Module 1.

Linear Algebra (Theory of Matrices)

* **Matrix:** A matrix is a set of mn numbers arranged in m rows and n columns. It is called an $m \times n$ matrix.

Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

we denote this matrix by $A = [a_{ij}]_{m \times n}$

* **square matrix:** If the number of rows of matrix is equal to the number of columns i.e. if $m = n$, then the matrix is called a square matrix.

for example: $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

are the square matrices of order 2 and 3

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* Diagonal elements : In a square matrix the elements lying along the diagonal of matrix. i.e the elements a_{ii} are called diagonal elements of the matrix.

for example: In the matrices $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 2 \\ 2 & -1 & -3 \end{bmatrix}$

2, -1 and 1, 2, -3 are the diagonal elements

* Diagonal matrix : A square matrix whose all non-diagonal elements are zero is called a diagonal matrix. i.e. All $a_{ij} = 0$, for $i \neq j$

for example. $[2]$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all diagonal matrices

* Trace of Matrix : The sum of all diagonal elements of a square matrix is called the trace of a matrix. It is Denoted by 'tr(A)'

i.e. if $A = [a_{ij}]_{n \times n}$ then $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

for example : ① If $A = [2]$ then $tr(A) = 2$

② If $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ then $tr(B) = 3 + 2 = 5$

③ If $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 2 \\ 0 & 5 & -2 \end{bmatrix}$ then $tr(C) = 1 + 3 + (-2) = 2$

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* Singular and non-singular Matrix :

let A be the square matrix

If determinant of A is zero (i.e. $|A| = 0$) then A is called singular matrix.

If determinant of A is non-zero (i.e. $|A| \neq 0$) then A is called non-singular matrix.

for examples :

$[0]$, $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ are singular matrices and

$[2]$, $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are non-singular matrices

* Transpose of a matrix :

A matrix obtained from a given matrix A by interchanging rows and columns is called transpose of a given matrix and is denoted by A^T or A'

for example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

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* Traingular matrices:

A matrix $A = [a_{ij}]$ is said to be Upper triangular if $a_{ij} = 0$ for all $i > j$ and is said to be lower triangular if $a_{ij} = 0$ for all $i < j$

for example: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ are upper triangular

and $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ are lower triangular.

* Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j (i.e. $A^T = A$)

for example, $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ are symmetric.

Note that: A is symmetric if $A^T = A$

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I) Eigenvalues :

Let A be any square matrix of order n, λ be any scalar and I be the unit matrix of order n then

* The determinant $|A - \lambda I|$ is called charecterstic polynomial of λ. and
* The Equation $|A - \lambda I| = 0$ is called charecterstic equation of the matrix A
* The roots of the charecterstic equation $|A - \lambda I| = 0$ is called Eigenvalues of matrix A

Note that : Eigenvalues is also called as charecterstics value or charecterstics roots

Example: 1. find charecterstic equation and eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

solution: Given: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

* For charecterstic equation :

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

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* For Eigenvalues :

consider, $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda^2 - 5\lambda + \lambda - 5 = 0$

$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$

$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda = 5, -1$

Therefore, 5, -1 are the eigenvalues of matrix A

Example. ② Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

solution: Given: $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

* for Eigenvalues:

consider, the charectestic equation

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\left[(-3-\lambda)(1-\lambda) - 8\right] + 8\left[4(1-\lambda) + 6\right] - 2\left[-16 - 3(-3-\lambda)\right] = 0$$

$$\Rightarrow (8-\lambda)\left[-3 + 2\lambda + \lambda^2 - 8\right] + 8\left[10 - 4\lambda\right] - 2\left[-7 + 3\lambda\right] = 0$$

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$\Rightarrow \quad (8-\lambda)(\lambda^2+2\lambda-11) + 80 - 32\lambda + 14 - 6\lambda = 0$

$\Rightarrow \quad 8\lambda^2 + 16\lambda - 88 - \lambda^3 - 2\lambda^2 + 11\lambda + 94 - 38\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \quad \lambda = 1, 2, 3$

Therefore, 1, 2, 3 are the Eigenvalues of A.

