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Integration of Complex Variable Functions

☆ Line (path or Contour) Integral of complex variable functions.

Let $C: z(t) = x(t) + iy(t)$, $a \leq t \leq b$ be a contour extending from $z_1 = z(a)$ to $z_2 = z(b)$ and $f(z) = u(x) + iV(x,y)$ be a complex function then,

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

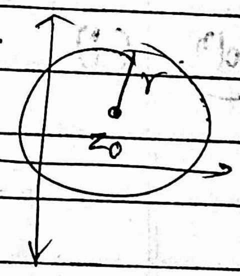
NOTE :

1) Parametric form of straight line joining 'z_1' and 'z_2' in positive sense is $z = (1-t)z_1 + tz_2$, $0 \leq t \leq 1$.

2) Parametric form of a circle with the centre 'z_0' and radius 'r' in anticlockwise direction (contour clockwise)

$$z(t) = z_0 + re^{it}, \quad 0 \leq t \leq 2\pi$$

$$z(\theta) = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$



3) $\bar{z}, |z|$ ~~they~~ they are always non-analytic at all points on complex plane.

4) Elementary Analytic Function → constant, z, polynomial in z, e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are analytic on all points.

5) If $f(z)$ and $g(z)$ are two analytic functions then -

(i) $f(z)+g(z) \to$ analytic (ii) $f(z)\cdot g(z) \to$ analytic

(iii) $f(g(z)) \to$ analytic (iv) $\frac{f(z)}{g(z)} \to$ analytic at all points except for $g(z)=0$

i.e $\frac{f(z)}{g(z)}$ is not analytic only at $g(z)=0$.

* Result / Theorem $\to$ If $f(z)$ is analytic on domain 'D' then for any curve extending from z_1 to z_2 and lying inside 'D' integral of $f(z)$ along the curve 'C' is independent of the path and depends only on initial & final point z_1 & z_2 i.e.

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = \left[\int f(z)\,dz\right]_{z=z_1}^{z=z_2}$$

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Evaluate the following -

① $\int_C \bar{z}\,dz$ where C : (i) curve $z=t^2+it$ from 0 to $4+2i$

(ii) straight line from 0 to $4+2i$

(iii) upper half of the circle $|z|$ starting from $z=1$.

Solⁿ. (i) $z = t^2+it$

$\bar{z} = t^2 - it$

$d\bar{z} = (2t+i)\,dt$

from 0 to $4+2i$ in terms of ~~[illegible]~~ t.

$z=0,\ t \Rightarrow 0$

$z = 4+2i,\ t \Rightarrow 2$

$$\therefore \int_0^2 (t^2-it)(2t+i)\,dt = \int_0^2 2t^3+t^2i-2t^2i+t\,dt$$

$$= \int_0^2 (2t^3 - t^2 i + t)\,dt$$

$$= \left(\frac{t^4}{2} - \frac{it^3}{3} + \frac{t^2}{2}\right)_0^2$$

$$= 8 - \frac{8}{3}i + 2 = 10 - \frac{8i}{3}$$

OR

$I = \int_C \bar{z}\,dz$

$z = x+iy$

$\bar{z} = x-iy$

$$= \int_C (x-iy)(dx+i\,dy)$$

$C: z = t^2+it$ from $z=0$ to $z=4+2i$

On comparing, $x=t^2,\ y=t$

$z=0,\ t\Rightarrow 0,\quad z=4+2i,\ t\Rightarrow 2$

$dx = 2t,\ dy = 1$

$$\therefore I = \int_0^2 (t^2-it)(2t+i)\,dt$$

OR

$$I = \int_C \bar{z}\,dz = \int (x-iy)(dx+i\,dy)$$

$C: z = t^2+it$ from 0 to $4+2i$

$x = t^2,\ y=t$

$x = y^2$

$dx = 2y\,dy$

$z=0,\ y\Rightarrow 0,\quad z=4+2i,\ y=2$

$$\int_0^2 (t^2-t)(2y+i)\,dy$$

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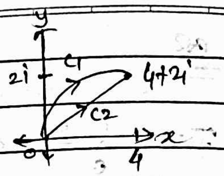
(ii) C_2: stgt. line from 0 to $4+2i$



~~$z = t^2 + it$~~

Parametric form

$z = (1-t)z_1 + tz_2$

$= (1-t)\times 0 + t\times(4+2i)$

$z = (4+2i)t \qquad 0\le t\le 1$

$\bar{z} = (4-2i)t$

$dz = (4+2i)dt \qquad I = \int \bar{z}\,dz$

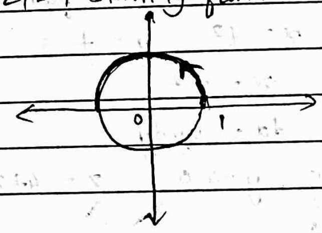
$I = \int_0^1 (4-2i)t\,(4+2i)\,dt$

$= \int_0^1 (16+4)t\,dt$

$= \underline{10}$

(iii) C : upper half of the circle $|z|=1$ starting from $z=1$.

$\boxed{|z-z_0| = r}$ — centre, radius



Centre $\equiv (0,0)$, $r=1$.

Parametric form of semi-circle

$z = 0 + 1e^{i\theta}$

$z = e^{i\theta} \qquad 0\le\theta\le\pi$

$\bar{z} = e^{-i\theta}$

$dz = ie^{i\theta}d\theta$

$\int_0^\pi e^{-i\theta}\, ie^{+i\theta}d\theta = \int_0^\pi i\,e^{\cancel{2i\theta}}\,d\theta = \left(\cancel{\frac{ie^{2i\theta}}{2i}}\right)_0^\pi = \underline{\pi i}$

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Q6

(2) $\int_0^{2+i} z^2 dz$ (i) along the line $x = 2y$ (ii) along the parabola $2y^2 = x$

Soln. $I = \int_C z^2 dz = \int (x+iy)^2 (dx+idy)$

$x = 2y$

(i) $\int (2y+iy)^2 (dx+idy)$

$dx = 2dy$.

$= \int (2y+iy)^2 (2+i)\,dy$.

$z=0,\ y=0$

$z=2+i,\ y=1$

$= \int y^2 (2+i)^3 dy$.

$= \int_0^1 y^2 (2+i)^3 dy$

$= \frac{(2+i)^3}{3} = \frac{2+11i}{3}$

(ii) $2y^2 = x$

$\int_C z^2 dz = \int (2y^2+iy)^2 (dx+idy)$

$z=0,\ y=0$

$z=2+i,\ y=1$

$dx = 4y\,dy$

$= \int_0^1 (2y^2+iy)^2 (4y+i)\,dy$

$= \int_0^1 y^2 (2y+i)^2 (4y+i)\,dy$

$= \int_0 y^2 (4y^2-1+4yi)(4y+i)\,dy$.

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$$= \int_0^1 y^2 (16y^3 + 4y^2 i - 4y - i + 16y^2 i - 4y)\, dy$$

$$= \int_0^1 (16y^5 + 4y^4 i - 4y^3 - y^2 i + 16y^4 i - 4y^3)\, dy$$

$$= \left(\frac{8}{3}y^6 + \frac{4}{5}y^5 i - y^4 - \frac{y^3 i}{3} + \frac{16}{5}y^5 i - \frac{4}{3}y^4 \right)_0^1$$

$$= \frac{8}{3} + \frac{4i}{5} - 1 - \frac{i}{3} + \frac{16i}{5} - \frac{4}{3} - 1$$

$$= \frac{2}{3} + \frac{11i}{3} = \frac{2+11i}{3}$$

OR

$4i - \frac{i}{3}$

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$f(z) = z^2$ is Analytic at all points on Complex plane

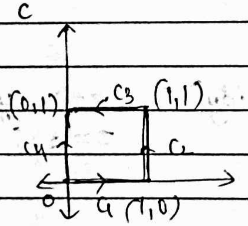
$\Rightarrow$ Integral is independent of the curve.

For both curves,

$$I = \int_0^{2+i} z^2 dz = \left[\frac{z^3}{3} \right]_{z=0}^{2+i} = \frac{(2+i)^3}{3}$$

Q3) $\int_C |z|^2 dz$ along the square with vertices $(0,0), (1,0), (1,1), (0,1)$

For a curve made of many curves evaluate separately.



