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Matrices

☆ Let A be equal to $(a_{ij})_n$ be a square matrix if there is a non-zero vector, $X \neq 0$ such that $AX = \lambda X$ for some scalar λ, then X is called an Eigen vector or characteristic vector or latent vector and the scalar λ is called Eigen value or characteristic value or latent value of matrix A.

Note-

$$X \rightarrow \boxed{A} \xrightarrow{\lambda X}$$

(1)

$$\Rightarrow AX = \lambda X$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0 \quad \text{——— (I)}$$

System of Equation (I) has non-zero solution.

$$|A - \lambda I| = 0 \quad \text{——— (II)}$$

Eq^n (II) is called characteristic equation of matrix A and solution of equation (II) gives all characteristic values (eigen values) of matrix A.

For a given eigen value 'λ', solution of Eq (I) gives the eigen vectors of A.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$\cancel{|A - \lambda I|}$ $|A - \lambda I| = 0$

$$\Leftrightarrow$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Note: (2) For 3x3 matrix,

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

Trace of A.

$\Rightarrow \lambda^3 - \left[\text{sum of all minors of order one along principal diagonal}\right]\lambda^2$

$+ \left[\text{sum of all minors of order two along principal diagonal}\right]\lambda$

$-$ sum of all minors of order three along principal diagonal. $|A|$

$= 0$.

* consider along the main diagonal

Similarly for higher powers. 2-smaller.

(Q) Find the eigen values and corresponding linearly independent eigen vectors.

(1) $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

characteristic eq^n of A is $|A - \lambda I| = 0$

2(4-1)

6

$$\lambda^3 - (2+2+2)\lambda^2 + (5+3+3)\lambda - |A| = 0$$

$$\lambda^3 - 6\lambda^2 + 13\lambda - |A| = 0.$$

$$\lambda^3 - 6\lambda^2 + 13\lambda - 6 = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

For $\lambda = 1$,

$(A - \lambda I)X = 0$

$(A - I)X = 0$.

$$\begin{bmatrix} 2-1 & -1 & 1 \\ 1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using row transformations,

$R_2 - R_1$

$R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0$

$2x_2 - 2x_3 = 0$.

case of infinite sol^n

$\Rightarrow x_2 = x_3, x_1 = 0$.

Aisa values nahi dalna hai ki 3 values 0 hojaye [eigen vector]

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ random no. dala hai.

Any one independent eigen vector dhundo

$\therefore$ There is exactly one eigen vector.

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(Q) Find the eigen values and all linearly independent values.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

characteristic eq^n of A is $|A - \lambda I| = 0$

$\lambda^3 - (7)\lambda^2 + (4+4+3)\lambda - |A| = 0$

$2(4) - 2(1) + 1(-1)$
$8 - 3$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\boxed{\lambda = 5, 1, 1}$

('1' calci me bad me aaya wo repeated tha)

(Derivative satisfy karke dekho single der. mtlb repeated 2 and so on)

For $\lambda = 5$,

$(A - 5I)X = 0$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 + x_3 = 0$

$4x_2 - 4x_3 = 0$

$\Rightarrow x_2 = x_3, \quad x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$\therefore x_1 = x_2 = x_3, \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 1$,

$(A - \lambda I)X = 0$.

$(A - I)X = 0$

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① nd-eigen vector not possible
repeated twice = 2 or less eigen vector.

$$\Rightarrow \begin{bmatrix}1&2&1\\1&2&1\\1&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0.$$

$x_1+2x_2+x_3=0$

$x_3=-x_1-2x_2.$

Any real values can be substituted & the eq will get satisfied.

1st conclusion - There are linearly independent eigen vectors.

$$X_2=\begin{bmatrix}1\\2\\-5\end{bmatrix},\quad X_3=\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

Smart way

$$X_2=\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad X_3=\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

(Q) Find eigen values,

$$A=\begin{bmatrix}4&6&6\\1&3&2\\-1&-5&-2\end{bmatrix}$$

$4(4)-6(-2+2)$
$+6(-5+3)$
$16-12$
④

$$\lambda^3-(5)\lambda^2+(6+4-2)\lambda-(4)=0$$

$$\lambda^3-5\lambda^2+8\lambda-4=0$$

$$\boxed{\lambda=1,2,2}$$

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For $\lambda=1$,

$(A-\lambda I)=0$

$(A-I)X=0.$

$$\begin{bmatrix}3&6&6\\1&2&2\\-1&-5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

R_1-3R_2 $\qquad$ R_3+R_2

$$\begin{bmatrix}0&0&0\\1&2&2\\-1&-5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad\begin{bmatrix}2&6&6\\1&1&2\\0&-4&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1+2x_2+2x_3=0$

$-x_1-5x_2-3x_3=0.$

$x_1+x_2+2x_3=0,\quad -4x_2-2x_3=0$

$x_3=-2x_2,\quad \therefore x_1=3x_2$

$\Rightarrow$ There is only one L.I eigen vector

$$X_2=\begin{bmatrix}3\\1\\-2\end{bmatrix}$$

(Q) $$A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$$

$$\lambda^3-6\lambda^2+12\lambda-8=0$$

$|A-\lambda I|=0$

$(2-\lambda)(2-\lambda)(2-\lambda)=0$

$$\boxed{\lambda=2,2,2.}$$

In case of triangular matrix the diagonal values will be the eigen values.

