* Eigenvector :

Suppose, A is square matrix and λ_1 is a eigenvalues of A then a non zero column matrix X is said to be eigenvector if $[A - \lambda_1 I] X = 0$

* Working Rule for Eigenvectors :

Suppose, A is matrix of order n

i> find the eigenvalues of matrix A says. $\lambda_1, \lambda_2, \cdots, \lambda_n$

ii> for $\lambda = \lambda_i$, $i = 1, 2, \cdots, n$

consider the system $[A - \lambda_i I] X = 0$

iii> Reduced the matrix $[A - \lambda_i I]$ to echolon form by using elementary row transformation

iv> write the system of equations and find X_i

Example ①

find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

solution:

* for eigenvalues:

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)(2-\lambda)-1] + 1[1(2-\lambda)+1] + 1[-1-1(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)[\lambda^2 - 4\lambda + 3] + 3 - \lambda + \lambda - 3 = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

∴ The eigenvalues of matrix A are 1, 2, 3

Now, To find eigenvector:

i) for $\lambda = \lambda_1 = 1$

$$[A - \lambda_1 I] X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ By Row echolon form of matrix.

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} x_1 - x_2 + x_3 = 0 \\ 2x_2 - 2x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} x_1 - x_2 + x_3 = 0 & -① \\ x_2 - x_3 = 0 & -② \end{array}$$

Clearly, x_3 is non-leading coefficient

$\therefore$ we put $x_3 = t$

$\Rightarrow$ $x_2 = t$, from ②

and $x_1 = 0$, from ①

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = X_1$

ii) for $\lambda = \lambda_2 = 2$.

$$[A - \lambda_2 I] X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_{12}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{matrix} x_1 - x_3 = 0 \\ -x_2 + x_3 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}\right\}$

clearly, x_3 is non-leading

$\therefore$ we put $x_3 = t$

$\Rightarrow$ $x_2 = t$ and $x_1 = t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, currespondiny to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = X_2$

iii) for $\lambda = \lambda_3 = 3$

$$[A - \lambda_3 I] X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{matrix} -x_1 - x_2 + x_3 = 0 \\ -2x_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 + x_2 - x_3 = 0 \\ x_2 = 0 \end{matrix}$

$\Rightarrow$ $x_1 - x_3 = 0$

$\therefore$ x_3 is non-leading

$\therefore$ put $x_3 = t$ $\Rightarrow$ $x_1 = t$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = X_3$

Hence, eigenvalues of A are $1, 2, 3$

and eigenvectors of A are $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Ex. (2) find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: for eigenvalues:

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - 2\left[1(2-\lambda) - 1\right] + 1\left[2 - 1(3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)\left[\lambda^2 - 5\lambda + 4\right] - 2(1-\lambda) + \lambda - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\Rightarrow \lambda = 1, 1, 5$$

$\therefore$ eigenvalues of A are $1, 1, 5$

Now To find eigenvectors:

i> for $\lambda = \lambda_1 = 1$

$$[A - \lambda_1 I] X = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_1 + 2x_2 + x_3 = 0$

$\therefore$ here, x_2 and x_3 are non-leading coefficient

$\therefore$ we put $x_2 = s$, $x_3 = t$

then $x_1 = -2s - t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Curresponding to $\lambda = 1$, the eigenvectors are $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

ii> for $\lambda = \lambda_2 = 5$

$$[A - \lambda_2 I] X = 0$$

$$\Longrightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_{13}} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ -4x_2 + 4x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$

here, x_3 is non-leading

$\therefore$ we put $x_3 = t$

$\Rightarrow x_2 = t$

and $x_1 = t$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ corresponding to $\lambda = 5$, the eigenvector is $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Hence, the eigenvalues of A are 1, 1, 5

and the eigenvectors of A are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example ③ find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

solution: for eigenvalues :

consider, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(2-\lambda)(2-\lambda) - 0] - 1[0-0] + 0[0-0] = 0$$

$$\Rightarrow \quad (2-\lambda)\left[(2-\lambda)^2\right] = 0$$

$$\Rightarrow \quad (2-\lambda)^3 = 0$$

$$\Rightarrow \quad \lambda = 2, 2, 2$$

$\therefore$ eigenvalues of A are $2, 2, 2$

Now. To find the eigenvector:

<u>for $\lambda = 2$</u>

$$[A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $x_2 = 0$ and $x_3 = 0$

and x_1 is free variable

$\therefore$ we put $x_1 = t$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ currresponding to eigenvalues $\lambda = 2$,

the eigenvector is $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

<u>Homework</u>:

<u>Que</u>. find the eigenvalues and eigenvectors of the following

1) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ 3) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$