QR Factorization:
If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

ex.1. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Soln:
<u>Construction an orthonormal basis for Col A</u>
The columns of A are the vectors $\{x_1, x_2, x_3\}$
Let $v_1 = x_1 = (1,1,1,1)$

$v_2 = x_2 - \frac{x_2.v_1}{v_1.v_1}v_1 = (0,1,1,1) - \frac{(0,1,1,1).(1,1,1,1)}{(1,1,1,1).(1,1,1,1)}(1,1,1,1)$

$= (0,1,1,1) - \frac{3}{4}(1,1,1,1) = (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

$v_3 = x_3 - \frac{x_3.v_1}{v_1.v_1}v_1 - \frac{x_3.v_2}{v_2.v_2}v_2$

$= (0,0,1,1) - \frac{(0,0,1,1).(1,1,1,1)}{(1,1,1,1).(1,1,1,1)}(1,1,1,1) - \frac{(0,0,1,1).(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})}{(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})}(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{2}{3}(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = (0, -\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$

$\therefore \{v_1, v_2, v_3\}$ forms an orthogonal basis of Col A .

$\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}), (0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}$ forms an orthonormal basis of Col A.

$\therefore Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$

<u>To construct an upper triangular invertible matrix</u>
We have $A = QR \implies Q^TA = Q^TQR \implies Q^TA = IR \implies Q^TA = R$ i.e., $R = Q^TA$.

$\therefore R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$

ex.2. Find a QR factorization of $A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$

Soln:
$\{x_1, x_2, x_3\}$ are the columns of the matrix A.
Let $v_1 = x_1 = (1,-1,-1,1,1)$

$v_2 = x_2 - \frac{x_2.v_1}{v_1.v_1}v_1 = (2,1,4,-4,2) - \frac{(2,1,4,-4,2).(1,-1,-1,1,1)}{(1,-1,-1,1,1).(1,-1,-1,1,1)}(1,-1,-1,1,1)$

$= (2,1,4,-4,2) - \frac{-5}{5}(1,-1,-1,1,1) = (3,0,3,-3,3)$

$v_3 = x_3 - \frac{x_3.v_1}{v_1.v_1}v_1 - \frac{x_3.v_2}{v_2.v_2}v_2$

$= (5,-4,-3,7,1) - \frac{(5,-4,-3,7,1).(1,-1,-1,1,1)}{(1,-1,-1,1,1).(1,-1,-1,1,1)}(1,-1,-1,1,1) - \frac{(5,-4,-3,7,1).(3,0,3,-3,3)}{(3,0,3,-3,3).(3,0,3,-3,3)}(3,0,3,-3,3)$

$= (5,-4,-3,7,1) - \dfrac{20}{5}(1,-1,-1,1,1) - \dfrac{-12}{36}(3,0,3,-3,3) = (2,0,2,2,-2).$
$\therefore \{(1,-1,-1,1,1),(3,0,3,-3,3),(2,0,2,2,-2)\}$ forms an orthogonal basis of Col A.
$\{(1/\sqrt{5},-1/\sqrt{5},-1/\sqrt{5},1/\sqrt{5},1/\sqrt{5}),(1/2,0,1/2,-1/2,1/2),(1/2,0,1/2,1/2,-1/2)\}$ forms an orthonormal basis of Col A.

$$\therefore Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

ex.3. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

Soln: The columns of A are the vectors $\{x_1, x_2, x_3\}$
where $x_1 = (3,1,-1,3), x_2 = (-5,1,5,-7), x_3 = (1,1,-2,8)$.
Let $v_1 = (3,1,-1,3)$
$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1} v_1 = (-5,1,5,-7) - \frac{(-5,1,5,-7).(3,1,-1,3)}{(3,1,-1,3).(3,1,-1,3)}(3,1,-1,3)$
$= (-5,1,5,-7) - \frac{-40}{20}(3,1,-1,3) = (1,3,3,-1)$
$v_3 = \dfrac{x_3.v_1}{v_1.v_1} v_1 - \dfrac{x_3.v_2}{v_2.v_2} v_2 = (1,1,-2,8) - \frac{(1,1,-2,8).(3,1,-1,3)}{(3,1,-1,3).(3,1,-1,3)}(3,1,-1,3) - \frac{(1,1,-2,8).(1,3,3,-1)}{(1,3,3,-1).(1,3,3,-1)}(1,3,3,-1)$
$= (1,1,-2,8) - \frac{30}{20}(3,1,-1,3) - \frac{-10}{20}(1,3,3,-1) = (-3,1,1,3)$
$\{(3,1,-1,3),(1,3,3,-1),(-3,1,1,3)\}$ is an orthogonal basis for the column space of the given matrix.

ex.4. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

Soln: The columns of A are the vectors $\{x_1, x_2, x_3\}$
where $x_1 = (-1,3,1,1), x_2 = (6,-8,-2,-4), x_3 = (6,3,6,-3)$.
Let $v_1 = (-1,3,1,1)$
$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1} v_1 = (6,-8,-2,-4) - \frac{(6,-8,-2,-4).(-1,3,1,1)}{(-1,3,1,1).(-1,3,1,1)}(-1,3,1,1)$
$= (6,-8,-2,-4) - \frac{-36}{12}(-1,3,1,1) = (3,1,1,-1)$
$v_3 = \dfrac{x_3.v_1}{v_1.v_1} v_1 - \dfrac{x_3.v_2}{v_2.v_2} v_2 = (6,3,6,-3) - \frac{(6,3,6,-3).(-1,3,1,1)}{(-1,3,1,1).(-1,3,1,1)}(-1,3,1,1) - \frac{(6,3,6,-3).(3,1,1,-1)}{(3,1,1,-1).(3,1,1,-1)}(3,1,1,-1)$
$= (6,3,6,-3) - \frac{6}{12}(-1,3,1,1) - \frac{30}{12}(3,1,1,-1) = (-1,-1,3,-1)$
$\{(-1,3,1,1),(3,1,1,-1),(-1,-1,3,-1)\}$ is an orthogonal basis for the column space of the given matrix.