$C = C_1 + C_2 + C_3 + C_4$

$C_1 \Rightarrow y = 0$ from $(0,0)$ to $(1,0)$

$dy = 0\, dx$

$\cancel{\int_{C_1} dz = \int_0^1 (iy)(dx)}$

$$I_1 = \int_{C_1} |z|^2 dz = \int_{C_1} (x^2+y^2)(dx)$$

$$= \int_0^1 (x^2+y^2)\, dx = \left(\frac{x^3}{3} \right)_0^1 = \frac{1}{3}$$

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$C_2 \Rightarrow x = 1 \quad 0 \le y \le 1$

~~$dx = 0dy$~~

$$I_2 = \int_{C_2} |z^2| dz = \int_0^1 (x^2+y^2)(i\,dy)$$

$$= \int_0^1 i(1+y^2)\, dy$$

$$I_2 = i\left(y + \frac{y^3}{3}\right)_0^1 = \frac{4i}{3}$$

$C_3 \Rightarrow y = 1 \quad 1 \le x \le 0$

$dy = 0dx$

$$I_3 = \int_{C_3} |z|^2 dz = \int_1^0 (x^2+y^2)(dx) = \left(\frac{x^3}{3} + x \right)_1^0$$

$$= -\frac{1}{3} - 1 = -\frac{4}{3}$$

$C_4 \Rightarrow x = 0 \quad 1 \le y \le 0$

$dx = 0dy$

$$I_4 = \int_{C_4} |z|^2 dz = \int_1^0 (x^2+y^2)(dy\, i)$$

$$= \left(\frac{iy^3}{3} \right)_1^0 = \frac{-i}{3}$$

$$C = \frac{1}{3} + \frac{4}{3}i - \frac{4}{3} - \frac{i}{3}$$

$$\boxed{C = -1 + i}$$

$-12-\frac{8}{3}$

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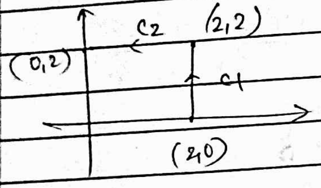
Q4) Evaluate $\int_C (z^2+3z)\,dz$ along the straight line from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$



$C = C_1 + C_2$.

$\cancel{\int_{2+0i}^{2+2i} (z^2+3z)\,dz = \left(\frac{z^3}{3}+\frac{3z^2}{2}\right)_{2+0i}^{2+2i}}$

$\cancel{= 8+6 = 14}$

$=$

$\cancel{\int_0^2 \left((x+iy)^2 + 3(x+iy)\right)(dx+idy)} \; \cancel{\int_0^2 (x^2-y^2+2xyi+3x+3iy)(dx+idy)}$

$\cancel{= \int_0^2 (x^2-y^2+3x+2xyi+3iy)\,dx + i\int_0^2 (x^2-y^2+3x+2xyi+3iy)}$

$\cancel{C_1:\ x=2,\ 0\le y\le 2,\ dx=0\,dy}$

$f(z) = z^2+3z$ is analytic.

$I = \int_C z^2+3z\,dz$

$= \int_2^{2i} z^2+3z\,dz = \left(\frac{z^3}{3}+\frac{3z^2}{2}\right)_2^{2i}$

$= \cancel{\left(\frac{-8i}{3} - 6i\right) - \left(\frac{8}{3}+6\right)}$

$= \left(\frac{-8i}{3} - 6\right) - \left(\frac{8}{3}+6\right) \cancel{= \frac{-14i}{3} - 14}$

$= -\frac{8i}{3} - \frac{44}{3}$

(zero) and −ve real axis → not analytic

3 possibilities for closed curves → Cannot → still I exists $=0$. analytic → on curve → I depends on curve. PT. → ✓ Exists not possible

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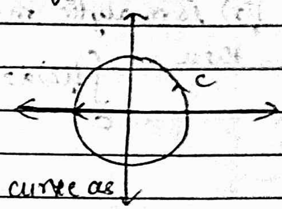
Q5) $\int_C \log z\,dz$ where C is the unit circle in the z-plane with centre as origin. ~~$f(z) = \log$~~

→ $f(z) = \log(z)$ is not analytic at 0 and −ve real axis.



∴ Integral depends on the given curve as $z=-1$ lies on the circle.

$C: z = 0 + 1e^{i\theta},\ 0 \le \theta \le 2\pi$

$\cancel{I = \int_0^{2\pi} e^{i\theta}\,d\theta = \left(\frac{e^{i\theta}}{i}\right)_0^{2\pi} = \frac{e^{2\pi i}}{i} - \frac{1}{i}}$

$= \frac{e^{2\pi i}-1}{i}$

$dz = ie^{i\theta}\,d\theta$

$\int_C \log_e(e^{i\theta})\,dz$

$= \int_C i\theta \times ie^{i\theta}\,d\theta$

$= -\int_0^{2\pi} \theta e^{i\theta}\,d\theta$

$\cancel{= -\theta \int_0^{2\pi} e^{i\theta}\,d\theta - \int_0}$

$= -\left[\theta\int e^{i\theta}d\theta - \int\left(\int e^{i\theta}\,d\theta\right) d\theta\right]_0^{2\pi}$

$= -\left[\frac{\theta e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$

$= -\left(\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 1\right)$

$= 1 - \frac{2\pi e^{2\pi i}}{i} - e^{2\pi i}$

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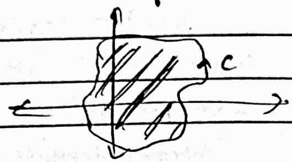
★ Cauchy Integral theorem

★ If $f(z)$ is analytic on and in the interior of a closed contour C, then $\int_C f(z)\,dz = 0$.



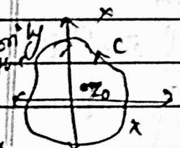
★ Cauchy Integral formula

★ If $f(z)$ is analytic on and in the interior of a closed contour 'c' except at z_0 lying in the interior of 'c' then $f(z) = \frac{g(z)}{(z-z_0)^n}$

where $g(z)$ is non-zero and analytic at $z = z_0$, then

$$\int_C f(z)\,dz = \int_C \frac{g(z)\,dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[g(z)]\Big|_{z=z_0}$$

compulsorily inside the curve



This formula is only for one not analytic pt lying inside.

$g(z) \to$ non-zero & not analytic at $z = z_0$.

(Q) Evaluate

$$\int_C \frac{2z-1}{z^4-2z^3-3z^2}\,dz \quad \text{where } c: (i)\ |z-1/2| = 1,\ (ii)\ |z+1| = 2$$

$(iii)\ |z-1| = 3,\ (iv)\ |z-2i| = 1$

circle.

1) Know whether fun^n is A or NA
2) then find where the pts are lying

Circle with centre z_0 and radius 'a'

$f(z) = \frac{2z-1}{z^4-2z^3-3z^2}$ $\therefore f(z)$ is not analytic at $z^4-2z^3-3z^2=0$

$\therefore z^2(z^2-2z-3) = 0$

$(z^2)(z+1)(z-3) = 0$

$z^2 = 0,\ z = -1,\ z = +3$

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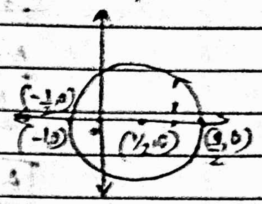
(i) $c: |z-1/2| = 1$

• Draw rough dg. & locate pts.



'0' non-analytic pt. lies inside the curve

$\therefore z=0$ lies inside;

$$f(z) = \frac{2z-1}{z^2(z^2-2z-3)}$$

From Cauchy Integral formula,

$$\int_C f(z)\,dz = \frac{2\pi i}{1}\frac{d}{dz}\left(\frac{2z-1}{z^2-2z-3}\right)\Big|_{z=0}$$

$$= 2\pi i\left(\frac{(2)(z^2-2z-3) - (2z-2)(2z-1)}{(z^2-2z-3)^2}\right)\Big|_{z=0}$$

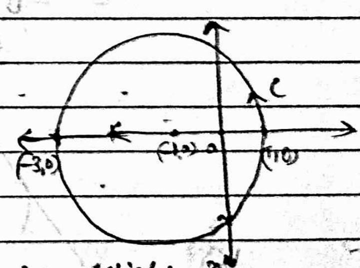
$$= 2\pi i\left(\frac{2z^2-4z-6-4z^2+2z+4z-2}{(z^2-2z-3)^2}\right)\Big|_{z=0}$$

$$= 2\pi i\left(\frac{-8}{9}\right)$$

$$= \frac{-16\pi i}{9}$$

(ii) $c: |z+1| = 2$

$\therefore z = 0, -1$ lies inside 'c'.



$$f(z) = \frac{2z-1}{z^2(z-3)(z+1)}$$

Taking only factor causing problm & using partial fract^n,

$$\frac{1}{z^2(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1}$$

$$\frac{1}{z^2(z+1)} = \frac{A(z)(z+1) + B(z+1) + C(z^2)}{z^2(z+1)}$$

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$1 = Az^2 + Az + Bz + B + Cz^2$