$(A-2I)X=0$

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$x_2=0,\ x_3=0,$

There is only one independent eigen vector.

$$X_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

(Q) $A=\begin{bmatrix}2&1&0\\0&2&0\\0&0&2\end{bmatrix}$

$(A-2I)$

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$x_2=0$

There are 2 L.I eigen vectors: x_1, x_2.

$$X_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad X_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$$

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Results

1) Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

$\lambda_1 \neq \lambda_2$, λ repeated two times
$\downarrow \quad \downarrow$
$x_1 \quad x_2$

1) A and A^T have the same eigen values.

2) Sum of all eigen values of a matrix A is the trace of A i.e sum of the principal elements of A.

3) Product of all the eigen values give the determinant of the matrix.

4) If λ is a eigen value of matrix A and X is an corresponding eigen vector then,

(i) $K\lambda \rightarrow KA \longrightarrow X$ and X is an corresponding eigen vector.

(ii) λ^n is an eigen value of A^n and X is the "

$\lambda^n \rightarrow A^n \rightarrow X$

(iii) For a polynomial,

$K_0+K_1\lambda+K_2\lambda^2+\dots\dots+K_n\lambda^n \longrightarrow K_0I+K_1A+K_2A^2+\dots K_nA^n \longrightarrow X$ (corr. vector)

(iv) $\lambda^{-1} \rightarrow A^{-1} \longrightarrow \underset{\text{(corr. ev)}}{X}$

provided A^{-1} exists. $(|A|\neq 0)$

(v) $\dfrac{|A|}{\lambda} \longrightarrow adj(A) \longrightarrow \underset{\text{(corr. ev)}}{X}$

provided $|A|\neq 0$

If $|A|=0$ then eigen value eigen vector of adj(A) can be found directly by first finding the adjoint

(vi) $\lambda = 0$ is an eigen value of matrix A if and only if matrix A is singular ($|A| = 0$)

(vii) The eigen values of a triangular matrix are just the diagonal elements of the matrix.

(viii) The eigen values of a Hermitian matrix are all real numbers.
(relle conjugate fir transpose)
$(\bar{A})^T = A^\theta$ or A^*
(when $A^\theta = A$ is Hermitian matrix)

as real nos. belonge to matrix symmetric

(ix) If A is a symmetric matrix then all the eigen values are real nos.

(x) The eigen values of an unitary matrix are of unit modulus.
$(A^\theta A = I)$ $\quad |\lambda| = 1$

$A^T A = I$ (orthogonal)

Eigen values of an Orthogonal matrix are always 1 or -1.

(xi) If A is a symmetric matrix then eigen vectors corresponding to 2 distinct eigen values are orthogonal.

$$\lambda_1 \neq \lambda_2$$
$$\downarrow \quad \downarrow$$
$$x_1 \quad x_2$$

$$X_1^T \cdot X_2 = 0$$

(Q) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & -2 & -1 & 0 \\ 5 & 2 & 3 & -1 \end{bmatrix}$, Find eigen values and eigen vectors of A^4, adj A, $A^3 + 2A^2 - 5I$, $7A^{-1}$.

Sol^n Since A is a lower triangular matrix, eigen values of A are $\lambda = 1, 2, -1, -1$.

For $\lambda = 1$,
$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & -2 & -2 & 0 \\ 5 & 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$
$$4x_1 - 2x_2 - 2x_3 = 0$$
$$5x_1 + 2x_2 + 3x_3 - 2x_4 = 0$$
$$\Rightarrow x_2 = -3x_1, \quad 10x_1 - 2x_3 = 0 \Rightarrow 5x_1 = x_3$$

$$5x_1 - 6x_1 + 15x_1 - 2x_4 = 0$$
$$\Rightarrow x_4 = 7x_1$$

$$X_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 7 \end{bmatrix}$$

For $\lambda = 2$
$$(A - 2I)X = 0$$
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & -2 & -3 & 0 \\ 5 & 2 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0 \ , \ 4x_1 - 2x_2 - 3x_3 = 0$

$\Rightarrow x_3 = -\frac{2x_2}{3}$

$5x_1 + 2x_2 + 3x_3 - 3x_4 = 0$

$2x_2 - 2x_2 - 3x_4 = 0$

$\Rightarrow x_4 = 0$

$$X_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

For $\lambda = -1$

$(A+I)X = 0$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 5 & 2 & 3 & 0 \end{bmatrix}$$

$x_1 = 0, \ 3x_1 + 3x_2 \Rightarrow x_2 = 0$

$5x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_3 = 0$

$\Rightarrow$ There is one L.I eigen vector

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(ii) Eigen values and eigen vectors of A^4 are -

$\lambda_1 = 1^4 = 1, \ X_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 7 \end{bmatrix}$

$\lambda_2 = 2^4 = 16, \quad X_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

$\lambda_3 = (-1)^4 = 1 \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_4 = 1$

(iii) Eigen values and eigen vectors of adj A are

$\lambda_1 = \frac{|A|}{\lambda} = \frac{2}{1} = 2, \ X_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 7 \end{bmatrix}$

$\lambda_2 = \frac{2}{2} = 1, \ X_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

$\lambda_3 = \frac{2}{-1} = -2 \quad , \ X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_4 = -2$

(iv) Eigen values and eigen vectors of $A^3 + 2A^2 - 5I$ are

$\lambda_1 = 1 + 2 - 5 = -2, \ X_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 7 \end{bmatrix}$

$\lambda_2 = 8 + 8 - 5 = 11, \quad X_2 = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

$\lambda_3 = -1 + 2 - 5 = -4 \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_4 = -4.$

(Q) Find the sum and product of all the eigen values of A.

$$A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{bmatrix}$$

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Q) Find the eigen values & eigen vectors of adj A.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(i) adj A $\rightarrow$ 6, 2, 3

(ii) adj (adj A) $\rightarrow \frac{6 \times 2 \times 3}{6}$ = 6, 18, 12