Example (3) Find eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: Given: $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

* for eigenvalues :

consider, the charecterstics equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[(2-\lambda)(2-\lambda) - 1] + 1[1(2-\lambda)+1] + 1[-1-(2-\lambda)] = 0$$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \quad \lambda = 1, 2, 3$

Therefore, 1, 2, 3 are the eigenvalues of A

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* Eigenvector:

Suppose, A is square matrix and λ_1 is a eigenvalues of A then a non zero column matrix X is said to be eigenvector if $[A - \lambda_1 I] X = 0$

* Working Rule for Eigenvectors:

Suppose, A is matrix of order n

i) find the eigenvalues of matrix A says. $\lambda_1, \lambda_2, \cdots, \lambda_n$

ii) for $\lambda = \lambda_i$, $i = 1, 2, \cdots, n$ consider the system $[A - \lambda_i I] X = 0$

iii) Reduced the matrix $[A - \lambda_i I]$ to echolon form by using elementary row transformation

iv) write the system of equations and find X_i

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Example ①

find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

solution:

* for eigenvalues:

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)(2-\lambda)-1\right] + 1\left[1(2-\lambda)+1\right] + 1\left[-1-1(2-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2 - 4\lambda + 3\right] + 3-\lambda + \lambda - 3 = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

$\therefore$ The eigenvalues of matrix A are 1, 2, 3

Now, To find eigenvector:

i) for $\lambda = \lambda_1 = 1$

$$[A - \lambda_1 I]\, X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$\therefore$ By Row echolon form of matrix.

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_2 - 2x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 - x_2 + x_3 &= 0 \quad -① \\ x_2 - x_3 &= 0 \quad -② \end{aligned}$$

clearly, x_3 is non-leading coefficient

$\therefore$ we put $x_3 = t$

$\Rightarrow$ $x_2 = t$, from ②

and $x_1 = 0$, from ①

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, currresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = X_1$

ii) for $\lambda = \lambda_2 = 2$.

$$[A - \lambda_2 I] X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_{12}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{aligned} x_1 - x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}\right\}$

clearly, x_3 is non-leading

$\therefore$ we put $x_3 = t$

$\Rightarrow$ $x_2 = t$ and $x_1 = t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, currresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = X_2$

iii) for $\lambda = \lambda_3 = 3$

$$[A - \lambda_3 I] X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{aligned} -x_1 - x_2 + x_3 &= 0 \\ -2x_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_2 &= 0 \end{aligned}$

$\Rightarrow$ $x_1 - x_3 = 0$

$\therefore$ x_3 is non-leading

$\therefore$ put $x_3 = t$ $\Rightarrow$ $x_1 = t$

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$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = X_3$

Hence, eigenvalues of A are 1, 2, 3

and eigenvectors of A are $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Ex. (2) find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: <u>for eigenvalues</u>:

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2\left[1(2-\lambda) - 1\right] + 1\left[2 - 1(3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2 - 5\lambda + 4\right] - 2(1-\lambda) + \lambda - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\Rightarrow \lambda = 1, 1, 5$$

$\therefore$ eigenvalues of A are 1, 1, 5

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Now To find eigenvectors!

i> for $\lambda = \lambda_1 = 1$

$$[A - \lambda_1 I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_1 + 2x_2 + x_3 = 0$

$\therefore$ here, x_2 and x_3 are non-leading coefficient

$\therefore$ we put $x_2 = s$, $x_3 = t$

then $x_1 = -2s - t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Curresponding to $\lambda = 1$, the eigenvectors are $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

ii> for $\lambda = \lambda_2 = 5$

$$[A - \lambda_2 I]X = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_{13}} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ -4x_2 + 4x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$

here, x_3 is non-leading

$\therefore$ we put $x_3 = t$

$\Rightarrow x_2 = t$

and $x_1 = t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ currsponding to $\lambda = 5$, the eigenvector is $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence, the eigenvalues of A are 1, 1, 5

and the eigenvectors of A are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example ③ find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

solution: for eigenvalues :

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)(2-\lambda) - 0] - 1[0-0] + 0[0-0] = 0$$

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$$\Rightarrow \quad (2-\lambda)\left[(2-\lambda)^2\right] = 0$$

$$\Rightarrow \quad (2-\lambda)^3 = 0$$

$$\Rightarrow \quad \lambda = 2, 2, 2$$

$\therefore$ eigenvalues of A are $2, 2, 2$

Now. To find the eigenvector:

for $\lambda = 2$

$$[A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_2 = 0$ and $x_3 = 0$

and x_1 is free variable

$\therefore$ we put $x_1 = t$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ corresponding to eigenvalues $\lambda = 2$,

the eigenvector is $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Homework:

