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QUADRATIC FORMS

A quadratic form in R^n is a fu^n Q defined on R^n whose value at a vector x in R^n can be computed by an expression of the form

$$\boxed{Q(x) = x^T A x}$$

where A is an $n \times n$ Symmetric matrix called matrix of the quadratic form.

EX:- Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Compute $X^T A X$ for the following matrices

(i) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

$$X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2$$

(ii) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$$X^T A X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_1x_2 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

2. For X in R^3, let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$. Write this quadratic form as X^TAX.

Sol^n :- Here x_1^2, x_2^2, x_3^2 go on the diagonal of A.

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$$Q(X) = X^T A X =$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3. Let $Q(X) = x_1^2 - 8x_1x_2 - 5x_2^2$

Compute the value of $Q(X)$ for

$X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \;\&\; \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$Q(-3,1) = (-3)^2 - 8(-3)(1) - 5(1)^2 = 28$$

$$Q(2,-2) = (2)^2 - 8(2)(-2) - 5(-2)^2 = 16$$

$$Q(1,-3) = (1)^2 - 8(1)(-3) - 5(-3)^2 = -20$$

4. Compute the quadratic form $X^T A X$ when $A = \begin{bmatrix} 5 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$ and

(a) $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (b) $X = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ (c) $X = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

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(5) Compute the quadratic form for

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \& \quad X = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

(6) Find the matrix of the quadratic form. Assume X is in R^2

(i) $10x_1^2 - 6x_1x_2 - 3x_2^2$

(ii) $20x_1^2 + 15x_1x_2 - 10x_2^2$

(iii) $5x_1^2 + 3x_1x_2$

(7) Find the matrix of the quadratic form. Assume X is in R^3.

(i) $8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x$

(ii) $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

(iii) $x_3^2 - 4x_1x_2 + 4x_2x_3$

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Classifying Quadratic forms

A quadratic form Q is

(i) Positive definite if $Q(x) > 0 \; \forall \; x \neq 0$

(ii) Negative definite if $Q(x) < 0 \; \forall \; x \neq 0$

(iii) Indefinite if $Q(x)$ assumes both positive & Negative values.

Quadratic Forms & Eigen Values

Let A be a $n \times n$ Symmetric matrix. Then a quadratic form $X^T A X$ is

(i) Positive definite if and only if the eigenvalues of A are all positive

(ii) Negative definite if and only if the eigenvalues of A are all negative

(iii) Indefinite if and only if A has both positive and negative eigen values.

Ex:- $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Eigen Values of A are 5, 2, −1
So, Q is an indefinite Quadratic form.

Q. Classify the quadratic forms

(i) $3x_1^2 - 4x_1x_2 + 6x_2^2$

(ii) $x_1^2 - 6x_1x_2 + 9x_2^2$

(iii) $-5x_1^2 + 4x_1x_2 - 2x_2^2$

(iv) $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$

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CONSTRAINED OPTIMIZATION

Often it is required to find the maximum or minimum value of a quadratic form $Q(x)$ for x in some specified set.

The requirement that a vector x in R^n be a unit vector

$$\|x\| = 1, \quad \|x\|^2 = 1, \quad X^T X = 1$$

and $\quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

Ex:- Find the maximum and minimum values of $Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $X^T X = 1$.

Sol^n:- Since x_2^2 and x_3^2 are nonnegative

$4x_2^2 \leq 9x_2^2$ and $3x_3^2 \leq 9x_3^2$

and hence
$$\begin{aligned} Q(x) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 9 \end{aligned}$$

whenever $x_1^2 + x_2^2 + x_3^2 = 1$.

Maximum value of $Q(x)$ cannot exceed 9 when x is a unit vector.

$\therefore$ $Q(x) = 9$ when $x = (1, 0, 0)$

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Thus 9 is the maximum value of $Q(x)$ for $x^T x = 1$.

To find the minimum Value of $Q(x)$

$9x_1^2 \geq 3x_1^2$, $4x_2^2 \geq 3x_2^2$

hence $Q(x) \geq 3x_1^2 + 3x_2^2 + 3x_3^2$

$= 3(x_1^2 + x_2^2 + x_3^2) = 3$

whenever $x_1^2 + x_2^2 + x_3^2 = 1$.

