Group Theoretic Approach to Quantum Field Theory

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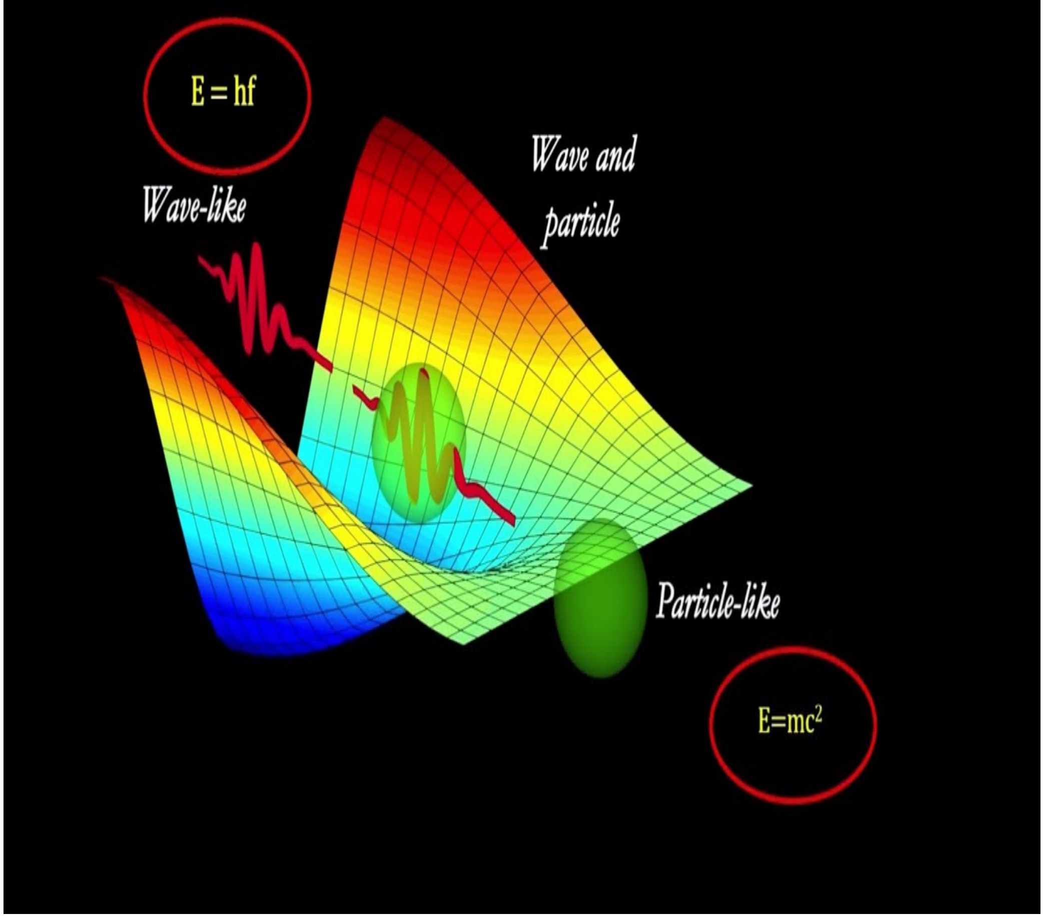

1 Groups

A group is a set G on which a multiplication operator $(*)$ is defined with the following properties:

- **Closure**: If $x, y \in G$ then $x * y \in G$;
- **Existence of Identity**: There exists an identity element (e) in G such that $a * e = e * a = a$ for all $a \in G$;
- **Existence of Inverse** For every element a in G, there exists an element a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$;
- **Associative Property**: Let $x, y, z \in G$, then $(x * y) * z = x * (y * z)$;

Hilbert Space

It is a linear vector space. It has an inner product operation that satisfies certain conditions:

- **Conjugate Symmetry:**

$$< \Psi_1, \Psi_2 > = < \Psi_2, \Psi_1 >$$

- **Linear w.r.t Second Vector:**

$$< \Psi_1, a\Psi_2 + b\Psi_3 > = a < \Psi_1, \Psi_2 > + b < \Psi_1, \Psi_3 >$$

- **Anti-linear w.r.t First Vector:**

$$< a\Psi_1 + b\Psi_2, \Psi_3 > = a^* < \Psi_1, \Psi_2 > + \ b^* < \Psi_2, \Psi_3 >$$

