* Diagonalizable and non- diagonalizable matrices :-

let A be the square matrix of order n then A is digonalizable if there exist a non- singular matrix p such that the matrix $P^{-1}AP$ is diagonal matrix

Note that If A is not diagonalizable then A is non- diagonalizable

* Important properties :

Let A be any square matrix of order 3 and $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A

— Algebric multiplicity of eigenvalue :-

The Number of Repitition of the eigenvalue is called Algebric multiplicity

— Geometric multiplicity :

The Number of Linearly independent eigenvector corresponding to λ eigenvalue is called geometric multiplicity.

— If Algebric multiplicity = Geometric multiplicity for every eigenvalues of A then A is diagonalizable.

Note that: If All eigenvalues of A are distinct then A is digonalizable.

Example ① show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

is diagonalisable. find the diagonal form D and the diagonalising matrix M

solution: Given matrix is $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

* For eigenvalues :

consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-9-\lambda)[(3-\lambda)(7-\lambda)-12] - 4[-8(7-\lambda)+64] + 4[-64+16(3-\lambda)] = 0$$

$$\Rightarrow (-9-\lambda)[\lambda^2 - 7\lambda + 16] - 4[-\lambda + 8] + 4[16 - 16\lambda] = 0$$

$$\Rightarrow -9\lambda^2 + 63\lambda - 144 - \lambda^3 + 7\lambda^2 - 16\lambda + 4\lambda - 32 + 64 - 64\lambda = 0$$

$$\Rightarrow -\lambda^3 - 2\lambda^2 - 13\lambda - 112 = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 + 13\lambda + 112 = 0$$

$$\Rightarrow \lambda = -1, -1, 3$$

$\therefore$ The eigenvalues of A are $-1, -1, 3$

* <u>for eigenvector</u> :

<u>for $\lambda = -1$</u>

consider $[A - \lambda I]X = 0$

$$\Rightarrow [A - (-1)I]X = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $-8x_1 + 4x_2 + 4x_3 = 0$

$$\Rightarrow 2x_1 - x_2 - x_3 = 0$$

here, x_2 and x_3 are non leading

$\therefore$ we put $x_2 = s$, $x_3 = t$.

$\therefore$ $2x_1 - s - t = 0$

$$\Rightarrow x_1 = \frac{s+t}{2} = \frac{s}{2} + \frac{t}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{s+t}{2} \\ s \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The curresponding to $\lambda = -1$ eigenvectors are $v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$

for $\lambda = 3$

consider, $[A - \lambda I]X = 0$

$\Rightarrow [A - 3I]X = 0$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[-\frac{R_2}{4}]{-\frac{R_1}{4}} \begin{bmatrix} 3 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $3x_1 - x_2 - x_3 = 0$ — ①

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ — ②

from ①, $3x_1 - x_1 - x_3 = 0 \Rightarrow 2x_1 - x_3 = 0$ — ③

Now here x_2 is non leading

$\therefore$ we put $x_2 = t$

$\therefore$ $x_1 = t$ and $x_3 = 2t$ (using ② & ③)

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ corresponding to $\lambda = 3$, the eigenvector is

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ The matrix P is

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

clearly Algebric multiplicity for $\lambda = -1$ is 2
geometric multiplicity for $\lambda = -1$ is 2
$\|^{ly}$ Algebric multiplicity for $\lambda = 3$ is 1
geometric multiplicity for $\lambda = 3$ is 1

$\therefore$ A is diagonalizable
and it can be write as

$$A = P^{-1} D P \quad , \text{ where } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$