Module 5. Linear programming problems (LPP)

* General Linear programming problem:

Maximize (minimize) $Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ —— ①

subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\le b_1 \ (\ge b_1) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\le b_2 \ (\ge b_2) \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\le b_m \ (\ge b_m) \end{aligned}\right\} \text{—— ②}$$

and $x_1, x_2, \dots x_n \ge 0$ (or Unristricted) —— ③

Note that * ① is called objective function

* ② are called constraints
* ③ are called Non-negativity restrictions.

* solution of Linear programming problem (LPP)

Any set of values $x_1, x_2, \dots x_n$ which satisfy the all constraints is called a solution of LPP.

* Feasible solution of LPP:

Any solution which satisfy the given non-negativity restriction is called feasible solution.

* optimal solution of LPP:

Any feasible solution which satisfy the objective function is called optimal solution of LPP.

* slack Variable :

If the constraints of LPP are of less than or equal to type ($\leq$) then we can add new non-negative variables so that the constraints can be expressed as equalities.

therefore that new non-negative variables are called as slack variables.

for ex.. ① suppose $x_1 + 2x_2 + x_3 \leq 5$

then we add s_1 ($s_1 \geq 0$) we get $x_1 + 2x_2 + x_3 + s_1 = 5$

* Surplus Variable :

If the constraints of LPP are of greater than or equal to type (≥ 0) then we can substract new non-negative variable so that the constraints can be expressed as equalities.

therefore, that new non-negative variable are called as surplus variables.

for ex. ① suppose, $x_1 - 2x_2 + 4x_3 \geq 7$

then we substract s_1 ($s_1 \geq 0$) we get $x_1 - 2x_2 + 4x_3 - s_1 = 7$

* Canonical and standard form of LPP :-

A general LPP

Maximise $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

with $x_i \ge 0$, $i = 1, 2, \cdots, n$

is called canonical form

And If we indroduce slack variables then the LPP

Maximise $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0s_1 + 0s_2 + \cdots + 0s_m$

subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 + 0s_2 + \cdots + 0s_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + 0s_1 + s_2 + 0s_3 + \cdots + 0s_m = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + 0s_1 + \cdots + 0s_{m-1} + s_m = b_m$$

with $x_1, x_2, \cdots x_n, s_1, s_2, \cdots s_m \ge 0$

is called standard form

for examples:

Ex. ① Convert the following LPP in the standard form.

Maximise $z = 3x_1 + 5x_2$

subject to
$$3x_1 + 2x_2 \le 15$$
$$2x_1 + 5x_2 \ge 12$$
$$x_1, x_2 \ge 0$$

Sol^n we introduce slack variable s_1 and surplus variable s_2 then the problem can be converted to standard form as

$$\text{Maximise } Z = 3x_1 + 5x_2 + 0s_1 + 0s_2$$

$$\text{subject to} \quad 3x_1 + 2x_2 + s_1 + 0s_2 = 15$$

$$2x_1 + 5x_2 + 0s_1 - s_2 = 12$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Ex. ② convert the following LPP to the standard form

$$\text{Maximise } z = 3x_1 + 2x_2 + 5x_3$$

$$\text{subject to} \quad 2x_1 - 3x_2 \leq 3$$

$$x_1 + 2x_2 + 3x_3 \geq 5$$

$$3x_1 + 2x_3 \leq 2$$

$$x_1, x_2 \geq 0$$

Solution: we introduce slack variable s_1, s_3 and surplus variable s_2

Also here, x_3 is unrestricted

$\therefore$ we put $x_3 = x_3' - x_3''$ and $x_3', x_3'' \geq 0$

$\therefore$ the standard form of LPP is

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3' - 5x_3'' + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to} \quad 2x_1 - 3x_2 + 0x_3' - 0x_3'' + s_1 + 0s_2 + 0s_3 = 3$$

$$x_1 + 2x_2 + 3x_3' - 3x_3'' + 0s_1 - s_2 + 0s_3 = 5$$

$$3x_1 + 0x_2 + 2x_3' - 2x_3'' + 0s_1 + 0s_2 + s_3 = 2$$

$$\text{with} \quad x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0$$

* Simplex Method :-

- Types of solution:

① Basic solution: A solution obtained by setting any n variables out of $m+n$ variables equal to zero and solving for remaining m variable, provided the determinant of the coefficient of these m variables is non zero is called a basic solution

Such m variable are called basic variables and the remaining n zero-valued variables are called non-basic variables

② Basic feasible solution :
A basic solution which also satisfies non-negativity restrictions is called basic feasible solution.

Note that : In the basic feasible solution obtained

i) All m values of basic variables are positive then it is non-degenerate basic F.S

ii) one or more values of m basic variable are zero then it is degenerate basic F.S.

Ex 1 Given that

Maximise $z = x_1 + 3x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$

find all basic solutions to the above problem. which of them are basic feasible, non-degenerate, infeasible basic and optimal basic feasible solution?

solution:

No. of Basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible?	Is the solution degenerate	value of z	Is the solution optimal?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\Rightarrow x_1 = 2, x_2 = 1$	yes	No.	$2+3(1)+0$ $= 5$	yes
2.	$x_2 = 0$	x_1, x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\Rightarrow x_1 = 1, x_2 = 1$	yes	No	$1+3(0)$ $+3(1)$ $= 4$	No
3.	$x = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\Rightarrow x_2 = -1, x_3 = 2$	No	No	-	-

Note that ! In the second solution, x_2 is the outgoing variable and x_3 is incoming variable Similarly, in the third solution x_1 is outgoing and x_2 is incomming variable

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Ex. ② consider the following problem

Maximise $z = 2x_1 - 2x_2 + 4x_3 - 5x_4$

subject to $x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$

$x_1, x_2, x_3, x_4 \ge 0$

determine i) All basic solution

ii) All feasible basic solution,

iii) optimal feasible basic solution.

