JOINT PROBABILITIES

Let X and Y be random variables on the same sample space S with respective range spaces $R_X = \{x_1, \cdots x_n\}$ and $R_Y = \{y_1, y_2, \cdots y_m\}$. The Joint distribution (OR) joint probability function of X and Y is the function f on the product space $R_X \times R_Y$ defined by

$$f(x_i, y_i) \text{ (or) } p(x_i, y_i) = P[X = x_i, Y = y_j] = P[\{s \in S;\ X(s) = x_i, Y(s) = y_i\}]$$

If i) $p(x_i, y_i) \geq 0$

ii) $\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$

JOINT distribution table

$X \downarrow \quad Y \rightarrow$	y_1	y_2	y_3	$\cdots$	y_m	$p_i = \sum_{j=1}^{m} p(x_i, y_j)$
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1m}	p_1
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2m}	p_2
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3m}	p_3
$\vdots$	$\vdots$				$\vdots$	$\vdots$
x_n	p_{m1}	p_{n2}	p_{n3}	$\cdots$	p_{nm}	p_n
$p_j = \sum_{i=1}^{n} p(x_i, y_j)$	p_1	p_2	p_3	$\cdots$	p_m	$\boxed{1}$

NOTE: $p(x_i, y_i) = P_{xy}(x, y)$

MARGINAL DISTRIBUTION OF X:

In the joint probability distribution, If the pmf of only X is taken then it is called Marginal distribution of X denoted by p_i (or) $p_X(x)$ (or) $p(x_i)$.

ie. $p_X(x) = p(x_i) = \sum_y p(x_i, y_j) = \sum_y p_{XY}(x, y)$

MARGINAL DISTRIBUTION OF Y:

In the joint probability distribution, If the pmf of only Y is taken then it is called Marginal distribution of Y denoted by p_j (or) $p_Y(y)$ (or) $p(y_j)$

i.e $p_Y(y) = p(y_j) = \sum_x p(x_i, y_j) = \sum_x p_{XY}(x, y)$.

Joint cummulative distribution function of X & Y

$$F(x,y) = F_{XY}(x,y) = P(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_j \le y} p(x_i, y_j)$$

Conditional probability Mass function of X given Y

$$p(y/x) = p_{Y/X}(y/x) = \frac{p_{XY}(x,y)}{p_X(x)} = \frac{p(x,y)}{p(x)}$$

Conditional probability Mass function of Y given X

$$p(x/y) = p_{X/Y}(x/y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{p(x,y)}{p(y)}$$

Expectation (Mean), ~~Variance~~ Correlation and Covariance

If X and Y are two discrete random variables having the joint probability function $p(x,y)$ then the expectations of X and Y are defined as follows.

$$\mu_X = E(X) = \sum x_i\, p(x_i) = \sum x\, p(x)$$

$$\mu_Y = E(Y) = \sum y_j\, p(y_j) = \sum y\, p(y)$$

$$\mu_{XY} = E(XY) = \sum x_i y_j\, p(x,y) = \sum x_i y_j\, p_{ij}$$

COVARIANCE :

Let X and Y be random variables with the joint distribution $p(x,y)$, respective means μ_X and μ_Y and respective variances σ_X^2 and σ_Y^2. The Covariance of X and Y denoted and defined as follows.

$$\begin{aligned}
\mathrm{COV}(X,Y) &= E[(X-E(X))(Y-E(Y))] \\
&= E[XY - X E(Y) - Y E(X) + E(X)E(Y)] \\
&= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)
\end{aligned}$$

$$\boxed{\mathrm{COV}(X,Y) = E(XY) - E(X)E(Y).}$$

NOTE: If X and Y are independent, then $\mathrm{COV}(X,Y) = 0$, so that $E(XY) = E(X)E(Y)$

Correlation:

The correlation of X and Y denoted by $\rho(X,Y)$ (or) ~~$r(X,Y)$~~ γ_{XY} and is defined as follows

$$\rho(X,Y) = \gamma_{XY} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} \quad ; \quad \begin{array}{l} \sigma_X = SD(X) = \sqrt{E(X^2) - [E(X)]^2} \\ \sigma_Y = SD(Y) = \sqrt{E(Y^2) - [E(Y)]^2} \end{array}$$

