7. Coordinates to index and Index to coordinates

7.1 2 D mapping

Given a 1 D raster scan the task here is to convert the same into 2D raster scanning. Consider the following

0	1	2	3	4	5	6	7	8	…..	n-2	n-1	n
0	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	……	(n-2)	(n-1)	(n)

To be converted into a general matrix2 Dimensional matrix $(L_1, L_2) = (k, m)$

L_1

L_2

0 (0,0)	1 (1,0)	2 (2,0)	3 (3,0)	4 (4,0)	…………	k-3 (k-3,0)	k-2 (k-2,0)	k-1 (k-1,0)
k (0,1)	k+1 (1,1)	k+2 (2,1)	k+3 (3,1)	k+4 (4,1)	…………	k+(k-3) (k-3,1)	k+(k-2) (k-2,1)	k+(k-1) (k-1,1)
2k (0,2)	2k+1 (1,2)	2k+2 (2,2)	2k+3 (3,2)	2k+4 (4,2)	…………	2k+ (k-3) (k-3,2)	2k+ (k-2) (k-2,2)	2k+ (k-1) (k-1,2)
: : :	: : :	: : :	: : :	: : :	…………	: : :	: : :	: : :
(m-1)*k (0,m-1)	(m-1)k +1 (1,m-1)	(m-1)k+ 2 (2,m-1)	(m-1)k+ 3 (3,m-1)	(m-1)k+ 4 (4,m-1)	…………	(m-1)k+ (k-3) (k-3,m- 1)	(m-1)k+ (k-2) (k-2,m- 1)	(m-1)k+ (k-1) (k-1,m- 1)

Deriving the mathematical proof for the above mapping we get:

<u>Postulate 1:</u> A single dimensional array of size n can be converted in to a 2D array of size (k,m) if

$n = k \times m$ —- (i)

This ensures that each elements of the 1D array maps to the 2D array of size $k \times m$ having km elements

<u>Postulate 2:</u> With a 2 dimensional matrix given by size $(L_1, L_2) = (k, m)$ there are k elements across each row and each column is repeated m times.

Thus with Postulate 2 the first row of the matrix will have k elements in total.
The second and the first row will have 2k elements in total

Similarly the total elements in row 1 to m will be $k \times m$ which is in accordance of postulate 1

Let i be an element of 1D array corresponding to l^{th} row and m^{th} column having index $((m-1),(l-1))$ (considering 0 indexing and the indexing pattern followed) of the 2D array

Total number of elements before l^{th} row are $(l-1)\times k$. (Postulate 2)

Starting l^{th} row to reach m^{th} column there will be (m-1) elements before the presence of the current element.

Thus total elements before the current element at $((m-1),(l-1))$ is $(l-1)\times k+(m-1)$

According to Postulate 1

$i=(l-1)\times k+(m-1)$ --(ii)

Getting the equivalence between $i=(l-1)\times k+(m-1)$ and $(L_1,L_2)=((m-1),(l-1))$

Thus
$L_1(index)=(m-1)$

$L_2(index)=(l-1)$ -- (iii)

Equation ii can be rewritten as

$i=(L_2(index))\times k+(L_1(index))$ -- (iv)

Postulate 3: Any number x can be written a multiple of any other number y (y<x) as

$x=p\times y+r$

where p is the quotient and r is the remainder.

Postulate 3 is analogous to Equation (iv)

Thus

$L_1(index)$ is the remainder of the division of i by k
$L_2(index)$ is the quotient of division of i by k

Let '\\' define the integer division operation and % define the modulo operation (operation that gives the remainder)

thus

i \\ k = $L_2(index)$ **and i%k =** $L_1(index)$ **--- (I)**

For equation (iv) any index i can be represented using 2 dimensional coordinates as

$i=(L_2(index))\times k+(L_1(index))$ **---- (II)**

7.2 D dimension

Extending 7.1 for d dimension. The goal here is to map a 1 dimensional array of indexes into a D dimensional array

