LINEAR ALGEBRA II

Let F be a field , V be a non-empty set. For every ordered pair $\alpha, \beta \in V$, let there be defined uniquely a sum $\alpha + \beta$ and for every $\alpha \in V$, and $c \ \ in \ F$ a scalar product $c.\alpha \in V$. The set V is called a vector space over the field F, if the following axioms are satisfied for every $\alpha, \beta, \gamma \in V$ and for every $c, c' \in F$.
(i) $\alpha + \beta \in V$,
(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,
(iii) Identity element w.r.t addition exists.
i.e, $\exists\ e \in V$ s.t. $\alpha + e = e + \alpha = \alpha$,
(iv) Inverse element w.r.t addition exists.
i.e, $\exists\ \alpha^{-1} \in V$ s.t. $\alpha + \alpha^{-1} = e = \alpha^{-1} + \alpha$,
(v) $\alpha + \beta = \beta + \alpha$,
(vi) $c.(\alpha + \beta) = c.\alpha + c.\beta$,
(vii) $(c + c').\alpha = c.\alpha + c'.\alpha$,
(viii) $(c.c').\alpha = c.(c'.\alpha)$
(ix) $1.\alpha = \alpha$, $\forall \alpha \in V$, where 1 is the unit element of F.

Examples:
1. Let F be a field and n be a positive integer. Let $V_n(F)$ be the set of all ordered n tupples of the elements of the field F.
i.e, $V_n(F) = \{(x_1, x_2, ..., x_n)/x_i \in F\}$.
Define addition and scalar multiplication as below:
(a) $\alpha + \beta = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
(b) $c.\alpha = c.(x_1, x_2, ..., x_n) = (c.x_1, c.x_2, ..., c.x_n)$, $\forall c \in F$.
Soln:
(i) $\alpha + \beta = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \in V_n(F)$
(ii) $(\alpha + \beta) + \gamma$
$= (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) + (z_1, z_2, ..., z_n)$
$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, ..., (x_n + y_n) + z_n)$
$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), ..., x_n + (y_n + z_n))$
$(x_1, x_2, ..., x_n) + (y_1 + z_1, y_2 + z_2, ..., y_n + z_n)$
$= \alpha + (\beta + \gamma)$
(iii) $\alpha + 0 = (x_1, x_2, ..., x_n) + (0, 0, ..., 0)$
$= (x_1, x_2, ..., x_n)$
$= (0, 0, ..., 0) + (x_1, x_2, ..., x_n)$
$= 0 + \alpha$
$\therefore 0 = (0, 0, ..., 0)$ is the identity.
(iv) $\alpha + (-\alpha) = (x_1, x_2, ..., x_n) + (-x_1, -x_2, ..., -x_n)$
$= (x_1 - x_1, x_2 - x_2, ..., x_n - x_n)$

$= (0, 0, ..., 0)$
$= 0$
$\therefore -\alpha = (-x_1, -x_2, ..., -x_n)$ is the additive inverse of $\alpha = (x_1, x_2, ..., x_n)$
(v) $\alpha + \beta = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)$
$= (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
$= (y_1 + x_1, y_2 + x_2, ..., y_n + x_n)$
$= (y_1, y_2, ..., y_n) + (x_1, x_2, ..., x_n)$
$= \beta + \alpha$
(vi) $c.(\alpha + \beta) = c.(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
$= (c.(x_1 + y_1), c.(x_2 + y_2), ..., c.(x_n + y_n))$
$= (c.x_1, c.x_2, ..., c.x_n) + (c.y_1, c.y_2, ..., c.y_n)$
$= c.(x_1, x_2, ..., x_n) + c.(y_1, y_2, ..., y_n)$
$= c.\alpha + c.\beta$
(vii) $(c + c')\alpha = (c + c')(x_1, x_2, ..., x_n)$
$= ((c + c')x_1, (c + c')x_2, ..., (c + c')x_n)$
$= (cx_1 + c'x_1, cx_2 + c'x_2, ..., cx_n + c'x_n)$
$= (c.x_1, c.x_2, ..., c.x_n) + (c'.x_1, c'.x_2, ..., c'.x_n)$
$= c.(x_1, x_2, ..., x_n) + c'.(x_1, x_2, ..., x_n)$
$= c\alpha + c'\alpha$
(viii) $(c.c').\alpha = (c.c').(x_1, x_2, ..., x_n)$
$= ((c.c').x_1, (c.c').x_2, ..., (c.c').x_n)$
$= (c.(c'.x_1), c.(c'.x_2), ..., c.(c'.x_n))$
$= c.(c'.x_1, c'.x_2, ..., c'.x_n)$
$= c.(c'.(x_1, x_2, ..., x_n)$
$= c.(c'.\alpha)$
(ix) $1.\alpha = 1.(x_1, x_2, ..., x_n)$
$= (1.x_1, 1.x_2, ..., 1.x_n)$
$= (x_1, x_2, ..., x_n)$
$= \alpha$

Thus $V_n(F)$ is a vector space over the field F.

Note:
(i) With $F = \mathbb{R}, V_1(\mathbb{R}), V_2(\mathbb{R}), V_3(\mathbb{R})$ are all vector spaces. They are also denoted as $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ respectively. The elements of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ are real numbers, plane vectors and space vectors respectively.
(ii) If $F = \mathbb{R}, V_n(\mathbb{R})$ is denoted as $\mathbb{R}^n$.
If $F = \mathbb{C}, V_n(\mathbb{C})$ is denoted as $\mathbb{C}^n$.

