Module 5. Linear programming problems (LPP)

* General Linear programming problem:

Maximize (minimize) $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ —— (1)

subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\le b_1 \ (\ge b_1) \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\le b_2 \ (\ge b_2) \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\le b_m \ (\ge b_m) \end{aligned}\right\} \text{—— (2)}$$

and $x_1, x_2, \dots x_n \ge 0$ (or Unristricted) —— (3)

Note that * (1) is called objective function

* (2) are called constraints
* (3) are called Non-negativity restrictions.

* solution of Linear programming problem (LPP)

Any set of values $x_1, x_2, \dots x_n$ which satisfy the all constraints is called a solution of LPP.

* Feasible solution of LPP:

Any solution which satisfy the given non-negativity restriction is called feasible solution.

* optimal solution of LPP:

Any feasible solution which satisfy the objective function is called optimal solution of LPP.

* slack Variable :

If the constraints of LPP are of less than or equal to type ($\leq$) then we can add new non-negative variables so that the constraints can be expressed as equalities.

therefore that new non-negative variables are called as slack variables.

for ex.. ① suppose $x_1 + 2x_2 + x_3 \leq 5$

then we add s_1 ($s_1 \geq 0$) we get $x_1 + 2x_2 + x_3 + s_1 = 5$

* Surplus Variable :

If the constraints of LPP are of greater than or equal to type (≥ 0) then we can substract new non-negative variable so that the constraints can be expressed as equalities.

therefore, that new non-negative variable are called as surplus variables.

for ex. ① suppose, $x_1 - 2x_2 + 4x_3 \geq 7$

then we substract s_1 ($s_1 \geq 0$) we get $x_1 - 2x_2 + 4x_3 - s_1 = 7$

* Canonical and standard form of LPP :-

A general LPP

$$\text{Maximise } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m$$

with $x_i \ge 0$, $i = 1, 2, \cdots, n$

is called canonical form

And If we indroduce slack variables then the LPP

$$\text{Maximise } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + 0s_1 + 0s_2 + \cdots + 0s_m$$

subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + s_1 + 0s_2 + \cdots + 0s_m = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n + 0s_1 + s_2 + 0s_3 + \cdots + 0s_m = b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n + 0s_1 + \cdots + 0s_{m-1} + s_m = b_m$$

with $x_1, x_2, \cdots x_n, s_1, s_2, \cdots s_m \ge 0$

is called standard form

for examples :

EX. ① Convert the following LPP in the standard form.

$$\text{Maximise } z = 3x_1 + 5x_2$$

subject to

$$3x_1 + 2x_2 \le 15$$

$$2x_1 + 5x_2 \ge 12$$

$$x_1, x_2 \ge 0$$

Sol^n we introduce slack variable s_1 and surplus variable s_2 then the problem can be converted to standard form as

Maximise $Z = 3x_1 + 5x_2 + 0s_1 + 0s_2$

subject to $3x_1 + 2x_2 + s_1 + 0s_2 = 15$

$2x_1 + 5x_2 + 0s_1 - s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$

Ex ② convert the following LPP to the standard form

Maximise $z = 3x_1 + 2x_2 + 5x_3$

subject to $2x_1 - 3x_2 \leq 3$

$x_1 + 2x_2 + 3x_3 \geq 5$

$3x_1 + 2x_3 \leq 2$

$x_1, x_2 \geq 0$

Solution: we introduce slack variable s_1, s_3 and surplus variable s_2

Also here, x_3 is unrestricted

$\therefore$ we put $x_3 = x_3' - x_3''$ and $x_3', x_3'' \geq 0$

$\therefore$ the standard form of LPP is

Maximize $z = 3x_1 + 2x_2 + 5x_3' - 5x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 - 3x_2 + 0x_3' - 0x_3'' + s_1 + 0s_2 + 0s_3 = 3$

$x_1 + 2x_2 + 3x_3' - 3x_3'' + 0s_1 - s_2 + 0s_3 = 5$

$3x_1 + 0x_2 + 2x_3' - 2x_3'' + 0s_1 + 0s_2 + s_3 = 2$

with $x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0$

* Simplex Method :-

- Types of solution:

1. Basic solution: A solution obtained by setting any n variables out of $m+n$ variables equal to zero and solving for remaining m variable, provided the determinant of the coefficient of these m variables is non zero is called a basic solution

   Such m variable are called basic variables and the remaining n zero-valued variables are called non-basic variables

2. Basic feasible solution :

   A basic solution which also satisfies non-negativity restrictions is called basic feasible solution.

