Tutorial Classes V 11 - fourier series

Engineering Mathematics (Lovely Professional University)

TUTORIAL NOTES OF ENGINEERING MATHEMATICS

Tutorial notes for Graduation/Post-graduation Students

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Dedicated to students

The author is not responsible for any kind loss due to mistakes in the text.

Contents

Chapter 1

Matrix Algebra

1.1 Elementary Row Operation

Definition 1.1.1

There are three basic operations on rows of a matrix:

1. Interchange of any two rows. $(R_i \leftrightarrow R_j)$.
2. Multiplication of all the elements of a row by a non-zero element. $(R_i \to kR_j, k \neq 0)$.
3. The addition to the elements of any row, the corresponding elements of any other row multiplied by any number$(R_i \to R_i + kR_j)$.

1.1.1 Practice Problems Based on Elementary Row operations

1. Find the determinant of following matrices

   (a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

2. Find the inverse of the following matrices using elementary row transformations:

   (a) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & 3 & -1 \\ 4 & 2 & 0 \\ 4 & 6 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & 3 & -1 \\ 4 & 2 & 0 \\ 4 & 6 & 2 \end{bmatrix}$

1.1.2 Hints 1.1.1

1. (a) 0

   (b) 88

2. (a) $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

   (c) $\begin{bmatrix} -1/15 & 1/5 & -1/30 \\ 2/15 & 1/10 & 1/15 \\ -4/15 & -7/10 & 11/30 \end{bmatrix}$

   (d) $\begin{bmatrix} 1/7 & 3/14 & -1/14 \\ -5/7 & 3/7 & -1/7 \\ -5/7 & -1/14 & 5/14 \end{bmatrix}$

1.2 Echelon Form and Rank of Matrix

Definition 1.2.1

A number r is called ***rank*** of a matrix A if

1. There exists atleast one minor of order r of A which does not vanish.
2. Every minor of order $r+1$, if any vanishes.

The rank of a matrix A is denoted by $\rho(A)$.

In other words, we can say that the rank of a matrix A is the largest order of any non-vanishing minor of the matrix.

Definition 1.2.2: Echelon Form

A matrix $A = [a_{ij}]$ is said to be in ***echelon form*** if

1. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.
2. The first non-zero entry in non-zero row is one.
3. The zero rows of A occurs below all the non-zero rows of A.

The rank of a matrix A is denoted by $\rho(A)$ = number of non-zero rows in echelon form of a given matrix. **An important remark here that first non zero entry in each row need NOT to be 1 for finding rank.** (Echelon type is suffiecient)

1.2.1 Problems of Finding Rank

1. Find x so that rank of the matrix $A = \begin{bmatrix} x & 0 & 1 \\ 1 & 2 & x \\ 1 & 2 & 3 \end{bmatrix}$ is less than 3. Also find the rank for these values of x.

2. Find the rank of the following matrices: (Try to apply both methods and see whether your answer is same!)

   (a) $A_1 = \begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -4 & 1 & 2 \\ 5 & 2 & 1 & 3 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 0 & 6 & 6 & 1 \\ -8 & 7 & 2 & 3 \\ -2 & 3 & 0 & 1 \\ -3 & 2 & 1 & 1 \end{bmatrix}$

   (c) $A_3 = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 6 & 3 & 0 & -7 \\ 3 & 1 & 3 & -2 \end{bmatrix}$

   (d) $A_4 = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

   (e) $A_5 = \begin{bmatrix} 0 & 1 & 3 & -1 & 4 \\ 2 & 0 & -4 & 1 & 2 \\ 1 & 4 & 2 & 0 & -1 \\ 3 & 4 & -2 & 1 & 1 \\ 6 & 9 & -1 & 1 & 6 \end{bmatrix}$

3. Convert the following matrices into echelon form also in normal form...

   (a) $\begin{bmatrix} 2 & 2 & -1 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 4 & -4 & 6 & 6 \\ 2 & -2 & 3 & 3 \\ 1 & -1 & 1 & 5 \end{bmatrix}$

1.2.2 Hints 1.2.1

1. $x = 0, 3$ rank is 2 for $x = 0, 3$.
2. (a) $\rho(A_1) = 3$. (b) $\rho(A_2) = 3$.
   (c) $\rho(A_3) = 3$. (d) $\rho(A_4) = 3$.
   (e) $\rho(A_5) = 3$.
3. (a) rank $= 3$ (b) rank $= 3$

1.3 Solving Linear system of Equations

1.3.1 Problems

1. Solve the following system of homogeneous system of equations $AX = 0$, where A is given by

(a) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}$

2. Does the following system of equations possess a non-zero solution?

$$x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$$

3. Find the value of k so that the equations

$$x - 2y + z = 0, 3x - y + 2z = 0, y + kz = 0 \text{ have}$$

(a) unique solution

(b) infinitely many solutions. Also find solutions for these values of k.