Solution:

No. of basic solutions	Non-basic variables $= 0$	Basic variables	Equations & the values of basic variables	Is the solution feasible	Is the solution degenerate	Value of Z	Is the solution optimal?
1	$x_3 = 0$ $x_4 = 0$	x_1, x_2	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $\Rightarrow x_1 = 0, x_2 = \frac{1}{2}$	yes	yes	$2(0) - 2(\frac{1}{2})$ $+ 4(0) - 5(0)$ $= -1.5$	No
2	$x_2 = 0$ $x_4 = 0$	x_1, x_3 outgoing x_2 incoming x_1	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $\Rightarrow x_1 = 8, x_3 = 3$	yes	No	$2(8) - 2(0)$ $+ 4(3) - 5(0)$ $= 28$	yes
3.	$x_1 = 0$ $x_4 = 0$	x_2, x_3 outgoing x_1 incoming x_2	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $\Rightarrow x_2 = \frac{1}{2}, x_3 = 0$	yes	yes	$2(0) - 2(\frac{1}{2}) +$ $4(0) - 5(0)$ $= -1$	No
4.	$x_2 = 0$ $x_3 = 0$	x_1, x_4 outgoing x_2 incoming x_1	$x_1 + 8x_4 = 2$ $-x_1 + 4x_4 = 1$ $\Rightarrow x_1 = 0, x_4 = \frac{1}{4}$	yes	yes	$2(0) - 2(0) + 4(0)$ $- 5(\frac{1}{4}) = -1.25$	No
5.	$x_1 = 0$ $x_3 = 0$	$x_2\ x_4$ outgoing x_1 incoming x_2	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ Unbounded	—	—	—	—
6.	$x_1 = 0$ $x_2 = 0$	x_3, x_4 outgoing x_2 incoming x_3	$-2x_3 + 8x_4 = 2$ $3x_3 + x_4 = 12$ $x_3 = 0, x_4 = \frac{1}{4}$	yes	yes	-12.5	NO

* Simplex Method :

Note : ① If problem is of Maximisation type then make All $C_j - Z_j \leq 0$

② If probem is of Minimisation type then make All $C_j - Z_j \geq 0$

Example ① solve the following LPP using simplex method

Maximize $Z = 3x_1 + 2x_2$

subject to
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Solution: first we convert given LPP into standard form

we introduce the slack variables S_1, S_2, S_3

∴ The standard form of LPP is

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to
$$3x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 = 18$$
$$x_1 + 0x_2 + 0S_1 + S_2 + 0S_3 = 4$$
$$0x_1 + x_2 + 0S_1 + 0S_2 + S_3 = 6$$

All $x_1, x_2, S_1, S_2, S_3 \geq 0$

* Initial iteration :

C_B	C_j / Basic variable	x_1 (3)	x_2 (2)	S_1 (0)	S_2 (0)	S_3 (0)	Solution	Ratio
0	S_1	3	2	1	0	0	18	$\frac{18}{3} = 6$
0	S_2	1	0	0	1	0	4	$\frac{4}{1} = 4$ ← Min key row
0	S_3	0	1	0	0	1	6	–
	$Z_j = \sum C_B B_j$	0	0	0	0	0		
	$C_j - Z_j$	3	2	0	0	0		

↑ Max. (key column)

here, 1 is pivot element

* First Iteration : (S_2 - outgoing , x_1 - incoming)

	C_B	C_j / Basic variable	x_1 (3)	x_2 (2)	S_1 (0)	S_2 (0)	S_3 (0)	Solution	Ratio
$R_1 - 3R_2$	0	S_1	0	2	1	-3	0	6	$\frac{6}{2} = 3$ ← Min (key row)
	3	x_1	1	0	0	1	0	4	–
	0	S_3	0	1	0	0	1	6	$\frac{6}{1} = 6$
		Z_j	3	0	0	3	0		
		$C_j - Z_j$	0	2	0	-3	0		

↑ Max (key column

∴ 2 is pivot element.

* Second Iteration :

(s_1 - outgoing, x_2 - incoming)

	C_j	3	2	0	0	0		
C_B	Basic Variable	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
2	x_2	0	1	$\frac{1}{2}$	$-\frac{3}{2}$	0	3	
3	x_1	1	0	0	1	0	4	
$R_3 - \frac{1}{2}R_1$ 0	S_3	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	3	
	z_j	3	2	1	0	0		
	$c_j - z_j$	0	0	-1	0	0		

here we can observe that

All $z_j - z_j \leq 0$

therefore,

$x_1 = 4$, $x_2 = 3$ is solution

and $Z = 3x_1 + 2x_2$

$= 3(4) + 2(3)$

$= 18$

$\therefore$ $\boxed{x_1 = 4 \;, \; x_2 = 3 \quad \text{and} \quad Z_{max} = 18}$

is Required solution of LPP

Ex 2: Solve the following L.P.P using simplex method.

$$\text{Maximise } z = 6x_1 - 2x_2 + 3x_3$$

Subject to
$$2x_1 - x_2 + 2x_3 \leq 2$$
$$x_1 + 4x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

Solution: First we convert given LPP into standard form

∴ we introduce the two slack variable s_1, s_2

∴ standard form of LPP is

$$\text{Maximize } Z = 6x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2$$

Subject to
$$2x_1 - x_2 + 2x_3 + s_1 + 0s_2 = 2$$
$$x_1 + 0x_2 + 4x_3 + 0s_1 + s_2 = 4$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

* Initial iteration:

C_B	C_j	6	-2	3	0	0		
	Basic variable	x_1	x_2	x_3	s_1	s_2	Solution	Ratio
0	s_1	2	-1	2	1	0	2	$\frac{2}{2} = 1$ ← Min
0	s_2	1	0	4	0	1	4	$\frac{4}{1} = 4$
	z_j	0	0	0	0	0		
	$C_j - z_j$	6 ↑	-2	3	0	0		

Maxi (key column)

∴ 2 is pivot element

* first Iteration : (s_1 - outgoing , x_1 - incoming)

	C_B	c_j / Basic variable	6 x_1	−2 x_2	3 x_3	0 s_1	0 s_2	solution	Ratio
$\frac{R_1}{2}$	6	x_1	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{-\frac{1}{2}} = -2$
$R_2 - \frac{1}{2}R_1$	0	s_2	0	$\frac{1}{2}$	3	$-\frac{1}{2}$	1	3	$\frac{3}{\frac{1}{2}} = 6$ ← Min (key row)
		z_j	6	−3	6	3	0		
		$c_j - z_j$	0	1	−3	−3	0		

