LINEAR ALGEBRA II

Linear Transformation:

Let U and V be two vector spaces over the same field F. The mapping $T : U \longrightarrow V$ is said to be a linear transformation, if
(i) $T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \quad \alpha, \beta \in U$
(ii) $T(c.\alpha) = c.T(\alpha) \quad \forall \quad c \in F, \alpha \in U.$
A Linear transformation $T : U \longrightarrow V$ is also called a linear map on U.

Theorem:

A mapping $T : U \longrightarrow V$ from the vector space $U(F)$ into $V(F)$ is a linear transformation iff $T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta) \quad \forall \quad c_1, c_2 \in F \quad \text{and} \quad \alpha, \beta \in U.$

Properties of Linear Transformation

Theorem

If $T : U \longrightarrow V$ is a linear transformation, then
(a) $T(0) = 0'$, where 0 and $0'$ are zero vectors of U and V respectively.
(b) $T(-\alpha) = -T(\alpha) \ \forall \ \alpha \in U$
(c) $T(c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + ... + c_nT(\alpha_n)$

Theorem

If $\beta_1, \beta_2, ..., \beta_m$ be any basis of vector space V and $\alpha_1, \alpha_2, ..., \alpha_m$ be any m vectors of the vector space W, then there exists one and only one linear transformation $T : V \longrightarrow W$ with $T(\beta_i) = \alpha_i$ for $i = 1, 2, ..., m$.

Problems:

1. If T is a mapping from $V_3(\mathbb{R})$ into $V_3(\mathbb{R})$ defined by $T(x_1, x_2, x_3) = (0, x_2, x_3)$. Show that T is a linear transformation.
Soln: Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in V_3(\mathbb{R})$
Consider, $T(\alpha + \beta) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$
$= T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$
$= T(0, x_2 + y_2, x_3 + y_3)$
$= (0, x_2, x_3) + (0, y_2, y_3)$
$= T(\alpha) + T(\beta)$
Consider, $T(c\alpha) = T(c(x_1, x_2, x_3))$
$= T(cx_1, cx_2, cx_3)$
$= (0, cx_2, cx_3)$
$= c(0, x_2, x_3)$
$= cT(x_1, x_2, x_3)$
$= cT(\alpha)$
$\therefore T$ is a linear transformation.

2. Find the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(1, 1) = (0, 1, 2), T(-1, 1) = (2, 1, 0)$.
Soln: $\{(1, 1), (-1, 1)\}$ forms a basis of $\mathbb{R}^2$.
Let $\alpha = (x, y) \in \mathbb{R}^2$
$(x, y) = c_1(1, 1) + c_2(-1, 1) \implies (x, y) = (c_1 - c_2, c_1 + c_2)$
$\implies x = c_1 - c_2, y = c_1 + c_2 \implies c_1 = \dfrac{x + y}{2}, c_2 = \dfrac{y - x}{2}$
$\therefore (x, y) = \dfrac{x + y}{2}(1, 1) + \dfrac{y - x}{2}(-1, 1)$
$\therefore$ the required transformation is

$T(x,y) = \frac{x+y}{2}(0,1,2) + \frac{y-x}{2}(2,1,0)$
$T(x,y) = (y-x, y, x+y)$

3. Is there a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which $T(2,2) = (4,-6)$ and $T(5,5) = (2,-3)$?
Soln: The vectors $(2,2)$ and $(5,5) \in \mathbb{R}^2$ are linearly dependent as $(5,5) = \frac{5}{2}(2,2)$
If T is linear, $T(5,5) = T(\frac{5}{2}(2,2)) = \frac{5}{2}T(2,2) = \frac{5}{2}(4,-6) = (10,-15)$
But $T(5,5) = (2,-3) \neq (10,-15)$.
$\therefore$ a linear map with the given data doesn't exist.