$z^2(A+C) = 0$

$A+B = 0$

$\boxed{B=1}$ $\boxed{C=1}$

$\boxed{A=-1}$

$$I = \int_c f(z)\,dz = \int_c \frac{-(2z-1)}{(z-3)z}\,dz + \int_c \frac{(2z-1)}{(z-3)z^2}\,dz + \int_c \frac{(2z-1)}{(z-3)(z+1)}\,dz$$

$$= 2\pi i\left[\frac{-(2z-1)}{(z-3)}\right]_{z=0} + \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{2z-1}{z-3}\right)\Bigg|_{z=0} + 2\pi i\left(\frac{2z-1}{z-3}\right)_{z=-1}$$

$$= \frac{2\pi i}{3} + 2\pi i\left(\frac{2z-6-2z+1}{(z-3)^2}\right)_{z=0} + 2\pi i\left(\frac{2z-1}{(z-3)}\right)_{z=-1}$$

$$= -\frac{6\pi i}{9} + 2\pi i\left(\frac{-5}{9}\right) + 2\pi i\left(\frac{+3}{4}\right)$$

$$= -\frac{16}{9}\pi i - \frac{10\pi i}{16} \quad -\frac{16\pi i}{9} + \frac{6\pi i}{4}$$

$$= \cancel{\left(\frac{256-90}{144}\right)i\pi} \quad \left(\frac{-64-54}{36}\right)\pi i$$

$$= \cancel{\frac{346}{144}} \quad = -\frac{5\pi i}{18}$$

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OR By complete partial,

$$f(z) = \frac{2z-1}{z^2(z-3)(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-3} + \frac{D}{z+1}$$

$A = -\frac{8}{9}$, $B = \frac{1}{3}$, $C = \frac{5}{36}$, $D = \frac{3}{4}$

$$I = -\frac{8}{9}\int_c \frac{1}{z}\,dz + \frac{1}{3}\int_c \frac{1}{z^2}\,dz + \frac{5}{36}\int_c \frac{1}{z-3}\,dz + \frac{3}{4}\int_c \frac{1}{z+1}$$

derivative = 0

$$= -\frac{16\pi i}{9} + \frac{3\pi i}{2}$$

$$= \frac{-32+27\pi i}{18} = -\frac{5\pi i}{18}$$

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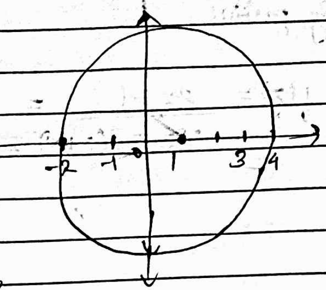
(iii) c: $|z-1| = 3$



$z = 0, -1, 3$ lies inside 'c'

$$\cancel{I = -\frac{8}{9}\cdot 2\pi i + \frac{5}{38}\cdot 2\pi i + \frac{3}{4}\cdot 2\pi i}$$

$$\cancel{= 2\pi i\left(\frac{-32+10+27}{78}\right)}$$

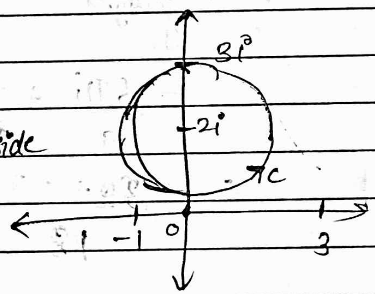
$$= 2\pi i\left(\frac{5}{18}\right) = \frac{5\pi i}{9}$$

$$I = -\frac{8}{9}\cdot 2\pi i + \frac{5}{36}\cdot 2\pi i + \frac{3}{4}\cdot 2\pi i$$

$$= \left[\frac{-16}{9} + \frac{5}{18} + \frac{3}{2}\right]\pi i$$

$$= 0$$

(iv) c: $|z-2i| = 1$



All $z = 0, -1, 3$ lies outside

$\Rightarrow f(z)$ is Analytic on and inside 'c'

$\therefore$ by Cauchy theorem

$$\int_c f(z)\,dz = 0$$

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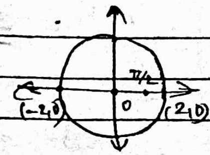
(Q) Evaluate

$$\int_c \frac{\sin z}{(z-\pi/2)^5}\,dz, \quad c: |z| = 2$$



~~For Analytic & not~~ $= 0$

$f(z) = \dfrac{\sin z}{(z-\pi/2)^5}$ is not analytic at $z = \pi/2$

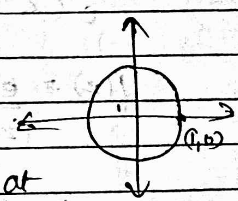
$z = \dfrac{\pi}{2}$ lies inside 'c'

$$\int_c f(z)\,dz = \frac{2\pi i}{4!}\frac{d^4(\sin z)}{dz^4}\bigg|_{z=\pi/2}$$

$$= \frac{2\pi i}{24}(\sin z)_{z=\pi/2}$$

$$= \frac{\pi i}{12}$$

(Q) $$\int_c \frac{\sin^6 z}{(z-\pi/6)^3}\,dz, \quad c: |z| = 1$$



$f(z) = \dfrac{\sin^6 z}{(z-\pi/6)^3}$ is not analytic at $z = \pi/6$

$$\cancel{\int_c f(z)\,dz = \frac{2\pi i}{5!}\frac{d^5}{dz^5}(720\cos z)\bigg|_{z=\pi/6}}$$

$$\cancel{= \frac{2\pi i}{120}\times 720\cos\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \cancel{\sqrt{3}\,6\pi i} \quad 6\sqrt{3}\pi i$$

$6\cos z \sin^5 z$ $\quad 6\cos z \sin \ldots$

$-30\sin^4 z \quad -6\sin^6 z + 30\cos^2 z \ldots$

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$\frac{d^2}{dz^2} = 30\sin^4 z - 36\sin^6 z$

$= \frac{2\pi i}{2}\left(30\sin^4 z - 36\sin^6 z\right)\Big|_{z=\pi/6}$

$= \pi i\left(30\times\frac{1}{16} - \frac{36\times 1}{64}\right)$

$= \pi i\left(\frac{120-36}{64}\right)$

$= \frac{84\pi i}{64}$

$= \frac{21\pi i}{16}$ //

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(Q) Q. $\int_C \frac{e^{3z}}{(z-i)^2}\,dz \qquad C: |z-2| + |z+2| = 6.$

$f(z) = \frac{e^{3z}}{(z-i)^2}$ is not analytic at $z=i$.

$|z-2| + |z+2| = |i-2| + |i+2|$

$= \sqrt{1+4} + \sqrt{1+4}$

$= 2\sqrt{5} < 6$

$\Rightarrow$ $z=i$ lies inside 'c'

$I = \int_C \frac{e^{3z}}{(z-i)^2}\,dz = \frac{2\pi i}{1!}\cdot\frac{d}{dz}\left(e^{3z}\right)\Big|_{z=i}$

$= 6\pi i e^{3i}$ //

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(Q) $\int_{C:|z|=1} \frac{Re(z)}{(z-a)}\,dz \;;\; 0<|a|<1$

$C: |z|=1 \Leftrightarrow |z|^2 = 1$

$z\cdot\bar{z} = 1$

$\bar{z} = \frac{1}{z}$

$z + \bar{z} = 2x$

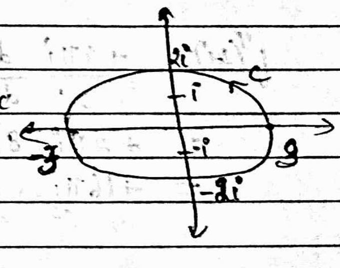
$x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

$\therefore I = \int_C \frac{z^2+1}{(z-a)2z}\,dz = \frac{2\pi i}{1}\left(\frac{z^2+1}{2z}\right)\Big|_{z=a} = 2\pi i\left(\frac{a^2+1}{2a}\right)$

$= \pi a i + \frac{\pi i}{a}$ //

(Q) If $f(a) = \int_C \frac{4z^2+z+5}{(z-a)}\,dz \quad C: 4x^2+9y^2=36$; find the values of $f(1)$, $f(-i)$, $f'(-3)$, $-2i)$, $f'(i)$; $f''(2+i)$

C: $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$



let $g(z) = \frac{4z^2+z+5}{(z-a)}$ is not analytic at $z=a$.

for $a=1$

$f(1) = 2\pi i(4z^2+z+5)\big|_{z=1}$

$= 20\pi i$

$f(-i) = 2\pi i(4z^2+z+5)\big|_{z=-i}$

$= 2\pi i(1-i)$

$= 2\pi i + 2\pi$

$f(-3,-2i) = 0$

$f'(a) = \frac{1}{da} f(a)$

$= \frac{1}{da} \int_c \frac{4z^2+z+5}{(z-a)} dz$

$\frac{1}{z-a} \xrightarrow{\text{Diff.}} \frac{1}{(z-a)^2}$

$f'(a) = \int_c \frac{4z^2+z+5}{(z-a)^2} dz$

$f'(i) = \cancel{\oint_c \frac{4\ 1+i}{(i-a)^2} dz} = \cancel{\int_c \frac{1+i}{a-1-2ia} dz}$

