LINEAR ALGEBRA II

Linear Combination

Let V be a vector space over the field F and $\alpha_1, \alpha_2, ..., \alpha_n$ be any n vectors of V. The vector of the form, $c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n$, where $c_1, c_2, ..., c_n \in F$, is called a linear combination of the vectors $\alpha_1, \alpha_2, ..., \alpha_n$.

Linear Span of S

Let S be a non empty subset of a vector space $V(F)$. The set of all linear combinations of finite number of elements of S is called the linear span of S and is denoted by $L[S]$.
i.e, $L[S] = \{c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n / c_i \in F, \alpha_i \in S, i = 1, 2, ..., n \text{ and } n \text{ is any positive integer}\}$
If $\alpha \in L[S]$, then α is of the form, $\alpha = c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n$, for some scalars $c_1, c_2, ..., c_n \in F$.

Theorem: Let S be a non-empty subset of a vector space $V[F]$. Then
(i) $L[S]$ is a subspace of V
(ii) $S \subseteq L[S]$
(iii) $L[S]$ is the smallest subspace of V containing S.

Linear Dependence and Independence

A set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly dependent if there exists scalars $c_1, c_2, ..., c_n \in F$, not all zero such that $c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n = 0$.
A set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly independent if $c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n = 0 \implies c_1 = c_2 = ... = c_n = 0$.

Problems:

1. Show that the vectors $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ...0), ...e_n = (0, 0, 0, ..., 1)$ of the vector space $V_n(\mathbb{R})$ are linearly independent.
Soln: Let $c_1, c_2, ..., c_n \in \mathbb{R}$
Consider $c_1e_1 + c_2e_2 + ... + c_ne_n = 0$
$\implies c_1(1, 0, 0, ..., 0) + c_2(0, 1, 0, ...0) + ... + c_n(0, 0, 0, ..., 1) = (0, 0, 0, ..., 0)$
$\implies (c_1, c_2, ..., c_n) = (0, 0, 0, ..., 0)$
$\implies c_1 = 0, c_2 = 0, ..., c_n = 0$
i.e., $e_1, e_2, e_3, ..., e_n$ are linearly independent.

2. Show that the set $S = \{(1, 0, 1), (1, 1, 0), (-1, 0, -1)\}$ is linearly dependent in $V_3(\mathbb{R})$.
Soln: Consider $c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(-1, 0, -1) = (0, 0, 0)$
$\implies (c_1 + c_2 - c_3, c_2, c_1 - c_3) = (0, 0, 0)$
$\implies c_1 + c_2 - c_3 = 0, c_2 = 0, c_1 - c_3 = 0$
$\implies c_1 = 1, c_2 = 0, c_3 = 1$
Thus there exists, not all zeros, scalars, such that $c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(-1, 0, -1) = (0, 0, 0)$
$\therefore S$ is linearly dependent.

Note:
1.The set $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ of vectors of the vector space $V_3(\mathbb{R})$ is linearly dependent iff $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$
2. Two vectors $\alpha, \beta \in V_2(\mathbb{R})$ are linearly dependent iff $\alpha = k\beta$ for some non zero $k \in \mathbb{R}$
3. A set of vectors of V, containing the zero vector is linearly dependent.
4. The set consisting of a single vector α of V is linearly independent iff $\alpha \neq 0$.

Basis

A subset B of a vector space $V[F]$ is called a basis of V if
(i) B is a linearly independent set
(ii) $L[B] = V$
i.e., a basis of a vector space $V[F]$ is linearly independent subset which spans the whole space.

Note: The zero vector cannot be an element of a basis of a vector space because a set of vectors with zero vector is always linearly dependent.

Problems:

1. Show that the vectors $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ...0), ...e_n = (0, 0, 0, ..., 1)$ of the vector space $V_n(\mathbb{R})$ form a basis of $V_n(\mathbb{R})$.

Soln: Consider $S = \{e_1, e_2, ..., e_n\}$

$c_1e_1 + c_2e_2 + ... + c_ne_n = 0$

$\implies c_1(1, 0, ..., 0) + c_2(0, 1, ..., 0) + ... + c_n(0, 0, ..., 1) = (0, 0, ..., 0)$

$\implies (c_1, c_2, ..., c_n) = (0, 0, ..., 0)$

$c_1 = 0, c_2 = 0, ..., c_n = 0$

Hence S is linearly independent.

