Joint Probability distribution of continuous random variables

Joint density function

The joint probability density function (pdf) of two continuous random variable (X,Y) is defined as a function $f(x,y)$ satisfying the following conditions:

(i) $f(x,y) \geq 0, \forall x, y$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$

Marginal density function of X and Y

The function $p_1(x) = g(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$ is called the marginal density function of X.

The function $p_2(y) = h(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$ is called the marginal density function of Y.

Independent Random variables

Two random variables X and Y are said to be independent or stochastically independent if

$$p_1(x)p_2(y) = f(x,y)$$

OR

$$E(XY) = E(X)E(Y)$$

Mean, Variance and Covariance

$$E(X) = \int_{-\infty}^{\infty} x\, p_1(x)dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, p_2(y)dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)dxdy$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)dxdy$$

$$covariance(X,Y) = E(XY) - E(X)E(Y)$$

$$r(X,Y) = \frac{covariance(X,Y)}{\sigma_X \sigma_Y}$$

where, $\sigma_X = \sqrt{E(X^2) - [E(X)]^2}$

$\sigma_Y = \sqrt{E(Y^2) - [E(Y)]^2}$

1. Let (X,Y) be continuous random variable with Joint PDF given by

$$\mathrm{f(x,y)} = \begin{cases} x^2 + \dfrac{xy}{3}, & 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0, & elsewhere \end{cases}$$

Is $f(x, y)$ a probability density function? (ii) Marginal pdf of X and Y.

Solution: The given $f(x, y) \ge 0, \forall x, y$

$$\int_{x=0}^{1} \int_{y=0}^{2} (x^2 + \frac{xy}{3})\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{6}\right) dx \quad between\ 0\ to\ 2$$

$$= \int_0^1 \left(2x^2 + \frac{4x}{6}\right) dx$$

$$= \frac{2x^3}{3} + \frac{4x^2}{12} \quad 0\ to\ 1$$

$$= 1$$

Therefore $f(x, y)$ is a probability density function.

Marginal density of X

$$p_1(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$p_1(x) = \int_0^2 (x^2 + \frac{xy}{3})\, dy$$

$$p_1(x) = x^2 y + \frac{xy^2}{6} \quad 0\ to\ 2$$

$$p_1(x) = 2x^2 + \frac{2x}{3}, \quad o \le x \le 1$$

Marginal density of Y

$$p_2(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

$$p_2(y) = \int_0^1 (x^2 + \frac{xy}{3})\, dx$$

$$p_2(y) = \frac{1}{3} + \frac{y}{6}, \quad 0 \le y \le 2$$

2. Find the constant *'k'* so that

$$\mathrm{h(x,y)} = \begin{cases} k(x+1)e^{-y}, & 0 < x < 1,\ y > 0 \\ 0, & elsewhere \end{cases}$$

is a joint probability density function. Are X and Y independent?

Solution: We observe that $h(x,y) \geq 0$ for $x, y,$ if $k \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, dx\, dy = \int_{y=0}^{\infty}\int_{x=0}^{1} h(x,y)\, dx\, dy$$

$$= k\left\{\int_0^1 (x+1)dx\right\}\left\{\int_0^{\infty} e^{-y}\, dy\right\}$$

$$1 = k\left\{\frac{3}{2}\right\}\{0+1\} = \frac{3}{2}k.$$

$$\Rightarrow k = \frac{2}{3}$$

Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, dx\, dy = 1$ if $\mathrm{k} = \frac{2}{3}$.

Therefore, $h(x,y)$ is a joint probability density function if $\mathrm{k} = \frac{2}{3}$.