4. Solve the following system of linear equations by matrix method:

a) $$\begin{aligned} x - 2y - 3z &= 0 \\ -2x + 3y + 5z &= 0 \\ 3x + y - 2z &= 0. \end{aligned}$$

b) $$\begin{aligned} x + 2y - 2z + 2s - t &= 0 \\ x + 2y - z + 3s - 2t &= 0 \\ 2x + 4y - 7z + s + t &= 0. \end{aligned}$$

c) $$\begin{aligned} 4x + 5y + 6z &= 0 \\ 5x + 6y + 7z &= 0 \\ 7x + 8y + 9z &= 0. \end{aligned}$$

d) $$\begin{aligned} x + y + z &= 0 \\ 2x - y - 3z &= 0 \\ 3x - 5y + 4z &= 0. \\ x + 17y + 4z &= 0 \end{aligned}$$

5. Solve the following system of linear equations by matrix method:

a) $$\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2. \end{aligned}$$

b) $$\begin{aligned} x - y + 3z &= 3 \\ 2x + 3y - z &= 2 \\ 3x + 2y + 4z &= 5. \end{aligned}$$

c) $$\begin{aligned} 2x + 3y + 4z &= 10 \\ x + 2y + 3z &= 14 \\ x + 4y + 7z &= 10. \end{aligned}$$

d) $$\begin{aligned} x + y + z &= 4 \\ 2x + 5y - 2z &= 3 \\ x + 7y - 7z &= -6. \end{aligned}$$

e) $$\begin{aligned} x + y + z &= 9 \\ 2x + 5y + 7z &= 52 \\ 2x + y - z &= 0. \end{aligned}$$

1.3.2 Hints to Problems 1.3.1

1. (a) $x = 0, y = 0, z = 0.$

   (b) Infinitely many solutions. $x = 4z + t, y = -3z - 2t, z, t$ are free variables.

2. No. *Only solution is zero.*

3. (a) $k \neq -\dfrac{1}{5}$

   (b) $k = -\dfrac{1}{5}, x = -\dfrac{3}{5}k, y = \dfrac{1}{5}k, z = k.$

4. (a) $x = k, y = -k, z = k.$

   (b) $z = -s + t$
   $x = -2y - 4s + 3t$

   (c) $x = -k, y = k, z = k$

   (d) $x = y = z = 0.$

5. (a) Unique solution $(2, -1, 1)$.

   (b) $x = 11/5, y = -4/5, z = 0.$

   (c) Inconsistent System.

   (d) Infinitely many. $x = -\frac{7}{3}k + \frac{17}{3}, y = \frac{4}{3}k - \frac{5}{3}, z = k.$

   (e) unique solution. $x = 1, y = 3, z = 5.$

1.4 Eigen Values and Eigen Vectors of a Matrix

Definition 1.4.1

Let A be a square matrix of order n over reals (complex) numbers. A real (complex) number λ is called an **eigen value** of A iff there exists a non-zero $n \times 1$ column matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ such that } AX = \lambda X.$$

The non-zero column matrix X is called the **eigen vector** of the matrix A corresponding to eigen value λ of A.

Following theorem provide a way for finding eigen values:

Theorem 1.4.1. *λ is an eigen value of matrix A iff $|A - \lambda I| = 0$.*

Remark 1.4.1. The equation $det(A - \lambda I) = 0$ is called **characteristic polynomial** or **characteristic equation** of A.

1. Find the characteristic equation and eigen values of the matrices:

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

(d) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. Find the eigen values and corresponding eigen vectors of the matrices:

(a) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$

1.4.1 Hints of Problems 1.4

1. (a) $\lambda = 0, 5.$
   (b) $-\lambda^3 + 13\lambda - 12 = 0.\lambda = 1, 3, -4$
   (c) $(1-\lambda)(-4-\lambda)(7-\lambda) = 0$
   (d) $\lambda = 0, 3, 15.$
   (e) $\lambda = -1, -1, 2.$
2. (a) $\lambda = 1, 2, 2, (1, 1, -1)^T; (2, 1, 0)^T$
   (b) $\lambda = -1, i, -i, (0, -1, 1)^T; (1+i, 1, 1)^T; (1-i, 1, 1)^T;$
   (c) $\lambda = 0, 1+\sqrt{3}, 1-\sqrt{3}(i, 0, -1)^T; (1, \sqrt{3}-1, -i)^T; (1, \sqrt{3}-1, -i)^T;$

1.5 Cayley-Hamilton Theorem

Theorem 1.5.1: Cayley-Hamilton Theorem

Every Matrix satisfies its characteristic equation.

Remark 1.5.1. The characteristics equaiton of a matrix square matrix A is $|A - \lambda I| = 0$.

1.5.1 Problems Verification of Caylay Hamilton theorem

1. Verify the Cayley-Hamilton theorem for the following matrices:

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

1.5.2 Hints to 1.5

1. (a) $\lambda = 0, 5.$
   (b) $-\lambda^3 + 13\lambda - 12 = 0.\lambda = 1, 3, -4$
   (c) $\lambda^3 + 3\lambda^2 - \lambda + 3I = 0$
   (d) $\lambda = 0, 3, 15.$
   (e) $\lambda = -1, -1, 2.$

1.5.3 Problems Finding inverse using Cayley Hamilton theorem

1. Verify Cayley-Hamilton theorem and using it find inverse of the following matrices:

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & i & i \\ i & 1 & i \\ i & i & 1 \end{bmatrix}$

1.5.4 Hints to 1.5.3

1.
(a) $A^3 - 3A^2 + A - 3I = 0$ and $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$

(b) $A^3 - 5A^2 + 9A - 13I = 0$ and $A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & -3 & -7 \\ 1 & 5 & 3 \\ 5 & -1 & 2 \end{bmatrix}$

(c) $A^3 - 3A^2 + 6A - (4-2i)I = 0$ and $A^{-1} = -\frac{1+3i}{10}\begin{bmatrix} i-1 & 1 & 1 \\ 1 & i-1 & 1 \\ 1 & 1 & i-1 \end{bmatrix}$

1.6 Special types of Matrices

Definition 1.6.1: Symmetric Matrix

A square matrix A is called ***symmetric matrix*** if

$$A^T = A.$$

In other words a matrix is symmetric if we interchange its rows and columns we will again get the same matrix. Condition for symmetric matrix is also written like $a_{ij} = a_{ji}$ for all i, j.