2. Show that $V = \{a + b\sqrt{2}/a, b \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of all rationals, is a vector space under usual addition and scalar multiplication.
Slon:
Let $\alpha = a_1 + b_1\sqrt{2}$, $\beta = a_2 + b_2\sqrt{2}$, $\gamma = a_3 + b_3\sqrt{2} \in V$ and $c, c' \in \mathbb{Q}$
(i) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) \in V$
(ii) $(\alpha + \beta) + \gamma = ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2}) = \alpha + (\beta + \gamma)$
(iii) 0 is the additive identity, as $0 + \alpha = \alpha = \alpha + 0$.
(iv) $-\alpha = -a_1 - b_1\sqrt{2}$ is the additive inverse of α as $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$.
(v) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \beta + \alpha$.
(vi) $c.(\alpha + \beta) = c.((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) = c.\alpha + c.\beta$.
(vii) $(c + c')\alpha = (c + c')(a_1 + b_1\sqrt{2}) = c.\alpha + c'\alpha$.
(viii) $(c.c')\alpha = (c.c')(a_1 + b_1\sqrt{2}) = c.(c'.\alpha)$.
(ix)$1.\alpha = 1.(a_1 + b_1\sqrt{2}) = a_1 + b_1\sqrt{2} = \alpha$.
Thus V is a vector space over $\mathbb{Q}$.

3. Let V be the set of all polynomials of degree $\leq n$, with coefficients in the field F, together with zero polynomial. Then show that V is a vector space under addition of polynomials and scalar multiplication of polynomials with the scalar $c \in F$ defined by $c(a_0 + a_1x + ... + a_nx^n) = ca_0 + ca_1x + ... + ca_nx^n$.
Soln:
(i) Sum of polynomials is again a polynomial
(ii) Sum of polynomials will be associative.
(iii) 0 is the additive identity.
(iv) If $\alpha = a_0 + a_1x + ... + a_nx^n$ then $-\alpha = -a_0 - a_1x - ... - a_nx^n$ is the additive inverse.
(v) Sum of polynomials is commutative.
(vi) $c.(\alpha + \beta) = c.\alpha + c.\beta$ will hold.
(vii) $(c + c').\alpha = c.\alpha + c'.\alpha$ will hold.
(viii) $(c.c').\alpha = c.(c'.\alpha)$ will hold.
(ix) $1.\alpha = 1.(a_0 + a_1x + ... + a_nx^n) = a_0 + a_1x + ... + a_nx^n = \alpha$.
Thus V is a vector space over F.

4. Let $V = \{\begin{bmatrix} x & y \\ -y & x \end{bmatrix}/x, y \in \mathbb{C}\}$ under usual addition and scalar multiplication, with field $\mathbb{C}$ of complex numbers. Show that V is a vector space.
Soln: Let $\alpha = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$, $\gamma = \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \in V$
and $c_1 = a_1 + b_1i, c_2 = a_2 + b_2i \in \mathbb{C}$.
(i) $\alpha + \beta \in V$,
(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,

(iii) $\alpha + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \alpha.$
$\therefore \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity.
(iv) $-\alpha = \begin{bmatrix} -x_1 & -y_1 \\ y_1 & -x_1 \end{bmatrix}$ is the additive inverse.
(v) $\alpha + \beta = \beta + \alpha$ will hold.
(vi) $c.(\alpha + \beta) = c.\alpha + c.\beta$ will hold.
(vii) $(c + c').\alpha = c.\alpha + c'.\alpha$ will hold.
(viii) $(c.c').\alpha = c.(c'.\alpha)$ will hold.
(ix) $1.\alpha = 1. \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = \alpha.$
Thus V is a vector space over $\mathbb{C}$.

5. Let $\mathbb{R}^+$ be the set of all positive integers. Define the operations of addition and scalar multiplication as below:
$\alpha + \beta = \alpha\beta, \ \forall \ \alpha, \beta \in \mathbb{R}^+$
$c.\alpha = \alpha^c$, $\alpha \in \mathbb{R}^+$ and $c \in \mathbb{R}$
Show that $\mathbb{R}^+$ is a vector space over the real field.
Soln:
(i) $\alpha + \beta = \alpha\beta \in \mathbb{R}^+$.
(ii) $(\alpha + \beta) + \gamma = (\alpha\beta) + \gamma = (\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha + \beta\gamma = \alpha + (\beta + \gamma)$.
(iii) $\alpha + 1 = \alpha.1 = \alpha = 1.\alpha = 1 + \alpha$.
(iv) $\alpha + \dfrac{1}{\alpha} = \alpha.\dfrac{1}{\alpha} = 1 = \dfrac{1}{\alpha}.\alpha = \dfrac{1}{\alpha} + \alpha.$
$\therefore \dfrac{1}{\alpha}$ is the additive inverse of α.
(v) $\alpha + \beta = \alpha.\beta = \beta.\alpha = \beta + \alpha$.
(vi) $c.(\alpha + \beta) = c.(\alpha\beta) = (\alpha\beta)^c = \alpha^c\beta^c = \alpha^c + \beta^c = c.\alpha + c.\beta$.
(vii) $(c + c')\alpha = \alpha^{(c+c')} = \alpha^c\alpha^{c'} = \alpha^c + \alpha^{c'} = c.\alpha + c'.\alpha$.
(viii) $(c.c').\alpha = \alpha^{(c.c')} = \alpha^{(c'.c)} = (\alpha^{c'})^c = c(\alpha^{c'}) = c.(c'\alpha)$
(ix) $1.\alpha = \alpha^1 = \alpha$, where 1 is the unit element of $\mathbb{R}^+$.