* Properties of eigenvalues :

(I) Let A be square matrix of order n and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A then

(a) $\lambda_1 + \lambda_2 + \dots + \lambda_n = tr(A)$ (i.e. trace of A)

i.e. Sum of All eigenvalues of A is equal to the trace of matrix A

(b) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \cdot \lambda_n = |A|$ (i.e. determinant of A)

i.e. Product of All eigenvalues of A is equal to the determinant of matrix A

for example: let $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

Note that eigenvalues of A are 1, 2, 3

Now, $tr(A) = 8 + (-3) + 1 = 6$

and sum of eigenvalues of A $= 1+2+3 = 6$

$\Rightarrow$ sum of All eigenvalues of A $= tr(A)$

and $|A| = 8(-3-8) + 8(4+6) - 2(-16+9)$

$= -88 + 80 + 14$

$= 6$

and the product of eigenvalues of A $= 1 \times 2 \times 3 = 6$

$\Rightarrow$ product of eigenvalues of A $= |A|$

(2) Any square matrix A and its transpose A^T will have same eigenvalues

for example. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

clearly, Both have same eigenvalues 1, 2

(3) let A be any square matrix

If A is either diagonal or triangular then eigenvalues of A are the diagonal elements of A

for example. (i) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then 2, 1 are eigenvalues of A

(ii) $B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix}$ then 3, -1, 4 are the eigenvalues of B

(4) The eigenvalues of symmetric matrix are always Real numbers

for example: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is symmetric

then eigenvalues are 0, 3, 15 (Real numbers)

(5) The eigenvalues of skew-symmetric matrix are either zero or purely imaginary

(A is skew-symmetric if $A^T = -A$)

for example: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric

then eigenvalues are $i, -i$ (purely imaginary)

(6) The eigenvalues of orthogonal matrix is either 1 or -1

(A is orthogonal if $AA^T = I$)

for example: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal

then eigenvalues are 1, -1

(7) let A be the square matrix of order n and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A then

(i) eigenvalues of KA are
$k\lambda_1, k\lambda_2, \dots, k\lambda_n$ (k is any scalar)

(ii) eigenvalues of A^m are
$\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ (m is positive integer)

(iii) eigenvalues of A^{-1} are
$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$

for example: let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then eigenvalues of A are $1, -1$

Now, $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $3, -3$

$A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $1^{10}, (-1)^{10}$ i.e. $1, 1$

and $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $\frac{1}{1}, \frac{1}{-1}$ i.e. $1, -1$

(8) let A be the square matrix of order n and $f(x)$ be an algebric polynomial in x then

i) if λ is eigenvalue of A then $f(\lambda)$ is an eigenvalue of $f(A)$

ii) If X is eigenvector corresponding to λ eigenvalue then X is also a eigenvector corresponding to $f(\lambda)$

(Algebric polynomial : $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$)

Ex (1) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $A^3 + 5A + 8I$

Solution:

Given: $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

First to find eigenvalues of A:

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 5 \\ 0 & 0 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)[(3-\lambda)(\lambda-2) - 0] - 2[0-0] + 3[0-0] = 0$$

$$\Rightarrow (-1-\lambda)(3-\lambda)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -1, 3, 2$$

$\therefore$ eigenvalues of A are $-1, 3, 2$

$\therefore$ eigenvalues of A^3 are $(-1)^3, (3)^3, (2)^3$

eigenvalues of $5A$ are $5(-1), 5(3), 5(2)$

eigenvalues of $8I$ are $8(1), 8(1), 8(1)$

Therefore, the eigenvalues of $A^3 + 5A + 8A$ are

$(-1)^3 + 5(-1) + 8(1) = 2$,

$(3)^3 + 5(3) + 8(1) = 50$,

$(2)^3 + 5(2) + 8(1) = -10$

Hence, eigenvalues of $A^3 + 5A + 8A$ are $2, 50, -10$

Ex ② Find the charecterstic root of $A^{30} - 9A^{28}$

where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Solution: first to find eigenvalues of A

$\therefore$ consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda = -1, 3$$

$\therefore$ eigenvalues of A are $-1, 3$

eigenvalues of A^{30} are $(-1)^{30}, (3)^{30}$

eigenvalues of $-9A^{28}$ are $-9(-1)^{28}, -9(3)^{28}$

$\therefore$ The eigenvalues of $A^{30} - 9A^{28}$ are

$(-1)^{30} - 9(-1)^{28} = (-1)^{28}[(-1)^2 - 9] = 1(-8) = -8$

and $(3)^{30} - 9(3)^{28} = 3^{28}(3^2 - 9) = 0$

Hence eigenvalues of $A^{30} - 9A^{28}$ are $\underline{0, -8}$

Homework:

① If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$, then find eigenvalues of $6A^{-1} + A^3 + 2I$

② If $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$, then find eigenvalues of $(A')^2 - 3A' + 4I$

③ find the eigenvalues of $A^3 - 3A^2 + A$

if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ (Ans: $-5, -1, 20$)