Functions

Mathematics for Engineering (6 ECTS)

Edunation Pathway Programme

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EDUNATION

Sets

$+\infty$

$1 \in A$

A *set* is a collection of *elements*, e.g.

$$\{1, 2, 3, 4, 5\}$$

$11 \notin A$

If we denote this set by A, we can write $1 \in A$ to express that A contains the element 1.

Some important sets

$2/1$

- $\mathbb{N}$ is the set of **natural numbers**, i.e. $\{0, 1, 2, \dots\}$
- $\mathbb{Z}$ is the set of **integers**, i.e. $\{\dots, -1, 0, 1, \dots\}$
- $\mathbb{Q}$ is the set of **rational numbers**, i.e. numbers $\frac{k}{n}$, where $k, n \in \mathbb{Z}$
- $\mathbb{R}$ is the set of **real numbers** ("the numberline")

0

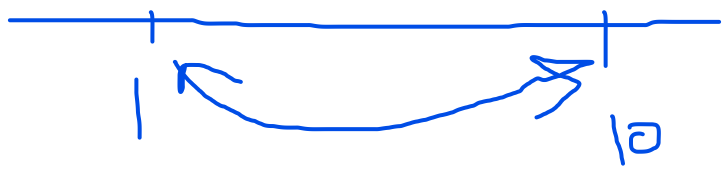
Sets

We can also write sets by giving some conditions for their elements. For example,

$$\{k \in \mathbb{Z} \mid k = 2n \text{ for some } n \in \mathbb{Z}\}$$

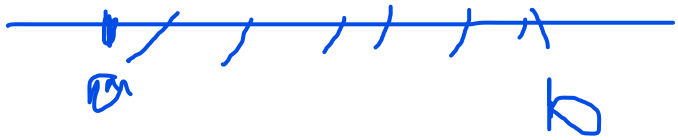
is the set of even integers, i.e. $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

Sets: Intervals



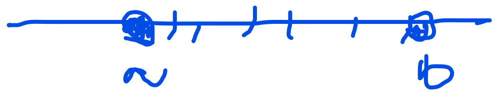
- An *open interval* is a set

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\}.$$



- A *closed interval* is a set

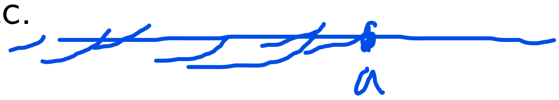
$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}.$$

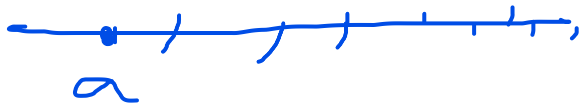


- We also use the following notations:

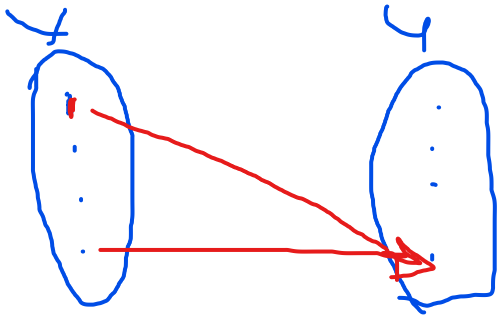
$$]-\infty, a] = \{x \in \mathbb{R} \mid x \le a\}, \quad [a, \infty[= \{x \in \mathbb{R} \mid x \ge a\}$$

etc.





Functions

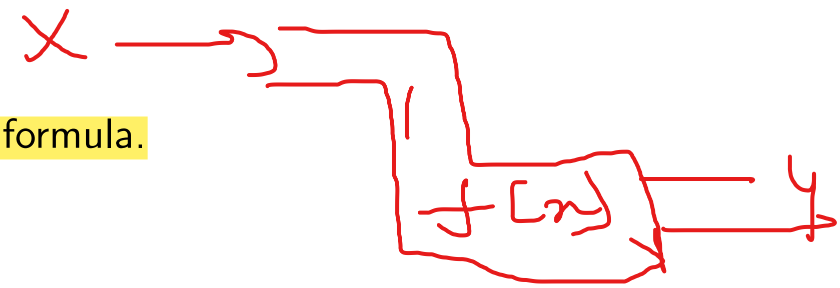


Definition

Let X and Y be sets. A *function* f from X to Y is a rule that maps every element in X to exactly one element in Y. It is denoted by $f : X \to Y$.

