Module: Z - Transform

* sequences : An orderd set of numbers is called a sequence and It is denoted by $\{f(k)\}$

for ex. i) $\{f(k)\} = \{\cdots, -3, -2, -1, \underset{\uparrow}{0}, 1, 2, 3, \cdots\}$

ii) $\{f(k)\} = \{\underset{\uparrow}{2^0}, 2^1, 2^2, 2^3, \cdots\}$

iii) $\{f(k)\} = \{\underset{\uparrow}{\frac{1}{1}}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

Note that : the arrow $\uparrow$ shows the 0^{th} position of sequence

* convergence and divergence of the sequence :-

— let $\{f(k)\}$ be the sequence of number if $f(k)$ tends to a finite real number L as k tends to infinity then the sequence $\{f(k)\}$ is called convergent sequence

— If sequence $\{f(k)\}$ is not convergent then the sequence $\{f(k)\}$ is called divergent for ex. i) $\{f(k)\} = \{\underset{\uparrow}{a}, a, a, a, \cdots\}$ converges to 'a'

ii) $\{f(k)\} = \{\underset{\uparrow}{\frac{1}{1}}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ converges to 0

iii) $\{f(k)\} = \{\underset{\uparrow}{1+\frac{1}{2^0}}, 1+\frac{1}{2^1}, 1+\frac{1}{2^2} \cdots\}$ converges to 1

iv) $\{f(k)\} = \{\underset{\uparrow}{1}, 2, 3, 4, \cdots\}$ diverges to ∞

v) $\{f(k)\} = \{\underset{\uparrow}{0}, 1, 0, 1, \cdots\}$ diverges (oscillates between 0 and 1)

vi) $\{f(k)\} = \{\underset{\uparrow}{-1}, -2, -3, \cdots\}$ diverges to $-\infty$

Note that: $\{f(k)\} = \{\cdots, f(-3), f(-2), f(-1), f(0), f(1), \cdots\}$

for ex: $\{f(k)\} = \{\cdots, 2^{-2}, 2^{-1}, \underset{\uparrow}{2^{0}}, 2^{1}, 2^{2}, \cdots\}$

here, $\ldots f(-2) = 2^{-2}$, $f(-1) = 2^{-1}$, $f(0) = 2^{0}$

$f(1) = 2^{1}$, $f(2) = 2^{2}$, $\cdots$

* z- transform :

let $\{f(k)\} = \{\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots\}$

be the sequence of terms (Numbers)

let $z = x + iy$ be a complex number then

$$Z\{f(k)\} = \cdots f(-3)z^{3} + f(-2)z^{2} + f(-1)z + f(0)z^{0} + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \cdots$$

$$= \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^{k}}$$

i-e
$$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^{k}}}$$

for ex. 1 If $\{f(k)\} = \{9, \underset{\uparrow}{6}, 3, 0, -3, -6, -9\}$

solution: given that $\{f(k)\} = \{9, \underset{\uparrow}{6}\ 3, 0, -3, -6, -9\}$

$\Rightarrow f(-1) = 9,\ f(0) = 6,\ f(1) = 3,\ f(2) = 0,\ f(3) = -3$
$f(4) = -6,\ f(5) = -9$

therefore the z-transform of $\{f(k)\}$ is

$$z\{f(k)\} = \sum_{k=-1}^{5} \frac{f(k)}{z^{k}} = \sum_{k=-1}^{5} f(k)\, z^{-k}$$

$$= 9z^{-(-1)} + 6z^{0} + 3z^{-1} + 0z^{-2} - 3z^{-3} - 6z^{-4} - 9z^{-5}$$

$$= 9z + 6z^{0} + 3z^{-1} + 0z^{-2} - 3z^{-3} - 6z^{-4} - 9z^{-5}$$

$$Z\{f(k)\} = 9z + 6 + \frac{3}{z} + 0 - \frac{3}{z^3} - \frac{6}{z^4} - \frac{9}{z^5}$$

*** Important Series and its ROC**

	Series		ROC
1>	$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$	$\rightarrow$	$\lvert z\rvert < 1$
2>	$1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z}$	$\rightarrow$	$\lvert z\rvert < 1$
3>	$1 - 2z + 3z^2 - 4z^3 + \cdots = \frac{1}{(1+z)^2}$	$\rightarrow$	$\lvert z\rvert < 1$
4>	$1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}$	$\rightarrow$	$\lvert z\rvert < 1$
5>	$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z$	$\rightarrow$	$\lvert z\rvert < \infty$
6>	$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sin z$	$\rightarrow$	$\lvert z\rvert < \infty$
7>	$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \cos z$	$\rightarrow$	$\lvert z\rvert < \infty$
8>	$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = \log(1+z)$	$\rightarrow$	$\lvert z\rvert < 1$

Ex. ② find the z-transform of a sequence

$$\{f(k)\} = \begin{cases} 4^k, & \text{for } k<0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$$

Solution:

Given: $$\{f(k)\} = \begin{cases} 4^k, & \text{for } k<0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$$

$$\Rightarrow \{f(k)\} = \{\ldots, 4^{-3}, 4^{-2}, 4^{-1}, \underset{\uparrow}{3^0}, 3^1, 3^2, 3^3, \ldots\}$$

$$\Rightarrow Z\{f(k)\} = \ldots + 4^{-3}z^{3} + 4^{-2}z^{2} + 4^{-1}z^{1} + 3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + \ldots$$

$$= \ldots + \frac{z^3}{4^3} + \frac{z^2}{4^2} + \frac{z}{4} + 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \ldots$$

$$= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \ldots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots\right]$$

$$= \frac{z}{4}\left[1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \ldots\right] + \left[1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \ldots\right]$$

$$= \frac{z}{4} \frac{1}{1-\left(\frac{z}{4}\right)} + \frac{1}{1-\left(\frac{3}{z}\right)} \quad \text{for } \left|\frac{z}{4}\right| < 1, \left|\frac{3}{z}\right| < 1$$

$$\left(\because 1 + z + z^2 + \ldots = \frac{1}{1-z}, \quad |z| < 1\right)$$

$$= \frac{z}{4} \frac{4}{4-z} + \frac{z}{z-3} \quad \text{for } |z| < 4, \ 3 < |z|$$

$$= \frac{z}{4-z} + \frac{z}{z-3} \quad , \quad \text{for } 3 < |z| \, , \; |z| < 4$$

$$= \frac{z(z-3) + z(4-z)}{(4-z)(z-3)} \quad , \quad \text{for } 3 < |z| < 4$$

$$= \frac{z}{(4-z)\,(z-3)} \quad , \quad \text{for } 3 < |z| < 4$$

$$\therefore \quad Z\{f(k)\} = \frac{z}{(4-z)\,(z-3)} \quad , \quad \text{if } 3 < |z| < 4$$

Ex. ② Find the z-transform of $k\,a^k$, $k \geq 0$

Solution: here, $f(k) = k\,a^k$, $k \geq 0$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} k\, a^k z^{-k} \qquad (\because k \geq 0)$$

$$= \sum_{k=0}^{\infty} k \frac{a^k}{z^k}$$

$$= \sum_{k=0}^{\infty} k \left(\frac{a}{z}\right)^k$$

$$= 0\cdot\left(\frac{a}{z}\right)^0 + 1\left(\frac{a}{z}\right) + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left(1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{a}{z}\left[\frac{1}{\left(1-\frac{a}{z}\right)^2}\right], \quad \left|\frac{a}{z}\right| < 1$$

$$\left(\because 1 + 2z + 3z^2 + \cdots = \frac{1}{(1-z)^2}, \quad |z| < 1\right)$$

$$\therefore Z\{f(k)\} = \frac{a}{z\left(1-\frac{a}{z}\right)^2}, \quad |a| < |z|$$

$$\boxed{Z\{k\,a^k\} = \frac{a}{z\left(1-\frac{a}{z}\right)^2}}, \quad |z| > |a|$$

In perticular,

if $a = 1$, then $\boxed{Z\{k\} = \frac{z}{(z-1)^2}}$, $|z| > 1$

if $k = 1$, $a = 1$ then $\boxed{Z\{1\} = \frac{z}{z-1}}$, $|z| > 1$

Ex ③ find $Z\{a^{|k|}\}$

Solution: Note that $a^{|k|} = \begin{cases} a^k & , k \geq 0 \\ a^{-k} & , k < 0 \end{cases}$

$$\therefore Z\{a^{|k|}\} = \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} z^{-k}$$

$$= \left[\cdots + a^{-(-3)} z^{-(-3)} + a^{-(-2)} z^{-(-2)} + a^{-(-1)} z^{-(-1)} \right]$$

$$+ \left[1 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots \right]$$

$$= \left[\cdots + a^3 z^3 + a^2 z^2 + az \right] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots \right]$$

$$= \left[az + (az)^2 + (az)^3 + \cdots \right] + \left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots \right]$$

$$= az \left[1 + (az) + (az)^2 + \cdots \right] + \left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots \right]$$

$$= az \left[\frac{1}{1-(az)} \right] + \left[\frac{1}{1-\left(\frac{a}{z}\right)} \right] \quad , \quad |az| < 1 , \quad \left|\frac{a}{z}\right| < 1$$

$$\left(\because 1 + z + z^2 + \cdots = \frac{1}{1-z} \quad , \quad |z| < 1 \right)$$

$$= \frac{az}{1-az} + \frac{z}{z-a} \quad , \quad |z| < \left|\frac{1}{a}\right| , \quad |a| < |z|$$

$$\Rightarrow \boxed{Z\{a^{|k|}\} = \frac{z(1-a^2)}{(1-az)(z-a)}} \quad , \quad a < |z| < \frac{1}{a}$$

$\therefore$ ROC is $\underline{a < |z| < \left(\frac{1}{a}\right)}$

* **some standard Z - transform :**

① $Z\{1\} = \frac{z}{z-1}$, $|z|>1$

② $Z\{k\} = \frac{z}{(z-1)^2}$, $|z|>1$

③ $Z\{ka^k\} = \frac{az}{(z-a)^2}$, $|z|>|a|$, $k \geq 0$

④ $Z\{{}^nC_k\} = \left(1+\frac{1}{z}\right)^n$, $0 \leq k \leq n$, $|z|>0$

⑤ $Z\{{}^kC_n\} = z^{-n}\left(1-\frac{1}{z}\right)^{-(n+1)}$, $|z|>1$

⑥ $Z\{{}^{k+n}C_n\} = \left(1-\frac{1}{z}\right)^{-(n+1)}$, $|z|>1$

⑦ $Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$, $|a|<|z|<\frac{1}{|a|}$