Definition 1.6.2: Skew-Symmetric Matrix

A square matrix A is called ***skew-symmetric matrix*** if

$$A^T = -A.$$

In other words a matrix is skew-symmetric if we interchange its rows and columns we will get the negative of given matrix. Condition for skew-symmetric matrix is also written like $a_{ij} = -a_{ji}$ for all i, j.

Definition 1.6.3: Harmitian Matrix

A square matrix A is called ***Harmitian Matrix*** if

$$A^\theta = A.$$

In other words a matrix is harmitian if we take conjugate and interchange its rows and columns we will again get the same matrix. Condition for symmetric matrix is also written like $\bar{a}_{ij} = a_{ji}$ for all i, j.

Definition 1.6.4: Skew-Harmitian Matrix

A square matrix A is called ***skew-harmitian matrix*** if

$$A^\theta = -A.$$

In other words a matrix is skew-harmitian if we take its conjugate and interchange its rows and columns we will get the negative of given matrix. Condition for skew-harmitian matrix is also written like $\bar{a}_{ij} = -a_{ji}$ for all i, j.

Definition 1.6.5: Orthogonal Matrix

A square matrix A is called ***Orthogonal matrix*** if

$$A^T A = AA^T = I \text{ or } A^{-1} = A^T$$

Definition 1.6.6: Unitary Matrix

A square matrix A is called ***Unitary matrix*** if

$$A^\theta A = AA^\theta = I \text{ or } A^{-1} = A^\theta$$

Definition 1.6.7: Normal Matrix

A square matrix A is called ***Normal matrix*** if

$$A^{\theta}A = AA^{\theta}.$$

If matrix contains real enteries, then

$$A^{T}A = AA^{T}.$$

1.6.1 Problems

1. Give two examples of 3x3 each of symmetric, skew symmetric matrices, harmitian and skew-harmitian matrices. (Don't give example like zero matrices, Identity matrix. Try to make some interesting examples.)

2. Express following matrices as sum of symmetric and skew symmetric matrices.

   (a) $A = \begin{bmatrix} 7 & 2 & 0 \\ -1 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -3 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

3. Check orthogonality ?

   (a) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

4. Prove that following matrices are unitary matrices.

   (a) $A = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$ (b) $A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & i-1 & 1+i \\ 0 & 2 & -2 \\ 2i & i+1 & 1-i \end{bmatrix}$

5. Find the value of x, y, z such that the matrix Q is Hermitian, where

$$Q = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$$

Chapter 2

Differential Equations

2.1 Degree, Order and Solution of Homogeneous Linear Differential Equations with constant coefficients

2.1.1 Problems

1. Find the order and degree of the following Differential Equations. State whether they are linear or non-linear

   (a) $y'' + 3y' + 4y = 0.$

   (b) $x^2y'' + xy' + 3y = 5x.$

   (c) $(y')^2 + 3xy' + y = 0.$

   (d) $\sqrt{1+2x^2}dx + \sqrt{1+2y^2}dy = 0$

   (e) $[1 + (y')^2]^{1/2} = x^2 + y.$

   (f) $yy'' + t^2y' + 4y = \cos t.$

   (g) $(y'')^2 + 3y' + x = 0$

   (h) $y'y'' + y' + 5y = \sin x$

   (i) $(1 + y')^{1/2} = y''.$

   (j) $y' = \sin y.$

2. Verify that the given function satisfies the differential equation.

   (a) $y = ce^{-x}; \;\; y' + 2xy = 0$

   (b) $y = x\log x - x; \;\; y' = \log x.$

   (c) $y = \sin^{-1} x; \;\; y'' = x/(1-x^2)^{3/2}.$

   (d) $y = \sec x + \tan x, \;\; (1-\sin^2 x)^2 y'' = \cos x$

3. Find all values of m for which $y = e^{mx}$ is solution of the following differential equations.

   (a) $y'' + 3y' + 2y = 0.$

   (b) $y''' - 6y'' + 11y' - 6y = 0.$

   (c) $y''' - 2y'' - y' + 2y = 0.$

   (d) $y'' - 4y' + y = 0.$

   (e) $y'' - 2y' + 4y = 0.$

4. From the following equations, find the constant coefficient and variable coefficient equations.

   (a) $y'' - a^2 y = 0$.
   (b) $y' = y/x$.
   (c) $y''' + 3y'' + 6y' + 12y = x^2$.
   (d) $x^3 y''' + 9x^2 y'' + 18xy' + 6y = 0$.
   (e) $(1-x)y'' + xy' - y = 0$.
   (f) $y'' - (1+x^2)y = 0$.

5. Verify that given functions are solution of associated differential equations.

   (a) $1, x, e^x; y''' - y'' = 0$.
   (b) $e^x, e^{-2x}; y'' + y' - 2y = 0$.
   (c) $e^{-x}\cos 2x, e^{-2x}\sin 2x; y''+2y'+5y = 0$.

6. Examine whether the following functions are linearly independant for $x \in (0, \infty)$.

   (a) $2x, 6x+3, 3x+2$.
   (b) $x^2 - x, 3x^2 + x + 1, 9x^2 - x + 2$.
   (c) $x^2 - 2x, 3x^2 + x + 2, 4x^2 - x + 1$.
   (d) $\sin x, \sin 2x, \sin 3x$
   (e) $1, \sin x, \cos x$.
   (f) $e^x, \sinh x, \cosh x$.
   (g) $x^2, 1/x^2$
   (h) $\ln x, \ln x^2, \ln x^3$.
   (i) $x - 1, x + 1, (x-1)^2$.