With $\mathrm{k} = \frac{2}{3}$, the marginal density functions are

$$p_1(x) = \int_{-\infty}^{\infty} h(x,y)\, dy, \quad 0 < x < 1$$

$$= \frac{2}{3}(x+1)\int_0^{\infty} e^{-y}\, dy$$

$$= \frac{2}{3}(x+1)(0+1).$$

$$p_1(x) = \frac{2}{3}(x+1), \quad 0 < x < 1$$

$$p_2(y) = \int_{-\infty}^{\infty} h(x,y)\, dx, \quad y > 0$$

$$p_2(y) = \frac{2}{3}e^{-y}\int_0^1 (x+1)\,dx = \frac{2}{3}e^{-y}\left\{\frac{2^2}{2} - \frac{1}{2}\right\} = \frac{2}{3}e^{-y}\frac{3}{2}$$

$$p_2(y) = e^{-y}, \; y > 0.$$

Therefore, $p_1(x)p_2(y) = h(x,y)$ and hence X and Y are stochastically independent.

3. Let X and Y be random variables having the joint density function

$$\mathrm{f(x,y)} = \begin{cases} 4xy, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & elsewhere \end{cases}$$

Verify that $E(X+Y) = E(X) + E(Y)$ and also find $E(XY)$.

Solution:

$$p_1(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$p_1(x) = \int_0^1 4xydy = 2x$$

$$p_2(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$p_2(y) = \int_0^1 4xydx = 2y$$

$$E(X) = \int_{-\infty}^{\infty} xp_1(x)dx = \int_0^1 2x^2dx = \frac{2}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} yp_2(y)dy = \int_0^1 2y^2dy = \frac{2}{3}$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)\,dx\,dy$$

$$E(X+Y) = \int_0^1\int_0^1 (x+y)4xy\,dx\,dy$$

$$E(X+Y) = \int_0^1\int_0^1 (4x^2y + 4xy^2)\,dx\,dy$$

$$E(X+Y) = \int_0^1 (\frac{4y}{3} + 2y^2)\, dy$$

$$E(X+Y) = \frac{4y^2}{6} + \frac{2y^3}{3}$$

$$E(X+Y) = \frac{2}{3} + \frac{2}{3}$$

Hence $E(X+Y) = E(X) + E(Y)$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f(x,y)\, dx\, dy$$

$$E(XY) = \int_0^1 \int_0^1 (xy) 4xy\, dx\, dy$$

$$E(XY) = \int_0^1 \int_0^1 4x^2y^2\, dx\, dy$$

$$E(XY) = \int_0^1 \frac{4y^2}{3}\, dy$$

$$E(XY) = \frac{4}{9}$$

Hence $E(XY) = E(X)E(Y)$

Therefore X and Y are independent.

4. Let X and Y be random variables having the joint density function

$$\mathrm{f(x,y)} = \begin{cases} \frac{xy}{96}, & 0 \le x \le 4, \qquad 1 \le y \le 5 \\ 0, & elsewhere \end{cases}$$

Evaluate the following

(i) $P(1 < x < 2, 2 < y < 3)$

(ii) $P(x \ge 3, y \le 2)$

(iii) $P(y \le x)$

(iv) $P(y > x)$

(v) $P(x + y \le 3)$

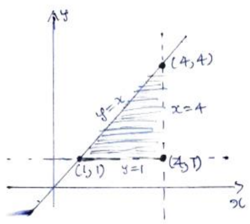
(vi) $P(x + y > 3)$

Solution:

(i)$P(1 < x < 2, 2 < y < 3) = \int_{x=1}^{2} \int_{y=2}^{3} \frac{xy}{96} dy\, dx = \frac{5}{128}$

(ii) $P(x \geq 3, y \leq 2) = \int_{x=3}^{4} \int_{y=1}^{2} \frac{xy}{96} dy\, dx = \frac{7}{128}$

