Ex. 3 Use Cayley - Hamilton theorem to find $2A^4 - 5A^3 - 7A + 6I$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

Solution: The charecterstic equation for A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow 2 - 3\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 2 = 0$$

Note that the Cayley-hamilton theorem states that A satisfy its charecterstic equation

$$\therefore A^2 - 3A - 2I = 0 \quad \text{—— (1)}$$

Now we divide $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$

$$\begin{array}{r|l}
 & 2\lambda^2 + \lambda + 7 \\
\lambda^2 - 3\lambda - 2 & 2\lambda^4 - 5\lambda^3 - 7\lambda + 6 \\
 & 2\lambda^4 - 6\lambda^3 - 4\lambda^2 \\
 & - \quad + \quad + \\
\hline
 & \lambda^3 + 4\lambda^2 - 7\lambda + 6 \\
 & \lambda^3 - 3\lambda^2 - 2\lambda \\
 & - \quad + \quad + \\
\hline
 & 7\lambda^2 - 5\lambda + 6 \\
 & 7\lambda^2 - 21\lambda - 14 \\
 & - \quad + \quad + \\
\hline
 & 16\lambda + 20
\end{array}$$

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therefore,

$$2\lambda^4 - 5\lambda^3 - 7\lambda + 6 = (\lambda^2 - 3\lambda - 2)(2\lambda^2 + \lambda + 7) + (16\lambda + 20)$$

$$\left(\because \ a \overline{) \ b \ } \ \ q, \ r \ \Rightarrow \ b = aq + r \right)$$

$$\therefore \ 2A^4 - 5A^3 - 7A + 6I = (A^2 - 3A - 2I)(2A^2 + A + 7I) + (16A + 20I)$$

$$\Rightarrow \ 2A^4 - 5A^3 - 7A + 6I = (0)(2A^2 + A + 7I) + (16A + 20I)$$

$$\Rightarrow \ 2A^4 - 5A^3 - 7A + 6I = 16A + 20I$$

$$= 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 32 \\ 32 & 32 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

$$\therefore \ 2A^4 - 5A^3 - 7A + 6I = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

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* simillarity of Matrices :-

let A and B be two square matrices of order n then we say B is simillar to A if there exist a non singular matrix p such that $B = p^{-1} A p$

* propeorties of simillary matrices :-

① If A and B are simillar matrices then $|A| = |B|$

② If A and B are simillar matrices then $tr(A) = tr(B)$

③ If A and B are simillar matrices then $rank(A) = rank(B)$

④ If A and B are simillar matrices then Both A and B have same charecterstic polynomial

⑤ If A and B are simillar matrices then Both A and B have same eigenvalues

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Example:

Determine whether the following matrices are simillar or not.

(1) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -8 \\ 1 & 5 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 0 & -7 \\ 5 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 15 & -18 & -2 \\ 17 & -17 & -4 \\ 7 & -22 & 4 \end{bmatrix}$

solution:

(1) To find the eigenvalues of matrix A and B first we consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-\lambda) - 1 = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda = 1, 3$$ which are eigenvalues of A

Now we consider $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & -8 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(5-\lambda) + 8 = 0$$

$$\Rightarrow -5 + \lambda - 5\lambda + \lambda^2 + 8 = 0$$

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$$\Rightarrow \quad \lambda^2 - 4\lambda + 3 = 0$$

$\Rightarrow \quad \lambda = 1, 3$ which are eigenvalues of B

that is Both the matrices A and B have same eigenvalues

Hence, A and B are simillar

② here,

$$A = \begin{bmatrix} 1 & 0 & -7 \\ 5 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & -18 & -2 \\ 17 & -17 & -4 \\ 7 & -22 & 4 \end{bmatrix}$$

Note that $tr(A) = 1+1+0 = 2$

$tr(B) = 15 - 17 + 4 = 2$

i.e. Both A and B have same Trace

Now, $|A| = 1(0-4) - 0(0+8) - 7(10+4)$

$= -4 + 0 - 98$

$= -102$

and $|B| = 15[-68 - 88] + 18[68 + 28] - 2[-374 + 119]$

$= -2340 + 1728 + 510$

$= -2340 + 2238$

$= -102$

ie Both A and B have same Determinant

$\therefore$ A and B are simillar.

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* Diagonalizable and non-diagonalizable matrices :-

let A be the square matrix of order n then A is digonalizable if there exist a non-singular matrix P such that the matrix $P^{-1}AP$ is diagonal matrix

Note that if A is not diagonalizable then A is non-diagonalizable

Example ①

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