Que. find the eigenvalues and eigenvectors of the following

1) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ 3) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

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* Properties of eigenvalues :

(I) Let A be square matrix of order n and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A then

(a) $\lambda_1 + \lambda_2 + \dots + \lambda_n = tr(A)$ (i.e. trace of A)

i.e. Sum of All eigenvalues of A is equal to the trace of matrix A

(b) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_n = |A|$ (i.e. determinant of A)

i.e. Product of All eigenvalues of A is equal to the determinant of matrix A

for example : let $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Note that eigenvalues of A are $1, 2, 3$

Now, $tr(A) = 8 + (-3) + 1 = 6$

and sum of eigenvalues of $A = 1 + 2 + 3 = 6$

$\Rightarrow$ sum of All eigenvalues of $A = tr(A)$

and $|A| = 8(-3-8) + 8(4+6) - 2(-16+9)$

$= -88 + 80 + 14$

$= 6$

and the product of eigenvalues of $A = 1 \times 2 \times 3 = 6$

$\Rightarrow$ Product of eigenvalues of $A = |A|$

(2) Any square matrix A and its transpose A^T will have same eigenvalues

for example. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

clearly, Both have same eigenvalues $1, 2$

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③ let A be any square matrix

If A is either diagonal or triangular then eigenvalues of A are the diagonal elements of A

for example. ① $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then 2, 1 are eigenvalues of A

(ii) $B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$ then 3, -1, 4 are the eigenvalues of B

④ The eigenvalues of symmetric matrix are always Real numbers

for example: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is symmetric

then eigenvalues are 0, 3, 15 (Real numbers)

⑤ The eigenvalues of skew-symmetric matrix are either zero or purely imaginary

(A is skew-symmetric if $A^T = -A$)

for example: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric

then eigenvalues are $i, -i$ (purely imaginary)

⑥ The eigenvalues of orthogonal matrix is either 1 or -1

(A is orthogonal if $AA^T = I$)

for example : $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal

then eigenvalues are 1, -1

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(7) let A be the square matrix of order n and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A then

(i) eigenvalues of KA are
$k\lambda_1, k\lambda_2, \cdots, k\lambda_n$ (k is any scalar)

(ii) eigenvalues of A^m are
$\lambda_1^m, \lambda_2^m, \cdots, \lambda_n^m$ (m is positive integer)

(iii) eigenvalues of A^{-1} are
$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$

for example: let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then eigenvalues of A are $1, -1$

Now, $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $3, -3$

$A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $1^{10}, (-1)^{10}$ i.e. $1, 1$

and $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $\frac{1}{1}, \frac{1}{-1}$ i.e. $1, -1$

(8) let A be the square matrix of order n and $f(x)$ be an algebric polynomial in x then

i) if λ is eigenvalue of A then $f(\lambda)$ is an eigenvalue of $f(A)$

ii) If X is eigenvector curresponding to λ eigenvalue then X is also a eigenvector curresponding to $f(\lambda)$

(Algebric polynomial : $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$)

Ex (1) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $A^3 + 5A + 8I$

Solution:

Given: $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

first to find eigenvalues of A :

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)[(3-\lambda)(\lambda-2) - 0] - 2[0-0] + 3[0-0] = 0$$

$$\Rightarrow (-1-\lambda)(3-\lambda)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -1, 3, 2$$

$\therefore$ eigenvalues of A are $-1, 3, 2$

$\therefore$ eigenvalues of A^3 are $(-1)^3, (3)^3, (2)^3$

eigenvalues of $5A$ are $5(-1), 5(3), 5(2)$

eigenvalues of $8I$ are $8(1), 8(1), 8(1)$

Therefore, the eigenvalues of $A^3 + 5A + 8A$ are

$(-1)^3 + 5(-1) + 8(1) = 2$,

$(3)^3 + 5(3) + 8(1) = 50$,

$(2)^3 + 5(2) + 8(1) = -10$

Hence, eigenvalues of $A^3 + 5A + 8A$ are $2, 50, -10$

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Ex (2) Find the charecterstic root of $A^{30} - 9A^{28}$

where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Solution: first to find eigenvalues of A

