AI and DS-2
BE-SEM VII

- The term **fuzzy** refers to things that are not clear or are vague. In the real world many times we encounter a situation when we can’t determine whether the state is true or false, their fuzzy logic provides very valuable flexibility for reasoning. In this way, we can consider the inaccuracies and uncertainties of any situation.
- Fuzzy Logic is a form of many-valued logic in which the truth values of variables may be any real number between 0 and 1, instead of just the traditional values of true or false.
- It is used to deal with imprecise or uncertain information and is a mathematical method for representing vagueness and uncertainty in decision-making.
- Fuzzy Logic is based on the idea that in many cases, the concept of true or false is too restrictive, and that there are many shades of gray
- in between. It allows for partial truths, where a statement can be partially true or false, rather than fully true or false.
- Fuzzy Logic is used in a wide range of applications, such as control systems, image processing, natural language processing, medical diagnosis, and artificial intelligence.

- The fundamental concept of Fuzzy Logic is the membership function, which defines the degree of membership of an input value to a certain set or category.
- The membership function is a mapping from an input value to a membership degree between 0 and 1, where 0 represents non-membership and 1 represents full membership.
- Fuzzy Logic is implemented using Fuzzy Rules, which are if-then statements that express the relationship between input variables and output variables in a fuzzy way.
- The output of a Fuzzy Logic system is a fuzzy set, which is a set of membership degrees for each possible output value.
- In summary, Fuzzy Logic is a mathematical method for representing vagueness and uncertainty in decision-making, it allows for partial truths, and it is used in a wide range of applications.
- It is based on the concept of membership function and the implementation is done using Fuzzy rules.
- In the boolean system truth value, 1.0 represents the absolute truth value and 0.0 represents the absolute false value. But in the fuzzy system, there is no logic for the absolute truth and absolute false value.
- But in fuzzy logic, there is an intermediate value too present which is partially true and partially false.

- Making decisions about processes that contain nonrandom uncertainty, such as the uncertainty in natural language, has been shown to be less than perfect.
- The idea proposed by Lotfi Zadeh suggested that set membership is the key to decision making when faced with uncertainty.
- In fact, Zadeh made the following statement in 1965: The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing.
- Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables

- This notion of set membership, then, is central to the representation of objects within a universe by sets defined on the universe.
- Classical sets contain objects that satisfy precise properties of membership;
- fuzzy sets contain objects that satisfy imprecise properties of membership, that is, membership of an object in a fuzzy set can be approximate.
- For example, the set of heights from 5 to 7 feet is precise (crisp); the set of heights in the region around 6 feet is imprecise, or fuzzy.
- To elaborate, suppose we have an exhaustive collection of individual elements (singletons) x, which make up a universe of information (discourse), X.
- Further, various combinations of these individual elements make up sets, say A, on the universe.
- For crisp sets, an element x in the universe X is either a member of some crisp set A or not. This binary issue of membership can be represented mathematically with the indicator function,
- $$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- Zadeh extended the notion of binary membership to accommodate various “degrees of membership” on the real continuous interval [0, 1], where the endpoints of 0 and 1 conform to no membership and full membership, respectively, just as the indicator function does for crisp sets, but where the infinite number of values in between the endpoints can represent various degrees of membership for an element x in some set on the universe.
- The sets on the universe X that can accommodate “degrees of membership” were termed by Zadeh as **fuzzy set**

$$\mu_{\underset{\sim}{A}}(x) \in [0, 1],$$

and the symbol $\mu_{\underset{\sim}{A}}(x)$ is the degree of membership of element x in fuzzy set $\underset{\sim}{A}$. Therefore, $\mu_{\underset{\sim}{A}}(x)$ is a value on the unit interval that measures the degree to which element x belongs to fuzzy set $\underset{\sim}{A}$; equivalently, $\mu_{\underset{\sim}{A}}(x) =$ degree to which $x \in \underset{\sim}{A}$.

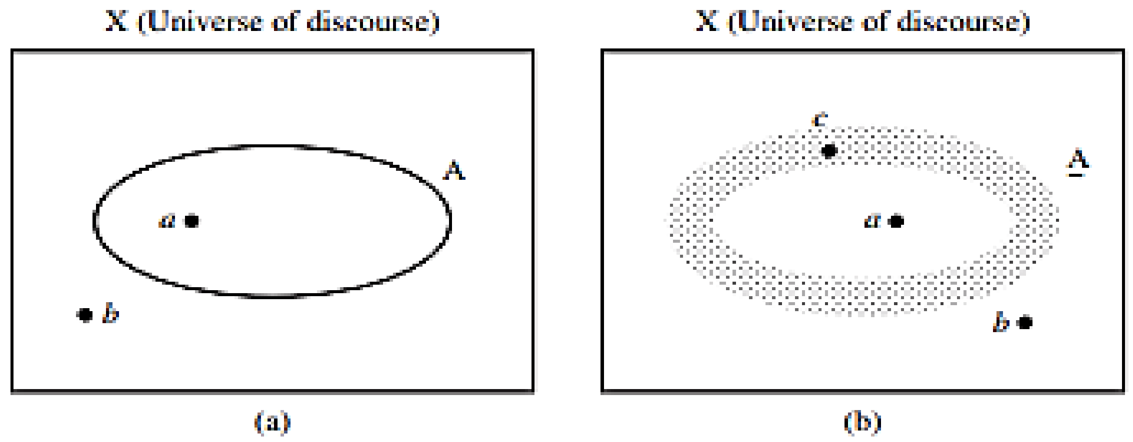
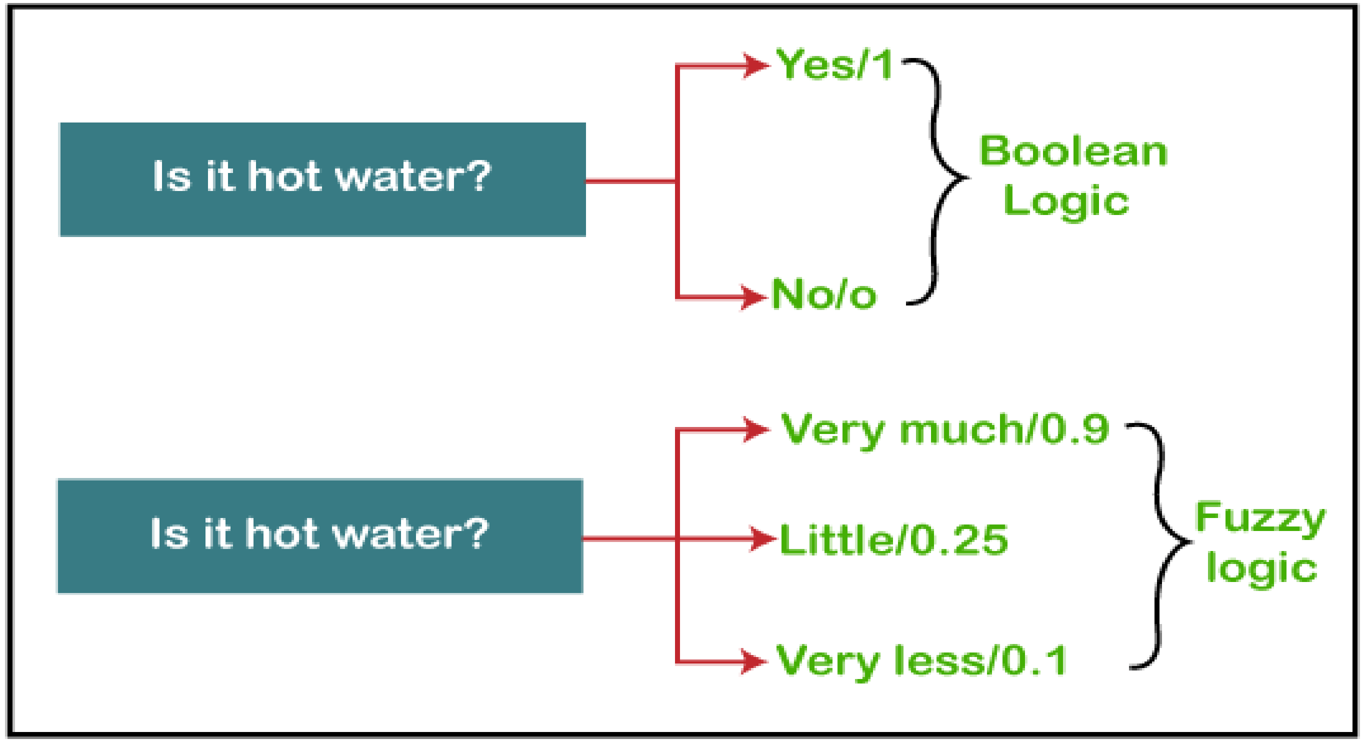


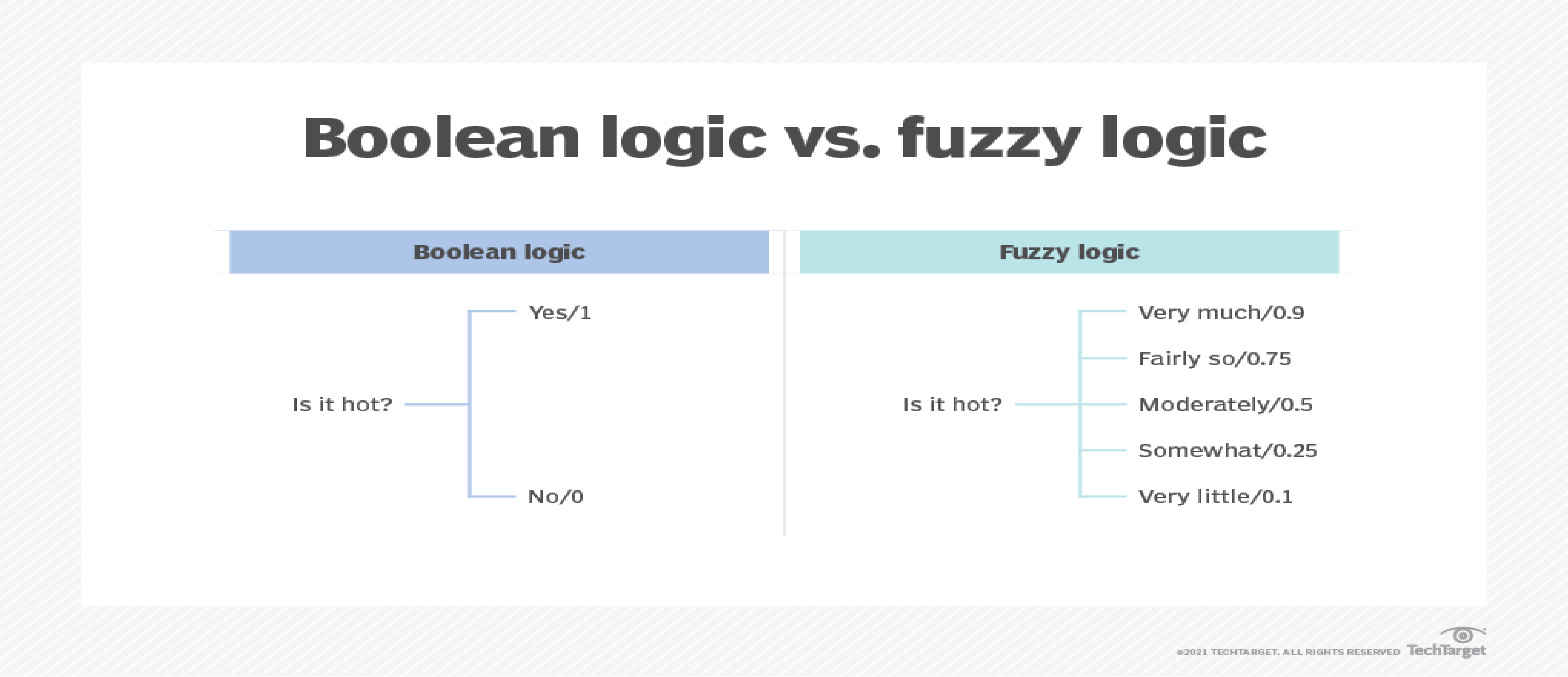
FIGURE
Diagrams for (a) crisp set boundary and (b) fuzzy set boundary.

the same universe X: the shaded boundary represents the boundary region of $\underset{\sim}{A}$. In the central (unshaded) region of the fuzzy set, point *a* is clearly a full member of the set. Outside the boundary region of the fuzzy set, point *b* is clearly not a member of the fuzzy set. However, the membership of point *c*, which is on the boundary region, is ambiguous. If complete membership in a set (such as point *a* in Figure b) is represented by the number 1, and no-membership in a set (such as point *b* in Figure . b) is represented by 0, then point *c* in Figure . b must have some intermediate value of membership (partial membership in fuzzy set $\underset{\sim}{A}$) on the interval [0,1]. Presumably, the membership of point *c* in A approaches a value of 1 as it moves closer to the central (unshaded) region in Figure .b of $\underset{\sim}{A}$ and the membership of point *c* in $\underset{\sim}{A}$ approaches a value of 0 as it moves closer to leaving the boundary region of $\underset{\sim}{A}$.

Fuzzy Logic


Is it hot water?
Yes/1
No/o
Boolean Logic
Is it hot water?
Very much/0.9
Little/0.25
Very less/0.1
Fuzzy logic

Fuzzy Logic



Boolean logic vs. fuzzy logic
Boolean logic
Fuzzy logic
Is it hot?
Yes/1
No/0
Is it hot?
Very much/0.9
Fairly so/0.75
Moderately/0.5
Somewhat/0.25
Very little/0.1
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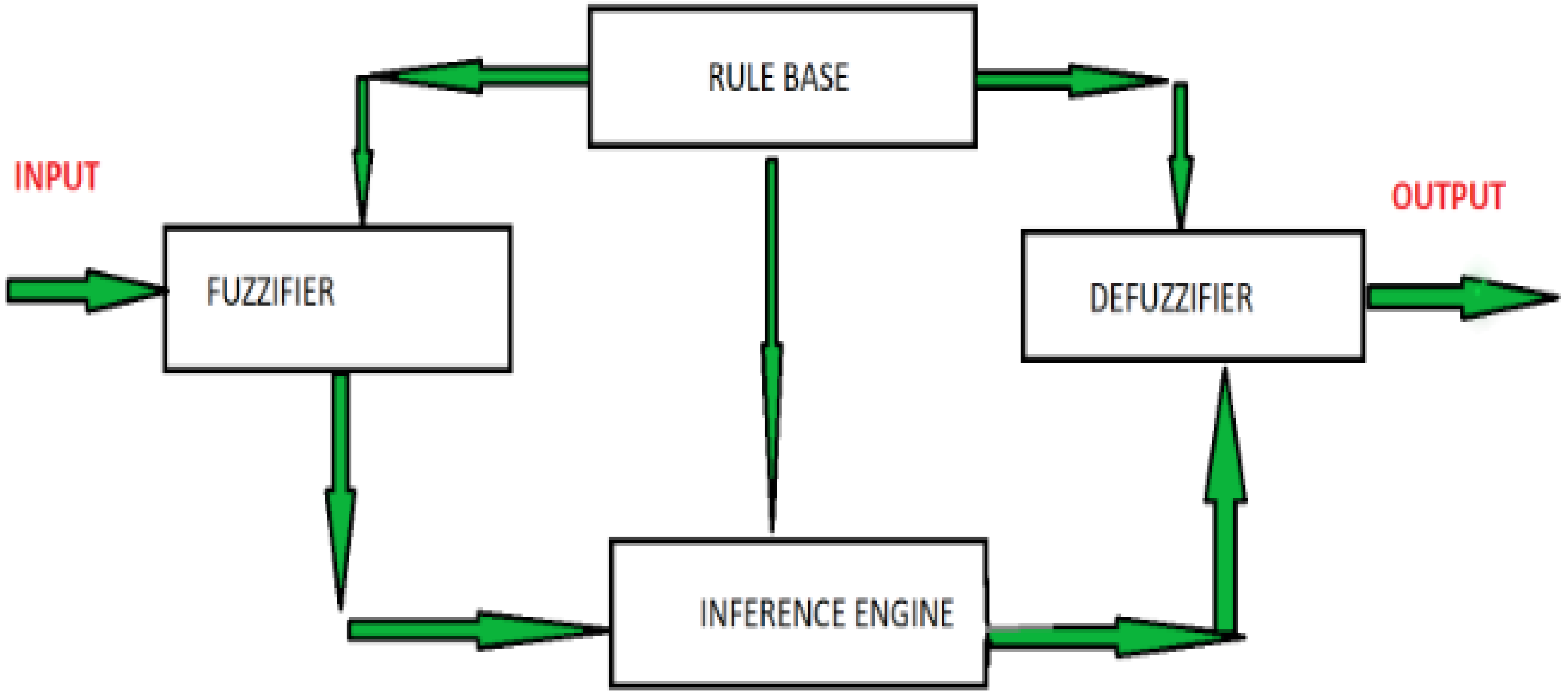
Fuzzy Logic

Fuzzy Logic Systems (FLS) produce acceptable but definite output in response to incomplete, ambiguous, distorted, or inaccurate (fuzzy) input

Fuzzy Logic (FL) is a method of reasoning that resembles human reasoning. The approach of FL imitates the way of decision making in humans that involves all intermediate possibilities between digital values YES and NO.

CERTAINLY YES
POSSIBLY YES
CANNOT SAY
POSSIBLY NO
CERTAINLY NO

- **ARCHITECTURE**
- Its Architecture contains four parts :
- **RULE BASE**: It contains the set of rules and the IF-THEN conditions provided by the experts to govern the decision-making system, on the basis of linguistic information. Recent developments in fuzzy theory offer several effective methods for the design and tuning of fuzzy controllers. Most of these developments reduce the number of fuzzy rules.
- **FUZZIFICATION**: It is used to convert inputs i.e. crisp numbers into fuzzy sets. Crisp inputs are basically the exact inputs measured by sensors and passed into the control system for processing, such as temperature, pressure, rpm’s, etc.
- **INFERENCE ENGINE**: It determines the matching degree of the current fuzzy input with respect to each rule and decides which rules are to be fired according to the input field. Next, the fired rules are combined to form the control actions.
- **DEFUZZIFICATION**: It is used to convert the fuzzy sets obtained by the inference engine into a crisp value. There are several defuzzification methods available and the best-suited one is used with a specific expert system to reduce the error.



FUZZY LOGIC ARCHITECTURE

- **What is Fuzzy Control?**
- It is a technique to embody human-like thinkings into a control system.
- It may not be designed to give accurate reasoning but it is designed to give acceptable reasoning.
- It can emulate human deductive thinking, that is, the process people use to infer conclusions from what they know.
- Any uncertainties can be easily dealt with the help of fuzzy logic

- **Advantages of Fuzzy Logic System**
- This system can work with any type of inputs whether it is imprecise, distorted or noisy input information.
- The construction of Fuzzy Logic Systems is easy and understandable.
- Fuzzy logic comes with mathematical concepts of set theory and the reasoning of that is quite simple.
- It provides a very efficient solution to complex problems in all fields of life as it resembles human reasoning and decision-making.
- The algorithms can be described with little data, so little memory is required.

- **Disadvantages of Fuzzy Logic Systems**
- Many researchers proposed different ways to solve a given problem through fuzzy logic which leads to ambiguity. There is no systematic approach to solve a given problem through fuzzy logic.
- Proof of its characteristics is difficult or impossible in most cases because every time we do not get a mathematical description of our approach.
- As fuzzy logic works on precise as well as imprecise data so most of the time accuracy is compromised

- **Application**
- It is used in the aerospace field for altitude control of spacecraft and satellites.
- It has been used in the automotive system for speed control, traffic control.
- It is used for decision-making support systems and personal evaluation in the large company business.
- It has application in the chemical industry for controlling the pH, drying, chemical distillation process.
- Fuzzy logic is used in Natural language processing and various intensive applications in Artificial Intelligence.
- Fuzzy logic is extensively used in modern control systems such as expert systems.
- Fuzzy Logic is used with Neural Networks as it mimics how a person would make decisions, only much faster. It is done by Aggregation of data and changing it into more meaningful data by forming partial truths as Fuzzy sets.

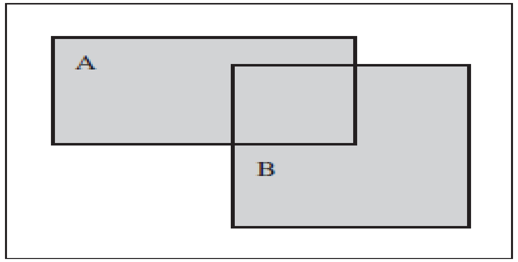
- Classical set

- Classical set is a collection of distinct objects. For example, a set of students passing grades.
- Each individual entity in a set is called a member or an element of the set.
- The classical set is defined in such a way that the universe of discourse is splitted into two groups members and non-members. Hence, In case classical sets, no partial membership exists.
- Let A is a given set. The membership function can be use to define a set A is given by:

$$\mu A(x)=\{\ 1 \text{ if } x \in A$$

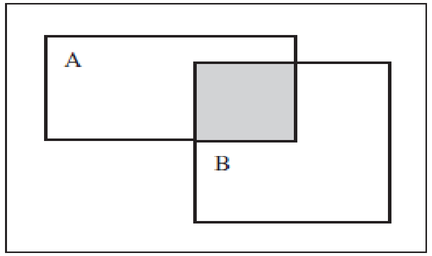
$$0 \text{ if } x \notin A\ \}$$

1. **Operations on classical sets**: For two sets A and B and Universe X:
   - **Union**:



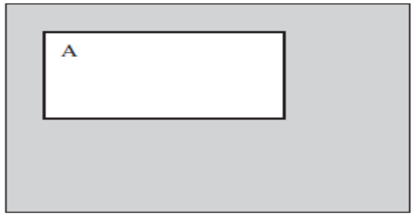
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

- This operation is also called **logical OR**.
- **Intersection**:



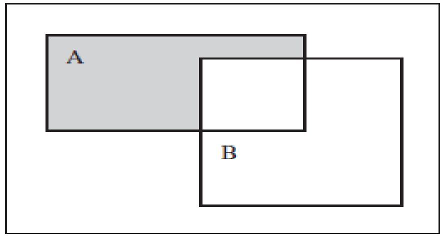
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

- This operation is also called **logical AND**.
- **Complement**:



$$A' = \{x \mid x \notin A,\ x \in X\}$$

- **Difference**:



$$A \mid B = \{x \mid x \in A \text{ and } x \notin B\}$$

1. **Properties of classical sets**: For two sets A and B and Universe X:
   - **Commutativity**:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- **Associativity**:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributivity**:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Idempotency:

$$A \cup A = A$$

$$A \cap A = A$$

- Identity:

$$A \cup \emptyset = A$$

$$A \cap X = A$$

$$A \cap \emptyset = \emptyset$$

$$A \cup X = X$$

- Transitivity:

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

- **Fuzzy set** is a set having **degrees of membership** between 1 and 0. Fuzzy sets are represented with tilde character(~). For example, Number of cars following traffic signals at a particular time out of all cars present will have membership value between [0,1].
- Partial membership exists when member of one fuzzy set can also be a part of other fuzzy sets in the same universe.
- The degree of membership or truth is not same as probability, fuzzy truth represents membership in vaguely defined sets.
- A fuzzy set A~ in the universe of discourse, U, can be defined as a set of ordered pairs and it is given by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

1. When the universe of discourse, U, is **discrete and finite**, fuzzy set A~ is given by

$$\tilde{A} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots + \frac{\mu_{\tilde{A}}(x_n)}{x_n}$$

$$\tilde{A} = \int \frac{\mu_{\tilde{A}}(x)}{x}$$

Definition : A fuzzy set A in the universe of discourse u can be defined as a set of **ordered pairs** and it is give by,

$$\underset{\sim}{A} = \{(x, \mu_{\underset{\sim}{A}}(x)) \mid x \in u\}$$

Where $\mu_{\underset{\sim}{A}}(x)$ is the degree of membership of x in $\underset{\sim}{A}$ and it is the degree that x belongs to $\underset{\sim}{A}$.

Note that $\mu_{\underset{\sim}{A}}(x) \in [0, 1]$

(A) When the universe of discourse U is discrete and finite, we write $\underset{\sim}{A}$ as :,

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}_1}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}_2}(x_2)}{x_2} + \frac{\mu_{\underset{\sim}{A}_3}(x_3)}{x_3} + \dots \right\}$$

$$= \left\{ \sum_{i=0}^{n} \frac{\mu_{\underset{\sim}{A}_i}(x_i)}{x_i} \right\} \text{; where n is finite value}$$

(B) When the universe of discourse U is continuous and infinite, then fuzzy set $\underset{\sim}{A}$ is given by,

$$\underset{\sim}{A} = \left\{ \int \frac{\mu_{\underset{\sim}{A}}(x)}{x} \right\}$$

(1) The horizontal bar in the above two representation of fuzzy sets for discrete and continuous universe is not a quotient (or division) but the representative of the corresponding variable, and called as a 'delimiter'

(2) The numerator in each representation is the membership value in set $\underset{\sim}{A}$.

(3) The summation symbol '+' in the representation of fuzzy set is not 'addition' but it is a discrete function union.