Also, $Q(x) = 3$ when $x_1 = 0$, $x_2 = 0$ & $x_3 = 1$

So 3 is the minimum value of $Q(x)$ when $x^T x = 1$.

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Problems

1. (a) Find the maximum value of $Q(x)$ subject to the constraint $x^T x = 1$
(b) a unit vector u where this maximum is attained.

(1) Given $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(3-\lambda)(4-\lambda) - 1\right] - 2\left[2(4-\lambda) - 1\right] + 1\left[2 - (3-\lambda)\right] = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$$

$$\Rightarrow -(\lambda-6)(\lambda-3)(\lambda-1) = 0$$

$\therefore$ The greatest Eigen Value is 6.

The Constrained maximum of $X^T A X$ is attained when X is a unit eigen Vector for $\lambda = 6$

solving $(A - 6I)X = 0$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x_2 + 5x_3 = 0 \Rightarrow x_2 = x_3$$

$$x_1 + x_2 - 2x_3 = 0 \Rightarrow x_1 = x_3$$

$$\therefore X = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

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(ii) $Q(X) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 6-\lambda & -2 \\ 0 & -2 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)\left[(6-\lambda)(7-\lambda) - 4\right] - 2\left[-4\right] = 0$$

$$\Rightarrow (5-\lambda)\left[42 - 7\lambda - 6\lambda + \lambda^2 - 4\right] + 8 = 0$$

$$\Rightarrow (5-\lambda)\left[\lambda^2 - 13\lambda + 38\right] + 8 = 0$$

$$\Rightarrow 5\lambda^2 - 65\lambda + 190 - \lambda^3 + 13\lambda^2 - 38\lambda + 8 = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - \overset{99}{103}\lambda + \overset{162}{198} = 0$$

$\lambda = 9,\ \cancel{\frac{9}{2} \pm 1.3229i}\ 3, 6$

The greatest eigen Value is 9.

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$[A - 9I]X = 0$

(iii)

$$\Rightarrow \begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(iv)

$$\begin{bmatrix} -4 & 2 & 0 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x_2 - 4x_3 = 0$$

$$\Rightarrow x_2 = -x_3$$

(v)

$$-4x_1 + 2x_2 = 0$$

$$\Rightarrow 4x_1 = -2x_3$$

$$x_3 = -2x_1$$

$$\therefore X = \begin{bmatrix} x_1 \\ 2x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\therefore u_1 = \begin{bmatrix} 1/\sqrt{9} \\ 2/3 \\ -2/3 \end{bmatrix}$$

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(iii) $Q(X) = 3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1 x_2 + 2x_1 x_3 + 4x_2 x_3$

(iv) $Q(X) = -2x_1^2 - x_2^2 + 4x_1 x_2 + 4x_2 x_3$

Ans:- $\lambda = 2, -1, -4$, $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

(v) $Q(X) = 7x_1^2 + 3x_2^2 + 3x_1 x_2$

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Mean and Covariance

1.

Let $[X_1 \ \cdots \ X_N]$ be a $p \times N$ matrix of observations. The Sample Mean M of the observation vectors $X_1, \ldots, X_N$ is given by

$$M = \frac{1}{N}(X_1 + \cdots + X_N)$$

The sample mean is the point in the center of the scatter plot.

Sol

For $k = 1, \ldots N$, let

$$\hat{X}_k = X_k - M$$

$$B = [\hat{X}_1 \ \hat{X}_2 \ \cdots \ \hat{X}_N]_{p \times N}$$

B is said to be Mean-Deviation form

The (Sample) Covariance Matrix is

$$\boxed{S = \frac{1}{N-1} BB^T}$$

Problem

1. Three measurements are made on each of four individuals in a random sample from a population. The observation vectors are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad X_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} \quad X_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

Compute the sample mean & the Covariance matrix.

