Module - 2

Complex Integration

* Revision:

→ complex number :

$z = x + iy$, where, x, y are Real numbers and $i = \sqrt{-1}$

$z = r(\cos\theta + i\sin\theta)$

$z = r e^{i\theta}$, where, $r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

→ The complex differential :

$dz = dx + i\,dy$

→ A curve in the complex plane :

$z(t) = x(t) + i\,y(t)$, $a \le t \le b$

→ A complex function :

$f(z) = u(x,y) + iv(x,y)$

→ If $z = x+iy$ and $z_0 = x_0 + iy_0$ then

$|z - z_0| = r$ Represent the equation of circle with centre at z_0 and radius r.

since, $|z-z_0| = r \Rightarrow |z-z_0|^2 = r^2$

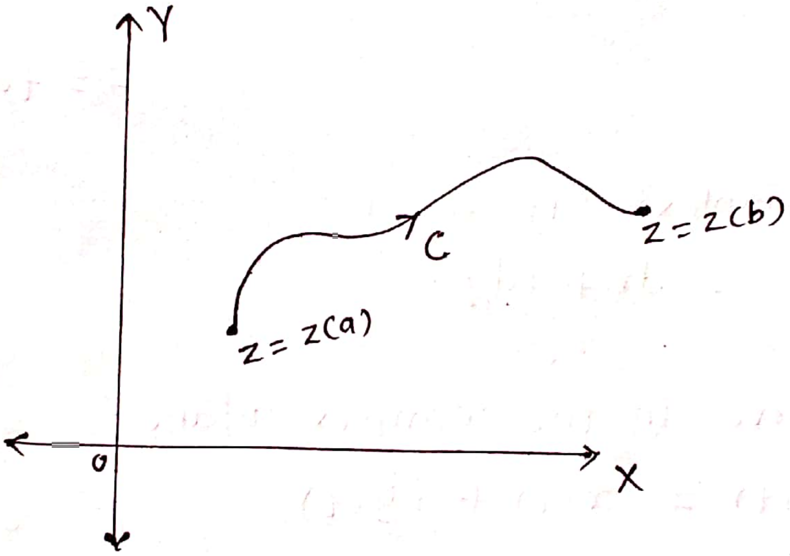
$\Rightarrow |x+iy - (x_0+iy_0)|^2 = r^2 \Rightarrow |(x-x_0) + i(y-y_0)|^2 = r^2$

$\Rightarrow (x-x_0)^2 + (y-y_0)^2 = r^2$ which is Equation of circle centre at $z_0 = (x_0, y_0)$ and radius r.

I) Line Integral : (Contour Integral / path integral)

let C be the curve from $z = z(a)$ to $z = z(b)$ and the function $f(z)$ be the piecewise contineous on C then the line integral of f over C is defined as

$$\int_C f(z)\,dz = \int_a^b f[z(t)] \cdot z'(t)\,dt, \quad a \le t \le b$$



Note that : ① $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

② $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

③ $\int_{C = C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

④ $\int_C k \cdot f(z)\,dz = k \int_C f(z)\,dz$, k - constant

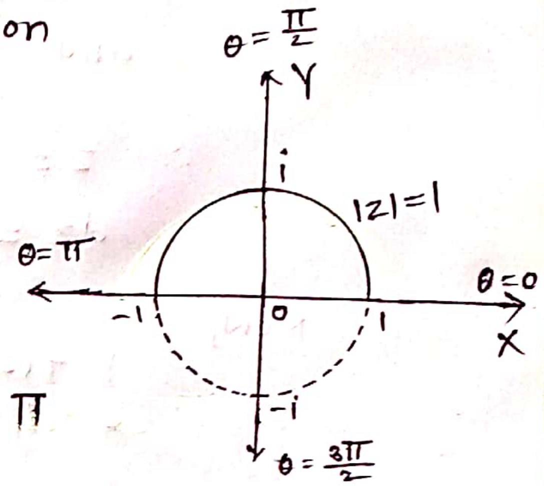
⑤ $\left| \int_C f(z)\,dz \right| \le \int_C |f(z)| \cdot |dz|$

* Examples on Line Integral :

Example ① Evaluate $\int_C \bar{z} \cdot dz$, where C is the upper half of the circle $|z|=1$

<u>Solution!</u> Given: C : $|z| = 1$

Clearly, $|z|=1$ is equation of circle with centre at origin and radius 1



let $z = re^{i\theta}$

since, $r = |z| = 1$, $0 \leq \theta \leq \pi$

$\Rightarrow \quad z = e^{i\theta}$

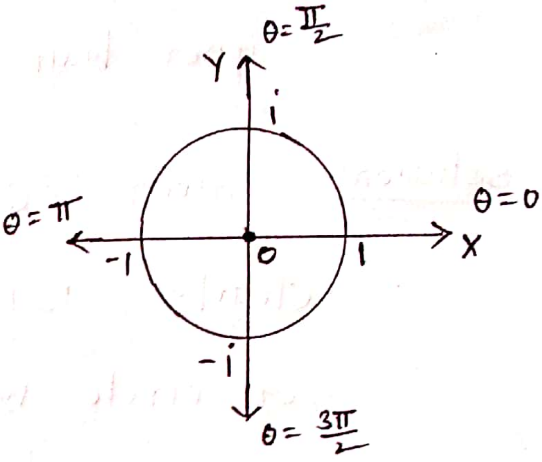
$\therefore \quad dz = e^{i\theta} \cdot i \, d\theta$ and $\bar{z} = e^{-i\theta}$

Now,
$$\int_C \bar{z}\, dz = \int_{\theta=0}^{\pi} e^{-i\theta} \cdot e^{i\theta} \cdot i \, d\theta$$
$$= \int_0^{\pi} i \, d\theta$$
$$= i\,[\theta]_0^{\pi}$$
$$= i\,[\pi - 0]$$
$$= \pi i$$

$$\therefore \quad \boxed{\int_C \bar{z}\, dz = \pi i}$$

Example ②. Evaluate $\int_C \log z\, dz$, where C is the unit circle in the z-plane

Solution: Given: $C : |z| = 1$



let $z = re^{i\theta}$

since, $r = |z| = 1$

and $0 \leq \theta \leq 2\pi$

$\therefore \quad z = e^{i\theta}$

$dz = e^{i\theta} \cdot i \cdot d\theta$

Now,

$$\int_C f(z)\, dz = \int_{\theta=0}^{2\pi} \log e^{i\theta} \cdot e^{i\theta} \cdot i \cdot d\theta$$

$$= \int_0^{2\pi} i\theta \cdot e^{i\theta} \cdot i \, d\theta$$

$$= -\int_0^{2\pi} \theta \cdot e^{i\theta}\, d\theta$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \int \frac{e^{i\theta}}{i} \cdot 1 \cdot d\theta\right]_0^{2\pi}$$

($\because$ using integration by part
$\int uv\,dx = u\int v\,dx - \int\left[\frac{\partial u}{\partial x}\int v\,dx\right]dx$)

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi}$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} + \frac{e^{i\theta}}{1}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi\, e^{2\pi i}}{i} + e^{2\pi i} - 0 - e^{0}\right]$$

$$= -\left[\frac{2\pi}{i} + 1 - 1\right] \qquad \left(\because e^{2\pi i} = 1\right)$$

$$= -\frac{2\pi}{i}$$

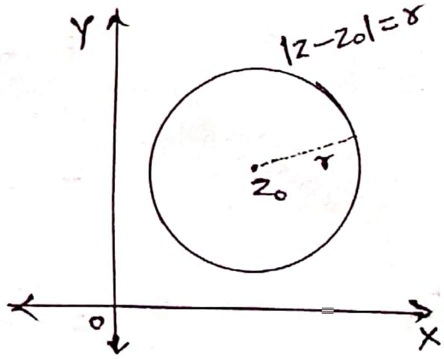
$$= 2\pi i \qquad \left(\because i = \frac{i \cdot i}{i} = -\frac{1}{i}\right)$$

$$\therefore \boxed{\int_C \log z \, dz = 2\pi i}$$

Example 3. Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$,

where, n is an integer and C is the circle $|z-z_0| = r$

Solution: Given: $C : |z-z_0| = r$



Clearly, $|z-z_0| = r$ is the equation of circle with centre at z_0 and radius r

let $z - z_0 = re^{i\theta}$, $0 \le \theta \le 2\pi$

$\Rightarrow dz = re^{i\theta} i \, d\theta$

Now, $$I = \int_C \frac{dz}{(z-z_0)^{n+1}} = \int_0^{2\pi} \frac{r \cdot e^{i\theta} \cdot i \, d\theta}{(re^{i\theta})^{n+1}}$$

$$= \frac{i}{r^n} \int_0^{2\pi} \frac{1}{e^{in\theta}} \, d\theta$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta} \, d\theta$$

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Case i) if $n=0$ then

$$I = \frac{i}{r^0}\int_0^{2\pi} e^0\, d\theta$$

$$= i\int_0^{2\pi} 1\, d\theta$$

$$= i\,[\theta]_0^{2\pi}$$

$$= i\,[2\pi - 0]$$

$$= 2\pi i$$

Case ii) if $n \neq 0$ then

$$I = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}\, d\theta$$

$$= \frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{r^n}\int_0^{2\pi} (i\cos n\theta + \sin n\theta)\, d\theta$$

$$= \frac{1}{r^n}\left[i\,\frac{\sin n\theta}{n} - \frac{\cos n\theta}{n}\right]_0^{2\pi}$$

$$= \frac{1}{r^n}\left[\frac{i\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n} - \frac{i\sin 0}{n} + \frac{\cos 0}{n}\right]$$

$$= 0$$

Therefore,

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

Home work: Example (4) Evaluate $\int_C \frac{2z+3}{z}\, dz$, where C is

i) the upper half of the circle $|z|=2$

ii) the lower half of the circle $|z|=2$

iii) The whole circle in anticlock-wise direction

* Examples on Line Integral :

→ If the Given contour is straight line or parabola:

Example 5: Evaluate the integral $\int_0^{1+i} (x-y+ix^2)\,dz$

i) along the line from $z=0$ to $z=1+i$

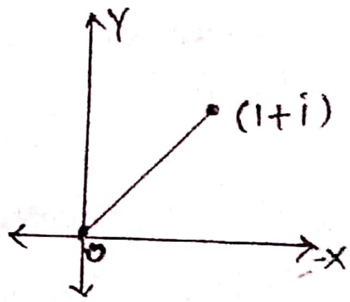
ii) along the real axis from $z=0$ to $z=1$

iii) along the parabola $y^2=x$

Solution:

i) given: C : straight line from $z=0$ to $z=i$

$\therefore$ Equation of line from $z=0$ to $z=1+i$



is
$$\frac{y-y_0}{y_1-y_0} = \frac{x-x_0}{x_1-x_0}$$

$$\Rightarrow \frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$\Rightarrow y = x \quad , \quad 0 \leq x \leq 1 \quad , \quad 0 \leq y \leq 1$$

