LINEAR ALGEBRA III

Diagonalization of a Matrix:

Suppose the n by n matrix A has n linearly independent eigen vectors. If these eigen vectors are the columns of a matrix P, then $P^{-1}AP$ is a diagonal matrix D. The eigen values of A are on the diagonal of D

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda_n \end{bmatrix}$$

Note:

1. Any matrix with distinct eigen values can be diagonalized.
2. The diagonalization matrix P is not unique.
3. Not all matrices posses n linearly independent eigen vectors, so not all matrices are diagonalizable.
4. Diagonalizability of A depends on enough eigen vectors.
5. Diagonalizability can fail only if there are repeated eigen values.
6. The eigen values of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$ and each eigen vector of A is still an eigen vector of A^k. $[D^k = D.D....D(k\,\text{times}) = (P^{-1}AP)(P^{-1}AP)...(P^{-1}AP) = P^{-1}A^kP]$.

ex.1. Find the diagonalization matrix for $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} \implies \lambda^2 - \lambda = 0 \implies \lambda = 0, \lambda = 1.$

with $\lambda = 1$ $(A - \lambda I)x = 0 \implies \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + x_2 = 0 \implies x_2 = x_1$

Letting $x_1 = 1 \implies x_2 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

with $\lambda = 0$ $(A - \lambda I)x = 0 \implies \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_2 = -x_1$

Letting $x_1 = 1 \implies x_2 = -1 \therefore x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

ex.2. Find the diagonalization matrix for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \implies \lambda^2 + 1 = 0 \implies \lambda = +i, \lambda = -i.$

with $\lambda = i$ $(A - \lambda I)x = 0 \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -ix_1 - x_2 = 0 \implies x_2 = -ix_1$

Letting $x_1 = 1 \implies x_2 = -i \therefore x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

with $\lambda = -i$ $(A - \lambda I)x = 0 \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies ix_1 - x_2 = 0 \implies x_2 = ix_1$

Letting $x_1 = 1 \implies x_2 = i \therefore x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Then $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix}$ $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\therefore A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix}$

ex.3. Express the matrix $A = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ in the form of PDP^{-1}.

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} 3-\lambda & 4 & 2 \\ 3 & 5-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 10\lambda^2 + 15\lambda = 0$

$\implies \lambda = 5 + \sqrt{10},\ 5 - \sqrt{10},\ 0$

with $\lambda = 5 + \sqrt{10}$ $|A - \lambda I| = 0 \implies \begin{bmatrix} -2-\sqrt{10} & 4 & 2 \\ 3 & -\sqrt{10} & 4 \\ 0 & 1 & -3-\sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{3\sqrt{10}+6} = \dfrac{-x_2}{-9-3\sqrt{10}} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2+\sqrt{10}} = \dfrac{x_2}{3+\sqrt{10}} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2+\sqrt{10} \\ 3+\sqrt{10} \\ 1 \end{bmatrix}$

with $\lambda = 5 - \sqrt{10}$ $|A - \lambda I| = 0 \implies \begin{bmatrix} -2+\sqrt{10} & 4 & 2 \\ 3 & \sqrt{10} & 4 \\ 0 & 1 & -3+\sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{-3\sqrt{10}+6} = \dfrac{-x_2}{-9+3\sqrt{10}} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2-\sqrt{10}} = \dfrac{x_2}{3-\sqrt{10}} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2-\sqrt{10} \\ 3-\sqrt{10} \\ 1 \end{bmatrix}$

with $\lambda = 0$ $|A - \lambda I| = 0 \implies \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{6} = \dfrac{-x_2}{6} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2} = \dfrac{x_2}{-2} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Hence $P = \begin{bmatrix} 2+\sqrt{10} & 2-\sqrt{10} & 2 \\ 3+\sqrt{10} & 3-\sqrt{10} & -2 \\ 1 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 5+\sqrt{10} & 0 & 0 \\ 0 & 5-\sqrt{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0.0581 & 0.1 & 0.0838 \\ -0.2581 & 0.1 & 0.7162 \\ 0.2 & -0.2 & 0.2 \end{bmatrix}$

$A = \begin{bmatrix} 2+\sqrt{10} & 2-\sqrt{10} & 2 \\ 3+\sqrt{10} & 3-\sqrt{10} & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5+\sqrt{10} & 0 & 0 \\ 0 & 5-\sqrt{10} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.0581 & 0.1 & 0.0838 \\ -0.2581 & 0.1 & 0.7162 \\ 0.2 & -0.2 & 0.2 \end{bmatrix}$

ex.4. Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 + 3\lambda^2 - 4 = 0$

$\implies \lambda = 1,\ -2,\ -2$

with $\lambda = 1$ $|A-\lambda I| = 0 \implies \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{9} = \dfrac{-x_2}{9} = \dfrac{x_3}{9} \implies \dfrac{x_1}{1} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} \therefore x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

with $\lambda = -2$ $|A-\lambda I| = 0 \implies \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + 3x_2 + 3x_3 = 0 \implies x_1 = -x_2 - x_3$

Letting $x_2 = k_1, x_3 = k_2 \implies x_1 = -k_1 - k_2 \therefore x = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

or $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are the linearly independent eigen vectors corresponding to $\lambda = -2$.

Hence $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

ex.5. Diagonalize the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Soln: $|A-\lambda I| = 0 \implies \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 + 3\lambda^2 - 4 = 0$

$\implies \lambda = 1,\ -2,\ -2$

with $\lambda = 1$ $|A-\lambda I| = 0 \implies \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{9} = \dfrac{-x_2}{9} = \dfrac{x_3}{9} \implies \dfrac{x_1}{1} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} \therefore x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

with $\lambda = -2$ $|A-\lambda I| = 0 \implies \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + 3x_2 + 3x_3 = 0 \implies x_1 = -x_2 - x_3$

$\implies \dfrac{x_1}{-3} = \dfrac{-x_2}{-3} = \dfrac{x_3}{0} \implies \dfrac{x_1}{-1} = \dfrac{x_2}{1} = \dfrac{x_3}{1} \therefore x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

There is only one linearly independent eigen vector corresponding to $\lambda = -2$ and only two linearly independent eigen vectors for the given matrix A. Hence the matrix A cannot be diagonalizable.

ex.6. Diagonalize the matrix $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

ex.7. Diagonalize the matrix $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$