point $(1, 2/3)$ is a point of relative minimum. The minimum value is $f(1, 2/3) = -4$.

On the boundary line OA, we have $y = 0$ and $f(x, y) = f(x, 0) = g(x) = 4x^2 - 8x + 4$, which is a function of one variable. Setting $dg/dx = 0$, we get $8x - 8 = 0$ or $x = 1$. Now $d^2g/dx^2 = 8 > 0$. Therefore, at $x = 1$, the function has a minimum. The minimum value is $g(1) = 0$. Also, at the corners $(0, 0)$, $(2, 0)$, we have $f(0, 0) = g(0) = 4$, $f(2, 0) = g(2) = 4$.

Similarly, along the other boundary lines, we have the following results:

$x = 2$: $h(y) = 9y^2 - 12y + 4$; $dh/dy = 18y - 12 = 0$ gives $y = 2/3$; $d^2h/dy^2 = 18 > 0$. Therefore, $y = 2/3$ is a point of minimum. The minimum value is $f(2, 2/3) = 0$. At the corner $(2, 3)$, we have $f(2, 3) = 49$.

$y = 3$: $g(x) = 4x^2 - 8x + 49$; $dg/dx = 8x - 8 = 0$ gives $x = 1$; $d^2g/dx^2 = 8 > 0$. Therefore, $x = 1$ is a point of minimum. The minimum value is $f(1, 3) = 45$. At the corner point $(0, 3)$, we have $f(0, 3) = 49$.

$x = 0$: $h(y) = 9y^2 - 12y + 4$, which is the same case as for $x = 2$.

Therefore, the absolute minimum value is -4 which occurs at $(1, 2/3)$ and the absolute maximum value is 49 which occurs at the points $(2, 3)$ and $(0, 3)$.

Example 2.36 Find the absolute maximum and minimum values of the function

$$f(x, y) = 3x^2 + y^2 - x \text{ over the region } 2x^2 + y^2 \le 1.$$

Solution We have $f_x = 6x - 1 = 0$ and $f_y = 2y = 0$. Therefore, the critical point is $(x, y) = (1/6, 0)$.

Now,
$$r = f_{xx} = 6, \quad s = f_{xy} = 0, \quad t = f_{yy} = 2, \quad rt - s^2 = 12 > 0.$$

Therefore, $(1/6, 0)$ is a point of minimum. The minimum value at this point is $f(1/6, 0) = -1/12$.

On the boundary, we have $y^2 = 1 - 2x^2$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$. Substituting in $f(x, y)$, we obtain

$$f(x, y) = 3x^2 + (1 - 2x^2) - x = 1 - x + x^2 = g(x)$$

which is a function of one variable. Setting $dg/dx = 0$, we get

$$\frac{dg}{dx} = 2x - 1 = 0, \text{ or } x = \frac{1}{2}. \text{ Also } \frac{d^2g}{dx^2} = 2 > 0.$$

For $x = 1/2$, we get $y^2 = 1 - 2x^2 = 1/2$ or $y = \pm 1/\sqrt{2}$. Hence, the points $(1/2, \pm 1/\sqrt{2})$ are points of minimum. The minimum value is $f(1/2, \pm 1/\sqrt{2}) = 3/4$. At the vertices, we have $f(1/\sqrt{2}, 0) = (3 - \sqrt{2})/2$, $f(-1/\sqrt{2}, 0) = (3 + \sqrt{2})/2$, $f(0, \pm 1) = 1$. Therefore, the given function has absolute minimum value $-1/12$ at $(1/6, 0)$ and absolute maximum value $(3 + \sqrt{2})/2$ at $(-1/\sqrt{2}, 0)$.

Example 2.37 Find the relative maximum/minimum values of the function

$$f(x, y, z) = x^4 + y^4 + z^4 - 4xyz.$$

Solution We have

$$f_x = 4x^3 - 4yz = 0, \quad f_y = 4y^3 - 4xz = 0, \quad f_z = 4z^3 - 4xy = 0.$$

Therefore,
$$x^3 = yz, \quad y^3 = xz, \quad z^3 = xy \quad \text{or} \quad x^3y^3z^3 = x^2y^2z^2 \quad \text{or} \quad x^2y^2z^2(xyz - 1) = 0.$$

Therefore, all points which satisfy $xyz = 0$ or $xyz = 1$ are critical points. The solutions of these equations are (0, 0, 0), (1, 1, 1), (±1, ±1, 1), (1, ±1, ±1), (±1, 1, ±1) with the same sign taken for the two coordinates. Now,

$$f_{xx} = 12x^2,\ f_{yy} = 12y^2,\ f_{zz} = 12z^2,\ f_{xy} = -4z,\ f_{xz} = -4y,\ f_{yz} = -4x.$$

At (0, 0, 0), all the second order partial derivatives are zero. Therefore, no conclusion can be drawn.