⑧ $Z\{a^k\} = \frac{z}{z-a}$, $|z|>|a|$, $k \geq 0$

⑨ $Z\{a^k\} = \frac{z}{a-z}$, $|z|<|a|$, $k<0$

⑩ $Z\left\{\frac{a^k}{k!}\right\} = e^{\frac{a}{z}}$, $k \geq 0$ ROC : All z-plane

⑪ $Z\{\cos(ak+b)\} = \frac{z(z\cos b - \cos(a-b))}{z^2 - 2z\cos a + 1}$, $|z|>1$

⑫ $Z\{\sin(ak+b)\} = \frac{z[\sin(a-b) + z\sin b]}{z^2 - 2z\cos a + 1}$, $|z|>1$

⑬ $Z\{\cosh ak\} = \frac{z(z-\cosh a)}{z^2 - 2z\cosh a + 1}$, All z-plane

⑭ $Z\{\sinh ak\} = \frac{z\sinh a}{z^2 - 2z\cosh a + 1}$, All z-plane

(15) $Z\{\delta(k)\} = 1$, RoC : All z-plane

where, $\delta(k)$ is dirac-delta function

(16) $Z\{U(k)\} = \frac{z}{z-1}$, $|z|>1$

where $U(k)$ is unit step function

Ex. find the z-transform of $\left(\frac{1}{3}\right)^{|k|}$

Solution: Note that $\left(\frac{1}{3}\right)^{|k|} = \begin{cases} \left(\frac{1}{3}\right)^{k}, & k \geq 0 \\ \left(\frac{1}{3}\right)^{-k}, & k < 0 \end{cases}$

Now,
$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[\cdots + \left(\frac{1}{3}\right)^{3} z^{3} + \left(\frac{1}{3}\right)^{2} z^{2} + \left(\frac{1}{3}\right) z\right]$$

$$+ \left[1 + \left(\frac{1}{3}\right) z^{-1} + \left(\frac{1}{3}\right)^{2} z^{-2} + \cdots\right]$$

$$= \left[\frac{z}{3} + \left(\frac{z}{3}\right)^{2} + \left(\frac{z}{3}\right)^{3} + \cdots\right]$$

$$+ \left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \frac{z}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^{2} + \cdots\right]$$

$$+ \left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \frac{z}{3}\left(\frac{1}{1-\left(\frac{z}{3}\right)}\right) + \left(\frac{1}{1-\frac{1}{3z}}\right), \left|\frac{z}{3}\right|<1, \left|\frac{1}{3z}\right|<1$$

$$\left(\because\ 1+z+z^2+\dots = \frac{1}{1-z}, \quad |z|<1\right)$$

$$= \frac{z}{3}\cdot\frac{3}{3-z} + \frac{3z}{3z-1}, \quad |z|<3, \ \frac{1}{3}<|z|$$

$$= \frac{z}{3-z} + \frac{3z}{3z-1}, \quad \frac{1}{3}<|z|<3$$

$$\Rightarrow \quad Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3}<|z|<3$$

Homework:

Ex. 1 find $Z\{3^{|k|}\}$

Ex. 2 find $Z\left\{\frac{\alpha^k}{k}\right\}, \quad k>1$

Ans: 1 $\frac{-8z}{(1-3z)(z-3)}$ 2 $-\log\left(1-\frac{\alpha}{z}\right)$

Scanned by CamScanner

* Properties of Z- transform :-

① Linearity :

$$Z\{a f(k) + b g(k)\} = a Z\{f(k)\} + b Z\{g(k)\}$$

② change of scale :-

If $Z\{f(k)\} = F(z)$, $R_1 < |z| < R_2$

then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$, $|a| R_1 < |z| < |a| R_2$

③ shifting property :

If $Z\{f(k)\} = F(z)$,

then $Z\{f(k+n)\} = z^n F(z)$

and $Z\{f(k-n)\} = z^{-n} F(z)$

④ Multiplication by 'k'

If $Z\{f(k)\} = F(z)$

then $Z\{k f(k)\} = -z \frac{d}{dz} F(z)$

⑤ Division by 'k'

If $Z\{f(k)\} = F(k)$

then $Z\left\{\frac{f(k)}{k}\right\} = -\int_{\infty}^{z} z^{-1} F(z)\, dz$

⑥ Initial and Final value :

If $Z\{f(k)\} = F(z)$, $k \geq 0$ then

$f(0) = \lim_{z \to \infty} F(z)$ and $\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$

* Examples based on properties:

Ex ① Find z - transform of $\left\{ \cos\left(\frac{k\pi}{3}+\alpha\right)\right\}$, $k \geq 0$