Sol^n:- The Sample mean is

$$M = \frac{1}{4}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix} \right\}$$

$$= \frac{1}{4}\begin{bmatrix} 20 \\ 16 \\ 20 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

$$\hat{X}_1 = X_1 - M = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix} ; \quad \hat{X}_2 = \begin{bmatrix} -1 \\ -2 \\ 8 \end{bmatrix}$$

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$$\hat{X}_3 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} \qquad \hat{X}_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

and $$B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$$

The sample Covariance matrix is

$$S = \frac{1}{3} \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

PRINCIPAL COMPONENT ANALYSIS

Assume the matrix $B = [\hat{X}_1 \ \hat{X}_2 \ \cdots \ \hat{X}_N]$ is already in mean-deviation form. The goal of PCA is to find an orthogonal $p \times p$ matrix $P = [u_1 \cdots u_p]$ that determines a change of variable,

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = [u_1 \ \cdots \ u_p] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

with the property that the new variables $y_1, \cdots y_p$ are uncorrelated & are arranged in order of decreasing variance.

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(1) The following table lists the weights and heights of five boys

Boy	1	2	3	4	5
Weight (lb)	120	125	125	135	145
Height (m)	61	60	64	68	72

(i) Find the Covariance matrix for the data

(ii) Make a principal Component analysis of the data to find a single size index that explains most of the variation in the data.

Sol^n:-

$$X_1 = \begin{bmatrix}120\\61\end{bmatrix} \quad X_2 = \begin{bmatrix}125\\60\end{bmatrix} \quad X_3 = \begin{bmatrix}125\\64\end{bmatrix}$$

$$X_4 = \begin{bmatrix}135\\68\end{bmatrix} \quad X_5 = \begin{bmatrix}145\\72\end{bmatrix}$$

$$M = \frac{1}{5}\left[X_1 + X_2 + X_3 + X_4 + X_5\right]$$

$$= \frac{1}{5}\left\{\begin{bmatrix}120\\61\end{bmatrix} + \begin{bmatrix}125\\60\end{bmatrix} + \begin{bmatrix}125\\64\end{bmatrix} + \begin{bmatrix}135\\68\end{bmatrix} + \begin{bmatrix}145\\72\end{bmatrix}\right\}$$

$$M = \begin{bmatrix}130\\65\end{bmatrix}$$

$$\hat{X}_1 = X_1 - M = \begin{bmatrix} 120 \\ 61 \end{bmatrix} - \begin{bmatrix} 130 \\ 65 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$$

$$\hat{X}_2 = X_2 - M = \begin{bmatrix} 125 \\ 60 \end{bmatrix} - \begin{bmatrix} 130 \\ 65 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

$$\hat{X}_3 = X_3 - M = \begin{bmatrix} 125 \\ 64 \end{bmatrix} - \begin{bmatrix} 130 \\ 65 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$\hat{X}_4 = X_4 - M = \begin{bmatrix} 135 \\ 68 \end{bmatrix} - \begin{bmatrix} 130 \\ 65 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\hat{X}_5 = X_5 - M = \begin{bmatrix} 145 \\ 72 \end{bmatrix} - \begin{bmatrix} 130 \\ 65 \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

$$B = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \hat{X}_3 & \hat{X}_4 & \hat{X}_5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}$$

$$S = \frac{1}{5-1} BB^T = \frac{1}{4} \begin{bmatrix} 400 & 190 \\ 190 & 100 \end{bmatrix}$$

$$S = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

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$|S - \lambda I| = 0$

$$\begin{vmatrix} 100-\lambda & 47.5 \\ 47.5 & 25-\lambda \end{vmatrix} = 0$$

$$(100-\lambda)(25-\lambda) - (47.5)^2 = 0$$

$$2500 - 25\lambda - 100\lambda + \lambda^2 - 2256.25 = 0$$

$$\lambda^2 - 125\lambda + 243.75 = 0$$

$$\lambda = 123.02, \quad 1.98$$

For $\lambda = 123.02$

$$[S - \lambda I]X = 0 \Rightarrow \begin{bmatrix} -23.02 & 47.5 \\ 47.5 & -98.02 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{-98.02} = \frac{-x_2}{47.5}$$

2

$$\therefore \quad u_1 = \begin{bmatrix} 98.02 \\ 47.5 \end{bmatrix}$$

$$\hat{u}_1 = \begin{bmatrix} \dfrac{98.02}{\sqrt{(98.02)^2+(47.5)^2}} \\ \dfrac{47.5}{\sqrt{(98.02)^2+(47.5)^2}} \end{bmatrix} = \begin{bmatrix} \dfrac{98.02}{108.92} \\ \dfrac{47.5}{108.92} \end{bmatrix} = \begin{bmatrix} 0.8999 \\ 0.4361 \end{bmatrix}$$