4. Let $M(\mathbb{R})$ be the vector space of all 2×2 matrices over $\mathbb{R}$ and B be a fixed non-zero element of $M(\mathbb{R})$. Show that the mapping $T : M(\mathbb{R}) \longrightarrow M(\mathbb{R})$ defined by $T(A) = AB - BA, \forall A \in M(\mathbb{R})$ is a linear map.
Soln: Let A and $C \in M(\mathbb{R})$ be arbitrary.
Consider, $T(A+C) = (A+C)B - B(A+C) = AB + CB - BA - BC = AB - BA + CB - BC$ $= T(A) + T(C)$
Let $c \in \mathbb{R}$ be any scalar.
Consider, $T(c.A) = (c.A)B - B(c.A) = c.(AB - BA) = c.T(A)$
$\therefore T$ is a linear transformation.

5. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T(1,0) = (1,1)$ & $T(0,1) = (-1,2)$, show that T maps the square with vertices $(0,0),(1,0),(1,1),(0,1)$ into a parallelogram.
Soln: $\{(1,0),(0,1)\}$ forms a basis of $\mathbb{R}^2$
$(x,y) = x(1,0) + y(0,1)$
$\therefore T(x,y) = xT(1,0) + yT(0,1) = x(1,1) + y(-1,2) = (x-y, x+2y)$
Now, $T(0,0) = (0,0) = A$, $T(1,0) = (1,1) = B$, $T(1,1) = (0,3) = C$, $T(0,1) = (-1,2) = D$.
To show that A,B,C,D are vertices of a parallelogram, we shall show that the diagonals AC and BD bisect each other.
Midpoint of $AC = (0, \frac{3}{2})$, Midpoint of $BD = (0, \frac{3}{2})$
Diagonals bisect each other. Hence $ABCD$ is a parallelogram.

6. If $T : V_1(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ is defined by $T(x) = (x, x^2, x^3)$, verify whether T is linear or not.
Soln: Let $x, y \in V_1(\mathbb{R})$
Consider, $T(x+y) = (x+y, (x+y)^2, (x+y)^3)$
$T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3)) = (x+y, x^2+y^2, x^3+y^3)$
We can see that, $T(x+y) \neq T(x) + T(y)$
$\therefore T$ is not a linear transformation.

7. Find the linear transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (1,1)$ and $f(0,1) = (-1,2)$.
Soln: Let $(x,y) \in \mathbb{R}^2$. Then $(x,y) = x(1,0) + y(0,1)$
Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = xf(1,0) + yf(0,1) = x(1,1) + y(-1,2)$
Hence $f(x,y) = (x-y, x+2y) \quad \forall \quad (x,y) \in \mathbb{R}^2$

8. Find the linear transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,1) = (0,1)$ and $f(-1,1) = (3,2)$.
Soln: $(1,1),(-1,1)$ of $\mathbb{R}^2$ forms a basis of $\mathbb{R}^2$
Let $\alpha = (x,y) \in \mathbb{R}^2$ be arbitrary.
Let$(x,y) = c_1(1,1) + c_2(-1,1) = (c_1 - c_2, c_1 + c_2)$
$\implies c_1 - c_2 = x, c_1 + c_2 = y \implies c_1 = \frac{x+y}{2}, c_2 = \frac{y-x}{2}$
$\therefore (x,y) = \frac{x+y}{2}(1,1) + \frac{y-x}{2}(-1,1)$

Hence the required transformation is $f(x,y)=\dfrac{x+y}{2}f(1,1)+\dfrac{y-x}{2}f(-1,1)$
$=\dfrac{x+y}{2}(0,1)+\dfrac{y-x}{2}(3,2)$
$\therefore f(x,y)=(\frac{3y-3x}{2},\frac{3y-x}{2})$.