Note that : In the basic feasible solution obtained

i) All m values of basic variables are positive then it is non-degenerate basic F.S

ii) one or more values of m basic variable are zero then it is degenerate basic F.S.

Ex 1 Given that

Maximise $z = x_1 + 3x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$

find all basic solutions to the above problem. which of them are basic feasible, non-degenerate, infeasible basic and optimal basic feasible solution?

solution:

No. of Basic solutions	Non-basic variables = 0	Basic Variables	Equations And the values of the basic variables	Is the solution feasible?	Is the solution degenerate	value of z	Is the solution optimal?
1.	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\Rightarrow x_1 = 2, x_2 = 1$	yes	No.	$2+3(1)+0$ $= 5$	yes
2.	$x_2 = 0$	x_1, x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\Rightarrow x_1 = 1, x_2 = 1$	yes	No	$1+3(0)+3(1)$ $= 4$	No
3.	$x = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\Rightarrow x_2 = -1, x_3 = 2$	No	No	-	-

Note that! In the second solution, x_2 is the outgoing variable and x_3 is incoming variable Similarly, in the third solution x_1 is outgoing and x_2 is incoming variable

Ex. ② consider the following problem

Maximise $z = 2x_1 - 2x_2 + 4x_3 - 5x_4$

subject to $x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$

$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$

$x_1, x_2, x_3, x_4 \ge 0$

determine i) All basic solution

ii) All feasible basic solution,

iii) optimal feasible basic solution.

Solution:

No. of basic solutions	Non-basic variables $=0$	Basic variables	Equations & the values of basic variables	Is the solution feasible	Is the solution degenerate	Value of Z	Is the solution optimal?
1	$x_3=0$ $x_4=0$	x_1, x_2	$x_1+4x_2=2$ $-x_1+2x_2=1$ $\Rightarrow x_1=0, x_2=\frac{1}{2}$	yes	yes	$2(0)-2(\frac{1}{2})+4(0)-5(0) = -1.5$	No
2	$x_2=0$ $x_4=0$	x_1, x_3 outgoing x_2 incoming x_1	$x_1-2x_3=2$ $-x_1+3x_3=1$ $\Rightarrow x_1=8, x_3=3$	yes	No	$2(8)-2(0)+4(3)-5(0) = 28$	yes
3.	$x_1=0$ $x_4=0$	x_2, x_3 outgoing x_1 incoming x_2	$4x_2-2x_3=2$ $2x_2+3x_3=1$ $\Rightarrow x_2=\frac{1}{2}, x_3=0$	yes	yes	$2(0)-2(\frac{1}{2})+4(0)-5(0) = -1$	No
4.	$x_2=0$ $x_3=0$	x_1, x_4 outgoing x_2 incoming x_1	$x_1+8x_4=2$ $-x_1+4x_4=1$ $\Rightarrow x_1=0, x_4=\frac{1}{4}$	yes	yes	$2(0)-2(0)+4(0)-5(\frac{1}{4}) = -1.25$	No
5.	$x_1=0$ $x_3=0$	x_2 x_4 outgoing x_1 incoming x_2	$4x_2+8x_4=2$ $2x_2+4x_4=1$ Unbounded	—	—	—	—
6.	$x_1=0$ $x_2=0$	x_3, x_4 outgoing x_2 incoming x_3	$-2x_3+8x_4=2$ $3x_3+x_4=12$ $x_3=0, x_4=\frac{1}{4}$	yes	yes	-12.5	NO