(4) A fuzzy set is universal fuzzy set if

$$\mu_u(x) = 1, \quad \forall x \in u$$

and it is empty set if

$$\mu_\phi(x) = 0, \quad \forall x \in u$$

(5) Two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are said to be equal if

$$\mu_{\underset{\sim}{A}}(x) = \mu_B(x), \quad \forall x \in u.$$

Also, $\because$ A $\subseteq$ u, $\quad \therefore \mu_A(x) \subseteq \mu_u(x) = 1, \quad \forall x \in u$

Following properties are frequently used :

(II)	**Commutitivity**	$\underline{A} \cup \underline{B} = \underline{B} \cup \underline{A}$; $\underline{A} \cap \underline{B} = \underline{B} \cap \underline{A}$
(III)	**Associativity**	$\underline{A} \cup (\underline{B} \cup \underline{C}) = (\underline{A} \cup \underline{B}) \cup \underline{C}$ $\underline{A} \cap (\underline{B} \cap \underline{C}) = (\underline{A} \cap \underline{B}) \cap \underline{C}$
(IV)	**Distributivity**	$\underline{A} \cup (\underline{B} \cap \underline{C}) = (\underline{A} \cup \underline{B}) \cap (\underline{A} \cup \underline{C})$ $\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cap (\underline{A} \cap \underline{C})$
(V)	**Idempotency**	$\underline{A} \cup \underline{A} = \underline{A}$ and $\underline{A} \cap \underline{A} = \underline{A}$
(VI)	**Identity**	$\underline{A} \cup \phi = \underline{A}$; $\underline{A} \cup u = u$ (universal set) $\underline{A} \cap \phi = \phi$ and $\underline{A} \cap u = \underline{A}$
(VII)	**Involution or double negation**	$\overline{\overline{\underline{A}}} = \underline{A}$
(VIII)	**Transitivity**	It $\underline{A} \subseteq \underline{B} \subseteq \underline{C}$ then $\underline{A} \subseteq \underline{C}$
(IX)	**De Morgan's laws**	$\overline{\underline{A} \cup \underline{B}} = \overline{\underline{A}} \cap \overline{\underline{B}}$; $\overline{\underline{A} \cap \underline{B}} = \overline{\underline{A}} \cup \overline{\underline{B}}$

1. Union Operation: The union operation of a fuzzy set is defined by:

$$\mu_{AUB}(x) = \max(\mu_A(x), \mu_B(x))$$

Example:

Let's suppose A is a set which contains following elements:

$A = \{(X_1, 0.6), (X_2, 0.2), (X_3, 1), (X_4, 0.4)\}$

And, B is a set which contains following elements:

$B = \{(X_1, 0.1), (X_2, 0.8), (X_3, 0), (X_4, 0.9)\}$

then,

$AUB = \{(X_1, 0.6), (X_2, 0.8), (X_3, 1), (X_4, 0.9)\}$

For X_1

$\mu_{A\cup B}(X_1) = \max(\mu_A(X_1), \mu_B(X_1))$

$\mu_{A\cup B}(X_1) = \max(0.6, 0.1)$

$\mu_{A\cup B}(X_1) = 0.6$

For X_2

$\mu_{A\cup B}(X_2) = \max(\mu_A(X_2), \mu_B(X_2))$

$\mu_{A\cup B}(X_2) = \max(0.2, 0.8)$

$\mu_{A\cup B}(X_2) = 0.8$

For X_3

$\mu_{A\cup B}(X_3) = \max(\mu_A(X_3), \mu_B(X_3))$

$\mu_{A\cup B}(X_3) = \max(1, 0)$

$\mu_{A\cup B}(X_3) = 1$

For X_4

$\mu_{A\cup B}(X_4) = \max(\mu_A(X_4), \mu_B(X_4))$

$\mu_{A\cup B}(X_4) = \max(0.4, 0.9)$

$\mu_{A\cup B}(X_4) = 0.9$

2. Intersection Operation: The intersection operation of fuzzy set is defined by:

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Example:

Let's suppose A is a set which contains following elements:

$A = \{(X_1, 0.3), (X_2, 0.7), (X_3, 0.5), (X_4, 0.1)\}$

And, B is a set which contains following elements:

$B = \{(X_1, 0.8), (X_2, 0.2), (X_3, 0.4), (X_4, 0.9)\}$

then,

$A\cap B = \{(X_1, 0.3), (X_2, 0.2), (X_3, 0.4), (X_4, 0.1)\}$

For X_1

$\mu_{A\cap B}(X_1) = \min(\mu_A(X_1), \mu_B(X_1))$

$\mu_{A\cap B}(X_1) = \min(0.3, 0.8)$

$\mu_{A\cap B}(X_1) = 0.3$

For X_2

$\mu_{A\cap B}(X_2) = \min(\mu_A(X_2), \mu_B(X_2))$

$\mu_{A\cap B}(X_2) = \min(0.7, 0.2)$

$\mu_{A\cap B}(X_2) = 0.2$

For X_3

$\mu_{A\cap B}(X_3) = \min(\mu_A(X_3), \mu_B(X_3))$

$\mu_{A\cap B}(X_3) = \min(0.5, 0.4)$

$\mu_{A\cap B}(X_3) = 0.4$

For X_4

$\mu_{A\cap B}(X_4) = \min(\mu_A(X_4), \mu_B(X_4))$

$\mu_{A\cap B}(X_4) = \min(0.1, 0.9)$

$\mu_{A\cap B}(X_4) = 0.1$

3. Complement Operation: The complement operation of fuzzy set is defined by:

$$\mu_{\bar{A}}(x) = 1-\mu_A(x),$$

Example:

Let's suppose A is a set which contains following elements:

$$A = \{(X_1, 0.3), (X_2, 0.8), (X_3, 0.5), (X_4, 0.1)\}$$

then,

$$\bar{A}= \{(X_1, 0.7), (X_2, 0.2), (X_3, 0.5), (X_4, 0.9)\}$$

For X_1

$\mu_{\bar{A}}(X_1) = 1-\mu_A(X_1)$
$\mu_{\bar{A}}(X_1) = 1 - 0.3$
$\mu_{\bar{A}}(X_1) = 0.7$

For X_2

$\mu_{\bar{A}}(X_2) = 1-\mu_A(X_2)$
$\mu_{\bar{A}}(X_2) = 1 - 0.8$
$\mu_{\bar{A}}(X_2) = 0.2$

For X_3

$\mu_{\bar{A}}(X_3) = 1-\mu_A(X_3)$
$\mu_{\bar{A}}(X_3) = 1 - 0.5$
$\mu_{\bar{A}}(X_3) = 0.5$

For X_4

$\mu_{\bar{A}}(X_4) = 1-\mu_A(X_4)$
$\mu_{\bar{A}}(X_4) = 1 - 0.1$
$\mu_{\bar{A}}(X_4) = 0.9$

: Consider two given fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{1}{2}+\frac{0.3}{4}+\frac{0.5}{6}+\frac{0.2}{8}\right\}, \underset{\sim}{B} = \left\{\frac{0.5}{2}+\frac{0.4}{4}+\frac{0.1}{6}+\frac{1}{8}\right\},$$

Perform union, intersection, difference and complement over the fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

Soln. :

(i) **Union :** $\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$

$$= \left\{\frac{1}{2}+\frac{0.4}{4}+\frac{0.5}{6}+\frac{1}{8}\right\}$$

> **Note :** The number 2, 4, 6, 8 in the denominator are delimiters, so to find maximum, numerators of first terms in $\underset{\sim}{A}$ and $\underset{\sim}{B}$, it is 1, and so on. Again '+' is a symbol and not an addition.

(ii) **Intersection**

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{\frac{0.5}{2}+\frac{0.3}{4}+\frac{0.1}{6}+\frac{0.2}{8}\right\}$$

(iii) **Complement**

$$\bar{\underset{\sim}{A}} = 1-\mu_{\underset{\sim}{A}}(x) = \left\{\frac{1-1}{2}+\frac{1-0.3}{4}+\frac{1-0.5}{6}+\frac{1-0.2}{8}\right\}$$

$$= \left\{\frac{0}{2}+\frac{0.7}{4}+\frac{0.5}{6}+\frac{0.8}{8}\right\}$$

and $\bar{\underset{\sim}{B}} = 1-\mu_{\underset{\sim}{B}}(x) = \left\{\frac{1-0.5}{2}+\frac{1-0.4}{4}+\frac{1-0.1}{6}+\frac{1-1}{8}\right\}$

$$= \left\{\frac{0.5}{2}+\frac{0.6}{4}+\frac{0.9}{6}+\frac{0}{8}\right\}$$

(iv) **Difference**

$$\underset{\sim}{A}/\underset{\sim}{B} = \underset{\sim}{A}-\underset{\sim}{B} = \underset{\sim}{A}\cap\bar{\underset{\sim}{B}} = \left\{\frac{0.5}{2}+\frac{0.3}{4}+\frac{0.5}{6}+\frac{0}{8}\right\}$$

And $\quad B/A = B-A = B\cap\bar{A} = \left\{\frac{0}{2}+\frac{0.4}{4}+\frac{0.1}{6}+\frac{0.8}{8}\right\}$

Ex. 3.3.2 : Given two fuzzy sets

$$B_1 = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}, \quad B_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$$

Find the following

(a) $B_1 \cup B_2$ (b) $B_1 \cap B_2$ (c) $\overline{\underline{B}_1}$ (d) $\overline{\underline{B}_2}$ (e) $\underline{B}_1 / \underline{B}_2$ (f) $\overline{\underline{B}_1 \cup \underline{B}_2}$

(g) $\overline{\underline{B}_1 \cap \underline{B}_2}$ (h) $\underline{B}_1 \cap \overline{B}_1$ (i) $\underline{B}_1 \cup \overline{\underline{B}_1}$ (j) $\underline{B}_2 \cap \overline{\underline{B}_2}$ (k) $\underline{B}_2 \cup \overline{\underline{B}_2}$

☑ **Soln. :** From the given fuzzy sets,

[Again note that the denominators of the terms of $\underline{B}_1$ and $\underline{B}_2$ are delimiters and '+' is a symbol and not arithmetic addition].

(a) $\underline{B}_1 \cup B_2 \max \{ \mu_{\underline{B}_1}(x), \mu_{\underline{B}_2}(x) \} = \left\{ \frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$

(b) $\underline{B}_1 \cap \underline{B}_2 = \left\{ \frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0} \right\}$

(c) $\overline{\underline{B}}_1 = 1 - \mu_{\underline{B}_1}(x) = \left\{ \frac{1-1}{1.0} + \frac{1-0.75}{1.5} + \frac{1-0.3}{2.0} + \frac{1-0.15}{2.5} + \frac{1-0}{3} \right\} = \left\{ \frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3} \right\}$

(d) $\overline{\underline{B}}_2 = 1 - \mu_{\underline{B}_2}(x) = \left\{ \frac{1-1}{1.0} + \frac{1-0.6}{1.5} + \frac{1-0.2}{2.0} + \frac{1-0.1}{2.5} + \frac{1-0}{3.0} \right\} = \left\{ \frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0} \right\}$

(e) $B_1 / B_2 = B_1 - B_2 = B_1 \cap \overline{B}_2 = \min (\mu_{\underline{B}_1}(x), \mu_{\underline{B}_2}(x) = \left\{ \frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0} \right\}$

(f) $\overline{\underline{B}_1 \cup \underline{B}_2} = \overline{B}_1 \cap \overline{B}_2$ (by De Morgan's laws)

$$= \min \{ \mu_{\overline{\underline{B}_1}}(x), \mu_{\overline{\underline{B}_2}}(x) \} = \left\{ \frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0} \right\}$$

(g) $\overline{\underline{B}_1 \cap \underline{B}_2} = \overline{\underline{B}}_1 \cup \overline{\underline{B}}_2$ (by De Morgan's laws)

$$= \max\{\mu_{\overline{\underline{B}_1}}(x), \mu_{\overline{\underline{B}_2}}(x)\} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$$

(h) $$\underline{B}_1 \cap \overline{\underline{B}}_1 = \min\{\mu_{\underline{B}_1}(x), \mu_{\overline{\underline{B}}_1}(x)\} = \left\{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\right\}$$

(i) $$\underline{B}_1 \cap \overline{\underline{B}}_1 = \max \cdot \{\mu_{\underline{B}_1}(x), \mu_{\overline{\underline{B}_1}}(x)\} = \left\{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\right\}$$

(j) $$\underline{B}_2 \cap \overline{\underline{B}}_2 = \min \cdot \{\mu_{\underline{B}_2}(x), \mu_{\overline{\underline{B}_2}}(x)\} = \left\{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\right\}$$

(k) $$\underline{B}_2 \cap \overline{\underline{B}}_2 = \max \cdot \{\mu_{\underline{B}_2}(x), \mu_{\overline{\underline{B}_2}}(x)\} = \left\{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\right\}$$

Consider two fuzzy sets :

$$\underset{\sim}{A} = \left\{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\} \text{ and } \underset{\sim}{B} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{1}{4}\right\}$$

Find the algebraic sum, algebraic product, bounded sum and bounded difference of the given fuzzy sets.

☑ **Soln. :**

(i) The algebraic sum is given by

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - [\mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)] \quad \text{...(i)}$$

$$\text{Now, } \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x) = \left\{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.05}{4}\right\} \quad \text{...(ii)}$$

$$\therefore \mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x)$$

$$= \left\{\frac{0.2+0.1-0.2}{1} + \frac{0.3+0.2-0.6}{2} + \frac{0.4+0.2-0.08}{3} + \frac{0.5+0.1-0.05}{4}\right\}$$

$$= \left\{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{0.55}{4}\right\}$$

(ii) Bounded sum

$$= \min \cdot [1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)]$$

$$= \min \cdot \left\{1, \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}$$

(iii) Bounded difference

$$\mu_{\underset{\sim}{A} \odot \underset{\sim}{B}}(x) = \max \{0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)\}$$

$$= \max \left\{0, \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}\right\}$$

$$= \left\{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\right\}$$

Prove De Morgan's theorem for the following fuzzy sets :

$$A = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} \right\} ; B = \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} \right\}$$

To show that $\overline{(A_1 \cup B_2)} = \overline{A} \cap \overline{B}$

☑ **Soln. :**

▶ **Step I :**

We have $A \cup B = \max \{\mu_A(x), \mu_B(x)\}$

$$\therefore \mu(A \cup B) = \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} \right\}$$

$$\overline{A \cup B} = 1 - \mu_{A \cup B} = 1 - \max\{\mu_A(x), \mu_B(x)\}$$

$$= 1 - \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} \right\}$$

$$= \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} \right\} \quad \text{...(i)}$$

Now, $\mu_{\overline{A}} = 1 - \mu_A = 1 - \left\{ \frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.85}{60} \right\}$

$$= \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} \right\}$$

$$\mu_{\overline{B}} = 1 - \mu_B = 1 - \left\{ \frac{0}{0} + \frac{0.45}{20} + \frac{0.6}{40} + \frac{0.8}{60} \right\}$$

$$= \left\{ \frac{1}{0} + \frac{0.55}{20} + \frac{0.4}{40} + \frac{0.2}{60} \right\}$$

$$\therefore \overline{A} \cap \overline{B} = \min\{\mu_{\overline{A}}, \mu_{\overline{B}}\}$$

$$= \left\{ \frac{1}{0} + \frac{0.5}{20} + \frac{0.35}{40} + \frac{0.15}{60} \right\} \quad \text{...(ii)}$$

From (i), (ii), the theorem is verified, Consider two fuzzy set given by

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.2}{3} + \frac{0.5}{4} \right\}, \underset{\sim}{B} = \left\{ \frac{0.9}{2} + \frac{0.4}{3} + \frac{0.8}{4} \right\}$$

Find (i) $\underset{\sim}{A} \cup \underset{\sim}{B}$ (ii) $A \cap B$ (iii) $\bar{A}$, (iv) $\bar{A} \cup B$ of the fuzzy sets

☑ **Soln. :**

We have $\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.2}{3} + \frac{0.5}{4} \right\}$; $\underset{\sim}{B} = \left\{ \frac{0.9}{2} + \frac{0.4}{3} + \frac{0.8}{4} \right\}$

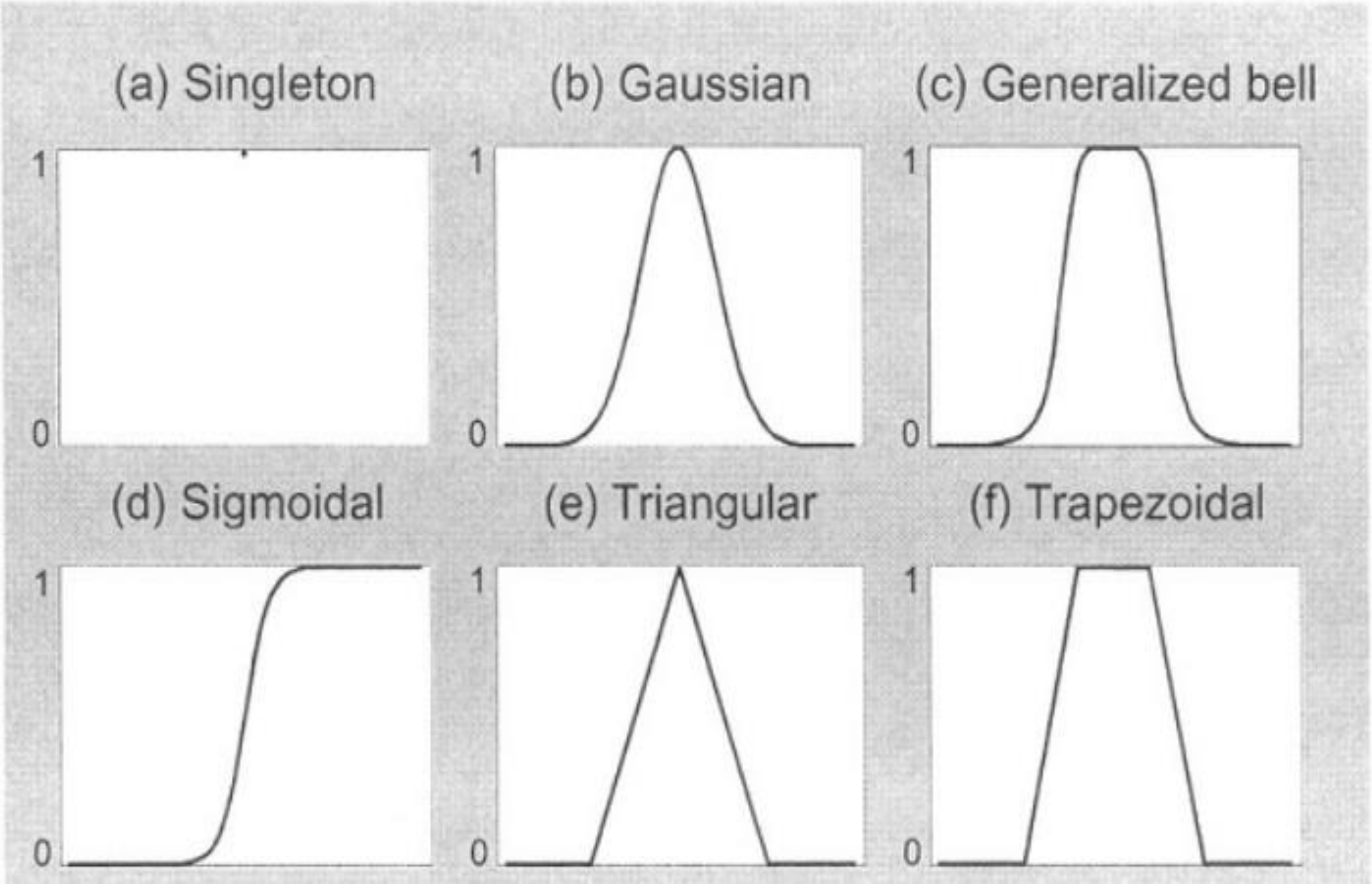
(i) $\underset{\sim}{A} \cup \underset{\sim}{B} = \max \{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{ \frac{1}{2} + \frac{0.4}{3} + \frac{0.8}{4} \right\}$

(ii) $\underset{\sim}{A} \cap \underset{\sim}{B} = \min \{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{ \frac{0.9}{2} + \frac{0.2}{3} + \frac{0.5}{4} \right\}$

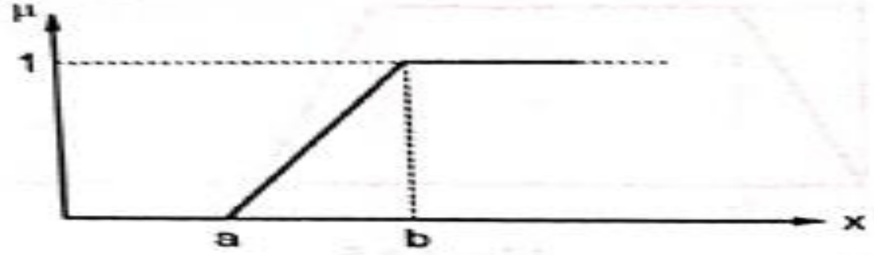
(iii) $\bar{\underset{\sim}{A}} = 1 - \underset{\sim}{A} = 1 - \left\{ \frac{1}{2} + \frac{0.2}{3} + \frac{0.5}{4} \right\} = \left\{ \frac{0}{2} + \frac{0.8}{3} + \frac{0.5}{4} \right\}$

(iv) $\bar{A} \cup B = \max \cdot \{\mu_{\bar{\underset{\sim}{A}}}(x), \mu_{\underset{\sim}{B}}(x)\} = \left\{ \frac{0.9}{2} + \frac{0.8}{3} + \frac{0.8}{4} \right\}$

Membership functions: Parameterization and Formulation 1) Triangular Membership function 2) Trapezoidal MF 3) Gaussian MF 4) Generalized bell MF 5) Sigmoid membership function

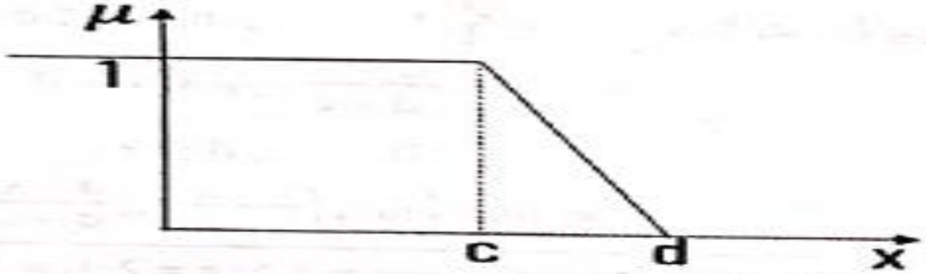


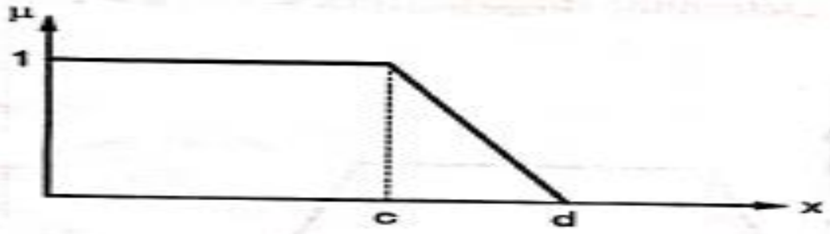
Increasing Membership Function (T Function) :



$$\mu_T(x; a, b) = \begin{cases} 0 & , x \le a \\ \dfrac{x-a}{b-a} & , a \le x \le b \\ 1 & , x \ge b \end{cases}$$

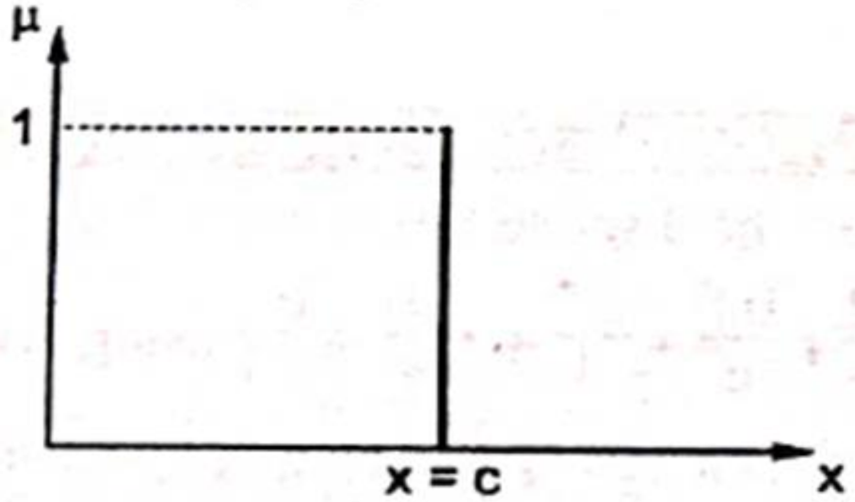
Decreasing Membership Function (L Function)





$$\mu_L(x; c, d) = \begin{cases} 1 & , x \le c \\ \dfrac{d-x}{d-c} & , c \le x \le d \\ 0 & , x \ge d \end{cases}$$

Singleton Membership Function : Singleton membership function assigns membership value 1 to particular value of x, and assigns value 0 to rest of all. It is represented by impulse function as shown.



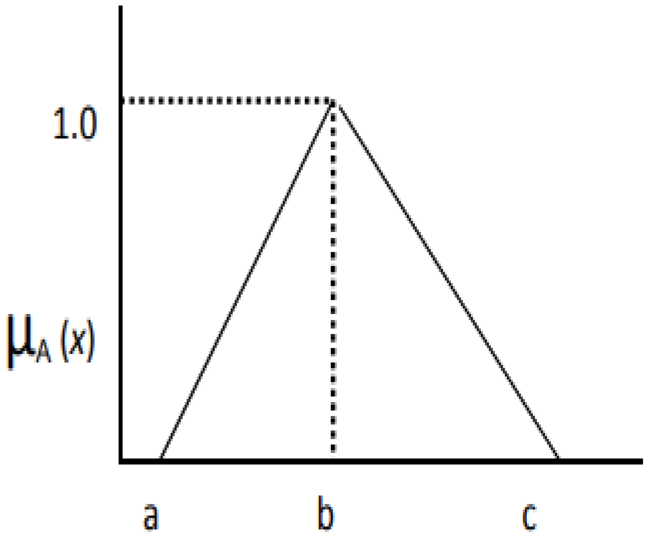
: Singleton Membership Function

Mathematically it is formulated as,

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}$$

Triangular Membership function:

Let a, b and c represent the x coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1).



$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ \dfrac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{if } x \ge c \end{cases}$$

$$\text{i.e. } \mu_{triangle}(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Ex. 3.4.1 : Determine $\mu_{triangle}$ $(x = 8; a = 2, b = 6, c = 10)$

☑ **Soln. :**

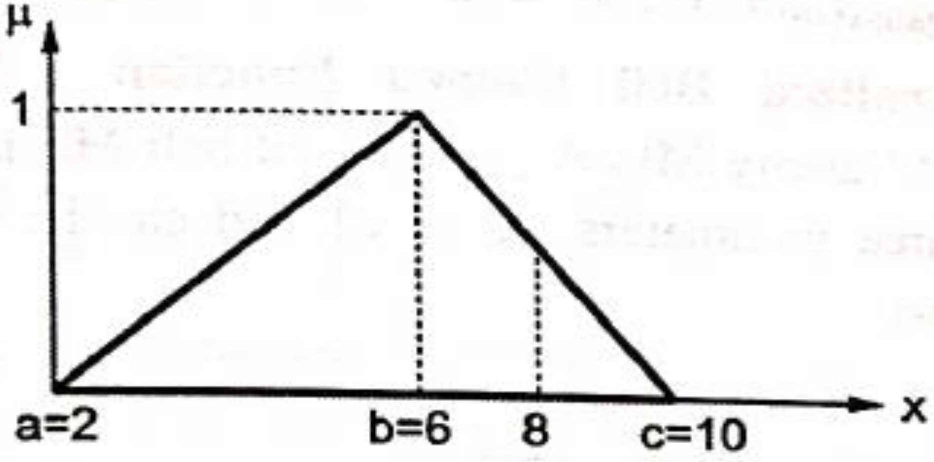


Fig. Ex. 3.4.1

$$\mu_{triangle}(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$\therefore$ $\mu_{triangle}$ $(x = 8; a = 2, b = 6, c = 10)$

$$= \max\left(\min\left(\frac{8-2}{6-2}, \frac{10-8}{10-6}\right), 0\right)$$

$$= \max\left(\min\left(\frac{3}{2}, \frac{1}{2}\right), 0\right) = \frac{1}{2}$$

Trapezoidal membership function:

Trapezoidal membership function:

Let a, b, c and d represents the x coordinates of the membership function. then

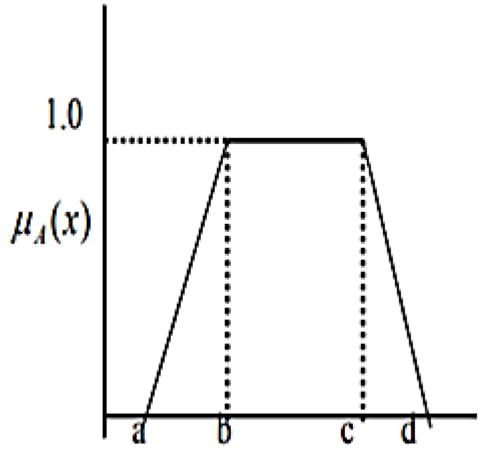
Trapezoid(x; a, b, c, d) = 0 if x ≤ a;

= (x-a)/ (b-a) if a ≤ x ≤ b

= 1 if b ≤ x ≤ c;

= (d-x)/ (d-c) if c ≤ x ≤ d;

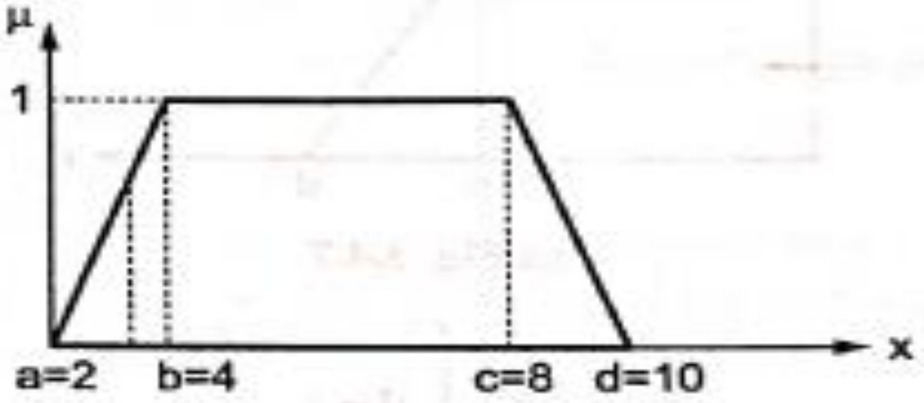
= 0, if d ≤ x.



$$\mu_{\text{trapezoid}} = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Ex. 3.4.2 : Determine $\mu_{trapezoidal}$ (x = 3.5; a = 2, b = 4, c = 8, d = 10).