NOTE:

(i) $E(X+Y) = E(X) + E(Y)$

(ii) $\rho(X,Y) = \rho(Y,X)$

iii) $-1 \le \rho \le 1$

Random Vectors:

Suppose that we have n random variables and it is convenient to put them in a vector $X = [X_1\ X_2 \cdots X_n]^T$ called as RANDOM VECTOR [i.e. $X : S \rightarrow \mathbb{R}^n$]

Expectation of a random vector is simply the expectation applied to each component

$$E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix}$$

Covariance Matrix:

The variance is generalized by the covariance matrix denoted by Σ & is the $n \times n$ matrix whose entries are given by $\Sigma_{ij} = \text{cov}[X_i, X_j]$ or $\text{cov}(X_i, X_j)$

i.e

$$\Sigma = \begin{bmatrix} \underbrace{\text{cov}(X_1, X_1)}_{Var(X_1)} & \cdots & \text{cov}(X_1, X_n) \\ \text{cov}(X_2, X_1) & \cdots & \text{cov}(X_2, X_n) \\ \vdots & \ddots & \vdots \\ \text{cov}(X_n, X_1) & \cdots & \underbrace{\text{cov}(X_n, X_n)}_{Var(X_n)} \end{bmatrix}$$

$$= \begin{bmatrix} E(X_1^2) - E(X_1)E(X_1) & \cdots & E(X_1 X_n) - E(X_1)E(X_n) \\ \vdots & & \\ E(X_n X_n) - E(X_n)E(X_1) & \cdots & E(X_n^2) - E(X_n)E(X_n) \end{bmatrix}$$

$$= \begin{bmatrix} E(x_1^2) & \cdots & E(x_1 x_n) \\ \vdots & \ddots & \vdots \\ E(x_n x_1) & \cdots & E(x_n^2) \end{bmatrix} - \begin{bmatrix} E(x_1)E(x_1) & \cdots & E(x_1)E(x_n) \\ \vdots & \ddots & \vdots \\ E(x_n)E(x_1) & \cdots & E(x_n)E(x_n) \end{bmatrix}$$

$$= E(XX^T) - E(X)E(X)^T$$

$$= E\left[(X - E(X))(X - E(X))^T\right] \quad //$$

<u>NOTE</u>: i) Σ^{-1} is called the precision matrix

ii) Σ is positive semi definite $\because x^T \Sigma x \geq 0$, for any x.

iii) $\Sigma = \Sigma^T$, ~~then~~ Σ is symmetric.

iv) $r_{12} = \dfrac{\operatorname{cov}(x_1, x_2)}{\sigma_1 \sigma_2}$ → Correlation coefficient btw x_1 and x_2

$\vdots$

$r_{1n} = \dfrac{\operatorname{cov}(x_1, x_n)}{\sigma_1 \sigma_n}$ → —"— x_1 & x_n.

<u>Example</u>:

$x_1 \to$ Age, $x_2 \to$ Height, $x_3 \to$ weight ; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

~~cov(X)~~ <u>Covariance matrix</u>

$$\Sigma = \begin{bmatrix} \operatorname{cov}(x_1, x_1) & \operatorname{cov}(x_1, x_2) & \operatorname{cov}(x_1, x_3) \\ \operatorname{cov}(x_2, x_1) & \operatorname{cov}(x_2, x_2) & \operatorname{cov}(x_2, x_3) \\ \operatorname{cov}(x_3, x_1) & \operatorname{cov}(x_3, x_2) & \operatorname{cov}(x_3, x_3) \end{bmatrix}$$

NOTE:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$cov(x,y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Trivariate data :

A statistical data with respect to three variables (x_1, x_2, x_3) is called trivariate data

Notations:

1. $\bar{x}_1 = \frac{\sum x_1}{n}$; $\bar{x}_2 = \frac{\sum x_2}{n}$; $\bar{x}_3 = \frac{\sum x_3}{n}$

