Module 6. Nonlinear programming problems

* **Definition:**

An optimisation problem in which either the objective function and/or some or all constraints are non-linear is called a non-linear programing problem.

For ex. ① optimise $z = x_1^2 + x_2^2 + x_3^2 - 100$

subject to

$$x_1 + x_2^2 + x_3 \le 10$$

$$x_1^2 - x_3 \ge 20$$

$$x_1^2 + x_1 x_2 + x_3 = 35$$

$$x_1, x_2 \ge 0$$

* Note that If $y = f(x)$ is differentiable function then $f'(x) = 0$ given the stationary points say $x = x_0$.

- If $f''(x_0) > 0$ then x_0 is a minima
- if $f''(x_0) < 0$ then x_0 is a maxima
- if x_0 is nither minima nor maxima then x_0 is inflection point (saddle point)

* NLPP with <u>one</u> equality constraint using the method of Lagrange's multipliers :

- Consider the non linear programming problem

optimise $z = f(x_1, x_2, \cdots, x_n)$

subject to $g(x_1, x_2, \cdots, x_n) = b$

$x_1, x_2, \cdots, x_n \geq 0$

given NLPP can be written as

optimise $z = f(x_1, x_2, \cdots, x_n)$

subject to $h(x_1, x_2, \cdots, x_n) = g(x_1, x_2, \cdots, x_n) - b = 0$

we construct a new function

$$L(x_1, x_2, \cdots x_n, \lambda) \equiv f(x_1, x_2, \cdots x_n) - \lambda h(x_1, x_2, \cdots, x_n)$$

is called Lagrangian function and

λ is called Lagrangian multiplier.

<u>Step 1</u> : consider, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\cdots$ $\frac{\partial L}{\partial x_n} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

solve this system of equations for $x_1, x_2, \cdots x_n$

say $X_0 = (x_1^0, x_2^0, \cdots, x_n^0)$

then X_0 is said to be stationary point.

Step 2 find All Δ_{n+1}

where,

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \cdots & \cdots & & \cdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{vmatrix}$$

Step 3: If the signs of all principle minor $\Delta_3, \Delta_4, \cdots$ are alternatively positive and negative (i.e. $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0$, $\cdots$) then the point X_0 is Maxima

and If All the principle minor $\Delta_3, \Delta_4, \cdots$ are negative then X_0 is minima

Step 4: find $f(X_0)$

Examples:

Ex. 1 Using the method of lagranges multiplier, solve the following NLPP.

optimise $Z = 6x_1^2 + 5x_2^2$

Subject to $x_1 + 5x_2 = 7$

$x_1, x_2 \geq 0$

<u>solution:</u> here, $f(x_1, x_2) = 6x_1^2 + 5x_2^2$

$h(x_1, x_2) = x_1 + 5x_2 - 7$

$\therefore$ The Lagrangian function is

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2)$$

i.e $$L(x_1, x_2, \lambda) = 6x_1^2 + 5x_2^2 - \lambda(x_1 + 5x_2 - 7)$$

<u>step 1.</u> consider, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$\Rightarrow$ $12x_1 - \lambda = 0$, $10x_2 - 5\lambda = 0$, $x_1 + 5x_2 - 7 = 0$

$\Rightarrow$ $\lambda = 12x_1$, $\lambda = 2x_2$, $x_1 + 5x_2 = 7$

$\Rightarrow$ $12x_1 = 2x_2$, $x_1 + 5x_2 = 7$

$\Rightarrow$ $6x_1 - x_2 = 0$

$x_2 + 5x_2 = 7$

$\Rightarrow$ $\boxed{x_1 = \frac{7}{31}, \quad x_2 = \frac{42}{31}}$

since, $\lambda = 12x_1 \Rightarrow \lambda = 12\left(\frac{7}{31}\right) = \frac{84}{31}$