Solution: Note that $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\therefore Z\{f(k)\} = Z\left\{ \cos\left(\frac{k\pi}{3}+\alpha\right)\right\}$$

$$= Z\left\{ \cos\frac{k\pi}{3} \cdot \cos\alpha - \sin\frac{k\pi}{3} \cdot \sin\alpha\right\}$$

$$= \cos\alpha \, Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\}$$

($\because$ By linearity property)

$$= \cos\alpha\left[\frac{z^2 - z\cos\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\left(\frac{\pi}{3}\right)+1}\right] - \sin\alpha\left[\frac{z\sin\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\left(\frac{\pi}{3}\right)+1}\right]$$

$$\left(\because \cos ak = \frac{z^2 - z\cos a}{z^2 - 2z\cos a + 1}\ ;\ \sin ak = \frac{z\sin a}{z^2 - 2z\cos a + 1}\right)$$

$$= \frac{z^2\cos\alpha - z\cos\alpha\cos\frac{\pi}{3} - z\sin\alpha\sin\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\frac{\pi}{3}+1}$$

$$= \frac{z\left\{z\cos\alpha - \left[\cos\left(\frac{\pi}{3}\right)\cdot\cos\alpha + \sin\left(\frac{\pi}{3}\right)\sin\alpha\right]\right\}}{z^2 - 2z\cos\left(\frac{\pi}{3}\right)+1}$$

$$= \frac{z\left[z\cos\alpha - \cos\left(\frac{\pi}{3}-\alpha\right)\right]}{z^2 - z + 1}$$

$$\therefore Z\left\{\cos\left(\frac{k\pi}{3}+\alpha\right)\right\} = \frac{z\left[z\cos\alpha - \cos\left(\frac{\pi}{3}-\alpha\right)\right]}{z^2 - z + 1}$$

Ex② find: $Z\{c^k \cos\alpha k\}$

Solution: Note that $Z\{\cos\alpha k\} = \dfrac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$

$\therefore$ By change of scale property,

$$Z\{c^k\cos\alpha k\} = F\left(\frac{z}{c}\right)$$

$$= \frac{\frac{z}{c}\left(\frac{z}{c}-\cos\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1}$$

$$= \frac{z(z-c\cos\alpha)}{z^2-2cz\cos\alpha+c^2}$$

Ex③ find $Z\{k^2\}$

Solution: Note that $Z\{k\} = \dfrac{z}{(z-1)^2}$, $|z|>1$

$\therefore$ By property of 'multiplication by k'

$$Z\{k f(k)\} = -z\frac{d}{dz}F(z)$$

$$\therefore \quad Z\{k^2\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

Ex. 4 find $Z\left\{\frac{1}{k+1}\right\}$, $k \geq 1$ and indicate the Radius of convergence.

Solution: Note that $Z\left\{\frac{1}{k}\right\} = -\log\left(1-\frac{1}{z}\right)$, $|z|>1$

$\therefore$ By shifting property,

if $Z\{f(k)\} = F(z)$ then $Z\{f(k+n)\} = z^n F(z)$

$$\therefore Z\left\{\frac{1}{k+1}\right\} = z^{(1)}\left[-\log\left(1-\frac{1}{z}\right)\right] \qquad (\text{here, } n=1)$$

$$= -z\log\left(1-\frac{1}{z}\right)$$

and ROC : $|z|>1$

* Convolution for Z-transform :- (convolution Theorem)

— let $\{f(k)\}$ and $\{g(k)\}$ be two sequences such that $Z\{f(k)\} = F(z)$ and

$$Z\{g(k)\} = G(z)$$

then

$$\boxed{Z\{f(k) * g(k)\} = F(z) \cdot G(z)}$$

where, $h(k) = f(k) * g(k) = \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)$

Proof: Note that $H(z) = Z\{h(k)\}$

$$= Z\left[\{f(k)\} * \{g(k)\}\right]$$

$$= Z\left[\sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\right]$$

$$= \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)\, z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(m) \cdot g(k-m)\, z^{-k+m-m}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{k=-\infty}^{\infty} g(k-m)\, z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{p=-\infty}^{\infty} g(p) \cdot z^{-p}$$

where $p = k-m$

$$= F(z) \cdot G(z)$$

$$\Rightarrow Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

Scanned by CamScanner

Example 1. If $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$, then find the Z-transform of $\{f(k) * g(k)\}$

Solution: Note that $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$

and $U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$\Rightarrow \quad \{f(k)\} = \{4^0, 4^1, 4^2, \cdots\}$ and

$\{g(k)\} = \{5^0, 5^1, 5^2, \cdots\}$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$$

$$= 4^0 z^0 + 4^1 z^{-1} + 4^2 z^{-2} + \cdots$$

$$= 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \left(\frac{4}{z}\right)}, \quad \left|\frac{4}{z}\right| < 1$$

$$\left(\because 1 + z + z^2 + \cdots = \frac{1}{1-z}, \; |z| < 1\right)$$

$$= \frac{z}{z-4}, \quad 4 < |z|$$

$$= F(z)$$

and $$Z\{g(k)\} = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= 5^0 z^0 + 5^1 z^{-1} + 5^2 z^{-2} + \cdots$$

$$= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \left(\frac{5}{z}\right)}, \quad \left|\frac{5}{z}\right| < 1$$

$$= \frac{z}{z-5}, \quad 5 < |z|$$

$$= G(z)$$

$\therefore$ By convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

$$= \frac{z}{z-4} \cdot \frac{z}{z-5}, \quad 5 < |z|$$

$$= \frac{z^2}{(z-4)(z-5)}, \quad |z| > 5$$

Ex. 2. find $Z\{f(k) * g(k)\}$

if $f(k) = \frac{1}{3^k}$ and $g(k) = \frac{1}{5^k}$, $k \geq 0$

Solution: here $f(k) = \frac{1}{3^k}$

$$\Rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{3^k} z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(3z)^k} = \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^k$$