7. Show that e^{2x} and xe^{2x} are solution of the equation $y'' - 4y + 4y = 0$ on any interval. Show that these solutions are independent.

2.1.2 Hints to Problems 2.1.1

1. (a) two, one, linear
   (b) two, one, linear
   (c) one, two, non-linear
   (d) one, one, non-linear
   (e) one, two, non-linear
   (f) two, one, non-linear
   (g) two, two, non-linear
   (h) two, one, non-linear
   (i) two, two, non-linear
   (j) one, one, linear

3. Find all values of m for which $y = e^{mx}$ is solution of the following differential equations.

(a) $m = -1, -2.$ (b) $m = 1, 2, 3$

(c) $m = -1, 1, 2$ (d) $m = 2 \pm \sqrt{3}.$

(e) $m = 1 + \sqrt{3}i$

4. (a) constant coeff. (b) variable coeff.

(c) constant coeff. (d) variable coeff.

(e) variable coeff. (f) variable coeff.

6. (a) Linearly dependant (b) Linearly dependant

(c) Linearly Independant, $W = 14$ (d) Linearly Independant, $W = -16\sin^6 x$

(e) Linearly Independant, W=1 (f) LD

(g) LI (h) LD

(i) LI

2.2 Solution of Second Order Homogeneous LDE with constant coefficients

Consider the linear homogeneous second order equation

$$ay'' + by' + cy = 0, \quad a, b, c \text{ are constants.} \tag{2.1}$$

$$ay'' + by' + cy = 0, \quad a, b, c \text{ are constants.}$$

In operator notation by taking $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}$ we write this equation as

$$aD^2y + bDy + cy = 0$$
$$(aD^2 + bD + c)y = 0.$$

The equation $aD^2 + bD + c = 0$ is known as auxiliary equation. The solution of equation (2.1) depends upon roots of auxiliary equation. Following table Highlights rules for writing solution:

Roots of Auxiliary Equation	Solution
$m_1 \neq m_2$ (real and distinct roots of AE)	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
$m_1 = m_2$ (real and same roots of AE)	$y = (c_1 + x c_2) e^{m_1 x}$
roots $= \alpha \pm i\beta$ (complex roots of AE)	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.
m_1, m_2, m_3 three distinct roots	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
$m_1 = m_2 = m_3$ same roots (For three degree equations)	$y = (c_1 + x c_2 + x^2 c_3) e^{m_1 x}$

Example 2.2.1. Solve the differential equation:

$$y'' + 4y' + 5y = 0$$

Solution: Using $D = \frac{d}{dx}$ and $D^2 = \dfrac{d^2}{dx^2}$, we have

$$D^2 y + 4Dy + 5y = 0$$
$$(D^2 + 4D + 5)y = 0$$

The auxiliary equation is given by

$$D^2 + 4D + 5 = 0$$
$$D = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

Therefore the solution of given equation is

$$y = e^{-2x}(A \cos x + B \sin x).$$

2.2.1 Problems (Solution of Homogeneous LDE with constant coefficients)

1. Find the general solution of the following system of linear equations:

   (a) $y'' - 4y = 0$.

   (b) $y'' - y' - 2y = 0$.

   (c) $y'' + y' - 2y = 0$.

   (d) $y'' - 4y' - 12y = 0$.

   (e) $y'' + 4y' + y = 0$.

   (f) $4y'' - 9y' + 2y = 0$.

   (g) $4y'' + 8y' - 5y = 0$.

   (h) $y'' + 2y' + y = 0$.

(i) $y'' + 2\pi y' + \pi^2 y = 0.$

(j) $9y'' - 12y' + 4y = 0.$

(k) $4y'' + 4y' + y = 0.$

(l) $25y'' - 20y' + 4y = 0.$

(m) $y'' + 25y = 0.$

(n) $y'' + 4y' + 5y = 0.$

(o) $y'' - 2y' + 2y = 0.$

(p) $(4D^2 - 4D + 17)y = 0.$

2. Show that in the following problems, $\{y_i(x)\}$ forms a set of fundamental solutions (basis) to the corresponding differential equation:

(a) $1, x^2, \; x^2y'' - xy' = 0, x > 0.$

(b) $e^{2x}\cos 3x, e^{2x}\sin 3x; 2y'' - 8y' + 26y = 0.$

(c) $e^x, e^x\cos x, e^x \sin x; y''' - 3y'' + 4y' - 2y = 0.$

(d) $x^{1/4}, x^{5/4}; 16x^2y'' - 8xy' + 5y = 0, x > 0.$

(e) $\sin(\ln x^2), \cos(\ln x^2); x^2y'' + xy' + 4y = 0, x > 0.$

2.3 Normal Differential equation

Theorem 2.3.1

If the functions $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the initial value problem

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n y = r(x), \tag{2.2}$$
$$y(x_0) = c_1, y'(x_0) = c_2, \ldots, y^{(n-1)}(x_0) \; = c_n.$$

where $x_0 \in I$, and $c_1, c_2, \ldots, c_n$ are n unknown constants.

Remark 2.3.1. If the condition of Theorem 2.3 are satisfied, then the Differential Equation 2.2 is called **normal on** I.

Remark 2.3.2. A point $x_o \in I$, for which $a_0(x) \neq 0$, called ***ordinary point*** or a ***regular point*** of the differential equation 2.2.

Example 2.3.1. Find the intervals in which the following differential equations are normal

1. $(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \; n$ is an integer.

2. $x^2y'' + xy' + (n^2 - x^2)y = 0$, n real.