$\therefore$ $L(x_1, x_2) = 6x_1^2 + 5x_2^2 - \frac{84}{31}(x_1 + 5x_2 - 7)$

$\Rightarrow$ $L(x_1, x_2) = 6x_1^2 + 5x_2^2 - \frac{84}{31}x_1 - \frac{420}{31}x_2 + \frac{588}{31}$

<u>step 2</u>

$\therefore$

$$\Delta_3 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial L}{\partial x_1^2} & \frac{\partial L}{\partial x_1 x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial L}{\partial x_2 x_1} & \frac{\partial L}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 5 \\ 1 & 12 & 0 \\ 5 & 0 & 10 \end{vmatrix}$$

$= 0\,(120-0) - 1\,(10-0) + 5(0-60)$

$= -10 - 300$

$= -310 < 0$

$\therefore \quad \Delta_3 < 0$

$\therefore \quad X_0 = \left(\frac{7}{31}, \frac{42}{31}\right)$ is a minima

Hence, $Z = 6x_1^2 + 5x_2^2$

$= 6\left(\frac{7}{31}\right)^2 + 5\left(\frac{42}{31}\right)^2$

$= \frac{294}{31}$

$$\therefore \quad \boxed{x_1 = \frac{7}{31}, \quad x_2 = \frac{42}{31}, \quad Z_{min.} = \frac{294}{31}}$$

Ex. 2 Using the method of Lagrange's multiplier, solve the following L.P.P.

optimise $Z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $x_1 + x_2 + x_3 = 7$, $x_i \geq 0$

Solution: here, $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

$h(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 7$

we construct the Lagrangian function

$L(x_1, x_2, x_3, \lambda) = (x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3) - \lambda(x_1 + x_2 + x_3 - 7)$

Step 1: $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$\Rightarrow \quad 2x_1 - 10 - \lambda = 0$, $2x_2 - 6 - \lambda = 0$, $2x_3 - 4 - \lambda = 0$, $x_1 + x_2 + x_3 - 7 = 0$

$\Rightarrow$ adding first 3 equation we get

$$2(x_1+x_2+x_3) - 20 - 3\lambda = 0$$

$\Rightarrow$ $2(7) - 20 - 3\lambda = 0$ (using equation ④)

$\Rightarrow$ $14 - 20 - 3\lambda = 0$

$\Rightarrow$ $-6 - 3\lambda = 0$

$\Rightarrow$ $\lambda = -2$

put this in equation ①, ② & ③ we get

$$2x_1 - 10 - (-2) = 0, \quad 2x_2 - 6 - (-2) = 0, \quad 2x_3 - 4 - (-2) = 0$$

$\Rightarrow$ $x_1 = 4$, $x_2 = 2$, $x_3 = 1$

$\therefore$ $X_0 = (4, 2, 1)$

step 2.

Now,

$$\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 x_2} & \frac{\partial^2 L}{\partial x_1 x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1\left[1(4-0)-0+0\right]+1\left[-1(4-0)+0-0\right]-1\left[1(4-0)-0+0\right]$$

$$= -4-4-4$$

$$= -12$$

i.e. $\Delta_4 = -12 < 0$

Now consider,

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0-1(2-0)+1(0-2)$$

$$= -2-2$$

$$= -4 < 0$$

i.e $\Delta_4 < 0$, $\Delta_3 < 0$

Hence X_0 is a minima

Now $z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

$$= (4)^2 + (2)^2 + (1)^2 - 10(4) - 6(2) - 4(1)$$

$$= 16+4+1-40-12-4$$

$$= -35$$

$\therefore$ $\boxed{x_1 = 4,\ x_2 = 2,\ x_3 = 1,\quad z_{min} = -35}$

is Required solution

H.W

EX. 3 Using the method of Lagrange's multipliers solve the following N.L.P.P.

optimise $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

Ans: $x_1 = 5$, $x_2 = 3$, $x_3 = 2$, $z_{max} = 35$