* Cayley - Hamilton Theorem :

Statement :

* Every square matrix satisfy its charecterstic equation.

Note that:

① If $P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$ is charecterstic equation of square matrix A

then $P(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$

Examples on cayley-Hamilton theorem

Ex ① Verify Cayley-Hamilton theorem for the matrix A and hence find A^{-1}, A^{-2} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution: * for the charecterstic equation :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)(1-\lambda) - 0\right] - 2\left[-1(1-\lambda) - 0\right] - 2\left[2 - 0\right] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 4\lambda + 3) + 2(1-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda + 2 - 2\lambda - 4 = 0$$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 9\lambda + 1 = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$

which is charecterstic equation of A

Now. By Cayley - Hamilton theorem,

A satisfy its charectestic equation.

$\therefore$ To show: $A^3 - 5A^2 + 9A - I = 0$ ——— (1)

$$* \; A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$* \; A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Now we consider,

$$A^3 - 5A^2 + 9A - I$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

that is $A^3 - 5A^2 + 9A - I = 0$

therefore, A satisfy its charecterstic equation

Hence, Cayley - Hamilton theorem is varified.

Now first to find A^{-1}:

Since, $A^3 - 5A^2 + 9A - I = 0$

Now, Multiplying both side by A^{-1} we get

$$A^3 \cdot A^{-1} - 5A^2 A^{-1} + 9AA^{-1} - IA^{-1} = 0 \cdot A^{-1}$$

$$\Rightarrow \quad A^2 - 5A + 9I - A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = A^2 - 5A + 9I \qquad \text{——— (2)}$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now To find : A^{-2}

Since, $A^3 - 5A^2 + 9A - I = 0$

Now multiplying both side by A^{-2}, we get

$$A^3 A^{-2} - 5A^2 A^{-2} + 9AA^{-2} - A^{-2} = 0 \cdot A^{-2}$$

$$\Rightarrow \quad A - 5I + 9A^{-1} - A^{-2} = 0$$

$$\Rightarrow \quad A^{-2} = A - 5I + 9A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 27 & 18 & 54 \\ 9 & 9 & 18 \\ 18 & 18 & 45 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

Now to find A^4 :

since, $A^3 - 5A^2 + 9A - I = 0$

multiplying both side by A, we get

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$\Rightarrow \quad A^4 = 5A^3 - 9A^2 + A$$

$$= 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} + \begin{bmatrix} 9 & -108 & 36 \\ 36 & -63 & -18 \\ -18 & 72 & -9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

Ex. (2) verify cayley - Hamilton theorem and hence find the matrix represented by

$$A^6 - 6A^5 + 9A^5 + 4A^3 - 12A^2 + 2A - I$$

where $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

solution: The charecterstic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(-3-\lambda)(7-\lambda)+20\right]-10\left[-2(7-\lambda)+12\right]+5\left[-10-3(-3-\lambda)\right]=0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2-4\lambda-1\right]+20(7-\lambda)-120+5(3\lambda-1)=0$$

$$\Rightarrow 3\lambda^2-12\lambda-3-\lambda^3+4\lambda^2+\lambda+140-20\lambda-120+15\lambda-5=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-16\lambda+12=0$$

$$\Rightarrow \lambda^3-7\lambda^2+16\lambda-12=0$$

which is charecterstic equation of A

Note that cayley-Hamilton theorem states that A satisfy its charecterstic equation

that is $A^3-7A^2+16A-12I=0$

Now, $$A^2=\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}=\begin{bmatrix}4&25&10\\-2&-31&-26\\20&50&44\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4&25&10\\-2&-31&-26\\20&50&44\end{bmatrix}\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}=\begin{bmatrix}-8&15&-10\\-52&-157&-118\\92&270&208\end{bmatrix}$$

considered, $A^3-7A^2+16A-12I$

$$=\begin{bmatrix}-8&15&-10\\-52&-157&-118\\92&270&208\end{bmatrix}-7\begin{bmatrix}4&25&10\\-2&-31&-26\\20&50&44\end{bmatrix}+16\begin{bmatrix}3&10&5\\-2&-3&-4\\3&5&7\end{bmatrix}-12\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}-8&15&-10\\-52&-157&-118\\92&270&208\end{bmatrix}+\begin{bmatrix}-28&-175&-70\\14&217&182\\-140&-350&-308\end{bmatrix}+\begin{bmatrix}48&160&80\\-32&-48&-64\\48&80&112\end{bmatrix}+\begin{bmatrix}-12&0&0\\0&-12&0\\0&0&-12\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

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that is A satisfy its charecteristic equation

Hence, cayley-Hamilton theorem is varified

Now To find: $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$

since, $A^3 - 7A^2 + 16A - 12I = 0$ ——— (1)

considerd, $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$

$$= (A^3 - 7A^2 + 16A - 12I)A^3 + (A^3 - 7A^2 + 16A - 12I)A^2$$

$$+ 2A - I$$

$$= (0)A^3 + (0)A^2 + 2A - I \qquad \text{(using (1))}$$

$$= 2A - I$$

$$= 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 20 & 10 \\ -4 & -6 & -8 \\ 6 & 10 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

$$\therefore A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$