Further, any vector $(x_1, x_2, ..., x_n) \in V_n(\mathbb{R})$ can be expressed as a linear combination of the elements of S, as $(x_1, x_2, ..., x_n) = x_1e_1 + x_2e_2 + ... + x_ne_n$. Hence $L[S] = V_n(\mathbb{R})$

$\therefore S$ is a basis of $V_n(\mathbb{R})$.

Standard Basis

The basis $S = \{e_1, e_2, ..., e_n\}$ of the vector space $V_n(\mathbb{R})$ is called the standard basis.
example: The vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ of $V_3(\mathbb{R})$ form a basis of $V_3(\mathbb{R})$, and is called the standard basis.

2. Show that the set $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a basis of the vector space $V_3(\mathbb{R})$.

Soln: Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$

Consider, $c_1(1, 1, 0) + c_2(1, 0, 1) + c_3(0, 1, 1) = (0, 0, 0)$

$\implies (c_1 + c_2, c + 1 + c_3, c_2 + c_3) = (0, 0, 0)$

$\implies c_1 + c_2 = 0, c_1 + c_3 = 0, c_2 + c_3 = 0$

$\implies c_1 = 0, c_2 = 0, c_3 = 0$

$\therefore B$ is linearly independent.

Let $(x_1, x_2, x_3) \in V_3(\mathbb{R})$ be arbitrary.

Let $c_1, c_2, c_3 \in \mathbb{R}$, such that

$(x_1, x_2, x_3) = c_1(1, 1, 0) + c_2(1, 0, 1) + c_3(0, 1, 1)$

$(x_1, x_2, x_3) = (c_1 + c_2, c_1 + c_3, c_2 + c_3)$

$\implies x_1 = c_1 + c_2, x_2 = c_1 + c_3, x_3 = c_2 + c_3$

$\implies c_1 = \dfrac{x_1 + x_2 - x_3}{2}, c_2 = \dfrac{x_1 - x_2 + x_3}{2}, c_3 = \dfrac{-x_1 + x_2 + x_3}{2}$

$\therefore (x_1, x_2, x_3) = \dfrac{x_1 + x_2 - x_3}{2}(1, 1, 0) + \dfrac{x_1 - x_2 + x_3}{2}(1, 0, 1) + \dfrac{-x_1 + x_2 + x_3}{2}(0, 1, 1)$

$\therefore L[B] = V_3(\mathbb{R})$

Dimension of a vector space V

The dimension of a finite dimensional vector space V over F is the number of elements in any basis of V and is denoted by $d[V]$.
example: $V_n(\mathbb{R})$ is a n dimensional space.
$V_3(\mathbb{R})$ is a 3 dimensional space.

Finite dimensional space

A vector space $V[F]$ is said to be a finite dimensional space if it has a finite basis.

Note:

(i) Any two bases of a finite dimensional vector space V have the same finite number of elements.
(ii) A vector space which is not finitely generated may be called an infinite dimensional space.
(iii) In an n dimensional vector space $V(F)$
(a) any $n+1$ elements of V are linearly dependent.
(b) no set of $n-1$ elements can span V.
(iv) In an n dimensional vector space $V(F)$ any set of n linearly independent vectors is a basis.
(v) Any linearly independent set of elememts of a finite dimensional vector space V is a part of a basis.
(vi) For n vectors of $n-$dimensional vector space V, to be a basis, it is sufficient that they span V or that they are Linearly independent.

Problems:

1. Let $A = \{(1, -2, 5), (2, 3, 1)\}$ be a linearly independent subset of $V_3(\mathbb{R})$. Extend this to a basis of $V_3(\mathbb{R})$.
Soln: Let $\alpha_1 = (1, -2, 5), \alpha_2 = (2, 3, 1)$
Let S be the subspace spanned by $\{\alpha_1, \alpha_2\}$
$\therefore S = \{c_1\alpha_1 + c_2\alpha_2 / c_1, c_2 \in \mathbb{R}\}$
$c_1\alpha_1 + c_2\alpha_2 = c_1(1, -2, 5) + c_2(2, 3, 1)$
$= (c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2)$
$\therefore S = \{(c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2)/c_1, c_2 \in \mathbb{R}\}$
Chose a vector of $V_3(\mathbb{R})$, outside of S.
$(1, 0, 0) \notin S$
$\therefore$ the set $A = \{(1, -2, 5), (2, 3, 1), (1, 0, 0)\}$ is a basis of $V_3(\mathbb{R})$.