☑ **Soln. :**



(1C24)**Fig. Ex. 3.4.2**

$$\mu_{trapezoidal}(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

$\therefore$ $\mu_{trapezoidal}$ (x = 3.5; a = 2, b = 4. C = 8, d = 10)

$$= \max\left(\min\left(\frac{3.5-2}{4-2}, 1, \frac{10-3.5}{10-8}\right), 0\right)$$

$$= \max\left(\min\left(\frac{3}{4}, 1, \frac{13}{4}\right), 0\right)$$

$$= \max(\min(0.75, 1, 3.25), 0)$$

$$= \max(0.75, 0) = 0.75$$

Gaussian membership function:

The Gaussian membership function is usually represented as Gaussian(x:c,s) where c, s represents the mean and standard deviation.

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

Here c represents centre, s represents width and m represents fuzzification factor.

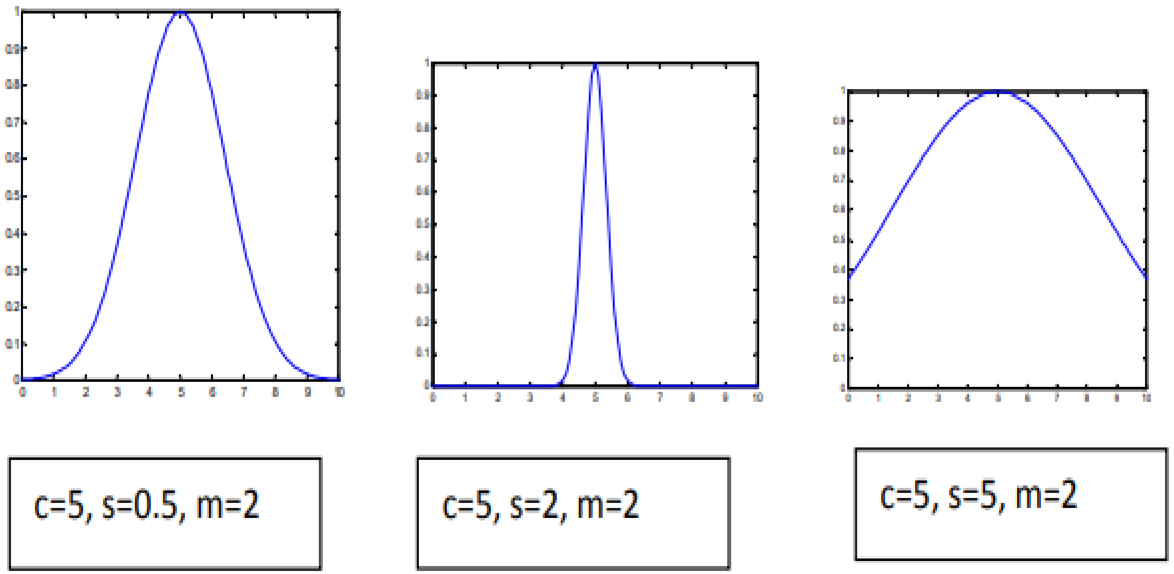
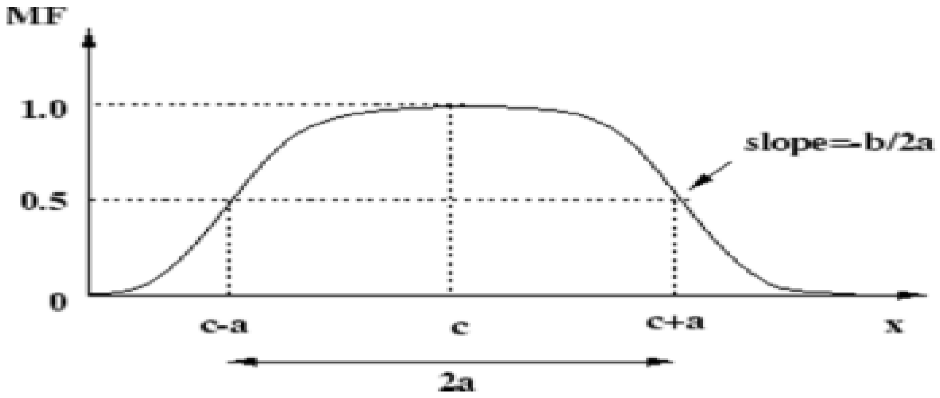


Figure : Different shapes of Gaussian MFs with different values of s and m.

Generalized Bell membership function:

A generalized bell membership function has three parameters: a –responsible for its widtl responsible for its center and b –responsible for its slopes. Mathematically,

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$



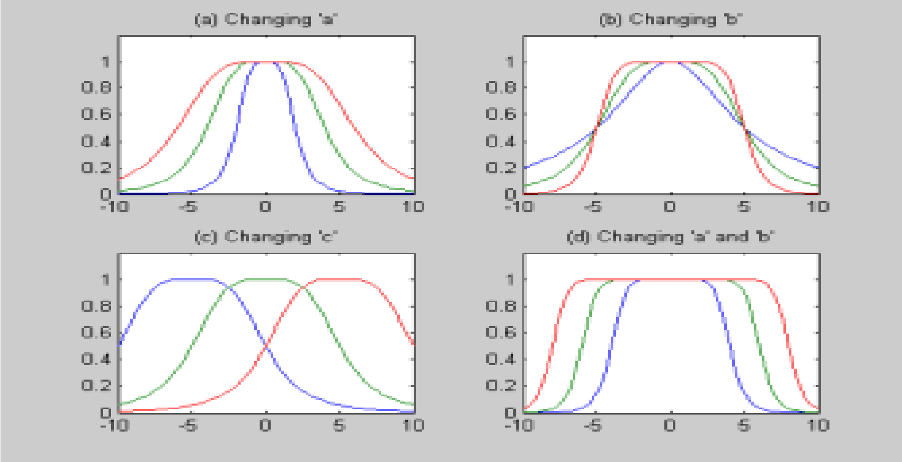


Figure ;: Different shapes of Gaussian MFs with different values of s and m.

Sigmoid Membership function:

A sigmoidal membership function has two parameters: a responsible for its slope at the crossover point x = c. The membership function of the sigmoid function can be represented as Sigmf (x:a, c) and it is

$$sigmf(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$$

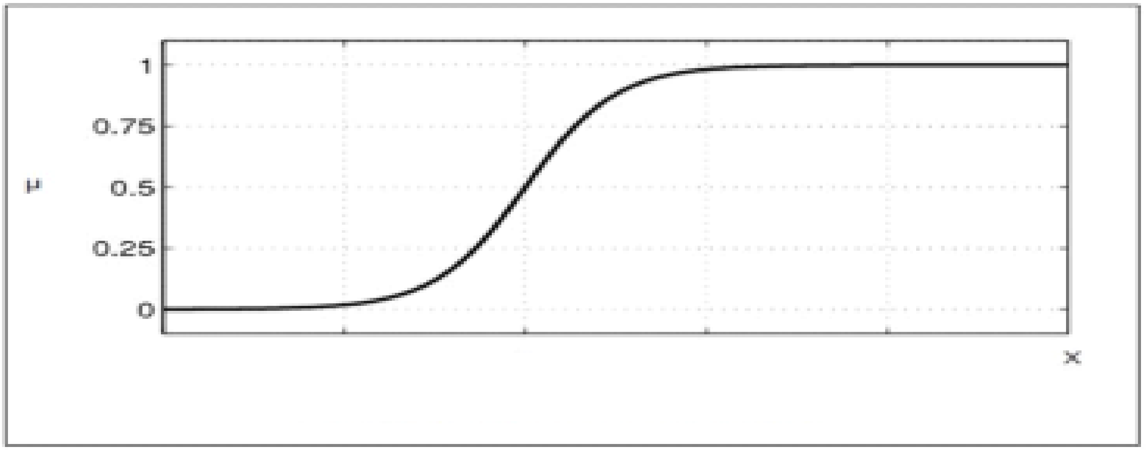


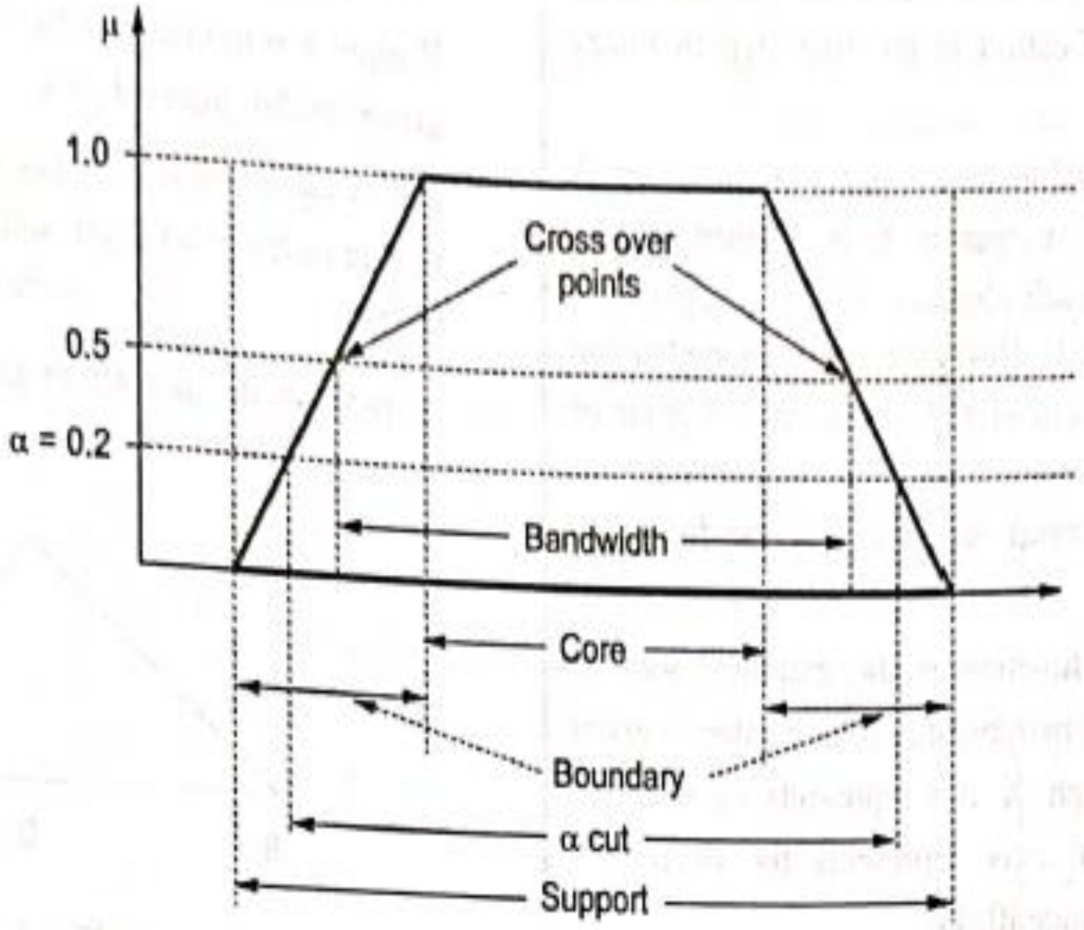
Figure : A general structures of sigmoid MF.

A sigmoidal MF is inherently open right or left & thus, it is appropriate for representing concepts such as "very large" or "very negative". Sigmoidal MF mostly used as activation function of artificial neural networks (NN).A NN should synthesize a close MF in order to simulate the behavior of a fuzzy inference system.

: Let $\underset{\sim}{A}$ = { (x_1, 0), (x_2, 0.2), (x_3, 0.5), (x_4, 1), (x_5, 1), (x_6, 1), (x_7, 0.5), (x_8, 0.2), (x_9, 0) }

Find support, core, crossover points, alpha cut and strong alpha cut for α = 0.2, boundary, bandwidth, normality of given fuzzy set.

☑ **Soln. :**



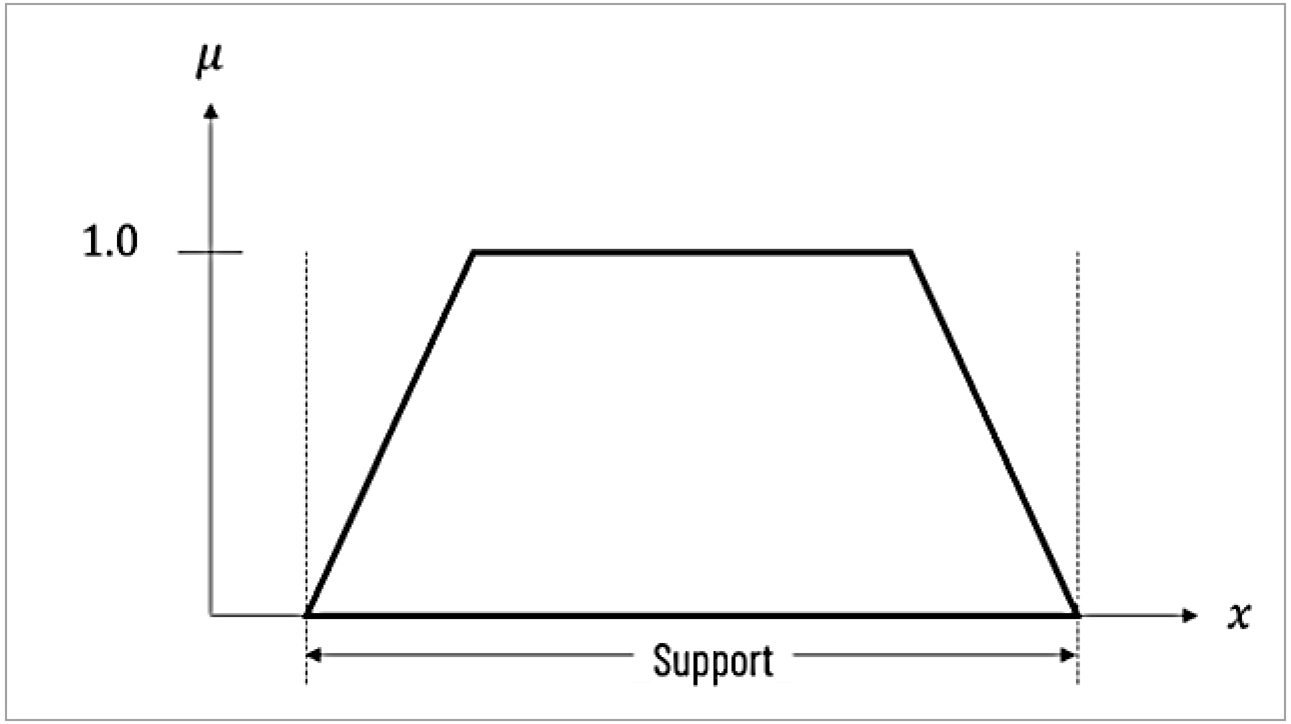
- **Support :** $\{(x_2, 0.2), (x_3, 0.5), (x_4, 1), (x_5, 1), (x_6, 1), (x_7, 0.5), (x_8, 0.2)\}$
- **Core :** $\{(x_4, 1), (x_5, 1), (x_6, 1)\}$
- **Crossover Points :** $\{(x_3, 0.5), (x_7, 0.5)\}$
- **Alpha Cut$_{0.2}$:** $\{(x_2, 0.2), (x_3, 0.5), (x_4, 1), (x_5, 1), (x_6, 1), (x_7, 0.5), (x_8, 0.2)\}$
- **Strong Alpha Cut$_{0.2}^{+}$:** $\{(x_3, 0.5), (x_4, 1), (x_5, 1), (x_6, 1), (x_7, 0.5)\}$
- **Boundary :** $\{(x_2, 0.2), (x_3, 0.5), (x_7, 0.5), (x_8, 0.2)\}$
- **Bandwidth :** $| x_7 - x_3 |$
- **Normality :** True

Support:

The **support** of a fuzzy set $\underline{A}$ is the set of all points $x \in X$ such that $\mu_A(x) > 0$

$$\text{Support}(\underline{A}) = \{ x \mid \mu_A(x) > 0, x \in X \}$$

Graphically, we can define support of fuzzy set as,



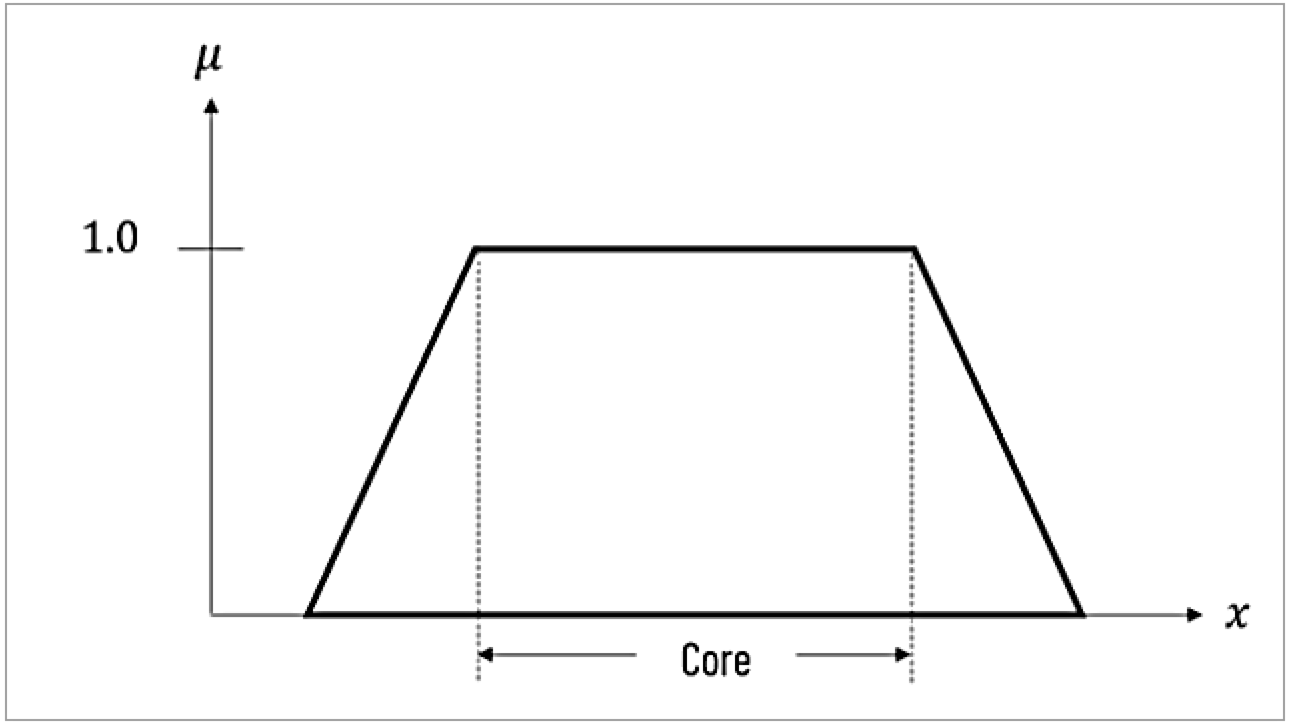
Support of fuzzy set

Core:

The **core** of a fuzzy set $\underline{A}$ is the set of all points $x \in X$ such that $\mu_A(x) = 1$

$$Core(\underline{A}) = \{ x \mid \mu_A(x) = 1, x \in X \}$$

All fuzzy sets might not have a core present in them.



The core of fuzzy set

Height of Fuzzy Set: It is defined as the largest membership value of the elements contained in that set. It may not be 1 always. If the core of the fuzzy set is non-empty, then the height of the fuzzy set is 1.

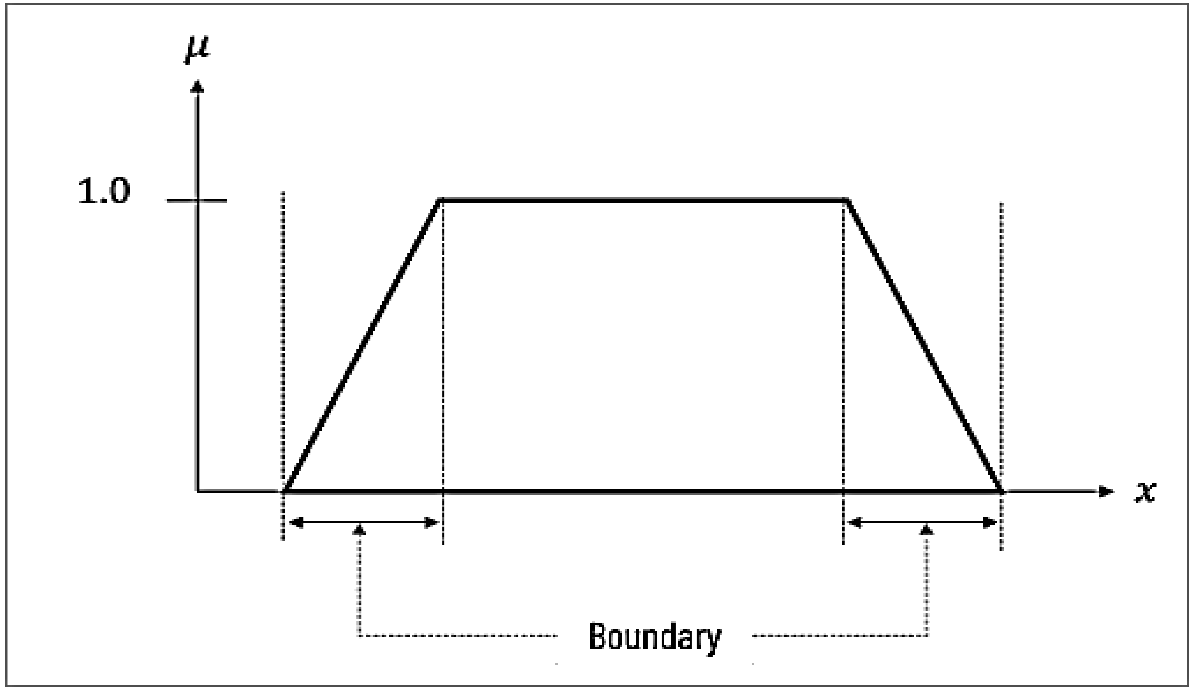
Boundary:

Boundary comprises those elements x of the universe such that $0 < \mu_A(x) < 1$

$$\text{Boundary}(\underline{A}) = \{ x \mid 0 < \mu_A(x) < 1 , x \in X \}$$

We can treat boundary as the difference between support and core.

Graphically, it is represented as



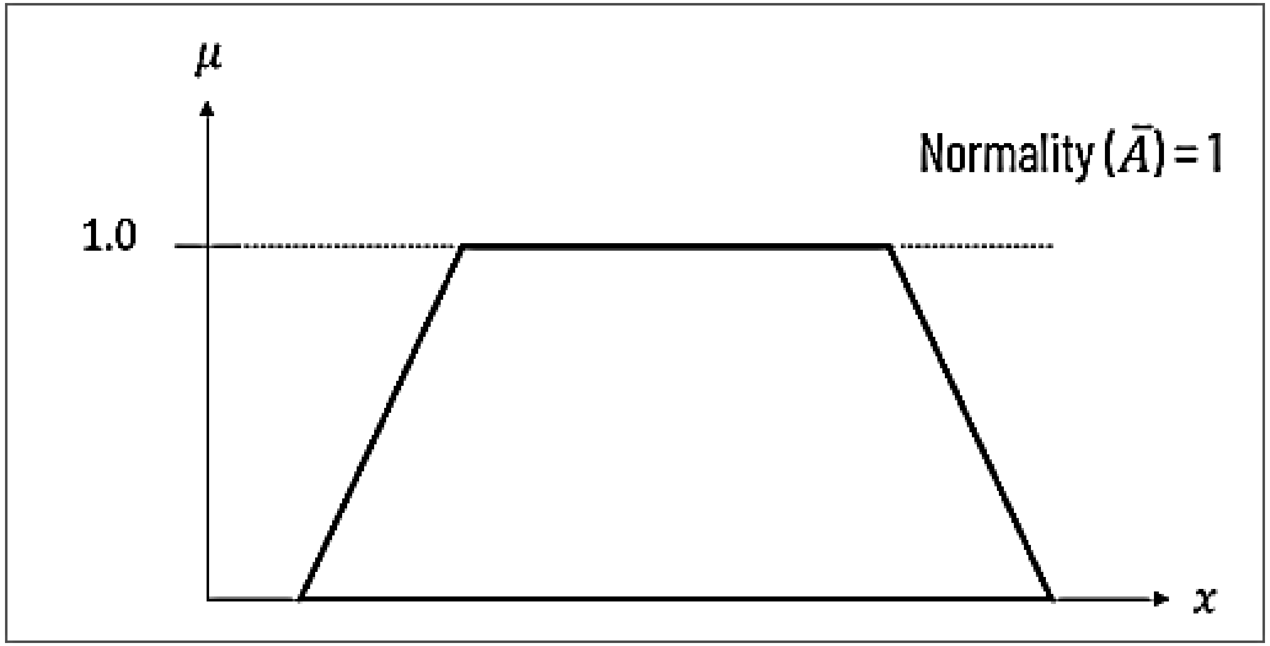
The boundary of fuzzy set

Normality:

A fuzzy set $\underline{A}$ is **normal** if its core is non-empty.