3. $\sqrt{x}y'' + 6xy' + 15y = \ln(x^4 - 256)$.

Solution:

1. Here $a_0(x) = (1-x^2)$, $a_1(x) = -2x$, and $a_2(x) = n(n+1)$. Now, a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = 1 - x^2 \neq 0$ for all $x \in (-\infty, \infty)$ except at the points $x = -1, 1$. Hence differential equation is normal on every subinterval of the open intervals $(-\infty, -1), (-1, 1), (1, \infty)$,

2. Here $a_0(x) = x^2$, $a_1(x) = x$, and $a_2(x) = (n^2 - x^2)$. a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = 1 - x^2 \neq 0$ for all $x \in (-\infty, \infty)$ except at the points $x = 0$. Hence differential equation is normal on every interval which does not contains 0.

3. Here $a_0(x) = \sqrt{x}$, $a_1(x) = 6x$, and $a_2(x) = 15, r(x) = \ln(x^4 - 256)$. a_0, a_1, a_2 and $r(x)$ are continuous for all $x > 4$. Also, $a_0(x) = \sqrt{x} \neq 0$ and real for all $x \in (0, \infty)$. Hence differential equation is normal on $(4, \infty)$.

2.3.1 Problems based on Normal differential equations

1. Find the intervals on which the following differential equations are normal.

   (a) $y' = 3y/x$.

   (b) $(1 + x^2)y'' + 2xy' + y = 0$

   (c) $x^2y'' - 4xy' + 6y = x$

   (d) $y'' + 3y' + \sqrt{x}y = \sin x$.

   (e) $y''' + 9y' + y = \log(x^2 - 9)$

   (f) $y'' + |x|y' + y = x \ln x$.

   (g) $x(1 - x)y'' - 3xy' - y = 0$.

   (h) $y'' + xy' + 6y = \ln \sin(\pi x/4)$.

2.3.2 Solution to Problems 2.3.1

1. (a) Any subinterval of $(-\infty, 0), (0, \infty)$.

   (b) Any subinterval of $(-\infty, \infty)$.

   (c) Any subinterval of $(-\infty, 0), (0, \infty)$.

   (d) Any subinterval of $[0, \infty)$.

   (e) Any subinterval of $(3, \infty)$.

   (f) Any subinterval of $(0, \infty)$.

   (g) Any subinterval of $(-\infty, 0), (0, 1), (1, \infty)$.

   (h) $4m < x < 4(m+1), m = 0, 2, 4, \ldots$

Chapter 3

Non-Homogeneous Differential Equations

CO2: understand the use of different methods for the solution of linear differential equations.

3.1 Solving Non-Homogeneous Linear differential equations with constant coefficients

Consider a non-homogeneous linear differential equation of order 2 with constant coefficients

$$c_0 \frac{d^2y}{dx^2} + c_1 \frac{dy}{dx} + c_2 y = r(x). \tag{3.1}$$

The solution of equation (3.1) is a function

$$y = y_c + y_p$$

where y_c is called ***complimentary solution (CS)*** is solution of the corresponding homogeneous equation

$$c_0 \frac{d^2y}{dx^2} + c_1 \frac{dy}{dx} + c_2 y = 0,$$

and y_p is called ***particular solution (PS)*** or ***particular integeral (PI)*** is given by

$$y_p = \frac{1}{c_0 D^2 + c_1 D + c_2} r(x).$$

The particular solution depends upon the function $r(x)$ in equation (3.1). Following picture gives a better illustration.

Solution of $c_0\dfrac{d^2y}{dx^2}+c_1\dfrac{dy}{dx}+c_2y=r(x)$.

$$\downarrow$$

$y=y_c+y_p$

y_c ***complimentary solution*** → y_c is solution of $c_0\dfrac{d^2y}{dx^2}+c_1\dfrac{dy}{dx}+c_2y=0$

y_p ***particular solution (PS)*** → $y_p=\dfrac{1}{c_0D^2+c_1D+c_2}r(x)$

Figure 3.1

Following are the formulas for finding particular Integral/Solution:

Rule. No.	Particular Solution/Particular Integral
1.	$\dfrac{1}{F(D)}e^{ax}=\dfrac{1}{F(a)}e^{ax};$ provided $F(a)\neq 0$ ***Sub-case 1:*** $F(a)=0, F'(a)\neq 0$ then $$\dfrac{1}{F(D)}e^{ax}=\dfrac{x}{F'(a)}e^{ax}$$ ***Sub-case 2:*** $F(a)=0, F'(a)=0, F''(a)\neq 0$ then $$\dfrac{1}{F(D)}e^{ax}=\dfrac{x^2}{F''(a)}e^{ax}$$

2.	$$\frac{1}{F(D^2)}\sin(ax+b)=\frac{1}{F(-a^2)}\sin(ax+b);\quad F(-a^2)\neq 0$$ ***Sub-case 1:*** $F(-a^2)=0, F'(-a^2)\neq 0$ then $$\frac{1}{F(D^2)}\sin(ax+b)=\frac{x}{F'(-a^2)}\sin(ax+b)$$
3.	$$\frac{1}{F(D^2)}\cos(ax+b)=\frac{1}{F(-a^2)}\cos(ax+b);\quad F(-a^2)\neq 0$$ ***Sub-case 1:*** $F(-a^2)=0, F'(-a^2)\neq 0$ then $$\frac{1}{F(D^2)}\cos(ax+b)=\frac{x}{F'(-a^2)}\cos(ax+b)$$
4.	$\frac{1}{F(D)}x^m=[F(D)]^{-1}x^m$ In this case, the following two formulas will be helpful: $$(1+X)^{-1}=1-X+X^2-X^3+X^4-\dots$$ $$(1-X)^{-1}=1+X+X^2+X^3+X^4+\dots$$
5.	$\frac{1}{F(D)}e^{ax}V(x)=e^{ax}\frac{1}{F(D+a)}V(x)$ After this, we will apply one of the rules from rule 1 to 4 depending upon type of function $V(x)$

Example. 3.1.1: Solve $y''+5y'+6y=e^x$.

