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Tutorial Sheet 5

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1. Find the Fourier series of the following three functions defined on the interval $[-\pi, \pi]$:

$$f_1(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ \sin(x) & \text{for } x \in [0, \pi] \end{cases}$$

and

$$f_2(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ \cos(x) & \text{for } x \in [0, \pi] \end{cases}$$

and

$$f_3(x) := \pi \qquad \text{for } x \in [-\pi, \pi].$$

Sol. First, let us look at the Fourier series of f_1.

$$f_1(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)dx$$

Thus, we get

$$a_0 = \frac{1}{\pi}$$

Now, since we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{\pi}^{\pi} f_1(x)\sin(nx)dx \qquad n \in \mathbb{N}$$

Thus,for $n \in \mathbb{N}$ we get

$$a_n = -\frac{1}{\pi}\frac{1+(-1)^n}{n^2-1}$$
$$b_n = 0$$

Thus, we get

$$f_1(x) = \frac{1}{\pi} + \sum_{n=even} \frac{-4}{\pi(n^2-1)}\cos(nx)$$

Now, doing the same for f_2, we will get

$$f_2(x) = \sum_{n=even} \frac{4n}{\pi(n^2-1)}\sin(nx)$$

The analysis for f_3 is trivial ■

2. Consider the following 2π-periodic function:

$$f(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ x^2 & \text{for } x \in [0, \pi) \end{cases}$$

Note that the 2π-periodicity of the above function essentially says that

$$f(\pi) := f(-\pi) = 0.$$

(2a) Find the Fourier series of the above function.

(2b) Employing the theorem of Dirichlet at the point $x = 0$ for the Fourier series obtained in Question (2a), show that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

(2c) Deduce from the sum obtained in Question (2b) that the following series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

sums to $\frac{\pi^2}{6}$.

Sol.

2a Let us note the following two results before proceeding ($n \in \mathbb{N}$)

$$\int_0^\pi x^2 \sin(nx)dx = \frac{(-1)^{n+1}}{n}\pi^2 + 2\frac{1-(-1)^n}{n^3}$$

and

$$\int_0^\pi x^2 \cos(nx)dx = \frac{2\pi(-1)^n}{n^2}$$

Thus, we obtain the Fourier series for $f(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ x^2 & \text{for } x \in [0, \pi) \end{cases}$ is

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2\pi(-1)^n}{n^2}\cos(nx) + \left[\frac{(-1)^{n+1}}{n}\pi^2 + 2\frac{1-(-1)^n}{n^3} \right] \sin(nx) \right)$$

2b Plugging in $x = 0$, we get

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2\pi(-1)^n}{n^2}$$

Thus, we get

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

2c Let

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Thus, we have

$$\frac{\pi^2}{12} + \frac{S}{2} = S$$

(Think about why this can be done) Thus, we get

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

3. Let $f : [-\pi, \pi] \to \mathbb{R}$ be such that

$$f(\pi) = f(-\pi) \quad \text{and} \quad \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty.$$

Consider its Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big)$$

with Fourier coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \qquad \text{for } n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \qquad \text{for } n = 1, 2, \ldots$$

(3a) Let $N \in \mathbb{N}$. Consider the truncated Fourier series of f given below:

$$S_N(x) := \frac{1}{2}a_0 + \sum_{n=1}^{N} \Big(a_n \cos(nx) + b_n \sin(nx) \Big).$$

Show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) S_N(x) \, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)$$

(3b) Consider the truncated Fourier series from Question (3a) and show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (S_N(x))^2 \, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)$$

(3c) By expanding out the square in the following integral

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x) - S_N(x))^2 \, dx$$

and using the results of Question (3a) and Question (3b) deduce that

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx$$

Sol.

3a This follows trivially from the expressions for the coefficients

3b The cross terms in S_N will cancel out in the integral, giving us the expression

3c Direct use of the results from 3a and 3b along with the fact that

$$0 \le \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x) - S_N(x))^2 \, dx$$

giving us

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx$$

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4. Which of the following functions is *even*, *odd* or *neither*:

$$x^5\sin(x),\quad x^2\sin(2x),\quad e^x,\quad (\sin(x))^3.\quad \cos(x+x^3),\quad x+x^2+x^3,\quad \ln\left(\frac{1+x}{1-x}\right)$$

Sol.

