Multivariate Gaussian Distribution

A vector-valued random variable $X = [X_1, X_2 \cdots X_n]^T$ is said to have multivariate normal (Gaussian) distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ ($n \times n$, symmetric positive definite, $x^T \Sigma x \geq 0$) if its probability density function is given by

$$f(x; \mu, \Sigma) = \frac{1}{(\sqrt{2\pi})^n |\Sigma|^{1/2}} e^{-\left[\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right]}$$

If $n=1$, the case of univariate normal

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\, \sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Note: 1 $-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = -\frac{1}{2\sigma^2}(x-\mu)^2$ is a quadratic ~~form~~ fun of the variable x

ARGUMENT of exponential function

$\Downarrow$

Geometrically, parabola points downwards

$\frac{1}{\sqrt{2\pi}\, \sigma}$; constant that does not depend on x

simply a normalized factor to ensure that

$$\frac{1}{\sqrt{2\pi}\, \sigma} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) = 1$$

Note 2: The diagonal Covariance matrix

$n = 2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$$f(x; \mu, \Sigma) = \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}^{1/2}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right\}$$

$$= \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}^{1/2}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right\}$$

$$= \frac{1}{2\pi \sigma_1 \sigma_2} \exp\left\{ -\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_1^2}(x_1-\mu_1) \\ \frac{1}{\sigma_2^2}(x_2-\mu_2) \end{bmatrix} \right\}$$

$$= \frac{1}{2\pi \sigma_1 \sigma_2} \exp\left\{ \frac{-1}{2\sigma_1^2}(x_1-\mu_1)^2 - \frac{1}{2\sigma_2^2}(x_2-\mu_2)^2 \right\}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left\{ -\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2 \right\} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{ -\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2 \right\}}$$

PRODUCT of two independent Gaussian densities with (μ_1, μ_2) & (σ_1^2, σ_2^2) in case of $\Sigma = \mathrm{dig}(\sigma_1^2, \sigma_2^2, \dots \sigma_n^2)$

NOTE 3 : Consider $f(x) = c$; $x \in R^2$, $c \in R$

LEVEL Curves (ISOCONTOURS)

$$\Rightarrow \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2 - \frac{1}{2\sigma_2^2}(x_2-\mu_2)^2 \right\} = c$$

⋮

$$1 = \frac{(x_1-\mu_1)^2}{r_1^2} + \frac{(x_2-\mu_2)^2}{r_2^2}$$

where $r_1 = \sqrt{2\sigma_1^2 \log\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}$; $r_2 = \sqrt{2\sigma_2^2 \log\left(\frac{1}{2\pi c\sigma_1\sigma_2}\right)}$

EQUATION of Ellipse (axis-aligned) ; vertex (centre) : (μ_1, μ_2)

NOTE 4: Σ = Non-diagonal case, (higher dimensions)

In this case level curves are simply ROTATED Ellipses. [Ellipsoids in R^n]

NOTE 4:

$Y = c_1 X_1 + c_2 X_2 + \cdots + c_K X_K$ is called the linear function of random variables $(X_1, X_2 \cdots X_K)$

MEAN:

$$E(Y) = c_1 E(X_1) + c_2 E(X_2) + \cdots + c_K E(X_K)$$

VARIANCE:

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \cdots + c_K^2 V(X_K) + 2 \sum_i \sum_j c_i c_j \operatorname{Cov}(X_i, X_j)$$

If $Y = c_1 X_1 + c_2 X_2$, then $E(Y) = c_1 E(X_1) + c_2 E(X_2)$

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + 2 c_1 c_2 \operatorname{Cov}(X_1, X_2).$$

If $X_1, X_2, \ldots X_K$ are independent, then $\operatorname{Cov}(X_1, X_2, \ldots X_K) = 0$, $V(Y) = c_1^2 V(X_1) + \cdots + c_K^2 V(X_K)$

If $\operatorname{Cov}(aX_1, bX_2) = ab \operatorname{Cov}(X_1, X_2)$.

Example 1

If $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix}$, obtain the bivariate normal density function.

Sol: Given $\mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{matrix} \to \mu_1 \\ \to \mu_2 \end{matrix}$; $\Sigma = \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix}$ (σ_{11}, σ_{12}, σ_{21}, σ_{22})

$\boxed{n = 2}$

$$|\Sigma|^{1/2} = \begin{vmatrix} 2 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{vmatrix}^{1/2} = \left(\frac{3}{2}\right)^{1/2} = \sqrt{\frac{3}{2}}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \operatorname{Adj}(\Sigma) = \frac{1}{\frac{3}{2}} \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 2 \end{bmatrix}$$

$$\therefore f(x_1, x_2) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{1/2}} e^{-\frac{1}{2}\left[(X-\mu)^T \Sigma^{-1} (X-\mu)\right]}$$

$$= \frac{1}{(2\pi)^{\frac{2}{2}} \sqrt{\frac{3}{2}}} e^{-\frac{1}{\cancel{2}}\left[\begin{bmatrix} x_1 - 0 & x_2 - 2 \end{bmatrix} \frac{\cancel{2}}{3} \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 - 0 \\ x_2 - 2 \end{bmatrix}\right]}$$

$$= \frac{1}{\sqrt{6}\,\pi} e^{-\frac{1}{3}\left\{x_1^2 - \sqrt{2}\, x_1 (x_2 - 2) + 2 (x_2 - 2)^2\right\}}$$

is the required density function.