In other words, a fuzzy set is normal if its height is 1

Sub-normal Fuzzy set: For a sub-normal fuzzy set, h($\underline{A}$) < 1, where h($\underline{A}$) represents the height of the fuzzy set / highest membership value in the fuzzy set.



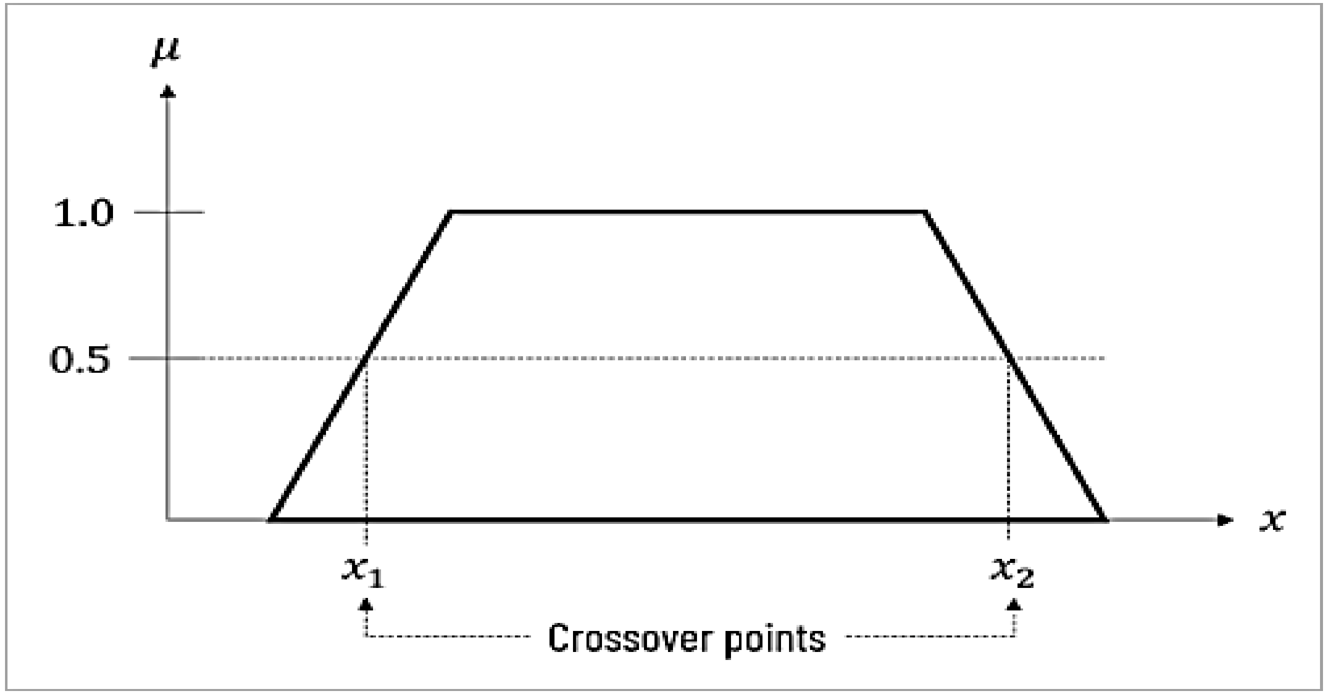
Normality of fuzzy set

Crossover points:

A crossover point of a fuzzy set $\underline{A}$ is a point $x \in X$ at which $\mu_A(x) = 0.5$

$$\text{Crossover}(\ \underline{A}\ ) = \{\ x \mid \mu_A(x) = 0.5\ \}$$

Graphically, we can represent it as



Cross-over points of fuzzy set

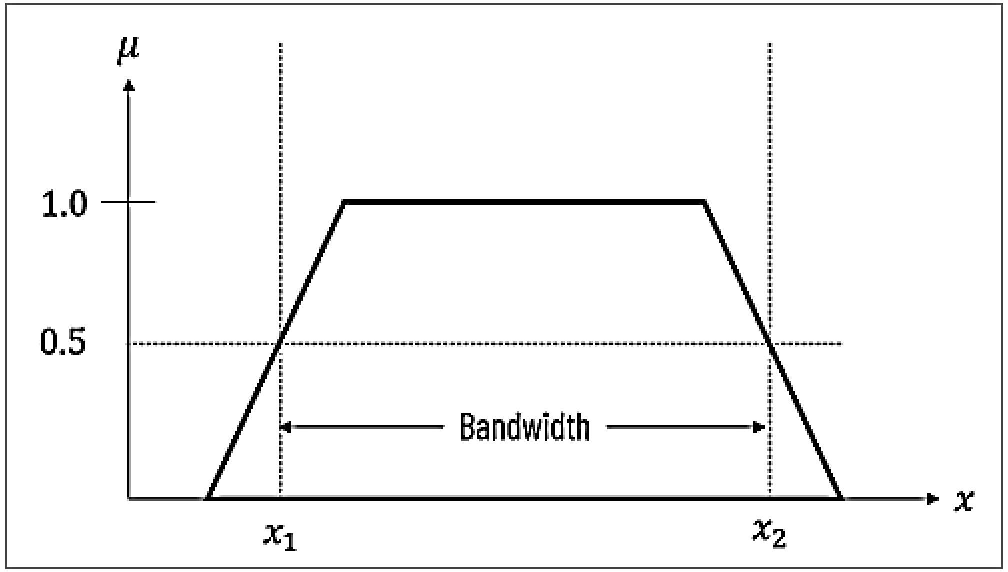
Bandwidth:

For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points.

Bandwidth($\underline{A}$) = $|x_1 - x_2|$

Where, $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Graphically,



The bandwidth of fuzzy set

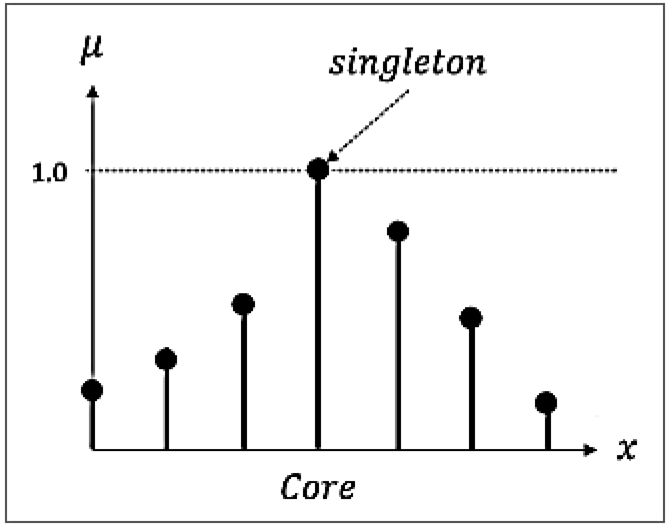
Fuzzy singleton:

A fuzzy set whose core is a single point in X with $\mu_A(x) = 1$, is called a fuzzy singleton.

In other words, if the fuzzy set is having only one element with a membership value of 1, then it is called a fuzzy singleton.

$|A| = \{ \mu_A(x) = 1 \}$

Graphically,



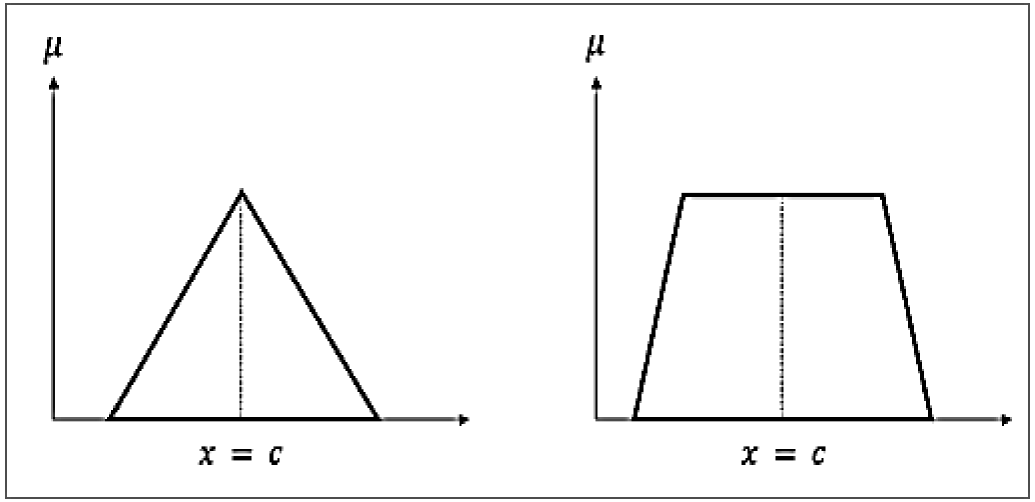
Fuzzy singleton

Symmetry:

Fuzzy set $\underline{A}$ is symmetric if its membership function around a centre point x = c is symmetric

i.e. $\mu_A(x + c) = \mu_A(x - c), \forall x \in X$

Triangular, Trapezoidal, Gaussian etc. are mostly symmetric. This is more natural to represent the membership than a non-symmetric shape.

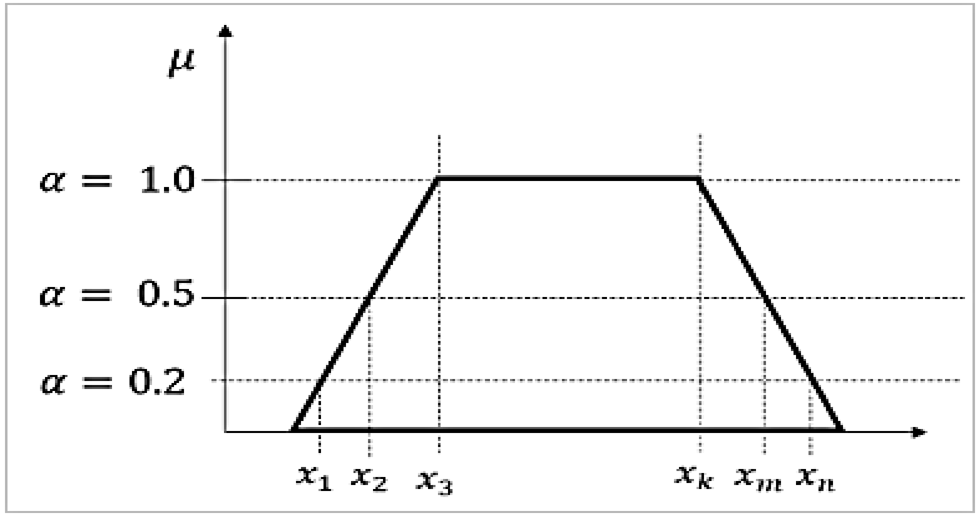


Symmetric fuzzy sets

Alpha cut:

The **α-cut** of a fuzzy set <u>A</u> is a crisp set defined by $A_\alpha = \{ x \mid \mu_A(x) \geq \alpha \}$

Strong α-cut of a fuzzy set <u>A</u> is a crisp set defined by $A_\alpha + = \{ x \mid \mu_A(x) > \alpha \}$

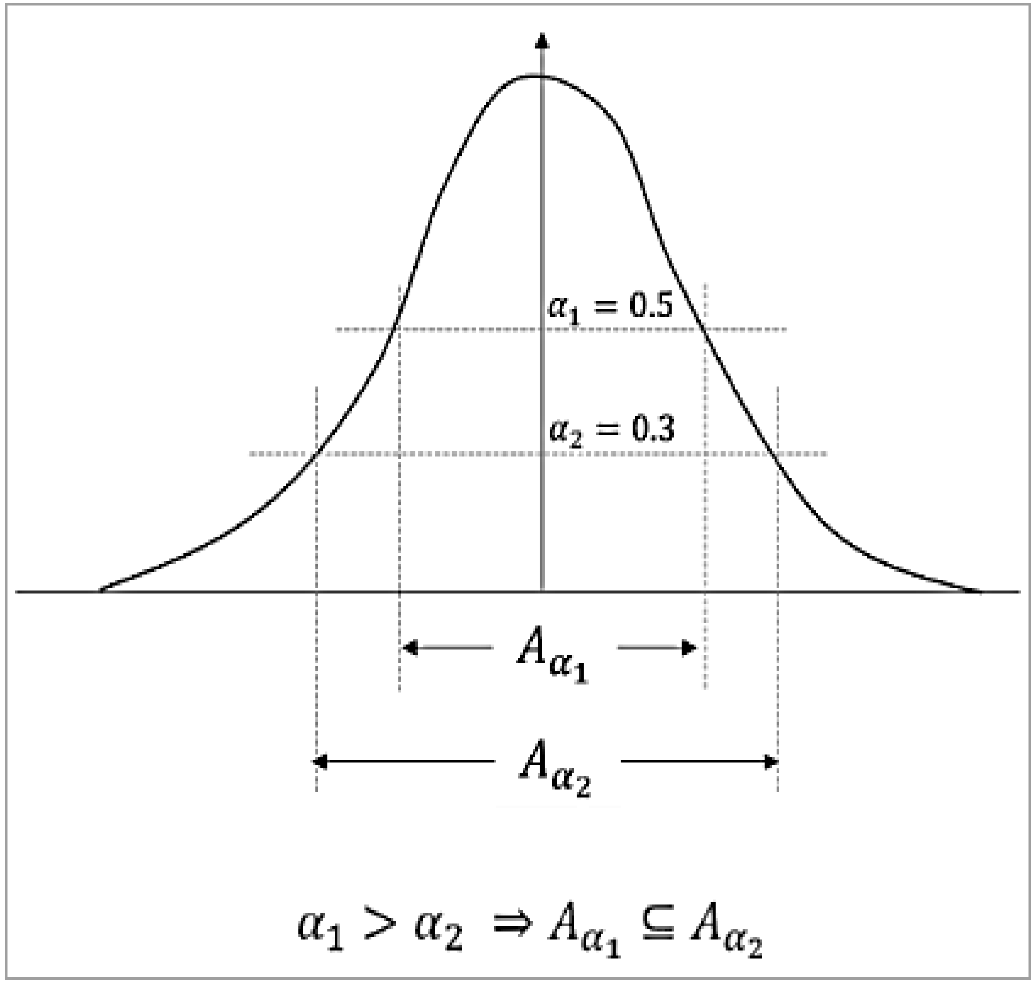


Alpha cut in fuzzy set

For the above diagram,

- The set $A_{\alpha=0.2}$ contains all the elements from x_1 to x_n, including both end values
- The set $A_{\alpha=0.5}$ contains all the elements from x_2 to x_m, including both end values
- The set $A_{\alpha=1.0}$ contains all the elements from x_3 to x_k, including both end values

For different values of α, we get different crisp sets. In general, if $\alpha_1 > \alpha_2$ then $A_{\alpha 1} \subseteq A_{\alpha 2}$



Relation between different α values

Cardinality:

Scalar cardinality:

Scalar cardinality is defined by the summation of membership values of all elements in the set. For the data given in the table,

$| \underline{A} | = \Sigma_{x \in X} \{ \mu_A(x) \}$

$|\text{Senior}| = 0.3 + 0.9 + 1 + 1 = 3.2$

Relative cardinality:

$|| \underline{A} || = | \underline{A} | / | X |$

$|| \text{Senior} || = 3.2 / 9 = 0.356$

Fuzzy cardinality:

$| \underline{A} |_F = \{ (\alpha , \mu_{A\alpha}(x)) \}$

$| \text{Senior} |_F = \{ (4, 0.3), (3, 0.9), (2, 1.0) \}$

Age	Infant	Young	Adult	Senior
5	0	0	0	0
15	0	0.2	0	0
25	0	0.8	0.8	0
35	0	1.0	0.9	0
45	0	0.6	1	0
55	0	0.5	1	0.3
65	0	0.1	1	0.9
75	0	0.0	1	1
85	0	0.0	1	1

$\alpha = \mu_{senior}$	$\lvert A_\alpha \rvert$
0.3	4
0.9	3
1	2

Open and Closed fuzzy sets:

Open left: As the name suggests, open left fuzzy sets have all the elements on left after a certain point have a membership value of 1, and all the elements on the right side after a certain point have a membership value of 0.

Open left:	if $\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 1$ and $\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 0$

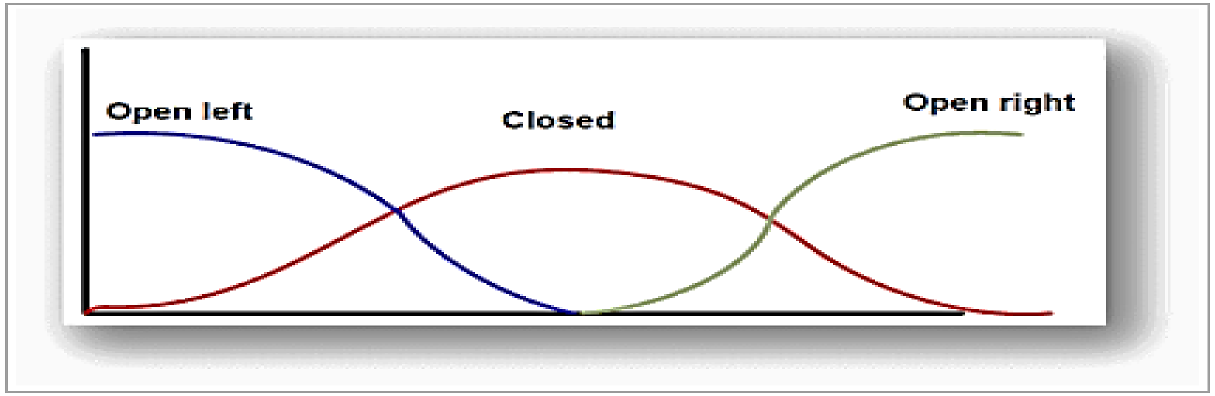
Open right: Open right fuzzy sets have all the elements on left after a certain point have a membership value of 0, and all the elements on the right side after a certain point have a membership value of 1.

Open right:	if $\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 0$ and $\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 1$

Closed: Closed fuzzy sets have all the elements on the left or right side after a certain point have a membership value of 0.

Closed:	if $\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 0$ and $\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 0$

The following diagram graphically demonstrates all three kinds of fuzzy sets.



Open and closed fuzzy sets

Convexity:

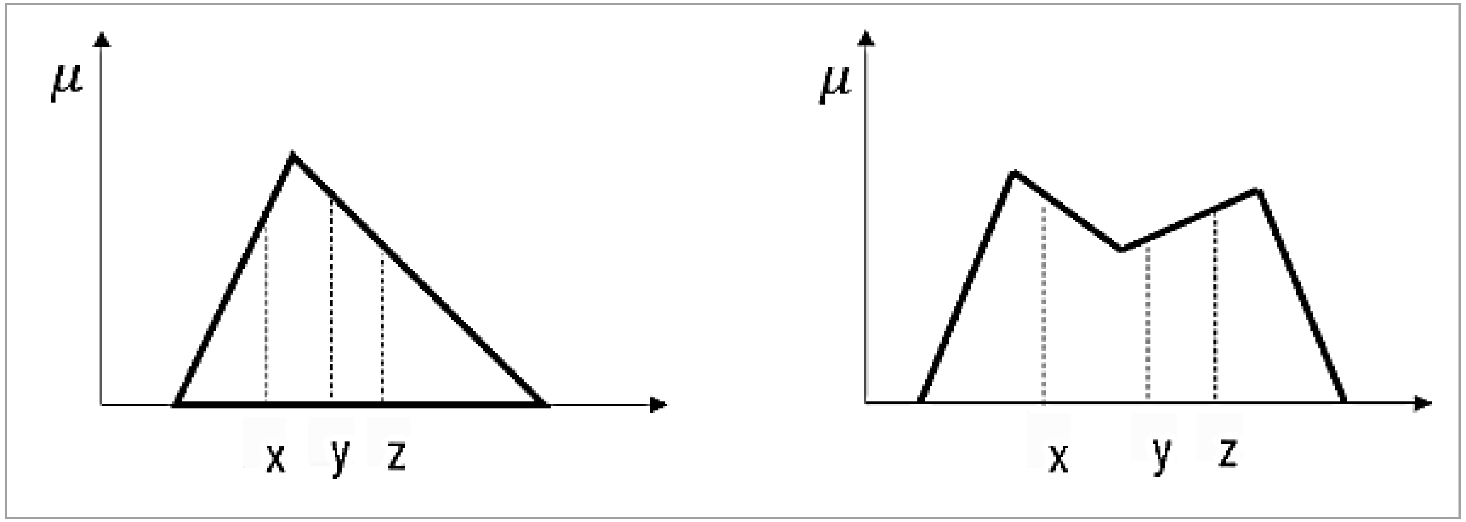
Crisp Set A is convex if $(\lambda x_1 + (1 - \lambda)\, x_2)$ in A, where $\lambda \in [0, 1]$

Fuzzy Set $\underline{A}$ is convex if $\mu_A(\lambda x_1 + (1 - \lambda)\, x_2)) \geq \min(\mu_A(x_1), \mu_A(x_2))$, where $x_1, x_2 \in X$

In other words, for any elements x, y and z in a fuzzy set $\underline{A}$, the relation $x < y < z$ implies that: $\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$. If this condition holds for all points, the fuzzy set is called a convex fuzzy set.

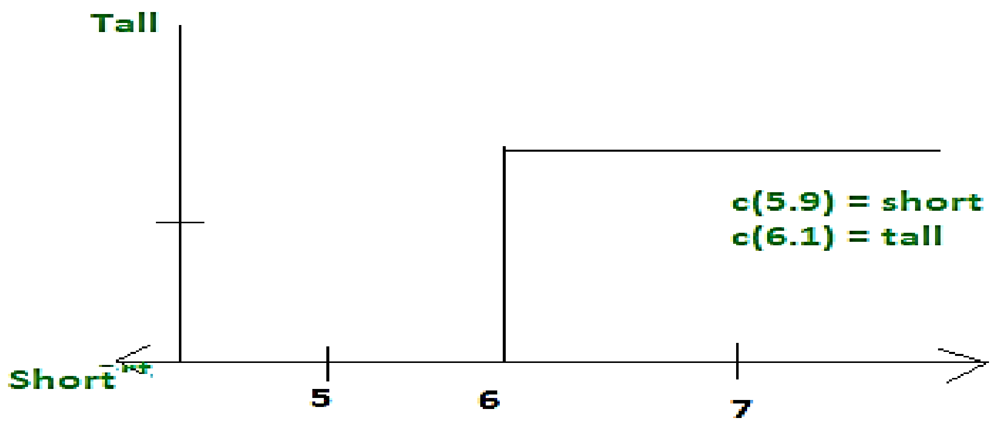
Convex fuzzy sets are strictly increasing and then strictly decreasing

$\underline{A}$ is convex if all its α-level sets are convex

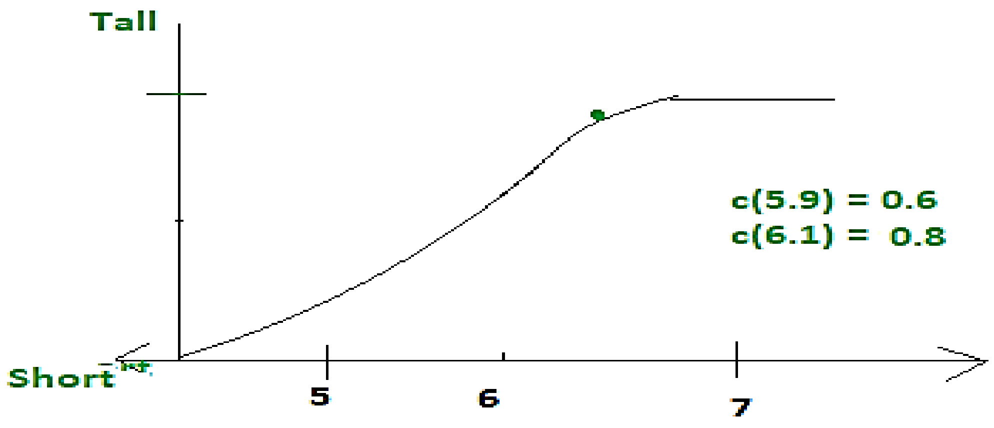


Convex fuzzy set (left), Non-convex fuzzy set (right)

S.No	Crisp Set	Fuzzy Set
1	Crisp set defines the value is either 0 or 1.	Fuzzy set defines the value between 0 and 1 including both 0 and 1.
2	It is also called a classical set.	It specifies the degree to which something is true.
3	It shows full membership	It shows partial membership.
4	Eg1. She is 18 years old. Eg2. Rahul is 1.6m tall	Eg1. She is about 18 years old. Eg2. Rahul is about 1.6m tall.
5	Crisp set application used for digital design.	Fuzzy set used in the fuzzy controller.
6	It is bi-valued function logic.	It is infinite valued function logic
7	Full membership means totally true/false, yes/no, 0/1.	Partial membership means true to false, yes to no, 0 to 1.



Crisp Set



Fuzzy Set

1. Fuzzy sets also satisfy every property of classical sets.
2. **Common Operations on fuzzy sets**: Given two Fuzzy sets A~ and B~
   - **Union** : Fuzzy set C~ is union of Fuzzy sets A~ and B~ :

$$\tilde{C} = \tilde{A} \cup \tilde{B},$$

- $$\mu_{\tilde{C}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- **Intersection**: Fuzzy set D~ is intersection of Fuzzy sets A~ and B~ :

$$\tilde{D} = \tilde{A} \cap \tilde{B}$$

- $$\mu_{\tilde{D}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- **Complement**: Fuzzy set E~ is complement of Fuzzy set A~ :

$$\tilde{E} = \mathbb{C}_{\tilde{A}} X$$

-

$$\mu_{\tilde{E}}(x) = 1 - \mu_{\tilde{A}}(x)$$

1. Some other useful operations on Fuzzy set:
   - **Algebraic sum**:

$$\mu A{+}B(x) = \mu A(x) + \mu B(x) - \mu A(x) .\ \mu B(x)$$

- **Algebraic product**:

$$\mu A.B(x) = \mu A(x) .\ \mu B(x)$$

- Bounded sum:

$$\mu A_{\oplus} B(x) = \min\{1, \mu A(x) + \mu B(x)\}$$

- Bounded difference:

$$\mu A \odot B(c) = \max\{0, \mu A(x) - \mu B(x)\}$$

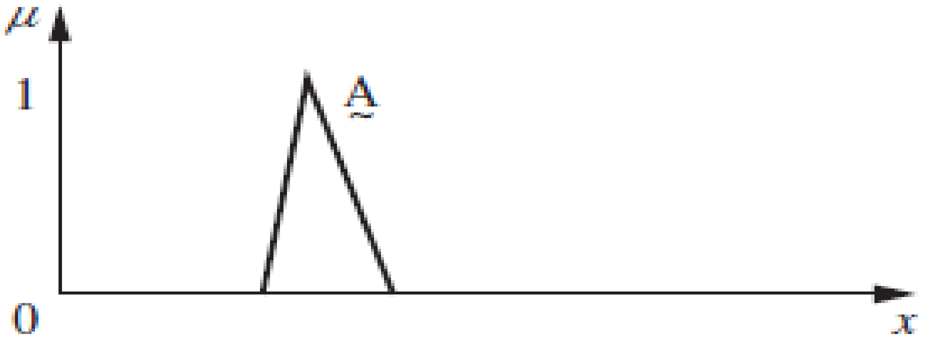


FIGURE
Membership function for fuzzy set $\underset{\sim}{A}$.