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$$\hat{u}_1 = \begin{bmatrix} 0.90 \\ 0.44 \end{bmatrix}$$

Set $y = 0.90\,\hat{w} + 0.44\,\hat{h}$.
where $\hat{w}$ & $\hat{h}$ are weight & height respectively, in mean-deviation form

The Variance of this index over the data set is 123.02, Because the total variance is $trace(S) = 100 + 25 = 125$, the size index accounts for practically all $\left(\frac{123.02}{125} = 98.5\%\right)$ of the variance of the data.

(2) Convert the matrix of observations to mean-deviation form, & Construct the sample covariance matrix. Also find the principal component of the data

$$X = \begin{bmatrix} 1 & 5 & 2 & 6 & 7 & 3 \\ 3 & 11 & 6 & 8 & 15 & 11 \end{bmatrix}$$

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Sol^n:- $M = \frac{1}{6}\left[\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}5\\11\end{bmatrix}+\begin{bmatrix}2\\6\end{bmatrix}+\begin{bmatrix}6\\8\end{bmatrix}+\begin{bmatrix}7\\15\end{bmatrix}+\begin{bmatrix}3\\11\end{bmatrix}\right]$

$$= \begin{bmatrix}4\\9\end{bmatrix}$$

$$B = \begin{bmatrix}-3 & 1 & -2 & 2 & 3 & -1\\ -6 & 2 & -3 & -1 & 6 & 2\end{bmatrix}$$

$$S = \frac{1}{5} BB^T = \frac{1}{5}\begin{bmatrix}28 & 40\\ 40 & 90\end{bmatrix}$$

$$= \begin{bmatrix}5.6 & 8\\ 8 & 18\end{bmatrix}$$

$$|S-\lambda I| = 0 \Rightarrow \lambda^2 - 23.6\lambda + 36.8 = 0$$
$$\lambda = 21.92, 1.68$$

For $\lambda = 21.92$ $\Rightarrow$ Corresponding eigenvector

$u_1 = \begin{bmatrix}3.92\\8\end{bmatrix}$ $\hat{u}_1 = \begin{bmatrix}0.44\\0.9\end{bmatrix}$

For $\lambda = 1.68$, $u_2 = \begin{bmatrix}16.32\\-8\end{bmatrix}$ $\hat{u}_2 = \begin{bmatrix}0.9\\-0.44\end{bmatrix}$

$\begin{bmatrix} 0.44 \\ 0.9 \end{bmatrix}$ is the first principal Component

$\begin{bmatrix} 0.9 \\ -0.44 \end{bmatrix}$ is the Second principal Component.

(3) Transform the provided matrix of observations: $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 2 & 3 \end{bmatrix}$, into mean deviation form.
Next, derive the covariance matrix & utilize it to identify the principal components. Additionally, determine the percentage of information captured by the first & second principal components.

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 2 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 3.5 \\ 2.5 \end{bmatrix}$$

$$B = \begin{bmatrix} -2.5 & -1.5 & -0.5 & 0.5 & 1.5 & 2.5 \\ 1.5 & 0.5 & -0.5 & -1.5 & -0.5 & 0.5 \end{bmatrix}$$

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$$S = \begin{bmatrix} 3.5 & -0.9 \\ -0.9 & 1.1 \end{bmatrix}$$

$$\lambda^2 - 4.6\lambda + 3.04 = 0$$

$$\lambda = 3.8,\ 0.8$$

$$\lambda = 3.8 \Rightarrow X = \begin{bmatrix} -2.7 \\ 0.9 \end{bmatrix} \sim \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\hat{u}_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$\lambda = 0.8,\ X = \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\hat{u}_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

Principal Components

% of information retrieved

$$\frac{3.8}{(3.8+0.8)} \times 100 = 82.61\%$$

$$\frac{0.8}{3.8+0.8} \times 100 = 17.39\%$$

Convert the matrix of observations to mean-deviation form & Construct the sample covariance matrix

(3)
$$\begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$$

Also find the principal components of the data.