* NLPP with inequality constraint :

* Kuhn - Tucker conditions :

— For one inequality constraint

consider the NLPP

Maximise $z = f(x_1, x_2, \cdots x_n)$

subject to $h(x_1, x_2, \cdots x_n) \leq 0$

$x_1, x_2, \cdots x_n \geq 0$

* Necessary condition :

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0 \qquad \text{—— (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 \qquad \text{—— (2)}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0 \qquad \text{—— (n)}$$

$$\lambda h(x_1, x_2, \cdots x_n) = 0 \qquad \text{—— (n+1)}$$

$$h(x_1, x_2, \cdots x_n) \leq 0 \qquad \text{—— (n+2)}$$

$$\lambda \geq 0 \qquad \text{—— (n+3)}$$

* Note that If problem is of Minimisation type then only (n+3) condition is change

i.e. $\lambda < 0$

Important : All the constraints in NLPP should be less than or equal to type '$\leq$'

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Ex (1) solve the following N.L.P.P

maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \le 5$

$x_1, x_2 \ge 0$

Solution: here, $f(x_1, x_2) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$h(x_1, x_2) = 2x_1 + x_2 - 5$

Now, kuhn-Tucker condition are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\lambda h(x_1, x_2) = 0$$

$$h(x_1, x_2) \le 0$$

$$\lambda \ge 0$$

therefore,

$10 - 4x_1 - 2\lambda = 0$ —— (1)

$4 - 2x_2 - \lambda = 0$ —— (2)

$\lambda (2x_1 + x_2 - 5) = 0$ —— (3)

$2x_1 + x_2 - 5 \le 0$ —— (4)

$x_1, x_2, \lambda \ge 0$ —— (5)

using equation (3) we get

either $\lambda = 0$ or $(2x + x_2 - 5) = 0$

case i) if $\lambda = 0$

Then from (1) & (2), $10 - 4x_1 = 0 \Rightarrow x_1 = \frac{5}{2}$

and $4 - 2x_2 = 0 \Rightarrow x_2 = 2$

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putting these value in equation ④

L.H.S = $2\left(\frac{5}{2}\right) + 2 - 5 = 2 \nleq 0$

$\therefore$ $x_1 = \frac{5}{2}$, $x_2 = 2$ not satisfy all condition of kuhn Tucker

$\therefore$ $\lambda = 0$ not gives the feasible solution

case ii) if $\lambda \neq 0$ then $2x_1 + x_2 - 5 = 0$ —— ⑥

$\therefore$ equation ①, ② and ⑥ can be written as

$$4x_1 + 0x_2 + 2\lambda = 10$$
$$0x_1 + 2x_2 + \lambda = 4$$
$$2x_1 + x_2 + 0\lambda = 0$$

implies that $x_1 = \frac{11}{6}$, $x_2 = \frac{4}{3}$, $\lambda = \frac{4}{3}$ (use calsi)

equation ④ becomes

L.H.S $2\left(\frac{11}{6}\right) + \frac{4}{3} - 5 = 0 \leq 0$

Hence, $x_1 = \frac{11}{6}$ and $x_2 = \frac{4}{3}$ satisfy all Necessary condition of kuhn Tucker

$\therefore$ The optimal solution is

$$x_1 = \frac{11}{6}, \quad x_2 = \frac{4}{3}$$

and $Z_{max} = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$$= 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2$$

$$= \frac{91}{6}$$

$\therefore$ $\boxed{x_1 = \frac{11}{6}, \quad x_2 = \frac{4}{3}, \quad Z_{max} = \frac{91}{6}}$

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Ex ② Use the Kuhn-Tucker condition to solve the following NLPP.

Minimise $z = x_1^3 - 4x_1 - 2x_2$

subject to $x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Solution: here, $f(x_1, x_2) = x_1^3 - 4x_1 - 2x_2$

$h(x_1, x_2) = x_1 + x_2 - 1$

The Kuhn-Tucker condition for minima are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \quad \lambda h(x_1, x_2) = 0,$$

$$h(x_1, x_2) \le 0, \quad \lambda < 0$$

implies that

$$3x_1^2 - 4 - \lambda = 0 \quad \text{--- ①}$$

$$-2 - \lambda = 0 \quad \text{--- ②}$$

$$\lambda(x_1 + x_2 - 1) = 0 \quad \text{--- ③}$$

$$x_1 + x_2 - 1 \le 0 \quad \text{--- ④}$$

$$\lambda < 0 \quad \text{--- ⑤}$$

$\therefore$ from ②, $\lambda = -2$ $(\because \lambda < 0)$

$\therefore$ from ①, $3x_1^2 - 4 + 2 = 0 \Rightarrow x_1 = \sqrt{\frac{2}{3}}$

from ③, $-2\left(\sqrt{\frac{2}{3}} + x_2 - 1\right) = 0 \Rightarrow x_2 = 1 - \sqrt{\frac{2}{3}}$

equation ④ becomes. L.H.S $= \sqrt{\frac{2}{3}} + 1 - \sqrt{\frac{2}{3}} - 1 = 0 \le 0$

That is the value $x_1 = \sqrt{\frac{2}{3}}$ and $x_2 = 1 - \sqrt{\frac{2}{3}}$ satisfy all condition of Kuhn-Tucker