$\therefore$ consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda = -1, 3$$

$\therefore$ eigenvalues of A are $-1, 3$

eigenvalues of A^{30} are $(-1)^{30}, (3)^{30}$

eigenvalues of $-9A^{28}$ are $-9(-1)^{28}, -9(3)^{28}$

$\therefore$ The eigenvalues of $A^{30} - 9A^{28}$ are

$(-1)^{30} - 9(-1)^{28} = (-1)^{28}[(-1)^2 - 9] = 1(-8) = -8$

and $(3)^{30} - 9(3)^{28} = 3^{28}(3^2 - 9) = 0$

Hence eigenvalues of $A^{30} - 9A^{28}$ are $\underline{\underline{0, -8}}$

Homework:

(1) If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$, then find eigenvalues of $6A^{-1} + A^3 + 2I$

(2) If $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$, then find eigenvalues of $(A')^2 - 3A' + 4I$

(3) find the eigenvalues of $A^3 - 3A^2 + A$

if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

(Ans: $-5, -1, 20$)

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* Cayley - Hamilton Theorem :

Statement :

* Every square matrix satisfy its charecterstic equation.

Note that:

① If $P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$ is charecterstic equation of square matrix A

then $P(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = 0$

Examples on cayley-Hamilton theorem

Ex ① Verify Cayley-Hamilton theorem for the matrix A and hence find A^{-1}, A^{-2} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution:

* for the charecterstic equation:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)(1-\lambda) - 0\right] - 2\left[-1(1-\lambda) - 0\right] - 2\left[2 - 0\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 4\lambda + 3) + 2(1-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda + 2 - 2\lambda - 4 = 0$$

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$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 9\lambda + 1 = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$

which is charecterstic equation of A

Now. By Cayley - Hamilton theorem,

A satisfy its charectestic equation.

$\therefore$ To show: $A^3 - 5A^2 + 9A - I = 0$ —— (1)

$$* \; A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$* \; A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Now we consider,

$A^3 - 5A^2 + 9A - I$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

that is $A^3 - 5A^2 + 9A - I = 0$

therefore, A satisfy its charecterstic equation

Hence, Cayley - Hamilton theorem is varified.

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Now first to find A^{-1} :

Since, $A^3 - 5A^2 + 9A - I = 0$

Now, multiplying both side by A^{-1} we get

$$A^3 \cdot A^{-1} - 5A^2 A^{-1} + 9AA^{-1} - IA^{-1} = 0 \cdot A^{-1}$$

$$\Rightarrow \quad A^2 - 5A + 9I - A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = A^2 - 5A + 9I \qquad \text{——} \quad (2)$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now To find : A^{-2}

Since, $A^3 - 5A^2 + 9A - I = 0$

Now multiplying both side by A^{-2}, we get

$$A^3 A^{-2} - 5A^2 A^{-2} + 9AA^{-2} - A^{-2} = 0 \cdot A^{-2}$$

$$\Rightarrow \quad A - 5I + 9A^{-1} - A^{-2} = 0$$

$$\Rightarrow \quad A^{-2} = A - 5I + 9A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 27 & 18 & 54 \\ 9 & 9 & 18 \\ 18 & 18 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

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Now to find A^4 :

since, $A^3 - 5A^2 + 9A - I = 0$

multiplying both side by A, we get

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$\Rightarrow \quad A^4 = 5A^3 - 9A^2 + A$$

$$= 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} + \begin{bmatrix} 9 & -108 & 36 \\ 36 & -63 & -18 \\ -18 & 72 & -9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

Ex. ② Verify Cayley - Hamilton theorem and hence find the matrix represented by

$$A^6 - 6A^5 + 9A^5 + 4A^3 - 12A^2 + 2A - I$$

where $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

solution: The charecterstic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

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$$\Rightarrow (3-\lambda)\left[(-3-\lambda)(7-\lambda)+20\right]-10\left[-2(7-\lambda)+12\right]+5\left[-10-3(-3-\lambda)\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-4\lambda-1\right]+20(7-\lambda)-120+5(3\lambda-1)=0$$

$$\Rightarrow 3\lambda^2-12\lambda-3-\lambda^3+4\lambda^2+\lambda+140-20\lambda-120+15\lambda-5=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-16\lambda+12=0$$

