Applied category theory, 18.S097 at MIT.

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I. Chapter 1

1.1 Lecture 1

Category theory is a fundamental part of mathematics, it has branched out into a variety of subjects like computer science and physics. Applied category theory is a relatively new field.

1.1.1 Generative/ Cascade effects

A set of different objects while might be observed to not have any common interactions, but when looked at differently might interact with each. Eg: contagion.

Definition I..1 *A set is a bag of dots.*

$$A = \{a, b, c\} \quad where, a, b, c \in A$$

Similarly, there are different sets of numbers :

$$\begin{aligned}
&\mathbb{N}, \quad the\, set\, of\, natural\, numbers\\
&\mathbb{Z}, \quad the\, set\, of\, integers\\
&\mathbb{R}, \quad the\, set\, of\, real\, numbers\\
&\mathbb{B}, \quad the\, set\, of\, booleans
\end{aligned}$$

Definition I..2 *Product Sets: Suppose A, B are sets then,*

$$A * B = \{(a, b) \mid a \in A, b \in B\}$$

In category theory, we think of objects in terms of the roles they play.

Definition I..3 *A relation, R, on sets A and B is defined by*

$$R \subset A * B$$

Every function is a relation. Properties like order, equivalence and tolerance are relations as well.

Definition I..4 *A function, f, from A to B, denoted $f : A \to B$ is a relation on A and B. $R \subset A * B$, satisfying*

- *For all $a \in A$, there exists an element $b \in B$ such that $(a, b) \in R$.*

- *For all a, b1, b2, if $(a, b1) \in R$ and $(a, b2) \in R$, then $b1 = b2$.*

Definition of injective (no two x'es are mapped to the same y) and surjective (for every y there exists an x in the mapping) functions.

We can order partitions, Say we have two partitions P1 and P2, then we say $P1 \leq P2$ if there is a function $P1 \to P2$ making the diagram commute.

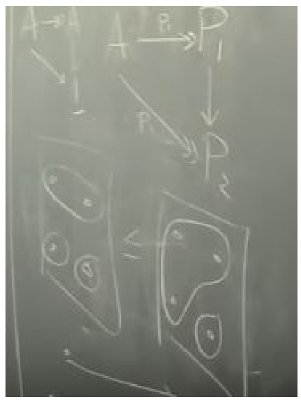

Figure 1: When is a partition lesser than another partititon

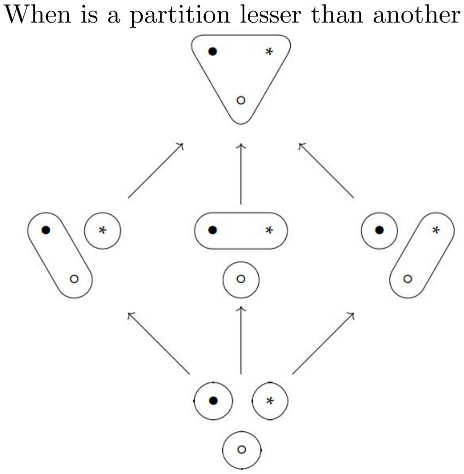

Figure 2: Can be thought of as a lattice theory structure/ poset

A *pre-order* is

1. a set S
2. a relation $"\leq" \subset S * S$

and also satisfying two properties i.e its reflexive and transitive.
Order creates join.

1.2 Lecture 2

Starting with the definition of pre-order, pre-order's are also called posets, A pre-order happens to be a fundamental thing to categories. A preorder is a category with at most one morphism between any two objects.
A preorder is a Bool-enriched category.
Preoder's enable meet and join.

Definition I..5 *Let $A \subset P$, then $p \in P$ is a*meet*of A if*

- *For all $a \in A$, $p \leq a$.*

- *For all $q \in P$, such that for all $a \in A, q \leq a, q \leq p$.*

A meet is the greatest lower bound. Similarly, a join is the least upper bound.
In elementary boolean logic, the truth table of meet would correspond to AND and the truth table of join would correspond to OR.
Consider a lattice of power sets, youtube video: 12:20, meet is equal to intersection, and join is equal to union.
Similar notions between preorder and set theory.
When you consider natural numbers as an order with less than to as the relation, then the meet is the minimum and the join is the maximum. When you consider natural numbers as an order with relation, $a \leq b$ if a/b then the meet is the greatest common divisor and the join is the lowest common multiple.
Remark 1: Meets/joins may not always exist. A set might not have a meet, if it doesnt have a lower bound. For example, the set of even integers, there is no lowest upper bound i.e the join does not exist.
Remark 2: Multiple meets/joins might exist. For example, an isomorphic point.