Now, put $x = t$

$\Rightarrow y = t$

$\therefore dx = dt$ and $dy = dt$

$\therefore dz = dx + i\,dy = dt + i\,dt = (1+i)\,dt$

$\therefore$ If $x=0$ then $t=0$
if $x=1$ then $t=1$

$\therefore$ t varies from $t=0$ to $t=1$

Now
$$\int_0^{1+i} (x-y+ix^2)\,dz = \int_{t=0}^{1} (t-t+it^2)(1+i)\,dt$$

$$= \int_0^1 i(1+i)\, t^2\, dt$$

$$= i(1+i)\left[\frac{t^3}{3}\right]_0^1$$

$$= i(1+i)\left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

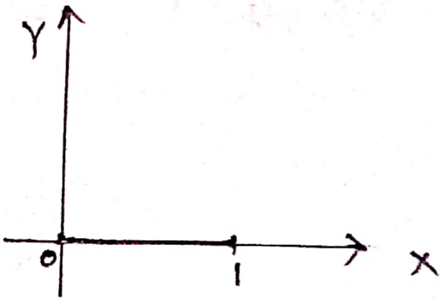
$$= \frac{i(1+i)}{3}$$

$$= \frac{1}{3}(i-1)$$

ii) Given: C : real axis from $z=0$ to $z=1$

$\therefore$ equation of real axis from $z=0$ to $z=1$ is

$y=0 \quad , \quad 0 \leq x \leq 1$



$\therefore \quad dy = 0$

$\therefore \quad dz = dx + i\,dy = dx + i0 = dx$

Now $\displaystyle\int_0^{1+i} (x-y+ix^2)\,dz = \int_0^1 (x-0+ix^2)\,dx$

$$= \int_0^1 (x+ix^2)\,dx$$

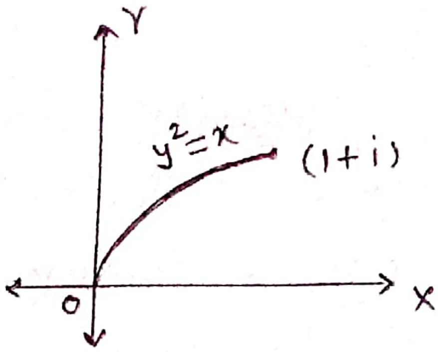
$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1^2}{2} + \frac{i(1)^3}{3}\right) - \left(\frac{0^2}{2} + \frac{i(0)^3}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{i}{3}$$

$$= \frac{1}{2} + i\frac{1}{3}$$

iii) Given: C : Parabola $y^2 = x$



put $y = t$

$\Rightarrow x = t^2$

$\therefore \quad dy = dt$

and $dx = 2t\,dt$

$\therefore \quad dz = dx + i\,dy = 2t\,dt + i\,dt$

$\Rightarrow \quad dz = (2t+i)\,dt$

if $y = 0 \Rightarrow t = 0$

if $y = 1 \Rightarrow t = 1$

$\therefore$ t varies from $t=0$ to $t=1$

$$\therefore \int_0^{1+i} (x - y + i x^2)\,dz = \int_0^1 \left(t^2 - t + i(t^2)^2\right)(2t+i)\,dt$$

$$= \int_0^1 (t^2 - t + i t^4)(2t+i)\,dt$$

$$= \int_0^1 (2t^3 + i t^2 - 2t^2 - i t + 2i t^5 - t^4)\,dt$$

$$= \left[2\frac{t^4}{4} + i\frac{t^3}{3} - 2\frac{t^3}{3} - i\frac{t^2}{2} + 2i\frac{t^6}{6} - \frac{t^5}{5}\right]_0^1$$

$$= \left[\frac{t^4}{2} + i\frac{t^3}{3} - 2\frac{t^3}{3} - i\frac{t^2}{2} + i\frac{t^6}{3} - \frac{t^5}{5}\right]_0^1$$

$$= \left[\frac{1}{2} + \frac{i}{3} - \frac{2}{3} - \frac{i}{2} + \frac{i}{3} - \frac{1}{5} - 0\right]$$

$$= -\frac{11}{30} + \frac{i}{6}$$

$$= -\frac{11}{30} + i\frac{1}{6}$$

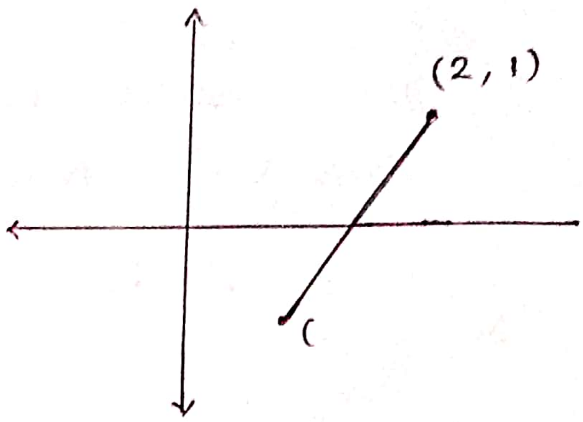
Example 2. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\, dz$ along the straight line joining $(1-i)$ to $(2+i)$

Solution:

Given: C : straight line from $(1-i)$ to $(2+i)$

$\therefore$ Equation of line from $(1,-1)$ to $(2,1)$ is

$$\frac{y-y_0}{y_1-y_0} = \frac{x-x_0}{x_1-x_0}$$



$$\Rightarrow \frac{y-(-1)}{1-(-1)} = \frac{x-1}{2-1}$$

$$\Rightarrow \frac{y+1}{2} = \frac{x-1}{1}.$$

$$\Rightarrow y = 2x-3$$

put $x = t$

$\Rightarrow y = 2t-3$

$\therefore dx = dt$ and $dy = 2dt$

$\therefore dz = dx + idy = dt + 2idt = (1+2i)\, dt$

if $x = 1$ then $t = 1$

if $x = 2$ then $t = 2$

$\therefore t$ varies from $t=1$ to $t=2$

Now, $$\int_{1-i}^{2+i} (2x+iy+1)\, dz = \int_1^2 (2t+i(2t-3)+1)(1+2i)\, dt$$

$$= (1+2i)\int_1^2 (2t+2it-3i+1)\, dt$$

$$= (1+2i)\left[2\frac{t^2}{2} + 2i\frac{t^2}{2} - 3it + t\right]_1^2$$

$$= (1+2i)\left[t^2 + it^2 - 3it + t\right]_1^2$$

$$= (1+2i)\left[(2)^2 + i(2)^2 - 3i(2) + 2 - (1+i-3i+1)\right]$$

$$= (1+2i)\left[4+4i-6i+2-2+2i\right]$$

$$= (1+2i)\cdot 4$$

$$\therefore \int_{1-i}^{2+i} (2x+iy+1)\,dz = 4(1+2i)$$

Example 6. Integrate the function $f(z) = 2x+iy+1$ from $A(1,-1)$ to $B(2,1)$ Along the curve $x = t+1$, $y = 2t^2-1$

Solution: Given: $f(z) = 2x+iy+1$

and C : $x = t+1$, $y = 2t^2-1$ from $A(1,-1)$ to $B(2,1)$

$$\therefore \quad dx = dt \quad , \quad dy = 4t\,dt$$

$$dz = dx + i\,dy = dt + 4it\,dy = (1+4it)\,dt$$

if $x = 1$ then $t = x-1 = 1-1 = 0$

if $x = 2$ then $t = x-1 = 2-1 = 1$

$\therefore$ t varies from $t=0$ to $t=1$

Now, $$\int_C f(z)\,dz = \int_C (2x+iy+1)\,dz$$

$$= \int_0^1 \left[2(t+1) + i(2t^2-1) + 1\right](1+4it)\,dt$$

$$= \int_0^1 (2t+2+2it^2-i+1)(1+4it)\,dt$$

$$= \int_0^1 (2t+2+2it^2-i+1+8it^2+8it-8t^3+4t + 4it)\,dt$$

$$= \int_0^1 (-8t^3 + 10it^2 + 12it + 6t + 3 - i)\, dt$$

$$= \left[-8\frac{t^4}{4} + 10i\frac{t^3}{3} + 12i\frac{t^2}{2} + 6\frac{t^2}{2} + 3t - it\right]_0^1$$

$$= \left[-2t^4 + \frac{10i}{3}t^3 + 6it^2 + 3t^2 + 3t - it\right]_0^1$$

$$= \left[-2 + \frac{10i}{3} + 6i + 3 + 3 - i\right]$$

$$= 4 + \frac{25}{3}i$$

$$\therefore \int_C f(z)\, dz = 4 + \frac{25}{3}i$$

Homework:

Example 8. Evaluate $\int_0^{1+i} z^2\, dz$ along

i) the line $y = x$

ii) the parabola $x = y^2$

iii) Is the line integral independent of the path?

(Hint: for iii) if value of Integral using i) and ii) are same then path independent otherwise, dependent)

$\boxed{\text{Ans: } \frac{2}{3}(i-1)}$

Example 9. Evaluate $\int f(z)\, dz$ along the parabola $y = 2x^2$ from $z = 0$ to $z = 3 + 18i$ where $f(z) = x^2 - 2iy$

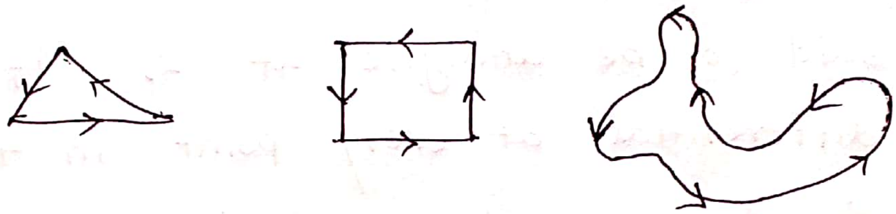
$\boxed{\text{Ans: } 333 + 45i}$

* Cauchy's Integral theorem for simple connected and multiply connected regions

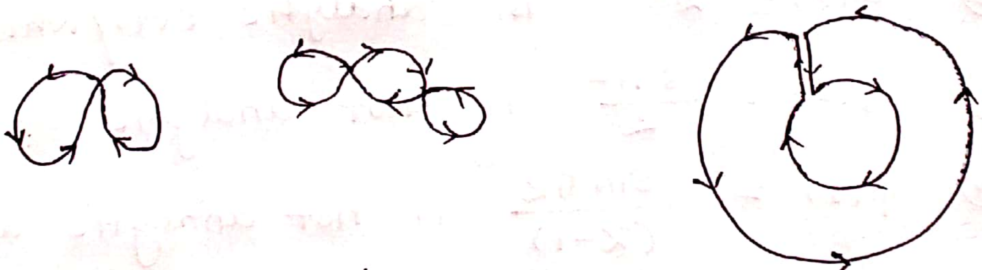
→ * Closed curve : closed curve is one in which end points coincide.

i.e. $\phi(a) = \phi(b)$, for some $a = b$

* simple closed curve :- A closed curve does not intersect itself is called simple closed curve



* multiple closed curve : A closed curve intersect itself is called multiple closed curve



* Simply connected domain :-

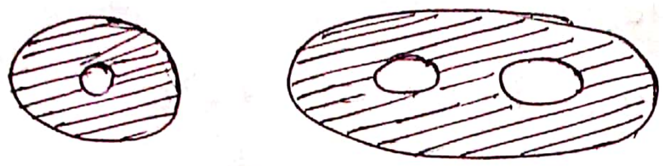
A domain D is said to be simply connected if every simple closed curve in D contains only points of D

<u>for example</u> : Interior of circle, rectangle

* **Multiply connected domain :**

A domain which is not simply connected is called multiply connected domain.