$= \frac{2\pi i}{1!} (8z+1)\Big|_{z=i}$

$= -16\pi + 2i\pi$

$\frac{1}{(z-a)^2}$

$f''(a) = \int_c \frac{+2(4z^2+z+5)}{(z-a)^3} dz \qquad \frac{-2\times(-1)}{(z-a)^3} = \frac{2}{}$

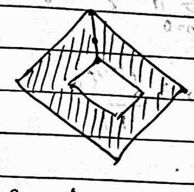
$f''(2+i) = \frac{+4\pi i}{2!} \frac{d^2}{dz^2}(4z^2+z+5)\Big|_{z=2+i}$

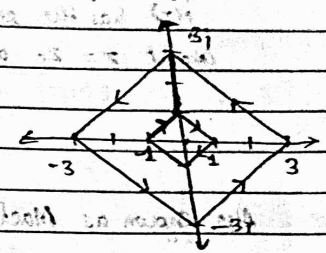
$= +2\pi i (8)$

$= +16\pi i$

(Q) $\int \frac{\sin z}{4z^2-8iz} dz$ where 'c' consists of boundaries of the squares with vertices $\{\pm 3, \pm 3i\}$ (anticlockwise)

Line segment from $3i$ to i and then square formed with vertices $\pm 1, \pm i$ (clockwise).





$4z^2 - 8iz = 0$

$4z(z-2i) = 0$

$z = 0, \quad z = 2i$

Not analytic at $z = 2i$

$\int \frac{\sin z}{4z(z-2i)} dz = \frac{2\pi i}{1} \left(\frac{\sin z}{4z}\right)\Big|_{z=2i}$

$= \frac{2\pi i (\sin 2i)}{8i}$

$= \frac{\pi \sin(2i)}{4}$

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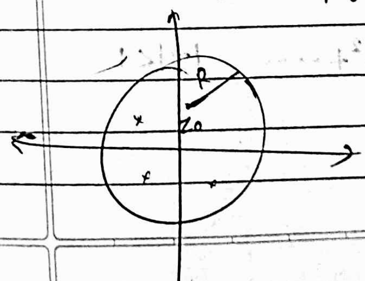
Taylor Series

If $f(z)$ is Analytic throughout an open disc $|z-z_0| < R$, then $f(z)$ has the series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \ \forall \ |z-z_0| < R \quad \text{where}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$



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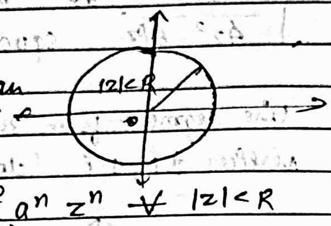
Note

1) $z_0 = 0$.

If $f(z)$ is analytic throughout an open disc $|z| < R$ then $f(z)$ has the series expansion about $z = z_0$ as $f(z) = \sum_{n=0}^{\infty} a^n z^n \quad \forall \ |z| < R$



where $a_n = \dfrac{f^{(n)}(a)}{n!}$

Also known as Maclaurin's series.

Laurent Series

If $f(z)$ is not analytic at $z = z_0$, but analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ then $f(z)$ has the series expansion about $z = z_0$ as,

with a hole

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \forall \ R_1 < |z - z_0| < R_2.$$

Basic Maclaurin series and their region (circle) of convergence.

① $\dfrac{1}{(1-z)} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad$ for $|z| < 1$

② $\dfrac{1}{(1+z)} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 + \dots \quad$ for $|z| < 1$.

③ $\dfrac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n = 1 + 2z + 3z^2 + \dots \quad |z| < 1$

④ $\dfrac{1}{(1+z)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n = 1 - 2z + 3z^2 - 4z^3 + \dots \quad$ for $|z| < 1$

⑤ $\dfrac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \dfrac{(n+1)(n+2)}{2!} z^n, \quad |z| < 1$

$\dfrac{1}{(1+z)^3} = \sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)}{2!} z^n, \quad |z| < 1$

⑥ $e^z = \sum_{n=0}^{\infty} \dfrac{1}{n!} z^n = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots \quad$ for $|z| < \infty$

$e^{-z} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!} z^n = 1 - z + \dfrac{z^2}{2!} - \dfrac{z^3}{3!} + \dots \quad$ for $|z| < \infty$

⑦ $\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots \quad$ for $|z| < \infty$

⑧ $\cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n!} z^{2n} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dots \quad$ for $|z| < \infty$

⑨ $\sinh z = \sum_{n=0}^{\infty} \dfrac{1}{(2n+1)!} z^{2n+1} = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dots \quad$ for $|z| < \infty$

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~~Basic Maclaurin's series and their region of convergence.~~

(10) $\cosh z = \sum_{n=0}^{\infty} \dfrac{1}{2n!} z^{2n} = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dots \quad$ for $|z| < \infty$

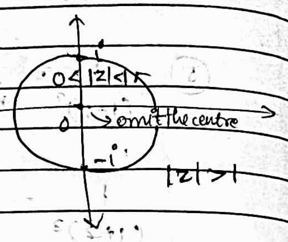
Q) Find all possible laurents series expansion and specify the region of convergence.

(1) $f(z) = \dfrac{1+2z}{z^3+z^5}$

→ $f(z)$ is not analytic at $z^3+z^5 = 0$

$z^3(z^2+1) = 0$

$z = 0, \quad z = \pm i$



$$f(z) = (1+2z)\frac{1}{z^3(z^2+1)} = \left(\frac{1}{z^3}+\frac{2}{z^2}\right)\frac{1}{1+z^2} \quad \text{— (1)}$$

$$= \left(\frac{1}{z^3}+\frac{2}{z^2}\right)\left(\frac{1}{1-z^2+z^4-z^6+\ldots}\right)$$

kaise aaya

ROC for $\frac{1}{1+z^2}$

$$\frac{1}{1+z^2} = 1-(z)^2+(z^2)^2-(z^2)^3+\ldots$$
$$= 1-z^2+z^4-z^6+\ldots \quad \text{for } |z^2|<1$$
$$\Rightarrow |z|^2<1$$
$$\Rightarrow |z|<1$$

$\Rightarrow \frac{1}{z^3} + \frac{2}{z^2}$ OR expansion

$$\frac{1}{z^2\left(1+\frac{1}{z^2}\right)} = \frac{1}{z^2}\left(1-\frac{1}{(z)^2}+\frac{1}{(z^2)^2}-\frac{1}{(z^2)^3}+\ldots\right)$$

$$= \frac{1}{z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}+\ldots\right)$$

$$= \frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}-\ldots$$

for $\left|\frac{1}{z^2}\right|<1 \Rightarrow \left|\frac{1}{z}\right|^2<1$

$\Rightarrow \left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$

(i) 1st domain $0<|z|<1$

$$f(z) = \left(\frac{1}{z^3}+\frac{2}{z^2}\right)\left(1-z^2+z^4-z^6+\ldots\right)$$

$$= \frac{1}{z^3}-\frac{1}{z}+z-z^3+\ldots+\frac{2}{z^2}-2+2z^2-2z^3+\ldots$$

(ii) for $|z|>1$

$$f(z) = \left(\frac{1}{z^3}+\frac{2}{z^2}\right)\left(\frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}+\ldots\right)$$

$$= \frac{1}{z^5}-\frac{1}{z^7}+\frac{1}{z^9}-\frac{2}{z^4}-\frac{2}{z^6}+\frac{2}{z^8}+\ldots$$

$$= \frac{2}{z^4}+\frac{1}{z^5}-\frac{2}{z^6}-\frac{1}{z^7}+\frac{2}{z^8}+\frac{1}{z^9}+\ldots$$

We have excluded the centre i.e laurent series is there and not purely Taylor. (Annular domain),

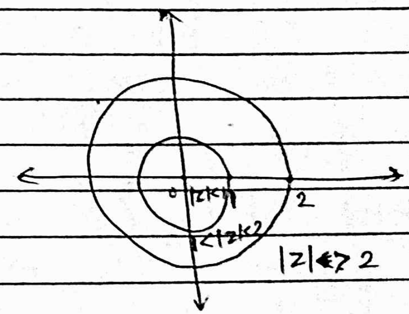
(2) $f(z) = \dfrac{z}{(z-1)(z-2)}$

$f(z)$ is not analytic at

$(z-1)(z-2) = 0$

$z=1, z=2$

Take partial fraction.