Fuzzy Set Operations

Define three fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ on the universe X. For a given element x of the universe, the following function-theoretic operations for the set-theoretic operations of union, intersection, and complement are defined for $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ on X:

Union $$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x).$$

Intersection $$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x).$$

Complement $$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x).$$

CARTESIAN PRODUCT

An ordered sequence of r elements, written in the form $(a_1, a_2, a_3, \ldots, a_r)$, is called an *ordered r-tuple*; an unordered r-tuple is simply a collection of r elements without restrictions on order. In a ubiquitous special case where $r = 2$, the r-tuple is referred to as an ordered *pair*. For crisp sets $A_1, A_2, \ldots, A_r$, the set of all r-tuples $(a_1, a_2, a_3, \ldots, a_r)$, where $a_1 \in A_1$, $a_2 \in A_2$, and $a_r \in A_r$, is called the *Cartesian product* of $A_1, A_2, \ldots, A_r$, and is denoted by $A_1 \times A_2 \times \cdots \times A_r$. The Cartesian product of two or more sets is *not* the same thing as the arithmetic product of two or more sets. The latter is dealt with in Chapter , when the extension principle is introduced.

When all the A_r are identical and equal to A, the Cartesian product $A_1 \times A_2 \times \cdots \times A_r$ can be denoted as A^r.

Example 3.1. The elements in two sets A and B are given as $A = \{0, 1\}$ and $B = \{a, b, c\}$. Various Cartesian products of these two sets can be written as shown:

$$A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}.$$

$$B \times A = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}.$$

$$A \times A = A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

$$B \times B = B^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

CRISP RELATIONS

A subset of the Cartesian product $A_1 \times A_2 \times \cdots \times A_r$ is called an *r-ary relation* over $A_1, A_2, \ldots, A_r$. Again, the most common case is for $r = 2$; in this situation, the relation

is a subset of the Cartesian product $A_1 \times A_2$ (i.e., a set of pairs, the first coordinate of which is from A_1 and the second from A_2). This subset of the full Cartesian product is called a *binary relation from* A_1 *into* A_2. If three, four, or five sets are involved in a subset of the full Cartesian product, the relations are called ternary, quaternary, and quinary, respectively. In this text, whenever the term *relation* is used without qualification, it is taken to mean a *binary relation*.

The Cartesian product of two universes X and Y is determined as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\},$$

which forms an ordered pair of every $x \in X$ with every $y \in Y$, forming *unconstrained* matches between X and Y. That is, every element in universe X is related completely to every element in universe Y. The *strength* of this relationship between ordered pairs of elements in each universe is measured by the characteristic function, denoted χ, where a value of unity is associated with *complete relationship* and a value of zero is associated with *no relationship*, that is,

$$\chi_{X\times Y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}.$$

One can think of this strength of relation as a mapping from ordered pairs of the universe, or ordered pairs of sets defined on the universes, to the characteristic function. When the universes, or sets, are finite the relation can be conveniently represented by a matrix, called a *relation matrix*. An r-ary relation can be represented by an r-dimensional relation matrix. Hence, binary relations can be represented by two-dimensional matrices (used throughout this text).

An example of the strength of relation for the unconstrained case is given in the Sagittal diagram shown in Figure (a Sagittal diagram is simply a schematic depicting points as elements of universes and lines as relationships between points, or it can be a pictorial of the elements as nodes which are connected by directional lines, as seen in Figure). Lines in the Sagittal diagram and values of unity in the *relation matrix*

$$
R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}
$$

correspond to the ordered pairs of mappings in the relation. Here, the elements in the two universes are defined as X = {1, 2, 3} and Y = {a, b, c}.

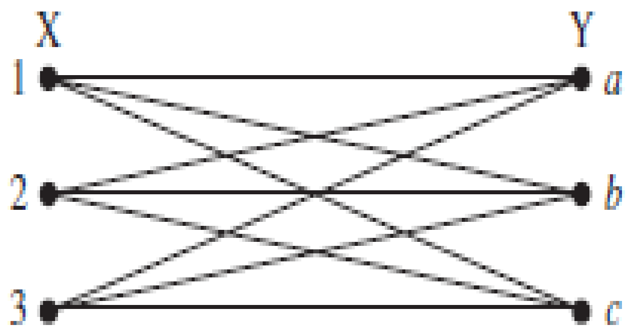


FIGURE
Sagittal diagram of an unconstrained relation.

A more general crisp relation, R, exists when matches between elements in two universes are *constrained*. Again, the characteristic function is used to assign values of relationship in the mapping of the Cartesian space X × Y to the binary values of (0, 1):

$$\chi_{\mathrm{R}}(x, y) = \begin{cases} 1, & (x, y) \in \mathrm{R} \\ 0, & (x, y) \notin \mathrm{R} \end{cases}.$$

Operations on Crisp Relations

Define R and S as two separate relations on the Cartesian universe X × Y, and define the null relation and the complete relation as the relation matrices **O** and E, respectively. An example of a 4 × 4 form of the **O** and E matrices is given here:

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The following function-theoretic operations for the two crisp relations (R, S) can now be defined.

Union $\quad \mathrm{R} \cup \mathrm{S} \longrightarrow \chi_{\mathrm{R}\cup\mathrm{S}}(x, y) : \chi_{\mathrm{R}\cup\mathrm{S}}(x, y) = \max[\chi_{\mathrm{R}}(x, y), \chi_{\mathrm{S}}(x, y)].$

Intersection $\quad \mathrm{R} \cap \mathrm{S} \longrightarrow \chi_{\mathrm{R}\cap\mathrm{S}}(x, y) : \chi_{\mathrm{R}\cap\mathrm{S}}(x, y) = \min[\chi_{\mathrm{R}}(x, y), \chi_{\mathrm{S}}(x, y)].$

Complement $\quad \overline{\mathrm{R}} \longrightarrow \chi_{\overline{\mathrm{R}}}(x, y) : \chi_{\overline{\mathrm{R}}}(x, y) = 1 - \chi_{\mathrm{R}}(x, y).$

Containment $\quad \mathrm{R} \subset \mathrm{S} \longrightarrow \chi_{\mathrm{R}\subset\mathrm{S}}(x, y) : \chi_{\mathrm{R}}(x, y) \leq \chi_{\mathrm{S}}(x, y).$

Identity $\quad \varnothing \longrightarrow \mathbf{O}$ and $\mathrm{X} \longrightarrow \mathrm{E}.$

Properties of Crisp Relations

The properties of commutativity, associativity, distributivity, involution, and idempotency all hold for crisp relations just as they do for classical set operations. Moreover, *De Morgan*'s *principles* and the *excluded middle axioms* also hold for crisp (classical) relations just as they do for crisp (classical) sets. The null relation, **O**, and the complete relation, E, are analogous to the null set, Ø, and the whole set, X, respectively, in the set-theoretic case

Composition

Let R be a relation that relates, or maps, elements from universe X to universe Y, and let S be a relation that relates, or maps, elements from universe Y to universe Z.

A useful question we seek to answer is whether we can find a relation, T, that relates the same elements in universe X that R contains to the same elements in universe Z that S contains. It turns out that we can find such a relation using an operation known as *composition*. From the Sagittal diagram in Figure 3.4, we see that the only "path" between relation R and relation S is the two routes that start at x_1 and end at z_2 (i.e., $x_1 - y_1 - z_2$ and $x_1 - y_3 - z_2$). Hence, we wish to find a relation T that relates the ordered pair (x_1, z_2), that is, $(x_1, z_2) \in T$. In this example,

$$R = \{(x_1, y_1), (x_1, y_3), (x_2, y_4)\}.$$

$$S = \{(y_1, z_2), (y_3, z_2)\}.$$

There are two common forms of the composition operation: one is called the *max–min composition* and the other the *max–product composition*. (Five other forms of the composition operator are available for certain logic issues; these are described at the end of this chapter.) The max–min composition is defined by the set-theoretic and membership function-theoretic expressions

$$\mathrm{T} = \mathrm{R} \circ \mathrm{S},$$
$$\chi_{\mathrm{T}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\chi_{\mathrm{R}}(x, y) \wedge \chi_{\mathrm{S}}(y, z)).$$

and the max–product (sometimes called *max–dot*) composition is defined by the set-theoretic and membership function-theoretic expressions

$$\mathrm{T} = \mathrm{R} \circ \mathrm{S},$$
$$\chi_{\mathrm{T}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\chi_{\mathrm{R}}(x, y) \bullet \chi_{\mathrm{S}}(y, z)).$$

Here, the symbol "•" is arithmetic product.

Example The matrix expression for the crisp relations shown in Figure can be found using the max–min composition operation. Relation matrices for R and S would be expressed as

$$\mathrm{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \mathrm{S} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \!\!\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}.$$

The resulting relation T would then be determined by max–min composition (Equation (3.9)) or max–product composition (Equation (3.10)). (In the crisp case, these forms of the composition operators produce identical results; other forms of this operator, such as those listed at the end of this chapter, will not produce identical results.) For example,

$$\mu_{\mathrm{T}}(x_1, z_1) = \max[\min(1, 0), \min(0, 0), \min(1, 0), \min(0, 0)] = 0,$$

$$\mu_{\mathrm{T}}(x_1, z_2) = \max[\min(1, 1), \min(0, 0), \min(1, 1), \min(0, 0)] = 1,$$

and for the rest,

$$\mathrm{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}.$$

FUZZY RELATIONS

Fuzzy relations also map elements of one universe, say X, to those of another universe, say Y, through the Cartesian product of the two universes. However, the "strength" of the relation between ordered pairs of the two universes is not measured with the characteristic function, but rather with a membership function expressing various "degrees" of strength of the relation on the unit interval [0,1]. Hence, a fuzzy relation $\underset{\sim}{R}$ is a mapping from the Cartesian space $X \times Y$ to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes, or $\mu_{\underset{\sim}{R}}(x, y)$.

Cardinality of Fuzzy Relations

Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

Operations on Fuzzy Relations

Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations (these are similar to the same operations on crisp sets, Equations (3.4)–(3.8)):

Union $$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$$

Intersection $$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)).$$

Complement $$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y).$$

Containment $$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y).$$

Properties of Fuzzy Relations

Just as for crisp relations, the properties of commutativity, associativity, distributivity, involution, and idempotency all hold for fuzzy relations. Moreover, De Morgan's principles hold for fuzzy relations just as they do for crisp (classical) relations, and the null relation, **O**, and the complete relation, **E**, are analogous to the null set and the whole set in set-theoretic form, respectively. Fuzzy relations are not constrained, as is the case for fuzzy sets in general, by the excluded middle axioms. Since a fuzzy relation $\underset{\sim}{R}$ is also a fuzzy set, there is overlap between a relation and its complement; hence,

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq \mathbf{E}.$$

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \mathbf{O}.$$

As seen in the foregoing expressions, the *excluded middle axioms* for fuzzy relations do not result, in general, in the null relation, **O**, or the complete relation, **E**.

Properties of Fuzzy Relations

Let $\underline{R}$, $\underline{S}$ and $\underline{T}$ be fuzzy relations defined on the universe $X \times Y$. Then, the properties of fuzzy relations are as below:

1.	**Commutativity**	$\underline{R} \cup \underline{S} = \underline{S} \cup \underline{R}$ $\underline{R} \cap \underline{S} = \underline{S} \cap \underline{R}$
2.	**Associativity**	$\underline{R} \cup (\underline{S} \cup \underline{T}) = (\underline{R} \cup \underline{S}) \cup \underline{T}$ $\underline{R} \cap (\underline{S} \cap \underline{T}) = (\underline{R} \cap \underline{S}) \cap \underline{T}$
3.	**Distributivity**	$\underline{R} \cup (\underline{S} \cap \underline{T}) = (\underline{R} \cup \underline{S}) \cap (\underline{R} \cup \underline{T})$ $\underline{R} \cap (\underline{S} \cup \underline{T}) = (\underline{R} \cap \underline{S}) \cup (\underline{R} \cap \underline{T})$
4.	**Idempotency**	$\underline{R} \cup \underline{R} = \underline{R}$ $\underline{R} \cap \underline{R} = \underline{R}$
5.	**Identity**	$\underline{R} \cup \phi = \underline{R}$ $\underline{R} \cap \phi = \phi$ $\underline{R} \cup X = X$ $\underline{R} \cap X = \underline{R}$ **where ϕ is null relation (null matrix) and X is complete relation (unit matrix)**
6.	**Involution**	$\bar{\bar{R}} = R$
7.	**De-Morgan's Law**	$\overline{\underline{R} \cap \underline{S}} = \underline{R} \cup \underline{S}$ $\overline{\underline{R} \cup \underline{S}} = \underline{R} \cap \underline{S}$
8.	**Law of excluded middle and law of contradiction are not satisfied**	$\underline{R} \cup \bar{\underline{R}} \neq X$ $\underline{R} \cap \bar{\underline{R}} \neq \phi$

Fuzzy Cartesian Product and Composition

Because fuzzy relations in general are fuzzy sets, we can define the Cartesian product to be a relation between two or more fuzzy sets. Let $\underset{\sim}{A}$ be a fuzzy set on universe X and $\underset{\sim}{B}$

be a fuzzy set on universe Y, then the Cartesian product between fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will result in a fuzzy relation $\underset{\sim}{R}$, which is contained within the full Cartesian product space, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y, \tag{3.15}$$

where the fuzzy relation $\underset{\sim}{R}$ has membership function

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)). \tag{3.16}$$

The Cartesian product defined by $\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R}$ (Equation (3.15)) is implemented in the same fashion as is the cross product of two vectors. Again, the Cartesian product is *not* the same operation as the arithmetic product. In the case of two-dimensional relations ($r = 2$), the former employs the idea of pairing of elements among sets, whereas the latter uses actual arithmetic products between elements of sets. Each of the fuzzy sets could be thought of as a vector of membership values; each value is associated with a particular element in each set. For example, for a fuzzy set (vector) $\underset{\sim}{A}$ that has four elements, hence column vector of size 4×1, and for a fuzzy set (vector) $\underset{\sim}{B}$ that has five elements, hence a row vector size of 1×5, the resulting fuzzy relation $\underset{\sim}{R}$ will be represented by a matrix of size 4×5, that is, $\underset{\sim}{R}$ will have four rows and five columns.

Example 3.5. Suppose we have two fuzzy sets, $\underset{\sim}{A}$ defined on a universe of three discrete temperatures, $X = \{x_1, x_2, x_3\}$, and $\underset{\sim}{B}$ defined on a universe of two discrete pressures, $Y = \{y_1, y_2\}$, and we want to find the fuzzy Cartesian product between them. Fuzzy set $\underset{\sim}{A}$ could represent the "ambient" temperature and fuzzy set $\underset{\sim}{B}$ the "near-optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature–pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \quad \text{and} \quad \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}.$$

Note that $\underset{\sim}{A}$ can be represented as a column vector of size 3×1 and $\underset{\sim}{B}$ can be represented by a row vector of 1×2. Then the fuzzy Cartesian product, using Equation (3.16), results in a fuzzy relation $\underset{\sim}{R}$ (of size 3×2) representing "efficient" conditions, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}.$$

Fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose $\underset{\sim}{R}$ is a fuzzy relation on the Cartesian space $X \times Y$, $\underset{\sim}{S}$ is a fuzzy relation on $Y \times Z$, and $\underset{\sim}{T}$ is a fuzzy relation on $X \times Z$, then fuzzy max–min composition is defined in terms of the set-theoretic notation and membership function-theoretic notation in the following manner:

$$\underset{\sim}{\mathrm{T}} = \underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}},$$

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \wedge \mu_{\underset{\sim}{\mathrm{S}}}(y, z)), \tag{3.17a}$$

and fuzzy max–product composition is defined in terms of the membership function-theoretic notation as

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \bullet \mu_{\underset{\sim}{\mathrm{S}}}(y, z)). \tag{3.17b}$$

It should be pointed out that neither crisp nor fuzzy compositions are commutative in general; that is,

$$\underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}} \neq \underset{\sim}{\mathrm{S}} \circ \underset{\sim}{\mathrm{R}}. \tag{3.18}$$

Equation (3.18) is general for any matrix operation, fuzzy or otherwise, that must satisfy consistency between the cardinal counts of elements in respective universes. Even for the case of square matrices, the composition converse, represented by Equation (3.18), is not guaranteed.

Example 3.6. Let us extend the information contained in the Sagittal diagram shown in Figure 3.4 to include fuzzy relationships for $X \times Y$ (denoted by the fuzzy relation $\underset{\sim}{R}$) and $Y \times Z$ (denoted by the fuzzy relation $\underset{\sim}{S}$). In this case, we change the elements of the universes to

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}.$$

Consider the following fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}.$$

Then, the resulting relation, $\underset{\sim}{T}$, which relates elements of universe X to elements of universe Z, that is, defined on Cartesian space $X \times Z$, can be found by max–min composition (Equation (3.17a)) to be, for example,

$$\mu_{\underset{\sim}{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7,$$

and the rest

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array},$$

and by max–product composition (Equation (3.17b)) to be, for example,

$$\mu_{\underset{\sim}{T}}(x_2, z_2) = \max[(0.8 \cdot 0.6), (0.4 \cdot 0.7)] = 0.48,$$

and the rest

$$\underset{\sim}{\mathrm{T}} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}.$$

We now illustrate the use of relations with fuzzy sets for three examples from the fields of medicine, electrical engineering, and civil engineering.

Example 3.7. A certain type of virus attacks cells of the human body. The infected cells can be visualized using a special microscope. The microscope generates digital images that medical doctors can analyze and identify the infected cells. The virus causes the infected cells to have a black spot, within a darker gray region (Figure 3.6).

A digital image process can be applied to the image. This processing generates two variables: the first variable, P, is related to black spot quantity (black pixels) and the second variable, S, is related to the shape of the black spot, that is, if they are circular or elliptic. In these images, it is often difficult to actually count the number of black pixels, or to identify a perfect circular cluster of pixels; hence, both these variables must be estimated in a linguistic way.

Suppose that we have two fuzzy sets: $\underset{\sim}{P}$ that represents the number of black pixels (e.g., none with black pixels, C_1, a few with black pixels, C_2, and a lot of black pixels, C_3) and $\underset{\sim}{S}$ that represents the shape of the black pixel clusters (e.g., S_1 is an ellipse and S_2 is a circle). So, we have

$$\underset{\sim}{P} = \left\{ \frac{0.1}{C_1} + \frac{0.5}{C_2} + \frac{1.0}{C_3} \right\} \quad \text{and} \quad \underset{\sim}{S} = \left\{ \frac{0.3}{S_1} + \frac{0.8}{S_2} \right\},$$

and we want to find the relationship between quantity of black pixels in the virus and the shape of the black pixel clusters. Using a Cartesian product between $\underset{\sim}{P}$ and $\underset{\sim}{S}$ gives

$$\underset{\sim}{R} = \underset{\sim}{P} \times \underset{\sim}{S} = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} \end{array}.$$

Now, suppose another microscope image is taken and the number of black pixels is slightly different; let the new black pixel quantity be represented by a fuzzy set, $\underset{\sim}{P}'$:

$$\underset{\sim}{P}' = \left\{ \frac{0.4}{C_1} + \frac{0.7}{C_2} + \frac{1.0}{C_3} \right\}.$$

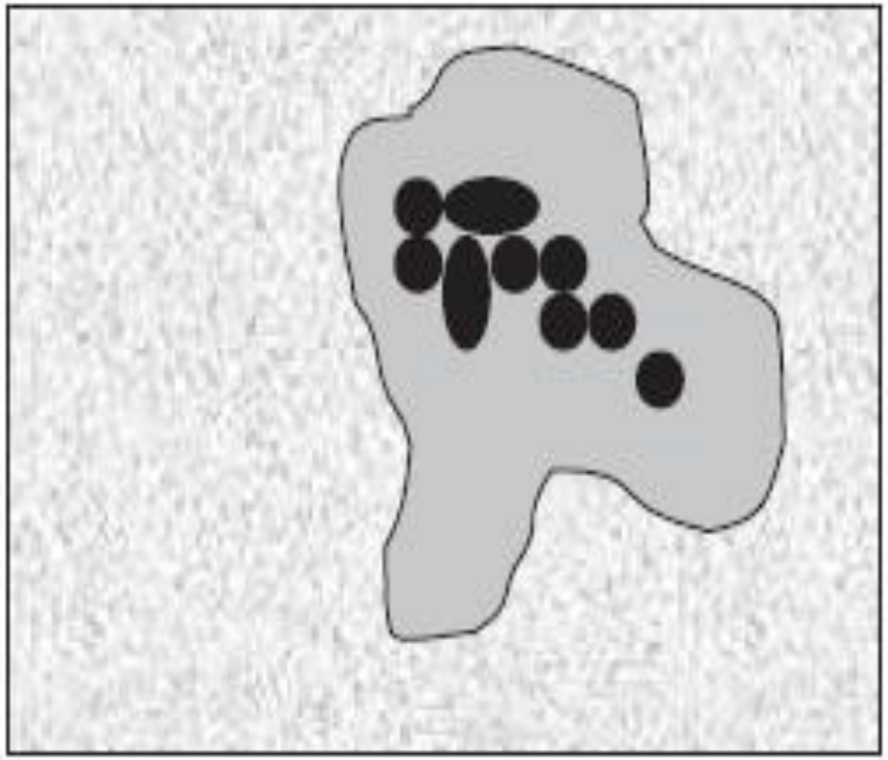


FIGURE 3.6
An infected cell shows black spots with different shapes in a micrograph.

Using max–min composition with the relation $\underset{\sim}{R}$ will yield a new value for the fuzzy set of pixel cluster shapes that are associated with the new black pixel quantity:

$$\underset{\sim}{S}' = \underset{\sim}{P}' \circ \underset{\sim}{R} = [0.4 \quad 0.7 \quad 1.0] \circ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} = [0.3 \quad 0.8].$$

Let $\underline{R}$ and $\underline{S}$ be two fuzzy relations defined as:

$\underline{R}$ =	y_1	y_2	y_3
x_1	0.0	0.2	0.8
x_2	0.3	0.6	1.0

$\underline{S}$ =	z_1	z_2	z_3
y_1	0.3	0.7	1.0
y_2	0.5	1.0	0.6
y_3	1.0	0.2	0.0

(a) Compute / Infer the result of $\underline{R} \cdot \underline{S}$ as a max-min composition.

(b) Compute / Infer the result of $\underline{R} \cdot \underline{S}$ as a max-product composition.

☑ **Soln. :**

1. Max – min Composition

$$\mu_{\underline{R} \cdot \underline{S}}(x, z) = \max_{y \in Y} \left(\min\left(\mu_{\underline{R}}(x, y), \mu_{\underline{S}}(y, z)\right)\right)$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_1) = \max(\min(0.0, 0.3), \min(0.2, 0.5), \min(0.8, 1.0)) = \max(0.0, 0.2, 0.8) = 0.8$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_2) = \max(\min(0.0, 0.7), \min(0.2, 1.0), \min(0.8, 0.2)) = \max(0.0, 0.2, 0.2) = 0.2$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_3) = \max(\min(0.0, 1.0), \min(0.2, 0.6), \min(0.8, 0.0)) = \max(0.0, 0.2, 0.0) = 0.2$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_1) = \max(\min(0.3, 0.3), \min(0.6, 0.5), \min(1.0, 1.0)) = \max(0.3, 0.5, 1.0) = 1.0$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_2) = \max(\min(0.3, 0.7), \min(0.6, 1.0), \min(1.0, 0.2)) = \max(0.3, 0.6, 1.2) = 0.6$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_3) = \max(\min(0.3, 1.0), \min(0.6, 0.6), \min(1.0, 0.0)) = \max(0.3, 0.6, 0.0) = 0.6$$

$T = \mu_{\underline{R} \cdot \underline{S}}(x, z)$	z_1	z_2	z_3
x_1	0.8	0.2	0.2
x_2	1.0	0.6	0.6

2. **Max – product Composition**

$$\mu_{\underline{R} \cdot \underline{S}}(x, z) = \max_{y \in Y} \left(\left(\mu_{\underline{R}}(x, y) \cdot \mu_{\underline{S}}(y, z) \right) \right)$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_1) = \max((0.0 \times 0.3), (0.2 \times 0.5), (0.8 \times 1.0)) = \max(0.0, 0.10, 0.8) = 0.8$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_2) = \max((0.0 \times 0.7), (0.2 \times 1.0), (0.8 \times 0.2)) = \max(0.0, 0.2, 0.16) = 0.2$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_1, z_3) = \max((0.0 \times 1.0), (0.2 \times 0.6), (0.8 \times 0.0)) = \max(0.0, 0.12, 0.0) = 0.12$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_1) = \max((0.3 \times 0.3), (0.6 \times 0.5), (1.0 \times 1.0)) = \max(0.09, 0.30, 1.0) = 1.0$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_2) = \max((0.3 \times 0.7), (0.6 \times 1.0), (1.0 \times 0.2)) = \max(0.21, 0.6, 0.2) = 0.6$$

$$\mu_{\underline{R} \cdot \underline{S}}(x_2, z_3) = \max((0.3 \times 1.0), (0.6 \times 0.6), (1.0 \times 0.0)) = \max(0.3, 0.36, 0.0) = 0.36$$

$T = \mu_{\underline{R} \cdot \underline{S}}(x, z)$

	z_1	z_2	z_3
x_1	0.8	0.2	0.12
x_2	1.0	0.6	0.36

Difference between Fuzzification and Defuzzification

Sr. No.	Key	Fuzzification	Defuzzification
1	Definition	Fuzzification is the process of transforming a crisp set to a fuzzy set.	Defuzzification is the process of reducing a fuzzy set into a crisp set or converting a fuzzy member into a crisp member.
2	Purpose	Fuzzification converts a precise data into imprecise data.	Defuzzification converts an imprecise data into precise data.
3	Example	Voltmeter.	Stepper motor, D/A converter.
4	Methods used	Inference, Rank ordering, Angular fuzzy sets, Neural network.	Maximum membership principle, Centroid method, Weighted average method, Centre of sums.
5	Complexity	Fuzzification is easy.	Defuzzification is quite complex to implement.
6	Approach	Fuzzification uses if-then rules to fuzzify the crisp value.	Defuzzification uses centre of gravity methods to get centroid of sets.

Among the many methods of defuzzification, we shall be studying seven methods for defuzzifying fuzzy output functions.

They are as follows :

1. Max-membership principle
2. Centroid method
3. Weighted average method
4. Mean-max membership
5. Centre of sums
6. Centre of largest area
7. First of maxima, last of maxima.

1. *Max membership principle*: Also known as the *height method*, this scheme is limited to peaked output functions. This method is given by the algebraic expression

$$\mu_{\underset{\sim}{C}}(z^*) \geq \mu_{\underset{\sim}{C}}(z), \qquad \text{for all } z \in Z, \tag{4.4}$$

where z^* is the defuzzified value, and is shown graphically in Figure 4.12.