2. $\sigma_1^2 = \frac{1}{n}\sum (x_1 - \bar{x}_1)^2 = \frac{1}{n}\sum x_1^2 - (\bar{x}_1)^2$,

$\sigma_2^2 = \frac{1}{n}\sum (x_2 - \bar{x}_2)^2 = \frac{1}{n}\sum x_2^2 - (\bar{x}_2)^2$

$\sigma_3^2 = \frac{1}{n}\sum (x_3 - \bar{x}_3)^2 = \frac{1}{n}\sum x_3^2 - (\bar{x}_3)^2$

3. $r_{12} = \frac{cov(x_1, x_2)}{\sigma_1 \sigma_2}$ → correlation coefficient btw x_1 & x_2

$r_{13} = \frac{cov(x_1, x_3)}{\sigma_1 \sigma_3}$ → correlation coefficient btw x_1 & x_3

$r_{23} = \frac{cov(x_2, x_3)}{\sigma_2 \sigma_3}$ → correlation coefficient btw x_2 & x_3.

In general;

$$\sigma_i^2 = \frac{1}{n}\sum X_i^2 \quad ; \quad X_i = x_i - \bar{x}_i$$

$$cov(x_i, x_j) = \frac{1}{n}\sum X_i X_j \quad ; \quad X_i = x_i - \bar{x}_i \quad X_j = x_j - \bar{x}_j$$

PRODUCT MOMENT Formula.

$$r_{ij} = \frac{cov(x_i, x_j)}{\sigma_i \sigma_j} = \frac{\sum X_i X_j}{n \sigma_i \sigma_j}$$

JOINT PROBABILITY DISTRIBUTION OF CONTINUOUS RANDOM VARIABLE :

1. **Joint density function**

The joint probability density function (pdf) of two-dimensional continuous random variable (X, Y) is defined as a function $f(x,y)$ satisfying the following conditions.

(i) $f(x,y) \geq 0, \quad \forall\, x, y$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$ ← TOTAL PROBABILITY.

2. **Marginal density function of X and Y**

The function $p_1(x) = g(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$ is called the marginal density function of X.

The function $p_2(y) = h(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$ is called the marginal density function of Y.

3. **Conditional probability density function of X given Y=y**

$$f(x/y) = \frac{f(x,y)}{p_2(y)} \quad ; \; p_2(y) > 0 \quad \Big| \quad \frac{f_{XY}(x,y)}{f_Y(y)}$$

Conditional probability density function of Y given X=x

$$f(y/x) = \frac{f(x,y)}{p_1(x)} \quad ; \; p_1(x) > 0 \quad \Big| \quad \frac{f_{XY}(x,y)}{f_X(x)}$$

4. The joint cummulative distribution function (cdf) is defined as

$$F(x,y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dx\, dy \quad ; -\infty < x < \infty, -\infty < y < \infty.$$

5. Two random variables X and Y are said to be independent (or) stochastically independent if

$$P_1(x) P_2(y) = f(x, y)$$

(or)

$$E(X) E(Y) = E(XY).$$

6. Mean, variance and Covariance

$$E(X) = \int_{-\infty}^{\infty} x \, p_1(x) \, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y \, p_2(y) \, dy$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f(x, y) \, dx \, dy$$

$$\text{cov}(X, Y) = E(XY) - E(X) E(Y)$$

7. Correlation of X and Y

$$\rho(X, Y) = r_{XY} = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

where $\sigma_X = SD(X) = \sqrt{V(X)}$

$\sigma_Y = SD(Y) = \sqrt{V(X)}$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

——— . ———

Problems:

1) Consider the following bivariate distribution $p(x,y)$ of discrete random variables X and Y

Y \ X	$x_1=-1$	$x_2=0$	$x_3=1$
$y_1=0$	$\frac{1}{15}$	$\frac{2}{15}$	$1/15$
$y_2=1$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
$y_3=2$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

Compute i) The marginal distributions $p(x)$ and $p(y)$

ii) The conditional distributions $p(X/Y=2)$, $p(Y/X=1)$.

iii) $P(X\le 1, Y=2)$, $P(X<0, Y\le 2)$

Sol:

Y \ X	-1	0	1	$p(y)$
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{4}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{6}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{5}{15}$
$p(x)$	$\frac{6}{15}$	$\frac{5}{15}$	$\frac{4}{15}$	$\boxed{1}$

i) Marginal distribution of X

$p(x=-1) = \frac{6}{15}$, $p(x=0) = \frac{5}{15}$, $p(x=1) = \frac{4}{15}$

Marginal distribution of Y

$p(y=0) = \frac{4}{15}$, $p(y=1) = \frac{6}{15}$, $p(y=2) = \frac{5}{15}$.

ii) Conditional distribution of X given $Y=2$

$$p\left[X=-1/Y=2\right] = \frac{p(X=-1 \cap Y=2)}{p(Y=2)} = \frac{2/15}{3/15} = \frac{2}{5}$$

$$p\left[X=0/Y=2\right] = \frac{p(X=0 \cap Y=2)}{p(Y=2)} = \frac{1/15}{3/15} = \frac{1}{3}$$

$$p\left[X=1|Y=2\right] = \frac{p(X=1 \cap Y=2)}{p(Y=2)} = \frac{2/15}{3/15} = \frac{2}{5}$$

Conditional distribution of Y given $X=1$

$$p\left[Y=0/X=1\right] = \frac{p(X=1 \cap Y=0)}{p(X=1)} = \frac{1/15}{4/15} = \frac{1}{4}$$

$$p\left[Y=1|X=1\right] =$$

$$p\left[Y=2|X=1\right] =$$

iii) $$p(X\leq 1, Y=2) = p(x=-1, y=2) + p(x=0, y=2) + p(x=1, y=2)$$

$$= \frac{2}{15} + \frac{1}{15} + \frac{2}{15} = \frac{1}{3}$$

$$p(X<0, Y\leq 2) = p(X=-1, Y=0) + p(X=-1, Y=1) + p(X=-1, Y=2)$$

$$= \frac{1}{15} + \frac{3}{15} + \frac{2}{15} = \frac{2}{5}$$

2) A coin is tossed three times, let X be equal to 0 or 1 according as a head (or) tail occurs on the FIRST Toss. Let Y be equal the total number of heads which occur. Determine

i) Marginal distribution of X and Y.

ii) The Joint probability distribution of X & Y

iii) $E(X)$, $E(Y)$, $E(X+Y)$.

Sol: Let $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Let X : 0 $\longrightarrow$ HEAD in the FIRST toss

1 $\longrightarrow$ TAIL in the FIRST toss

Y : No of heads appeared

i) The Joint probability distribution is given by

$p_{11} = P(X=0, Y=0) = \{\ \} = 0$ $\longrightarrow m/n$

$p_{12} = P(X=0, Y=1) = P\{HTT\} = \frac{1}{8}$

$p_{13} = P(X=0, Y=2) = P\{HHT, HTH\} = \frac{2}{8}$

$p_{14} = P(X=0, Y=3) = P\{HHH\} = \frac{1}{8}$

$p_{21} = P(X=1, Y=0) = P\{TTT\} = \frac{1}{8}$

$p_{22} = p(X=1, Y=1) = p\{TTH, THT\} = \frac{2}{8}$

$p_{23} = p(X=1, Y=2) = p\{THH\} = \frac{1}{8}$

$p_{24} = p(X=1, Y=3) = \{\ \} = 0$

X \ Y	0	1	2	3	$p(x)$
0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
$p(y)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	1

ii)

iii) $E(X) = \sum x\, p(x) = \frac{1}{2}$

$E(Y) = \sum y\, p(y) = 1.625$

$E(X+Y) = E(X) + E(Y) = 2.125$

2. A Joint probability distribution is given by the following table.

X \ Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find (i) Marginal distribution of X and Y

(ii) μ_X, μ_Y, σ_X, σ_Y

(iii) correlation coefficient.