Hence, optimal solution are

$$x_1 = \sqrt{\frac{2}{3}}, \quad x_2 = 1 - \sqrt{\frac{2}{3}}$$

and $Z_{min} = \left(\sqrt{\frac{2}{3}}\right)^3 - 4\left(\sqrt{\frac{2}{3}}\right) - 2\left(1 - \sqrt{\frac{2}{3}}\right) = -3.093$

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H.W

**Ex 3** Use the Kuhn Tucker condition to solve the following NLPP

Maximise $Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Subject to $2x_1 + 5x_2 \leq 98$

$x_1, x_2 \geq 0$

Ans: $x_1 = 44$
$x_2 = 2$
$Z_{max} = 4900$

**Ex 4** Use Kuhn - Tucker condition to solve the following N.L.P.P.

Maximise $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Ans: $x_1 = \frac{4}{13}$
$x_2 = \frac{33}{13}$
$Z_{max} = 21.3$

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- self learning

* The Kuhn-Tucker condition for General NLPP

consider the NLPP.

Maximise $z = f(x_1, x_2, \cdots x_n)$

subject to
$$h_1(x_1, x_2, \cdots x_n) \leq 0$$
$$h_2(x_1, x_2, \cdots x_n) \leq 0$$
$$\vdots$$
$$h_m(x_1, x_2, \cdots x_n) \leq 0$$
$$x_1, x_2, \cdots x_n \geq 0$$

* Necessary condition

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_2} = 0$$

$$\vdots \qquad \cdots\cdots$$

$$\frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial h_1}{\partial x_n} - \lambda_2 \frac{\partial h_2}{\partial x_n} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_n} = 0$$

$$\lambda_1 h_1(x_1, x_2, \cdots x_n) = 0$$
$$\lambda_2 h_2(x_1, x_2, \cdots x_n) = 0$$
$$\vdots$$
$$\lambda_m h_m(x_1, x_2, \cdots x_n) = 0$$

$$\lambda_1, \lambda_2, \cdots, \lambda_m \geq 0$$

Note that if the problem is of Minimisation Type then

$$\lambda_1, \lambda_2, \cdots, \lambda_m < 0$$

Important: All the constraint in NLPP should be of less than or equal to type '$\leq$'

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Ex ① Using the kuhn - Tucker condition, solve the following N.L.P.P.

Maximise $z = x_1^2 + x_2^2$

subject to $x_1 + x_2 - 4 \leq 0$

$2x_1 + x_2 - 5 \leq 0$

$x_1, x_2 \geq 0$

solution: here, $f(x_1, x_2) = x_1^2 + x_2^2$

& $h_1(x_1, x_2) = x_1 + x_2 - 4$

$h_2(x_1, x_2) = 2x_1 + x_2 - 5$

Note that the kuhn - Tucker conditions

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$\lambda_1 h_1(x_1, x_2) = 0$$

$$\lambda_2 h_2(x_1, x_2) = 0$$

$$h_1(x_1, x_2) \leq 0$$

$$h_2(x_1, x_2) \leq 0$$

$$\lambda_1, \lambda_2 \geq 0$$

implies that

$$2x_1 - \lambda_1 - 2\lambda_2 = 0 \quad \text{—— ①}$$

$$2x_1 - \lambda_1 - \lambda_2 \quad \text{—— ②}$$

$$\lambda_1 (x_1 + x_2 - 4) = 0 \quad \text{—— ③}$$

$$\lambda_2 (2x_1 - x_2 - 5) = 0 \quad \text{—— ④}$$

$$x_1 + x_2 - 4 \leq 0 \quad \text{—— ⑤}$$

$$2x_1 - x_2 - 5 \leq 0 \quad \text{—— ⑥}$$

$$x_1, x_2 \; \lambda_1, \lambda_2 \geq 0 \quad \text{—— ⑦}$$

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we consider the cases depends on $\lambda_1, \lambda_2$

case i) If $\lambda_1 = 0$, $\lambda_2 = 0$

from ① & ② $x_1 = 0$, $x_2 = 0$

which is trivial solution

case ii) If $\lambda_1 = 0$, $\lambda_2 \neq 0$

from ① and ② $2x_1 = 2\lambda$ and $2x_2 = \lambda_2$

$\Rightarrow$ $x_1 = \lambda_2$ and $2x_2 = \lambda_2$

and from ④, $2x_1 + x_2 = 5$

$\therefore$ $2(\lambda_2) + \frac{\lambda_2}{2} = 5 \Rightarrow \lambda_2 = 5$

$\therefore$ $x_1 = 5$ and $x_2 = \frac{5}{2}$

from eqⁿ ③ L.H.S $5 + \frac{5}{2} - 4 \nleq 0$

$\therefore$ $x_1 = 5$ and $x_2 = \frac{5}{2}$ cannot be feasible solution

case iii) If $\lambda_1 \neq 0$, $\lambda_2 = 0$

from ① & ②, $2x_1 = \lambda_1$, $2x_2 = \lambda_1 \Rightarrow x_1 = x_2$

from ③ $x_1 + x_2 = 4$

$\Rightarrow$ $x_1 + x_1 = 4 \Rightarrow x_1 = 2$

$\Rightarrow x_2 = 2$

clearly, $x_1 = 2$, $x_2 = 2$ satisfy all condition

$\therefore$ $Z_{max} = x_1^2 + x_2^2 = 4 + 4 = 8$

$\therefore$ optional solution is

$$\boxed{x_1 = 2, \quad x_2 = 2 \quad \text{and} \quad Z_{max} = 8}$$

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