Notice that f has three components:

- The *domain* X
- The *codomain* Y
- Some rule, often given as a formula.



range

Functions

The diagram below shows how every $x \in X$ has a unique *image* $f(x) \in Y$.

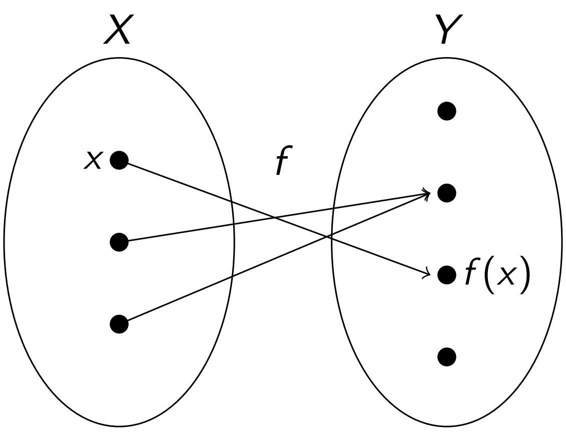
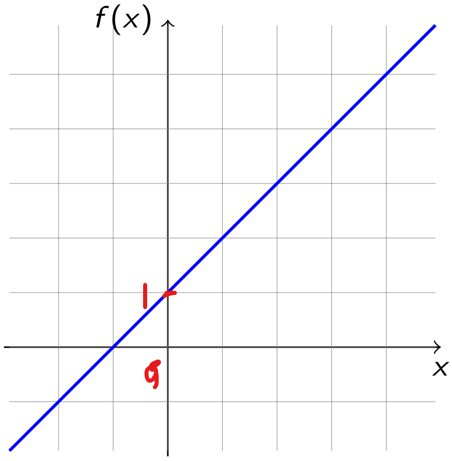


Figure: A function with its domain X and codomain Y.

Example

$y = x + 1$ $\quad$ $y = mx + c$

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$.
The rule $f(x) = x + 1$ can also be written as $x \mapsto x + 1$.



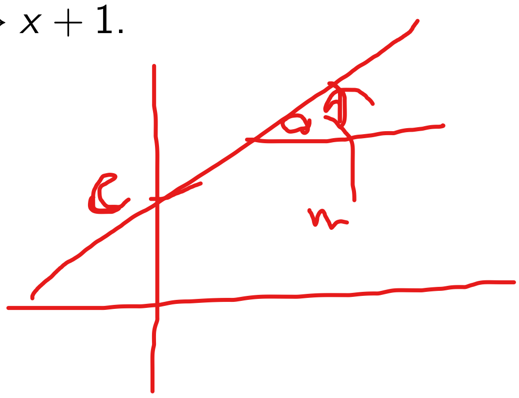


Figure: The graph of the function f.

Question

Pause the video to answer the following question.
Are f and g functions $\mathbb{R} \to \mathbb{R}$? Explain why or why not.

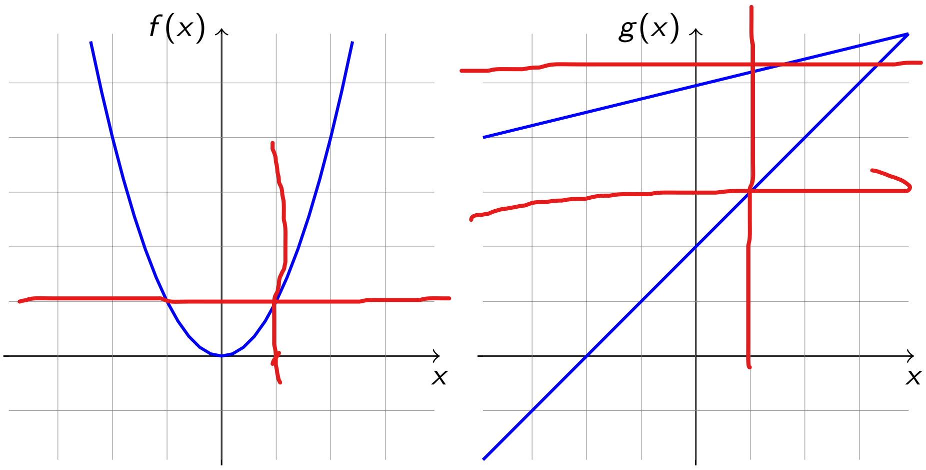
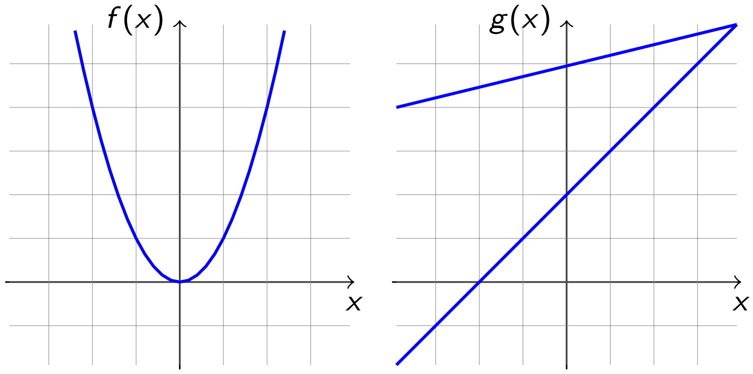