Sol: Given

X \ Y	-3	2	4	P(x)
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
p(y)	0.4	0.3	0.3	$\boxed{1}$

(i) Marginal distribution of X

$p(x=1) = 0.5$, $p(x=3) = 0.5$

Marginal distribution of Y

$p(y=-3) = 0.4$, $p(y=2) = 0.3$, $p(y=4) = 0.3$

(ii) $\mu_X = E(X) = \Sigma x\, p(x) = (1 \times 0.5) + (3 \times 0.5) = 2$

$\mu_Y = E(Y) = \Sigma y\, p(y) = (-3 \times 0.4) + (2 \times 0.3) + (4 \times 0.3) = 0.6$

$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 p(x) = 1^2 \times 0.5 + (3^2) \times 0.5 = 5$

$\therefore \cancel{\sigma_X^2 = 5 - (2)^2}$

$\therefore \sigma_X^2 = 5 - (2)^2 = 1. \Rightarrow \boxed{\sigma_X = \sqrt{1} = 1}$

$V(Y) = \sigma_Y^2 = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \sum y^2 p(y)$

$= (-3)^2 \times 0.4 + 2^2 \times 0.3 + 4^2 \times 0.3$

$= 9.6$

$\therefore \sigma_Y^2 = (9.6) - (0.6)^2 = 9.24$

$\sigma_Y = \sqrt{9.24} = 3.0397$

(iii) $\text{COV}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$E(XY) = \sum xy\, p(x,y)$

$= 1 \times (-3) \times 0.1 + (1 \times 2 \times 0.2) + (1 \times 4 \times 0.2) + (3 \times 4 \times 0.1)$
$+ 3 \times 0.3 \times (-3) + (3 \times 2 \times 0.1) = 0$

$\therefore \text{COV}(X,Y) = 0 - 2 \cdot (0.6) = -1.2$

$\therefore \rho(X,Y) = \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y} = \frac{-1.2}{1 \times 3.039} = -0.395$

[1] The joint pmf is represented by the following table, where the number in each square (x, y) gives the value of $P_{X,Y}(x, y)$

(i) Compute the marginal pmf $P_X(x)$ and $P_Y(y)$.

(ii) $E(X)$, $E(Y)$. If $Z = X + 2Y$, compute pmf of Z $P_Z(z)$, $E(Z)$ and verify that $E(Z) = E(X) + 2E(Y)$.

y				
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
3	1/20	1/20	1/20	0
	1	2	3	4 x

Sol:

Y \ X	1	2	3	4	$P_Y(y)$
1	1/20	1/20	1/20	0	3/20
2	1/20	2/20	3/20	1/20	7/20
3	1/20	2/20	3/20	1/20	7/20
4	0	1/20	1/20	1/20	3/20
$P_X(x)$	$\frac{3}{20}$	6/20	8/20	3/20	

(i) Marginal distribution of X

$P_X(x=1) = 3/30$, $P_X(x=2) = 6/20$, $P_X(x=3) = 8/20$, $P_X(x=4) = 3/20$

(ii) Marginal distribution of Y

$p_Y(y=1) = 3/20$, $p_Y(y=2) = 7/20 = p_Y(y=3)$, $p_Y(y=4) = 3/20$.

$$E(X) = \sum_x x\, p(x) = 1\cdot\frac{3}{20} + 2\cdot\frac{6}{20} + 3\cdot\frac{8}{20} + 4\cdot\frac{3}{10} = \frac{51}{20}$$

$$E(Y) = \sum_y y\, p(y) = 1\cdot\frac{3}{20} + 2\cdot\frac{7}{20} + 3\cdot\frac{7}{20} + 4\cdot\frac{3}{20} = \frac{50}{20}$$

Pmf of Z is calculated by

$$p_Z(z) = \sum_{\{(x,y) \mid z = x+2y\}} p_{X,Y}(x,y).$$

From the table we have

$p_Z(3) = \frac{1}{20}$, $p_Z(4) = \frac{1}{20}$, $p_Z(5) = \frac{2}{20}$, $p_Z(6) = \frac{2}{20}$

$x + 2y = z$

1 1

$p_Z(7) = \frac{4}{20}$, $p_Z(8) = \frac{3}{20}$, $p_Z(9) = \frac{3}{20}$, $p_Z(10) = \frac{2}{20}$