$$\Rightarrow \lambda^3-7\lambda^2+16\lambda-12=0$$

which is charecterstic equation of A

Note that Cayley-Hamilton theorem states that A satisfy its charecterstic equation

that is $A^3-7A^2+16A-12I=0$

Now, $$A^2=\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}=\begin{bmatrix}4 & 25 & 10\\ -2 & -31 & -26\\ 20 & 50 & 44\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4 & 25 & 10\\ -2 & -31 & -26\\ 20 & 50 & 44\end{bmatrix}\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}=\begin{bmatrix}-8 & 15 & -10\\ -52 & -157 & -118\\ 92 & 270 & 208\end{bmatrix}$$

considered, $A^3-7A^2+16A-12I$

$$=\begin{bmatrix}-8 & 15 & -10\\ -52 & -157 & -118\\ 92 & 270 & 208\end{bmatrix}-7\begin{bmatrix}4 & 25 & 10\\ -2 & -31 & -26\\ 20 & 50 & 44\end{bmatrix}+16\begin{bmatrix}3 & 10 & 5\\ -2 & -3 & -4\\ 3 & 5 & 7\end{bmatrix}-12\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-8 & 15 & -10\\ -52 & -157 & -118\\ 92 & 270 & 208\end{bmatrix}+\begin{bmatrix}-28 & -175 & -70\\ 14 & 217 & 182\\ -140 & -350 & -308\end{bmatrix}+\begin{bmatrix}48 & 160 & 80\\ -32 & -48 & -64\\ 48 & 80 & 112\end{bmatrix}+\begin{bmatrix}-12 & 0 & 0\\ 0 & -12 & 0\\ 0 & 0 & -12\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

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that is A satisfy its charecteristic equation

Hence, Cayley-Hamilton theorem is varified

Now To find: $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$

since, $A^3 - 7A^2 + 16A - 12I = 0$ ——— (1)

considerd, $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$

$$= (A^3 - 7A^2 + 16A - 12I)A^3 + (A^3 - 7A^2 + 16A - 12I)A^2 + 2A - I$$

$$= (0)A^3 + (0)A^2 + 2A - I \qquad \text{(using (1))}$$

$$= 2A - I$$

$$= 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 20 & 10 \\ -4 & -6 & -8 \\ 6 & 10 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

$$\therefore A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

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Ex. 3 Use Cayley - Hamilton theorem to find $2A^4 - 5A^3 - 7A + 6I$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

solution: The charecteristic equation for A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow 2 - 3\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 2 = 0$$

Note that the cayley-hamilton theorem states that A satisfy its charecteristic equation

$$\therefore \quad A^2 - 3A - 2I = 0 \quad \text{——} \quad (1)$$

Now we divide $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$

$$\begin{array}{r|l}
 & 2\lambda^2 + \lambda + 7 \\
\lambda^2 - 3\lambda - 2 & 2\lambda^4 - 5\lambda^3 - 7\lambda + 6 \\
 & \underline{2\lambda^4 - 6\lambda^3 - 4\lambda^2} \\
 & \,\,\,\overset{-}{}\quad \overset{+}{}\quad \overset{+}{} \\
 & \lambda^3 + 4\lambda^2 - 7\lambda + 6 \\
 & \underline{\lambda^3 - 3\lambda^2 - 2\lambda} \\
 & 7\lambda^2 - 5\lambda + 6 \\
 & \underline{7\lambda^2 - 21\lambda - 14} \\
 & 16\lambda + 20
\end{array}$$

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therefore,

$$2\lambda^4 - 5\lambda^3 - 7\lambda + 6 = (\lambda^2 - 3\lambda - 2)(2\lambda^2 + \lambda + 7) + (16\lambda + 20)$$

$$\left(\because \begin{array}{r|l} a & \overline{b} \\ & \overline{r} \end{array} \; q \Rightarrow b = aq + r \right)$$

$$\therefore \; 2A^4 - 5A^3 - 7A + 6I = (A^2 - 3A - 2I)(2A^2 + A + 7I) + (16A + 20I)$$

$$\Rightarrow 2A^4 - 5A^3 - 7A + 6I = (0)(2A^2 + A + 7I) + (16A + 20I)$$

$$\Rightarrow 2A^4 - 5A^3 - 7A + 6I = 16A + 20I$$

$$= 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 32 \\ 32 & 32 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

$$\therefore \; 2A^4 - 5A^3 - 7A + 6I = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