Figure: f (on the left) and g (on the right).

Solution

- f **is a function**. Every $x \in \mathbb{R}$ is mapped to exactly one $f(x) \in \mathbb{R}$. Here f is defined by $f(x) = x^2$. This graph is called a parabola.
- g **is not a function**. We see that there are $x \in \mathbb{R}$ that do not have unique value $g(x) \in \mathbb{R}$.
  For example, in the graph $g(0) = 2$ and $g(0) = 5$.



$g \cdot f(x) = x + 1$

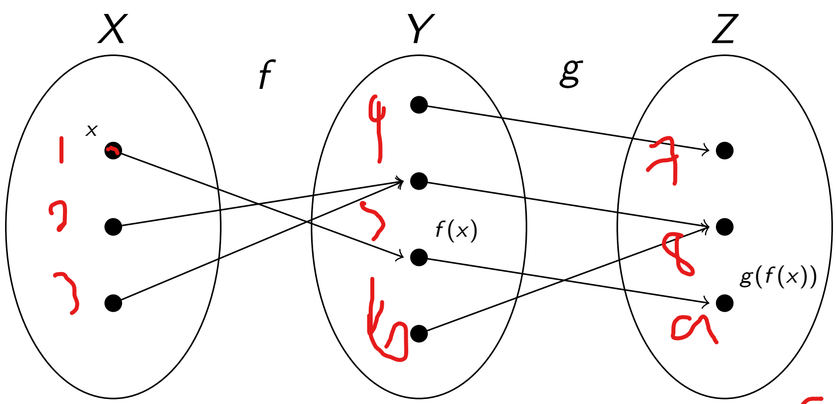
Composite Functions

$f(2)$

Definition

Let $f: X \to Y$ and $g: Y \to Z$ be functions. The *composite function* of f and g, denoted by $g \circ f$, is a function $X \to Z$ defined by

$$\boxed{(g \circ f)(x) = g(f(x)).}$$



$f(1) = 5$

$g(5) = 9$

Figure: The composite function $g \circ f$. $g(f(1))$

Example

$g \circ f = g(f(x)) = g(x+1) = (x+1)^2$

Consider the functions

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x + 1$$

and

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2.$$

Let us determine the functions $g \circ f$ and $f \circ g$.

$f \circ g = f(g(x)) = f(x^2) = x^2 + 1$

Example

$a + b = b + a$

$5 + 2 = 2 + 5$

- For $f \circ g$, we get

$$f(g(x)) = f(x^2) = x^2 + 1.$$

- For $g \circ f$, we get

$$g(f(x)) = g(x+1) = (x+1)^2 = x^2 + 2x + 1.$$

Remark

- The composition of functions **is not** *commutative* (see Example).
- However, the composition of functions is *associative*:

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Identity Function

$A = \{1, 2, 3, 4, 5\}$

$y = x$

Definition

Let A be a set. A function $f: A \to A$ is called an *identity function*, if

$$f(x) = x \quad \text{for all} \quad x \in A.$$

It is often denoted by id_A.

Notice that id has a similar role as 0 in addition or 1 in multiplication.
If $f: X \to Y$ is a function, then

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_Y \circ f = f.$$

Inverse Functions

Definition

Let $f: X \to Y$ and $g: Y \to X$. We say that g is the *inverse function* of f, if

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y.$$

The inverse of f is denoted by f^{-1}.

Notice that the inverse does not always exist.

Proof of Uniqueness

Theorem

If a function has an inverse function, then it is unique.