1.2.1 Monotone Maps

Definition I..6 *A monotone map $f : (P, \leq) \to (q, \leq)$ is a function. $f : P \to Q$ such that if $p \leq_p p'$ in P, then $f(p) \leq_q f(p')$ in Q.*

Example, The tree of life. When you take the example of lion(species), panthera(genus), carnivore, mammal(class), and sapiens(species), homo(genus) then you see even though there are two preorders, there's a universal map(given below) which connects them.

Species → *Genus* → *Family* → *order* → *class* → *phylum* → *kingdom*.

Cardinality is a monotone map. Contagion under Boolean is a monotone map.

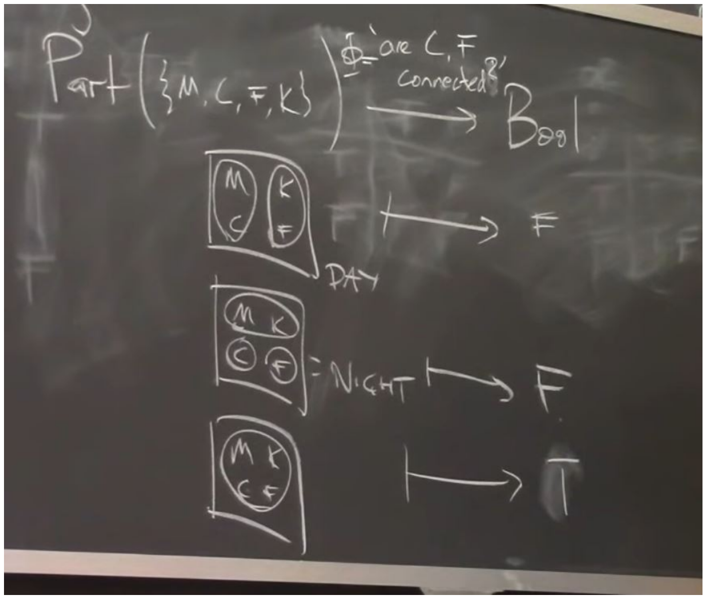

Figure 3: Contagion under boolean is a monotone map.

Definition I..7 *A monotone map* $f : P \to Q$ *preserves joins if* $A \subset P$, $f(\vee A) = \vee\{f(a) | a \in A\}$

Remark: A monotone map will preserve order, but the universal things like meet and join need not be preserved.

1.2.2 Galois Connections

Definition I..8 *A galois connection is a pair of posets p and q and monotone map f from p to q and a monotone map g from q to p, i.e*

Given P, Q are preorders and f, g are monotone maps such that,

$\forall p \in P, q \in Q$

$f(p) \leq q$ *iff* $p \leq g(q)$

where f is the left adjoint and g is the right adjoint.

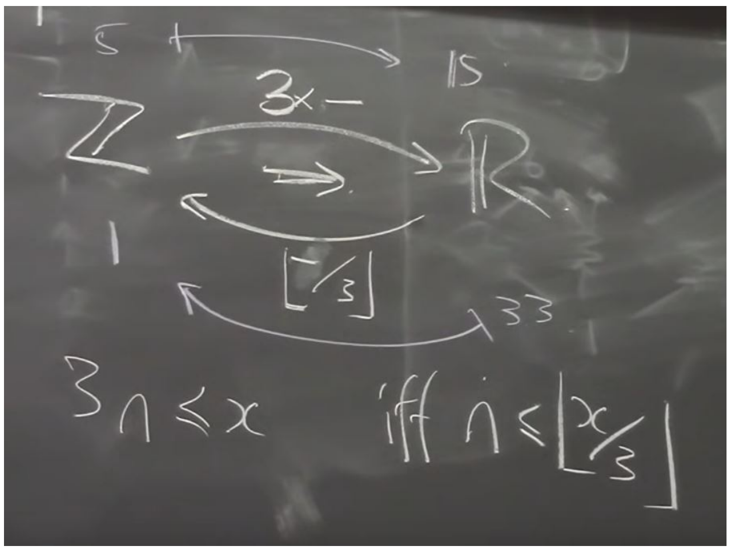

Figure 4: Example of a galois connection

You'll see throughout these notes, that there are these abstract/ universal constructions and you see some sort of important structure follow through. For example, In the above example of figure 4, the floor is a structure that follows because of the structure given in the left adjoint.

Definition I..9 *Adjoint functor theorem for preoders: A monotone map f is a left adjoint iff it preserves joins. A monotone map g is a right adjoint iff it preserves meets.*

1.3 Additional Chapter Notes