9. $f:V_3(\mathbb{R})\longrightarrow V_2(\mathbb{R})$ is defined by $f(x,y,z)=(x+y,y+z)$, show that f is a linear transformation.
Soln: Let $\alpha=(x_1,y_1,z_1)$ and $\beta=(x_2,y_2,z_2)$ be any two elements of $V_3(\mathbb{R})$.
Consider, $f(\alpha+\beta)=f(x_1+x_2,y_1+y_2,z_1+z_2)$
$=(x_1+x_2+y_1+y_2,y_1+y_2+z_1+z_2)=((x_1+y_1)+(x_2+y_2),(y_1+z_1)+(y_2+z_2))$
$=(x_1+y_1,y_1+z_1)+(x_2+y_2,y_2+z_2)=f(x_1,y_1,z_1)+f(x_2,y_2,z_2)$
$\therefore f(\alpha+\beta)=f(\alpha)+f(\beta)$.
Consider, $f(c.\alpha)=f(cx_1,cy_2,cz_2)$
$=(cx_1+cy_1,cy_1+cz_2)=c(x_1+y_1,y_1+z_1)=cf(x_1,y_1,z_1)$
$\therefore f(c.\alpha)=cf(\alpha)$
Hence f is a linear transformation.

10. If T is a mapping from $V_2(\mathbb{R})$ into $V_2(\mathbb{R})$ defined by $T(x,y)=(x\cos\theta-y\sin\theta,x\sin\theta+y\cos\theta)$, show that T is a linear transformation.
Soln: Let $\alpha=(x_1,x_2),\beta=(y_1,y_2)\in V_2(\mathbb{R})$
Consider, $T(\alpha+\beta)=T(x_1+y_1,x_2+y_2)$
$=((x_1+y_2)\cos\theta-(x_2+y_2)\sin\theta,(x_1+y_2)\sin\theta-(x_2+y_2)\cos\theta)$
$=((x_1\cos\theta-x_2\sin\theta)+(y_1\cos\theta-y_2\sin\theta),(x_1\sin\theta+x_2\cos\theta)+(y_1\sin\theta+y_2\cos\theta))$
$=(x_1\cos\theta-x_2\sin\theta,x_1\sin\theta+x_2\cos\theta)+(y_1\cos\theta-y_2\sin\theta,y_1\sin\theta+y_2\cos\theta)$
$=T(x_1,x_2)+T(y_1,y_2)\therefore T(\alpha+\beta)=T(\alpha)+T(\beta)$
Consider, $T(c.\alpha)=T(cx_1,cx_2)$
$=(cx_1\cos\theta-cx_2\sin\theta,cx_1\sin\theta+cx_2\cos\theta)=c(x_1\cos\theta-x_2\sin\theta,x_1\sin\theta+x_2\cos\theta)$
$=cT(x_1,x_2)$
$\therefore T(c.\alpha)=cT(\alpha)$
Hence T is a linear transformation.

Ordered Basis:
Let V be a vector space over a field F and $d[V]=n>0$.
Choose a basis $\alpha_1,\alpha_2,...,\alpha_n$ of V written in this order. This basis $B=\{\alpha_1,\alpha_2,...,\alpha_n\}$ is called an ordered basis of V.

Coordinate Vector:
Let $B=\{\alpha_1,\alpha_2,...,\alpha_n\}$ be an ordered basis of the vector space. Then $\alpha\in V$ can be uniquely written as $\alpha=c_1\alpha_1+c_2\alpha_2+...+c_n\alpha_n;c_i\in F$. The scalars $c_1,c_2,...,c_n$ are called the coordinates of the vector α w.r.t the ordered basis B. The n-tupple $(c_1,c_2,...,c_n)$ is called the coordinate vector of V relative to the basis B.