(iii) $P(y \leq x) = \int_{x=1}^{4} \int_{y=1}^{x} \frac{xy}{96} dy\, dx$



x: 1 to 4
y: 1 to x

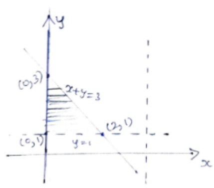
$$P(y \leq x) = \int_{x=1}^{4} \frac{xy^2}{192} dx \quad between\ 1\ to\ x$$

$$P(y \leq x) = \frac{1}{192} \int_{x=1}^{4} (x^3 - x) dx$$

$$P(y \leq x) = \frac{75}{256}$$

(iv) $P(y > x) = 1 - \ P(y \leq x) = 1 - \frac{75}{256} = \frac{181}{256}$

(v) $P(x + y \leq 3) = \int_{x=0}^{2} \int_{y=1}^{3-x} \frac{xy}{96} dy\, dx$



x: 0 to 2
y: 1 to 3−x

$$P(x + y \leq 3) = \int_{x=0}^{2} \frac{xy^2}{192} dx \quad between\ 1\ to\ 3 - x$$

$$P(x + y \leq 3) = \int_{x=0}^{2} [x(3 - x)^2 - x] dx$$

$$P(x+y \leq 3) = \int_{x=0}^{2} [x(9+x^2-6x)-x]dx$$

$$P(x+y \leq 3) = \int_{x=0}^{2} [8x+x^3-6x^2]dx$$

$$P(x+y \leq 3) = \frac{1}{48}$$

(vi) $P(x+y > 3) = 1 - P(x+y \leq 3) = 1 - \frac{1}{48} = \frac{47}{48}$

5. Verify that $f(x, y) = \begin{cases} e^{-(x+y)}, & x \geq 0, \ \ y \geq 0 \\ 0, & elsewhere \end{cases}$ is a density function of a joint probability distribution. Then evaluate the following:

(i) $P\left(\frac{1}{2} < x < 2, 0 < y < 4\right)$ (ii) $P(x < 1)$ (iii) $P(x > y)$ (iv) $P(x + y \leq 1)$.

Solution: Given $f(x, y) \geq 0$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x+y)}dx\,dy = \int_{-\infty}^{\infty} e^{-x}\,dx \int_{0}^{\infty} e^{-y}\,dy$$

$$= (0+1)(0+1) = 1.$$

Therefore, $f(x, y)$ is a density function.

(i) $P\left(\frac{1}{2} < x < 2,\ 0 < y < 4\right) = \int_{1/2}^{2}\int_{0}^{4} f(x,y)\,dy\,dx = \int_{1/2}^{2}\int_{0}^{4} e^{-(x+y)}dy\,dx$

$$= \int_{1/2}^{2} e^{-x}\,dx \int_{0}^{4} e^{-y}dy = \left(e^{-1/2} - e^{-2}\right)(1 - e^{-4}).$$

(ii) The marginal density function of x is

$$p_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{\infty} e^{-(x+y)}\,dy = e^{-x}\int_{0}^{\infty} e^{-y}\,dy = e^{-x}$$

Therefore, $P(x < 1) = \int_{0}^{1} h_1(x)\,dx = \int_{0}^{1} e^{-x}\,dx = 1 - \frac{1}{e}$.

(iii) $P(x \leq y) = \int_{0}^{\infty}\left\{\int_{0}^{y} f(x,y)\,dx\right\}dy = \int_{0}^{\infty}\left\{\int_{0}^{y} e^{-(x+y)}\,dx\right\}dy$

$$= \int_{0}^{\infty} e^{-y}\left(\int_{0}^{y} e^{-x}\,dx\right)dy = \int_{0}^{\infty} e^{-y}(1-e^{-y})\,dy$$

$$= \int_{0}^{\infty} (e^{-y} - e^{-2y})\,dy = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, $P(x > y) = 1 - P(x \leq y) = 1 - \frac{1}{2} = \frac{1}{2}$.

(iv) $P(x + y \leq 1) = \iint_A f(x,y)dA$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} f(x,y)\, dy\, dx = \int_0^1 \left\{ \int_0^{1-x} e^{-(x+y)}\, dy \right\} dx$$

$$= \int_0^1 e^{-x} \left\{ \int_0^{1-x} e^{-y}\, dy \right\} dx = \int_0^1 e^{-x} \{1 - e^{-(1-x)}\}\, dx$$

$$= \int_0^1 (e^{-x} - e^{-1})\, dx = 1 - \frac{2}{e}.$$