* Convolution Method : (Inverse Z-transform)

Let $F(z)$ and $G(z)$ be the function of z

such that $Z^{-1}[F(z)] = f(k)$ and

$Z^{-1}[G(z)] = g(k)$

then $Z^{-1}\{F(z) \cdot G(z)\} = f(k) * g(k)$

$$= \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m) = h(k)$$

Example 1: find Inverse z-transform of

$$\frac{z^2}{(z-a)(z-b)}$$

Solution: let $F(z) = \frac{z}{z-a}$, $G(z) = \frac{z}{z-b}$

$\Rightarrow$ $Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z-a}\right] = a^k = f(k)$, $k \geq 0$

and $Z^{-1}[G(z)] = Z^{-1}\left[\frac{z}{z-b}\right] = b^k = g(k)$, $k \geq 0$

$\therefore$ By convolution method

$Z^{-1}\{F(z) \cdot G(z)\} = f(k) * g(k)$

$$\Rightarrow Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)$$

$$= \sum_{m=-\infty}^{\infty} a^m \cdot b^{(k-m)} \quad , \quad k \geq 0$$

$$= \sum_{m=-\infty}^{\infty} a^m . b^k . b^{-m} , \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{b^m} . b^k , \quad k \geq 0$$

$$= b^k \sum_{m=0}^{\infty} \left(\frac{a}{b}\right)^m , \quad k \geq 0$$

$$= b^k \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots \right] , \quad k \geq 0$$

$$= b^k \left[\frac{\left(\frac{a}{b}\right)^{k+1} - 1}{\left(\frac{a}{b}\right) - 1} \right] , \quad k \geq 0 \quad \left(\because \text{Geometric series } 1 + r + r^2 + \cdots = \frac{r^{k+1} - 1}{r - 1} \right)$$

$$= b^k \left[\frac{\frac{a^{k+1} - b^{k+1}}{b^{k+1}}}{\frac{a - b}{b}} \right] , \quad k \geq 0$$

$$= b^k \left[\frac{a^{k+1} - b^{k+1}}{b^{k+1}} \times \frac{b}{a - b} \right] , \quad k \geq 0$$

$$= \frac{b^k}{b^k} \left[\frac{a^{k+1} - b^{k+1}}{a - b} \right] , \quad k \geq 0$$

$$= \frac{a^{k+1} - b^{k+1}}{a - b} , \quad k \geq 0$$

$$\therefore \quad Z^{-1} \left[\frac{z^2}{(z-b)\,(z-b)} \right] = \frac{a^{k+1} - b^{k+1}}{a - b} , \quad k \geq 0$$

Ex (2) Find the inverse Z transform of

$$\frac{z}{(z-1)(z-2)}, \quad |z|>2 \text{ By convolution method.}$$

Solution: let $F(z) = \frac{1}{z-1}$ and $G(z) = \frac{z}{z-2}$

$$\Rightarrow Z^{-1}[F(z)] = Z^{-1}\left[\frac{1}{z-1}\right] = 1^{k-1} = 1 = f(k)$$

and $Z^{-1}[G(z)] = Z^{-1}\left[\frac{1}{z-2}\right] = 2^k = g(k), \quad k \geq 0$

$\therefore$ By convolution method

$$Z^{-1}[F(z) \cdot G(z)] = f(k) * g(k)$$

$$= \sum_{m=0}^{\infty} f(m) \cdot g(k-m), \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} 1 \cdot 2^{k-m}, \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} 2^k \cdot 2^{-m}, \quad k \geq 0$$

$$= 2^k \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m, \quad k \geq 0$$

$$= 2^k \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right], \quad k \geq 0$$

$$= 2^k \left[\frac{\left(\frac{1}{2}\right)^{k+1} - 1}{\left(\frac{1}{2}\right) - 1}\right], \quad k \geq 0 \quad \text{(By Geometric series)}$$

$$= 2^k \left[\frac{\frac{1-2^{k+1}}{2^{k+1}}}{\frac{1-2}{2}}\right], \quad k \geq 0$$

$$= 2^k \left[\frac{1-2^{k+1}}{2^{k+1}} \times \frac{2}{1-2}\right], \quad k \geq 0$$

$$= \frac{2^k}{2^k}\left[\frac{1-2^{k+1}}{1-2}\right] \quad , \quad k \geq 0$$

$$= 2^{k+1} - 1 \quad , \quad k \geq 0$$

$$= 2^k - 1 \quad , \quad k \geq 1$$

$$\therefore \quad Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^k - 1 \quad , \quad k \geq 1$$

Homework:

1) find inverse z- transform of

$\frac{1}{(z-5)^2}$, $|z| > 5$ by convolution method.

2) find inverse z- transform of

$\frac{1}{(z-2)(z-3)}$ by convolution method.