We have
$$\mathbf{A} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{bmatrix}$$

Depending on whether **A** or **B** = – **A** is positive definite, we can decide the points of minimum or maximum. The leading minors are

$$M_1 = 12x^2,\ M_2 = \begin{vmatrix} 12x^2 & -4z \\ -4z & 12y^2 \end{vmatrix} = 16\,(9x^2y^2 - z^2)$$

and
$$M_3 = |\,\mathbf{A}\,| = 192x^2\,(9y^2z^2 - x^2) - 192z^4 - 64\,xyz - 64x\,yz - 192\,y^4$$
$$= 192\,[9x^2y^2z^2 - (x^4 + y^4 + z^4)] - 128xyz.$$

At all points (1, 1, 1), (±1, ±1, 1), (±1, 1, ±1), (1, ±1, ±1) with the same sign taken for two coordinates, we find that $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$. Hence, **A** is a positive definite matrix and the given function has relative minimum at all these points, since $f_{xx} > 0,\ f_{yy} > 0,$ and $f_{zz} > 0$. The relative minimum value at all these points is same and is given by $f(1, 1, 1) = -1$.

Conditional maximum/minimum

In many practical problems, we need to find the maximum/minimum value of a function $f(x_1, x_2, \ldots, x_n)$ when the variables are not independent but are connected by one or more constraints of the form

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ \ i = 1, 2, \ldots, k$$

where generally $n > k$. We present the Lagrange method of multipliers to find the solution of such problems.

2.5.1 Lagrange Method of Multipliers

We want to find the extremum of the function $f(x_1, x_2, \ldots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ \ i = 1, 2, \ldots, k. \tag{2.75}$$

We construct an auxiliary function of the form

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i\, \phi_i(x_1, x_2, \ldots, x_n) \tag{2.76}$$

where λ_i's are undetermined parameters and are known as *Lagrange multipliers*. Then, to determine the stationary points of F, we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_n}$$

which give the equations

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0, \; j = 1, 2, \ldots, n. \tag{2.77}$$

From Eqs. (2.75) and (2.77), we obtain $(n + k)$ equations in $(n + k)$ unknowns $x_1, x_2, \ldots, x_n$, $\lambda_1, \lambda_2, \ldots, \lambda_k$. Solving these equations, we obtain the required stationary points $(x_1, x_2, \ldots, x_n)$ at which the function f has an extremum. Further investigation is needed to determine the exact nature of these points.

Example 2.38 Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda (xyz - a^3) .$$

We obtain the necessary conditions for extremum as

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0, \quad \frac{\partial F}{\partial y} = 2y + \lambda xz = 0, \quad \frac{\partial F}{\partial z} = 2z + \lambda xy = 0.$$

From these equations, we obtain

$$\lambda yz = -2x \text{ or } \lambda xyz = -2x^2$$
$$\lambda xz = -2y \text{ or } \lambda xyz = -2y^2$$
$$\lambda xy = -2z \text{ or } \lambda xyz = -2z^2.$$

Therefore, $x^2 = y^2 = z^2$. Using the condition $xyz = a^3$, we obtain the solutions as (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$. At each of these points, the value of the given function is $x^2 + y^2 + z^2 = 3a^2$.

Now, the arithmetic mean of x^2, y^2, z^2 is $AM = (x^2 + y^2 + z^2)/3$

the geometric mean of x^2, y^2, z^2 is $GM = (x^2y^2z^2)^{1/3} = a^2$.

Since, $AM \geq GM$, we obtain $x^2 + y^2 + z^2 \geq 3a^2$.