↑ Max (key column)

$\therefore$ $\frac{1}{2}$ is pivot element

* second iteration:

(s_2 - outgoing , x_2 - incoming)

	C_B	c_j / Basic variable	6 x_1	−2 x_2	3 x_3	0 s_1	0 s_2	solution	Ratio.
$R_1 - R_2$	6	x_1	1	0	4	0	1	4	
$R_2 \times 2$	−2	x_2	0	1	6	−1	2	6	
		z_j	6	−2	12	2	10		
		$c_j - z_j$	0	0	−9	−2	−10		

here, All $c_j - z_j \leq 0$

$\therefore$ solution is $x_1 = 4$, $x_2 = 6$, $x_3 = 0$

$\therefore$ $Z = 6x_1 - 2x_2 + 3x_3$

$= 6(4) - 2(6) + 3(0)$

$= 12$

$\therefore$ optimal solution is $x_1 = 4$, $x_2 = 6$, $x_3 = 0$, $Z_{max} = 12$

Ex (3) Solve the following LPP using simplex method

Minimise $z = x_1 - 3x_2 + x_3$

Subject $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \geq -12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \geq 0$

* Duality : The phenomenon occuring in linear programming that given a problem there exists another closely related problem with the same set of data and with the same solution is called the duality.

* Working Rule to write duel from primal

1> convert the inequality of All constraints to either '$\geq$' or '$\leq$' (by using (-1))

2> if primal is of minimisation type then convert it to maximisation and vice versa.

Ex ① write the dual of following LPP.

Maximise $z = 2x_1 - x_2 + 4x_3$

subject to $x_1 + 2x_2 - x_3 \le 5$

$2x_1 - x_2 + x_3 \le 6$

$x_1 + x_2 + 3x_3 \le 10$

$4x_1 + x_3 \le 12$

$x_1, x_2, x_3 \ge 0$

Solution: Given LPP can be written as

Maximise $z = 2x_1 - x_2 + 4x_3$

subject to $x_1 + 2x_2 - x_3 \le 5$

$2x_1 - x_2 + x_3 \le 6$

$x_1 + x_2 + 3x_3 \le 10$

$4x_1 + 0x_2 + x_3 \le 12$

$x_1, x_2, x_3 \ge 0$

Therefore the dual of given LPP is

Minimise $z = 5y_1 + 6y_2 + 10y_3 + 12y_4$

subject to $y_1 + 2y_2 + y_3 + 4y_4 \ge 2$

$2y_1 - y_2 + y_3 + 0y_4 \ge -1$

$-y_1 + y_2 + 3y_3 + y_4 \ge 4$

Ex② obtain the dual of the following LPP

Minimise $z = 3x_1 - 2x_2 + x_3$

subject to
$$2x_1 - 3x_2 + x_3 \le 5$$
$$4x_1 - 2x_2 \ge 9$$
$$-8x_1 + 4x_2 + 3x_3 = 8$$
$$x_1, x_2 \ge 0, \quad x_3 \text{ unrestricted.}$$

Solution: given LPP can be written as

Minimise $z = 3x_1 - 2x_2 + x_3$

subject to
$$-2x_1 + 3x_2 - x_3 \ge -5$$
$$4x_1 - 2x_2 + 0x_3 \ge 9$$
$$-8x_1 + 4x_2 + 3x_3 \ge 8$$
$$8x_1 - 4x_2 - 3x_3 \ge -8$$
$$x_1, x_2 \ge 0, \quad x_3 \text{ unrestricted}$$

since, x_3 is unrestricted

$\therefore$ we put $x_3 = x_3' - x_3''$, $x_3' \ge 0$, $x_3'' \ge 0$

$\therefore$ Minimise $z = 3x_1 - 2x_2 + x_3' - x_3''$

subject to
$$-2x_1 + 3x_2 - x_3' + x_3'' \ge -5$$
$$4x_1 - 2x_2 + 0x_3' - 0x_3'' \ge 9$$
$$-8x_1 + 4x_2 + 3x_3' - 3x_3'' \ge 8$$
$$8x_1 - 4x_2 - 3x_3' + 3x_3'' \ge -8$$

with All $x_1, x_2, x_3', x_3'' \ge 0$

$\therefore$ The Dual of given LPP is

Maximise $z = -5y_1 + 9y_2 + 8y_3' - 8y_3''$

subject to
$$-2y_1 + 4y_2 - 8y_3' + 8y_3'' \le 3$$
$$3y_1 - 2y_2 + 4y_3' - 4y_3'' \le -2$$
$$-y_1 + 0y_2 + 3y_3' - 3y_3'' \le 1$$
$$y_1 + 0y_2 - 3y_3' + 3y_3'' \le -1$$

All $y_1, y_2, y_3', y'' \ge 0$

It can be written as (by Replacing $y_3' - y_3''$ by y_3)

Maximise $w = -5y_1 + 9y_2 + 8y_3$

subject to
$$-2y_1 + 4y_2 - 8y_3 \leq 3$$
$$3y_1 - 2y_2 + 4y_3 \leq -2$$
$$-y_1 + 3y_3 = 1$$
$$y_1, y_2 \geq 0, \quad y_3 \text{ unrestricted.}$$

Ex. 3 obtain the dual of following LPP

Maximise $z = -3x_1 - 2x_2$

Subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

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* Dual simplex method to solve the LPP:

Ex① Use the dual simplex method to solve the LPP.

minimize $z = 6x_1 + x_2$

Subject to $2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$

Solution: The dual of given LPP is

Maximize $z = -6x_1 - x_2$

Subject to $-2x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq 0$

$\therefore$ The standard form of above LPP is

Maximise $z = -6x_1 - x_2 + 0s_1 + 0s_2$

Subject to $-2x_1 - x_2 + s_1 + 0s_2 = -3$

$-x_1 + x_2 + 0s_1 + s_2 = 0$

$x_1, x_2, s_1, s_2 \geq 0$

Initial iteration:

C_B	C_j / Basic variable	-6 / x_1	-1 / x_2	0 / s_1	0 / s_2	Solution	Ratio
0	s_1	-2	-1	1	0	-3	$\frac{-3}{-1} = 3$ ← Min (key Row)
0	s_2	-1	1	0	1	0	$\frac{0}{1} = 0$
	Z_j	0	0	0	0		
	$C_j - Z_j$	-6	-1	0	0		

↑ Max (key column)

-1 is pivot element.