<u>for ex.</u> Annulus region, regions with holes.



* **Analytic function:** A function $f(z)$ is said to be analytic at z_0 if $f(z)$ is differential at every point in the neighbourhood of z_0.

<u>for. ex:</u> ① $f(z) = \frac{1}{z}$ is not analytic at $z=0$

② $f(z) = z^2$ is analytic everywhere

③ $f(z) = \frac{\sin z}{z}$ is not analytic at $z=0$

④ $f(z) = \frac{\sin \pi z}{(z-1)}$ is not analytic at $z=1$

Imp ⇒ **Cauchy Integral Theorem for simply connected region (Domain)**

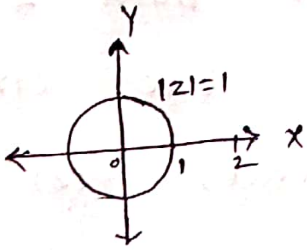
<u>statement</u> :- let $f(z)$ be analytic on and inside a simple closed contour C and let $f'(z)$ also continuous on and inside C,

then $$\oint_C f(z)\, dz = 0$$

Examples:

Example ① Evaluate $\int_C \frac{1}{z-2}\,dz$, where C is the circle $|z|=1$

Solution: given: $C : |z|=1$



let $f(z) = \frac{1}{z-2}$

clearly, $f(z)$ is not analytic at $z=2$

But $z=2$ lies outside the circle $|z|=1$

Hence, $f(z)$ is analytic everywhere on and inside C

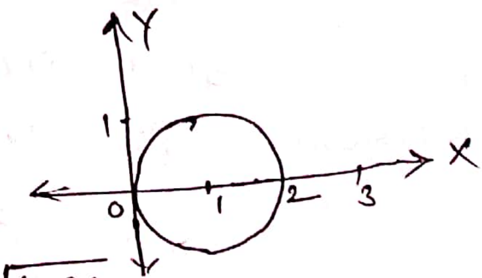
$\therefore$ By cauchy integral theorem,

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \boxed{\int_C \frac{1}{z-2}\,dz = 0}$$

Example ② Evaluate $\int_C \frac{z+3}{z^2-2z+5}\,dz$, where C is the circle $|z-1|=1$

Solution: given: $C : |z-1|=1$



let $f(z) = \frac{z+3}{z^2-2z+5}$

Note that $z^2-2z+5=0$ gives $z = \frac{2 \pm \sqrt{4-20}}{2}$

i.e. $z = 1 \pm 2i$

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$\therefore\ f(z) = \dfrac{z+3}{z^2-2z+5} = \dfrac{z+3}{[z-(1+2i)][z-(1-2i)]}$

clearly, $f(z)$ is not analytic at $z = 1+2i$ and $z = 1-2i$

But Both $z = 1+2i$ and $z = 1-2i$ lies Outside the circle $|z-1| = 1$

Hence, $f(z)$ is analytic everywhere on and inside the circle $|z-1| = 1$

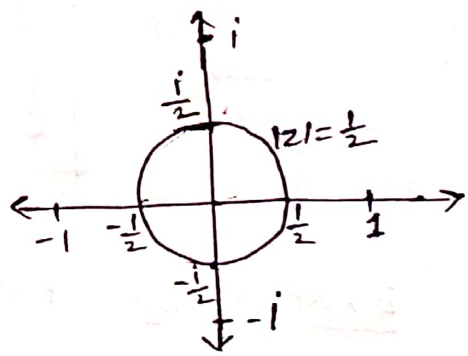
$\therefore$ By cauchy's Integral theorem,

$$\int_C f(z)\, dz = 0$$

$$\Rightarrow \oint_C \frac{z+3}{z^2-2z+5}\, dz = 0$$

Example ③. Evaluate $\int_C \tan z\, dz$, where C is $|z| = \frac{1}{2}$

solution: Given: $C : |z| = \frac{1}{2}$.



let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

Note that $\cos z = 0 \Rightarrow z = \pm \frac{\pi}{2}$

$\therefore$ $f(z)$ is not analytic at $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$

But Both $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ lies outside the given circle C i.e. $|z| = \frac{1}{2}$

Hence, $f(z)$ is Analytic everywhere on and inside C

$\therefore$ By cauchy integral theorem,

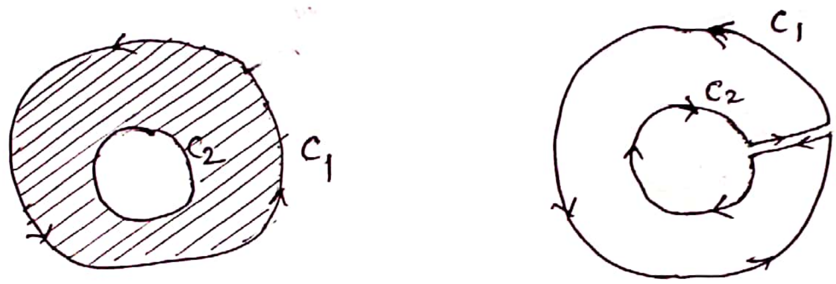
$$\int_C f(z)\, dz = 0$$

$$\Rightarrow \boxed{\int_C \tan z\, dz = 0}$$

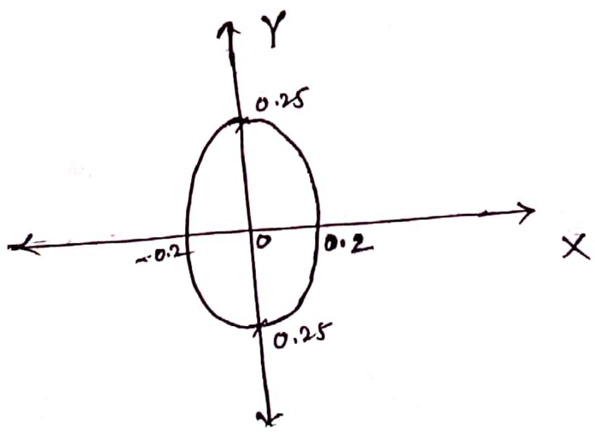
* Cauchy Integral Theorem for multiply connected region:-

__Statement__ : let C_1 and C_2 be simple closed curves such that C_2 is interior to C_1. if $f(z)$ is analytic on C_1 and C_2 and $f(z)$ is analytic on each point that is interior to C_1 and exterior to C_2, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$



__Practice Example.__ Evaluate $\int_C \frac{z^2+z+2}{z^2-7z+2}\,dz$, where C is the ellipse $25x^2+16y^2=1$



* **Cauchy Integral formula :-**

Statement : Let $f(z)$ be the analytic everywhere on and inside a simple closed contour C taken in the anticlockwise direction.

If z_0 is any point interior to C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz$$

that is

$$\boxed{\int_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)}$$

⇒ Important Note:

The General Cauchy Integral formula is

$$\boxed{\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$

In perticular,

* $\displaystyle\int_C \frac{f(z)}{(z-z_0)^2}\, dz = \frac{2\pi i}{1!} f'(z_0)$, for $n=1$

* $\displaystyle\int_C \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i}{2!} f''(z_0)$, for $n=2$

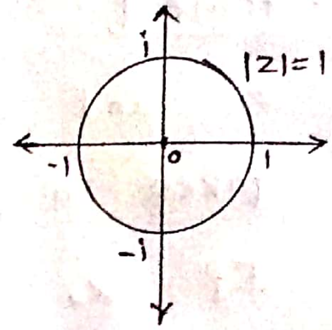
⋮

and so on.

* Examples on Cauchy Integral formula :

Example ① Evaluate $\int_C \frac{z^3-6}{3z-i}\,dz$, where C is $|z|=1$

Solution! Given : $C: \ |z|=1$



let $f(z) = z^3 - 6$

Note that $\int_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\int_C \frac{z^3-6}{\left(z-\frac{i}{3}\right)}$ — ①

clearly, $f(z) = z^3-6$ is analytic everywhere on and inside C

only $z_0 = \frac{i}{3}$ lie inside C with order 1

∴ By cauchy integral formula

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\Rightarrow \int_C \frac{z^3-6}{z-\frac{i}{3}}\,dz = 2\pi i\left(\left(\frac{i}{3}\right)^3-6\right)$$

$$= 2\pi i\left(\frac{-i}{27}-6\right)$$

$$= -2\pi i\left(\frac{i}{27}+6\right)$$

∴ equation ① becomes

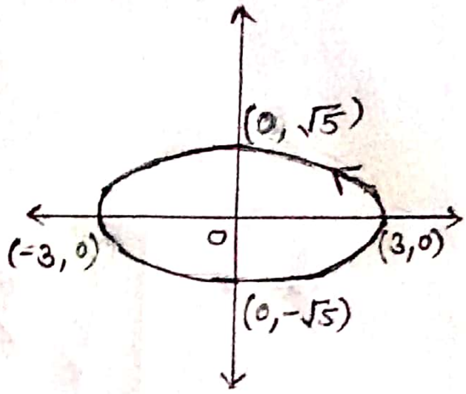
$$\int_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\int_C \frac{z^3-6}{z-\frac{i}{3}}\,dz$$

$$= \frac{1}{3}\left[-2\pi i\left(\frac{i}{27}+6\right)\right]$$

$$= \frac{-2\pi i}{3}\left(\frac{i}{27}+6\right)$$

i.e. $$\boxed{\int_C \frac{z^3-6}{3z-i}\,dz = -\frac{2\pi i}{3}\left(\frac{i}{27}+6\right)}$$

Example ② Evaluate $\int_C \frac{e^{3z}}{z-i}\, dz$, where C is the curve $|z-2|+|z+2|=6$