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* simillarity of Matrices :-

let A and B be two square matrices of order n then we say B is simillar to A if there exist a non singular matrix p such that $B = p^{-1} A p$

* properties of simillary matrices :-

① If A and B are simillar matrices then $|A| = |B|$

② If A and B are simillar matrices then $tr(A) = tr(B)$

③ If A and B are simillar matrices then $rank(A) = rank(B)$

④ If A and B are simillar matrices then Both A and B have same charecterstic polynomial

⑤ If A and B are simillar matrices then Both A and B have same eigenvalues

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Example :

Determine whether the following matrices are simillar or not.

① $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -8 \\ 1 & 5 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 0 & -7 \\ 5 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 15 & -18 & -2 \\ 17 & -17 & -4 \\ 7 & -22 & 4 \end{bmatrix}$

solution:

① To find the eigenvalues of matrix A and B first we consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda) - 1 = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$\Rightarrow \lambda = 1, 3$ which are eigenvalues of A

Now we consider $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & -8 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(5-\lambda) + 8 = 0$$

$$\Rightarrow -5 + \lambda - 5\lambda + \lambda^2 + 8 = 0$$

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$\Rightarrow \quad \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow \quad \lambda = 1, 3$ which are eigenvalues of B

that is Both the matrices A and B have same eigenvalues

Hence, A and B are simillar

② here,

$$A = \begin{bmatrix} 1 & 0 & -7 \\ 5 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & -18 & -2 \\ 17 & -17 & -4 \\ 7 & -22 & 4 \end{bmatrix}$$

Note that $tr(A) = 1+1+0 = 2$

$tr(B) = 15 - 17 + 4 = 2$

i.e. Both A and B have same Trace

Now, $|A| = 1(0-4) - 0(0+8) - 7(10+4)$

$= -4 + 0 - 98$

$= -102$

and $|B| = 15[-68 - 88] + 18[68 + 28] - 2[-374 + 119]$

$= -2340 + 1728 + 510$

$= -2340 + 2238$

$= -102$

i.e Both A and B have same Determinant

$\therefore$ A and B are simillar.

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* Diagonalizable and non-diagonalizable matrices :-

let A be the square matrix of order n then A is digonalizable if there exist a non-singular matrix P such that the matrix $P^{-1}AP$ is diagonal matrix

Note that if A is not diagonalizable then A is non-diagonalizable

Example ①

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* Diagonalizable and non-diagonalizable matrices :-

let A be the square matrix of order n then A is digonalizable if there exist a non-singular matrix p such that the matrix $p^{-1}AP$ is diagonal matrix

Note that If A is not diagonalizable then A is non-diagonalizable

* Important properties :

Let A be any square matrix of order 3 and $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A

— Algebric multiplicity of eigenvalue :-
The Number of Repitition of the eigenvalue is called Algebric multiplicity

— Geometric multiplicity :
The Number of Linearly independent eigenvector corresponding to λ eigenvalue is called geometric multiplicity.

— If Algebric multiplicity = Geometric multiplicity for every eigenvalues of A then A is diagonalizable.

Note that: If All eigenvalues of A are distinct then A is digonalizable.

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Example ① show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

is diagonalisable. find the diagonal form D and the diagonalising matrix M

solution: Given matrix is $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

* For eigenvalues :

consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-9-\lambda)[(3-\lambda)(7-\lambda)-32] - 4[-8(7-\lambda)+64] + 4[-64+16(3-\lambda)] = 0$$

$$\Rightarrow (-9-\lambda)[\lambda^2 - 7\lambda + 16] - 4[-\lambda + 8] + 4[16 - 16\lambda] = 0$$

$$\Rightarrow -9\lambda^2 + 63\lambda - 144 - \lambda^3 + 7\lambda^2 - 16\lambda + 4\lambda - 32 + 64 - 64\lambda = 0$$

$$\Rightarrow -\lambda^3 - 2\lambda^2 - 13\lambda - 112 = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 + 13\lambda + 112 = 0$$

$$\Rightarrow \lambda = -1, -1, 3$$

$\therefore$ The eigenvalues of A are $-1, -1, 3$

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* <u>for eigenvector</u> :

<u>for $\lambda = -1$</u>

consider $[A - \lambda I] X = 0$

$$\Rightarrow [A - (-1) I] X = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $-8x_1 + 4x_2 + 4x_3 = 0$

$$\Rightarrow 2x_1 - x_2 - x_3 = 0$$

here, x_2 and x_3 are non leading

$\therefore$ we put $x_2 = s$, $x_3 = t$.