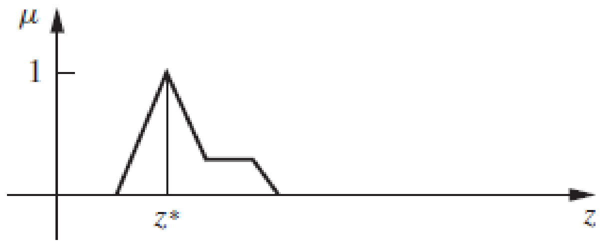


FIGURE 4.12
Max membership defuzzification method.

2. *Centroid method*: This procedure (also called *center of area* or *center of gravity*) is the most prevalent and physically appealing of all the defuzzification methods (Sugeno, 1985; Lee, 1990); it is given by the algebraic expression

$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z \, dz}{\int \mu_{\underset{\sim}{C}}(z) \, dz}, \tag{4.5}$$

where $\int$ denotes an algebraic integration. This method is shown in Figure 4.13.

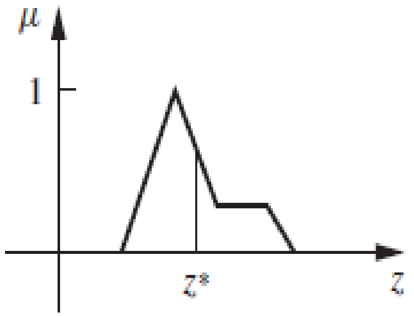


FIGURE 4.13
Centroid defuzzification method.

3. *Weighted average method*: The weighted average method is the most frequently used in fuzzy applications since it is one of the more computationally efficient methods. Unfortunately, it is *usually* restricted to symmetrical output membership functions. It is given by the algebraic expression

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\overline{z}) \cdot \overline{z}}{\sum \mu_{\underset{\sim}{C}}(\overline{z})}, \tag{4.6}$$

where $\sum$ denotes the algebraic sum and where $\overline{z}$ is the centroid of each symmetric membership function. This method is shown in Figure 4.14. The weighted average method is formed by weighting each membership function in the output by its

respective maximum membership value. As an example, the two functions shown in Figure 4.14 would result in the following general form for the defuzzified value:

$$z^* = \frac{a(0.5) + b(0.9)}{0.5 + 0.9}.$$

Since the method can be limited to symmetrical membership functions, the values a and b are the means (centroids) of their respective shapes. This method is sometimes applied to unsymmetrical functions and various scalar outputs (see Sugeno, 1985).

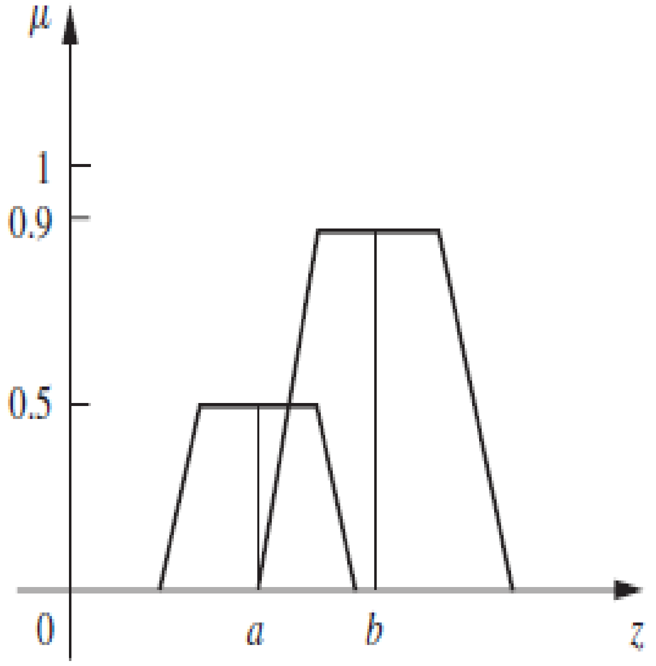


FIGURE 4.14
Weighted average method of defuzzification.

4. *Mean max membership*: This method (also called *middle-of-maxima*) is closely related to the first method, except that the locations of the maximum membership can be nonunique (i.e., the maximum membership can be a plateau rather than a single point). This method is given by the expression (Sugeno, 1985; Lee, 1990)

$$z^* = \frac{a + b}{2} \tag{4.7}$$

where a and b are as defined in Figure 4.15.

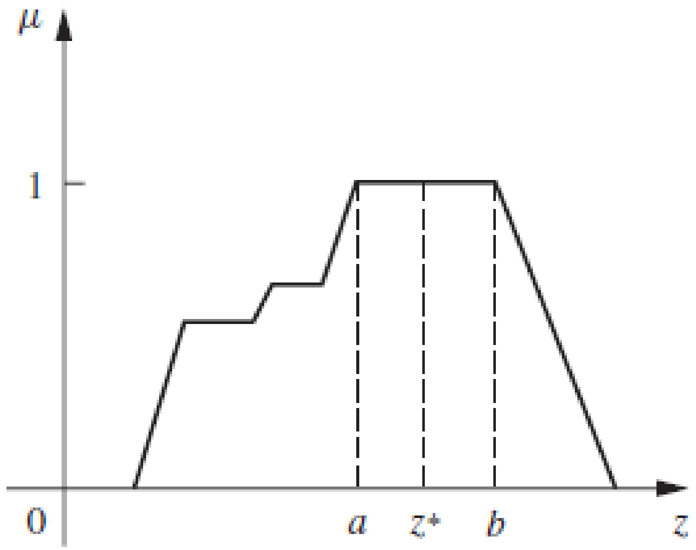


FIGURE 4.15
Mean max membership defuzzification method.

5. *Center of sums*: This is faster than many defuzzification methods that are currently in use, and the method is not restricted to symmetric membership functions. This process involves the algebraic sum of individual output fuzzy sets, say $\underset{\sim}{C}_1$ and $\underset{\sim}{C}_2$, instead of their union. Two drawbacks to this method are that the intersecting areas are added twice, and the method also involves finding the centroids of the individual membership functions. The defuzzified value z^* is given as follows:

$$z^* = \frac{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z \bar{z}\, dz}{\sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z) \int_z dz}, \tag{4.8}$$

where the symbol $\bar{z}$ is the distance to the centroid of each of the respective membership functions.

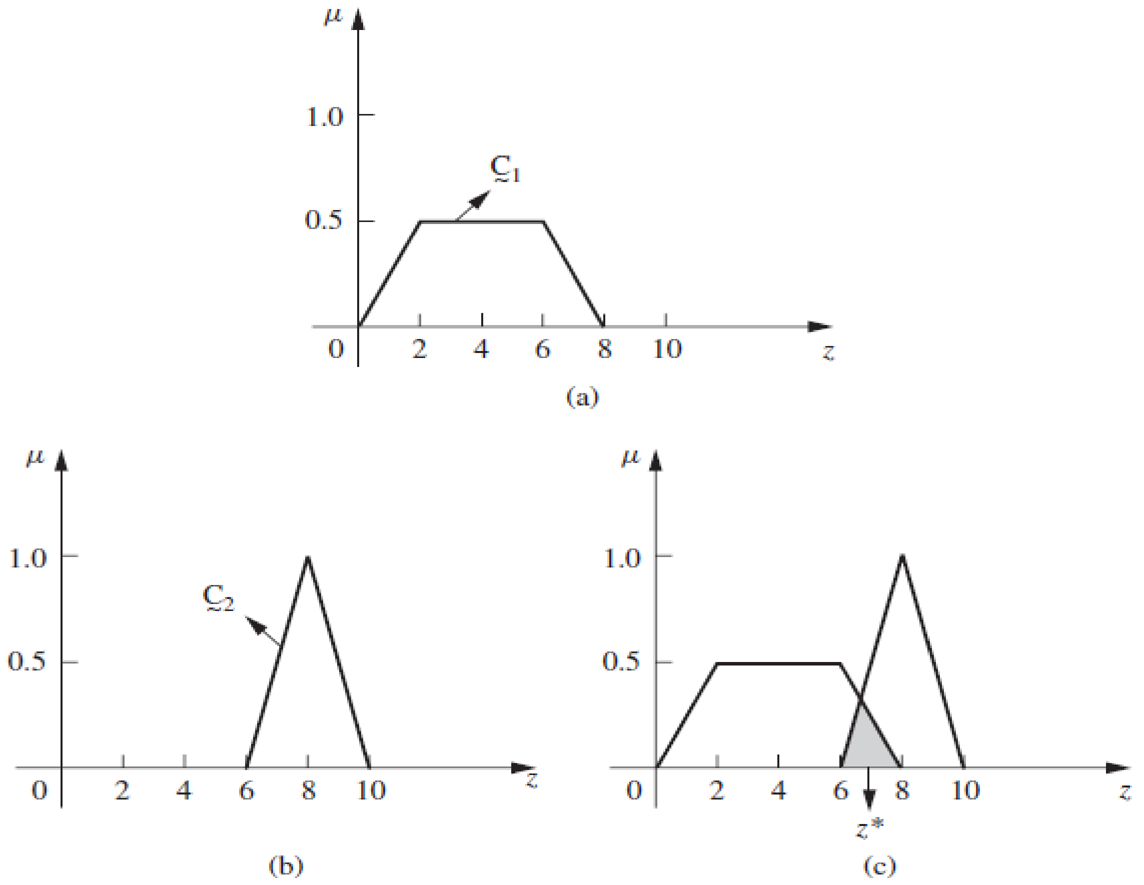


FIGURE 4.28
Center of sums method: (a) first membership function; (b) second membership function; and (c) defuzzification step.

This method is similar to the weighted average method, Equation (4.6), except that in the center of sums method the weights are the areas of the respective membership functions whereas in the weighted average method the weights are individual membership values. Figure 4.28 is an illustration of the center of sums method.

6. *Center of largest area*: If the output fuzzy set has at least two convex subregions, then the center of gravity (i.e., z^* is calculated using the centroid method, Equation 4.5) of the convex fuzzy subregion with the largest area is used to obtain the defuzzified value z^* of the output. This is shown graphically in Figure 4.29, and given algebraically as

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z) z \, dz}{\int \mu_{\underset{\sim}{C}_m}(z) \, dz}, \tag{4.9}$$

where $\underset{\sim}{C}_m$ is the convex subregion that has the largest area making up $\underset{\sim}{C}_k$. This condition applies in the case when the overall output $\underset{\sim}{C}_k$ is nonconvex. And, in the case when $\underset{\sim}{C}_k$ is convex, z^* is the same quantity as determined by the centroid method or the center of largest area method (because then there is only one convex region).

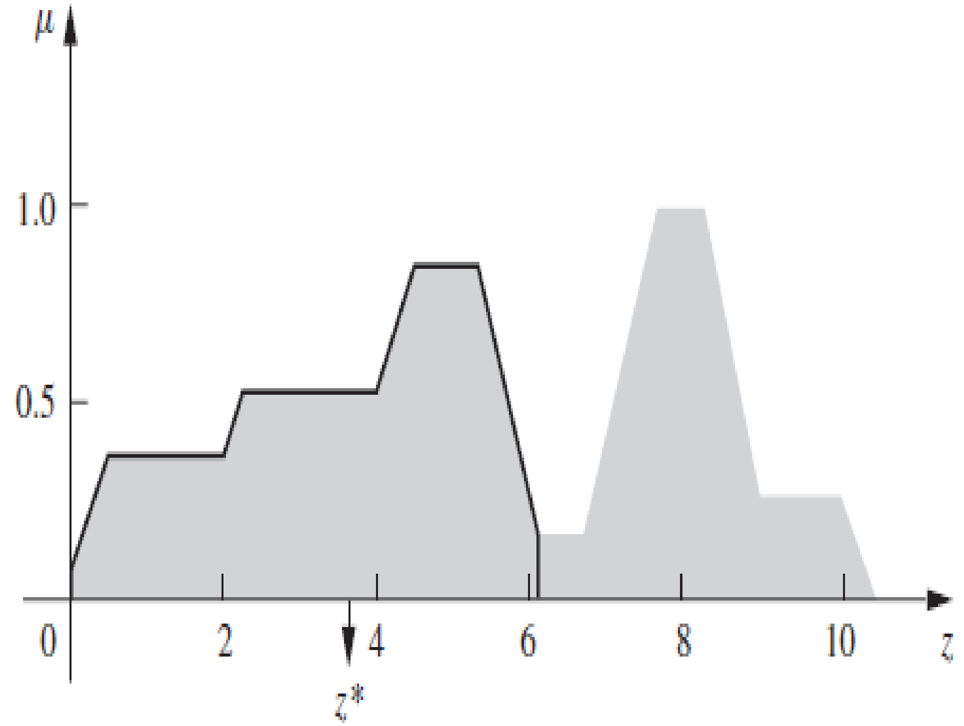


FIGURE 4.29
Center of largest area method (outlined with bold lines), shown for a nonconvex $\underset{\sim}{C}_k$.

7. *First (or last) of maxima*: This method uses the overall output or union of all individual output fuzzy sets $\underset{\sim}{C}_k$ to determine the smallest value of the domain with maximized membership degree in $\underset{\sim}{C}_k$. The equations for z^* are as follows.

First, the largest height in the union (denoted $\mathrm{hgt}(\underset{\sim}{C}_k)$) is determined,

$$\mathrm{hgt}(\underset{\sim}{C}_k) = \sup_{z \in Z} \mu_{\underset{\sim}{C}_k}(z). \tag{4.10}$$

Then, the first of the maxima is found,

$$z^* = \inf_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}_k}(z) = \mathrm{hgt}(\underset{\sim}{C}_k)\}. \tag{4.11}$$

An alternative to this method is called the *last of maxima*, and it is given as

$$z^* = \sup_{z \in Z}\{z \in Z | \mu_{\underset{\sim}{C}_k}(z) = \mathrm{hgt}(\underset{\sim}{C}_k)\}. \tag{4.12}$$

In Equations (4.10)–(4.12) the supremum (sup) is the least upper bound and the infimum (inf) is the greatest lower bound. Graphically, this method is shown in Figure 4.30, where, in the case illustrated in the figure, the first max is also the last max and, because it is a distinct max, is also the mean max. Hence, the methods presented in Equations (4.4) (max or height), (4.7) (mean max), (4.11) (first max), and (4.12) (last max) all provide the same defuzzified value, z^*, for the particular situation illustrated in Figure 4.30.

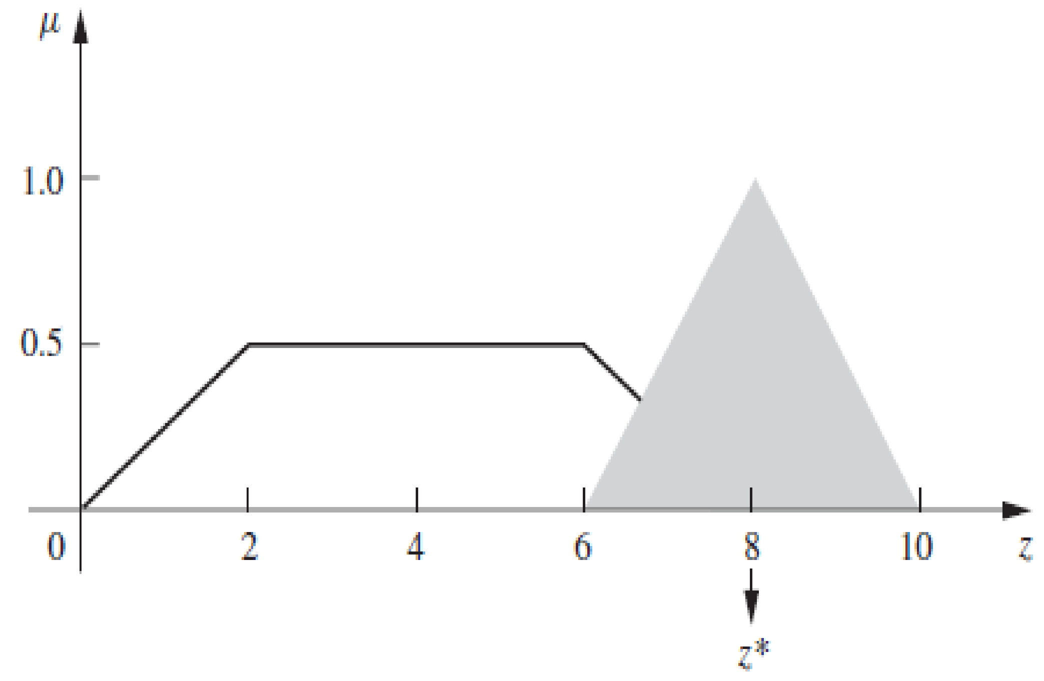


FIGURE 4.30
First of max (and last of max) method.

Fuzzy logic applications

1> In automobiles, fuzzy logic is used for gear selection and is based on factors such as engine load, road conditions and style of driving.

2> In dishwashers, fuzzy logic is used to determine the washing strategy and power needed, which is based on factors such as the number of dishes and the level of food residue on the dishes.

3> In copy machines, fuzzy logic is used to adjust drum voltage based on factors such as humidity, picture density and temperature.

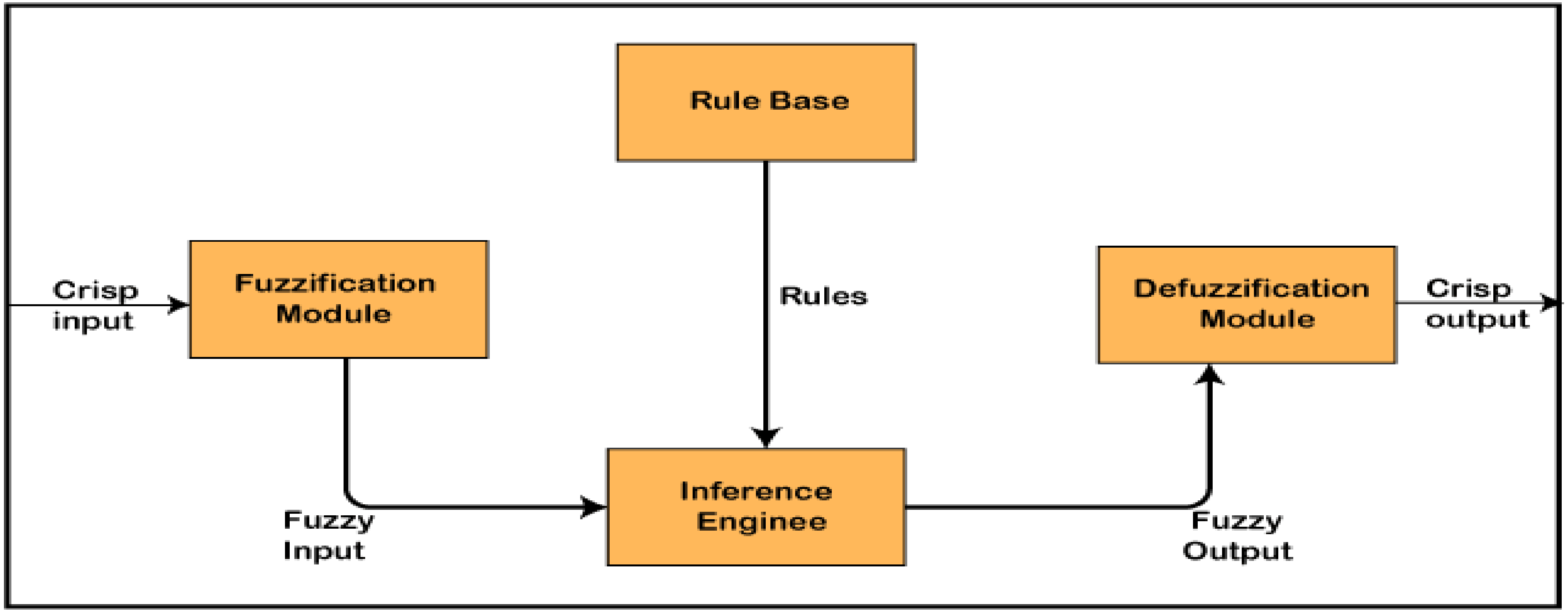
4> In aerospace, fuzzy logic is used to manage altitude control for satellites and spacecrafts based on environmental factors.

5> In medicine, fuzzy logic is used for computer-aided diagnoses, based on factors such as symptoms and medical history.

Architecture of Mamdani type Fuzzy Controller

Fuzzy Controller for Washing Machine

Structure of Mamdani Fuzzy control System

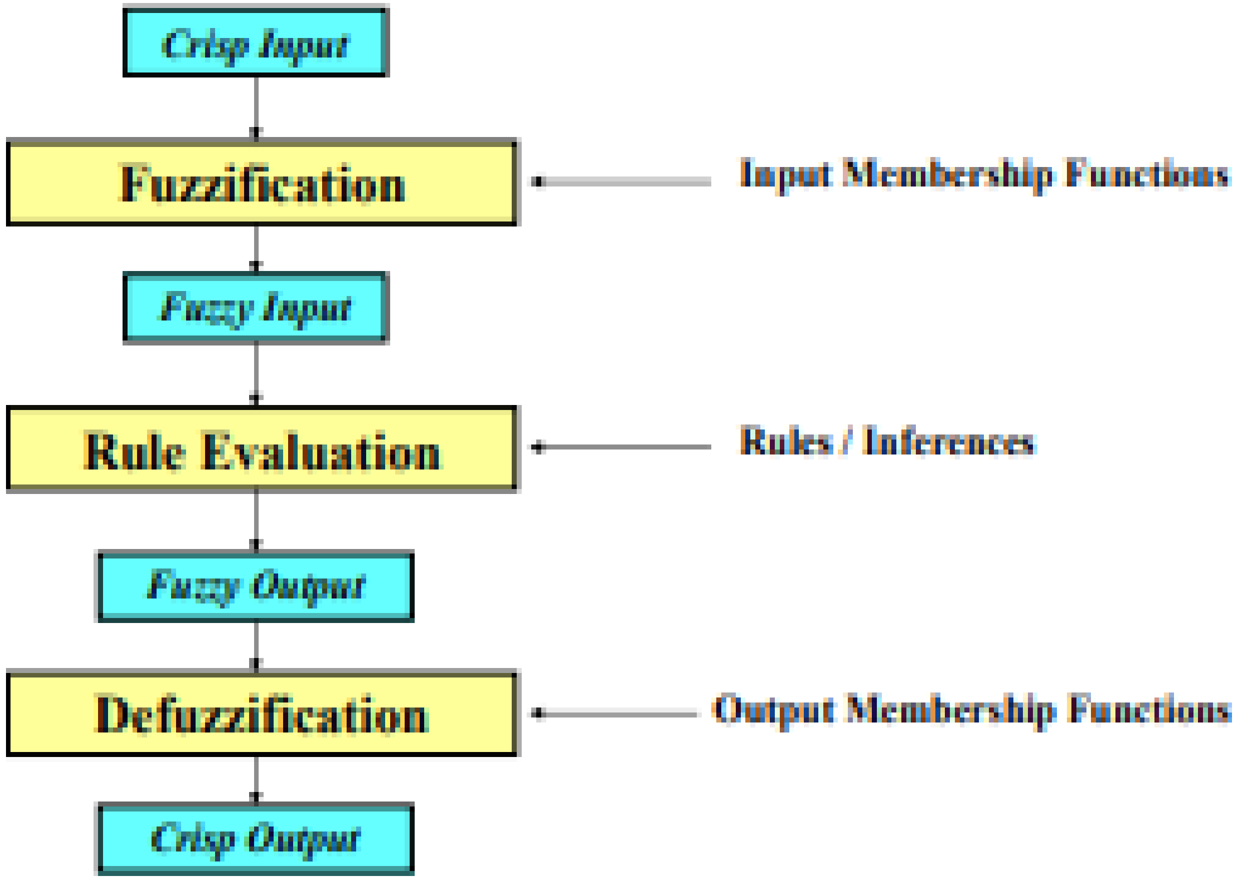


Mamdani fuzzy inference

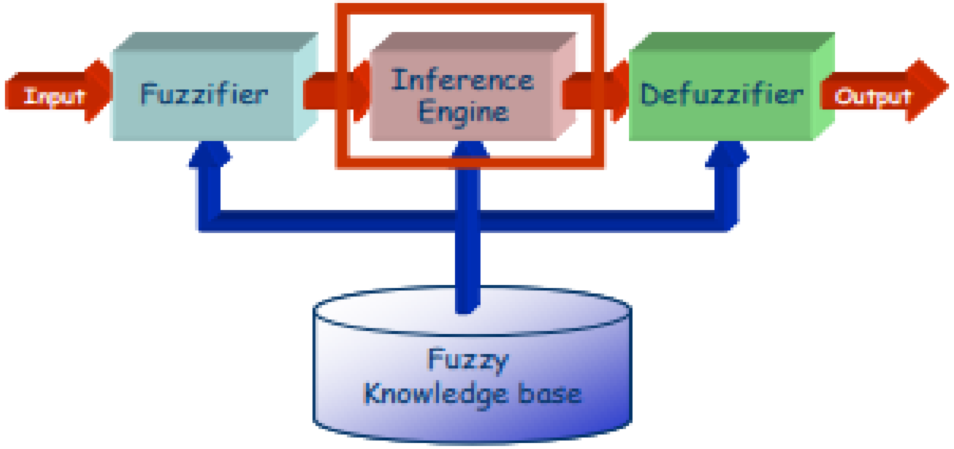
The Mamdani-style fuzzy inference process is performed in four steps:

1. Fuzzification of the input variables,
2. Rule evaluation;
3. Aggregation of the rule outputs, and finally
4. De-fuzzification.