Solution: Given: $C: |z-2|+|z+2|=6$

Note that $|z-2|+|z+2|=6$

$$\Rightarrow |(x+iy)-2|+|(x+iy)+2|=6$$

$$\Rightarrow |(x-2)+iy|+|(x+2)+iy|=6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6$$

if $y=0$ then $\sqrt{(x-2)^2}+\sqrt{(x+2)^2}=6$

$$\Rightarrow x-2+x+2=6 \Rightarrow x=3$$

if $x=0$ then $\sqrt{(-2)^2+y^2}+\sqrt{(2)^2+y^2}=6$

$$\Rightarrow 2\sqrt{y^2+4}=6$$

$$\Rightarrow \sqrt{y^2+4}=3$$

$$\Rightarrow y^2+4=9$$

$$\Rightarrow y^2=5$$

$$\Rightarrow y=\pm\sqrt{5}$$

$\therefore$ Intersection points of ellipse with x-axis are $(-3,0)$, $(3,0)$

and Intersection points of ellipse with y axis are $(0,\sqrt{5})$, $(0,-\sqrt{5})$

Now, let $f(z)=e^{3z}$

then clearly, $f(z)=e^{3z}$ is analytic everywhere on and inside C

only $z_0=i$ lie inside C with order 1

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$\therefore$ By Cauchy integral formula,

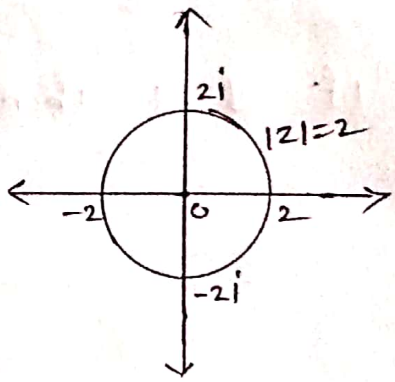
$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(i)$$

$$= 2\pi i\, e^{3i}$$

$$\Rightarrow \boxed{\int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i\, e^{3i}}$$

Example ③ Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z|=2$



Solution: $C: |z|=2$

Note that $\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{3z^2+z}{(z+1)(z-1)}$

here, Both $z=1$, $z=-1$ lie inside C

$\therefore$ we use partial fraction

Consider, $\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$

$$\Rightarrow \frac{1}{(z+1)(z-1)} = \frac{(z-1)A + (z+1)B}{(z+1)(z-1)}$$

$$\Rightarrow A(z-1) + B(z+1) = 1$$

Now, If $z=-1$ then $A(-1-1) + B(0) = 1 \Rightarrow A = -\frac{1}{2}$

If $z=1$ then $A(0) + B(1+1) = 1 \Rightarrow B = \frac{1}{2}$

$$\therefore \frac{1}{(z+1)(z-1)} = \frac{-1}{2(z+1)} + \frac{1}{2(z-1)}$$

i.e. $\dfrac{1}{(z+1)(z-1)} = \dfrac{1}{2(z-1)} - \dfrac{1}{2(z+1)}$

$\Rightarrow \quad \dfrac{3z^2+z}{(z+1)(z-1)} = \dfrac{3z^2+z}{2(z-1)} - \dfrac{3z^2+z}{2(z+1)}$

$\Rightarrow \quad \displaystyle\int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{(z-1)}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{(z+1)}\,dz$

clearly, $f(z) = 3z^2+z$ is analytic everywhere on and inside C

$\therefore$ By cauchy integral formula,

$$\int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\,2\pi i\, f(1) - \frac{1}{2}\,2\pi i\, f(-1)$$

$$= \frac{1}{2}\,2\pi i\left(3(1)^2+1\right) - \frac{1}{2}\,2\pi i\left(3(-1)^2-1\right)$$

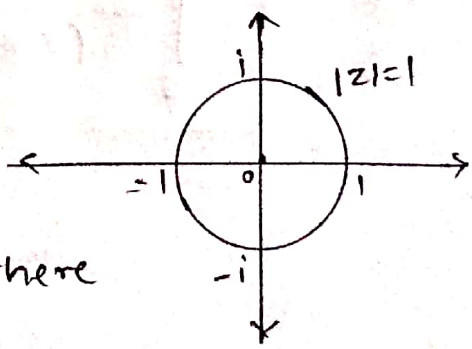
$$= \frac{1}{2}\,2\pi i\,(4) - \frac{1}{2}\,2\pi i\,(2)$$

$$= 4\pi i - 2\pi i$$

$$= 2\pi i$$

i.e. $$\boxed{\int_C \frac{3z^2+z}{z^2-1}\,dz = 2\pi i}$$

Example (4) Evaluate $\displaystyle\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$ where C is $|z|=1$



Solution: Given: $C: |z|=1$

let $f(z) = \sin^6 z$

clearly, $f(z)$ is analytic everywhere on and inside C

only $z_0 = \frac{\pi}{6}$ lie inside C with order 3

$\therefore$ By genral Cauchy integral formula for $n=3$ is

$$\int_C \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

$$= \pi i\, f''\left(\frac{\pi}{6}\right) \quad \text{——— (1)}$$

since,
$$f(z) = \sin^6 z$$
$$f'(z) = 6\sin^5 z \cdot \cos z$$
$$f''(z) = 6\left[5\sin^4 z \cdot \cos^2 z + \sin^5 z \cdot (-\sin z)\right]$$
i.e. $$f''(z) = 6\left[5\sin^4 z \cdot \cos^2 z - \sin^6 z\right]$$
$$\Rightarrow \quad f''\left(\frac{\pi}{6}\right) = 6\left[5 \cdot \sin^4\left(\frac{\pi}{6}\right) \cdot \cos^2\left(\frac{\pi}{6}\right) - \sin^6\left(\frac{\pi}{6}\right)\right]$$
$$= 6\left[5 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right]$$
$$= 6\left[5 \cdot \frac{1}{16} \cdot \frac{3}{4} - \frac{1}{64}\right]$$
$$= 6\left[\frac{15}{64} - \frac{1}{64}\right]$$
$$= \frac{21}{16}$$

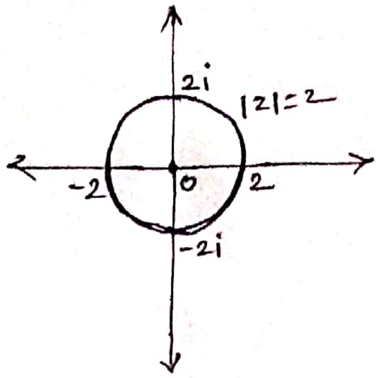
$\therefore$ equation (1) becomes,

$$\int_C \frac{f(z)}{(z-z_0)^3}\,dz = \pi i \, \frac{21}{16}$$

i.e. $$\boxed{\int_C \frac{\sin^6 z}{(z-z_0)^3}\,dz = \frac{21\pi i}{16}}$$

Example 5 Evaluate $\int_c \frac{dz}{z^2(z+4)}$, where c is the circle $|z| = 2$

Solution: Given: $C: |z| = 2$



Note that:

$$\int_C \frac{dz}{z^2(z+4)} = \int_C \frac{\left(\frac{1}{z+4}\right)}{z^2} dz$$

let $f(z) = \frac{1}{z+4}$

here, $z_0 = 0, z_1 = -4$ are singular points

clearly, $f(z)$ is analytic everywhere on and inside C

$\therefore$ only $z_0 = 0$ lie inside C

$\therefore$ By general cauchy integral formula, for n=2

$$\int_C \frac{f(z)}{(z-z_0)^2} dz = \frac{2\pi i}{1!} \cdot f'(z_0)$$

$$\Rightarrow \int_C \frac{\frac{1}{z+4}}{z^2} dz = 2\pi i\, f'(0)$$

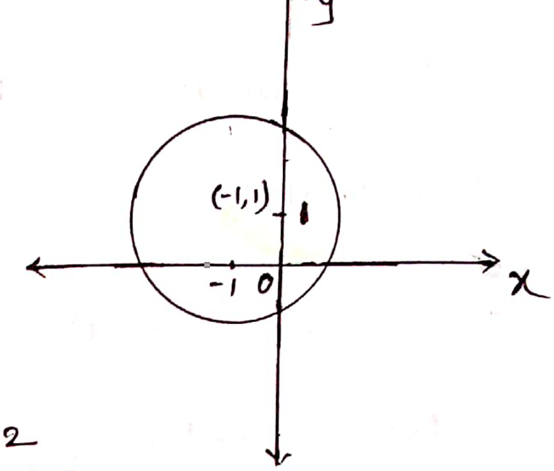
$$= 2\pi i\left(-\frac{1}{(0+4)^2}\right) \qquad \left(\because f'(z) = -\frac{1}{(z+4)^2}\right)$$

$$= 2\pi i\left(-\frac{1}{16}\right)$$

$$= -\frac{\pi i}{8}$$

i.e. $$\boxed{\int_c \frac{1}{z^2(z+4)} dz = -\frac{\pi i}{8}}$$

Example ⑥ Evaluate $\int_C \frac{z+4}{z^2+2z+5}\, dz$ where C is the circle $|z+1-i| = 2$



Solution: Given: $C: |z+1-i| = 2$

Note that $|z+1-i| = 2$

$\Rightarrow |z-(-1+i)| = 2$

which is equation of circle with centre $(-1, 1)$ and radius 2

Now, $z^2 + 2z + 5 = 0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{(2)^2 - 20}}{2} = \frac{-2 \pm 4i}{2}$$

$\Rightarrow z = -1-2i, \; -1+2i$

Clearly, the point $z = -1+2i = (-1, 2)$ lies inside C and the point $z = -1-2i = (-1,-2)$ lies outside C

Now,
$$\int_C \frac{z+4}{z^2+2z+5}\, dz = \int_C \frac{z+4}{[z-(-1-2i)][z-(-1+2i)]}\, dz$$

$$= \int_C \frac{\frac{z+4}{[z-(-1-2i)]}}{z-(-1+2i)}\, dz$$

$\therefore$ we take $f(z) = \frac{z+4}{z-(-1-2i)}$

$\Rightarrow$ $f(z)$ is analytic everywhere on and inside C and $z_0 = -1+2i$ lies inside C. with order 1

$\therefore$ By Cauchy integral formula,

$$\int_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$$

$$\Rightarrow \int_C \frac{\frac{z+4}{[z-(-1-2i)]}}{z-(-1+2i)}\, dz = 2\pi i\, f(-1+2i)$$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5}\, dz = 2\pi i \left[\frac{(-1+2i)+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i \left[\frac{3+2i}{4i}\right]$$

$$= \frac{\pi}{2}(3+2i)$$

$$\therefore \boxed{\int_C \frac{z+4}{z^2+2z+5}\, dz = \frac{\pi}{2}(3+2i)}$$

* Taylor's and Laurent's Series :-

→ power series in complex number :-

the power series in powers of $(z-a)$ is of the form $\sum_{n=1}^{\infty} C_n (z-a)^n$ —— ①

where, z is complex number and C_n's are constants.

<u>Note that</u> : The power series ① is convergent for $|z-a| < R$, for some real Number R.

therefore the number 'R' is called as <u>Radius of convergence</u>.

