* Inverse Z- transform :

Note that if $Z\{f(k)\} = F(z)$ then the inverse z-transform of $F(z)$ is denoted by '$Z^{-1}\{F(z)\}$' and is given by

$$Z^{-1}\{F(z)\} = f(k)$$

Example (1) find the inverse Z-transform of

$$F(z) = \frac{1}{z-a} \quad \text{when i) } |z|<|a| \quad \text{ii) } |z|>|a|$$

Solution: i) $|z| < |a|$

$$\Rightarrow \quad \left|\frac{z}{a}\right| < 1$$

$\therefore$ consider,

$$F(z) = \frac{1}{z-a}$$

$$= \frac{1}{a\left(\frac{z}{a}-1\right)}$$

$$= \frac{1}{-a\left(1-\left(\frac{z}{a}\right)\right)}$$

$$= -\frac{1}{a}\left[\frac{1}{1-\left(\frac{z}{a}\right)}\right]$$

$$= -\frac{1}{a}\left[1+\frac{z}{a}+\left(\frac{z}{a}\right)^2+\cdots+\left(\frac{z}{a}\right)^k+\cdots\right]$$

$$= -\left[\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\cdots+\frac{z^k}{a^{k+1}}+\cdots\right]$$

$$= -\left[a^{-1}+a^{-2}z+a^{-3}z^2+\cdots+a^{-(k+1)}z^k+\cdots\right]$$

$\therefore$ The coefficient of $z^{k} = -a^{-(k+1)}$, $k \geq 0$

$\Rightarrow$ The coefficient of $z^{-k} = -a^{-(-k+1)}$, $k \leq 0$ $\left(\begin{array}{l}\text{Replace}\\ k \text{ by } -k\end{array}\right)$

that is the coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{-a^{k-1}\} \quad , \quad k \leq 0$$

$$\Rightarrow \quad z^{-1}\left[\frac{1}{z-a}\right] = \{-a^{k-1}\} \quad , \quad k \leq 0$$

ii) $|z| > |a| \Rightarrow 1 > \left|\frac{a}{z}\right|$ i.e $\left|\frac{a}{z}\right| < 1$

$$\therefore \quad F(z) = \frac{1}{z-a}$$

$$= \frac{1}{z(1-\frac{a}{z})} = \frac{1}{z}\left[\frac{1}{1-(\frac{a}{z})}\right]$$

$$= \frac{1}{z}\left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots + \left(\frac{a}{z}\right)^k + \cdots\right]$$

$$= z^{-1}\left[1 + a z^{-1} + a^2 z^{-2} + \cdots + a^k . z^{-k} + \cdots\right]$$

$$= \left[z^{-1} + a z^{-2} + a^2 z^{-3} + \cdots + a^k z^{-(k+1)} + \cdots\right]$$

$$= \left[a^0 z^{-1} + a z^{-2} + \cdots + a^{k-1} z^{-k} + a^k z^{-(k+1)} + \cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = a^{k-1}$, $k \geq 1$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{a^{k-1}\} \quad , \quad k \geq 1$$

$\therefore$

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* Inverse z-transform by partial fraction:-

Ex ① Find inverse z-transform of

$$F(z) = \frac{z}{(z-1)(z-2)}, \quad |z| > 2$$

Solution: consider,
$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \text{--- ①}$$

$$\Rightarrow \frac{z}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow z = A(z-2) + B(z-1)$$

$\Rightarrow$ if $z=1$ then $1 = A(1-2) + B(0)$

$\Rightarrow A = -1$

if $z=2$ then $2 = A(0) + B(2-1)$

$B = 2$

$\therefore$ equation ① becomes.

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{\;}{z-2}$$

$$\Rightarrow F(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

Since, $|z| > 2 \Rightarrow 1 > \left|\frac{2}{z}\right| \Rightarrow \boxed{\left|\frac{2}{z}\right| < 1}$

and $|z| > 2 > 1 \Rightarrow |z| > 1 \Rightarrow 1 > \left|\frac{1}{z}\right| \Rightarrow \boxed{\left|\frac{1}{z}\right| < 1}$

$$\therefore F(z) = \left[\frac{2}{z-2}\right] - \left[\frac{1}{z-1}\right]$$

$$= \frac{2}{z\left[1-\left(\frac{2}{z}\right)\right]} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{2}{z}\left[\frac{1}{1-\left(\frac{2}{z}\right)}\right] - \frac{1}{z}\left[\frac{1}{1-\left(\frac{1}{z}\right)}\right]$$

$$= \frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^k+\cdots\right]$$

$$- \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)+\cdots\right]$$

$$= 2z^{-1}\left[1+2z^{-1}+2^2z^{-2}+\cdots+2\;2^{-k}+\cdots\right]$$

$$- z^{-1}\left[1+z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-1}+2^2z^{-2}+2^3z^{-3}+\cdots+2^k\;2^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-2}+2^2z^{-2}+\cdots+2^k\;z^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = 2^k - 1 \quad , \quad k \geq 1$

$\therefore \quad Z^{-1}[F(z)] = \{f(k)\} = \{2^k - 1\}$

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Ex ② find inverse z-transform of

$$F(z) = \frac{1}{(z-3)(z-2)} , \quad 2 < |z| < 3$$

Solution: Note that $F(z) = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$

(By partial fraction method)

since, $2 < |z| < 3$

if $2 < |z|$ then $\left|\frac{2}{z}\right| < 1$

if $|z| < 3$ then $\left|\frac{z}{3}\right| < 1$

$$\therefore F(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$$

$$= -\frac{1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\cdots+\left(\frac{z}{3}\right)^k+\cdots\right] - \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\left(\frac{2}{z}\right)^k+\cdots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\frac{1}{3^3}z^2+\cdots+\frac{1}{3^{k+1}}z^k+\cdots\right] - \left[z^{-1}+2z^{-2}+\cdots+2^k z^{-(k+1)}+\cdots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\cdots+\frac{1}{3^{k+1}}z^k+\cdots\right] - \left[2^0 z^{-1}+2z^{-2}+\cdots+2^{k-1}z^{-k}+\cdots\right]$$

$$= -\left[3^{-1}z^0+3^{-2}z^1+\cdots+3^{-k-1}z^k+\cdots\right] - \left[2^0 z^{-1}+2z^{-2}+\cdots+2^{k-1}z^{-k}+\cdots\right]$$

from the first series the coefficient of $z^k = -3^{-k-1}$ $\quad k \geq 0$

i.e The coefficient of $z^{-k} = -3^{k-1}$, $k \leq 0$

& from the second series the coefficient of $z^{-k} = 2^{k-1}$

$k \geq 1$

$$\therefore z^{-1}\left[F(z)\right] = \{f(k)\} = \{-3^{k-1}\} , \quad k \leq 0$$

$$= \{-2^{k-1}\} , \quad k \geq 1$$

Homework:

① find inverse z-transform of $\frac{1}{(z-3)(z-2)}$

if ROC is $|z|>3$

② find inverse z-transform of

$$F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}, \quad 3<z<4$$