$$= \frac{1}{1} + \frac{1}{3z} + \left(\frac{1}{3z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \left(\frac{1}{3z}\right)}, \quad \left|\frac{1}{3z}\right| < z$$

$$\left(\because 1 + z + z^2 + \cdots = \frac{1}{1-z}, \quad |z| < 1\right)$$

$$= \frac{3z}{3z-1} \quad , \quad \frac{1}{3} < |z|$$

$$= F(z)$$

and $\quad g(k) = \frac{1}{5^k}$

$$\therefore \quad Z\{g(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{5^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{5z}\right)^k$$

$$= 1 + \frac{1}{5z} + \left(\frac{1}{5z}\right)^2 + \cdots$$

$$= \frac{1}{1-\left(\frac{1}{5z}\right)} \quad , \quad \left|\frac{1}{5z}\right| < 1$$

$$= \frac{5z}{5z-1} \quad , \quad \frac{1}{5} < |z|$$

$$= G(z)$$

$\therefore$ By convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

$$= \frac{3z}{3z-1} \cdot \frac{5z}{5z-1} \quad , \quad \frac{1}{5} < \frac{1}{3} < |z|$$

$$= \frac{15z^2}{(3z-1)(5z-1)} \quad , \quad \frac{1}{3} < |z|$$

<u>Homework</u>! find $Z\{f(k) * g(k)\}$

if $\quad f(k) = \frac{1}{5^k} \quad$ and $\quad g(k) = \frac{1}{7^k}$

* Inverse Z-transform :

Note that if $Z\{f(k)\} = F(z)$ then the inverse z-transform of $F(z)$ is denoted by '$Z^{-1}\{F(z)\}$' and is given by

$$Z^{-1}\{F(z)\} = f(k)$$

Example ① find the inverse Z-transform of

$$F(z) = \frac{1}{z-a} \quad \text{when i) } |z|<|a| \quad \text{ii) } |z|>|a|$$

Solution: i) $|z| < |a|$

$$\Rightarrow \quad \left|\frac{z}{a}\right| < 1$$

∴ consider, $F(z) = \dfrac{1}{z-a}$

$$= \frac{1}{a\left(\frac{z}{a}-1\right)}$$

$$= \frac{1}{-a\left(1-\left(\frac{z}{a}\right)\right)}$$

$$= -\frac{1}{a}\left[\frac{1}{1-\left(\frac{z}{a}\right)}\right]$$

$$= -\frac{1}{a}\left[1+\frac{z}{a}+\left(\frac{z}{a}\right)^2+\cdots+\left(\frac{z}{a}\right)^k+\cdots\right]$$

$$= -\left[\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\cdots+\frac{z^k}{a^{k+1}}+\cdots\right]$$

$$= -\left[a^{-1}+a^{-2}z+a^{-3}z^2+\cdots+a^{-(k+1)}z^k+\cdots\right]$$

Scanned by CamScanner

$\therefore$ The coefficient of $z^{k} = -a^{-(k+1)}$, $k \geq 0$

$\Rightarrow$ The coefficient of $z^{-k} = -a^{-(-k+1)}$, $k \leq 0$ $\left(\begin{array}{l}\text{Replace} \\ k \text{ by } -k\end{array}\right)$

that is the coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{-a^{k-1}\} \quad , \quad k \leq 0$$

$$\Rightarrow \quad z^{-1}\left[\frac{1}{z-a}\right] = \{-a^{k-1}\} \quad , \quad k \leq 0$$

ii> $|z| > |a| \quad \Rightarrow \quad 1 > \left|\frac{a}{z}\right| \quad$ i.e $\quad \left|\frac{a}{z}\right| < 1$

$$\therefore \quad F(z) = \frac{1}{z-a}$$

$$= \frac{1}{z\left(1-\frac{a}{z}\right)} \qquad = \frac{1}{z}\left[\frac{1}{1-\left(\frac{a}{z}\right)}\right]$$

$$= \frac{1}{z}\left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^{2} + \cdots + \left(\frac{a}{z}\right)^{k} + \cdots\right]$$

$$= z^{-1}\left[1 + a z^{-1} + a^{2} z^{-2} + \cdots + a^{k} \cdot z^{-k} + \cdots\right]$$

$$= \left[z^{-1} + a z^{-2} + a^{2} z^{-3} + \cdots + a^{k} z^{-(k+1)} + \cdots\right]$$

$$= \left[a^{0} z^{-1} + a z^{-2} + \cdots + a^{k-1} z^{-k} + a^{k} z^{-(k+1)} + \cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = a^{k-1}$, $k \geq 1$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{a^{k-1}\} \quad , \quad k \geq 1$$

$\therefore$

Scanned by CamScanner

* Inverse z-transform by partial fraction:-

Ex ① Find inverse z-transform of

$$F(z) = \frac{z}{(z-1)(z-2)}, \quad |z|>2$$

Solution: consider,
$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \text{—①}$$

$$\Rightarrow \frac{z}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow z = A(z-2) + B(z-1)$$

$\Rightarrow$ if $z=1$ then $1 = A(1-2) + B(0)$

$\Rightarrow A = -1$

if $z=2$ then $2 = A(0) + B(2-1)$

$B = 2$

$\therefore$ equation ① becomes.