* To find the Radius of convergence :-

consider the power series $\sum_{n=1}^{\infty} C_n (z-a)^n$

① $$R = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|$$

② $$R = \lim_{n\to\infty} (C_n)^{-\frac{1}{n}}$$

Example Find the radius of convergence of following

(i) $\sum_{n=0}^{\infty} \frac{z^n}{3^n+1}$ (ii) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \cdot z^n$

solution:

(1) Given power series is $\sum_{n=0}^{\infty} \frac{1}{3^n+1} z^n$

compare with $\sum_{n=0}^{\infty} C_n z^n$, we get

$$C_n = \frac{1}{3^n+1}$$

$$\therefore R \doteq \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{3^n+1} \times \frac{3^{n+1}+1}{1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{3^{n+1}+1}{3^n+1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{3^n\left(3+\frac{1}{3^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)}\right|$$

$$= \frac{3+0}{1+0}$$

$$= 3.$$

$\therefore$ Radius of convergence : $\underline{\underline{R = 3.}}$

(2) Given power series is $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \cdot z^n$

compare with $\sum_{n=1}^{\infty} C_n z^n$, we get

$$C_n = \left(1+\frac{1}{n}\right)^{n^2}$$

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$$\therefore \quad R = \lim_{n\to\infty} (c_n)^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= e^{-1} = \frac{1}{e}$$

$\therefore$ Radius of convergence : $\underline{\underline{R = \frac{1}{e}}}$

<u>Homework</u>.

Example : find the Radius of convergence of following

① $\sum_{n=0}^{1} \frac{n+1}{(n+2)(n+3)} z^n$. ② $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$

* **Taylor's series Expansion:**

let C be the circle with centre at z_0 and $f(z)$ be analytic everywhere inside C then Taylor's series expansion of $f(z)$ is

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots \quad \text{—(1)}$$

Note that: (1) Above series of $f(z)$ is convergent at every point inside C

(2) If we put $z = z_0 + h$ then equation (1) becomes

$$f(z_0+h) = f(z_0) + h f'(z_0) + \frac{h^2}{2!} f''(z_0) + \cdots$$

(3) If we put $z_0 = 0$ then equation (1) becomes

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

is known as Maclaurin's series.

* **Important power series of the function:-**

→ $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$, where $|z|<1$

→ $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$, where $|z|<1$

→ $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$, where $|z|<1$

→ $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$, where $|z|<1$

* Examples on Taylor's series:

Example ① obtain Taylor's expansion of

$$f(z) = \frac{z+2}{(z-1)(z-4)} \quad \text{at} \quad z=2$$

solution: given function is $f(z) = \frac{z+2}{(z-1)(z-4)}$

clearly, degree of Numerators is less than degree of Denomenators

Now, consider the partial fraction

$$\frac{z+2}{(z-1)(z-4)} = \frac{A}{z-1} + \frac{B}{z-4}$$

$$\Rightarrow \frac{z+2}{(z-1)(z-4)} = \frac{A(z-4)+B(z-1)}{(z-1)(z-4)}$$

$$\Rightarrow A(z-4)+B(z-1) = z+2$$

if $z=1$ then $A(1-4)+0 = 1+2 \Rightarrow A=-1$

if $z=4$ then $A(0)+B(4-1) = 4+2 \Rightarrow B=2$

$$\therefore \frac{z+2}{(z-1)(z-4)} = -\frac{1}{z-1} + \frac{2}{z-4}$$

we have to find series expansion in the power of $(z-2)$

$$\frac{z+2}{(z-1)(z-4)} = -\frac{1}{(z-2)+1} + \frac{2}{(z-2)-2}$$

$$= -\frac{1}{[1+(z-2)]} + \frac{2}{-2[1-(\frac{z-2}{2})]}$$

$$= -\frac{1}{1+(z-2)} - \frac{1}{1-(\frac{z-2}{2})}$$

$$= -[1+(z-2)]^{-1} - [1-(\frac{z-2}{2})]^{-1}$$

we know, $[1+z]^{-1} = 1-z+z^2-z^3+\cdots$ and $[1-z]^{-1} = 1+z+z^2+z^3+\cdots$

$$\Rightarrow \frac{z+2}{(z-1)(z-4)} = -\left[1-(z-2)+(z-2)^2-(z-2)^3+\cdots\right]$$

$$-\left[1+\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2+\left(\frac{z-2}{2}\right)^3+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}(-1)^n(z-2)^n - \sum_{n=0}^{\infty}\left(\frac{z-2}{2}\right)^n$$

$$= -\sum_{n=0}^{\infty}\left[(-1)^n+\frac{1}{2^n}\right](z-2)^n$$

i.e. $$f(z) = -\sum\left[(-1)^n+\frac{1}{2^n}\right](z-2)^n, \quad |z-2|<1$$

Example ②. obtain Taylor's expansion of $f(z) = \frac{1-z}{z^2}$ in the powers of $(z-1)$

solution: Given: $f(z) = \frac{1-z}{z^2}$

$$\Rightarrow f(z) = -\frac{(z-1)}{(z-1+1)^2}$$

$$= -\frac{(z-1)}{[1+(z-1)]^2}$$

$$= -(z-1)\left[1+(z-1)\right]^{-2}$$

$$= -(z-1)\left[1-2(z-1)+3(z-1)^2-4(z-1)^3+\cdots\right]$$

$$\left(\because [1+z]^{-2} = 1-2z+3z^2-4z^3+\cdots\right)$$

$$= \left[-(z-1)+2(z-1)^2-3(z-1)^3+4(z-1)^4-\cdots\right]$$

$$= \sum_{n=1}^{\infty}(-1)^n\, n\,(z-1)^n$$

i.e. $$f(z) = \sum_{n=1}^{\infty}(-1)^n . n\,(z-1)^n$$

* Laurent's Series Expansion:

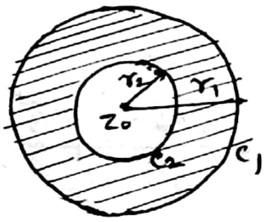
let C_1 and C_2 be two circle of radii r_1 and r_2 with centre z_0

let $f(z)$ be analytic on C_1, C_2 and between C_1 and C_2 then Laurent's series Expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where, $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

and $$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-z_0)^{-n+1}}\, dw$$



* some Important power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad , \quad |z| < \infty$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \quad , \quad |z| < \infty$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \quad , \quad |z| < \infty$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \quad , \quad |z| < \infty$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \quad , \quad |z| < \infty$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \quad , \quad |z| < 1$$

Example ① find Laurent's series for $f(z) = \dfrac{e^{3z}}{(z-1)^3}$ about $z=1$

Solution: Given: $f(z) = \dfrac{e^{3z}}{(z-1)^3}$

since, we want expansion around $z=1$, therefore we have to obtain Laurent's series in the power of $(z-1)$.

$$\therefore f(z) = \frac{e^{3z}}{(z-1)^3} = \frac{e^{3(z-1)+3}}{(z-1)^3}$$

$$= \frac{e^3 \cdot e^{3(z-1)}}{(z-1)^3}$$

$$= \frac{e^3}{(z-1)^3}\left[e^{3(z-1)}\right]$$

$$= \frac{e^3}{(z-1)^3}\left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \frac{3^3(z-1)^3}{3!} + \frac{3^4(z-1)^4}{4!} + \cdots\right]$$

$$\left(\because e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)$$

$$= e^3\left[\frac{1}{(z-1)^3} + \frac{3(z-1)}{(z-1)^3} + \frac{3^2(z-1)^2}{2!\,(z-1)^3} + \frac{3^3(z-1)^3}{3!\,(z-1)^3} + \frac{3^4(z-1)^4}{4!\,(z-1)^3} + \cdots\right]$$

$$= e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2!(z-1)} + \frac{3^3}{3!} + \frac{3^4}{4!}\cdot(z-1) + \cdots\right]$$

i.e
$$\boxed{f(z) = e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2!(z-1)} + \frac{3^3}{3!} + \frac{3^4}{4!}\cdot(z-1) + \cdots\right]}$$

Example ② Find Laurent's series which represent the function $f(z) = \frac{1}{z(z+1)(z-2)}$ when

i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$

solution: Given: $f(z) = \frac{1}{z(z+1)(z-2)}$

consider, $\frac{1}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$

$\Rightarrow \frac{1}{z(z+1)(z-2)} = \frac{A(z+1)(z-2) + Bz(z-2) + Cz(z+1)}{z(z+1)(z-2)}$

$\Rightarrow A(z+1)(z-2) + Bz.(z-2) + Cz(z+1) = 1$

if $z=0$ then $A(0+1)(0-2) + B(0) + C(0) = 1 \Rightarrow A = -\frac{1}{2}$

if $z=-1$ then $A(0) + B(-1)(-1-2) + C(0) = 1 \Rightarrow B = \frac{1}{3}$

if $z=2$ then $A(0) + B(0) + C(2)(2+1) = 1 \Rightarrow C = \frac{1}{6}$

$\therefore f(z) = \frac{1}{z(z+1)(z-2)} = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$

case i) When $0<|z|<1$

$\therefore f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$

$\Rightarrow f(z) = -\frac{1}{2z} + \frac{1}{3(1+z)} - \frac{1}{12[1-(\frac{z}{2})]}$

since, $|z|<1$ i.e. $|z|<1<2 \Rightarrow |z|<2$

$\therefore |z|<2 \Rightarrow |\frac{z}{2}|<1$

therefore,

$f(z) = -\frac{1}{2z} + \frac{1}{3}[1+z]^{-1} - \frac{1}{12}[1-(\frac{z}{2})]^{-1}$

$\Rightarrow f(z) = -\frac{1}{2z} + \frac{1}{3}[1-z+z^2-z^3+z^4-\cdots]$
$- \frac{1}{12}[1+(\frac{z}{2})+(\frac{z}{2})^2+(\frac{z}{2})^3+\cdots]$

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$\left(\because [1-z]^{-1} = 1+z+z^2+\cdots \text{ and } [1+z]^{-1} = 1-z+z^2-z^3+\cdots \right)$

case ii) when $1 < |z| < 2$

i.e. $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$ and

$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{3z\left[1+\left(\frac{1}{z}\right)\right]} - \frac{1}{12\left[1-\left(\frac{z}{2}\right)\right]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1+\left(\frac{1}{z}\right)\right]^{-1} - \frac{1}{12}\left[1-\left(\frac{z}{2}\right)\right]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$- \frac{1}{12}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

case iii) when $|z| > 2$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{3z\left(1+\frac{1}{z}\right)} + \frac{1}{6z\left(1-\frac{2}{z}\right)}$$

since, $|z| > 2$ i.e. $|z| > 2 > 1 \Rightarrow |z| > 1$

$\therefore \quad 1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$ and

$2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1+\left(\frac{1}{z}\right)\right]^{-1} + \frac{1}{6z}\left[1-\left(\frac{2}{z}\right)\right]^{-1}$$

$$\Rightarrow \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$+ \frac{1}{6z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