- $x^5\sin(x)$: even
- $x^2\sin(2x)$: odd
- e^x : neither
- $(\sin(x))^3$: odd
- $\cos(x+x^3)$: even
- $x+x^2+x^3$: neither
- $\ln\left(\frac{1+x}{1-x}\right)$: odd

■

5. Show that any function $f:[-\pi,\pi]\to\mathbb{R}$ can be written as the sum of an even function and an odd function.
Sol.

$$f(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}$$

The first term is even while the second is odd ■

6. Recall that the Fourier transform of a function $f\in L^1(\mathbb{R})$ is defined by

$$\widehat{f}(\omega):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx\qquad\text{for }\omega\in\mathbb{R}.$$

Find the Fourier transforms of the following three functions defined on the whole of $\mathbb{R}$:

$$g_1(x):=\begin{cases}x^2 & \text{for } x\in(0,1)\\ 0 & \text{otherwise}\end{cases}$$

and

$$g_2(x):=\begin{cases}|x| & \text{for } x\in(-1,1)\\ 0 & \text{otherwise}\end{cases}$$

and

$$g_3(x):=\begin{cases}xe^{-x} & \text{for } x\in(0,\infty)\\ 0 & \text{otherwise}\end{cases}$$

Sol. $g_1(x)$:

$$\begin{aligned}\widehat{f_1}(\omega)&=\frac{1}{\sqrt{2\pi}}\int_0^1 x^2e^{-i\omega x}\,dx\\ &=\frac{1}{\sqrt{2\pi}}\left(\frac{\omega x(i\omega x+2)-2i}{\omega^3}e^{-i\omega x}\Big|_0^1\right)\\ &=\frac{1}{\sqrt{2\pi}}\left(\frac{\omega(i\omega+2)-2i}{\omega^3}e^{-i\omega}+\frac{2i}{\omega^3}\right)\end{aligned}$$

g_2:

$$\begin{aligned}\widehat{f_2}(\omega) &= -\frac{1}{\sqrt{2\pi}}\int_{-1}^{0} xe^{-i\omega x}\,dx + \int_{0}^{1} xe^{-i\omega x}\,dx\\ &= -\frac{(i\omega x+1)}{\omega^2}e^{-i\omega x}\big|_{-1}^{0} + \frac{(i\omega x+1)}{\omega^2}e^{-i\omega x}\big|_{0}^{1}\\ &= \frac{-i\omega+1}{\omega^2}e^{i\omega} + \frac{i\omega+1}{\omega^2}e^{-i\omega} - \frac{2}{\omega^2}\end{aligned}$$

g_3:

$$\begin{aligned}\widehat{f_2}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-x}e^{-i\omega x}\,dx\\ &= \frac{i\omega x+x+1}{(i\omega+1)^2}e^{-(i\omega+1)x}\big|_{0}^{\infty}\\ &= -\frac{\omega^2+2i\omega-1}{\omega^4+2\omega^2+1}\end{aligned}$$

■

7. Consider the following initial value problem:

$$\begin{cases}\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}\\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}.\end{cases}$$

In the lectures, we derived the following representation formula for the solution:

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}u_0(y)\,dy \qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}$$

(7a) Show that the solution $u(t,x) \geq 0$ for all $(t,x) \in (0,\infty)\times\mathbb{R}$ if the initial datum $u_0(x) \geq 0$ for all $x \in \mathbb{R}$.

(7b) Find the solution to the above initial value problem when

$$u_0(x) = \frac{1}{1+x^2}$$

and when

$$u_0(x) := \begin{cases}1 & \text{for } x \in (-1,1)\\ 0 & \text{otherwise}\end{cases}$$

Sol.

7a Showing this is trivial

7b Direct substitution will give definite integrals for both cases. However, the integrals are best evaluated numerically. The values for both will be available in standard tables

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8. Let $a > 0$. Applying the Fourier transform in the x variable, show the solution $u(t,x)$ to the following first-order partial differential equation:

$$\begin{cases}\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}\\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}.\end{cases}$$

is

$$u(t,x) = u_0(x-at) \qquad \text{for } (t,x) \in [0,\infty)\times\mathbb{R}.$$

Sol. On taking the Fourier transform of the given differential equation, we get

$$\frac{d}{dt}\hat{u} + i\omega a\hat{u} = 0$$

Let the Fourier transform of the initial condition be denoted as $\hat{u}_0(\omega)$, given by

$$\hat{u}_0(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(x)e^{-i\omega x}dx$$

The solution of the differential equation is clearly

$$\hat{u}(t,\omega) = \hat{u}_0(\omega)e^{-ia\omega t}$$

Taking its inverse Fourier transform gives

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}_0(\omega)e^{i\omega(x-at)}dx \\ &= u_0(x-at) \end{aligned}$$

Hence proved! ■