First Iteration:

(s_1 - outgoing, x_2 - incoming)

C_B	C_j / Basic variable	-6 / x_1	-1 / x_2	0 / s_1	0 / s_2	Solution	Ratio
-1	x_2	2	1	-1	0	3	$\frac{3}{2} = 1.5$
0	s_2	-3	0	1	1	-3	$\frac{-3}{-3} = 1$ ← Min key row
	Z_j	-2	-1	1	0		
	$C_j - Z_j$	-4	0	-1	0		

↑ Max (key column)

$\therefore$ -3 is pivot element.

* Second iteration:

(s_2 - outgoing , x_2 - incoming)

	C_B	C_j / Basic variable	-6 x_1	-1 x_2	0 s_1	0 s_2	solution	Ratio
$R_1 + \frac{2}{3}R_2$	-6	x_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	1	
$\frac{R_2}{-3}$	-1	x_1	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1	
		z_j	-1	-6	$\frac{7}{3}$	$\frac{-11}{3}$		
		$c_j - z_j$	-5	5	$-\frac{7}{3}$	$\frac{11}{3}$		

$\therefore$ $x_1 = 1, \; x_2 = 1$ is solution of given LPP

and $z = 6x_1 + x_2$

$= 6(1) + 1$

$= 7$

$\therefore$ optimal solution is

$x_1 = 1, \; x_2 = 1, \; Z_{min} = 7$

HW.

Ex. 2. Use simplex method to solve the following LPP.

minimize $Z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

* Big - M method : (charne's method)

<u>Note</u> ! In this method we use Artificial. Variable denoted as A_i $(i=1,2,\dots)$

<u>Ex ①</u> Use Big-M method to solve the following LPP.

Minimise $Z = 2x_1 + 3x_2$

Subject to $x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

$x_1, x_2 \geq 0$

<u>Solution</u>: we introduce the surplus variable s_1, s_2 are artifficial variable A_1, A_2

∴ The standard form of given LPP is

Minimise $Z = 2x_1 + 3x_2 + 0s_1 + 0s_2$

s.t $x_1 + x_2 - s_1 + 0s_2 + A_1 + 0A_2 = 5$

$x_1 + 2x_2 + 0s_1 - s_2 + 0A_1 + A_2 = 6$

All $x_1, x_2, s_1, s_2 \geq 0$

Initial iteration:

C_B	C_j	2	3	0	0	M	M		
	Basic variable	x_1	x_2	s_1	s_2	A_1	A_2	solution	Ratio
M	A_1	1	1	-1	0	1	0	5	$\frac{5}{1} = 5$
M	A_2	1	2	0	-1	0	1	6	$\frac{6}{2} = 3$ ← min (key row)
	Z_j	2M	3M	-M	-M	M	M		
	$C_j - Z_j$	2-2M	3-3M	M	M	0	0		

↑ max. (key column)

first iteration:

(A_2 - outgoing , x_2 - incoming)

	C_B	C_j / Basic variable	2 x_1	3 x_2	0 s_1	0 s_2	M A_1	M A_2	sol^n	Ratio
$R_1 - \frac{1}{2}R_2$	M	A_1	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	2	4 ← Min (key row)
$\frac{R_2}{2}$	3	x_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	6
		Z_j	$\frac{M}{2}+\frac{3}{2}$	3	$-M$	$\frac{M}{2}-\frac{3}{2}$	M	$-\frac{M}{2}+\frac{1}{2}$		
		$C_j - Z_j$	$\frac{1}{2}-\frac{M}{2}$	0	M	$\frac{3}{2}-\frac{M}{2}$	0	$\frac{3M}{2}-\frac{1}{2}$		

↑ Max (key column)

$\therefore$ $\frac{1}{2}$ is pivot element

* second iteration: (A_1 - outgoing , x_1 - incoming)

	C_B	C_j / Basic variable	2 x_1	3 x_2	0 s_1	0 s_2	M A_1	M A_2	Solution	Ratio.
$2R_1$	2	x_1	1	0	-2	1	2	-1	4	
$R_2 - R_1$	3	x_2	0	1	1	-1	-1	1	1	
		Z_j	2	3	-1	-1	1	1		
		$C_j - Z_j$	0	0	1	1	$M-1$	$M-1$		

$\therefore$ $x_1 = 4$, $x_2 = 1$ is feasible solution

Now
$$Z = 2x_1 + 3x_2$$
$$= 2(4) + 3(1)$$
$$= 8 + 3$$
$$= 11$$

$\therefore$ optimal solution are

$x_1 = 4$, $x_2 = 1$, $Z_{min} = 11$

(H.W)

Ex (2) use Big-M method to solve the following LPP.

Minimise $z = 2x_1 + x_2$

subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

(Hint : use two Artificial variable A_1, A_2 for first & second constraint.)

Module 6. Nonlinear programming problems

* **Definition:**

An optimisation problem in which either the objective function and/or some or all constraints are non-linear is called a non-linear programing problem.

for ex. (1) optimise $z = x_1^2 + x_2^2 + x_3^2 - 100$

subject to
$$x_1 + x_2^2 + x_3 \leq 10$$
$$x_1^2 - x_3 \geq 20$$
$$x_1^2 + x_1 x_2 + x_3 = 35$$
$$x_1, x_2 \geq 0$$

* Note that If $y = f(x)$ is differentiable function then $f'(x) = 0$ given the stationary points say $x = x_0$.

- If $f''(x_0) > 0$ then x_0 is a minima
- if $f''(x_0) < 0$ then x_0 is a maxima
- if x_0 is nither minima nor maxima then x_0 is inflection point (saddle point)

* NLPP with one equality constraint using the method of Lagrange's multipliers :

- Consider the non linear programming problem

optimise $z = f(x_1, x_2, \cdots, x_n)$

subject to $g(x_1, x_2, \cdots, x_n) = b$

$x_1, x_2, \cdots, x_n \geq 0$

given NLPP can be written as

optimise $z = f(x_1, x_2, \cdots, x_n)$

subject to $h(x_1, x_2, \cdots, x_n) = g(x_1, x_2, \cdots, x_n) - b = 0$

we construct a new function

$$L(x_1, x_2, \cdots x_n, \lambda) \equiv f(x_1, x_2, \cdots x_n) - \lambda h(x_1, x_2, \cdots, x_n)$$

is called Lagrangian function and

λ is called Lagrangian multiplier.