Example ③ Expand $\dfrac{z^2-1}{z^2+5z+6}$ around $z=0$

solution: let $f(z) = \dfrac{z^2-1}{z^2+5z+6}$

here, degree of the numerator is not less than degree of denominator

$\therefore$ we divide the numerator by denominator.

$$\therefore \quad f(z) = 1 - \frac{5z+7}{z^2+5z+6}$$

$$\begin{array}{r} 1 \\ z^2-1 \,\big)\, z^2+5z+6 \\ \underline{z^2-1} \\ 5z+7 \end{array}$$

$$= 1 - \frac{5z+7}{(z+3)(z+2)}$$

$$= 1 + \frac{(-5z-7)}{(z+3)(z+2)}$$

Now let $\dfrac{-5z-7}{(z+3)(z+2)} = \dfrac{A}{z+3} + \dfrac{B}{z+2}$

$$\Rightarrow \quad \frac{-5z-7}{(z+3)(z+2)} = \frac{A(z+2) + B(z+3)}{(z+3)(z+2)}$$

$$\Rightarrow \quad A(z+2) + B(z+3) = -5z-7$$

if $z=-3$ then $A(-3+2) + B(0) = -5(-3)-7$

$\Rightarrow \quad A = -8$

if $z=-2$ then $A(0) + B(-2+3) = -5(-2)-7$

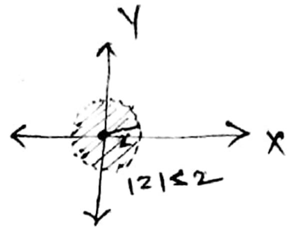
$\Rightarrow \quad B = 3$

$$\therefore \quad \frac{-5z-7}{(z+3)(z+2)} = \frac{-8}{z+3} + \frac{3}{z+2}$$

$$\therefore \quad f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

case i) when $|z| < 2$

$$\therefore \quad f(z) = 1 - \frac{8}{3\left[1 + \left(\frac{z}{3}\right)\right]} + \frac{3}{2\left[1 + \left(\frac{z}{2}\right)\right]}$$



clearly, $|z| < 2 < 3$

$\therefore \quad |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$ and

$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

$$\therefore \quad f(z) = 1 - \frac{8}{3}\left[1 + \left(\frac{z}{3}\right)\right]^{-1} + \frac{3}{2}\left[1 + \left(\frac{z}{2}\right)\right]^{-1}$$

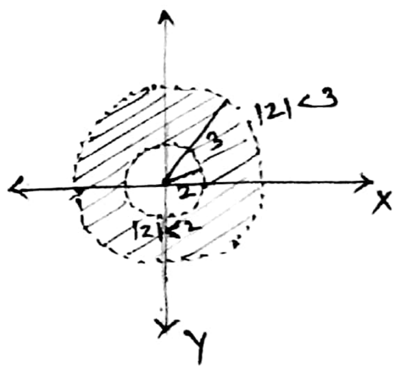
$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right] + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \cdots\right]$$

$$\left(\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots\right)$$

case ii) when $2 < |z| < 3$

i.e. $2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

and $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$



$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

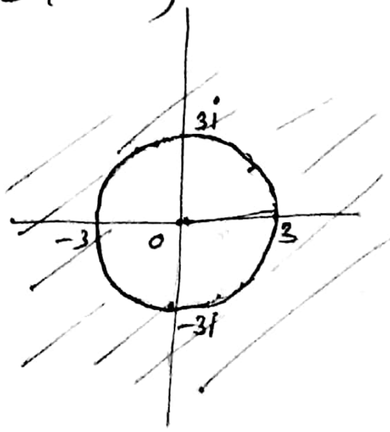
$$= 1 - \frac{8}{3\left[1 + \left(\frac{z}{3}\right)\right]} + \frac{3}{z\left[1 + \left(\frac{2}{z}\right)\right]}$$

$$= 1 - \frac{8}{3}\left[1 + \left(\frac{z}{3}\right)\right]^{-1} + \frac{3}{z}\left[1 + \frac{2}{z}\right]^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right] + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \cdots\right]$$

$$\left(\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots\right)$$

case iii) when $|z| > 3$



$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

since, $|z| > 3 > 2 \Rightarrow |z| > 2$

$\therefore \quad |z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1$

and $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore\ f(z) = 1 - \frac{8}{z\left(1+\frac{3}{z}\right)} + \frac{3}{z\left(1+\frac{2}{z}\right)}$$

$$= 1 - \frac{8}{z}\left[1+\left(\frac{3}{z}\right)\right]^{-1} + \frac{3}{z}\left[1+\left(\frac{2}{z}\right)\right]^{-1}$$

$$= 1 - \frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\cdots\right] + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\cdots\right]$$

Example ④ Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region

i) $1 < |z-1| < 2$ ii) $1 < |z-3| < 2$ iii) $|z| < 1$

Solution: Given: $f(z) = \frac{1}{(z-1)(z-2)}$

Consider,
$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\Rightarrow \quad A(z-2) + B(z-1) = 1$$

if $z = 1 \Rightarrow A(1-2) + B(0) = 1 \Rightarrow A = -1$

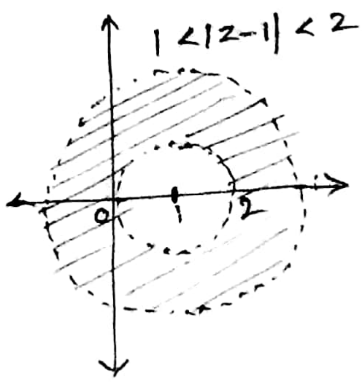
if $z = 2 \Rightarrow A(0) + B(2-1) = 1 \Rightarrow B = 1$

$$\therefore\ \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}$$

i) when $1 < |z-1| < 2$

$\therefore\ 1 < |z-1| \Rightarrow \left|\frac{1}{z-1}\right| < 1$

and $|z-1| < 2 \Rightarrow \left|\frac{z-1}{2}\right| < 1$



$$f(z) = -\frac{1}{z-1} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{z-1} - \frac{1}{1-(z-1)}$$

$$= -\frac{1}{z-1} - \left[1-(z-1)\right]^{-1}$$

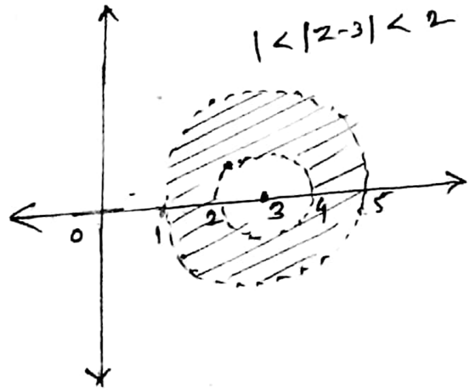
$$\therefore \quad f(z) = -\frac{1}{z-1} - \left[1 + (z-1) + (z-1)^2 + (z+1)^3 + \cdots\right]$$

$$\left(\because (1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots\right)$$

ii) When $1 < |z-3| < 2$

$$\therefore \quad 1 < |z-3| \Rightarrow \left|\frac{1}{z-3}\right| < 1$$

and $|z-3| < 2 \Rightarrow \left|\frac{z-3}{2}\right| < 1$



$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{(z-3)+2} + \frac{1}{(z-3)+1}$$

$$= -\frac{1}{2\left[1 + \left(\frac{z-3}{2}\right)\right]} + \frac{1}{(z-3)\left[1 + \left(\frac{1}{z-3}\right)\right]}$$

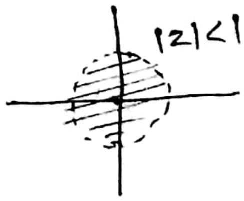
$$= \frac{-1}{2}\left[1 + \left(\frac{z-3}{2}\right)\right]^{-1} + \frac{1}{(z-3)}\left[1 + \left(\frac{1}{z-3}\right)\right]^{-1}$$

$$= -\frac{1}{2}\left[1 - \left(\frac{z-3}{2}\right) + \left(\frac{z-3}{2}\right)^2 - \left(\frac{z-3}{2}\right)^3 + \cdots\right]$$

$$+ \frac{1}{(z-3)}\left[1 - \left(\frac{1}{z-3}\right) + \left(\frac{1}{z-3}\right)^2 - \left(\frac{1}{z-3}\right)^3 + \cdots\right]$$

iii) when $|z| < 1 \Rightarrow |z| < 1 < 2 \Rightarrow |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$



$$= [1-z]^{-1} + \frac{1}{-2\left[1 - \frac{z}{2}\right]}$$

$$= [1-z]^{-1} - \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$$

$$= \left[1 + z + z^2 + z^3 + \cdots\right] - \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

Homework :

Ex. ① obtain Taylor's or Laurent's series of the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$

when i) $|z| < 1$ ii) $|z| < 2$

Ex ② Expand $f(z) = \dfrac{z^2 - 1}{z^2 + 5z + 6}$ around $z = 1$

Ex. ③ Find all possible Laurent's expansion of the function. $f(z) = \dfrac{7z - 2}{z(z - \text{[illegible]})(z+1)}$ about $z = -1$

* Zeros of an analytic function :-

Definition: let $f(z)$ be the analytic function then a point z_0 is said to be zeros of $f(z)$ if $f(z_0) = 0$

Note that :

① If $f(z_0) = 0$, $f'(z_0) \neq 0$ then z_0 is called simple zero or zero of order 1

② If $f(z_0) = 0$, $f'(z_0) = 0$, $f''(z_0) \neq 0$ then z_0 is called zero of order 2

③ If $f(z_0) = 0$, $f'(z_0) = 0, \cdots, f^{(n-1)}(z_0) = 0$, $f^{(n)}(z_0) \neq 0$ then z_0 is called zero of order 'n'

Example ① find zeros of $f(z) = (z-1)e^{z}$

solution: Given: $f(z) = (z-1)e^{z}$

clearly, $f(z) = 0$ if $(z-1)e^{z} = 0$ if $z = 1$

$\Rightarrow z = 1$

$\therefore z = 1$ is zero of $f(z)$

Now, $f'(z) = (z-1)e^{z} + e^{z}$

$\therefore f'(1) = (1-1)e^{1} + e^{1} = e$

i.e. $f'(1) \neq 0$

Hence, $z = 1$ is simple zero of $f(z)$

Example ② find the zeros and their order of

$f(z) = z^2 \sin z$

Solution: consider $f(z) = 0$

$\Rightarrow \quad z^2 \sin z = 0$

$\Rightarrow \quad z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

Therefore, $z = 0, \pm\pi, \pm 2\pi, \dots$ are all the zeros of $f(z)$

Now, $f'(z) = z^2 \cos z + 2z \sin z$

$\Rightarrow \quad f'(z) = 0$, for only $z = 0$

But $f'(z) \neq 0$ for every $z = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