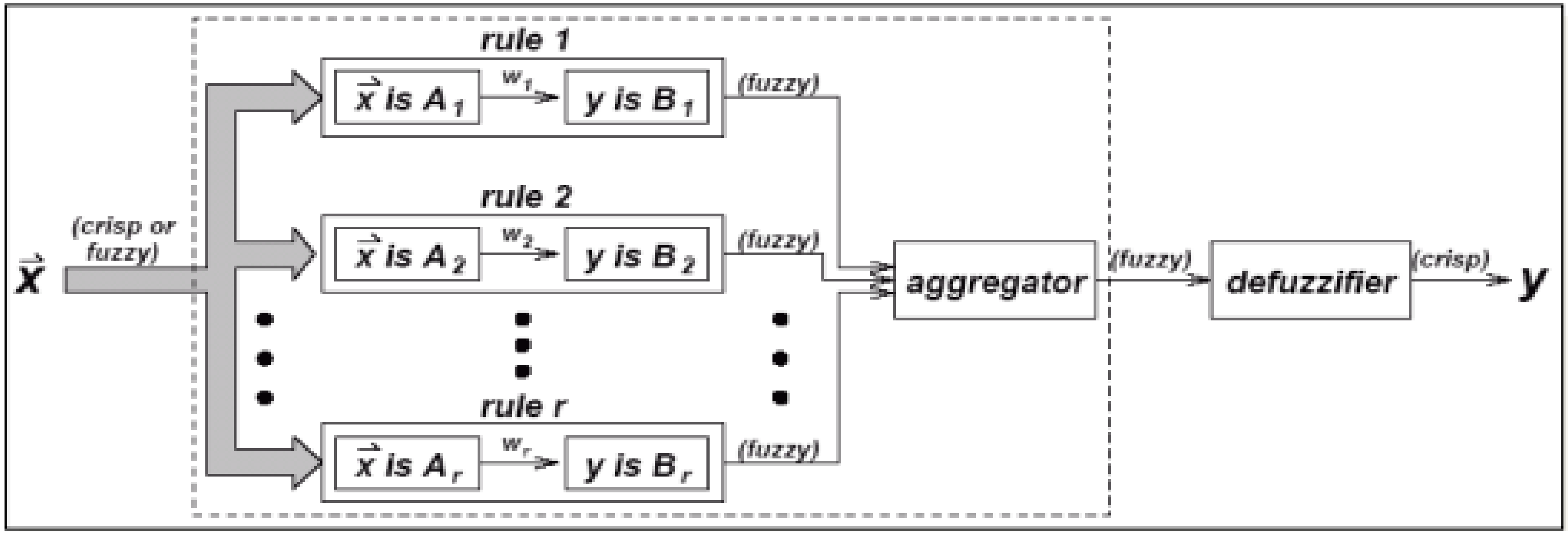
Operation of Fuzzy System



Inference Engine

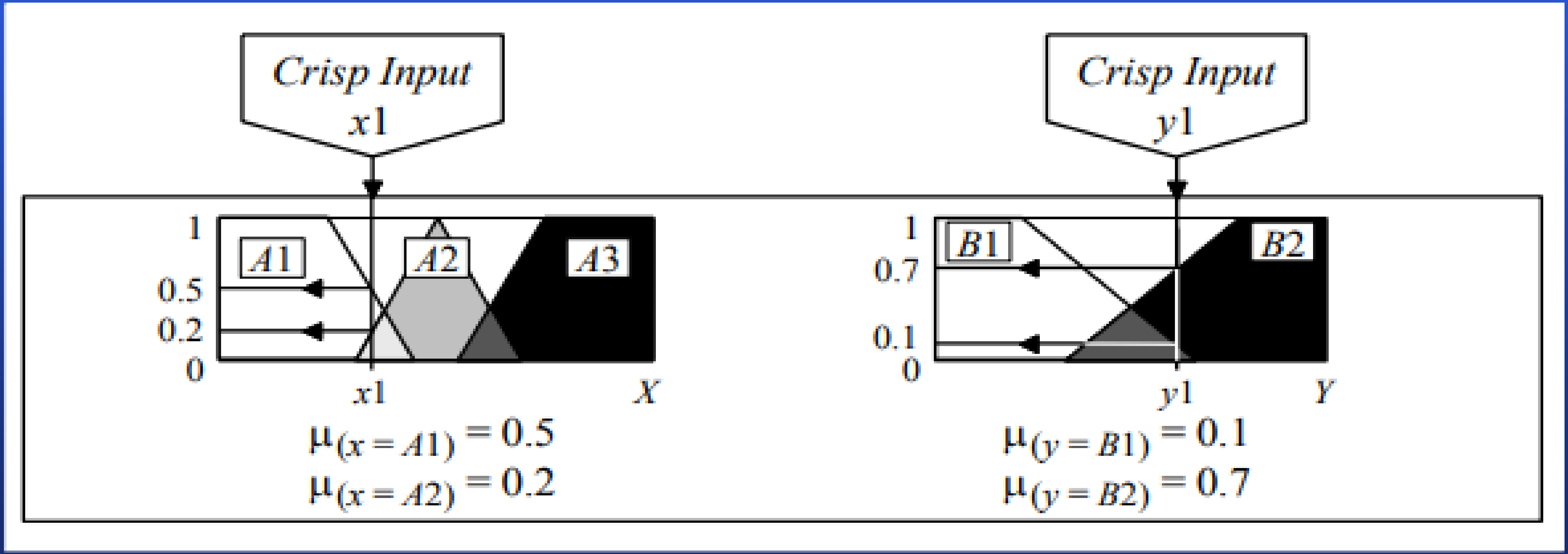


Using If-Then type fuzzy rules converts the fuzzy input to the fuzzy output.



Step 1: Fuzzification

- Take the crisp inputs, $x1$ and $y1$ (*project funding* and *project staffing*)
- Determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



project funding *project staffing*

Step 2: Rule Evaluation

- take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$

- apply them to the antecedents of the fuzzy rules.

- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation. This number (the truth value) is then applied to the consequent membership function.

Step 2: Rule Evaluation

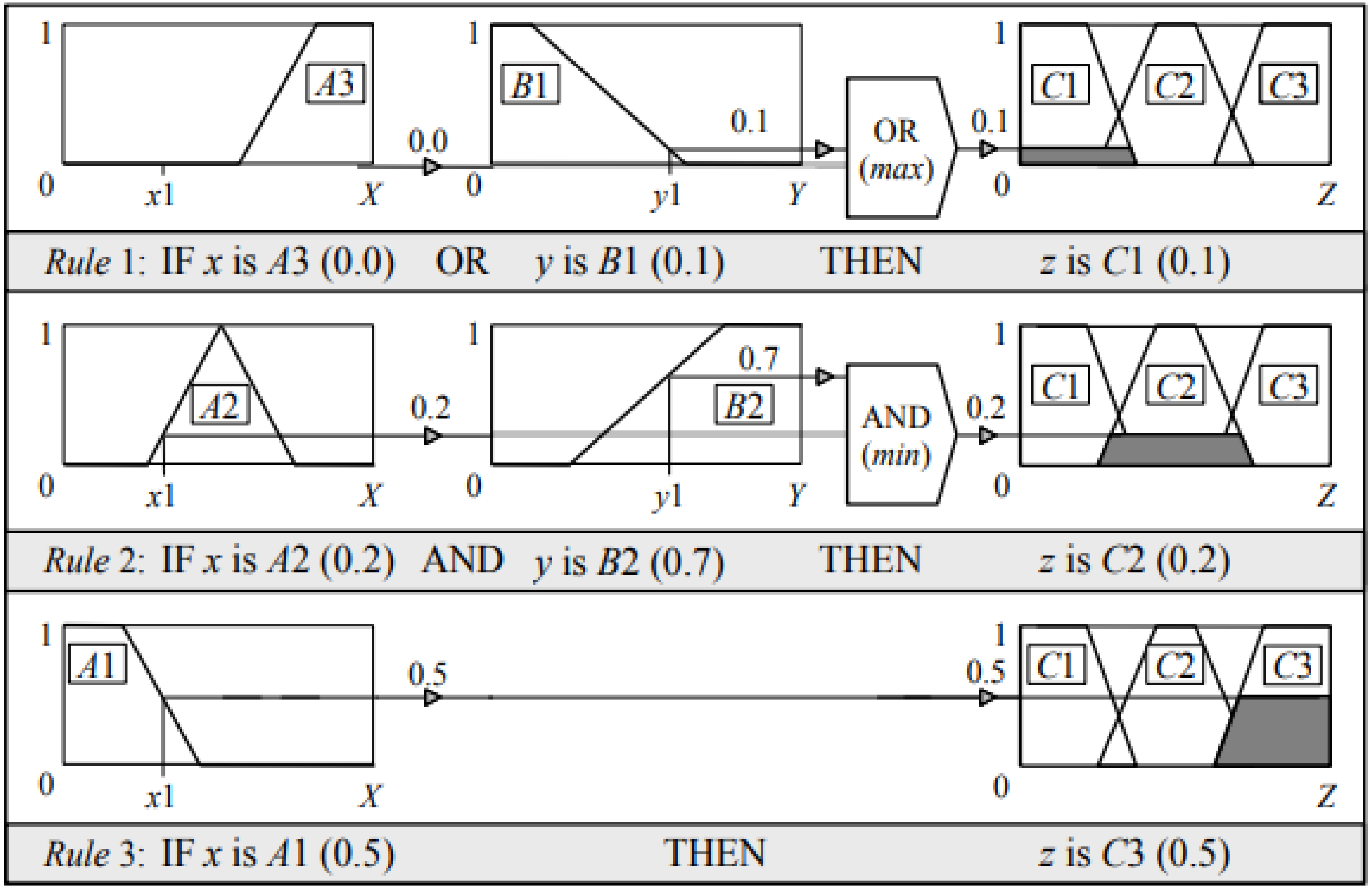
To evaluate the disjunction of the rule antecedents, we use the **OR fuzzy operation**. Typically, fuzzy expert systems make use of the classical fuzzy operation **union**:

$$\mu_{A \cup B}(x) = \boldsymbol{max}\ [\mu_A(x),\ \mu_B(x)]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND fuzzy operation intersection**:

$$\mu_{A \cap B}(x) = \boldsymbol{min}\ [\mu_A(x),\ \mu_B(x)]$$

Mamdani-style rule evaluation



Step 3: Aggregation of The Rule Outputs

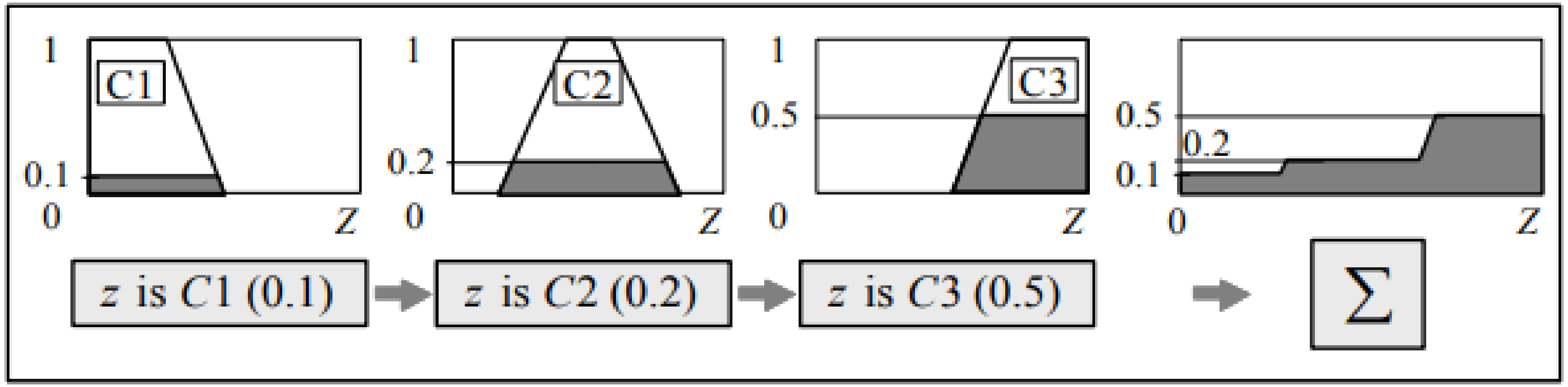
- Aggregation is the process of unification of the outputs of all rules.

- We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.

- There are several defuzzification methods, but probably the most popular one is the **centroid technique**.
- It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity (COG)** can be expressed as:

$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

Aggregation of the rule outputs



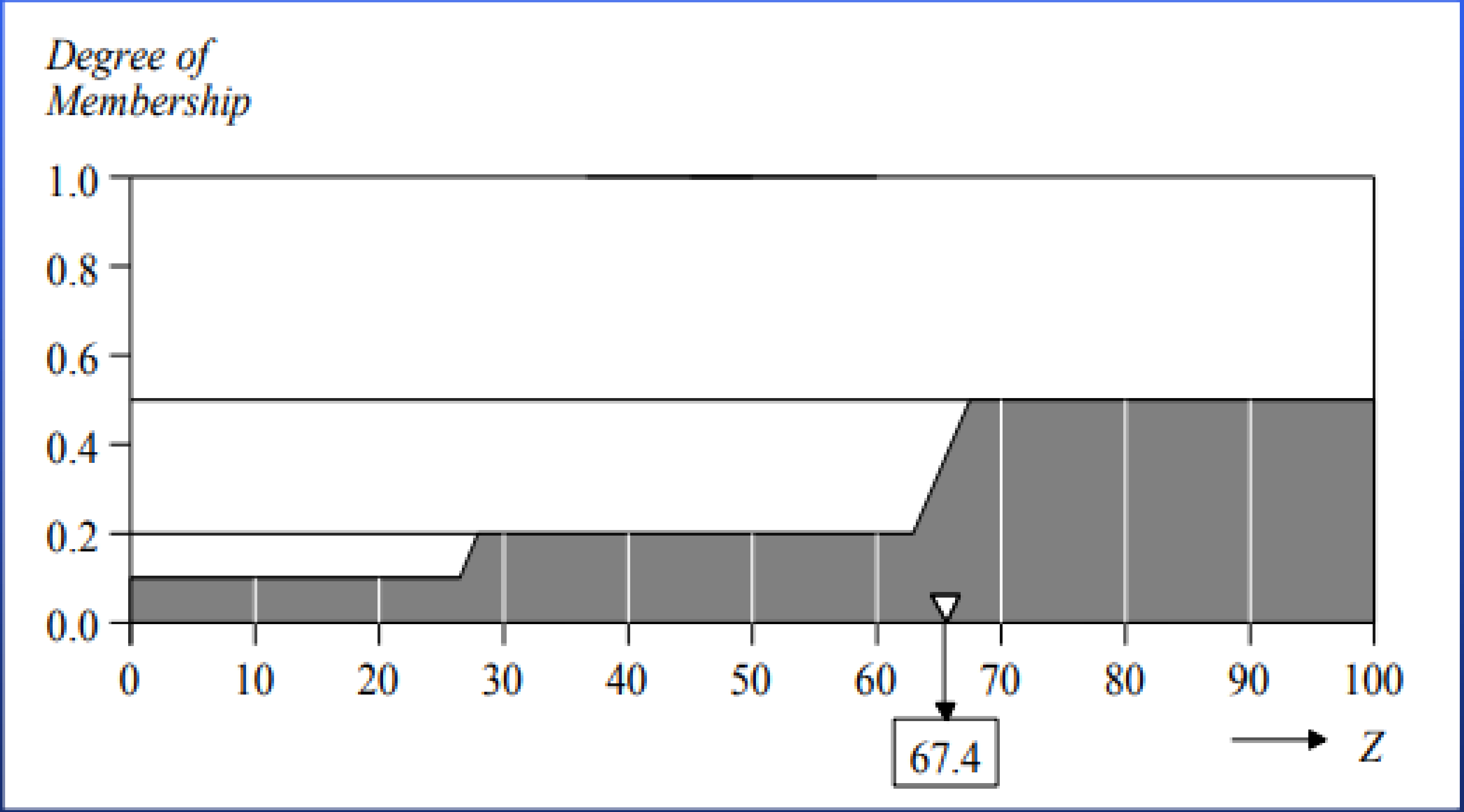
Step 4: Defuzzification

- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.

- The input for the defuzzification process is the aggregated output fuzzy set and the output is a single number.

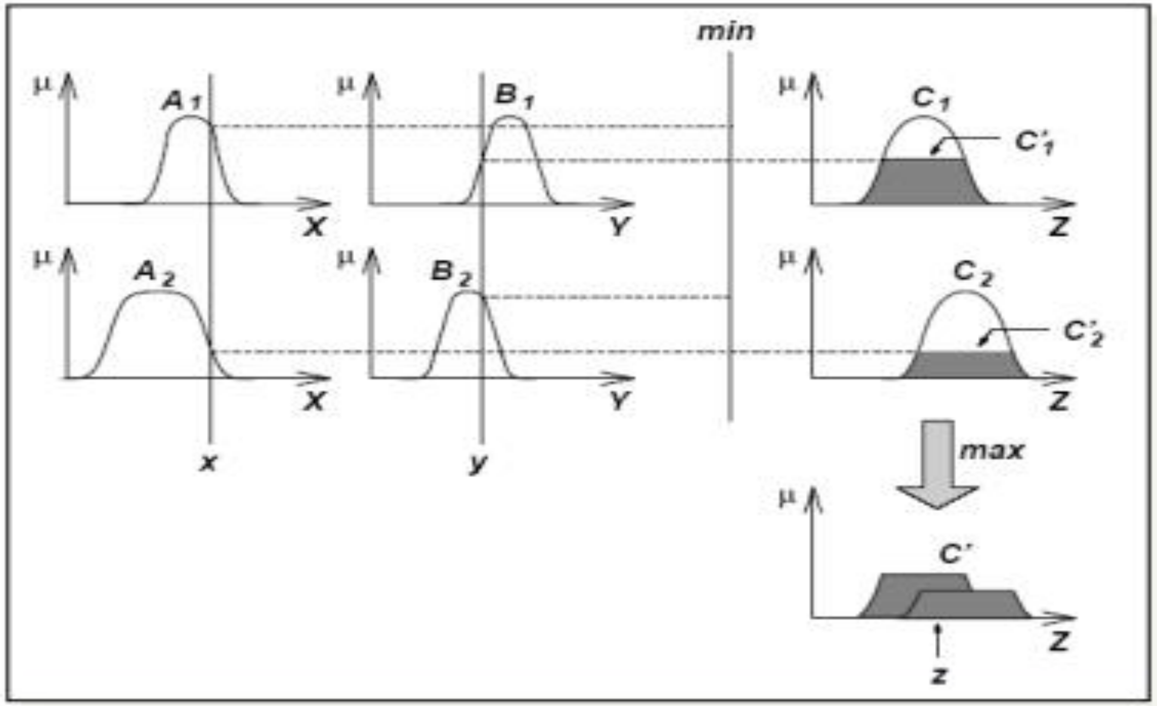
Centre of gravity (COG):

$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$



Max-Min Composition is used.

The Reasoning Scheme



We examine a simple two-input one-output problem that includes three rules:

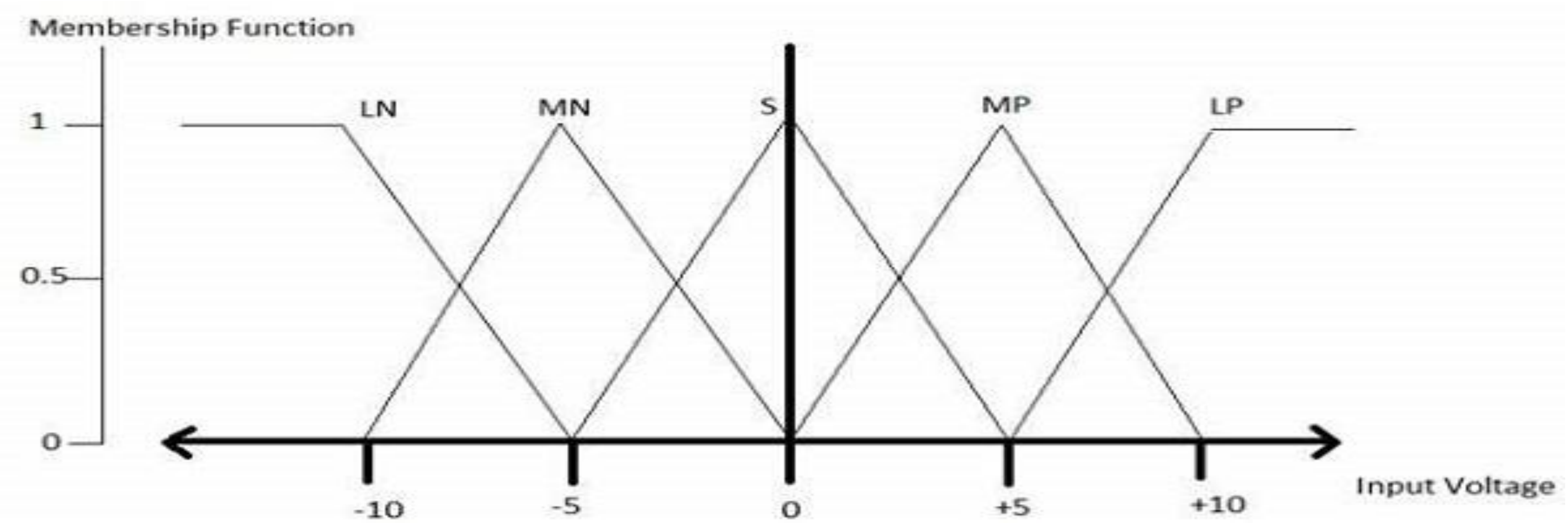
Rule: 1	Rule: 1
IF x is $A3$	IF *project_funding* is *adequate*
OR y is $B1$	OR *project_staffing* is *small*
THEN z is $C1$	THEN *risk* is *low*
Rule: 2	Rule: 2
IF x is $A2$	IF *project_funding* is *marginal*
AND y is $B2$	AND *project_staffing* is *large*
THEN z is $C2$	THEN *risk* is *normal*
Rule: 3	Rule: 3
IF x is $A1$	IF *project_funding* is *inadequate*
THEN z is $C3$	THEN *risk* is *high*

Membership Function

A membership function for a fuzzy *set A* on the universe of discourse X is defined as μA:X → [0,1].

x axis represents the universe of discourse.

y axis represents the degrees of membership in the [0, 1] interval.



Fuzzy Control system

Design a controller to determine the wash time of a domestic washing machine. Assume the input is dirt & grease on cloths. Use three descriptors for input variables and five descriptor for output variable. Derive the set of rules for controller action and defuzzification. The design should be supported by figure wherever possible. Show that if the cloths are solid to a larger degree the wash time will be more and vice versa.

Steps to solve

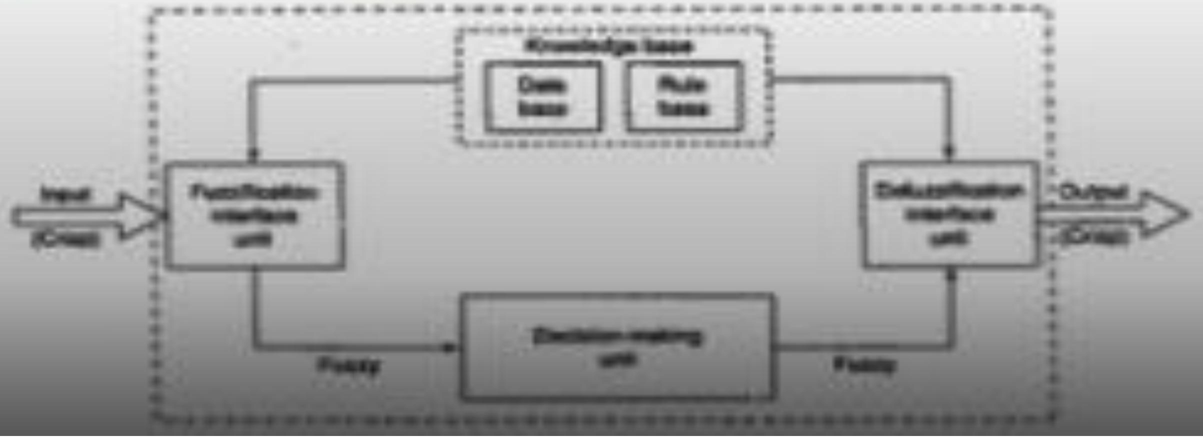
Step01: Identify input and output variables and decide descriptor for the same.

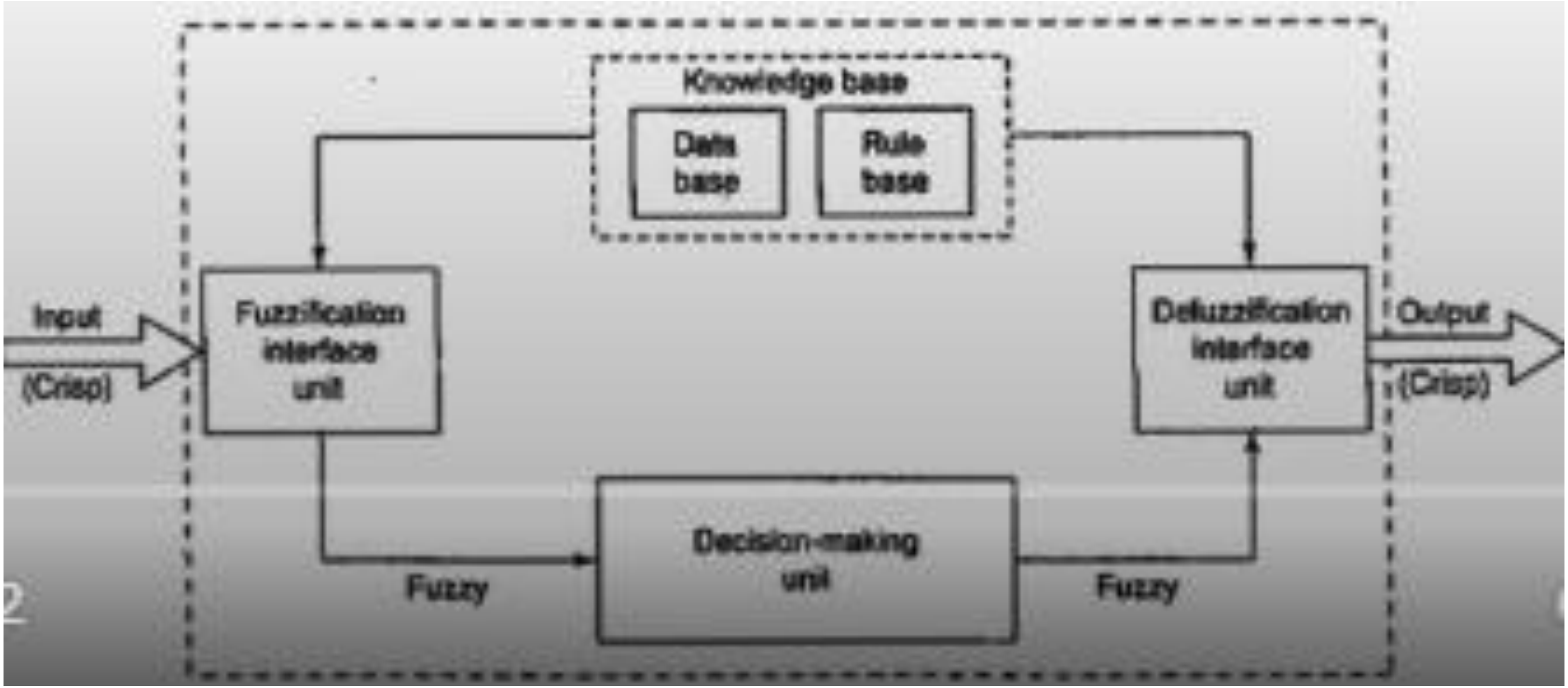
Step02: Define membership functions for each of input and output variables

Step03: Form a rule base

Step04: Rule Evaluation

Step05: Defuzzification




Knowledge base
Data base
Rule base
Input
(Crisp)
Fuzzification interface unit
Defuzzification interface unit
Output
(Crisp)
Decision-making unit
Fuzzy
Fuzzy

Step01: Identify input and output variables and decide descriptor for the same.

- Here inputs are "dirt" and "grease". Assume they are in %
- Output is "wash time" measured in minute.