The 1 D array containing n elements are represented as $(0,1,2,3,4,5,6,7,8,9,....,n-1)$

A D dimensional array can be represented as $D_{array} = (l_0, l_1, l_2, l_3, \ldots, l_d)$
where
$l_0 = (0,1,2,..,(l'_0 - 1)), l_1 = (0,1,2,..,(l'_1 - 1)), ... , l_d = (0,1,2,..,(l'_d - 1))$

Postulate 1: A single dimensional array of size n can be converted in to a d Dimensional array of $D_{array} = (l_1, l_2, l_3, \ldots, l_d)$
.
$n = l_1 \times l_2 \times l_3 \ldots \times l_d$ —- (i)

In order to map an 1 dimensional matrix into a d dimensional matrix, the total number of elements must match.

Corollary 1: Each row in the d dimension array contains l_0 elements. Each row is repeated l_1 times. Each of these two dimensional matrix is repeated l_3 for the 3rd dimension. The pattern continues for all the previous dimension array till the last dimension l_d

Thus each of the l_d element matrix can be represented as

$n(u_d) = l_{d-1} \times l_{d-2} \times \ldots l_2 \times \ l_1$ elements
Similarly

$n(u_{d-1}) = l_{d-2} \times l_{d-3} \ldots . \times l_2 \times l_1$ elements
.
.
.
$n(u_3) = l_2 \times l_1$ elements
$n(u_2) = l_1$ elements

where $n(u_i)$ represents the number of element in unit i dimension matrix

Proof
Considering any i^{th} element in a d dimensional array given by $D_{matrix} = (l_1, l_2, l_3, \ldots, l_d)$

have coordinates $(x_1, x_2, x_3 \ldots, x_d)$

The representation of the d dimensional array into one dimension will result in representing the number of elements that occurred before the current element specifically.

$i(1^d) =$ total number of elements in d dimensional array till coordinate $(x_0, x_1, x_2, x_3 \ldots, x_d)$

By Corollary 1 we have

Number of elements in x_d complete dimensions are

$$n(x_d) = x_d \times n(u_d) = x_d \times l_{d-1} \times \ldots \times l_2 \times l_1$$

The number of elements in x_{d-1}
is
$$n(x_{d-1}) = x_{d-1} \times n(u_{d-1}) = x_{d-1} \times l_{d-2} \times \ldots \times l_2 \times l_1$$

On similar Co relation we have

$$n(x_3) = x_3 n(u_3) = x_3 \times l_2 \times l_1$$

$$n(x_2) = x_2 n(u_2) = x_2 \times l_1$$

We have x_1 extra elements at dimension l_1

Thus the total number of elements at till the matrix having coordinate $(x_1, x_2, x_3 \ldots, x_d)$ is

$$i(1^d) = n(x_d) + n(x_{d-1}) + n(x_{d-2}) \ldots + n(x_3) + n(x_2) + x_0$$

$i(1^d)$ =
$$x_d \times (l_{d-1} \times l_{d-2} \times \ldots l_3 \times l_2 \times l_1) + x_{d-1} \times (l_{d-2} \times \ldots l_3 \times l_2 \times l_1) + \ldots + x_3 \times l_2 \times l_1 + x_2 \times l_1 + x_1$$

let $l_0 = 1$
$i(1^d)$ =
$$x_d \times (l_{d-1} \times l_{d-2} \times \ldots l_3 \times l_2 \times l_1 \times l_0) + x_{d-1} \times (l_{d-2} \times \ldots l_3 \times l_2 \times l_1 \times l_0) + \ldots + x_3 \times l_2 \times l_1 \times l_0 + x_2 \times l_1 \times l_0 + x_1 \times l_0$$

$$= x_d \prod_{i=0}^{d-1} l_i + x_{d-1} \prod_{i=0}^{d-2} l_i + \ldots + x_3 \prod_{i=0}^{3-1} l_i + x_2 \prod_{i=0}^{2-1} l_i + x_1 \times l_0$$

$= \sum_{i=1}^{d} x_i \prod_{j=0}^{i-1} l_j$ (where l_j represents the dimension of the matrix for $d \geq j \geq 1$ and $l_j = 0$) …….. (i)

The equation I gives the mapping between the number of elements in d dimensional array and the corresponding 1D index.