Problems

1. Find the co-ordinate vector of $(3,-2,1)$ relative to
(i) ordered standard basis
(ii) the ordered basis $\{(1,1,1),(1,0,0),(1,1,0)\}$ of $\mathbb{R}^3$.
Soln: (i) Let $(3,-2,1)=c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(c_1,c_2,c_3)$
$\implies c_1=3,c_2=-2,c_3=1$
$\therefore$ co-ordinates of $(3,-2,1)$ relative to standard basis is $(3,-2,1)$.
(ii) Let $(3,-2,1)=c_1(1,1,1)+c_2(1,0,0)+c_3(1,1,0)=(c_1+c_2+c_3,c_1+c_3,c_1)$

$\Longrightarrow c_1 = 1, c_2 = 5, c_3 = -3$
$\therefore$ co-ordinates of $(3, -2, 1)$ relative to the basis $\{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$ is $(1, 5, -3)$.

Matrix of a Linear transformation

Let V and W be vector spaces of dimension m and n respectively, over the same field F.
Let $B_1 = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $B_2 = \{\beta_1, \beta_2, ..., \beta_n\}$ be ordered bases of V and W respectively.
Let $T : V \longrightarrow W$ be a linear transformation. T can be completely determined by $T(\alpha_1), T(\alpha_2), ..., T(\alpha_m)$.
Let
$T(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + ... + a_{n1}\beta_n$
$T(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + ... + a_{n2}\beta_n$
$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$
$T(\alpha_m) = a_{1m}\beta_1 + a_{2m}\beta_2 + ... + a_{nm}\beta_n$

The transpose of the coefficient matrix is called the matrix of the linear transformation relative to bases B_1 and B_2. The matrix is denoted by A_T.

$$i.e., A_T = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{pmatrix}$$

is the matrix of the linear transformation T relative to bases B_1 and B_2.

Note:

1. The columns of the matrix A_T are the coordinates of m vectors $T(\alpha_1), T(\alpha_2), ..., T(\alpha_m)$ w.r.t the basis B_2 of W.
2. The order of the matrix associated with the linear transformation T from $m-$dimensional space to $n-$dimensional space is $n \times m$ and its entries are the elements of the field F.

Hence to every linear transformation $T : V \longrightarrow W$ there is associated a $n \times m$ matrix.

Problems

1. Find the matrix of the linear transformation $T : V_2(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ defined by $T(x, y) = (x + y, x, 3x + y)$ w.r.t the standard bases.
Soln: Let $\{e_1, e_2\}$ and $\{f_1, f_2, f_3\}$ be the ordered standard basis of $V_2(\mathbb{R})$ and $V_3(\mathbb{R})$ respectively.
Consider, $T(e_1) = T(1, 0) = (1, 1, 3) = c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (c_1, c_2, c_3)$
$\Longrightarrow c_1 = 1, c_2 = 1, c_3 = 3$
$i.e., T(1, 0) = 1(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$
and
$T(e_2) = T(0, 1) = (1, 0, -1) = c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (c_1, c_2, c_3)$
$\Longrightarrow c_1 = 1, c_2 = 0, c_3 = -1$
$i.e., T(0, 1) = 1(1, 0, 0) + 0(0, 1, 0) - 1(0, 0, 1)$
$\therefore A_T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 3 & -1 \end{pmatrix}$

2. Find the matrix of the linear transformation defined by $T(x, y) = (x + y, x, 3x - y)$, relative to $B_1 = \{(1, 1), (3, 1)\}$, $B_2 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ of $V_2(\mathbb{R})$ and $V_3(\mathbb{R})$ respectively.
Soln: $T(1, 1) = (2, 1, 2) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + c_2, c_1)$
$\Longrightarrow c_1 = 2, c_2 = -1, c_3 = 1$
$\therefore T(1, 1) = 2(1, 1, 1) - 1(1, 1, 0) + 1(1, 0, 0)$
$T(3, 1) = (4, 3, 8) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + c_2, c_1)$
$\Longrightarrow c_1 = 8, c_2 = -5, c_3 = 1$
$\therefore T(3, 1) == 8(1, 1, 1) - 5(1, 1, 0) + 1(1, 0, 0)$.

