LINEAR ALGEBRA II

Subspaces: A non empty subset W of a vector space V over a field F is called a subspace of V, if W is itself a vector space over F, under the same operations of addition and scalar multiplication as defined in V.
examples
(i) The set $\{0\}$ consisting of zero vector of V is a subspace of V.
(ii) The whole vector space V, itself is a subspace of V.
These two subspaces are called trivial or improper subspaces of V.
Any subspace W of V different from $\{0\}$ and V is called a proper subspace of V.

Theorem 1 A non empty subset W of a vector space V over a field F is a subspace of V, if and only if (i) $\forall \alpha, \beta \in W$, $\alpha + \beta \in W$ (ii) $\forall c \in F, \alpha \in W, c.\alpha \in W$.

Problems
1. Verify whether $W = \{f(x)/2f(0) = f(1)\}$ over $0 \leq x \leq 1$, is a subspace of $V = \{\text{all functions}\}$ over the field $\mathbb{R}$.
Soln: Let $f_1, f_2 \in W$.
Thus $2f_1(0) = f_1(1)$ and $2f_2(0) = f_2(1)$
Consider, $2(f_1 + f_2)(0) = 2[f_1(0) + f_2(0)]$
$= 2f_1(0) + 2f_2(0)$
$= f_1(1) + f_2(1)$
$= (f_1 + f_2)(1)$
Thus, $f_1 + f_2 \in W$. $i.e., W$ is closed under vector addition.
Consider, $2(cf_1)(0) = (2c)f_1(0)$
$= c.2f_1(0)$
$= c.f_1(1)$
$= (cf_1)(1)$.
Thus $cf_1 \in W$ $i.e., W$ is closed under scalar multiplication.
Hence W is a subspace.

2. Is the subset $W = \{(x_1, x_2, x_3)/x_1^2 + x_2^2 + x_3^3 \leq 1\}$ of $V_3(\mathbb{R})$ a subspace of $V_3(\mathbb{R})$?
Soln: Let $\alpha = (1, 0, 0)$, where $1^2 + 0^2 + 0^2 = 1$ and $\beta = (0, 1, 0)$, where $0^2 + 1^2 + 0^2 = 1$ be two vectors in W.
Consider $\alpha + \beta = (1, 0, 0) + (0, 1, 0) = (1, 1, 0)$, where, $1^2 + 1^2 + 0^2 = 2 \nleq 1$.
Hence $\alpha + \beta \notin W$. $\therefore W$ is not a subspace.

3. Show that the subset $W = \{(x_1, x_2, x_3)/x_1 + x_2 + x_3 = 0\}$ of the vector space $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$.
Soln: Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3)$ be any two elements of W.
Then, $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$.
Consider, $c_1\alpha + c_2\beta = c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3)$
$= (c_1x_1, c_1x_2, c_1x_3) + (c_2y_1, c_2y_2, c_2y_3)$
$= (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3)$
To show that $c_1\alpha + c_2\beta \in W$, we have to show that the sum of the components of $c_1\alpha + c_2\beta$ is zero.
$\therefore$ consider $c_1x_1 + c_2y_1 + c_1x_2 + c_2y_2 + c_1x_3 + c_2y_3$
$= c_1(x_1 + x_2 + x_3) + c_2(y_1 + y_2 + y_3)$
$= c_1.0 + c_2.0$
$= 0$
$\therefore c_1\alpha + c_2\beta \in W$, hence W ia a subspace of $V_3(\mathbb{R})$.

4.Verify whether $W = \{$ Polynomial of degreee three $\}$ defined on $0 \leq x \leq 1$ is a subspace of the vector space $V = \{$ all polynomials $\}$ over $\mathbb{R}$.
Soln: The set of all polynomials of degree three is not a subspace, as the sum of two polynomials of degree three need not be of degree three.
$\because f_1(x) = 3x^3 - 4x^2 + 2x + 1, f_2(x) = -3x^3 + 3x^2 + 2x + 5$
$\implies f_1(x) + f_2(x) = -x^2 + 4x + 6$ which is not a polynomial of degree three.
Hence W is not a subspace of V

5. Let $V = \mathbb{R}^3$, the vector space of all ordered triplets of real numbers, over the field of real numbers. Show that the subset $W = \{(x, 0, 0)/x \in \mathbb{R}\}$ is a subspace of $V = \mathbb{R}^3$.
Soln: The element $\mathbf{0} = (0, 0, 0) \in W$.
Thus W is non-empty.
Let $\alpha_1 = (x_1, 0, 0)$ and $\alpha_2 + (x_2, 0, 0)$ be any two elements of W.
Then $\alpha_1 + \alpha_2 = (x_1, 0, 0) + (x_2, 0, 0) = (x_1 + x_2, 0, 0) \in W$.
$\therefore W$ is closed under addition.
Again, for any scalar $c \in \mathbb{R}$,
$c.\alpha_1 = c.(x_1, 0, 0) = (cx_1, 0, 0) \in W$
$\therefore W$ is closed under scalar multiplication.
Hence W is a subspace of $\mathbb{R}^3$.