$p_Z(11) = \frac{1}{20}$, $p_Z(12) = \frac{1}{20}$.

$$E(Z) = \sum_z z\, p(z) = 3\cdot\frac{1}{20} + \cdots + 12\cdot\frac{1}{20} = 7.55 \quad (1)$$

$\Downarrow$

$$E(X+2Y) = E(X) + 2E(Y)$$

$$= \frac{51}{20} + 2\cdot\frac{50}{20} = 7.55 \quad (2)$$

$\therefore$ From (1) & (2) $\underline{E(Z) = E(X) + 2E(Y)}$

Ex1 Compute the covariance Matrix for the two rv X_1 & X_2 given by the following table.

$x_1 \backslash x_2$	0	1	$p_1(x_1)$
−1	0.24	0.06	0.3
0	0.16	0.14	0.3
1	0.40	0.00	0.4
$p_2(x_2)$	0.8	0.2	$\boxed{1}$

$\to p_1$

Joint probabilities

Marginal probabilities

$\to \Sigma p_1 = \Sigma p_2$

p_2

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Soln:

$$\mu_1 = E(X_1) = \sum_{\text{all } x_1} x_1\, p_1(x_1)$$

$$= (-1 \times 0.3) + (0 \times 0.3) + (1 \times 0.4)$$

$$= 0.1$$

$$\mu_2 = E(X_2) = \sum_{\text{all } x_2} x_2\, p_2(x_2)$$

$$= 0 \times 0.8 + 1 \times 0.2 = 0.2$$

$$\therefore \text{Mean vector} = \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} //.$$

$$\sigma_{11} = Var(X_1) = E(X_1-\mu_1)^2 = \sum_{\text{all } x_1} (x_1-\mu_1)^2 \, \underline{p_1(x_1)}$$

variance

$$= (\overset{x_1}{-1} - \overset{\mu_1}{0.1})^2 \, \overset{p_1(x_1)}{(0.3)} + (0-0.1)^2(0.3) + (1-0.1)^2(0.4) = \boxed{0.69}$$

$$\sigma_{22} = Var(X_2) = E(X_2-\mu_2)^2 = \sum_{\text{all } x_2} (x_2-\mu_2)^2 \, \underline{p_2(x_2)}$$

$$= (0-0.2)^2(0.8) + (1-0.2)^2(0.2)$$

$$= \boxed{0.16}$$

$$\sigma_{12} = Cov(X_1, X_2) = E[(X_1-\mu_1)(X_2-\mu_2)]$$

$$= \sum_{\text{all pairs } (x_1, x_2)} (x_1-0.1)(x_2-0.2) \, p_{12}(x_1, x_2)$$

Covariance

Joint probabilities

$$= (-1-0.1)(0-0.2)(0.24) +$$
$$(-1-0.1)(1-0.2)(0.06) + \cdots$$

$$\cdots$$

$$= -0.08$$

$$\therefore \underline{\sigma_{21}} = E[(X_2-\mu_2)(X_1-\mu_1)] = \underline{\sigma_{12}}$$

$$\therefore \underbrace{\Sigma}_{\text{Covariance Matrix}} = E\left[(X-\mu)(X-\mu)^T\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0.69 & -0.08 \\ -0.08 & 0.16 \end{bmatrix}$$

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} = \frac{Cov(x_1, x_2)}{\cancel{\sqrt{SD}}\ SD(X_1)\ SD(X_2)}$$

$$= \frac{-0.08}{\sqrt{0.69}\sqrt{0.16}} = -0.24$$

rho

$$\therefore \underbrace{\rho}_{\text{Correlation Matrix}} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.24 \\ -0.24 & 1 \end{bmatrix}$$

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Ex 2 Compute the Correlation Matrix given that covariance matrix $\Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix}$

obtain $V^{1/2}$ & ρ

Sol: Given $\Sigma = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

Covariance Matrix

$\therefore \quad V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Standard deviation Matrix

$$(V^{1/2})^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

In Calculator

MATA

MATB

$\therefore \quad \rho = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$

Correlation Matrix

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{5} \\ \frac{1}{6} & 1 & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & 1 \end{bmatrix}$$ //.