Descriptor for INPUT variable

Dirt

SD: Small dirt

MD: Medium dirt

LD: Large dirt

{**SD**, **MD**, **LD**}

Grease

NG: No Grease

MG: Medium Grease

LG: Large Grease

{**NG**, **MG**, **LD**}

SISTec

Descriptor for OUTPUT variable

Wash Time

VS: Very Short

S: Short

M: Medium

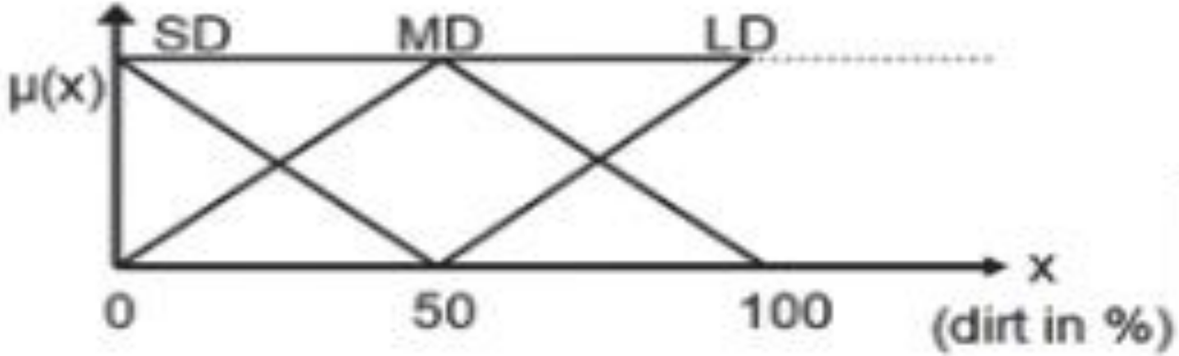
L: Large

VL: Very Large

{**VS**, **S**, **M**, **L**, **VL**}

Step02: Define membership functions for each of input and output variables (We use triangular MF's)

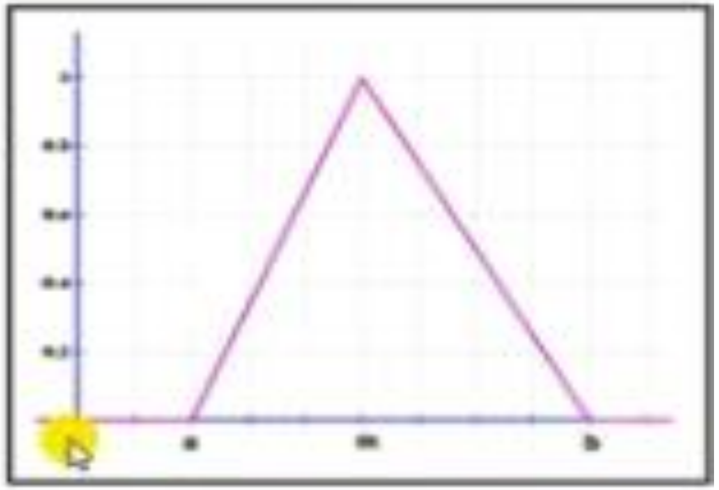
(1) Membership function for dirt:



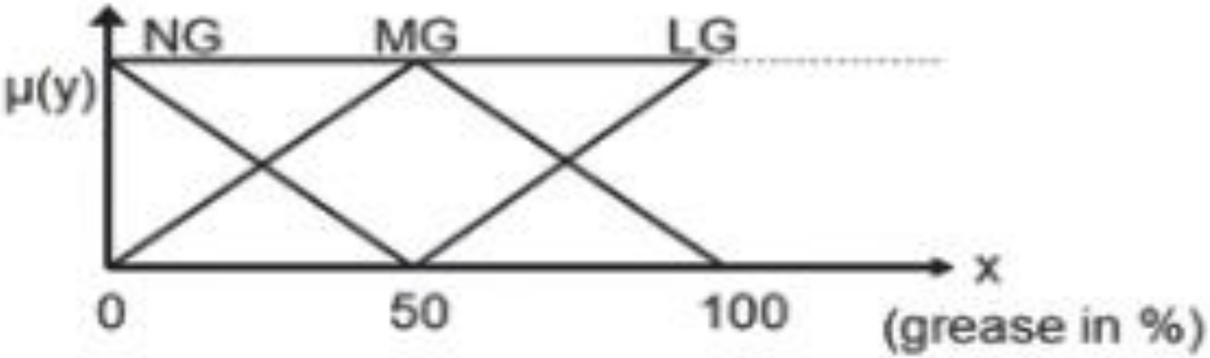
$$\mu_{SD}(x) = \frac{50 - x}{50}, 0 \le x \le 50$$

$$\mu_{MD}(x) = \begin{cases} \frac{x}{50}, & 0 \le x \le 50 \\ \frac{100 - x}{50}, & 50 \le x \le 100 \end{cases}$$

$$\mu_{LD}(x) = \frac{x - 50}{50}, 50 \le x \le 100$$



(2) Membership function for grease:

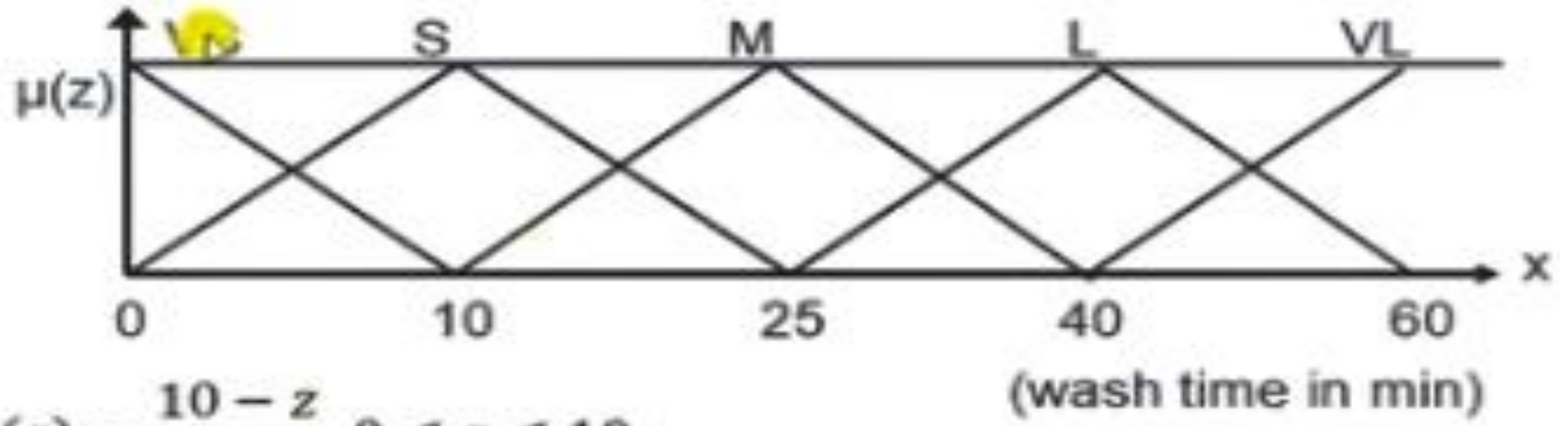


$$\mu_{NG}(y) = \frac{50-y}{50}, 0 \le y \le 50$$

$$\mu_{MG}(y) = \begin{cases} \frac{y}{50}, & 0 \le y \le 50 \\ \frac{100-y}{50}, & 50 \le y \le 100 \end{cases}$$

$$\mu_{LG}(y) = \frac{y-50}{50}, 50 \le y \le 100$$

(3) Membership function for Wash time:



$$\mu_{VS}(z) = \frac{10 - z}{10}, 0 \le z \le 10$$

$$\mu_S(z) = \begin{cases} \frac{z}{10}, & 0 \le y \le 10 \\ \frac{25 - Z}{15}, & 10 \le z \le 25 \end{cases}$$

$$\mu_M(z) = \begin{cases} \frac{z - 10}{15}, 10 \le Z \le 25 \\ \frac{40 - z}{15}, 25 \le z \le 40 \end{cases}$$

$$\mu_L(z) = \begin{cases} \frac{z - 25}{15}, 25 \le z \le 40 \\ \frac{60 - z}{20}, 40 \le z \le 60 \end{cases}$$

$$\mu_{VL}(z) = \frac{z - 40}{20}, 40 \le z \le 60$$

Step03: Form a rule base

x \ y	NG	MG	LG
SD	VS	M	L
MD	S	M	L
LD	M	L	VL

Step04: Rule Evaluation

Assume Dirt = 60%, Grease= 70%

Dirt=60% maps two MFs of dirt

$$\mu_{MD}(x) = \frac{100 - x}{50} \mid \mu_{LD}(x) = \frac{x - 50}{50}$$

Grease=70% maps 2 MFs

$$\mu_{MG}(y) = \frac{100 - y}{50} \mid \mu_{LG}(y) = \frac{y - 50}{50}$$

Evaluate:

$$\mu_{MD}(60) = \frac{100 - 60}{50} = \frac{4}{5}$$

$$\mu_{LD}(60) = \frac{60 - 50}{50} = \frac{1}{5}$$

$$\mu_{MG}(70) = \frac{100 - 70}{50} = \frac{3}{5}$$

$$\mu_{LG}(70) = \frac{70 - 50}{50} = \frac{2}{5}$$

The above four equation leads to 4 rules need to evaluate:

1. Dirt is Medium and Grease is Medium
2. Dirt is Medium and Grease is Large
3. Dirt is Large and Grease is Medium
4. Dirt is Large and Grease is Large

Since the antecedent part of each of the above rule is connected by and operator we use min operator to evaluate strength of each rule.

Strength of Rule 1 DMGM

$$S1 = \min(\mu_{MD}(60), \mu_{MG}(70))$$
$$= \min\left(\tfrac{4}{5}, \tfrac{3}{5}\right)$$
$$= \tfrac{3}{5}$$

Strength of Rule 2 DMGL

$$S2 = \min(\mu_{MD}(60), \mu_{GL}(70))$$
$$= \min\left(\tfrac{4}{5}, \tfrac{2}{5}\right)$$
$$= \tfrac{2}{5}$$

Strength of Rule 3 DLGM

$$S3 = \min(\mu_{LD}(60), \mu_{MG}(70))$$
$$= \min\left(\tfrac{1}{5}, \tfrac{3}{5}\right)$$
$$= \tfrac{1}{5}$$

Strength of Rule 4 DLGL

$$S4 = \min(\mu_{LD}(60), \mu_{LG}(70))$$
$$= \min\left(\tfrac{1}{5}, \tfrac{2}{5}\right)$$
$$= \tfrac{1}{5}$$

SISTec

Dirt \ Grease		MG	LG
	X	X	X
MD	X	M	L
LD	X	L	VL

MAX Membership Function

Dirt \ Grease		MG	LG
	X	X	X
MD	X	3/5	2/5
LD	X	1/5	1/5

Step05: Defuzzification

Since we use "Mean of Max" defuzzification technique

Maximum strength = Max(S1, S2, S3, S4)

= Max(3/5, 2/5, 1/5, 1/5)

= 3/5

- This corresponds to rule 1
- Rule 1: Dirt is medium and Grease is medium has maximum strength (3/5)
- To find out the final defuzzified value, we now take average (mean) of $\mu_M(z)$.

$$\mu_M(z) = \frac{z-10}{15}$$

$$\frac{3}{5} = \frac{z-10}{15}$$

$$\mu_M(z) = \frac{40-z}{15}$$

$$\frac{3}{5} = \frac{40-z}{15}$$

Watch Time

$\therefore z = 19$

$\therefore z = 31$

$$\therefore Z = \frac{19 + 31}{2}$$

$$\boxed{Z = 25\ min}$$

Using Mamdani Fuzzy model, design a fuzzy logic controller to regulate the temperature of a domestic shower. Assume that the input is the position of mixer tap. Use five descriptors for both input and output variable. Derive necessary membership function and required fuzzy rules for the application.

☑ **Soln. :**

- **Step 1: Identify input and output variables and their descriptors.**

Given that the input is the position of mixer tap. We assume that the position of mixer tap is measured in degrees (0° to 180°). 0° indicates tap is closed and 180° indicates tap is fully opened.

The output is the temperature of water measured in °C as per the position of mixer tap.

We use five descriptors for each input and output variables.

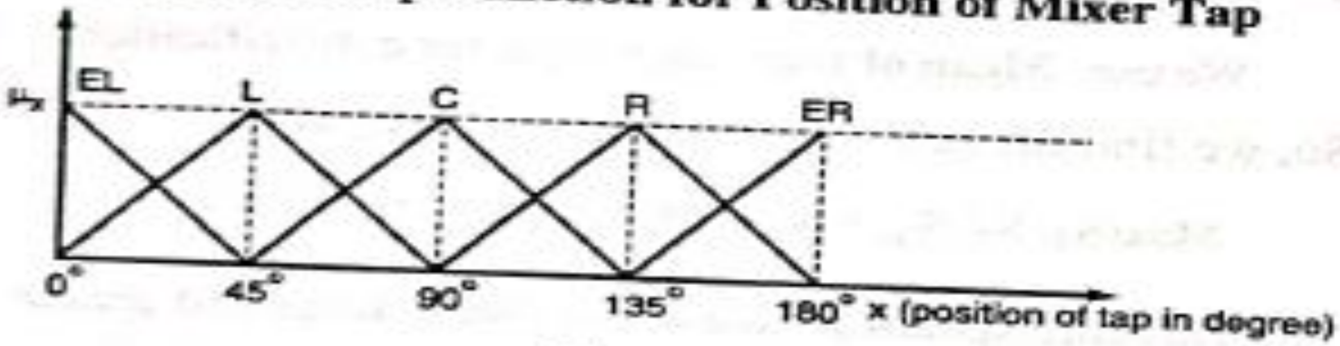
Descriptor for position of mixer tap : Extreme Left (EL), Left (L), Centre (C), Right ®, Extreme Right (ER)

Descriptor for temperature : Very Cold (VC), Cold (C), Medium (M), Hot (H), very Hot (VH).

- **Step 2 : Define membership functions for input and output variables.**

We use triangular membership functions.

1. Membership Function for Position of Mixer Tap



(1C47)**Fig. Ex. 3.8.2**

$$\mu_{EL}(x) = \frac{45 - x}{45}, \quad 0 \le x \le 45$$

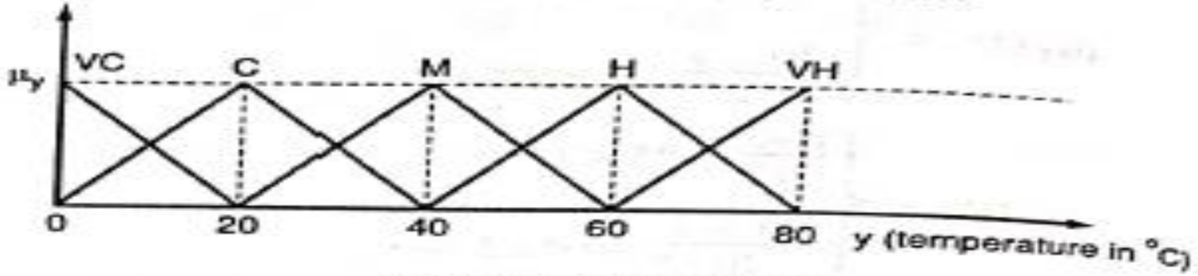
$$\mu_L(x) = \begin{cases} \frac{x}{45}, & 0 \le x \le 45 \\ \frac{90 - x}{45}, & 45 < x \le 90 \end{cases}$$

$$\mu_C(x) = \begin{cases} \frac{x - 45}{45}, & 45 \le x \le 90 \\ \frac{135 - x}{45}, & 90 < x \le 135 \end{cases}$$

$$\mu_R(x) = \begin{cases} \dfrac{x-90}{45}, & 90 \le x \le 135 \\ \dfrac{180-x}{45}, & 135 < x \le 180 \end{cases}$$

$$\mu_{ER}(x) = \frac{x-135}{45}, 135 \le x \le 180$$

2. Membership Function for Temperature



(1C48)**Fig. Ex. 3.8.2(a)**

$$\mu_{VC}(y) = \frac{20-y}{20}, 0 \le y \le 20$$

$$\mu_C(y) = \begin{cases} \dfrac{y}{20}, & 0 \le y \le 20 \\ \dfrac{40-y}{20}, & 20 < y \le 40 \end{cases}$$

$$\mu_M(y) = \begin{cases} \dfrac{y-20}{20}, & 20 \le y \le 40 \\ \dfrac{60-y}{20}, & 40 < y \le 60 \end{cases}$$

$$\mu_H(y) = \begin{cases} \dfrac{y-40}{20}, & 40 \le y \le 60 \\ \dfrac{80-y}{20}, & 60 < y \le 80 \end{cases}$$

$$\mu_{VH}(y) = \frac{y-60}{20}, 60 \le y \le 80$$

Step 3 : Form a rule base.

Input Position of Mixer Tap	Output Temperature of Water
EL	VC
L	C
C	M
R	H
ER	VH

There are total five rules defined in above table. For example, "If position of mixer tap is left then temperature of water is cold." Similarly, we can also define all rules using if – then.

Step 4 : Rule Evaluation

Assume position of mixer tap is 80°.

This value $x = 80°$ maps to the following two MFs.

$$\mu_L(x) = \frac{90 - x}{45} \quad \text{and} \quad \mu_C(x) = \frac{x - 45}{45}$$

Evaluate $\mu_L(x)$ and $\mu_C(x)$ for $x = 80$.

$$\mu_L(80) = \frac{90 - 80}{45} = \frac{2}{9}$$

$$\mu_C(80) = \frac{80 - 45}{45} = \frac{7}{9}$$

Step 5 : Defuzzification

We use "**Mean of max**" technique for defuzzification. So, we find the rule with maximum strength.

$$\text{Max}\,(\mu_L(x), \mu_C(x)) = \max\left(\frac{2}{9}, \frac{7}{9}\right) = \frac{7}{9}$$

This corresponds to rule 3 in rule base, i.e. If the position of mixer tap is left, temperature of water is medium.

We have following two equations for medium water temperature.

$$\mu_M(y) = \frac{y-20}{20} \quad \text{and} \quad \mu_M(y) = \frac{60-y}{20}$$

The strength of rule is $\frac{7}{9}$

$$\mu_M(y) = \frac{y-20}{20}$$

$$\therefore \quad \frac{7}{9} = \frac{y-20}{20} \Rightarrow y = \frac{7\times 20}{9} + 20 = 35.55$$

$$\mu_M(y) = \frac{60-y}{20}$$

$$\therefore \quad \frac{7}{9} = \frac{60-y}{20} \Rightarrow y = 60 - \frac{7\times 20}{9} = 44.44$$

To find the final defuzzified value, we now take the average of $\mu_M(y)$.

$$\therefore \quad y^* = \frac{35.55 + 44.44}{2} = 39.995$$

$$\therefore \quad y^* = 40°C$$

Using Mamdani fuzzy model, design a fuzzy logic controller to the feed amount of purifier for the water purification plant. Assume input as water temperature and grade of water. Use three descriptors for both input and output variables. Derive necessary membership function and required fuzzy rules for the application.

Soln. :

- **Step 1 : Identify input and output variables and their descriptors.**

Given that the input variables are water temperature and grade of water.

We assume that the temperature of water measured in °C and the grade of water is measured in percentage (%).

Also, we assume that amount of purifier is measured in grams.

We use three descriptors for each input and output variables.

Descriptor for water temperature : Cold (C), Medium (M), High (H)

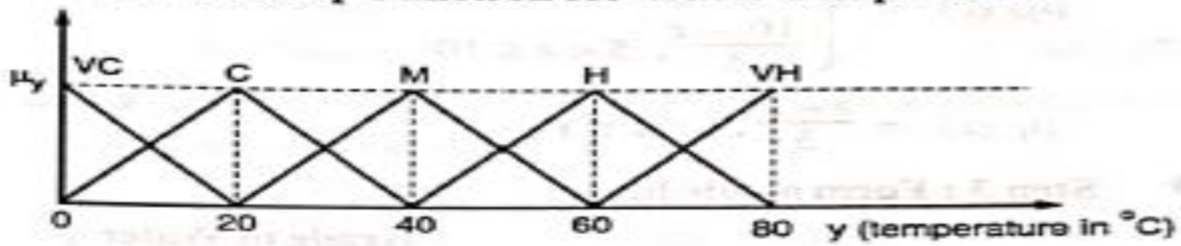
Descriptor for grade : Low (L), Medium (M), High (H)

Descriptor for amount of purifier : Small (S), Medium (M), Large (L)

- **Step 2 : Define membership functions for input and output variables.**

We use triangular membership functions.

1. Membership Function for Water Temperature



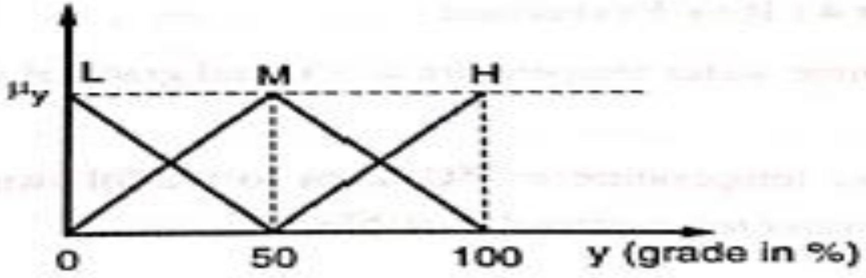
(1C49)**Fig. Ex. 3.8.3**

$$\mu_c(x) = \frac{50-x}{50}, 0 \le x \le 50$$

$$\mu_M(x) = \begin{cases} \frac{x}{50}, & 0 \le x \le 50 \\ \frac{100-x}{50}, & 50 < x \le 100 \end{cases}$$

$$\mu_H(x) = \frac{x-50}{50}, 50 \le x \le 100$$

2. Membership Function for Grade of Water

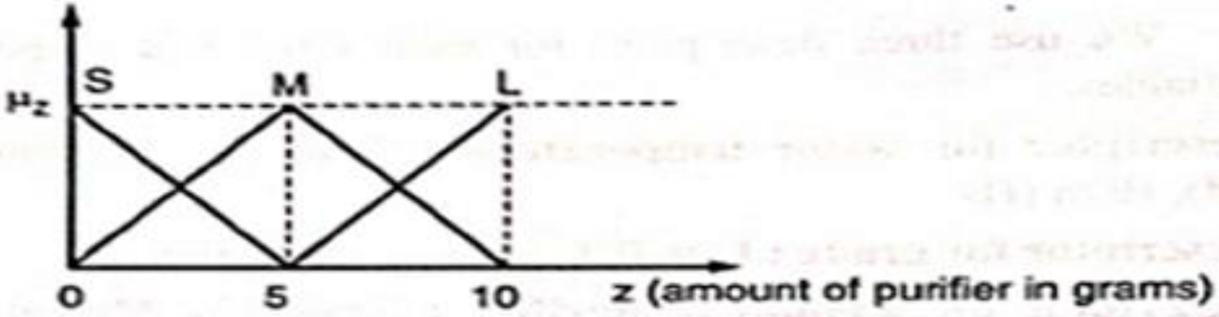


$$\mu_L(y) = \frac{50-y}{50}, 0 \le y \le 50$$

$$\mu_M(y) = \begin{cases} \frac{y}{50}, & 0 \le y \le 50 \\ \frac{100-y}{50}, & 50 < y \le 100 \end{cases}$$

$$\mu_H(y) = \frac{y-50}{50}, 50 \le y \le 100$$

3. Membership Function for Amount of Purifier



$$\mu_S(z) = \frac{5-z}{5}, 0 \le z \le 5$$

$$\mu_M(z) = \begin{cases} \frac{z}{5}, & 0 \le z \le 5 \\ \frac{10-z}{5}, & 5 < z \le 10 \end{cases}$$

$$\mu_L(z) = \frac{z-5}{5}, 5 \le z \le 10$$

Step 3 : Form a rule base.

		Grade of Water		
		L	**M**	**H**
Temperature of water	**C**	L	M	S
	M	L	M	M
	H	M	S	S

The above matrix represents total nine rules. For example, "**If temperature is cold and grade of water is low then amount of purifier required is large**". Similarly, we can define all the rules using if - then.

Step 4 : Rule Evaluation

Assume water temperature = 5°C and grade of water = 30%.

Water temperature = 5°C maps to the following two MFs of "water temperature" variable.

$$\mu_c(x) = \frac{50-x}{50} \text{ and } \mu_M(x) = \frac{x}{50}$$

Evaluate $\mu_c(x)$ and $\mu_M(x)$ for $x = 5$.

$$\mu_c(5) = \frac{50-5}{50} = \frac{9}{10} \quad \ldots(1)$$

$$\mu_M(5) = \frac{5}{50} = \frac{1}{10} \quad \ldots(2)$$

Similarly, grade = 30% maps to the following two MFs of "grade" variable.

$$\mu_L(y) = \frac{50-y}{50} \text{ and } \mu_M(y) = \frac{y}{50}$$

Evaluate $\mu_L(y)$ and $\mu_M(y)$ for $y = 30$.

$$\mu_L(30) = \frac{50-30}{50} = \frac{2}{5} \quad \ldots(3)$$

$$\mu_M(30) = \frac{30}{50} = \frac{3}{5} \quad \ldots(4)$$

The above four equations represent the following four rules that we need to evaluate.

1. If temperature is **cold** and grade is **low**.
2. If temperature is **cold** and grade is **medium**.
3. If temperature is **medium** and grade is **low**.
4. If temperature is **medium** and grade is **medium**.

The antecedent part of each of the above rule is connected by **and** operator. So, we use **min** operator to evaluate the strength of each rule.

Strength of rule 1: $S_1 = \min (\mu_c (5), \mu_L (30))$

$$= \min \left(\frac{9}{10}, \frac{2}{5}\right) = \frac{2}{5}$$

Strength of rule 2: $S_2 = \min (\mu_c (5), \mu_M (30))$

$$= \min \left(\frac{9}{10}, \frac{3}{5}\right) = \frac{3}{5}$$

Strength of rule 3: $S_3 = \min (\mu_M (5), \mu_L (30))$

$$= \min \left(\frac{1}{10}, \frac{2}{5}\right) = \frac{1}{10}$$

Strength of rule 4: $S_4 = \min (\mu_M (5), \mu_M (30))$

$$= \min \left(\frac{1}{10}, \frac{3}{5}\right) = \frac{1}{10}$$

- **Step 5 : Defuzzification**

We use "**Mean of max**" technique for defuzzification.

So, we find the rule with maximum strength.

$$\text{Max}(S_1, S_2, S_3, S_4) = \text{Max}\left(\frac{2}{5}, \frac{3}{5}, \frac{1}{10}, \frac{1}{10}\right) = \frac{3}{5}$$

This corresponds to rule 2, i.e. Temperature is **cold** and grade is **medium**.

From rule base, If Temperature is **cold and** grade is medium, then amount of purifier is **medium.**

We have following two equations for medium water temperature.

$$\mu_M(z) = \frac{z}{5} \quad \text{and} \quad \mu_M(z) = \frac{10-z}{5}$$

The strength of rule is $\frac{3}{5}$.

$$\mu_M(z) = \frac{z}{5}$$

$$\therefore \quad \frac{3}{5} = \frac{z}{5} \Rightarrow z = 3$$

$$\mu_M(z) = \frac{10-z}{5}$$

$$\therefore \quad \frac{3}{5} = \frac{10-z}{5} \Rightarrow z = 7$$

To find the final defuzzified value, we now take the average of $\mu_M(z)$.

$$\therefore \quad z^* = \frac{3+7}{2} = 5$$

$$\therefore \quad z^* = 5 \text{ gms}$$