<u>step 1</u> : consider, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\cdots$ $\frac{\partial L}{\partial x_n} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

solve this system of equations for $x_1, x_2, \cdots x_n$

say $X_0 = (x_1^0, x_2^0, \cdots, x_n^0)$

then X_0 is said to be stationary point.

Step 2 find All Δ_{n+1}

where,

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \cdots & \cdots & & \cdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{vmatrix}$$

Step 3: If the signs of all principle minor $\Delta_3, \Delta_4, \cdots$ are alternatively positive and negative (i.e. $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0$, $\cdots$) then the point X_0 is Maxima

and If All the principle minor $\Delta_3, \Delta_4, \cdots$ are negative then X_0 is minima

Step 4: find $f(X_0)$

Examples:

Ex. 1 Using the method of lagranges multiplier, solve the following NLPP.

optimise $Z = 6x_1^2 + 5x_2^2$

Subject to $x_1 + 5x_2 = 7$

$x_1, x_2 \geq 0$

Solution: here, $f(x_1, x_2) = 6x_1^2 + 5x_2^2$

$h(x_1, x_2) = x_1 + 5x_2 - 7$

$\therefore$ The Lagrangian function is

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\, h(x_1, x_2)$$

i.e $$L(x_1, x_2, \lambda) = 6x_1^2 + 5x_2^2 - \lambda(x_1 + 5x_2 - 7)$$

Step 1. consider, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$\Rightarrow \quad 12x_1 - \lambda = 0, \quad 10x_2 - 5\lambda = 0, \quad x_1 + 5x_2 - 7 = 0$

$\Rightarrow \quad \lambda = 12x_1, \quad \lambda = 2x_2, \quad x_1 + 5x_2 = 7$

$\Rightarrow \quad 12x_1 = 2x_2, \quad x_1 + 5x_2 = 7$

$\Rightarrow \quad 6x_1 - x_2 = 0$

$x_2 + 5x_2 = 7$

$\Rightarrow \quad \boxed{x_1 = \frac{7}{31}, \quad x_2 = \frac{42}{31}}$

since, $\lambda = 12x_1 \Rightarrow \lambda = 12\left(\frac{7}{31}\right) = \frac{84}{31}$

$\therefore L(x_1, x_2) = 6x_1^2 + 5x_2^2 - \frac{84}{31}(x_1 + 5x_2 - 7)$

$\Rightarrow L(x_1, x_2) = 6x_1^2 + 5x_2^2 - \frac{84}{31}x_1 - \frac{420}{31}x_2 + \frac{588}{31}$

Step 2

$\therefore$

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial L}{\partial x_1^2} & \frac{\partial L}{\partial x_1 x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial L}{\partial x_2 x_1} & \frac{\partial L}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix}$$

$$= 0\,(120-0) - 1\,(10-0) + 5\,(0-60)$$

$$= -10 - 300$$

$$= -310 < 0$$

$\therefore \quad \Delta_3 < 0$

$\therefore \quad X_0 = \left(\frac{7}{31}, \frac{42}{31}\right)$ is a minima.

Hence, $\quad Z = 6x_1^2 + 5x_2^2$

$$= 6\left(\frac{7}{31}\right)^2 + 5\left(\frac{42}{31}\right)^2$$

$$= \frac{294}{31}$$

$\therefore \quad \boxed{x_1 = \frac{7}{31}, \quad x_2 = \frac{42}{31}, \quad Z_{min.} = \frac{294}{31}}$

Ex. ② Using the method of Lagrange's multiplier, solve the following L.P.P.

optimise $\quad Z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $\quad x_1 + x_2 + x_3 = 7, \quad x_i \geq 0$

Solution: here, $\quad f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

$$h(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 7$$

We construct the Lagrangian function

$$L(x_1, x_2, x_3, \lambda) = (x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3) - \lambda(x_1 + x_2 + x_3 - 7)$$

Step 1. $\quad \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$

$\Rightarrow \quad 2x_1 - 10 - \lambda = 0, \quad 2x_2 - 6 - \lambda = 0, \quad 2x_3 - 4 - \lambda = 0, \quad x_1 + x_2 + x_3 - 7 = 0$

$\Rightarrow$ adding first 3 equation we get

$$2(x_1+x_2+x_3) - 20 - 3\lambda = 0$$

$\Rightarrow$ $2(7) - 20 - 3\lambda = 0$ (using equation ④)

$\Rightarrow$ $14 - 20 - 3\lambda = 0$

$\Rightarrow$ $-6 - 3\lambda = 0$

$\Rightarrow$ $\lambda = -2$

put this in equation ①, ② & ③ we get

$$2x_1 - 10 - (-2) = 0, \quad 2x_2 - 6 - (-2) = 0, \quad 2x_3 - 4 - (-2) = 0$$

$\Rightarrow$ $x_1 = 4, \quad x_2 = 2, \quad x_3 = 1$

$\therefore$ $X_0 = (4, 2, 1)$

Step 2.