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{}{z-2}$$

$$\Rightarrow F(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

Since, $|z|>2 \Rightarrow 1 > \left|\frac{2}{z}\right| \Rightarrow \boxed{\left|\frac{2}{z}\right|<1}$

and $|z|>2>1 \Rightarrow |z|>1 \Rightarrow 1>\left|\frac{1}{z}\right| \Rightarrow \boxed{\left|\frac{1}{z}\right|<1}$

$$\therefore F(z) = \left[\frac{2}{z-2}\right] - \left[\frac{1}{z-1}\right]$$

$$= \frac{2}{z\left[1-\left(\frac{2}{z}\right)\right]} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{2}{z}\left[\frac{1}{1-\left(\frac{2}{z}\right)}\right] - \frac{1}{z}\left[\frac{1}{1-\left(\frac{1}{z}\right)}\right]$$

$$= \frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^k+\cdots\right]$$

$$- \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)^k+\cdots\right]$$

$$= 2z^{-1}\left[1+2z^{-1}+2^2z^{-2}+\cdots+2\ 2^{-k}+\cdots\right]$$

$$- z^{-1}\left[1+z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-1}+2^2z^{-2}+2^3z^{-3}+\cdots+2^k\ 2^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-2}+2^2z^{-2}+\cdots+2^k\ z^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = 2^k - 1, \quad k \geq 1$

$$\therefore \quad Z^{-1}[F(z)] = \{f(k)\} = \{2^k - 1\}$$

Ex ② find inverse z-transform of

$$F(z) = \frac{1}{(z-3)(z-2)}, \quad 2<|z|<3$$

Solution: Note that $F(z) = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$

(By partial fraction method)

since, $2<|z|<3$

if $2<|z|$ then $\left|\frac{2}{z}\right|<1$

if $|z|<3$ then $\left|\frac{z}{3}\right|<1$

$$\therefore F(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$$

$$= -\frac{1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\cdots+\left(\frac{z}{3}\right)^k+\cdots\right] - \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\left(\frac{2}{z}\right)^k+\cdots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\frac{1}{3^3}z^2+\cdots+\frac{1}{3^{k+1}}z^k+\cdots\right] - \left[z^{-1}+2z^{-2}+\cdots+2^k z^{-(k+1)}+\cdots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\cdots+\frac{1}{3^{k+1}}z^k+\cdots\right] - \left[2^0 z^{-1}+2z^{-2}+\cdots+2^{k-1}z^{-k}+\cdots\right]$$

$$= -\left[3^{-1}z^0+3^{-2}z^1+\cdots+3^{-k-1}z^k+\cdots\right] - \left[2^0 z^{-1}+2z^{-2}+\cdots+2^{k-1}z^{-k}+\cdots\right]$$

from the first series the coefficient of $z^k = -3^{-k-1}$ $k\geq 0$

i.e The coefficient of $z^{-k} = -3^{k-1}$, $k\leq 0$

& from the second series the coefficient of $z^{-k} = 2^{k-1}$

$k\geq 1$

$$\therefore z^{-1}[F(z)] = \{f(k)\} = \{-3^{k-1}\}, \quad k\leq 0$$

$$= \{-2^{k-1}\}, \quad k\geq 1$$

Homework!

① find inverse z-transform of $\frac{1}{(z-3)(z-2)}$

if ROC is $|z|>3$

② find inverse z-transform of

$$F(z) = \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} \quad , \quad 3 < z < 4$$

* Inverse Z- transform :

Note that if $Z\{f(k)\} = F(z)$ then the inverse z-transform of $F(z)$ is denoted by '$Z^{-1}\{F(z)\}$' and is given by

$$Z^{-1}\{F(Z)\} = f(k)$$

Example (1) find the inverse Z-transform of

$$F(Z) = \frac{1}{Z-a} \quad \text{when i) } |Z|<|a| \quad \text{ii) } |Z|>|a|$$

Solution: i) $|Z| < |a|$

$$\Rightarrow \quad \left|\frac{Z}{a}\right| < 1$$

$\therefore$ consider,
$$F(Z) = \frac{1}{Z-a}$$

$$= \frac{1}{a\left(\frac{Z}{a}-1\right)}$$

$$= \frac{1}{-a\left(1-\left(\frac{Z}{a}\right)\right)}$$

$$= -\frac{1}{a}\left[\frac{1}{1-\left(\frac{Z}{a}\right)}\right]$$

$$= -\frac{1}{a}\left[1+\frac{Z}{a}+\left(\frac{Z}{a}\right)^2+\cdots+\left(\frac{Z}{a}\right)^k+\cdots\right]$$

$$= -\left[\frac{1}{a}+\frac{Z}{a^2}+\frac{Z^2}{a^3}+\cdots+\frac{Z^k}{a^{k+1}}+\cdots\right]$$

$$= -\left[a^{-1}+a^{-2}z+a^{-3}Z^2+\cdots+a^{-(k+1)}z^k+\cdots\right]$$

$\therefore$ The coefficient of $z^{k} = -a^{-(k+1)}$, $k \geq 0$

$\Rightarrow$ The coefficient of $z^{-k} = -a^{-(-k+1)}$, $k \leq 0$ (Replace k by $-k$)

