Module 1.

Linear Algebra (Theory of Matrices)

* **Matrix:** A matrix is a set of mn numbers arranged in m rows and n columns. It is called an $m \times n$ matrix.

Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

we denote this matrix by $A = [a_{ij}]_{m \times n}$

* **square matrix:** If the number of rows of matrix is equal to the number of columns i.e. if $m = n$, then the matrix is called a square matrix.

for example: $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

are the square matrices of order 2 and 3

* Diagonal elements : In a square matrix the elements lying along the diagonal of matrix. i.e the elements a_{ii} are called diagonal elements of the matrix.

for example: In the matrices $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 2 \\ 2 & -1 & -3 \end{bmatrix}$

2, -1 and 1, 2, -3 are the diagonal elements

* Diagonal matrix : A square matrix whose all non-diagonal elements are zero is called a diagonal matrix. i.e. All $a_{ij} = 0$, for $i \neq j$

for example. $[2]$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all diagonal matrices

* Trace of Matrix : The sum of all diagonal elements of a square matrix is called the trace of a matrix. It is Denoted by 'tr(A)'

i.e. if $A = [a_{ij}]_{n \times n}$ then $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

for example : ① If $A = [2]$ then $tr(A) = 2$

② If $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ then $tr(B) = 3 + 2 = 5$

③ If $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 2 \\ 0 & 5 & -2 \end{bmatrix}$ then $tr(C) = 1 + 3 + (-2) = 2$

* Singular and non-singular Matrix :

let A be the square matrix

If determinant of A is zero (i.e. $|A| = 0$) then A is called singular matrix.

If determinant of A is non-zero (i.e. $|A| \neq 0$) then A is called non-singular matrix.

for examples :

$[0]$, $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ are singular matrices and

$[2]$, $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are non-singular matrices

* Transpose of a matrix :

A matrix obtained from a given matrix A by interchanging rows and columns is called transpose of a given matrix and is denoted by A^T or A'

for example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

* Traingular matrices :

A matrix $A = [a_{ij}]$ is said to be Upper triangular if $a_{ij} = 0$ for all $i > j$ and is said to be lower triangular if $a_{ij} = 0$ for all $i < j$

for example: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ are upper triangular

and $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ are lower triangular.

* Symmetric Matrix :

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j (i.e. $A^T = A$)

for example, $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ are symmetric.

Note that : A is symmetric if $A^T = A$

I) **Eigenvalues** :

Let A be any square matrix of order n, λ be any scalar and I be the unit matrix of order n then

* The determinant $|A - \lambda I|$ is called charecterstic polynomial of λ. and
* The Equation $|A - \lambda I| = 0$ is called charecterstic equation of the matrix A
* The roots of the charecterstic equation $|A - \lambda I| = 0$ is called Eigenvalues of matrix A

Note that : Eigenvalues is also called as charecterstics value or charecterstics roots

Example: 1. find charecterstic equation and eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

solution: Given: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

* For charecterstic equation :

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

* For Eigenvalues :

consider, $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda^2 - 5\lambda + \lambda - 5 = 0$

$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$

$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda = 5, -1$

Therefore, 5, -1 are the eigenvalues of matrix A

<u>Example.</u> (2) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

<u>solution:</u> Given: $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

* for Eigenvalues:

consider, the charectestic equation

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(-3-\lambda)(1-\lambda) - 8] + 8[4(1-\lambda) + 6] - 2[-16 - 3(-3-\lambda)] = 0$$

$$\Rightarrow (8-\lambda)[-3 + 2\lambda + \lambda^2 - 8] + 8[10 - 4\lambda] - 2[-7 + 3\lambda] = 0$$

$$\Rightarrow \quad (8-\lambda)(\lambda^2+2\lambda-11)+80-32\lambda+14-6\lambda=0$$

$$\Rightarrow \quad 8\lambda^2+16\lambda-88-\lambda^3-2\lambda^2+11\lambda+94-38\lambda=0$$

$$\Rightarrow \quad -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\Rightarrow \quad \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\Rightarrow \quad \lambda = 1, 2, 3$$

Therefore, 1, 2, 3 are the Eigenvalues of A.

Example (3) Find eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: Given: $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

* for eigenvalues :

consider, the charecterstics equation

$$|A-\lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[(2-\lambda)(2-\lambda)-1]+1[1(2-\lambda)+1] + 1[-1-(2-\lambda)] = 0$$

$$\Rightarrow \quad \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\Rightarrow \quad \lambda = 1, 2, 3$$

Therefore, 1, 2, 3 are the eigenvalues of A