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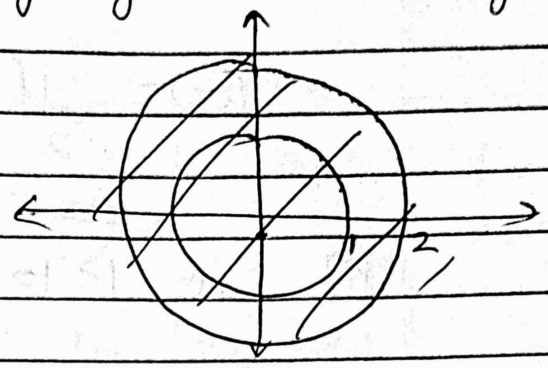
(2) $f(z) = \dfrac{z-1}{z^2-2z-3}$. Nothing is given so take abt origin



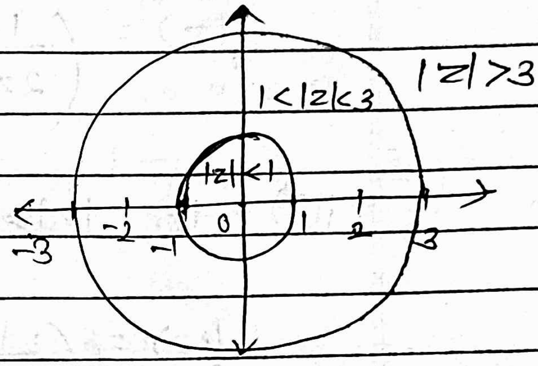
$f(z)$ is not analytic at

$z^2-2z-3=0$

$(z+1)(z-3)=0$

$z=-1, 3$

$f(z) = \dfrac{z-1}{(z+1)(z-3)}$



$\dfrac{z-1}{(z+1)(z-3)} = \dfrac{A}{z+1} + \dfrac{B}{z-3}$

$z-1 = z(\cancel{-3}A+B)z + (-3A+B)$

$A+B=1$

$-3A+B=-1$

$4A=2$

$A=\frac{1}{2}$ $\quad B=\frac{1}{2}$

$f(z) = \dfrac{1}{2(z+1)} + \dfrac{1}{2(z-3)}$ —— (1)

$\dfrac{1}{2z(1+\frac{1}{z})} = \dfrac{1}{2z}\left(1-\dfrac{1}{z}+\dfrac{1}{z^2}-\cdots\right)$ for $\left|\frac{1}{z}\right|<1$ $\quad |z|>1$

OR $\dfrac{1}{2(z+1)}$ $\quad \dfrac{1}{2}\left(1-z+z^2\cdots\right)$ for $|z|<1$

2 ways
-3 common
z common

$\dfrac{1}{2(z-3)} = \dfrac{1}{2z(1-3/z)} = \dfrac{1}{2z}\left(1+\dfrac{3}{z}+\dfrac{9}{z^2}+\cdots\right)$

for $\left|\dfrac{3}{z}\right|<1$

or

$|z|>3$

$\dfrac{1}{2(z-3)} = \dfrac{1}{-6(1-z/3)} = -\dfrac{1}{6}\left(1+\dfrac{z}{3}+\dfrac{z^2}{9}+\cdots\right)$

for $\left|\dfrac{z}{3}\right|<1$

$|z|<3$

(i) for $|z|<1$

$$f(z) = \frac{1}{2}\left(z - z^2 + z^3 \cdots\right) - \frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right)$$

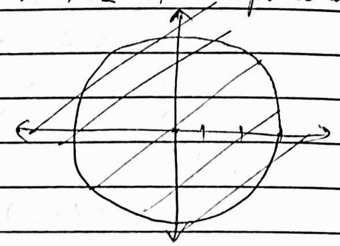
(ii) For $1<|z|<3$

$$f(z) = \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \cdots\right) - \frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right)$$

iii) For $|z|>3$

$$f(z) = \left(\frac{1}{2z} + \frac{3}{2z^2} + \frac{9}{2z^3} + \cdots\right) + \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \cdots\right)$$

(II) $f(z) = \dfrac{2z-3}{z^2-4z+3}$ about $z=4$. $\rightarrow$ $z-4$ me express kama.



$$f(z) = \frac{2z-3}{(z-1)(z-3)}$$

$f(z)$ is not analytic at 1, 3

$z^2 - z - 3z + 3$

$(z-1) - 3(z-1)$

$$\frac{2z-3}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

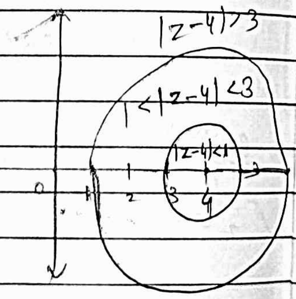
$2z-3 = (A+B)z + (-3A-B)$

$A+B = 2$

$-3A - B = -3$

$-2A = -1$

$A = 1/2$ $\quad$ $B = 3/2$



$$f(z) = \frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

(z-4) me karna hai.

$$\frac{1}{2(z-1)} = \frac{1}{2(\underbrace{z-4+3}_{z})} = \frac{1}{6\left(1+\frac{z-4}{3}\right)} = \frac{1}{6}\left(1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 - \cdots\right)$$

For $\left|\frac{z-4}{3}\right|<1$

$|z-4|<3$

$$\frac{1}{2(z-4+3)} = \frac{1}{2(z-4)\left(1+\frac{3}{z-4}\right)} = \frac{1}{2(z-4)}\left(1 - \left(\frac{3}{z-4}\right) + \left(\frac{3}{z-4}\right)^2 - \cdots\right)$$

For $\left|\frac{3}{z-4}\right|<1$

$|z-4|>3$

$$\cancel{\frac{3}{2(z-3)} = \frac{-1}{2\left(1-\frac{z}{3}\right)} = \frac{-1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)}$$

~~For $\left|\frac{z}{3}\right|<1$~~

~~$|z|<3$~~

$$\cancel{\frac{3}{2(z-3)} = \frac{3}{2z(1-3/z)} = \frac{3}{2z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\cdots\right)}$$

~~For $\left|\frac{3}{z}\right|<1$~~

~~$\therefore |z|>3$.~~

$$\frac{3}{2(z-4+1)} = \frac{3}{2}\left(1 - (z-4) + (z-4)^2 - \cdots\right)$$

For $|z-4|<1$

$$\frac{3}{2(z-4+1)} = \frac{3}{2(z-4)\left(1+\frac{1}{z-4}\right)} = \frac{3}{2(z-4)}\left(1 - \left(\frac{1}{z-4}\right) + \left(\frac{1}{z-4}\right)^2 - \cdots\right)$$

For $\left|\frac{1}{z-4}\right|<1 \Rightarrow |z-4|>1$

Dependise, Annulas.

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(i) for $|z-4| < 1$

$$f(z) = \frac{1}{6}\left(1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 \cdots\right) +$$

$$\frac{3}{2}\left(1 - (z-4) + (z-4)^2 - \cdots\right)$$

ii) for $1 < |z-4| < 3$

$$f(z) = \frac{3}{2(z-4)}\left(1 - \left(\frac{1}{z-4}\right) + \left(\frac{1}{z-4}\right)^2 - \cdots\right) +$$

$$+ \frac{1}{6}\left(1 - \left(\frac{z-4}{3}\right) + \left(\frac{z-4}{3}\right)^2 \cancel{+} \cdots\right)$$

iii) for $|z-4| > 3$

$$f(z) = \frac{3}{2(z-4)}\left(1 - \left(\frac{1}{z-4}\right) + \left(\frac{1}{z-4}\right)^2 + \cdots\right) +$$

$$\frac{1}{2(z-4)}\left(1 - \left(\frac{3}{z-4}\right) + \left(\frac{3}{z-4}\right)^2 - \cdots\right)$$

$$-\frac{1}{2} + \frac{1}{3}$$

$$\frac{-3+2}{6}$$

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(Q) Expand $f(z) = \dfrac{1}{z(z-1)(z+2)}$ in the given region of convergence: (i) $0 < |z| < 1$ (ii) $|z| > 2$ iii) $1 < |z-1| < 2$ iv) $1 < |z+2| < 2$

→ $f(z)$ is not analytic at $z = 0, 1, -2$

$$\frac{1}{z(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$\cancel{= z(A+B+C)}$$
$$= (z-1)(z+2)A + B(z)(z+2) + C(z)(z-1)$$
$$= (z^2+z-2)A + B(z^2+2z) + C(z^2-z)$$
$$= z^2(A+B+C) + z(A+2B-C) - 2A$$

$-2A = 1$

$A = -1/2$

$A+B+C = 0$

$A+2B-C = 0$

$2A+3B = 0$

$C = \frac{1}{6}$

$$\boxed{B = \frac{-2A}{3} = \frac{1}{3}}$$

$$f(z) = \frac{-1}{2z} + \frac{1}{3(z-1)} + \frac{1}{6(z+2)} \quad \text{—— (1)}$$

about 0.