Now,

$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 x_2} & \frac{\partial^2 L}{\partial x_1 x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{vmatrix}.$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1\left[1(4-0)-0+0\right]+1\left[-1(4-0)+0-0\right]-1\left[1(4-0)-0+0\right]$$

$$= -4-4-4$$

$$= -12$$

i.e. $\Delta_4 = -12 < 0$

Now consider,

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0-1(2-0)+1(0-2)$$

$$= -2-2$$

$$= -4 < 0$$

i.e $\Delta_4 < 0$, $\Delta_3 < 0$

Hence X_0 is a minima

Now $Z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

$$= (4)^2 + (2)^2 + (1)^2 - 10(4) - 6(2) - 4(1)$$

$$= 16+4+1-40-12-4$$

$$= -35$$

$\therefore$ $\boxed{x_1 = 4,\ x_2 = 2,\ x_3 = 1,\ Z_{min} = -35}$

is Required solution

H.W

EX. 3 Using the method of Lagrange's multipliers solve the following N.L.P.P.

optimise $Z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

Ans: $x_1 = 5,\ x_2 = 3,\ x_3 = 2,\ Z_{max} = 35$

* Non linear programming problem with two equality Constraints

- working Rule:

Step 1. Consider NLPP

optimise $Z = f(x_1, x_2, \cdots x_n)$

subject to $h_1(x_1, x_2, \cdots, x_n) = 0$

$h_2(x_1, x_2, \cdots, x_n) = 0$

$x_1, x_2, \cdots, x_n \geq 0$

construct a Lagrangian equation

$$L(x_1, x_2, \cdots x_n, \lambda_1, \lambda_2) = f(x_1, x_2, \cdots x_n) - \lambda_1 h(x_1, x_2, \cdots x_n) - \lambda_2 h(x_1, x_2, \cdots x_n)$$

Step 2. consider,

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \cdots, \quad \frac{\partial L}{\partial x_n} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0$$

by solving this system, we get $x_1, x_2, \cdots x_n$ values

i.e the stationary point $X_0 = (x_1, x_2, \cdots x_n)$

Step 3:

case i) If number of variable $n > 2$

then consider the matrix

$$H^B = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & Q \end{array}\right]$$

where, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \end{bmatrix}$

P^T = transpose of matrix P and

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

If $\det(H^B) > 0$ then X_0 is Minima

otherwise X_0 is Maxima

Case ii) If number of variables $n = 2$

considered, $H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix}$

If $A_1 = \frac{\partial^2 z}{\partial x_1^2}$ is Negative and $\det(H)$ is positive

then X_0 is Maxima

If Both $A_1 = \frac{\partial^2 z}{\partial x_1^2}$ and $\det(H)$ are Negative

then X_0 is Minima

Step 4: find maximum (Z_{max}) or Minimum (Z_{min}) value.

Ex ① Using the method of Lagrangian multiplier solve the following NLPP.

optimise $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

solution: here, $f(x_1, x_2, x_3) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 15$

$h_2(x_1, x_2, x_3) = 2x_1 - x_2 + 2x_3 - 20$

Step 1. consider, Lagranges equation

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h(x_1, x_2, x_3)$$

$$\Rightarrow \quad L = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

Step 2. consider $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$$\Rightarrow \quad 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{——— ①}$$

$$4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{——— ②}$$

$$2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{——— ③}$$

$$x_1 + x_2 + x_3 = 15 \quad \text{——— ④}$$

$$2x_1 - x_2 + 2x_3 = 20 \quad \text{——— ⑤}$$

multiply ③ by 4 and add to ①

$$8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + 8x_3 - 4\lambda_1 - 8\lambda_2 = 0$$

$$\Rightarrow \quad 4(2x_1 - x_2 + 2x_3) = 5\lambda_1 + 10\lambda_2$$

$$\Rightarrow \quad 4(20) = 5\lambda_1 + 10\lambda_2$$

$$\Rightarrow \quad 5\lambda_1 + 10\lambda_2 = 80 \quad \text{———} \quad (6)$$

Now multiply ① by 2, ② by 3 and ③ by 2 and add

$$16x_1 - 8x_2 - 2\lambda_1 - 4\lambda_2 + 12x_2 - 12x_1 - 3\lambda_1 + 3\lambda_2 + 4x_3 - 2\lambda_1 - 4\lambda_2 = 0$$

$$\Rightarrow \quad 4(x_1 + x_2 + x_3) - 7\lambda_1 - 5\lambda_2 = 0$$

$$\Rightarrow \quad 4(15) - 7\lambda_1 - 5\lambda_2 = 0$$

$$\Rightarrow \quad 7\lambda_1 + 5\lambda_2 = 60 \quad \text{———} \quad (7)$$

solving ⑥ & ⑦ we get

$$\lambda_1 = \frac{40}{9}, \quad \lambda_2 = \frac{52}{9}$$

Now, adding ① & ② we get

$$4x_1 = 2\lambda_1 + \lambda_2$$

$$\Rightarrow \quad 4x_1 = 2\left(\frac{40}{9}\right) + \frac{52}{9}$$

$$\Rightarrow \quad x_1 = \frac{11}{3}$$

Now, multiply ② by 2 and adding in ① we get

$$4x_2 = 3\lambda_1$$

$$x_2 = \frac{3}{4}\left(\frac{40}{9}\right) = \frac{10}{3}$$

Now, from ③, $2x_3 = \lambda_1 + 2\lambda_2$.

$$\Rightarrow \quad 2x_3 = \frac{40}{9} + 2\left(\frac{52}{9}\right)$$

$$\Rightarrow \quad x_3 = 8$$

$\therefore \quad X_0 = (x_1, x_2, x_3) = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is stationary point

Now, to find H^B

Note that $H^B = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & Q \end{array}\right]$

$$* \quad P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$* \quad P^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$* \quad Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \quad H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\therefore \det(H^B) = (-1)^{3+4+1} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$+ (-1)^{3+5+1} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} + (-1)^{4+5+1} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}$$

$\Rightarrow \quad \det(H^B) = -(-3)(6) - (0) - (3)(24)$

$= 54$

$\therefore$ $\det(H^B)$ is positive

$\therefore \quad X_0 = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is Minima

$\therefore \quad z = 4x_1^2 + 2x_2^2 - x_3^2 - 4x_1x_2$

$= 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + (8)^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right)$

$= \frac{820}{9}$

$\therefore$ solution is

$x_1 = \frac{11}{3}$, $x_2 = \frac{10}{3}$, $x_3 = 8$ and $Z_{min} = \frac{820}{9}$

Ex. (2) Using method of Lagranges multiplies solve the following NLPP

Maximize $\quad z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Subject to

$$4x_1 + 3x_2 = 16$$

$$3x_1 + 5x_2 = 15$$

$$x_1, x_2 \geq 0$$

solution: here, $f(x_1, x_2) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$h_1(x_1, x_2) = 4x_1 + 3x_2 - 16$

