* **Properties of Z- transform :-**

① **Linearity :**

$$Z\{ a\, f(k) + b\, g(k)\} = a\, Z\{f(k)\} + b\, Z\{g(k)\}$$

② **change of scale :-**

If $\quad Z\{f(k)\} = F(z)$, $\quad R_1 < |z| < R_2$

then $\quad Z\{ a^k\, f(k)\} = F\left(\frac{z}{a}\right)$, $\quad |a|R_1 < |z| < |a|R_2$

③ **shifting property :**

If $\quad Z\{f(k)\} = F(z)$,

then $\quad Z\{f(k+n)\} = z^n\, F(z)$

and $\quad Z\{f(k-n)\} = z^{-n}\, F(z)$

④ **Multiplication by 'k'**

If $\quad Z\{f(k)\} = F(z)$

then $\quad Z\{k\, f(k)\} = -z\, \frac{d}{dz}\, F(z)$

⑤ **Division by 'k'**

If $\quad Z\{f(k)\} = F(k)$

then $\quad Z\left\{\frac{f(k)}{k}\right\} = -\int_{\infty}^{z} z^{-1}\, F(z)\, dz$

⑥ **Initial and Final value :**

If $\quad Z\{f(k)\} = F(z)$, $\quad k \geq 0 \quad$ then

$$f(0) = \lim_{z \to \infty} F(z) \qquad \text{and} \qquad \lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1)\, F(z)$$

* Examples based on properties:

Ex ① Find z - transform of $\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\}$, $k \geq 0$

Solution: Note that $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\therefore Z\{f(k)\} = Z\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\}$$

$$= Z\left\{\cos\frac{k\pi}{3} \cdot \cos\alpha - \sin\frac{k\pi}{3} \cdot \sin\alpha\right\}$$

$$= \cos\alpha \; Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\}$$

$$(\because \text{By linearity property})$$

$$= \cos\alpha\left[\frac{z^2 - z\cos\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\left(\frac{\pi}{3}\right) + 1}\right] - \sin\alpha\left[\frac{z\sin\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\left(\frac{\pi}{3}\right) + 1}\right]$$

$$\left(\because \cos ak = \frac{z^2 - z\cos a}{z^2 - 2z\cos a + 1} \; ; \; \sin ak = \frac{z\sin a}{z^2 - 2z\cos a + 1}\right)$$

$$= \frac{z^2\cos\alpha - z\cos\alpha\cos\frac{\pi}{3} - z\sin\alpha\sin\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\frac{\pi}{3} + 1}$$

$$= \frac{z\left\{z\cos\alpha - \left[\cos\left(\frac{\pi}{3}\right)\cdot\cos\alpha + \sin\left(\frac{\pi}{3}\right)\sin\alpha\right]\right\}}{z^2 - 2z\cos\left(\frac{\pi}{3}\right) + 1}$$

$$= \frac{z\left[z\cos\alpha - \cos\left(\frac{\pi}{3} - \alpha\right)\right]}{z^2 - z + 1}$$

$$\therefore Z\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\} = \frac{z\left[z\cos\alpha - \cos\left(\frac{\pi}{3} - \alpha\right)\right]}{z^2 - z + 1}$$

Ex① find: $\quad Z\{c^k \cos\alpha k\}$

Solution: Note that $Z\{\cos\alpha k\} = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$

∴ By change of scale property,

$$Z\{c^k \cos\alpha k\} = F\left(\frac{z}{c}\right)$$

$$= \frac{\frac{z}{c}\left(\frac{z}{c} - \cos\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1}$$

$$= \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$$

Ex③ find $Z\{k^2\}$

Solution: Note that $Z\{k\} = \dfrac{z}{(z-1)^2}$, $\quad |z| > 1$

∴ By property of 'multiplication by k'

$$Z\{k f(k)\} = -z\frac{d}{dz} F(z)$$

$$∴ \quad Z\{k^2\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

Ex. ④ find $Z\left\{\frac{1}{k+1}\right\}$, $k \geq 1$ and indicate the Radius of convergence.

Solution: Note that $Z\left\{\frac{1}{k}\right\} = -\log\left(1 - \frac{1}{z}\right)$, $|z| > 1$

∴ By shifting property,

if $Z\{f(k)\} = F(z)$ then $Z\{f(k+n)\} = z^n F(z)$

∴ $Z\left\{\frac{1}{k+1}\right\} = z^{(1)}\left[-\log\left(1 - \frac{1}{z}\right)\right]$ (here, $n = 1$)

$\qquad\qquad = -z\log\left(1 - \frac{1}{z}\right)$

and ROC ∶ $|z| > 1$