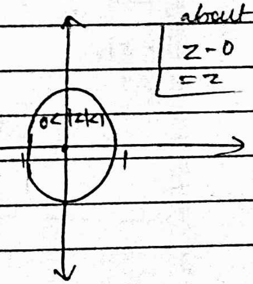
$z-0$

$= z$

(i) $0 < |z| < 1$

$$\frac{1}{3(z-1)} = \frac{-1}{3(1-z)} = \frac{-1}{3}\left(1+z+z^2 \cdots\right)$$

for $|z| < 1$



$$\frac{1}{6(z+2)} = \frac{1}{12(1+z/2)} = \frac{1}{12}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\dots\right)$$

for $\left|\frac{z}{2}\right|<1$

$|z|<2$

Laurent z:

$$\boxed{f(z) = \frac{1}{2z} + \frac{1}{12}\left(1-\frac{z}{2}+\frac{z^2}{4}-\dots\right) - \frac{1}{3}\left(z+z^2+\dots\right)}$$

(ii) $|z|>2$

$$\frac{1}{3(z-1)} = \frac{1}{3z(1-1/z)} = \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

for $\left|\frac{1}{z}\right|<1$

$|z|>1$

$$\frac{1}{6(z+2)} = \frac{1}{6z(1+2/z)} = \frac{1}{6z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\dots\right)$$

for $\left|\frac{2}{z}\right|<1$

$|z|>2$

$$\boxed{\therefore f(z) = \frac{1}{2z} + \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) + \frac{1}{6z}\left(1-\frac{2}{z}+\frac{4}{z^2}+\dots\right)}$$

(iii) $1<|z-1|<2$

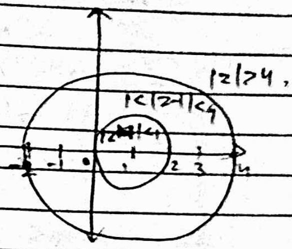
$$f(z) = \frac{1}{2z} + \frac{1}{3(z-1)} + \frac{1}{6(z+2)}$$

About $|z-1|=0$

$z=1$

$$\frac{-1}{2z} = \frac{1}{2(z-1+1)} = \frac{1}{2(z-1)\left(1+\frac{1}{z-1}\right)}$$

$$= \frac{1}{2(z-1)}\left(1-\left(\frac{1}{z-1}\right)+\left(\frac{1}{z-1}\right)^2-\dots\right)$$



for $|z-1|>1$

$$\frac{1}{3(z-1)}$$

$$\cancel{\frac{1}{6(z-1+3)} = \frac{1}{6(z-1)\left(3+\frac{3}{z-1}\right)} = \frac{1}{6(z-1)}}$$

$$\frac{1}{6(z-1+3)} = \frac{1}{18\left(1+\frac{z-1}{3}\right)} = \frac{1}{18}\left(1-\left(\frac{z-1}{3}\right)+\left(\frac{z-1}{3}\right)^2-\dots\right)$$

for $\left|\frac{z-1}{3}\right|<1$

$|z-1|<3$

$$\boxed{f(z) = \frac{-1}{2(z-1)}\left(1-\left(\frac{1}{z-1}\right)+\frac{1}{(z-1)^2}-\dots\right) + \frac{1}{3(z-1)} \quad \frac{1}{18}\left(1-\left(\frac{z-1}{3}\right)+\left(\frac{z-1}{3}\right)^2+\dots\right)}$$

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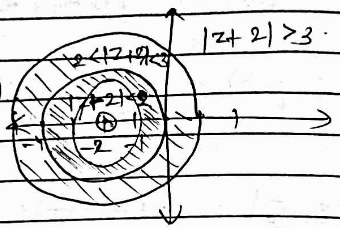
iv) $1 < |z+2| < 2$

$\Rightarrow$ $(0 < |z+2| < 2)$ explaination



$\cancel{\dfrac{1}{2(z+2-2)} = \dfrac{1}{2(z+2)\left(1-\frac{2}{z+2}\right)}}$

$\cancel{= \dfrac{1}{2(z+2)}\left(1+\left(\dfrac{2}{z+2}\right)+\left(\dfrac{2}{z+2}\right)^2+\dots\right)}$

$\cancel{\text{for } \left|\dfrac{2}{z+2}\right| < 1}$

$\cancel{|z+2| >}$

$\dfrac{-1}{2(z+2-2)} = \dfrac{1}{-4\left(1-\left(\frac{z+2}{2}\right)\right)} = +\dfrac{1}{4}\left(1+\left(\dfrac{z+2}{2}\right)+\left(\dfrac{z+2}{2}\right)^2+\dots\right)$

For $\left|\dfrac{z+2}{2}\right| < 1$

$|z+2| < 2$

$\dfrac{1}{3(z+2-3)} = \dfrac{1}{-9\left(\cancel{z}1-\left(\frac{z+2}{3}\right)\right)} = \dfrac{-1}{9}\left(z+\left(\dfrac{z+2}{3}\right)+\left(\dfrac{z+2}{3}\right)^2+\dots\right)$

For $|z+2| < 3$

$\dfrac{1}{6(z+2)}$

$f(z) = \dfrac{1}{4}\left(1+\left(\dfrac{z+2}{2}\right)+\left(\dfrac{z+2}{2}\right)^2+\dots\right) - \dfrac{1}{9}\left(z+\left(\dfrac{z+2}{3}\right)+\left(\dfrac{z+2}{3}\right)^2+\dots\right)$

$+ \dfrac{1}{6(z+2)}$

(−2 will not be included because not analytic.)

$\cos 2x = 1 - 2\sin^2 x$

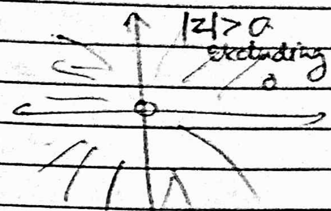
$\dfrac{1-\cos 2x}{2}$

Q) Expand $f(z) = ze^{1/z^2}$ as a Laurent Series.

$f(z)$ is not analytic at $z^2 = 0 \Rightarrow z = 0$.

$z = 0$

$f(z) = ze^{1/z^2}$



$e^z = 1 + z + \dfrac{z^2}{2!} + \dots$ for $|z| < \infty$

$e^{1/z^2} = 1 + \dfrac{1}{z^2} + \dfrac{1}{z^4 2!} + \dots$ for $\left|\dfrac{1}{z^2}\right| < \infty$

$|z|^2 > 0$

$|z| > 0$.

$$\boxed{f(z) = ze^{1/z^2} = z + \frac{1}{z} + \frac{1}{z^3 2!} + \frac{1}{z^7 3!} + \dots}$$

Q) $\dfrac{\sin^2 z}{z}$ $f(z)$ is not analytic at $z=0$ $|z|>0$

$\dfrac{1-\cos 2z}{2z} =$

$1-\cos 2z$

$\cos 2z = 1 - \dfrac{(2z)^2}{2!} + \dfrac{(2z)^4}{4!} - \dots$

$\dfrac{1-\cos 2z}{2z} = \cancel{\dfrac{1}{2z}} + \cancel{\dfrac{1}{2z}} - \dfrac{2z}{2!} + \dfrac{(2z)^3}{4!} - \dots$

$= \dfrac{1}{z} - \dfrac{2z}{2!} + \dfrac{(2z)^3}{4!} - \dots$ //

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(Q) Expand $\frac{\sin^2 z}{z}$ as Laurent series.

→ $\sin^2 z = \frac{1}{2}[1-\cos 2z]$

$= \frac{1}{2}\left[1-\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+\cdots\right)\right]$

$= \frac{1}{2}\left(\frac{(2z)^2}{2!}-\frac{(2z)^4}{4!}+\frac{(2z)^6}{6!}-\cdots\right)$

for $|2z|<\infty \Rightarrow |z|<\infty$

$f(z) = \frac{1}{2z}\left(\quad \right)$

$= \frac{2z}{2!} - \frac{2^3 z^3}{4!} + \frac{2^6 z^5}{6!} - \cdots$ for $0<|z|<\infty$

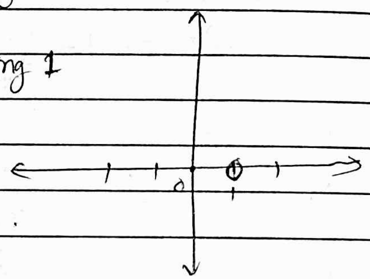
There is no Laurent for this

(Q) Expand $\frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent series.