NOTE: Quadratic form: $ax^2 + 2bxy + cy^2$

$\Updownarrow$

Matrix form: $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$x \to x_1$, $y \to x_2 - 2$

$$ax^2 + 2bxy + cx^2 = \begin{bmatrix} x_1 & x_2 - 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - 2 \end{bmatrix}$$

where $a = 1$, $b = -\frac{\sqrt{2}}{2}$, $c = 1$

Example 2

Let $X \sim N_3(\mu, \Sigma)$, $\mu = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{bmatrix}$

Compute i> $P[X_1 > 6]$ ii> $P[5X_2 + 4X_3 > 70]$

iii> $P[4X_1 - 3X_2 + 5X_3 < 80]$

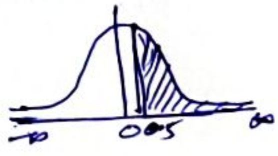
Soln: Given $\mu = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix} \begin{matrix} \to \mu_1 \\ \to \mu_2 \\ \to \mu_3 \end{matrix}$, $\Sigma = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{bmatrix}$

σ_{11} (or) σ_1, σ_{12}, σ_{13}, σ_{23}, σ_{33} (or) σ_3, σ_{22} (or) σ_2

(i) $X_1 \sim N(\mu_1, \sigma_1) \sim N(5, 4)$

$$z = \frac{X_1 - \mu_1}{\sigma_{11}} = \frac{6-5}{2} = 0.5$$

$\therefore P[X_1 > 6] = P[z > 0.5] = 0.3085$



ii) $E[5X_2 + 4X_3] = 5E(X_2) + 4E(X_3) = 5 \times \mu_2 + 4 \times \mu_3 = 43$

$V[5X_2 + 4X_3] = 5^2 V(X_2) + 4^2 V(X_3) + 2 \times 5 \times 4 \, Cov(X_2, X_3)$

$= 25 \times 4 + 16 \times 9 + 2 \times 5 \times 4 \times 2$

σ_{22} $\quad \sigma_{23}$

$= 324.$

$$z = \frac{(X_2+X_3) - E[5X_2+4X_3]}{SD[5X_2+4X_3]} = \frac{70-43}{\sqrt{324}} = 1.5$$

$\therefore P[5X_2 + 4X_3 > 70] = P[z > 1.5]$

$= A(0,\infty) - A(0, 1.5)$

$= 0.5 - A(1.5)$ From Table

$= 0.0668$

(iii)

$$E[4X_1 - 3X_2 + 5X_3] = 4E(X_1) - 3E(X_2) + 5E(X_3)$$

$$= 46$$

$$V[\underset{a}{\underset{\downarrow}{4}}X_1 \underset{b}{\underset{\downarrow}{-3}}X_2 + \underset{c}{\underset{\downarrow}{5}}X_3] = 16\,V(X_1) + 9\,V(X_2) + 25\,V(X_3) + 2ab\,\mathrm{cov}(X_1, X_2) + 2ac\,\mathrm{cov}(X_1, X_3) + 2bc\,\mathrm{cov}(X_2, X_3)$$

$$\cancel{= 16\times 4 + 9\times 4 + 25\times 9}$$

$$= 16\times 4 + 9\times 4 + 25\times 9 + 2\times 4\times(-3)(-1) + 2\times 4\times(5)\times 0 + 2\times(-3)\times 5\times 2$$

$$= 289$$

(OR)

$$V(4X_1 - 3X_2 + 5X_3)$$

$$= A^T \Sigma A$$

$$= \underset{\text{Coefficient Matrix}}{\underset{\downarrow}{\begin{bmatrix} 4 & -3 & 5 \end{bmatrix}}} \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

$$\therefore Z = \frac{X-\mu}{\sigma} = \frac{80-46}{\sqrt{289}} = 2.0$$

$$\therefore P[4X_1 - 3X_2 + 5X_3 < 80] = P[Z < 2]$$

$$= A(-\infty, 2)$$

$$= A(-\infty, 0) + A(0, 2)$$

$$= 0.5 + \underbrace{A(2)}_{0.4772}$$

$$= 0.9772$$