**Singular Value Decomposition:**

Any $m \times n$ matrix $A$ can be factored into $A = U\Sigma V^T =$ (orthogonal)(diagonal)(orthogonal). The columns of $U(m \,\text{by}\, m)$ are eigen vectors of $AA^T$, and the columns of $V(n \,\text{by}\, n)$ are eigen vectors of $A^TA$. The $r$ singular values on the diagonal of $\Sigma(m \,\text{by}\, n)$ are the square roots of the non-zero eigen values of both $AA^T$ and $A^TA$.

**Note:**

The diagonal(but rectangular) matrix $\Sigma$ has eigen values from $A^TA$. These positive entries(also called sigma) will be $\sigma_1, \sigma_2, ..., \sigma_r$,such that $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. They are the singular values of $A$.

When $A$ multiplies a column $v_j$ of $V$, it produces $\sigma_j$ times a column of $U(A = U\Sigma V^T \implies AV = U\Sigma)$.

**ex.1.**Decompose $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ as $U\Sigma V^T$, where $U$ and $V$ are orthogonal matrices.

**Soln:**

$AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 4-\lambda & 4 \\ -2 & 4 & 4-\lambda \end{vmatrix} = 0$

$\implies \lambda^3 - 9\lambda^2 = 0 \implies \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 9$

with $\lambda = 9, [AA^T - \lambda I]x = 0 \implies$

$\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -8x_1 - 2x_2 - 2x_3 = 0, -18x_2 + 18x_3 = 0$

$\implies x_1 = -(1/2)x_3, x_2 = x_3 \implies x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

with $\lambda = 0, [AA^T - \lambda I]x = 0 \implies$

$\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 2x_2 + 2x_3 \implies x = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

Hence $U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$

$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$

$|A^TA - \lambda I| = 0 \implies |9 - \lambda| = 0 \implies \lambda = 9$

Then $A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

Let $x_1 = 1 \therefore x = \begin{bmatrix} 1 \end{bmatrix}$

Hence $V = \begin{bmatrix} 1 \end{bmatrix}$ or $V^T = \begin{bmatrix} 1 \end{bmatrix}$

9 is an eigen value of both $AA^T$ and $A^TA$.

And rank of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is $r = 1$.

$\therefore \Sigma$ has only $\sigma_1 = \sqrt{9} = 3$. $\therefore \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ the SVD of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$

**ex.2.** Obtain the SVD of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

**Soln:**

$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$\implies \lambda^2 - 3\lambda + 1 = 0 \implies \lambda_1 = \dfrac{3-\sqrt{5}}{2}, \lambda_2 = \dfrac{3+\sqrt{5}}{2}$

with $\lambda = \dfrac{3-\sqrt{5}}{2}$, $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & 1 \\ 1 & \dfrac{-1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{1+\sqrt{5}}{2} x_1 + x_2 = 0$ Letting $x_1 = -1$, then $x_2 = \dfrac{1+\sqrt{5}}{2}$

$\therefore x = \begin{bmatrix} -1 \\ \dfrac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

with $\lambda = \dfrac{3+\sqrt{5}}{2}$, $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} & 1 \\ 1 & \dfrac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{1-\sqrt{5}}{2} x_1 + x_2 = 0$ Letting $x_1 = -1$, then $x_2 = \dfrac{1-\sqrt{5}}{2}$

$\therefore x = \begin{bmatrix} -1 \\ \dfrac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \beta \end{bmatrix}$, where $\beta = \dfrac{1-\sqrt{5}}{2}$.

Hence $U = \begin{bmatrix} \dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{-1}{\sqrt{1+\beta^2}} \\ \dfrac{\alpha}{\sqrt{1+\alpha^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$

As $A^TA = AA^T$ $\quad V^T = \begin{bmatrix} \dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{\alpha}{\sqrt{1+\alpha^2}} \\ \dfrac{-1}{\sqrt{1+\beta^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$.

**ex.3.** Obtain the SVD of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

**Soln:** $AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 4\lambda + 3 = 0$

$\implies \lambda_1 = 1, \lambda_2 = 3$

with $\lambda = 3$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_1 = -x_2$

Letting $x_2 = 1 \implies x_1 = -1 \therefore x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

with $\lambda = 1$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 - x_2 = 0 \implies x_1 = x_2$

Letting $x_2 = 1 \implies x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$|A^T A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 4\lambda^2 + 3\lambda = 0$

$\implies \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

with $\lambda = 0$ $(A^T A - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies x_1 - x_2 = 0, x_2 - x_3 = 0 \implies x_1 = x_2, x_2 = x_3$

Letting $x_3 = 1 \implies x_2 = 1, x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

with $\lambda = 1$ $(A^T A - \lambda I)x = 0 \implies \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies -x_1 + x_2 - x_3 = 0, x_2 = 0 \implies x_1 = -x_3, x_2 = 0$

Letting $x_3 = 1 \implies x_2 = 0, x_1 = -1 \therefore x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

with $\lambda = 3$ $(A^T A - \lambda I)x = 0 \implies \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies -2x_1 - x_2 = 0, x_2 + 2x_3 = 0 \implies 2x_1 = x_2, x_2 = -2x_3$

Letting $x_3 = 1 \implies x_2 = -2, x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence $U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

$V^T = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

**ex.4.** Obtain the SVD of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.