→ $f(z)$ is not analytic at $(z-1)^3=0$

$\therefore z=1$

excluding 1



$e^{2z} = e^{2(z-1+1)}$

$= e^{2(z-1)} \cdot e^2$

$e^{2(z-1)} = 1+2(z-1)+\frac{(2(z-1))^2}{2!}+\cdots$

$\Rightarrow e^{2(z-1)} \cdot e^2 = e^2\left(1+2(z-1)+\frac{(2(z-1))^2}{2!}+\cdots\right)$ for $|2(z-1)|<\infty$, $|z-1|<\infty$

~~f(z) = e^{2(z-1)}·e^2~~

Opposite
Singular X Regular (Analytic)

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$f(z) = \frac{e^2}{(z-1)^3}\left(1+2(z-1)+\frac{2^2(z-1)^2}{2!}+\cdots\right)$

$= e^2\left(\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{z-1}+\cdots\right)$

☆ Singular point

☆ If $f(z)$ is not analytic at z_0 and analytic at atleast one point in every open disc around z_0 then 'z_0' is called a singular point of $f(z)$.

☆ Isolated singular point – A singular point 'z_0' is called isolated if there exists an open disc around 'z_0' where $f(z)$ is analytic except at z_0.

☆ For simple understanding, If $f(z)$ is not analytic at finite or uncountably infinite number of points then all are isolated singular points.

☆ Types of Isolated Singular points

☆ Let z_0 be an isolated singular point of $f(z)$ then $f(z)$ is analytic in some deleted neighbourhood (punctured disc) $0<|z-z_0|<R$ and therefore $f(z)$ admits a Laurent's series expansion about $z=z_0$ as $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$

for $0<|z-z_0|<R$ — (I)

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1) If all $b_n = 0$, i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ then 'z_0' is called a removable singular point of $f(z)$.

2) If finite no. of b_n's are not equal to zero i.e $b_m \neq 0, b_{m+1} = 0, b_{m+2} = 0 \dots$ i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \dots + \frac{b_m}{(z-z_0)^m}$
then z_0 is called a pole of order 'm'.

3) If infinite no. of b_n's are non-zero i.e $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$
then z_0 is called an essential singular point.

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Note
1) If z_0 is a removable point of $f(z)$ then it can be identified only by series expansion of $f(z)$ about the point $z = z_0$ in the ROC $0 < |z - z_0| < R$

2) If z_0 is a pole then it can be identified using some more results.

Results.
1) z_0 is a pole of order 'm' of $f(z)$ if $f(z) = \frac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic & non-zero at $z = z_0$.

2) If $f(z) = \frac{g(z)}{h(z)}$ where $g(z)$ is Analytic and non-zero at $z = z_0$ and, $h(z_0) = 0$, $h'(z_0) = 0$, $\dots$ $h^{(m-1)}(z_0) = 0$;
and $h^{(m)}(z_0) \neq 0$
then $z = z_0$ is a pole of order m of $f(z)$.

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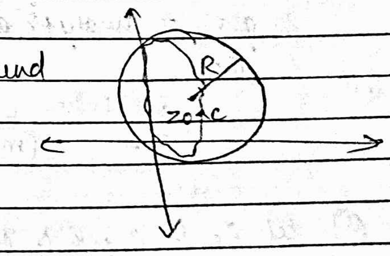
Note- If $f(z) = \frac{g(z)}{h(z)}$ where $g(z)$ is analytic and non-zero at $z = z_0$ and $h(z)$ is a polynomial, if z_0 is a root of equation $h(z) = 0$ and repeated m times then z_0 is a pole of order m of $f(z)$.

Residue of $f(z)$ at an isolated singular point -

Let z_0 be an isolated singular point of $f(z)$ then $f(z)$ has Laurent series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} \dots$
for $0 < |z - z_0| < R$.

Let 'C' be any closed contour around z_0 and lying completely inside $0 < |z - z_0| < R$;



$\int_C f(z)dz = 2\pi i b_1$
we define,
Residue of $f(z)$ at $z = z_0 \Rightarrow b_1 \equiv$ coefficient of $\frac{1}{(z-z_0)}$ term in the Laurent series expansion of $f(z)$ about $z = z_0$ in the domain $0 < |z - z_0| < R$.

Also,
$\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$

$2\pi i \rightarrow$ never a residue
it is an integral multiple.

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Note - If z_0 is a removable singular point of $f(z)$ then

(1) Removable;

$$\operatorname*{Res}_{z=z_0} f(z) = 0$$

(2) If z_0 is an essential singular point of $f(z)$ then the residue can be obtained only from series expansion.

Removable & essential me residue hoga jisme hum series expansion karna hai.

(3) If z_0 is a pole, then residue can be obtained using some more results -

Residue at poles

(1) Let z_0 be a pole of order m of $f(z)$ and $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic and non-zero at $z = z_0$,

Then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[g(z) \right] \Big|_{z=z_0}.$$

(2) Let z_0 be a pole of order one (simple pole) of $f(z)$, and $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z)$ is analytic and non-zero at $z = z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z)}{h'(z)} \Big|_{z=z_0}$$

★ If $f(z)$ is analytic on and in the interior of a closed contour 'c' except at finite number of points $z_1, z_2, \dots z_n$ lying completely inside 'c'.

Then,

$$\int_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \dots \operatorname*{Res}_{z=z_n} f(z) \right]$$

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(Q) Evaluate

$$\int_C \frac{2z-1}{z^4 - 2z^3 - 3z^2}\,dz \ ; \quad \text{(i) } C: |z - \tfrac{1}{2}| = 1 \quad \text{(ii) } |z+1| = 2 \quad \text{(iii) } |z-1| = 3 \quad \text{(iv) } |z - 2i| = 1.$$

Sol^n $I = \displaystyle\int \frac{2z-1}{z^4 - 2z^3 - 3z^2}\,dz.$

$f(z) = \dfrac{2z-1}{z^4 - 2z^3 - 3z^2}$ is not analytic at $z^4 - 2z^3 - 3z^2 = 0$.

1) It will always be a pole
2) root agar repeated hai to (0,0) ⇒ order of pole

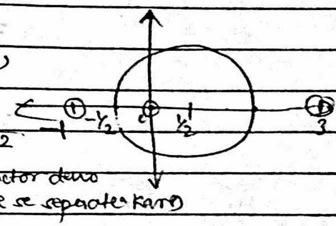
"Pole"

$$z^2(z^2 - 2z - 3) = 0$$
$$z^2(z+1)(z-3)$$
$$\Rightarrow z = 0, 0, z = -1, 3$$

(i) $C: |z - \tfrac{1}{2}| = 1$

$z = 0$ lies completely inside 'c'

$z = 0$ is a pole of order $= 2$.



$(z-0)^2$
ye factor deno me se separate karo

$$\operatorname*{Res}_{z=0} f(z) = \operatorname*{Res}_{z=0} \frac{2z-1}{z^4 - 2z^3 - 3z^2}$$

$$= \operatorname*{Res}_{z=0} \frac{2z-1}{z^2(z^2 - 2z - 3)}$$

Jab ek hi pt curve ke inside hai Cauchy & Residue done me diff nhi hai.

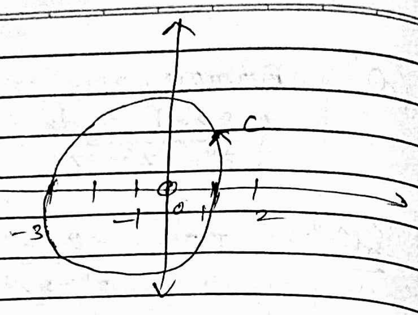
$$= \frac{1}{1!} \frac{d}{dz} \left(\frac{2z-1}{z^2 - 2z - 3} \right) \Big|_{z=0} = \frac{-8}{9}$$

$$\int_C f(z)\,dz = 2\pi i \left(\frac{-8}{9} \right) = \frac{-16\pi i}{9}$$

Q. types
- Find non-analytic pts.
- Find types of singularity of those pts.
- Simplify the residue

(ii) $|z+1| = 2$

$z=0, -1$ lies inside 'c'.