that is the coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{-a^{k-1}\} , \quad k \leq 0$$

$$\Rightarrow \quad z^{-1}\left[\frac{1}{z-a}\right] = \{-a^{k-1}\} , \quad k \leq 0$$

ii) $|z| > |a| \Rightarrow 1 > \left|\frac{a}{z}\right|$ i.e $\left|\frac{a}{z}\right| < 1$

$$\therefore \quad F(z) = \frac{1}{z-a}$$

$$= \frac{1}{z\left(1-\frac{a}{z}\right)} = \frac{1}{z}\left[\frac{1}{1-\left(\frac{a}{z}\right)}\right]$$

$$= \frac{1}{z}\left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots + \left(\frac{a}{z}\right)^k + \cdots\right]$$

$$= z^{-1}\left[1 + a z^{-1} + a^2 z^{-2} + \cdots + a^k . z^{-k} + \cdots\right]$$

$$= \left[z^{-1} + a z^{-2} + a^2 z^{-3} + \cdots + a^k z^{-(k+1)} + \cdots\right]$$

$$= \left[a^0 z^{-1} + a z^{-2} + \cdots + a^{k-1} z^{-k} + a^k z^{-(k+1)} + \cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = a^{k-1}$, $k \geq 1$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{a^{k-1}\} , \quad k \geq 1$$

$\therefore$

Scanned by CamScanner

* Inverse z-transform by partial fraction:-

Ex ① Find inverse z-transform of

$$F(z) = \frac{z}{(z-1)(z-2)} , \quad |z| > 2$$

Solution: consider,
$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \text{—①}$$

$$\Rightarrow \quad \frac{z}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow \quad z = A(z-2) + B(z-1)$$

$\Rightarrow$ if $z=1$ then $1 = A(1-2) + B(0)$

$\Rightarrow A = -1$

if $z=2$ then $2 = A(0) + B(2-1)$

$B = 2$

$\therefore$ equation ① becomes.

$$\frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2}$$

$$\Rightarrow \quad F(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

since, $|z| > 2 \Rightarrow 1 > \left|\frac{2}{z}\right| \Rightarrow \boxed{\left|\frac{2}{z}\right| < 1}$

and $|z| > 2 > 1 \Rightarrow |z| > 1 \Rightarrow 1 > \left|\frac{1}{z}\right| \Rightarrow \boxed{\left|\frac{1}{z}\right| < 1}$

$$\therefore \quad F(z) = \left[\frac{2}{z-2}\right] - \left[\frac{1}{z-1}\right]$$

$$= \frac{2}{z\left[1-\left(\frac{2}{z}\right)\right]} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{2}{z}\left[\frac{1}{\left[1-\left(\frac{2}{z}\right)\right.}\right] - \frac{1}{z}\left[\frac{1}{1-\left(\frac{1}{z}\right)}\right]$$

$$= \frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^k+\cdots\right]$$

$$- \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)^k+\cdots\right]$$

$$= 2z^{-1}\left[1+2z^{-1}+2^2z^{-2}+\cdots+2^k z^{-k}+\cdots\right]$$

$$- z^{-1}\left[1+z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-1}+2^2z^{-2}+2^3z^{-3}+\cdots+2^k z^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-k}+\cdots\right]$$

$$= \left[2^1z^{-1}+2^2z^{-2}+\cdots+2^k z^{-k}+\cdots\right]$$

$$- \left[z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots\right]$$

$\therefore$ The coefficient of $z^{-k} = 2^k - 1 \quad , \quad k \geq 1$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{2^k - 1\}$$

Scanned by CamScanner

Ex (2) find inverse z-transform of

$$F(z) = \frac{1}{(z-3)(z-2)}, \qquad 2<|z|<3$$

solution: Note that $F(z) = \dfrac{1}{(z-3)(z-2)} = \dfrac{1}{z-3} - \dfrac{1}{z-2}$

(By partial fraction method)

since, $2<|z|<3$

if $2<|z|$ then $\left|\frac{2}{z}\right| < 1$

if $|z|<3$ then $\left|\frac{z}{3}\right| < 1$

$$\therefore F(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$$

$$= -\frac{1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\dots+\left(\frac{z}{3}\right)^k+\dots\right] - \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\dots\left(\frac{2}{z}\right)^k+\dots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\frac{1}{3^3}z^2+\dots+\frac{1}{3^{k+1}}z^k+\dots\right] - \left[z^{-1}+2z^{-2}+\dots+2^k z^{-(k+1)}+\dots\right]$$

$$= -\left[\frac{1}{3}+\frac{1}{3^2}z+\dots+\frac{1}{3^{k+1}}z^k+\dots\right] - \left[2^0 z^{-1}+2z^{-2}+\dots+2^{k-1}z^{-k}+\dots\right]$$

$$= -\left[3^{-1}z^0+3^{-2}z^1+\dots+3^{-k-1}z^k+\dots\right] - \left[2^0 z^{-1}+2z^{-2}+\dots+2^{k-1}z^{-k}+\dots\right]$$

from the first series the coefficient of $z^k = -3^{-k-1}$ $k \geq 0$

i.e The coefficient of $z^{-k} = -3^{k-1}$, $k \leq 0$

& from the second series the coefficient of $z^{-k} = 2^{k-1}$ $k \geq 1$

$$\therefore \quad z^{-1}[F(z)] = \{f(k)\} = \{-3^{k-1}\}, \quad k \leq 0$$

$$= \{-2^{k-1}\}, \quad k \geq 1$$

Homework:

1. find inverse z-transform of $\frac{1}{(z-3)(z-2)}$

   if ROC is $|z|>3$

2. find inverse z-transform of

$$F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} \quad , \quad 3<z<4$$

* Convolution Method : (Inverse Z-transform)

Let $F(z)$ and $G(z)$ be the function of z

such that $z^{-1}[F(z)] = f(k)$ and

$z^{-1}[G(z)] = g(k)$

then $z^{-1}\{F(z) \cdot G(z)\} = f(k) * g(k)$

$$= \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m) = h(k)$$

Example 1: find Inverse z-transform of

$$\frac{z^2}{(z-a)(z-b)}$$

Solution: let $F(z) = \frac{z}{z-a}$, $G(z) = \frac{z}{z-b}$

$\Rightarrow$ $z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-a}\right] = a^k = f(k)$, $k \geq 0$

and $z^{-1}[G(z)] = z^{-1}\left[\frac{z}{z-b}\right] = b^k = g(k)$, $k \geq 0$

$\therefore$ By convolution method

$z^{-1}\{F(z) \cdot G(z)\} = f(k) * g(k)$

$\Rightarrow$ $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \sum_{m=-\infty}^{\infty} f(m) \cdot g(k-m)$

$$= \sum_{m=-\infty}^{\infty} a^m \cdot b^{(k-m)} \quad , \quad k \geq 0$$

$$= \sum_{m=-\infty}^{\infty} a^m . b^k . b^{-m} , \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} \frac{a^m}{b^m} . b^k , \quad k \geq 0$$

$$= b^k \sum_{m=0}^{\infty} \left(\frac{a}{b}\right)^m , \quad k \geq 0$$

$$= b^k \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots \right] , \quad k \geq 0$$

$$= b^k \left[\frac{\left(\frac{a}{b}\right)^{k+1} - 1}{\left(\frac{a}{b}\right) - 1} \right] , \quad k \geq 0 \quad \left(\because \text{Geometric series } 1 + r + r^2 + \cdots = \frac{r^{k+1} - 1}{r - 1} \right)$$

$$= b^k \left[\frac{\frac{a^{k+1} - b^{k+1}}{b^{k+1}}}{\frac{a - b}{b}} \right] , \quad k \geq 0$$

$$= b^k \left[\frac{a^{k+1} - b^{k+1}}{b^{k+1}} \times \frac{b}{a-b} \right] , \quad k \geq 0$$

$$= \frac{b^k}{b^k} \left[\frac{a^{k+1} - b^{k+1}}{a-b} \right] , \quad k \geq 0$$

$$= \frac{a^{k+1} - b^{k+1}}{a-b} , \quad k \geq 0$$

$$\therefore \quad Z^{-1} \left[\frac{z^2}{(z-b)(z-b)} \right] = \frac{a^{k+1} - b^{k+1}}{a-b} , \quad k \geq 0$$

Ex ② Find the inverse Z transform of

$$\frac{z}{(z-1)(z-2)}, \quad |z|>2$$ By convolution method.

solution: let $F(z) = \frac{1}{z-1}$ and $G(z) = \frac{z}{z-2}$

$\Rightarrow \; z^{-1}[F(z)] = z^{-1}\left[\frac{1}{z-1}\right] = 1^{k-1} = 1 = f(k)$

and $z^{-1}[G(z)] = z^{-1}\left[\frac{1}{z-2}\right] = 2^{k} = g(k), \quad k \geq 0$

$\therefore$ By convolution method

$$z^{-1}[F(z) \cdot G(z)] = f(k) * g(k)$$

$$= \sum_{m=0}^{\infty} f(m) \cdot g(k-m), \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} 1 \cdot 2^{k-m}, \quad k \geq 0$$

$$= \sum_{m=0}^{\infty} 2^{k} \cdot 2^{-m}, \quad k \geq 0$$

$$= 2^{k} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m}, \quad k \geq 0$$

$$= 2^{k}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^{2} + \cdots\right], \quad k \geq 0$$

$$= 2^{k}\left[\frac{\left(\frac{1}{2}\right)^{k+1} - 1}{\left(\frac{1}{2}\right) - 1}\right], \quad k \geq 0 \quad \text{(By Geometric series)}$$

$$= 2^{k}\left[\frac{\frac{1-2^{k+1}}{2^{k+1}}}{\frac{1-2}{2}}\right], \quad k \geq 0$$

$$= 2^{k}\left[\frac{1-2^{k+1}}{2^{k+1}} \times \frac{2}{1-2}\right], \quad k \geq 0$$

$$= \frac{2^k}{2^k}\left[\frac{1-2^{k+1}}{1-2}\right], \quad k \geq 0$$

$$= 2^{k+1} - 1, \quad k \geq 0$$

$$= 2^k - 1, \quad k \geq 1$$

$$\therefore \quad Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^k - 1, \quad k \geq 1$$

Homework:

1) find inverse z-transform of

$$\frac{1}{(z-5)^2}, \quad |z| > 5$$ by convolution method.

2) find inverse z-transform of

$$\frac{1}{(z-2)(z-3)}$$ by convolution method.