Hence $A_T = \begin{pmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{pmatrix}$

3. Let $R_3[x]$ denote the vector space of all real polynomials. Define $T : R_3[x] \longrightarrow R_3[x]$ by $T(p) = p^{'}$ denotes the differential coefficient of p. Find the matrix of the linear transformation T relative to the basis $\{1, x, x^2, x^3\}$.

Soln:
$T(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$
$T(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$
$T(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$
$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$

Hence the matrix of T is $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. Given the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$. Determine the linear transformation $T : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ relative to bases B_1 and B_2 given by
(i) B_1 and B_2 are standard bases of $V_3(\mathbb{R})$ and $V_2(\mathbb{R})$ respectively.
(ii) $B_1 = \{(1, 1, 1), (1, 2, 3), (1, 0, 0)\}$, $B_2 = \{(1, 1), (1, -1)\}$

Soln: (i) B_1 and B_2 are standard bases.
Define $T : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ by
$T(1, 0, 0) = 1(1, 0) + 3(0, 1) = (1, 3)$
$T(0, 1, 0) = -1(1, 0) + 1(0, 1) = (-1, 1)$
$T(0, 0, 1) = 2(1, 0) + 0(0, 1) = (2, 0)$
Now, $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$
Then $T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) = x(1, 3) + y(-1, 1) + z(2, 0)$
$\therefore T(x, y, z) = (x - y + 2z, 3x + y)$ is the required transformation.
(ii) $B_1 = \{(1, 1, 1), (1, 2, 3), (1, 0,)\}$, $B_2 = \{(1, 1), (1, -1)\}$
Define $T : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ by
$T(1, 1, 1) = 1(1, 1) + 3(1, -1) = (4, -2)$
$T(1, 2, 3) = -1(1, 1) + 1(1, -1) = (0, -1)$
$T(1, 0, 0) = 2(1, 1) + 0(1, -1) = (2, 2)$
Now, $(x, y, z) = c_1(1, 1, 1) + c_2(1, 2, 3) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + 2c_2, c_1 + 3c_2)$
$\implies c_1 + c_2 + c_3 = x, c_1 + 2c_2 = y, c_1 + 3c_2 = z \implies c_1 = 3y - 2z, c_2 = z - y, c_3 = x - 2y + z$
$\therefore (x, y, z) = (3y - 2z)(1, 1, 1) + (z - y)(1, 2, 3) + (x - 2y + z)(1, 0, 0)$
$T(x, y, z) = (3y - 2z)T(1, 1, 1) + (z - y)T(1, 2, 3) + (x - 2y + z)T(1, 0, 0)$
$= (3y - 2z)(4, -2) + (z - y)(0, -2) + (x - 2y + z)(2, 2)$
Hence $T(x, y, z) = (2x + 8y - 6z, 2x - 8y + 4z)$ is the required transformation.

5. For the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find the corresponding linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ w.r.t the basis $\{(1, 0), (1, 1)\}$

Soln: $B_1 = \{(1, 0), (1, 1)\}$, $B_2 = \{(1, 0), (1, 1)\}$
Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by
$T(1, 0) = 1(1, 0) + 3(1, 1) = (4, 3)$
$T(1, 1) = 2(1, 0) + 4(1, 1) = (6, 4)$.
Now, $(x, y) = c_1(1, 0) + c_2(1, 1) = (c_1 + c_2, c_2)$
$\implies c_1 + c_2 = x, c_2 = y \implies c_1 = x - y, c_2 = y$
$\therefore (x, y) = (x - y)(1, 0) + y(1, 1)$
Then $T(x, y) = (x - y)T(1, 0) + yT(1, 1) = (x - y)(4, 3) + y(6, 4) = (4x + 2y, 3x + y)$

Hence $T(x, y) = (4x + 2y, 3x + y)$ is the required transformation.