$\therefore$ $z = \pm\pi, \pm 2\pi, \pm 3\pi$ are zeros of order 1 or simple

Now, $f''(z) = 2z\cos z - z^2 \sin z + 2z\cos z + 2\sin z$

i.e. $f''(z) = 4z\cos z - z^2\sin z + 2\sin z$

$\Rightarrow \quad f''(z) = 0$ for $z = 0$

Now, $f'''(z) = 4\cos z - 4z\sin z - z^2\cos z - 2z\cos z + 2\cos z$

i.e. $f'''(z) = 6\cos z - 4z\sin z - z^2\cos z - 2z\cos z$

$\Rightarrow \quad f'''(z) \neq 0$, for $z = 0$

therefore, $z = 0$ is zero of order 3

Homework:

* find the zeros and its order of following function

① $f(z) = z \tan z$ ② $(z^2 - 1)(z^3 + 3z + 2)$

* Singular point:

The point z_0 is said to be singular point of $f(z)$ if

i) $f(z)$ is not analytic at z_0

ii) $f(z)$ is analytic at every point in the neighbourhood of z_0

for example: (1) $f(z) = \frac{z^2}{z-2}$

$\therefore$ $z=2$ is singular point of $f(z)$

(2) $f(z) = \frac{z}{z(z+1)}$

$\therefore$ $z=0$ and $z=-1$ are singular points of $f(z)$

→ Negative term in Laurent's Series:-

Note that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$

that is $f(z) = \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right]$

$$+ \left[b_1 \frac{1}{z-z_0} + b_2 \frac{1}{(z-z_0)^2} + b_3 \frac{1}{(z-z_0)^3} + \cdots\right]$$

In the above power series the terms

$\frac{1}{z-z_0}$, $\frac{1}{(z-z_0)^2}$, $\frac{1}{(z-z_0)^3}$, ... are called

as Negative terms.

* Types of singularity :

① **Pole :-** The singular point z_0 of $f(z)$ is said to be pole If

the Laurent's series of $f(z)$ around $z = z_0$ contains only finite number of Negative terms

Note that: If z_0 is pole of $f(z)$ of order 1 then z_0 is called simple pole of $f(z)$

② Removable singularity :

The singular point z_0 of $f(z)$ is said to be Removable If

The Laurent's series of $f(z)$ around $z = z_0$ does not contains a Negative term

③ Essential singularity :

The singular point z_0 of $f(z)$ is said to be Essential If

The Laurent's series of $f(z)$ around $z = z_0$ contains infinite number of Negative term

Examples:

Determine the nature of singularities of following function.

① $\dfrac{e^z}{z^3}$ ② $\dfrac{\sin z}{z}$ ③ $e^{\frac{1}{z}}$

④ $\dfrac{\cot \pi z}{(z-a)^3}$

Solution: ① Given $f(z) = \dfrac{e^z}{z^3}$

$$\therefore \quad f(z) = \frac{1}{z^3}\left[e^z\right]$$

$$= \frac{1}{z^3}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\cdot z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots$$

that is Laurent's series expansion of $f(z)$ contains only finite Negative terms (3 Negative terms)

Hence, $z = 0$ is pole of order 3

② Given: $f(z) = \dfrac{\sin z}{z}$

clearly, $z=0$ is singularity of $f(z)$

Now, $$f(z) = \frac{1}{z}\left[\sin z\right]$$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

that is Laurents series of $f(z)$ contains no Negative term

Hence, $z=0$ is Removable singularity.

③ Given: $f(z) = e^{\frac{1}{z}}$

Clearly, $z = 0$ is singular point of $f(z)$

Since, $e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$\therefore f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \cdots$

that is laurent's series of $f(z)$ contains infinite number of Negative terms

Hence, $z = 0$ is a essential singularity.

④ Given: $f(z) = \frac{\cot \pi z}{(z-a)^3}$

$\Rightarrow f(z) = \frac{\cos \pi z}{\sin \pi z \cdot (z-a)^3}$

$\therefore$ The points $z = a$, $z = 0, \pm 1, \pm 2, \cdots$ are singular points.

$\therefore$ here, $z = a$ is pole of order 3

and $z = 0, \pm 1, \pm 2, \cdots$ are simple Poles

Homework:

Determine the Nature of singularity

① $z e^{\frac{1}{z^2}}$ ② $\frac{1 - e^{2z}}{z^3}$ ③ $\frac{1 - e^{z}}{z}$

④ $\frac{1 - \cos 2z}{z}$

* Residues :

If z_0 be the singular point of $f(z)$ then the coefficient of $\frac{1}{z-z_0}$ in the laurent series expansion of $f(z)$ is called Residue of $f(z)$ at z_0

that is

$$\boxed{\text{Residue of } f(z) \left(\text{at } z=z_0\right) = b_1 = \text{coefficient of } \frac{1}{z-z_0}}$$

Note that :

① If z_0 is simple pole of $f(z)$ then

$$\boxed{\text{Residue of } f(z) \text{ at } z_0 = \lim_{z \to z_0} (z-z_0) \cdot f(z)}$$

② If z_0 is pole of order 'm' then

$$\boxed{\text{Residue of } f(z) \text{ at } z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z)\right]}$$

Examples :

Find the Residues at each pole of the following function

① $\dfrac{e^z}{(1-z)^3}$ ② $\dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$

Solution:

(1) Given: $f(z) = \dfrac{e^z}{(z-1)^3}$

clearly, the point $z=1$ is pole of $f(z)$ of order 3

$\therefore$ Residue of $f(z)$ at $z = z_0 = 1$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z) \right] \quad \left(\begin{array}{l}\text{for pole}\\ \text{of order } m\end{array}\right)$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} \left[(z-1)^3 \cdot \frac{e^z}{(z-1)^3} \right] \quad \left(\begin{array}{l}\text{if } m=3\\ \& \ z_0 = 1\end{array}\right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} (e^z)$$

$$= \frac{1}{2} \lim_{z \to 1} e^z$$

$$= \frac{1}{2} \cdot e^1$$

$\therefore$ Residue of $f(z)$ at $z=1$ is $\frac{1}{2} e$

(2) Given: $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$

clearly, $z=1$ is simple pole of $f(z)$ and $z=2$ is pole of $f(z)$ of order 2

* Residue of $f(z)$ at $z = z_0 = 1$

$$= \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2} \quad (\because z_0 = 1)$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)^2}$$

$$= \frac{\sin \pi + \cos \pi}{(1-2)^2}$$

$$= -1$$

* Residue at f(z) at z = 2

$$= \frac{1}{(-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$= \frac{1}{1!} \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 \cdot \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2} \right]$$

$$= \lim_{z \to 2} \frac{d}{dz} \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1} \right]$$

$$= \lim_{z \to 2} \left[\frac{(z-1)\left[(\cos \pi z^2).(2\pi z) - (\sin \pi z^2)(2\pi z) \right] - (\sin \pi z^2 + \cos \pi z^2)(}{(z-1)^2} \right]$$

$$\left(\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right)$$

$$= \frac{(2-1)\left[\cos 4\pi \cdot (4\pi) - \sin 4\pi \, (4\pi) \right] - (\sin 4\pi + \cos 4\pi)}{(2-1)^2}$$

$$= \frac{((1)\, 4\pi - 0) - (0 + 1)}{1}$$

$$= 4\pi - 1$$

Ex (3) Find the residue of $f(z) = \frac{1}{z - \sin z}$ at its singularity using laurent's series expansion

Solution: Given: $f(z) = \frac{1}{z - \sin z}$

clearly, $z=0$ is pole of $f(z)$

Now,
$$f(z) = \frac{1}{z - \sin z}$$

$$= \frac{1}{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{1}{\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}\left[1 - \frac{z^2}{4\times5} + \frac{z^4}{4\times5\times6\times7} - \cdots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}} \cdot \left[1 - \frac{z^2}{20} + \frac{z^4}{840} - \cdots\right]^{-1}$$

$$= \frac{3!}{z^3}\left[1 - \left(\frac{z^2}{20} + \frac{z^4}{840} - \cdots\right)\right]^{-1}$$

$$= \frac{6}{z^3}\left[1 + \frac{z^2}{20} + \frac{z^4}{840} \cdots\right] \quad \left(\because [1-z]^{-1} = 1 + z + z^2 + \cdots\right)$$

$$f(z) = \frac{6}{z^3} + \frac{6}{20}\frac{1}{z} + \frac{6z}{840} + \cdots$$

$\therefore$ Residue of $f(z)$ (at $z=0$) $= b_1 =$ coefficient of $\frac{1}{z}$

$$= \frac{6}{20}$$

$$= \frac{3}{10}$$

Ex. (4) find the sum of all the residues at singular point of $f(z) = \dfrac{z}{(z-1)^2 (z^2-1)}$

solution: Given: $f(z) = \dfrac{z}{(z-1)^2(z^2-1)} = \dfrac{z}{(z-1)^3 \cdot (z+1)}$

clearly, $z=1$ is pole of $f(z)$ of order 3 and $z=-1$ is a simple pole of $f(z)$

Now, Residue of $f(z)$ at $z = z_0 = 1$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z) \right] \quad \left(\because \text{for pole of order } m \right)$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} \left[(z-1)^3 \frac{z}{(z-1)^3 (z+1)} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} \left(\frac{z}{z+1} \right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} \left[\frac{(z+1)(1) - z(1)}{(z+1)^2} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} \left(\frac{1}{(z+1)^2} \right)$$

$$= \frac{1}{2} \lim_{z \to 1} \left[- \frac{2}{(z+1)^3} \right]$$

$$= \frac{1}{2} \left[- \frac{2}{(1+1)^3} \right]$$

$$= -\frac{1}{8}$$

Now, Residue of $f(z)$ at $z = z_0 = -1$

$$= \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to -1} (z+1) \frac{z}{(z-1)^3 (z+1)} = \frac{-1}{(-1-1)^3} = \frac{1}{8}$$

Therefore, Sum of the Residues $= -\frac{1}{8} + \frac{1}{8} = \underline{\underline{0}}$

Homework:

* Find the residues of following function at its singularities.

