Unit 1P3 Non-Homogeneous differential equation

Engineering Mathematics (Lovely Professional University)

ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be | A |.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Than $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

The matrix formed by the co-factors of the elements in

$$|A| \text{ is } \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

where $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$, $A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1c_3 + b_3c_1$

$A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$, $B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2c_3 + a_3c_2$

$B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$, $B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1c_2 + a_2c_1$

$C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$, $C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1b_3 + a_3b_1$

$$C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Then the transpose of the matrix of co-factors

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}^T$ is called the adjoint of the matrix A and is written as adj A.

Ex. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

PROPERTY OF ADJOINT MATRIX

(1) $A.adj(A) = adj(A).A = |A|.I_n$ where, A is a square matrix, I is an identity matrix of same order.

(2) if A is invertible square matrix

$$|adj(A)| = |A|^{n-1}$$

(3) if A is invertible square matrix

$$adj(adj(A)) = |A|^{n-2}.A$$

if $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is adjoint of 3×3 matrix with $|A| = 4$ then the value of α is

a. 2
b. 11
c. 13
d. 1/3

$|adj A| = |A|^{n-1}$

$1(0) - \alpha(-2) + 3(-2) = (4)^2$

$2\alpha - 6 = 16 \Rightarrow 2\alpha = 22$

$\alpha = 11$

(4) $adj(AB) = adj(B).adj(A)$

(5) $(adj(A))^T = adj(A^T)$

(6) $adj(kA) = k^{n-1} adj(A)$

INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix) then B is called the inverse of A i.e. B = A^{-1} and A is the inverse of B.

To find the inverse of the Matrix A we use

$$A^{-1} = \frac{1}{|A|}(Adj(A)), \qquad if\ |A| \neq 0$$

Properties of inverse of the matrix

1. Inverse of the matrix is unique
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is an invertible square matrix; Then $(A)^T$ is also invertible and $(A^T)^{-1} = (A^{-1})^T$
4. The inverse of an invertible symmetric matrix is a symmetric matrix.
5. $|A^{-1}| = |A|^{-1}$ $i.e.$ $|A^{-1}| = \frac{1}{|A|}$

Solution of $n \times n$ linear system of equation

Consider the system of n equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n.
\end{aligned}
$$

In matrix form we can write this system as $Ax = b$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

Note:

1. A is the coefficient matrix, b the right hand side, and x is the solution vector.

2. If b not equal to zero system is called non-homogeneous.

3. If b is zero its call homogeneous.

4. The system of equations is called consistent if it has at least one solution.

   otherwise the system is inconsistent.

Homogeneous system of equations:

Consider the homogeneous system of equations $Ax = 0$
Trivial solution $x = 0$ is always a solution of this system.

If A is non singular, then $x = A^{-1}0 = 0$ is the solution.

Thus $Ax = 0$ is the always consistent .

We conclude that non-trivial solution for $Ax = 0$ exist if and only if A is singular, in this case this system has infinite solutions.

Ex. Solve the system of the equation using matrix method $x - y + z = 0,\quad 2x + y - 3z = 0,\quad x + y + z = 0$

Ex. If the system of the equations $x - ky - z = 0, kx - y - z = 0, x + y - z = 0$ has non-zero solution then values of K are

a. -1,2

b. 0,1

c. 1,1

d. -1,1

Ex. If the system of the equations k$x + y + z = 0, -x + ky + z = 0, -x - y + kz = 0$ has non-zero solution then value of K is

a. 0

b. 1

c. -1

d. 2

Solution of Non-homogeneous system of equations

The non-homogeneous system of equations $Ax = b$ can be solved by the following methods

(i) Matrix method

(ii) Cramer's Rule

(i) Matrix method:

Let A be non-singular, then pre-multiplying $Ax = b$ by

A^{-1}, we obtain
$x = A^{-1}b$

Ex. Solve the system of the equation using matrix method $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$

Ex. Solve the system of the equation using matrix method $-x + y + 2z = 2$, $3x - y + z = 3$, $-x + 3y + 4z = 6$

Ex. Solve the system of the equation using matrix method 2x-z=1, 5x+y=7, y+3z=5

(ii) Cramer's Rule:

(ii) Cramer's Rule:

Let A be a non-singular matrix then by Cramer's rule solution of Ax=b is given by

$$x_i = \frac{|A_i|}{|A|}, \qquad i = 1,2,3,\dots,n$$

Where $|A_i|$ is the determinant of the matrix $|A_i|$ obtained by replacing the ith column of A by the right hand side column vector b.

