* Non linear programming problem with two equality Constraints

- working Rule:

Step 1. Consider NLPP

optimise $Z = f(x_1, x_2, \cdots x_n)$

subject to $h_1(x_1, x_2, \cdots, x_n) = 0$

$h_2(x_1, x_2, \cdots, x_n) = 0$

$x_1, x_2, \cdots, x_n \geq 0$

construct a Lagrangian equation

$$L(x_1, x_2, \cdots x_n, \lambda_1, \lambda_2) = f(x_1, x_2, \cdots x_n) - \lambda_1 h(x_1, x_2, \cdots x_n) - \lambda_2 h(x_1, x_2, \cdots x_n)$$

Step 2. consider,

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \cdots, \quad \frac{\partial L}{\partial x_n} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0$$

by solving this system, we get $x_1, x_2, \cdots x_n$ values

i.e the stationary point $X_0 = (x_1, x_2, \cdots x_n)$

Step 3:

case i) If number of variable $n > 2$

then consider the matrix

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right]$$

Scanned by CamScanner

where, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \end{bmatrix}$

P^T = transpose of matrix P and

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

If $\det(H^B) > 0$ then X_0 is Minima

otherwise X_0 is Maxima

Case ii) If number of variables $n = 2$

considered, $H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix}$

If $A_1 = \frac{\partial^2 z}{\partial x_1^2}$ is Negative and $\det(H)$ is positive

then X_0 is Maxima

If Both $A_1 = \frac{\partial^2 z}{\partial x_1^2}$ and $\det(H)$ are Negative

then X_0 is Minima

Step 4: find maximum (Z_{max}) or Minimum (Z_{min}) value.

Ex ① Using the method of Lagrangian multiplier solve the following NLPP.

optimise $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Solution: here, $f(x_1, x_2, x_3) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 15$

$h_2(x_1, x_2, x_3) = 2x_1 - x_2 + 2x_3 - 20$

Step 1. consider, Lagranges equation

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h(x_1, x_2, x_3)$$

$$\Rightarrow \quad L = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

Step 2. consider $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$$\Rightarrow \quad 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{—— ①}$$

$$4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{—— ②}$$

$$2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{—— ③}$$

$$x_1 + x_2 + x_3 = 15 \quad \text{—— ④}$$

$$2x_1 - x_2 + 2x_3 = 20 \quad \text{—— ⑤}$$

multiply ③ by 4 and add to ①

$$8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + 8x_3 - 4\lambda_1 - 8\lambda_2 = 0$$

$$\Rightarrow \quad 4(2x_1 - x_2 + 2x_3) = 5\lambda_1 + 10\lambda_2$$

$$\Rightarrow \quad 4(20) = 5\lambda_1 + 10\lambda_2$$

$$\Rightarrow \quad 5\lambda_1 + 10\lambda_2 = 80 \quad \text{———} \quad ⑥$$

Now multiply ① by 2, ② by 3 and ③ by 2 and add

$$16x_1 - 8x_2 - 2\lambda_1 - 4\lambda_2 + 12x_2 - 12x_1 - 3\lambda_1 + 3\lambda_2 + 4x_3 - 2\lambda_1 - 4\lambda_2 = 0$$

$$\Rightarrow \quad 4(x_1 + x_2 + x_3) - 7\lambda_1 - 5\lambda_2 = 0$$

$$\Rightarrow \quad 4(15) - 7\lambda_1 - 5\lambda_2 = 0$$

$$\Rightarrow \quad 7\lambda_1 + 5\lambda_2 = 60 \quad \text{———} \quad ⑦$$

solving ⑥ & ⑦ we get

$$\lambda_1 = \frac{40}{9}, \quad \lambda_2 = \frac{52}{9}$$

Now, adding ① & ② we get

$$4x_1 = 2\lambda_1 + \lambda_2$$

$$\Rightarrow \quad 4x_1 = 2\left(\frac{40}{9}\right) + \frac{52}{9}$$

$$\Rightarrow \quad x_1 = \frac{11}{3}$$

Now, multiply ② by 2 and adding in ① we get

$$4x_2 = 3\lambda_1$$

$$x_2 = \frac{3}{4}\left(\frac{40}{9}\right) = \frac{10}{3}$$

Now, from ③, $2x_3 = \lambda_1 + 2\lambda_2$.