Unitary Operator: A bounded linear operator $U : H \to H$ on a Hilbert space H which satisfies the following conditions:

- U is **surjective**
- U preserves the inner product of the Hilbert space, i.e., for all vectors x, y in H, we have

$$< Ux, Uy >_H = < x, y >_H$$

In quantum mechanical systems $\to$ transformations are associated with unitary operators in Hilbert space. For each element x in G there exists a $D(x)$ which is a unitary operator which satisfies the multiplication law: $D(x)D(y) = D(x * y)$ for all x, y in G.

Representation

A mapping which satisfies the above condition is called a **representation** of the group.

Permutation Group:
( ): (a,b,c) $\to$ (a,b,c)
(1,2): (a,b,c) $\to$ (b,a,c)
(2,3): (a,b,c) $\to$ (a,c,b)
(1,3): (a,b,c) $\to$ (c,b,a)
(1,2,3): (a,b,c) $\to$ (c,a,b)
(3,2,1): (a,b,c) $\to$ (b,c,a)

$$D(\;) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad D(12) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$D(13) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \qquad D(23) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix},$$

$$D(321) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}, \qquad D(123) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}.$$

Figure 1.1: Representation of Permutation Group

Abelian and Non-Abelian Groups

Commutative groups are known as **Abelian groups**, whereas the non-commutative groups are called **Non-Abelian groups**.

The minimal set using the elements of which, the group under consideration can be constructed, provided the relations are all specified is called the **generating set** of the group. eg: $\{a\}$ with the relation $a^n = e$ generates the group $\{e, a, a^2, ...a^{n-1}\}$

A group whose generating set contains a single element is called a **cyclic group**.

Reducible and Irreducible representations

A representation which can be reduced to a block diagonal form using symmetry transformations is called a reducible representation.

$$D'(x) = SD(x)S^{-1} = \begin{bmatrix} D'_1(x) & 0 \\ 0 & D'_2(x) \end{bmatrix}$$

D' can be written as the **direct sum** of D'_1 and D'_2, i.e.,

$$D' = D'_1 \oplus D'_2$$

A representation is irreducible if it is not equivalent to any block diagonal matrix.

§ Chapter 1 Exercises

I.A If $x \in G$, then prove that the inverse element $x^{-1} \in G$ is unique.

Let's assume that x_1^{-1} and x_2^{-1} are two different inverses of $x \in G$. They must satisfy:

$$x * x_1^{-1} = x_1^{-1} * x = e \tag{1.1}$$
$$x * x_2^{-1} = x_2^{-1} * x = e \tag{1.2}$$

We take the relation $x * x_1^{-1} = e$, and pre-multiply x_2^{-1}, which gives

$$x_2^{-1} * x * x_1^{-1} = x_2^{-1} * e \tag{1.3}$$

Using Eq. (1.2), we get

$$x_1^{-1} = x_2^{-1} \tag{1.4}$$

Contradiction!!!

Hence proved.

I.B Find the multiplication law for a group having three elements and prove that it is unique.

Consider a group of three elements $\{e, a, b\}$. Multiplication law should be such that it satisfies closure. Let's take the elements one by one and find the possible values of their products with all elements.

	e	a	b
e	e	a	b
a	a	a*a	a*b
b	b	b*a	b*b

For $*$ operator to define a valid multiplication law, each row and each column must have every element of the group occurring once. $a * a$ can take two values e and b. If takes e then $b * a$ automatically becomes b. This makes two elements on row three equal, which is not something we want.

Hence, $a * a$ must be b. which implies $b * a = e$. Thus, a and b are inverses of each other. Therefore, $a * b = e$. This renders $b * b = a$. Thus the final multiplication law is

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

I.C Show that the representation of $S(3)$ is reducible.

Clearly, all the matrices which define the permutation representation of 3 objects, have a common eigen-vector $[1\ 1\ 1]^T$. This means, the solution spaces of each of the matrices have a common basis vector $[111]^T$. Thus, this representation can be decompes into a 1×1 and a 2×2 representation. Hence the permutation representation of 3 objects is reducible!