Hence, all the above points are the points of constrained minimum and the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

Example 2.39 Find the extreme values of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = 2x + 3y + z + \lambda_1 (x^2 + y^2 - 5) + \lambda_2 (x + z - 1).$$

For the extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0; \quad \frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0, \quad \frac{\partial F}{\partial z} = 1 + \lambda_2 = 0.$$

From these equations, we get

$$\lambda_2 = -1, \; 3 + 2\lambda_1 y = 0 \text{ and } 1 + 2\lambda_1 x = 0$$

or

$$x = -1/(2\lambda_1) \text{ and } y = -3/(2\lambda_1).$$

Substituting in the constraint $x^2 + y^2 = 5$, we get

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5 \quad \text{or} \quad \lambda_1^2 = \frac{1}{2} \quad \text{or} \quad \lambda_1 = \pm\frac{1}{\sqrt{2}}.$$

For $\lambda_1 = 1/\sqrt{2}$, we get $x = -\sqrt{2}/2,\ y = -3\sqrt{2}/2,\ z = 1 - x = (2 + \sqrt{2})/2$

and $f(x, y, z) = -\sqrt{2} - \dfrac{9\sqrt{2}}{2} + \dfrac{2+\sqrt{2}}{2} = \dfrac{2 - 10\sqrt{2}}{2} = 1 - 5\sqrt{2}.$

For $\lambda_1 = -1/\sqrt{2}$, we get $x = \sqrt{2}/2,\ y = 3\sqrt{2}/2,\ z = 1 - x = (2 - \sqrt{2})/2$.

and $f(x, y, z) = \sqrt{2} + \dfrac{9\sqrt{2}}{2} + \dfrac{2-\sqrt{2}}{2} = \dfrac{2 + 10\sqrt{2}}{2} = 1 + 5\sqrt{2}.$

Example 2.40 Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $(x^2/4) + (y^2/9) = 1$.

Solution Let (x, y) be a point on the ellipse and (u, v) be a point on the line. Then, the shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2$$

subject to the constraints

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad \text{and} \quad \phi_2(u, v) = 2u + v - 10 = 0.$$

We define the auxiliary function as

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10).$$

For extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2(x - u) + \frac{x}{2}\lambda_1 = 0, \quad \text{or} \quad \lambda_1 x = 4(u - x)$$

$$\frac{\partial F}{\partial y} = 2(y - v) + \frac{2y}{9}\lambda_1 = 0, \quad \text{or} \quad \lambda_1 y = 9(v - y)$$

$$\frac{\partial F}{\partial u} = -2(x - u) + 2\lambda_2 = 0, \quad \text{or} \quad \lambda_2 = x - u$$

$$\frac{\partial F}{\partial v} = -2(y - v) + \lambda_2 = 0, \quad \text{or} \quad \lambda_2 = 2(y - v).$$

Eliminating λ_1 and λ_2 from the above equations, we get

$$4(u - x)y = 9(v - y)x \quad \text{and} \quad x - u = 2(y - v).$$

Dividing the two equations, we obtain $8y = 9x$. Substituting in the equation of the ellipse, we get

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1, \quad \text{or} \quad x^2 = \frac{64}{25}.$$

Therefore, $x = \pm 8/5$ and $y = \pm 9/5$. Corresponding to $x = 8/5$, $y = 9/5$, we get

$$\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right), \quad \text{or} \quad 2v - u = 2, \quad \text{or} \quad u = 2v - 2.$$

Substituting in the equation of the line $2u + v - 10 = 0$, we get $u = 18/5$ and $v = 14/5$. Hence, an extremum is obtained when $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$. The distance between the two points is $\sqrt{5}$.

Corresponding to $x = -8/5$, $y = -9/5$, we get $u - 2v = 2$. Substituting in the equation $2u + v - 10 = 0$, we obtain $u = 22/5$, $v = 6/5$. Hence, another extremum is obtained when $(x, y) = (-8/5, -9/5)$ and $(u, v) = (22/5, 6/5)$. The distance between these two points is $3\sqrt{5}$.

Hence, the shortest distance between the line and the ellipse is $\sqrt{5}$.

Exercise 2.4

Test the following functions for relative maximum and minimum.

1. $xy + (9/x) + (3/y)$.

2. $\sqrt{a^2 - x^2 - y^2}$ $a > 0$.

3. $x^2 + 2bxy + y^2$.

4. $x^2 + xy + y^2 + (1/x) + (1/y)$.

5. $x^2 + 2/(x^2 y) + y^2$.

6. $\cos 2x + \cos y + \cos(2x + y)$, $0 < x, y < \pi$.

7. $4x^2 + 4y^2 - z^2 + 12xy - 6y + z$.

8. $18xz - 6xy - 9x^2 - 2y^2 - 54z^2$.

9. $x^4 + y^4 + z^4 + 4xyz$.

10. $2\ln(x + y + z) - (x^2 + y^2 + z^2)$, $x + y + z > 0$.

Find the relative and absolute maximum and minimum values for the following functions in the given closed region R in problemes **11** to **20**.

11. $x^2 - y^2 - 2y$, R: $x^2 + y^2 \le 1$.