$$\Rightarrow \quad 2x_3 = \frac{40}{9} + 2\left(\frac{52}{9}\right)$$

$$\Rightarrow \quad x_3 = 8$$

$\therefore \quad X_0 = (x_1, x_2, x_3) = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is stationary point

Now, to find H^B

Note that $$H^B = \left[\begin{array}{c|c} 0 & P \\ \hline P^T & Q \end{array}\right]$$

$$* \quad P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$* \quad P^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$* \quad Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\therefore \det(H^B) = (-1)^{3+4+1} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$+ (-1)^{3+5+1} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & -4 \\ 1 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix} + (-1)^{4+5+1} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & 8 \\ 1 & -1 & -4 \\ 1 & 2 & 0 \end{vmatrix}$$

$\Rightarrow \quad \det(H^B) = -(-3)(6) - (0) - (3)(24)$

$= 54$

$\therefore \quad \det(H^B)$ is positive

$\therefore \quad X_0 = \left(\frac{11}{3}, \frac{10}{3}, 8\right)$ is Minima

$\therefore \quad z = 4x_1^2 + 2x_2^2 - x_3^2 - 4x_1x_2$

$= 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + (8)^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right)$

$= \frac{820}{9}$

$\therefore$ solution is

$x_1 = \frac{11}{3}, \; x_2 = \frac{10}{3}, \; x_3 = 8$ and $Z_{min} = \frac{820}{9}$

Ex.(2) Using method of Lagranges multiplies solve the following NLPP

Maximize $\quad z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Subject to

$4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

Solution: here, $f(x_1, x_2) = 6x_1 + 8x_2 - x_1^2 - x_2^2$

$h_1(x_1, x_2) = 4x_1 + 3x_2 - 16$

$h_2(x_1, x_2) = 3x_1 + 5x_2 - 15$

Step 1. construct Lagranges equation

$L = f - \lambda h_1 - \lambda_2 h_2$

$\Rightarrow \quad L = 6x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$

step 2. consider $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$$\Rightarrow \quad 6 - 2\lambda_1 - 4\lambda_1 - 3\lambda_2 = 0 \quad \text{——} ①$$

$$8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0 \quad \text{——} ②$$

$$4x_1 + 3x_2 = 16 \quad \text{——} ③$$

$$3x_1 + 5x_2 = 15 \quad \text{——} ④$$

solving ③ & ④ we get

$$x_1 = \frac{35}{11}, \quad x_2 = \frac{12}{11}$$

$\therefore$ $X_0 = \left(\frac{35}{11}, \frac{12}{11}\right)$ is stationary point

here, number of variable = 2

$\therefore$ we consider $H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix}$

$$\Rightarrow \quad H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\therefore$ $A_1 = \frac{\partial^2 z}{\partial x_1^2} = -2$ is negative

and $\det(H) = 4$ is positive

Hence, $X_0 = \left(\frac{35}{11}, \frac{12}{11}\right)$ is maxima

$$\therefore \quad Z_{max} = 6x_1 + 8x_2 - x_1^2 - x_2^2 = 6\left(\frac{35}{11}\right) + 8\left(\frac{12}{11}\right) - \left(\frac{31}{11}\right)^2 - \left(\frac{12}{11}\right)^2$$

$$= 16.504$$

$\therefore$ solution is

$$Z_{max} = 16.504, \quad x_1 = \frac{35}{11}, \quad x_2 = \frac{12}{11}$$

Ex ③ Using the method of Lagrangian multiplies solve the following NLPP.

optimise. $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$