Proof. Let $f : X \to Y$ be a function. Assume that f has inverse functions g and h. Now it follows that

$$g = g \circ \mathrm{id}_Y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_X \circ h = h.$$

Since $g = h$, the inverse of f is unique.

Question

$y = 2x + 1$

$x = \frac{y-1}{2}$

$= \frac{x-1}{2}$

Pause the video to answer the following question.

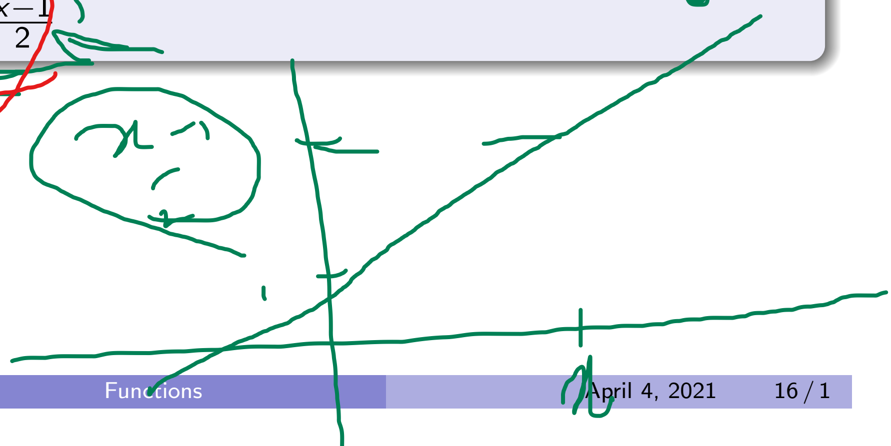
Choose the right answer.

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1$. Which one is the inverse f^{-1}?

1. $g_1 : \mathbb{R} \to \mathbb{R}, \quad g_1(x) = \frac{x}{2} - 1$
2. $g_2 : \mathbb{R} \to \mathbb{R}, \quad g_2(x) = -2x - 1$
3. $g_3 : \mathbb{R} \to \mathbb{R}, \quad g_3(x) = \frac{x-1}{2}$

$2x+1$

$\frac{x-1}{2}$



Solution

$f\left(\frac{x-1}{2}\right) = x-1+1$
$= x$

The inverse of f is g_3.
We can check that

$$
\begin{aligned}
(f \circ g_3)(x) &= f(g_3(x)) \\
&= 2\left(\frac{x-1}{2}\right) + 1 \\
&= \frac{2(x-1)}{2} + 1 \\
&= x - 1 + 1 \\
&= x,
\end{aligned}
$$

so $f \circ g_3 = \mathrm{id}_{\mathbb{R}}$. Next we will check that also $g_3 \circ f = \mathrm{id}_{\mathbb{R}}$.

Solution

$g_3(f(x)) = \frac{2x+1-1}{2} = x$

We see that

$$
\begin{aligned}
(g_3 \circ f)(x) &= g_3(f(x)) \\
&= \frac{(2x+1)-1}{2} \\
&= \frac{2x}{2} \\
&= x,
\end{aligned}
$$

so $g_3 \circ f = \mathrm{id}_{\mathbb{R}}$. Hence $f^{-1} = g_3$.

How to Find the Inverse Function

Suppose we were only given the formula $f(x) = 2x + 1$ in the previous question. How could we find f^{-1}?

Answer

Write $y = f(x)$ and solve for x. Now $x = f^{-1}(y)$.