Solution: Using $D=\frac{d}{dx}$ and $D^2=\frac{d^2}{dx^2}$, we have

$$D^2y+5Dy+6y=e^x.$$

$$(D^2+5D+6)y=e^x.$$

For finding the complimentary solution: the auxiliary equation is given by

$$D^2 + 5D + 6 = 0$$
$$D = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2} = -3, -2.$$

Therefore the complimentary solution of given equation is

$$y_c = c_1 e^{-3x} + c_2 e^{-2x}.$$

Now we will find particular solution (PS or PI)

$$(D^2 + 5D + 6)y = e^x.$$
$$y_p = \frac{1}{D^2 + 5D + 6} e^x$$
$$y_p = \frac{1}{1^2 + 5(1) + 6} e^x \qquad \text{Rule 1, Putting } D = 1$$
$$y_p = \frac{1}{12} e^x$$

Therefore complete solution or general solution is given by

$$y = y_c + y_p = c_1 e^{-3x} + c_2 e^{-2x} + \frac{1}{12} e^x$$

Sr. No.	Formula
1.	$\sinh x = \frac{e^x - e^{-x}}{2}$
2.	$\cosh x = \frac{e^x + e^{-x}}{2}$
3.	$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
4.	$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Example. 3.1.2: Solve $y''' - 3y' + 2y = e^{-2x} + 2\sinh x$.

Solution: Using $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$, we have

$$D^3 y - 3Dy + 2y = e^{-2x} + 2\sinh x.$$
$$(D^3 - 3D + 2)y = e^{-2x} + e^x - e^{-x}.$$

For finding the complimentary solution: the auxiliary equation is given by

$$D^3 - 3D + 2 = 0 \tag{3.2}$$

Cubic equation, we will apply HIT & TRIAL, $D = 0$ not satisfying eqn. (3.2), $D = 1$ is satisfying eqn(3.2). Therefore, one root is $D = 1$. Now we apply synthetic division to find other two roots:

	D^3	D^2	D	constant
1	1	0	-3	2
	0	1	1	-2
	1	1	-2	**0**

Therefore other two roots are given by:

$$\begin{aligned}
&D^2 + D - 2D = 0.\\
&(D-1)(D+2) = 0\\
&\therefore \quad D = 1, 1, -2
\end{aligned}$$

Therefore the complimentary solution of given equation is

$$y_c = (c_1 + xc_2)e^x + c_3 e^{-2x}.$$

Now we will find PI

$$\begin{aligned}
(D^3 - 3D + 2)y_p &= e^{-2x} + e^x - e^{-x}.\\
y_p &= \frac{1}{D^3 - 3D + 2}\left(e^{-2x} + e^x - e^{-x}\right)\\
y_p &= \frac{1}{D^3 - 3D + 2}e^{-2x} + \frac{1}{D^3 - 3D + 2}e^x - \frac{1}{D^3 - 3D + 2}e^{-x}
\end{aligned}$$

Let us apply rule 1:

$$\begin{aligned}
y_p &= \frac{1}{(-2)^3 - 3(-2) + 2}e^{-2x} + \frac{1}{1^3 - 3(1) + 2}e^x - \frac{1}{(-1)^3 - 3(-1) + 2}e^{-x}\\
y_p &= \frac{1}{0}e^{-2x} + \frac{1}{0}e^x - \frac{1}{4}e^{-x}
\end{aligned}$$

We have to evaluate first two terms separately as rule 1 fails on them. We will apply Rule 1 Sub-case 1 for first term

$$\frac{1}{D^3 - 3D + 2}e^{-2x} = \frac{x}{3D^2 - 3}e^{-2x} = \frac{x}{3(-2)^2 - 3}e^{-2x} = \frac{xe^{-2x}}{9}.$$

We have to apply rule 1 sub-case 2 as 1 is repeated root.

$$\frac{1}{D^3-3D+2}e^x = \frac{x^2}{6.D}e^x = \frac{x^2}{6(1)}e^x = \frac{x^2}{6}e^x.$$

$$y_p = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}.$$

Therefore complete solution is

$$y = y_c + y_p$$
$$y = (c_1 + xc_2)e^x + c_3e^{-2x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}.$$

Example. 3.1.3: Find PI of $(D^3+1)y = \cos(2x-1)$.

Solution: $(D^3+1)y = \cos(2x-1)$.

$$PI = \frac{1}{(D^3+1)}\cos(2x-1) = \frac{1}{(D^2.D+1)}\cos(2x-1)$$
$$y_p = \frac{1}{((-2^2).D+1)}\cos(2x-1) \quad [\text{ Rule 3 Putting } D^2 = -2^2]$$
$$y_p = \frac{1}{(-4D+1)}\cos(2x-1)$$
$$= (1+4D)\frac{1}{(1+4D)(1-4D)}\cos(2x-1)$$
$$= (1+4D)\frac{1}{(1-16D^2)}\cos(2x-1)$$
$$= (1+4D)\frac{1}{(1-16(-2^2))}\cos(2x-1)$$
$$= \frac{1}{65}(1+4D)\cos(2x-1)$$
$$= \frac{1}{65}\left(\cos(2x-1) + 4\frac{d}{dx}\cos(2x-1)\right)$$
$$= \frac{1}{65}\left(\cos(2x-1) - 8\sin(2x-1)\right) \quad \square$$

3.1.1 Problems based on solving non-homo. LDE with constant coeff.

1. Solve the following differential equations:

   (a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2.$

(b) $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+5y=-2\cosh x$. Also find y when $y=0, \dfrac{dy}{dx}=0$ at $x=0$.