$h_2(x_1, x_2) = 3x_1 + 5x_2 - 15$

Step 1. construct Lagranges equation

$L = f - \lambda h_1 - \lambda_2 h_2$

$\Rightarrow \quad L = 6x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$

step2. consider $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$$\Rightarrow \quad 6 - 2\lambda_1 - 4\lambda_1 - 3\lambda_2 = 0 \quad \text{— (1)}$$

$$8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0 \quad \text{— (2)}$$

$$4x_1 + 3x_2 = 16 \quad \text{— (3)}$$

$$3x_1 + 5x_2 = 15 \quad \text{— (4)}$$

solving (3) & (4) we get

$$x_1 = \frac{35}{11}, \quad x_2 = \frac{12}{11}$$

$\therefore$ $X_0 = \left(\frac{35}{11}, \frac{12}{11}\right)$ is stationary point

here, number of variable $= 2$

$\therefore$ we consider $H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix}$

$$\Rightarrow \quad H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\therefore$ $A_1 = \frac{\partial^2 z}{\partial x_1^2} = -2$ is negative

and $\det(H) = 4$ is positive

Hence, $X_0 = \left(\frac{35}{11}, \frac{12}{11}\right)$ is maxima

$$\therefore Z_{max} = 6x_1 + 8x_2 - x_1^2 - x_2^2 = 6\left(\frac{35}{11}\right) + 8\left(\frac{12}{11}\right) - \left(\frac{31}{11}\right)^2 - \left(\frac{12}{11}\right)^2$$

$$= 16.504$$

$\therefore$ solution is

$$Z_{max} = 16.504, \quad x_1 = \frac{35}{11}, \quad x_2 = \frac{12}{11}$$

Ex ③ Using the method of Lagrangian multiplies solve the following NLPP.

optimise. $Z = x_1^2 + x_2^2 + x_3^2$

subject to

$$x_1 + x_2 + 3x_3 = 2$$

$$5x_1 + 2x_2 + x_3 = 5$$

$$x_1, x_2, x_3 \geq 0$$

* NLPP with inequality constraint :

* Kuhn - Tucker Conditions :

— For one inequality constraint

consider the NLPP

Maximise $z = f(x_1, x_2, \cdots x_n)$

subject to $h(x_1, x_2, \cdots x_n) \leq 0$

$x_1, x_2, \cdots x_n \geq 0$

* Necessary condition :

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 \quad \text{--- (2)}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0 \quad \text{--- (n)}$$

$$\lambda h(x_1, x_2, \cdots x_n) = 0 \quad \text{--- (n+1)}$$

$$h(x_1, x_2, \cdots x_n) \leq 0 \quad \text{--- (n+2)}$$

$$\lambda \geq 0 \quad \text{--- (n+3)}$$

* Note that If problem is of Minimisation type then only (n+3) condition is change

i.e. $\lambda < 0$

Important : All the constraints in NLPP should be less than or equal to type '$\leq$'

Ex ① Solve the following N.L.P.P.

maximise $z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

Solution: here, $f(x_1, x_2) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$h(x_1, x_2) = 2x_1 + x_2 - 5$

Now, Kuhn-Tucker condition are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\lambda h(x_1, x_2) = 0$$

$$h(x_1, x_2) \leq 0$$

$$\lambda \geq 0$$

therefore,

$$10 - 4x_1 - 2\lambda = 0 \quad \text{—— ①}$$

$$4 - 2x_2 - \lambda = 0 \quad \text{—— ②}$$

$$\lambda(2x_1 + x_2 - 5) = 0 \quad \text{—— ③}$$

$$2x_1 + x_2 - 5 \leq 0 \quad \text{—— ④}$$

$$x_1, x_2, \lambda \geq 0 \quad \text{—— ⑤}$$

using equation ③ we get

either $\lambda = 0$ or $(2x + x_2 - 5) = 0$

case i) if $\lambda = 0$

Then from ① & ②, $10 - 4x_1 = 0 \Rightarrow x_1 = \frac{5}{2}$

and $4 - 2x_2 = 0 \Rightarrow x_2 = 2$

putting these value in equation 4

L.H.S = $2\left(\frac{5}{2}\right)+2-5 = 2 \nleq 0$

$\therefore$ $x_1 = \frac{5}{2}$, $x_2 = 2$ not satisfy all condition of kuhn Tucker

$\therefore$ $\lambda = 0$ not gives the feasible solution

case ii) if $\lambda \neq 0$ then $2x_1 + x_2 - 5 = 0$ —— 6

$\therefore$ equation 1, 2 and 6 can be written as

$$4x_1 + 0x_2 + 2\lambda = 10$$
$$0x_1 + 2x_2 + \lambda = 4$$
$$2x_1 + x_2 + 0\lambda = 0$$

implies that $x_1 = \frac{11}{6}$, $x_2 = \frac{4}{3}$, $\lambda = \frac{4}{3}$ (use calsi)

equation 4 becomes

L.H.S $2\left(\frac{11}{6}\right) + \frac{4}{3} - 5 = 0 \leq 0$

Hence, $x_1 = \frac{11}{6}$ and $x_2 = \frac{4}{3}$ satisfy all Necessary condition of kuhn Tucker

$\therefore$ The optimal solution is

$$x_1 = \frac{11}{6}, \quad x_2 = \frac{4}{3}$$

and $Z_{max} = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

$$= 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2$$

$$= \frac{91}{6}$$

$\therefore$ $\boxed{x_1 = \frac{11}{6}, \quad x_2 = \frac{4}{3}, \quad Z_{max} = \frac{91}{6}}$

Ex ② Use the kuhn-Tucker condition to solve the following NLPP.