In the previous example, we would write

$$y = 2x + 1$$

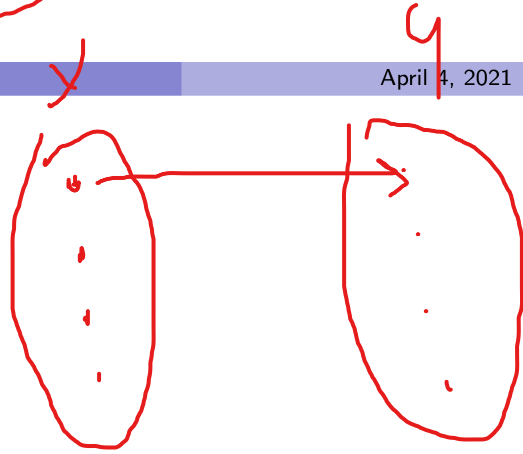
which is equivalent to

$$y - 1 = 2x.$$

Divide the both sides by 2 and we get

$$x = \frac{y-1}{2}.$$

$y = \frac{x-1}{2}$



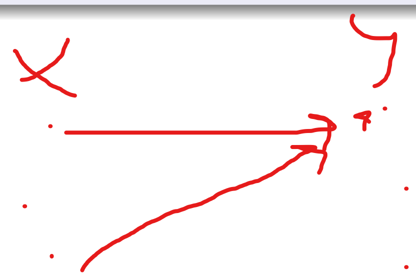
Injections and Surjections

Definition

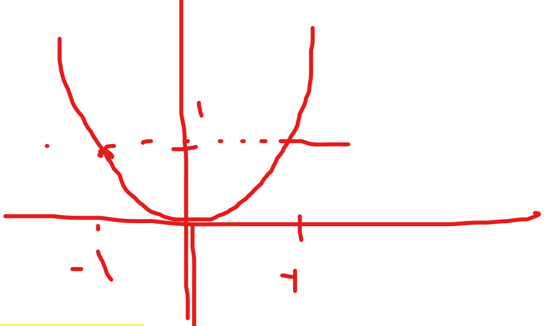
A function $f : X \to Y$ is *injective* (or *one-to-one*), if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for every $x_1, x_2 \in X$.

Definition

A function $f : X \to Y$ is *surjective* (or *onto*), if for every $y \in Y$ there is $x \in X$ such that $y = f(x)$.



Example



The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not injective. For example, we have

$$f(-1) = 1 = f(1), \quad \text{but} \quad -1 \neq 1.$$

We also notice that f is not surjective. For example, for $-1 \in \mathbb{R}$ (in the codomain), there is no $x \in \mathbb{R}$ such that

$$f(x) = x^2 = -1.$$

Example

If we define $g: [0, \infty[\to \mathbb{R}$, $g(x) = x^2$, then g is injective.
Notice that the domain of f was $\mathbb{R}$.

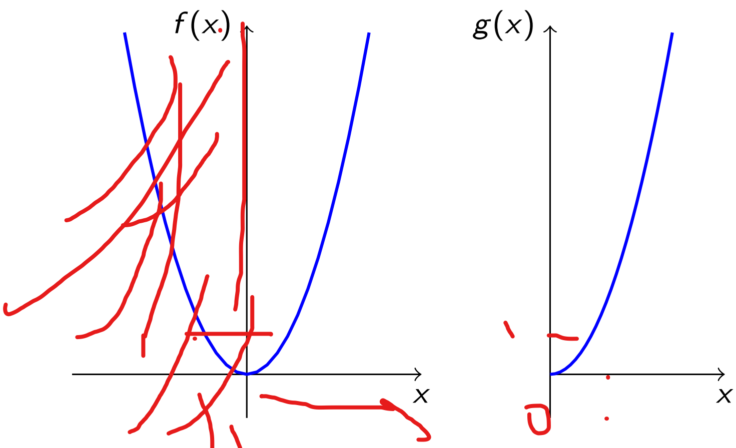


Figure: Functions f and g.

How would you change the codomain of g to get a surjective function?

$g: [0, \infty] \to [0, \infty]$

Bijections

Definition

If a function is both injective and surjective, then it is called *bijective*.

Recall that there are functions that do not have the inverse function. Now we can state the following theorem.

Theorem

A function has the inverse if and only if it is bijective.

Example

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x+1}{2}$. We will show that f is bijective.
First, we show that f is injective. We assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$. Our goal is to deduce that $x_1 = x_2$. We can write

$$f(x_1) = f(x_2)$$

in the form

$$\frac{x_1 + 1}{2} = \frac{x_2 + 1}{2}.$$

If we multiply both sides by 2, we get

$$x_1 + 1 = x_2 + 1.$$

Now we subtract 1 from both sides and we are left with

$$x_1 = x_2.$$

This proves the injectivity.

Example

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x+1}{2}$. We will show that f is bijective.
Next, we show that f is surjective. For this purpose, let $y \in \mathbb{R}$ (codomain). Our goal is to find $x \in \mathbb{R}$ (domain) such that $y = f(x)$.
Let $x = 2y - 1$. Clearly $x \in \mathbb{R}$. Now we see that

$$f(x) = \frac{x+1}{2} = \frac{(2y-1)+1}{2} = \frac{2y}{2} = y.$$

This proves the surjectivity.
Since f is injective and surjective, f is bijective.