(c) $\dfrac{d^2y}{dx^2}+n^2y=k\cos(nx+\alpha)$. n, α are constants.

(d) $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+3x=\sin t$.

(e) $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=4\cos^2 x$.

(f) $(D^2-4D+3)y=\sin 3x\cos 2x$.

(g) $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^{-x}+\sin 2x$.

(h) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=e^{2x}-\cos^2 x$.

(i) $(D^3-5D^2+7D-3)y=e^{2x}\cosh x$.

(j) $\dfrac{d^2y}{dx^2}-y=e^x+x^2e^x$.

(k) $(D^3-D)y=2x+1+4\cos x+2e^x$.

(l) $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+25y=e^{2x}+\sin x+x$.

3.1.2 Hints to Problems 3.1.1

1. (a) $y=(c_1+c_2x)e^{3x}+3x^2e^{3x}+\dfrac{7}{25}e^{-2x}-\dfrac{1}{9}\log 2$.

   (b) $y=\frac{3}{5}e^{-2x}(\cos x+3\sin x)-\frac{e^x}{10}-\frac{e^{-x}}{2}$

   (c) $y=c_1\cos nx+c_2\sin nx+\frac{kx}{2n}\sin(nx+\alpha)$

   (d) $y=e^{-x}(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x)+\frac{1}{4}(\sin x-\cos x)$.

   (e) $y=c_1e^{-x}+c_2e^{-2x}+1+\frac{1}{10}(3\sin 2x-\cos 2x)$

   (f) $y=c_1e^x+c_2e^{3x}+\frac{1}{884}(10\cos 5x-11\sin 5x)+\frac{1}{20}(\sin x+2\cos x)$.

   (g) $y=c_1+(c_2+c_3x)e^{-x}-\frac{x^2}{2}e^{-x}+\frac{3}{50}\cos 2x-\frac{2}{25}\sin 2x$.

   (h) $y=(c_1+c_2x)e^{-x}+\frac{1}{2}+\frac{1}{5}(2\sin 2x+\cos 2x)$.

   (i) $y=(c_1+c_2x)e^x+c_3e^{3x}+\frac{1}{8}(xe^{3x}-x^2e^x)$.

   (j) $y=c_1e^x+c_2e^{-x}+\frac{e^x}{12}(2x^3-3x^2+9x)$.

   (k) $y=c_1+c_2e^x+c_3e^{-x}+xe^x-(x^2+x)-2\sin x$.

   (l) $y=e^{3x}(c_1\cos 4x+c_2\sin 4x)+\frac{1}{17}e^{2x}+\frac{1}{565}(23\sin x+6\cos x)+\frac{x}{25}+\frac{6}{625}$.

Chapter 4

MCQ Question UNIT-1 to 3

4.1 MCQ from UNIT-3 (LDE)

1. The complementary function of $(D^4 - a^4)y = 0$ is ....

   (a) $c_1e^{ax} + c_2e^{-ax}$ (b) $(c_1 + xc_2)e^{ax}$

   (c) $(c_1 + xc_2 + x^2c_3 + x^3c_4)e^{ax}$ (d) $c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$.

2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....

   (a) $-\frac{1}{25}(3 \sin 2x + 4 \cos 2x)$ (b) $\frac{1}{25}(3 \sin 2x + 4 \cos 2x)$

   (c) $-\frac{1}{25}(4 \sin 2x + 3 \cos 2x)$ (d) $-\frac{1}{25}(\sin 2x + \cos 2x)$

3. PI of $y'' - 3y' + 2y = 12$ is ....

   (a) 12 (b) 1/12 (c) 6 (d) None of these

4. The Wronskian x and e^x is ....

   (a) $e^x(x - 1)$ (b) $e^{-x}(x - 1)$

   (c) $e^x(x + 1)$ (d) $e^{-x}(x + 1)$

5. The CF of $y'' - 2y' + y = xe^x \sin x$ is

   (a) $c_1e^x + c_2e^{-x}$ (b) $(c_1x + c_2)e^x$ (c) $(c_1 + c_2x)e^{-x}$ (d) None of these

6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ...

7. The particular integral of $(D^2 + a^2)y = \sin ax$ is

(a) $-\frac{x}{2a}\cos ax$ (b) $\frac{x}{2a}\cos ax$ (c) $-\frac{ax}{2}\cos ax$ (d) $\frac{ax}{2}\cos ax$

8. Solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....?

9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\frac{\pi}{2}) = 2$, is ...

(a) $\cos x + \sin x$ (b) $\cos x - \sin x$

(c) $\cos x$ (d) None of these

10. $e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + c_3 e^{2xf}$ is the general solution of

(a) $\frac{d^3y}{dx^3} + 4y = 0$ (b) $\frac{d^3y}{dx^3} - 8y = 0$

(c) $\frac{d^3y}{dx^3} + 8y = 0$ (d) $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2 = 0$

11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ...

