* Convolution for Z-transform :- (Convolution Theorem)

— let $\{f(k)\}$ and $\{g(k)\}$ be two sequences such that $Z\{f(k)\} = F(z)$ and

$Z\{g(k)\} = G(z)$

then

$$\boxed{Z\{f(k) * g(k)\} = F(z) \cdot G(z)}$$

where, $h(k) = f(k) * g(k) = \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)$

Proof: Note that $H(z) = Z\{h(k)\}$

$$= Z[\{f(k)\} * \{g(k)\}]$$

$$= Z\left[\sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\right]$$

$$= \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\, z^{-k}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(m) \cdot g(k-m)\, z^{-k+m-m}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{k=-\infty}^{\infty} g(k-m)\, z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{p=-\infty}^{\infty} g(p) \cdot z^{-p}$$

where $p = k-m$

$$= F(z) \cdot G(z)$$

$\Rightarrow Z\{f(k) * g(k)\} = F(z) \cdot G(z)$

Example 1. If $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$, then find the Z-transform of $\{f(k) * g(k)\}$

solution: Note that $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$

and $U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$\Rightarrow \quad \{f(k)\} = \{4^0, 4^1, 4^2, \cdots\}$ and

$\{g(k)\} = \{5^0, 5^1, 5^2, \cdots\}$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$$

$$= 4^0 z^0 + 4^1 z^{-1} + 4^2 z^{-2} + \cdots$$

$$= 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \left(\frac{4}{z}\right)}, \quad \left|\frac{4}{z}\right| < 1$$

$$\left(\because 1 + z + z^2 + \cdots = \frac{1}{1-z}, \ |z| < 1\right)$$

$$= \frac{z}{z-4}, \quad 4 < |z|$$

$$= F(z)$$

and
$$Z\{g(k)\} = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= 5^0 z^0 + 5^1 z^{-1} + 5^2 z^{-2} + \cdots$$

$$= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \left(\frac{5}{z}\right)}, \quad \left|\frac{5}{z}\right| < 1$$

$$= \frac{z}{z-5}, \quad 5 < |z|$$

$$= G(z)$$

$\therefore$ By Convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

$$= \frac{z}{z-4} \cdot \frac{z}{z-5}, \quad 5 < |z|$$

$$= \frac{z^2}{(z-4)(z-5)}, \quad |z| > 5$$

Ex. ②. find $Z\{f(k) * g(k)\}$

if $f(k) = \frac{1}{3^k}$ and $g(k) = \frac{1}{5^k}$, $k \geq 0$

Solution: here $f(k) = \frac{1}{3^k}$

$$\Rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{3^k} z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(3z)^k} = \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^k$$

$$= \frac{1}{1} + \frac{1}{3z} + \left(\frac{1}{3z}\right)^2 + \cdots$$

$$= \frac{1}{1-\left(\frac{1}{3z}\right)}, \quad \left|\frac{1}{3z}\right| < z$$

$$\left(\because 1+z+z^2+\cdots = \frac{1}{1-z}, \quad |z|<1\right)$$

$$= \frac{3z}{3z-1} \quad , \quad \frac{1}{3} < |z|$$

$$= F(z)$$

and $\quad g(k) = \frac{1}{5^k}$

$$\therefore \quad Z\{g(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{5^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{5z}\right)^k$$

$$= 1 + \frac{1}{5z} + \left(\frac{1}{5z}\right)^2 + \cdots$$

$$= \frac{1}{1-\left(\frac{1}{5z}\right)} \quad , \quad \left|\frac{1}{5z}\right| < 1$$

$$= \frac{5z}{5z-1} \quad , \quad \frac{1}{5} < |z|$$

$$= G(z)$$

$\therefore$ By convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

$$= \frac{3z}{3z-1} \cdot \frac{5z}{5z-1} \quad , \quad \frac{1}{5} < \frac{1}{3} < |z|$$

$$= \frac{15z^2}{(3z-1)(5z-1)} \quad , \quad \frac{1}{3} < |z|$$

<u>Homework!</u> find $Z\{f(k) * g(k)\}$

if $f(k) = \frac{1}{5^k}$ and $g(k) = \frac{1}{7^k}$