Range of a Linear transformation:
Let $T : V \longrightarrow W$ be a linear transformation. The range of T is the set $R(T) = \{T(\alpha)/\alpha \in V\}$

Kernel(or null space) of a Linear transformation:
Let $T : V \longrightarrow W$ be a linear transformation. The kernel(or null space) of T is the set $N(T) = \{\alpha \in V/T(\alpha) = 0\}$, where 0 is the zero vector of W.

Note:
(i) For the identity map $I : V \longrightarrow V$ the range is the entire space V and the kernel is the zero subspace.
(ii) For the zero linear map $T : V \longrightarrow W$ defined by $T(\alpha) = 0 \forall \alpha \in V$, the range $R(T) = \{0\} =$ zero space of V and the null space $N(T) = V$.

Theorem
Let $T : V \longrightarrow W$ be a linear transformation. Then
(a) $R(T)$ is a subspace of W.
(b) $N(T)$ is a subspace of V.
(c) T is one-one iff $N(T) = \{0\}$, where 0 is the zero vector of W.

Theorem
Let $T : V \longrightarrow W$ be a linear transformation. If the vectors $\alpha_1, \alpha_2, ..., \alpha_n$ generate V, then the vectors $T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)$ generates $R(T)$.

Rank and Nullity of a Linear Transformation:
Let $T : V \longrightarrow W$ be a linear transformation. The dimension of the range space $R(T)$ is called the rank of the linear transformation T and is denoted by $r(T)$. The dimension of the null space $N(T)$ is called the nullity of the linear transformation T and is denoted by $n(T)$.

Rank-Nullity theorem
Let $T : V \longrightarrow W$ be a linear transformation and V be a finite dimensional vector space. Then $r(T) + n(T) = d[V](d[R(T)] + d[N(T)] = d[V])$.

Problems:

1. Let $T : V \longrightarrow W$ be a linear transformation defined by $T(x, y, z) = (x + y, x - y, 2x + z)$. Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.
Soln: $T(e_1) = T(1, 0, 0) = (1, 1, 2) = \alpha_1$
$T(e_2) = T(0, 1, 0) = (1, -1, 0) = \alpha_2$
$T(e_3) = T(0, 0, 1) = (0, 0, 1) = \alpha_3$
$\{\alpha_1, \alpha_2, \alpha_3\}$ generates $R(T)$
Consider $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $|A| = -2 \neq 0$
$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent. Thus it is a basis of $R(T)$.
$d[R(T)] = 3$.
Let $\alpha \in R(T)$
Then $\alpha = c_1(\alpha_1) + c_2(\alpha_2) + c_3(\alpha_3) = c_1(1, 1, 2) + c_2(1, -1, 0) + c_3(0, 0, 1) = (c_1 + c_2, c_1 - c_2, 2c_1 + c_3)$
$\therefore R(T) = \{(c_1 + c_2, c_1 - c_2, 2c_1 + c_3)/c_1, c_2, c_3 \in \mathbb{R}\}$
Suppose $T(x, y, z) = (0, 0, 0)$
$\implies (x + y, x - y, 2x + z) = (0, 0, 0) \implies x + y = 0, x - y = 0, 2x + z = 0 \implies x = 0, y = 0, z = 0$

$\therefore N(T) = \{(0,0,0)\}$
$d[N(T)] = 0$
rank + nullity $= 3 + 0 = 3 = d[V_3(\mathbb{R})]$.