<u>Finding the reverse Mapping:</u>

From equation 1 we have

$$i(1^d) = x_d \prod_{i=0}^{d-1} l_i + x_{d-1} \prod_{i=0}^{d-2} l_i + \ldots + x_3 \prod_{i=0}^{3-1} l_i + x_2 \prod_{i=0}^{2-1} l_i + x_1 \times l_0$$

We know $i(1^d)$ and l_j and are solving for x_i $\forall x_i \in [1,d]$

i.e. Given the 1 D mapping we can also find the corresponding d dimensional mapping

Let $Q_i = \prod_{j=0}^{j} l_i$

Thus

$$i(1^d) = x_d Q_{d-1} + x_{d-1} Q_{d-2} + \ldots + x_3 Q_2 + x_2 Q_1 + x_1 Q_0$$

When d = 1

we have

$i(1^d) = x_1 Q_0 = x_1$ (There is one to one mapping since $Q_0 = 1$) —- (a)

When d = 2

$$i(1^d) = x_2 Q_1 + x_1 Q_0$$

Performing integer division(division followed by floor operation) by Q_1

$i(1^d)//Q_1 = x_2 + (x_1 Q_0)//Q_1$ ——————— (b_1)

$x_2 = i(1^d)//Q_1$ (since $x_1 Q_0 < Q_1$ by Corollary 1) —— (b_2)

$(i(1^d) - x_2 Q_1) = x_1 Q_0$ ————- (b_3)

b_1 is similar to equation (a)

$$x_1 = (i(1^d) - x_2 Q_1)$$

When d = 3

$$i(1^d) = x_3 Q_2 + x_2 Q_1 + x_1 Q_0$$

Performing integer division(division followed by floor operation) by Q_2

$$i(1^d)//Q_2 = x_3 + (x_2 Q_1 + x_1 Q_0)//Q_2$$

$x_3 = i(1^d)//Q_2$ —- —— —- —- (c_1)

$i(1^d) - x_3 Q_2 = x_2 Q_1 + x_1 Q_0$ —- —— —- —- (c_2)

c_2 is similar to b_1

Thus from c_2

$x_2 = (i(1^d) - x_3 Q_2)//Q_1$ —————————— (c_3)

$x_1 = i(1^d) - x_3 Q_2 - x_2 Q_1$ —————————— (c_4)

For an n dimensional

$$i(1^d) = \sum_{i=1}^{n} x_i \prod_{j=0}^{i-1} l_i$$

$$i(1^d) = x_n Q_{n-1} - \sum_{i=1}^{n-1} x_i Q_{i-1}$$

Similar to equation a, b_1,b_2, c_1, c_2,c_3,c_4

$$x_n = i(1^d)//Q_{n-1}$$

$$x_{n-1} = (i(1^d) - x_n Q_{n-1})//Q_{n-1}$$

$$x_{n-2} = (i(1^d) - x_n Q_{n-1} - x_{n-1} Q_{n-2})//Q_{n-3}$$

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$$x_j = (i(1^d) - \sum_{k=(j-1)}^{n-1} x_k Q_{k-1})//Q_{j-1}$$

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.
.
.

$$x_3 = (i(1^d) - \sum_{k=4}^{n-1} x_k Q_{k-1})//Q_2$$

$$x_2 = (i(1^d) - \sum_{k=3}^{n-1} x_k Q_{k-1})//Q_1$$

$$x_1 = (i(1^d) - \sum_{k=2}^{n-1} x_k Q_{k-1})//Q_0 = (i(1^d) - \sum_{k=2}^{n-1} x_k Q_{k-1})$$

The above set of equations give generalised representation of an n-d reverse mapping starting from finding n^{th} dimension all the way till finding the 1^{st}