$z=0$ is a pole of order 2

$z=-1$ is a pole of order 1

$\underset{z=0}{\text{Res}}\, f(z) = \frac{-8}{9}$

$\underset{z=-1}{\text{Res}}\, f(z) = \left.\frac{2z-1}{4z^3-6z^2-6z}\right|_{z=-1} = \frac{3}{4}$

Only for pole of order 1 case, otherwise normal method

$\int_C f(z)dz = 2\pi i\left(\frac{-8}{9}+\frac{3}{4}\right)$

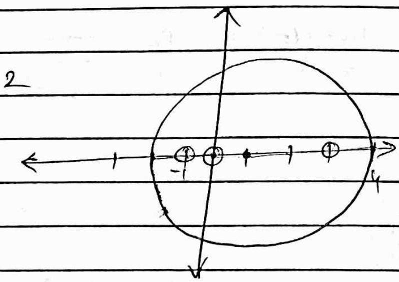
$= 2\pi i\left(\frac{-5}{36}\right)$

$= \frac{-5\pi i}{18}$

(iii) $|z-1| = 3$

$z=0$ is a pole of order 2

$z=-1$ is pole of order 1



$\underset{z=0}{\text{Res}}\, f(z) = \frac{-8}{9}$

$\underset{z=-1}{\text{Res}}\, f(z) = \frac{3}{4}$

$z=3$ is a pole of order 1

$\underset{z=3}{\text{Res}}\, f(z) = \left.\frac{2z-1}{4z^3-6z^2-6z}\right|_{z=3} = \frac{5}{36}$

$\int_C f(z)dz = 2\pi i\left(\frac{-8}{9}+\frac{3}{4}+\frac{5}{36}\right)$

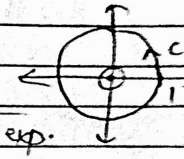
$= \frac{2\pi i(-32+27+5)}{36} = 0$

(iv) $\oint$ $|z-2i| = 1$

All pts are outside the curve

$\therefore$ Residue $= 0$

(c) Evaluate $\int_C z^2 e^{1/z} dz$, $c: |z|=1$



$f(z) = z^2 e^{1/z}$ is not analytic at $z=0$

$C: |z|=1$

$\therefore z=0$ lies inside 'c'

not coming from a poly. in denominator

$\therefore$ directly judge nahi kar sakte ki konse type ka singular pt. hai.

series exp. $z-0=z$

$f(z) = z^2 e^{1/z}$

~~$= z^2\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)$~~

~~$= z^2+z^3+\frac{z^4}{2!}+\frac{z^5}{3!}+\dots$~~

$e^{1/z} = 1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}\dots$

$= z^2\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\dots\right)$

For $\left|\frac{1}{z}\right| < \infty$ $\therefore |z|>0$ } ROC

$f(z) = z^2+z+\frac{1}{2!}+\frac{1}{3!z}+\dots$ $\to$ for $|z|>0$

$z=0$, is an essential singular point.

$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{3!}$ } Coefficient of $\frac{1}{z}$ = Res $f(z)$

$\int_C f(z)dz = 2\pi i \times \frac{1}{3!} = \frac{\pi i}{3}$

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Q) Evaluate $I = \int_C \tan 2\pi z \, dz$; $C: |z|=1$

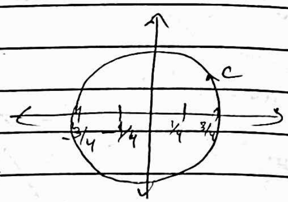
$f(z) = \tan 2\pi z = \dfrac{\sin 2\pi z}{\cos 2\pi z}$

$f(z)$ is not analytic at $\cos 2\pi z = 0$ i.e. $2\pi z = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2} \ldots$

$\therefore z = \pm\dfrac{1}{4}, \pm\dfrac{3}{4}, \pm\dfrac{5}{4}, \ldots$

$C: |z| = 1$

$z = \pm\dfrac{1}{4}, \pm\dfrac{3}{4}$ lies inside 'C'



$h(z) = \cos 2\pi z$

$h'(z) = -2\pi \sin 2\pi z$

$h'(z) \neq 0$ at all $z = \pm 1/4, \pm 3/4$

1st. singularity
pole korrow.
Avoid series
expansion.

All $z = \pm 1/4, \pm 3/4$ are poles of order one.

$\operatorname{Res}_{z=+1/4} f(z) = \left.\dfrac{\sin 2\pi z}{\cos 2\pi z}\right|_{z=1/4} = \left.\dfrac{\sin 2\pi z}{-2\pi \sin 2\pi z}\right|_{z=+1/4} = -\dfrac{1}{2\pi}.$

$\operatorname{Res}_{z=-1/4} f(z) = \left.\dfrac{\sin 2\pi z}{-2\pi \sin 2\pi z}\right|_{z=-1/4} = -\dfrac{1}{2\pi}$

⋮

Similarly $\operatorname{Res}_{z=+1/4} f(z) = \operatorname{Res}_{z=-1/4} f(z) = \operatorname{Res}_{z=3/4} f(z) = \operatorname{Res}_{z=-3/4} f(z) = -\dfrac{1}{2\pi}$

$\therefore I = \int_C \tan 2\pi z \, dz = 2\pi i \left[-\dfrac{1}{2\pi} - \dfrac{1}{2\pi} - \dfrac{1}{2\pi} - \dfrac{1}{2\pi}\right]$

$= -4i$

$\sinh z = i\sin \ldots$

$\dfrac{e^z + e^{-z}}{2} = \dfrac{2}{e^z - e^{-z}} = \dfrac{2e^z}{e^{2z}-1}.$

$\sin iz = i\sinh z$

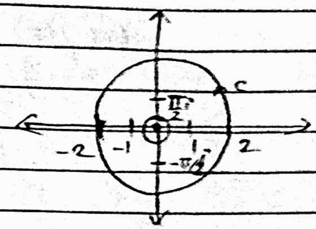
$\sinh iz = i\sin hz$

Q) $\int_C \operatorname{cosech} 2z \, dz$, $|z| = 2$

$I = \int_C \operatorname{cosech} 2z \, dz = \dfrac{2}{e^{2z} - e^{-2z}} = \dfrac{2e^{2z}}{e^{4z}-1}$

$\Rightarrow \dfrac{1}{\sinh 2z}.$

$\therefore f(z)$ is not analytic at $z = 0$



$\dfrac{1}{i}\sin 2iz = 0.$

$\sin i 2z = 0.$

$2iz = 0, \pm\pi, \pm 2\pi \ldots$

$z = 0, \pm\dfrac{\pi}{2i}, \pm\dfrac{\pi}{i} + \ldots$

or

$z = 0, \mp\dfrac{\pi i}{2}, \mp \pi i, \ldots$

$\therefore z = 0, \pm\dfrac{\pi i}{2}$ lie inside circle.

$\cancel{\operatorname{Res}_{z=0} f(z) = \left.\dfrac{1}{\sinh 2z}\right|_{z=0} = \left.\dfrac{1}{2\cosh 2z}\right|_{z=0}} \Rightarrow \dfrac{1}{2}.$

$\cancel{\operatorname{Res}_{z=\pi i/2} f(z) = \dfrac{1}{2\cosh 2z}}$

$h(z) = \sinh 2z$

$h'(z) = 2\cosh 2z \neq$ at all $z = 0, \pm\dfrac{\pi i}{2}$

$=$

$\therefore z = 0, \dfrac{\pi i}{2}, -\dfrac{\pi i}{2}$ are poles of order one.

$\operatorname{Res}_{z=0} f(z) = \left.\dfrac{1}{\sinh 2z}\right|_{z=0} = \left.\dfrac{1}{2\cosh 2z}\right|_{z=0} = \dfrac{1}{2}$

$$\operatorname*{Res}_{z=\frac{\pi}{2}i} f(z) = \frac{1}{\sinh 2z}\Bigg|_{z=\frac{\pi}{2}i} = \frac{1}{2\cosh 2z}\Bigg|_{z=\frac{\pi}{2}i}$$

$$= \frac{1}{2\cosh \pi i} = \frac{-1}{2}$$

$$\operatorname*{Res}_{z=\frac{\pi}{2}i} f(z) = \frac{-1}{2}$$

$$\therefore I = \int_c \operatorname{cosech} 2z\,dz = \left(\frac{1}{2} + \left(\frac{-1}{2}\right) - \frac{1}{2}\right) 2\pi i$$

$$= -\pi i$$

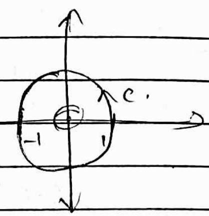
(Q) Evaluate $\int_c \frac{1}{z - \sin z}\,dz$, $c : |z| = 1$.

$1 - \cos z$

$\sin z$

$\cos z$

$f(z) = \frac{1}{z - \sin z}$ is not analytic at $z = 0$.



It is a pole of order 3

(3^{rd} derivative of deno. non-zero)

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{-\cos z}\Bigg|_{z=0} = -1$$

$$\int_c f(z)\,dz = -2\pi i$$

OR

By expansion.

$$f(z) = \frac{1}{z - \sin z} = \frac{1}{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)}$$

$$= \frac{1}{\frac{z^3}{3!} - \frac{z^5}{5!} + \cdots}$$

$$= \frac{1}{z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\right)}$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2!} \frac{d^2}{dz^2}\left[\frac{1}{\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots}\right]_{z=0}$$