12. The particular integral of $\frac{d^2y}{dx^2} + y = \cosh 3x$ is ...

13. The solution of $x^2 y'' + xy' = 0$ is ...

14. The general solution of $(D^2 - 2)^2 y = 0$ is ...

15. P.I. of $(D + 1)^2 y = xe^{-x}$ is ...

(a) $\frac{1}{6}x^3e^{-x}$ (b) $\frac{1}{6}x^2e^{-x}$ (c) $\frac{1}{6}xe^{-x}$ (d) None of these

16. If $f(D) = D^2 - 2, \frac{1}{f(D)}e^{2x} = \ldots$

(a) $\frac{1}{4}e^{2x}$ (b) $\frac{1}{4}e^{-2x}$ (c) $\frac{1}{2}e^{2x}$ (d) $\frac{1}{2}e^{-2x}$

17. If $f(D) = D^2 + 5, \frac{1}{f(D)}\sin 2x = \ldots$

(a) $\sin 2x$ (b) $\cos 2x$ (c) $-\sin 2x$ (d) $-\cos 2x$

18. The particular integral of $(D + 1)^2 y = e^{-x}$ is ...

(a) $\frac{1}{2}x^3e^{-x}$ (b) $\frac{1}{2}x^2e^{x}$ (c) $\frac{1}{2}xe^{-x}$ (d) None of these

19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ...

20. P.I. of $(D^2 + 4)y = \cos 2x$ is ...?

    (a) $\frac{1}{2}\sin 2x$ (b) $\frac{1}{2}x\sin 2x$ (c) $\frac{1}{4}\sin 2x$ (d) $\frac{1}{2}x\cos 2x$

21. By method of undetermined coefficients y_p of $y'' + 3y' + 2y = 12x^2$ is of the form

    (a) $a + bx + cx^2$ (b) $a + bx$

    (c) $ax + bx^2 + cx^3$ (d) None of these

22. In the equation $\frac{dx}{dt} + y = \sin t + 1, \frac{dy}{dt} + x = \cos t$ if $y = \sin t + 1 + e^{-t}$, then $x = ...$?

23. $(x^2D^2 + xD + 7)y = 2/x$ converted to a linear differential equation with constant coefficients is ...

24. The PI of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ is

    (a) $\frac{x^2}{3} + 4x$ (b) $\frac{x^3}{3} + 4$ (c) $\frac{x^3}{3} + 4x$ (d) $\frac{x^3}{3} + 4x^2$

25. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is given by

    (a) $C_1e^x + C_2e^{2x} + \frac{1}{2}e^{3x}$ (b) $C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}e^{3x}$

    (c) $C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (d) $C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{-3x}$

26. The particular integral of the differential equation $(D^3 - D)y = e^x + e^{-x}, D = \frac{d}{dx}$ is

    (a) $\frac{1}{2}(e^x + e^{-x})$ (b) $\frac{1}{2}x(e^x + e^{-x})$

    (c) $\frac{1}{2}x^2(e^x + e^{-x})$ (d) $\frac{1}{2}x^2(e^x - e^{-x})$

27. The complimentary function of the differential equation $x^2y'' - xy' + y = \log x$ is .....

28. The homogeneous linear differential equation whose auxiliary equation has roots 1, -1 is ....

29. The particular integral of the differential equation $(D^2 - 6D + 9)y = \log 2$ is ...

30. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficients, put $x = ...$

31. The particular integral of $(D^2-4)y = \sin 3x$ is

(a) 1/4 (b) $-1/13$ (c) 1/5 (d) None of these.

32. The solution of $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 0$ is ...

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ...

34. Complimentary function of $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 2x\log x$ is

(a) $(c_1+c_2x)e^x$ (b) $(c_1+c_2\log x)x$

(c) $(c_1+c_2x)\log x$ (d) $(c_1+c_2\log x)e^x$

35. The general solution of $(D^2-D-2)x = 0$ is $x = c_1e^t + c_2e^{-2t}$

(a) True (b) False

36. $\dfrac{1}{f(D)}x^2e^{ax} = \dfrac{1}{f(D+a)}e^{ax}x^2$

(a) True (b) False

4.2 Hints to 4.1

1. $c_1e^{ax}+c_2e^{-ax}+c_3\cos ax+c_4\sin ax.$
2. $-\dfrac{1}{25}(3\sin 2x+4\cos 2x)$
3. 1/6
4. $e^x(x-1)$
5. (b)
6. $y = c_1+(c_2+c_3x+c_4x^2)e^{2x}$
7. (a)
8. $y = e^x[(c_1+c_2x)\cos 2x+(c_3+c_4x)\sin 2x]$
9. $y = \cos x + 2\sin x$
10. (b)
11. $y = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x$
12. $\dfrac{1}{10}\cosh 3x.$
13. $y = a\log x + 6.$
14. $y = (c_1+c_2x)e^{\sqrt{2}x}+(c_3+c_4x)e^{-\sqrt{2}x}.$
15. $\dfrac{1}{6}x^3e^{-x}.$
16. $y = \dfrac{1}{2}e^{2x}.$
17. $\sin 2x$
18. $\dfrac{1}{2}x^2e^{-x}$

19. $y = (c_1 + c_2 x)e^{-x/2} + c_3$
20. (c)
21. (a)
22. xe^{-x}
23. $\dfrac{d^2y}{dt^2} + 7y = 2e^t$
24. (c)
25. (a)
26. (b)
27. $y = (c_1 + c_2 \log x)x$
28. $x^2y'' + xy' - y = 0$
29. $\dfrac{1}{9}\log 2$
30. e^t
31. (d)
32. $y = c_1e^{-x} + c_2e^{2(1+\sqrt{2})x} + c_3e^{2(1-\sqrt{2})x}$.
33. $(D^3 + 2D^2 + D)y = 0.$
34. ...
35. False
36. False