Ex. Solve the system of the equation using $x - y + z = 4$,
$2x + y - 3z = 0$, $x + y + z = 2$

Note: We have the following cases in this method

Case 1: when $|A| \neq 0$, the system is consistent and the unique solution is obtained by using the above method.

Case 2: When $|A| = 0$, and one or more of $|A_i|, i = 1,2,3,\dots,n$

are not zero then the system of the equations has no solution that is the system is inconsistent.

Case 3: When $|A| = 0$, and all $|A_i| = 0$, $\quad i = 1,2,3, \dots, n,$ then the system of equations is consistent and has infinite number of solutions . The system of equations has at least a one-parameter family of solutions.

Ex. Solve the system of the equation using $4x + 9y + 3z = 6,$
$2x + 3y + z = 2,\ 2x + 6y + 2z = 7$

Ex. Solve the system of the equation using matrix method
$x - y + 3z = 3, \qquad 2x + 3y + z = 2, \qquad 3x + 2y + 4z = 5$

Ex. The system of linear equations $x + y + z = 2, 2x + 3y + 2z = 5, 2x + 3y + (a^2 - 1)z = a + 1$

a. Is inconsistent for $a = 4$

b. Has unique solution for $a = \sqrt{3}$

c. Has infinite solution for $a = 4$

b. Has unique solution for $a = \sqrt{3}$
c. Has infinite solution for $a = 4$
d. Inconsistent for $a = \sqrt{3}$

Ex.

If the system of linear equation x- 4y+7z = g, 3y-5z= h, -2x +5y -9z = k is consistent, then:

A g+h+k=0

B 2g+h +k= 0

C g+h+2k=0

D g+2h+2k=0

(iii)Gauss elimination Method for Non-homogeneous System

Let we have the non-homogeneous system Ax=b where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Now we write the augmented matrix of order $m \times (n+1)$

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Now we can reduce this matrix in to row echelon form using elementary operations

$$
(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccccc|c}
a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & b_1 \\
0 & \bar{a}_{22} & \cdots & \bar{a}_{2r} & \cdots & \bar{a}_{2n} & \bar{b}_2 \\
\vdots & & & & & & \vdots \\
0 & 0 & \cdots & a_{rr}^* & \cdots & a_{rn}^* & b_r^* \\
0 & 0 & \cdots & 0 & \cdots & 0 & b_{r+1}^* \\
\vdots & & & & & & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & b_m^*
\end{array}\right].
$$

Ex. Solve following system using gauss elimination

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix},$

Note:

1. Let $r < m$ and one or more elements $b^*_{r+1}, b^*_{r+2}, \dots, b^*_m$ are not zero. Then $rank(A) \neq rank(A|b)$ and the system of equation has no solution .
2. Let $m \geq n$ and $r = n$ (the number of columns in A) and $b^{*}{}_{+1}, b^{*}{}_{+2}, \dots, b^{*}{}_{\imath}$ are all zero. In this case $rank(A) = rank(A|b) = n$ and the system of equations has unique solution.
3. Let $r < n$ and $b^*_{r+1}, b^*_{r+2}, \dots, b^*_m$are all zero. In this case $x_1, x_2, \dots, x_r$ can be determined in term of remaining (n-r) unknowns $x_{r+1}, x_{r+2}, \dots, x_n$.

(ii) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$,

$$\text{(iii)} \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

(iv) Gauss-Jordan method

Ex. Using gauss-Jordan method solve the system of equations Ax=b, where

$$[\mathbf{A} \mid \mathbf{b}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{c}]$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -3 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -5 & 4 \\ 0 & 2 & 0 & 1 \end{array}\right] R_2/3$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ R_3 - 2R_2 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 10/3 & -5/3 \end{array}\right] R_3/(10/3)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 4/3 \\ 0 & 1 & -5/3 & 4/3 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \begin{array}{l} R_1 + 2R_3/3 \\ R_2 + 5R_3/3 \end{array} \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right].$$

$$\mathbf{x} = [1 \quad 1/2 \quad -1/2]^T.$$

Ex. Using Gauss-Jordan method find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{I}] \xrightarrow[\text{row operations}]{\text{Elementary}} [\mathbf{I} \mid \mathbf{A}^{-1}]$$