2. Given two linearly independent vectors $(2, 1, 4, 3)\&(2, 1, 2, 0)$, find a basis of $V_4(\mathbb{R})$ that includes these two vectors.
Soln: $\alpha_1 = (2, 1, 4, 3), \alpha_2 = (2, 1, 2, 0)$
$S = \{c_1\alpha_1 + c_2\alpha_2 / c_1, c_2 \in \mathbb{R}\}$
Choose $\alpha_3 = (1, 0, 0, 0) \,\&\, \alpha_4 = (0, 1, 0, 0) \notin S$
$\therefore \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $V_4(\mathbb{R})$.

The non-zero rows of a row-reduced echelon form of a matrix are linearly independent.

3. Test the following set of vectors for linear dependence in $V_3(\mathbb{R})$.
$\{(1, 0, 1), (0, 2, 2), (3, 7, 1)\}$. Do they form a basis?
Soln: Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{pmatrix}$
$|A| = 1(2 - 14) - 0(0 - 6) + 1(0 - 6) = -18 \neq 0.$
Therefore the given set in linearly independent.
Any three vectors in $V_3(\mathbb{R})$ which are linearly independent is a basis of $V_3(\mathbb{R})$.

4. Does the set $S = \{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$ form a basis of $\mathbb{R}^3$.
Soln: Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{pmatrix}$
$|A| = 1(3 - 0) - 2(9 + 0) + 3(3 + 2) = 0.$
$\therefore S$ is linearly dependent and hence is not a basis of $\mathbb{R}^3$.

5. Show that the vectors $(1,1,2,4),(2,-1,-5,2),(1,-1,-4,0),(2,1,1,6)$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.

Soln: Consider $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix}$

$R_2 = R_2 - 2R_1; R_3 = R_3 - R_1; R_4 = R_4 - 2R_1$ implies

$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{pmatrix}$

$R_2 = R_2/(-3)$ implies

$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{pmatrix}$

$R_3 = R_3 + 2R_2; R_4 = R_4 + R_2$ implies

$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The final matrix is in echelon form and the rank of A is 2. Therefore the given vectors are linearly dependent.

The corresponding non-zero rows of the initial matrix are $(1,1,2,4)\,\&(2,-1,-5,2)$, which are linearly independent.

The dimension of the subspace spanned by the vectors is 2. These two vectors form a basis of the subspace.

6. Let S be the subspace of $\mathbb{R}^3$ defined by $S = \{(a,b,c)/a+b+c=0\}$. Find a basis and dimension of S.

Soln: $S \neq \mathbb{R}^3$ $[\because (1,2,3) \in \mathbb{R}^3 \text{but} (1,2,3) \notin S]$

$\alpha = (1,0,-1)\,\&\,\beta = (1,-1,0) \in S$ and further they are independent.

$\therefore d[S] = 2$ and hence $\{\alpha, \beta\}$ forms a basis of S.

7. Show that the field $\mathbb{C}$ of complex numbers is a vector space over the field $\mathbb{R}$ of reals. What is its dimension?

Soln: $\mathbb{C} = \{a + ib/a, b \in \mathbb{R}\}$

$\mathbb{C}$ is closed under $'+'$.

$\mathbb{C}$ is associative under $'+'$.

$0 + i0$ is the identity w.r.t $'+'$.

$-a - ib$ is the inverse of $a + ib$.

$\mathbb{C}$ is commutative.

Hence $(\mathbb{C}, +)$ is an abelian group.