2. Find the range, null space, rank, nullity of the linear transformation $T : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ defined by $T(x, y, z) = (y - x, y - z)$ and hence verify the rank-nullity theorem.
Soln:
$T(1,0,0) = (-1,0) = \alpha_1$
$T(0,1,0) = (1,1) = \alpha_2$
$T(0,0,1) = (0,-1) = \alpha_3$.
$R(T) = L\{\alpha_1, \alpha_2, \alpha_3\}$
Consider $\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$, $R_1 = -R_1 \implies \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$, $R_2 = R_2 - R_1 \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$,
$R_3 = R_3 + R_2 \implies \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\therefore d[R(T)] = 2$ and basis of $R(T) = \{(-1,0), (1,1)\}$.
Let $\alpha \in R(T)$
Then $\alpha = c_1\alpha_1 + c_2 + \alpha_2 = c_1(-1,0) + c_2(1,1) = (-c_1 + c_2, c_2)$
$\therefore R(T) = \{(-c_1 + c_2, c_2)/c_1, c_2 \in \mathbb{R}\}$.
Suppose $T(x,y,z) = (0,0) \implies (y - x, y - z) = (0,0) \implies y = x, y = z \implies x = y = z$
$\therefore N(T) = \{(a,a,a)/a \in \mathbb{R}\}$ and basis of $N(T) = \{(1,1,1)\}$. $\therefore d[N(T)] = 1$.
Hence rank + nullity $= 2 + 1 = 3 = d[V_3(\mathbb{R})]$.

3. Find the linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ whose range space is spanned by $(1,0,-1)$ and $(1,2,2)$.
Soln: Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that
$f(1,0,0) = (1,0,-1)$
$f(0,1,0) = (1,2,2)$
$f(0,0,1) = (0,0,0)$
Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ generates $\mathbb{R}^3$, the image vectors $\{(1,0,-1), (1,2,2)\}$ generate $R(f)$.
Let $(x,y,z) \in \mathbb{R}^3$.
$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$
$\therefore f(x,y,z) = xf(1,0,0) + yf(0,1,0) + zf(0,0,1) = x(1,0,-1) + y(1,2,2) + z(0,0,0)$
$= (x + y, 2y, -x + 2y)$
Hence $f(x,y,z) = (x + y, 2y, -x + 2y)$ is the required transformation.

4. Find a linear transformation $T : V_3(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ whose kernel is spanned by $(1,1,-1)$ and $(1,2,2)$.
Soln: $(1,1,-1)$ and $(1,2,2)$ are L.I.
Let S be the subspace spanned by the vectors $\alpha_1 = (1,1,-1)$ and $\alpha_2 = (1,2,2)$
$\therefore S = \{c_1\alpha_1 + c_2\alpha_2/c_1, c_2 \in \mathbb{R}\}$
Now $c_1\alpha_1 + c_2\alpha_2 = c_1(1,1,-1) + c_2(1,2,2) = (c_1 + c_2, c_1 + 2c_2, 2c_2 - c_1)$
$\therefore S = \{(c_1 + c_2, c_1 + 2c_2, 2c_2 - c_1)/c_1, c_2 \in \mathbb{R}\}$.
Choose a vector of $V_3(\mathbb{R})$ outside this set S.
$(1,0,0)$ is not in S.
Hence the set $B = \{(1,1,-1), (1,2,2), (1,0,0)\}$ is a basis of $V_3(\mathbb{R})$.
Now define T by
$T(1,1,-1) = (0,0,0)$
$T(1,2,2) = (0,0,0)$

$T(1,0,0) = (0,0,1)$.
Let $\alpha \in V_3(\mathbb{R})$
$\alpha = (x_1, x_2, x_3) = c_1(1,1,-1) + c_2(1,2,2) + c_3(1,0,0) = (c_1 + c_2 + c_3, c_1 + 2c_2, -c_1 + 2c_2)$
$\implies c_1 = \frac{x_2 - x_3}{2}, c_2 = \frac{x_2 + x_3}{4}, c_3 = \frac{4x_1 - 3x_2 + x_3}{4}$
$\therefore (x_1, x_2, x_3) = \frac{x_2 - x_3}{2}(1,1,-1) + \frac{x_2 + x_3}{4}(1,2,2) + \frac{4x_1 - 3x_2 + x_3}{4}(1,0,0)$
Hence $T(x_1, x_2, x_3) = (0, 0, \frac{4x_1 - 3x_2 + x_3}{4})$ is the required transformation.