$\therefore$ $2x_1 - s - t = 0$

$$\Rightarrow x_1 = \frac{s+t}{2} = \frac{s}{2} + \frac{t}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{s+t}{2} \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The curresponding to $\lambda = -1$ eigenvectors are $v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

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for $\lambda = 3$

consider, $[A - \lambda I]X = 0$

$$\Rightarrow [A - 3I]X = 0$$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[-\frac{R_2}{4}]{-\frac{R_1}{4}} \begin{bmatrix} 3 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $3x_1 - x_2 - x_3 = 0$ — ①

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ — ②

from ①, $3x_1 - x_1 - x_3 = 0 \Rightarrow 2x_1 - x_3 = 0$ — ③

Now here x_2 is non leading

$\therefore$ we put $x_2 = t$

$\therefore$ $x_1 = t$ and $x_3 = 2t$ (using ② & ③)

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

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$\therefore$ corresponding to $\lambda = 3$, the eigenvector is

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ The matrix P is

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

clearly Algebric multiplicity for $\lambda = -1$ is 2
geometric multiplicity for $\lambda = -1$ is 2
lly Algebric multiplicity for $\lambda = 3$ is 1
geometric multiplicity for $\lambda = 3$ is 1

$\therefore$ A is diagonalizable
and it can be write as

$$A = P^{-1} D P \quad , \text{ where } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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Ex ② Prove that the matrix A is diagonalisable. Also find diagonal matrix and the transforming matrix.

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Solution: * for eigenvalues of A:

we consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(4-\lambda)(-3-\lambda) - 12\right] + 6(0-0) - 4(0-0) = 0$$

$$\Rightarrow (1-\lambda)\left[-12 - \lambda + \lambda^2 + 12\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - \lambda) = 0$$

$$\Rightarrow \lambda^2 - \lambda - \lambda^3 + \lambda^2 = 0$$

$$\Rightarrow -\lambda^3 + 2\lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\Rightarrow \lambda = 0, 1, 1$$

$\therefore$ eigenvalues of A are $0, 1, 1$

Now to find eigenvectors :-

for $\lambda = 0$

consider $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\xrightarrow[R_3 \to \frac{R_3}{3}]{R_2 \to \frac{R_2}{2}} \begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -6 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_1 - 6x_2 - 4x_3 = 0$

$2x_2 + x_3 = 0$

here, x_3 is non-leading

$\therefore$ we put $x_3 = 2t$

then $x_2 = -t$ and $x_1 = 2t$

$$\therefore X = \begin{bmatrix} 2t \\ -t \\ 2t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ The curresponding eigenvector to $\lambda = 0$ is $V_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

<u>for $\lambda = 1$</u>

consider $[A - \lambda I] X = 0$

$$\Rightarrow \begin{bmatrix} 0 & -6 & -4 \\ 0 & 3 & 2 \\ 0 & -6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 + \frac{1}{2}R_1 \\ R_3 \to R_3 - R_1}]{R_1 \to \frac{R_1}{-2}} \begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $3x_2 + 2x_3 = 0$

here x_1 is free variable & x_3 is non-leading

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$\therefore$ we put $x_1 = t$ and $x_3 = -3s$

$\therefore$ $x_2 = 2s$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2s \\ -3s \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$\therefore$ The corresponding eigenvector to $\lambda = 1$ are

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$\therefore$ The matrix P is

$$P = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$$

Clearly, Algebric multiplicity for $\lambda = 0$ is 1
and geometric multiplicity for $\lambda = 0$ is also 1

lly Algebric multiplicity for $\lambda = 1$ is 2
and geometric multiplicity for $\lambda = 1$ is also 2

Hence A is diagonalizable

and it can be written as

$$A = P^{-1} D P$$

where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & -3 \end{bmatrix}$

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Homework :

Ex. show that following matrices are diagonalizable.

(1) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

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