Ex ② Prove that the matrix A is diagonalisable. Also find diagonal matrix and the transforming matrix.

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Solution: * for eigenvalues of A :

we consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(4-\lambda)(-3-\lambda) - 12\right] + 6(0-0) - 4(0-0) = 0$$

$$\Rightarrow (1-\lambda)\left[-12 - \lambda + \lambda^2 + 12\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - \lambda) = 0$$

$$\Rightarrow \lambda^2 - \lambda - \lambda^3 + \lambda^2 = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\Rightarrow \lambda = 0, 1, 1$$

$\therefore$ eigenvalues of A are 0, 1, 1

Now to find eigenvectors :-

for $\lambda = 0$

consider $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to \frac{R_3}{3}]{R_2 \to \frac{R_2}{2}} \begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_1 - 6x_2 - 4x_3 = 0$

$2x_2 + x_3 = 0$

here, x_3 is non-leading

$\therefore$ we put $x_3 = 2t$

then $x_2 = -t$ and $x_1 = 2t$

$$\therefore X = \begin{bmatrix} 2t \\ -t \\ 2t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ the curresponding eigenvector to $\lambda = 0$ is $V_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

<u>for $\lambda = 1$</u>

consider $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{R_1}{-2}} \quad R_2 \to R_2 + \tfrac{1}{2}R_1 \quad R_3 \to R_3 - R_1 \quad \begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $3x_2 + 2x_3 = 0$

here x_1 is free variable & x_3 is non-leading

$\therefore$ we put $x_1 = t$ and $x_3 = -3s$

$\therefore$ $x_2 = 2s$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2s \\ -3s \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$\therefore$ The corresponding eigenvector to $\lambda = 1$ are

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$\therefore$ The matrix P is

$$P = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$$

Clearly, Algebric multiplicity for $\lambda = 0$ is 1 and geometric multiplicity for $\lambda = 0$ is also 1

lly Algebric multiplicity for $\lambda = 1$ is 2 and geometric multiplicity for $\lambda = 1$ is also 2

Hence A is diagonalizable

and it can be written as

$$A = P^{-1} D P$$

where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$

Homework :

Ex. Show that following matrices are diagonalizable.

(1) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$