Module: Z - Transform

* sequences : An orderd set of numbers is called a sequence
and It is denoted by $\{f(k)\}$

for ex. i> $\{f(k)\} = \{\cdots, -3, -2, -1, \underset{\uparrow}{0}, 1, 2, 3, \cdots\}$

ii> $\{f(k)\} = \{\underset{\uparrow}{2^0}, 2^1, 2^2, 2^3, \cdots\}$

iii> $\{f(k)\} = \{\underset{\uparrow}{\frac{1}{1}}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

Note that : the arrow $\uparrow$ shows the 0^{th} position of sequence

* convergence and divergence of the sequence :-

— let $\{f(k)\}$ be the sequence of number if $f(k)$ tends to a finite real number L as k tends to infinity then the sequence $\{f(k)\}$ is called convergent sequence

— If sequence $\{f(k)\}$ is not convergent then the sequence $\{f(k)\}$ is called divergent

<u>for ex.</u> i) $\{f(k)\} = \{a, a, a, a, \cdots\}$ converges to 'a'

ii) $\{f(k)\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ converges to 0

iii) $\{f(k)\} = \{1+\frac{1}{2^0}, 1+\frac{1}{2^1}, 1+\frac{1}{2^2} \cdots\}$ converges to 1

iv) $\{f(k)\} = \{1, 2, 3, 4, \cdots\}$ diverges to ∞

v) $\{f(k)\} = \{0, 1, 0, 1, \cdots\}$ diverges (oscillates between 0 and 1)

vi) $\{f(k)\} = \{-1, -2, -3, \cdots\}$ diverges to $-\infty$

<u>Note that</u>: $\{f(k)\} = \{\cdots, f(-3), f(-2), f(-1), f(0), f(1), \cdots\}$

<u>for ex</u>: $\{f(k)\} = \{\cdots, 2^{-2}, 2^{-1}, 2^{0}, 2^{1}, 2^{2}, \cdots\}$

here, $\ldots f(-2) = 2^{-2}$, $f(-1) = 2^{-1}$, $f(0) = 2^0$

$f(1) = 2^1$, $f(2) = 2^2$, $\cdots$

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* z- transform :

let $\{f(k)\} = \{\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots\}$

be the sequence of terms (Numbers)

let $z = x + iy$ be a complex number then

$$Z\{f(k)\} = \cdots f(-3)z^3 + f(-2)z^2 + f(-1)z + f(0)z^0 + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \cdots$$

$$= \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

i-e
$$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}}$$

for ex. 1 If $\{f(k)\} = \{9, \underset{\uparrow}{6}, 3, 0, -3, -6, -9\}$

solution: given that $\{f(k)\} = \{9, \underset{\uparrow}{6}\ 3, 0, -3, -6, -9\}$

$\Rightarrow f(-1) = 9,\ f(0) = 6,\ f(1) = 3,\ f(2) = 0,\ f(3) = -3$
$f(4) = -6,\ f(5) = -9$

therefore the z-transform of $\{f(k)\}$ is

$$z\{f(k)\} = \sum_{k=-1}^{5} \frac{f(k)}{z^k} = \sum_{k=-1}^{5} f(k)\, z^{-k}$$

$$= 9z^{-(-1)} + 6z^{0} + 3z^{-1} + 0z^{-2} - 3z^{-3} - 6z^{-4} - 9z^{-5}$$

$$= 9z + 6z^{0} + 3z^{-1} + 0z^{-2} - 3z^{-3} - 6z^{-4} - 9z^{-5}$$

$$Z\{f(k)\} = 9z + 6 + \frac{3}{z} + 0 - \frac{3}{z^3} - \frac{6}{z^4} - \frac{9}{z^5}$$

* Important Series and its ROC

	Series		ROC
1)	$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$	$\rightarrow$	$\lvert z\rvert < 1$
2)	$1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z}$	$\rightarrow$	$\lvert z\rvert < 1$
3)	$1 - 2z + 3z^2 - 4z^3 + \cdots = \frac{1}{(1+z)^2}$	$\rightarrow$	$\lvert z\rvert < 1$
4)	$1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}$	$\rightarrow$	$\lvert z\rvert < 1$
5)	$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z$	$\rightarrow$	$\lvert z\rvert < \infty$
6)	$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sin z$	$\rightarrow$	$\lvert z\rvert < \infty$
7)	$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \cos z$	$\rightarrow$	$\lvert z\rvert < \infty$
8)	$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = \log(1+z)$	$\rightarrow$	$\lvert z\rvert < 1$

Ex. ② find the z-transform of a sequence

$$\{f(k)\} = \begin{cases} 4^k, & \text{for } k<0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$$

Solution:

Given: $$\{f(k)\} = \begin{cases} 4^k, & \text{for } k<0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$$

$$\Rightarrow \{f(k)\} = \{\dots, 4^{-3}, 4^{-2}, 4^{-1}, \underset{\uparrow}{3^0}, 3^1, 3^2, 3^3, \dots\}$$

$$\Rightarrow Z\{f(k)\} = \dots + 4^{-3}z^{3} + 4^{-2}z^{2} + 4^{-1}z^{1} + 3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + \dots$$

$$= \dots + \frac{z^3}{4^3} + \frac{z^2}{4^2} + \frac{z}{4} + 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \dots$$

$$= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \dots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots\right]$$

$$= \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \dots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots\right]$$

$$= \frac{z}{4}\left[1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \dots\right] + \left[1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \dots\right]$$

$$= \frac{z}{4}\,\frac{1}{1-\left(\frac{z}{4}\right)} + \frac{1}{1-\left(\frac{3}{z}\right)} \qquad \text{for } \left|\frac{z}{4}\right|<1,\ \left|\frac{3}{z}\right|<1$$

$$\left(\because\ 1 + z + z^2 + \dots = \frac{1}{1-z},\quad |z|<1\right)$$

$$= \frac{z}{4}\,\frac{4}{4-z} + \frac{z}{z-3} \qquad \text{for } |z|<4,\ 3<|z|$$

$$= \frac{z}{4-z} + \frac{z}{z-3} \quad , \quad \text{for } 3 < |z| \, , \; |z| < 4$$

$$= \frac{z(z-3) + z(4-z)}{(4-z)(z-3)} \quad , \quad \text{for } 3 < |z| < 4$$

$$= \frac{z}{(4-z)\,(z-3)} \quad , \quad \text{for } 3 < |z| < 4$$

$$\therefore \quad Z\{f(k)\} = \frac{z}{(4-z)(z-3)} \quad , \quad \text{if } 3 < |z| < 4$$