12. xy, R: $x^2 + y^2 \le 1$.

13. $x + y$, R: $4x^2 + 9y^2 \le 36$.

14. $4x^2 + y^2 - 2x + 1$, R: $2x^2 + y^2 \le 1$.

15. $x^2 + y^2 - x - y + 1$, R: rectangular region; $0 \le x \le 2$, $0 \le y \le 2$.

16. $2x^2 + y^2 - 2x - 2y - 4$, R: triangular region bounded by the lines $x = 0$, $y = 0$ and $2x + y = 1$.

17. $x^3 + y^3 - xy$, R: triangular region bounded by the lines $x = 1$, $y = 0$ and $y = 2x$.

18. $4x^2 + 2y^2 + 4xy - 10x - 2y - 3$, R: rectangular region; $0 \le x \le 3$, $-4 \le y \le 2$.

19. $\cos x + \cos y + \cos(x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.

20. $\cos x \cos y \cos(x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.

21. Show that the necessary condition for the existence of an extreme value of $f(x, y)$ such that $\phi(x, y) = 0$ is that x, y satisfy the equation $f_x \phi_y - f_y \phi_x = 0$.

22. Find the smallest and the largest value of xy on the line segment $x + 2y = 2$, $x \ge 0$, $y \ge 0$.

23. Find the smallest and the largest value of $x + 2y$ on the circle $x^2 + y^2 = 1$.

24. Find the smallest and the largest value of $2x - y$ on the curve $x - \sin y = 0$, $0 \le y \le 2\pi$.

25. Find the extreme value of $x^2 + y^2$ when $x^4 + y^4 = 1$.

26. Find the points on the curve $x^2 + xy + y^2 = 16$, which are nearest and farthest from the origin.

27. Find the rectangle of constant perimeter whose diagonal is maximum.

28. Find the triangle whose perimeter is constant and has largest area.

29. Find a point on the plane $Ax + By + cz = D$ which is nearest to origin.

30. Find the extreme value of xyz, when $x + y + z = a$, $a > 0$.

31. Find the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ such that $x^{-1} + y^{-1} + z^{-1} = 1$, where $a > 0$, $b > 0$, $c > 0$.
32. Find the extreme value of $x^p + y^p + z^p$ on the surface $x^q + y^q + z^q = 1$, where $0 < p < q$, $x > 0$, $y > 0$, $z > 0$.
33. Find the extreme value of $x^3 + 8y^3 + 64z^3$, when $xyz = 1$.
34. Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to the coordinate axes that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.
35. Divide a number into three parts such that the product of the first, square of the second and cube of the third is maximum.
36. Find the dimensions of a rectangular parallelopiped of fixed total edge length with maximum surface area.
37. Find the dimensions of a rectangular parallelopiped of greatest volume having constant surface area S.
38. A rectangular box without top is to have a given volume. How should the box be made so as to use the least material.
39. Find the dimensions of a right circular cone of fixed lateral area with minimum volume.
40. A tent is to be made in the form of a right circular cylinder surmounted by a cone. Find the ratios of the height H of the cylinder and the height h of the conical part to the radius r of the base, if the volume V of the tent is maximum for a given surface area S of the tent.
41. Find the maximum value of xyz under the constraints $x^2 + z^2 = 1$ and $y - x = 0$.
42. Find the extreme value of $x^2 + 2xy + z^2$ under the constraints $2x + y = 0$ and $x + y + z = 1$.
43. Find the extreme value of $x^2 + y^2 + z^2 + xy + xz + yz$ under the constraints $x + y + z = 1$ and $x + 2y + 3z = 3$.
44. Find the points on the ellipse obtained by the intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + y^2 + 2z^2 = 1$ which are nearest and farthest from the origin.
45. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.

2.6 Multiple Integrals

In the previous chapter, we studied methods for evaluating the definite integral $\int_a^b f(x)dx$, where the integrand $f(x)$ is piecewise continuous on the interval $[a, b]$. In this section, we shall discuss methods for evaluating the double and triple integrals, that is integrals of the forms

$$\iint_R f(x, y)dx\,dy \quad \text{and} \quad \iiint_T f(x, y, z)dx\,dy\,dz.$$

We assume that the integrand f is continuous at all points inside and on the boundary of the region R or T. These integrals are called *multiple integrals*. The multiple integral over $\mathbb{R}^n$ is written as

$$\iint_R \cdots \int f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n.$$

2.6.1 Double Integrals

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve C (Fig. 2.5).