$c.((a_1 + ib_1) + (a_2 + ib_2)) = c.(a_1 + ib_1) + c.(a_2 + ib_2) \in \mathbb{C}$

$(c_1 + c_2).(a_1 + ib_1) = c_1.(a_1 + ib_1) + c_2.(a_1 + ib_1) \in \mathbb{C}$

$(c.c^{'}).(a_1 + ib_1) = c.(c^{'}.(a_1 + ib_1)) \in \mathbb{C}$

$'1'$ is the unity

Therefore $\mathbb{C}$ is a vector space over $\mathbb{R}$

Let $\alpha \in \mathbb{C}, \alpha = a + ib \quad \text{s.t.} \quad a, b \in \mathbb{R}$

$\therefore \alpha = 1.a + i.b = a.1 + b.i$

i.e., every element of $\mathbb{C}$ is a linear combination of the elements $1, i$. That is $\{1, i\}$ generates $\mathbb{C}$.
Further $c_1.1 + c_2.i = 0 \implies c_1 = 0 \& c_2 = 0$
$\therefore \{1, i\}$ is linearly independent.
$\therefore \{1, i\}$ is a basis of $\mathbb{C}$ and $d[\mathbb{C}] = 2$.

8. Let V be the vector space of 2×2 symmetric matrices over the field F. Show that $d[V] = 3$.
Soln: Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V, a, b, c \in F$.
Set $a = 1, b = 0, c = 0; a = 0, b = 1, c = 0; a = 0, b = 0, c = 1$
We get three matrices $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
We shall show that these elements of V form a basis.
Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in V$ be arbitrary.
Then, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Thus $\{E_1, E_2, E_3\}$ generates V.
Suppose $c_1E_1 + c_2E_2 + c_3E_3 = 0$, $c_1, c_2, c_3 \in F$
$\implies c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\implies \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies c_1 = c_2 = c_3 = 0.$
$\therefore \{E_1, E_2, E_3\}$ is linearly independent.
Hence $\{E_1, E_2, E_3\}$ is a basis of V and $d[V] = 3$.

9. Find the basis and dimension of the subspace spanned by the subset
$S = \{\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -5 & 7 \end{pmatrix}, \begin{pmatrix} 1 & -7 \\ -5 & 1 \end{pmatrix}\}$ of the vector space of all 2×2 matrices over $\mathbb{R}$.
Soln: Let $\alpha, \beta, \gamma, \delta$ are the matrices of S.
Then the coordinates of $\alpha, \beta, \gamma, \delta$ w.r.t standard basis are
$(1, -5, -4, 2), (1, 1, -1, 5), (2, -4, -5, 7), (1, -7, -5, 1)$.
Consider $\begin{pmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{pmatrix}$

$R_2 = R_2 - R_1, R_3 = R_3 - 2R_1, R_4 = R_4 - R_1 \implies$

$$\begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{pmatrix}$$

$R_3 = R_3 - R_2, R_4 = 3R_4 + R_2 \implies$

$$\begin{pmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The final matrix has two non-zero rows.
$\therefore$ d(subspace) $= 2$.
Further the matrices corresponding to the non-zero rows in the final matrix are $\begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 3 & 3 \end{pmatrix}$.

10. In a vector space $V_3(\mathbb{R})$, let $\alpha = (1,2,1), \beta = (3,1,5) \,\&\, \gamma = (-1,3,-3)$. Show that the subspace spanned by $\{\alpha,\beta\} \,\&\, \{\alpha,\beta,\gamma\}$ are the same.

Soln: Consider, $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{vmatrix} = 1(-3-15) - 2(-9+5) + 1(9+1) = -18 + 8 + 10 = 0$

$\therefore \{\alpha,\beta,\gamma\}$ is linearly dependent.

Let $\gamma = c_1\alpha + c_2\beta$

$(-1,3,-3) = c_1(1,2,1) + c_2(3,1,5)$

$\implies (-1,3,-3) = (c_1 + 3c_2, 2c_1 + c_2, c_1 + 5c_2)$

$\implies c_1 + 3c_2 = -1, 2c_1 + c_2 = 3, c_1 + 5c_2 = -3$

Solving these equations, we get $c_1 = 2, c_2 = -1$.

$\therefore \gamma \in$ subspace spanned by $\{\alpha,\beta\}$.

$\therefore$ the subspace spanned by $\{\alpha,\beta\}$ & $\{\alpha,\beta,\gamma\}$ are the same.