Ex. ② Find the z-transform of $k\, a^k$, $k \geq 0$

Solution: here, $f(k) = k\, a^k$, $k \geq 0$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} k\, a^k z^{-k} \qquad (\because k \geqslant 0)$$

$$= \sum_{k=0}^{\infty} k \frac{a^k}{z^k}$$

$$= \sum_{k=0}^{\infty} k \left(\frac{a}{z}\right)^k$$

$$= 0 \cdot \left(\frac{a}{z}\right)^0 + 1\left(\frac{a}{z}\right) + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left(1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{a}{z}\left[\frac{1}{\left(1 - \frac{a}{z}\right)^2}\right], \quad \left|\frac{a}{z}\right| < 1$$

$$\left(\because 1 + 2z + 3z^2 + \cdots = \frac{1}{(1-z)^2}, \quad |z| < 1\right)$$

$$\therefore Z\{f(k)\} = \frac{a}{z\left(1 - \frac{a}{z}\right)^2}, \quad |a| < |z|$$

$$\boxed{Z\{k\, a^k\} = \frac{a}{z\left(1 - \frac{a}{z}\right)^2}}, \quad |z| > |a|$$

In perticular,

if $a = 1$, then $\boxed{Z\{k\} = \frac{z}{(z-1)^2}}$, $|z| > 1$

if $k = 1$, $a = 1$ then $\boxed{Z\{1\} = \frac{z}{z-1}}$, $|z| > 1$

Ex ③ find $Z\{a^{|k|}\}$

Solution: Note that $a^{|k|} = \begin{cases} a^{k} & , k \geq 0 \\ a^{-k} & , k < 0 \end{cases}$

$$\therefore Z\{a^{|k|}\} = \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} z^{-k}$$

$$= \left[\cdots + a^{-(-3)} z^{-(-3)} + a^{-(-2)} z^{-(-2)} + a^{-(-1)} z^{-(-1)}\right]$$

$$+ \left[1 + a^{1} z^{-1} + a^{2} z^{-2} + a^{3} z^{-3} + \cdots\right]$$

$$= \left[\cdots + a^{3} z^{3} + a^{2} z^{2} + az\right] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right]$$

$$= \left[az + (az)^2 + (az)^3 + \cdots\right] + \left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= az\left[1 + (az) + (az)^2 + \cdots\right] + \left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= az\left[\frac{1}{1-(az)}\right] + \left[\frac{1}{1-\left(\frac{a}{z}\right)}\right], \quad |az| < 1, \quad \left|\frac{a}{z}\right| < 1$$

$$\left(\because 1 + z + z^2 + \cdots = \frac{1}{1-z}, \quad |z| < 1\right)$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \quad |z| < \left|\frac{1}{a}\right|, \quad |a| < |z|$$

$$\Rightarrow \boxed{Z\{a^{|k|}\} = \frac{z(1-a^2)}{(1-az)(z-a)}}, \quad a < |z| < \frac{1}{a}$$

$\therefore$ ROC is $\underline{a < |z| < \left(\frac{1}{a}\right)}$

* **some standard Z - transform :**

① $Z\{1\} = \frac{z}{z-1}$, $|z|>1$

② $z\{k\} = \frac{z}{(z-1)^2}$, $|z|>1$

③ $z\{ka^k\} = \frac{az}{(z-a)^2}$, $|z|>|a|$, $k\geq 0$

④ $z\{{}^nC_k\} = \left(1+\frac{1}{z}\right)^n$, $0\leq k\leq n$, $|z|>0$

⑤ $z\{{}^kC_n\} = z^{-n}\left(1-\frac{1}{z}\right)^{-(n+1)}$, $|z|>1$

⑥ $z\{{}^{k+n}C_n\} = \left(1-\frac{1}{z}\right)^{-(n+1)}$, $|z|>1$

⑦ $z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$, $|a|<|z|<\frac{1}{|a|}$

⑧ $z\{a^k\} = \frac{z}{z-a}$, $|z|>|a|$, $k\geq 0$

⑨ $z\{a^k\} = \frac{z}{a-z}$, $|z|<|a|$, $k<0$

⑩ $z\left\{\frac{a^k}{k!}\right\} = e^{\frac{a}{z}}$, $k\geq 0$ ROC : All z-plane

⑪ $z\{\cos(ak+b)\} = \frac{z(z\cos b - \cos(a-b))}{z^2 - 2z\cos a + 1}$, $|z|>1$

⑫ $z\{\sin(ak+b)\} = \frac{z[\sin(a-b) + z\sin b]}{z^2 - 2z\cos a + 1}$, $|z|>1$

⑬ $z\{\cosh ak\} = \frac{z(z-\cosh a)}{z^2 - 2z\cosh a + 1}$, All z- plane

⑭ $z\{\sinh ak\} = \frac{z\sinh a}{z^2 - 2z\cosh a + 1}$, All z- plane

(15) $Z\{\delta(k)\} = 1$, RoC : All z-plane

where, $\delta(k)$ is dirac-delta function

(16) $Z\{U(k)\} = \frac{z}{z-1}$, $|z| > 1$

where $U(k)$ is unit step function

Ex. find the z-transform of $\left(\frac{1}{3}\right)^{|k|}$

Solution: Note that $\left(\frac{1}{3}\right)^{|k|} = \begin{cases} \left(\frac{1}{3}\right)^{k}, & k \geq 0 \\ \left(\frac{1}{3}\right)^{-k}, & k < 0 \end{cases}$

Now, $$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[\cdots + \left(\frac{1}{3}\right)^{3} z^{3} + \left(\frac{1}{3}\right)^{2} z^{2} + \left(\frac{1}{3}\right) z\right]$$

$$+ \left[1 + \left(\frac{1}{3}\right) z^{-1} + \left(\frac{1}{3}\right)^{2} z^{-2} + \cdots\right]$$

$$= \left[\frac{z}{3} + \left(\frac{z}{3}\right)^{2} + \left(\frac{z}{3}\right)^{3} + \cdots\right]$$

$$+ \left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \frac{z}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^{2} + \cdots\right]$$

$$+ \left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \frac{z}{3}\left(\frac{1}{1-\left(\frac{z}{3}\right)}\right) + \left(\frac{1}{1-\frac{1}{3z}}\right), \quad \left|\frac{z}{3}\right|<1, \ \left|\frac{1}{3z}\right|<1$$

$$\left(\because \ 1+z+z^2+\cdots = \frac{1}{1-z}, \quad |z|<1\right)$$

$$= \frac{z}{3}\cdot\frac{3}{3-z} + \frac{3z}{3z-1}, \quad |z|<3, \ \frac{1}{3}<|z|$$

$$= \frac{z}{3-z} + \frac{3z}{3z-1}, \quad \frac{1}{3}<|z|<3$$

$$\Rightarrow \quad Z\left\{\left(\tfrac{1}{3}\right)^{|k|}\right\} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3}<|z|<3$$

Homework :

Ex. ① find $Z\{3^{|k|}\}$

Ex. ② find $Z\left\{\frac{\alpha^k}{k}\right\}, \quad k>1$

Ans: ① $\frac{-8z}{(1-3z)(z-3)}$ ② $-\log\left(1-\frac{\alpha}{z}\right)$