Minimise $z = x_1^3 - 4x_1 - 2x_2$

subject to $x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Solution: here, $f(x_1, x_2) = x_1^3 - 4x_1 - 2x_2$

$h(x_1, x_2) = x_1 + x_2 - 1$

The kuhn-Tucker condition for minima are

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$, $\lambda h(x_1, x_2) = 0$,

$h(x_1, x_2) \le 0$, $\lambda < 0$

implies that

$3x_1^2 - 4 - \lambda = 0$ —— ①

$-2 - \lambda = 0$ —— ②

$\lambda(x_1 + x_2 - 1) = 0$ —— ③

$x_1 + x_2 - 1 \le 0$ —— ④

$\lambda < 0$ —— ⑤

$\therefore$ from ②, $\lambda = -2$ $(\because \lambda < 0)$

$\therefore$ from ①, $3x_1^2 - 4 + 2 = 0 \Rightarrow x_1 = \sqrt{\frac{2}{3}}$

from ③, $-2\left(\sqrt{\frac{2}{3}} + x_2 - 1\right) = 0 \Rightarrow x_2 = 1 - \sqrt{\frac{2}{3}}$

equation ④ becomes. L.H.S $= \sqrt{\frac{2}{3}} + 1 - \sqrt{\frac{2}{3}} - 1 = 0 \le 0$

That is the value $x_1 = \sqrt{\frac{2}{3}}$ and $x_2 = 1 - \sqrt{\frac{2}{3}}$ satisfy all condition of kuhn-Tucker

Hence, optional solution are

$x_1 = \sqrt{\frac{2}{3}}$, $x_2 = 1 - \sqrt{\frac{2}{3}}$

and $Z_{min} = \left(\sqrt{\frac{2}{3}}\right)^3 - 4\left(\sqrt{\frac{2}{3}}\right) - 2\left(1 - \sqrt{\frac{2}{3}}\right) = -3.093$

H.W

Ex 3 Use the kuhn Tucker condition to solve the following NLPP

Maximise $Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Subject to $2x_1 + 5x_2 \leq 98$

$x_1, x_2 \geq 0$

Ans: $x_1 = 44$
$x_2 = 2$
$Z_{max} = 4900$

Ex 4 Use kuhn - Tucker condition to solve the following N.L.P.P.

Maximise $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Ans: $x_1 = \frac{4}{13}$
$x_2 = \frac{33}{13}$
$Z_{max} = 21.3$

— self learning

* The Kuhn - Tucker condition for General NLPP.

Consider the NLPP.

Maximise $z = f(x_1, x_2, \cdots x_n)$

subject to
$$h_1(x_1, x_2, \cdots x_n) \leq 0$$
$$h_2(x_1, x_2, \cdots x_n) \leq 0$$
$$\vdots$$
$$h_m(x_1, x_2, \cdots x_n) \leq 0$$
$$x_1, x_2, \cdots x_n \geq 0$$

* Necessary condition

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_2} = 0$$
$$\vdots \quad \cdots\cdots$$
$$\frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial h_1}{\partial x_n} - \lambda_2 \frac{\partial h_2}{\partial x_n} - \cdots - \lambda_m \frac{\partial h_m}{\partial x_n} = 0$$

$$\lambda_1 h_1(x_1, x_2, \cdots x_n) = 0$$
$$\lambda_2 h_2(x_1, x_2, \cdots x_n) = 0$$
$$\vdots$$
$$\lambda_m h_m(x_1, x_2, \cdots x_n) = 0$$
$$\lambda_1, \lambda_2, \cdots, \lambda_m \geq 0$$

Note that if the problem is of Minimisation Type then

$$\lambda_1, \lambda_2, \cdots, \lambda_m < 0$$

Important: All the constraint in NLPP should be of less than or equal to type '$\leq$'

Ex ① Using the kuhn - Tucker condition, solve the following N.L.P.P.

Maximise $z = x_1^2 + x_2^2$

subject to $x_1 + x_2 - 4 \leq 0$

$2x_1 + x_2 - 5 \leq 0$

$x_1, x_2 \geq 0$

solution: here, $f(x_1, x_2) = x_1^2 + x_2^2$

& $h_1(x_1, x_2) = x_1 + x_2 - 4$

$h_2(x_1, x_2) = 2x_1 + x_2 - 5$

Note that the kuhn - Tucker conditions

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$\lambda_1 h_1(x_1, x_2) = 0$$

$$\lambda_2 h_2(x_1, x_2) = 0$$

$$h_1(x_1, x_2) \leq 0$$

$$h_2(x_1, x_2) \leq 0$$

$$\lambda_1, \lambda_2 \geq 0$$

implies that

$2x_1 - \lambda_1 - 2\lambda_2 = 0$ — ①

$2x_1 - \lambda_1 - \lambda_2$ — ②

$\lambda_1 (x_1 + x_2 - 4) = 0$ — ③

$\lambda_2 (2x_1 - x_2 - 5) = 0$ — ④

$x_1 + x_2 - 4 \leq 0$ — ⑤

$2x_1 - x_2 - 5 \leq 0$ — ⑥

$x_1, x_2 \ \lambda_1, \lambda_2 \geq 0$ — ⑦

we consider the cases depends on λ_1, λ_2

case i) If $\lambda_1 = 0$, $\lambda_2 = 0$

from ① & ② $x_1 = 0$, $x_2 = 0$

which is trivial solution

case ii) If $\lambda_1 = 0$, $\lambda_2 \neq 0$

from ① and ② $2x_1 = 2\lambda$ and $2x_2 = \lambda_2$

$\Rightarrow$ $x_1 = \lambda_2$ and $2x_2 = \lambda_2$

and from ④, $2x_1 + x_2 = 5$

$\therefore$ $2(\lambda_2) + \frac{\lambda_2}{2} = 5 \Rightarrow \lambda_2 = 5$

$\therefore$ $x_1 = 5$ and $x_2 = \frac{5}{2}$

from eq^n ③ L.H.S $5 + \frac{5}{2} - 4 \nleq 0$

$\therefore$ $x_1 = 5$ and $x_2 = \frac{5}{2}$ cannot be feasible solution

case iii) If $\lambda_1 \neq 0$, $\lambda_2 = 0$

from ① & ②, $2x_1 = \lambda_1$, $2x_2 = \lambda_1$ $\Rightarrow$ $x_1 = x_2$

from ③ $x_1 + x_2 = 4$

$\Rightarrow$ $x_1 + x_1 = 4$ $\Rightarrow$ $x_1 = 2$

$\Rightarrow$ $x_2 = 2$

clearly, $x_1 = 2$, $x_2 = 2$ satisfy all condition

$\therefore$ $Z_{max} = x_1^2 + x_2^2 = 4 + 4 = 8$

$\therefore$ optional solution is

$$\boxed{x_1 = 2, \quad x_2 = 2 \quad \text{and} \quad Z_{max} = 8}$$