① $\dfrac{1-e^{2z}}{z^3}$ ② $\dfrac{1-z}{1-\cos z}$ ③ $\dfrac{\sin \pi z}{(z-1)^2 (z-2)}$

④ find the sum of the Residues at singular point of the function $\dfrac{z}{z^3+1}$

* Cauchy Residue Theorem:

Statement: let $z_1, z_2, \cdots, z_n$ be the singular point of $f(z)$ lies inside a simple closed curve C. If $f(z)$ is analytic on and inside a simple closed curve C except the points $z_1, z_2, \cdots, z_n$ then

$$\boxed{\oint_C f(z)\,dz = 2\pi i\left[\text{sum of residues at } z_1, z_2, \cdots z_n\right]}$$

In perticular,

① If z_1 is only the singular point of $f(z)$ lie inside C then

$$\boxed{\oint_C f(z)\,dz = 2\pi i\left[\text{Residues of } f(z) \text{ at } z_1\right]}$$

Examples: Using Cauchy residue theorem evaluate $\oint_C \frac{z^2+3}{z^2-1}\,dz$ where C is the circle $|z-1| = 1$

Solution: let $f(z) = \frac{z^2+3}{z^2-1} = \frac{z^3+3}{(z+1)(z-1)}$

Note that $z^2-1=0 \Rightarrow z = 1, -1$

$\therefore$ clearly, $z = 1$ and $z = -1$ are the simple pole of $f(z)$

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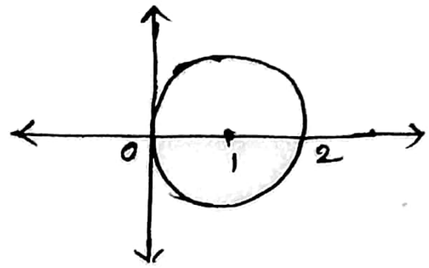
since, $C: |z-1| = 1$ which equation of circle with centre $(1,0)$ and radius 1

$\therefore$ only a pole $z = 1$ lies inside C while $z = -1$ lies outside C



Now, Residue at $(z=1)$

$$= \lim_{z\to 1} (z-1)\cdot f(z)$$

$$= \lim_{z\to 1} (z-1) \frac{z^3+3}{(z+1)(z-1)}$$

$$= \lim_{z\to 1} \frac{z^3+3}{z+1}$$

$$= \frac{(1)^3+3}{1+1}$$

$$= 2$$

$\therefore$ By Cauchy Residue theorem,

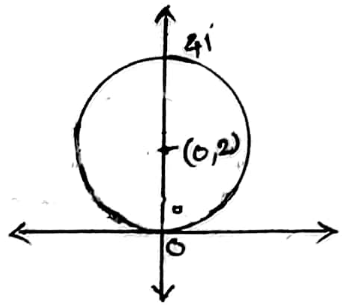
$$\oint_C f(z)\,dz = 2\pi i\,[\text{Residue of } f(z) \text{ at } z_1]$$

$$\Rightarrow \oint_C \frac{z^3+1}{z^2-1}\,dz = 2\pi i\,[2] = 4\pi i$$

i.e $$\oint_C \frac{z^3+1}{z^2-1}\,dz = 4\pi i$$

Ex. 2 Using residue theorem evaluate

$$\oint_C \frac{e^{2z}}{(z-\pi i)^3}\, dz \quad \text{where } C \text{ is } |z-2i| = 2$$



Solution: let $f(z) = \dfrac{e^{2z}}{(z-\pi i)^3}$

Given: $C = |z-2i| = 2$ which is circle with centre $(0,2)$ and radius 2

Note that $(z-\pi i)^3 = 0 \Rightarrow z = \pi i$

clearly, $z = \pi i$ is pole of $f(z)$ of order 3 and $z = \pi i = (0, \pi)$ lies inside C

Now, Residue at $z = z_0 = \pi i$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m . f(z) \right] \quad \left(\begin{array}{c}\text{if } z_0 \text{ is} \\ \text{Pole of order} \\ 'm'\end{array}\right)$$

$$= \frac{1}{2!} \lim_{z \to \pi i} \frac{d^2}{dz^2} \left[(z-\pi i)^3 . \frac{e^{2z}}{(z-\pi i)^3} \right]$$

$$= \frac{1}{2} \lim_{z \to \pi i} \frac{d^2}{dz^2} \left[e^{2z} \right]$$

$$= \frac{1}{2} \lim_{z \to \pi i} \frac{d}{dz} \left(2e^{2z} \right)$$

$$= \frac{1}{2} \lim_{z \to \pi i} 4e^{2z} = \frac{1}{2} 4e^{2(\pi i)} = 2$$

$$(\because e^{2\pi i} = 1)$$

$\therefore$ By cauchy residue theorem.

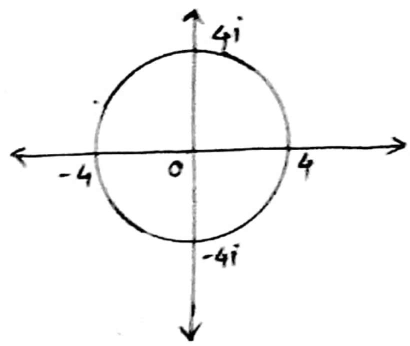
$$\int_C f(z)\, dz = 2\pi i \left[\text{Residue of } f(z) \text{ at } z_0 \right]$$

$$\Rightarrow \int_C f(z)\, dz = 2\pi i\,(2) = 4\pi i$$

Ex (8) Using residue theorem evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is $|z|=4$

Solution: let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

Given: $C: |z|=4$ which is equation of circle with centre origin and radius 4



Note that $(z^2+\pi^2)^2 = 0$

$\Rightarrow (z^2+\pi^2)(z^2+\pi^2) = 0$

$\Rightarrow z = \pi i, \pi i, -\pi i, -\pi i$ all lies inside C

Therefore, $z = \pi i$ is pole of $f(z)$ of order 2

and $z = -\pi i$ is pole of $f(z)$ of order 2

Now, Residue of $f(z)$ at $z = z_0 = \pi i$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$= \frac{1}{1!} \lim_{z \to \pi i} \frac{d}{dz} \left[(z-\pi i)^2 \frac{e^z}{(z^2+\pi^2)^2} \right]$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left[(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right]$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left[\frac{e^z}{(z+\pi i)^2} \right]$$

$$= \lim_{z \to \pi i} \left[\frac{(z+\pi i)^2 \cdot e^z - e^z \, 2(z+\pi i)}{(z+\pi i)^4} \right]$$ ($\because$ By division Rule of derivative)

$$= \lim_{z \to \pi i} \left[\frac{e^z (z+\pi i - 2)}{(z+\pi i)^3} \right]$$

$$= \frac{e^{\pi i}(\pi i + \pi i - 2)}{(\pi i + \pi i)^3}$$

$$= \frac{e^{\pi i} \cdot 2i(\pi + i)}{(2\pi i)^3}$$

$$= \frac{e^{\pi i}\, 2i(\pi + i)}{-8\pi^3 i}$$

$$= \frac{\pi + i}{4\pi^3} \qquad (\because e^{\pi i} = \cos\pi + i\sin\pi = -1)$$

Residue of $f(z)$ at $z = z_0 = -\pi i$

$$= \frac{1}{(2-1)!} \lim_{z \to -\pi i} \frac{d}{dz}\left[(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2}\right]$$

$$= \lim_{z \to -\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$$= \lim_{z \to -\pi i} \left[\frac{(z-\pi i)^2 e^z - e^z \cdot 2(z-\pi i)}{(z-\pi i)^4}\right]$$

$$= \lim_{z \to -\pi i} \frac{e^z (z - \pi i - 2)}{(z-\pi i)^3}$$

$$= \frac{e^{-\pi i}(-\pi i - \pi i - 2)}{(-\pi i - \pi i)^3} = \frac{-e^{-\pi i}\, 2i(\pi - i)}{-8\pi^3 i^3}$$

$$= \frac{-e^{-\pi i}(\pi - i)}{4\pi^3} = \frac{\pi - i}{4\pi^3} \qquad (\because e^{-\pi i} = \cos\pi - i\sin\pi = -1)$$

$\therefore$ By Residue Theorem,

$$\oint_c f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

$$= 2\pi i\left[\frac{\pi + i}{4\pi^3} + \frac{\pi - i}{4\pi^3}\right] = 2\pi i\left[\frac{2\pi}{4\pi^3}\right]$$

$$\therefore \int \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}$$

Ex. (4) Using Residue theorem evaluate

$$\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz \quad \text{where } C \text{ is } |z|=1$$

Solution: let $f(z) = e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right)$

Clearly, $z = 0$ is singular point of $f(z)$ which lies inside C

Now, $f(z) = e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right)$

$$\Rightarrow \quad f(z) = \left[1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \cdots\right]\left[\frac{1}{z} - \frac{1}{3!\,z^3} + \cdots\right]$$

$$\left(\because e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \& \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$\Rightarrow e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \cdots$$

which is Laurent's series expansion around $z = 0$

$\therefore$ Residue of $f(z)$ at $z = 0$ = coefficient of $\frac{1}{z}$

$= 1$

$\therefore$ By Cauchy Residue theorem,

$$\int_C f(z)\, dz = 2\pi i \left[\text{Residues at } z = 0\right]$$

$$\Rightarrow \int_C f(z)\, dz = 2\pi i\,(1)$$

$$\Rightarrow \int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz = 2\pi i$$

Homework :

Ex. 1 Using cauchy residue theorem evaluate $\oint_C \frac{z^3+3}{z^2-1}\,dz$ where C is the circle $|z+1| = 1$

Ans : $-4\pi i$

Ex 2 Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z| = 2$

Ans: $\frac{\pi i}{32}$

Module : Complex Integration

Evaluation of Integration

Types of function 'f(z)'	Types of contor 'C'	Solving Criterion.
f(z) : 'polynomial' (straight forword)	C : 'straight line' or 'parabola'	$z = x+iy$ put $x = t$ or $y = t$
f(z) : Not involving trignometric, exponential, logarithm	C : circle $\|z\| = r$	put $z = re^{i\theta}$
f(z) : $\frac{p(z)}{(z-z_1)(z-z_2)}$	C : circle ellipse, square	
i) z_1, z_2 Both Not lies inside C	"	cauchy's theorem $\int_C f(z)\,dz = 0$
ii) z_1 lies inside z_2 not lies inside C	"	cauchy integral formula OR cauchy Residue theorem
iii) z_1, z_2 Both lies inside C	"	cauchy integral formula using partial fraction OR cauchy Residue theorem
f(z) : involves trignometric, exponential, logarithmic terms	C : circle, ellipse, square	cauchy Residue theorem (if z_0 is singular point then find Residue using laurent's series expansion $z=z_0$)

Taylors or Laurent series Expansion

standard form : $f(z) = \dfrac{p(z)}{q(z)}$

Step 1. degree of $p(z)$ < degree of $q(z)$

Step 2. $f(z) = \dfrac{p(z)}{(z-z_1)(z-z_2)}$

Step 3. partial fraction

$$f(z) = \frac{A}{(z-z_1)} + \frac{B}{(z-z_2)}$$

Step 4. i) $|z| < z_1$ (ROC) , $z_1 < z_2$

ii) $z_1 < |z| < z_2$ (ROC) , $z_1 < z_2$

iii) $|z| > z_2$ (ROC) , $z_1 < z_2$