MA 207 Autumn 2022
Tutorial Sheet 4

Rishabh Ravi & Anurag Pendse & Shashwat Chakraborty

Autumn 2022

1. We say that a solution to a differential equation is non-trivial if it is NOT identically zero everywhere. Let $\kappa > 0$ be a constant.

(1a) Find two linearly independent solutions to the following differential equation and hence write down its general solution:

$$y''(x) = \kappa y(x) \qquad \text{for } x \in (0, 1).$$

(1b) Suppose the above differential equation is supplemented with one of the following boundary conditions:

- Dirichlet: $y(0) = 0$ and $y(1) = 0$
- Neumann: $y'(0) = 0$ and $y'(1) = 0$
- Mixed-I: $y'(0) = 0$ and $y(1) = 0$
- Mixed-II: $y(0) = 0$ and $y'(1) = 0$

For which of the above boundary conditions, the general solution found in Question (1a) remains non-trivial? Justify your answer.

Sol.

(1a) In order to find the linearly independent solutions, one can use the power series method and it is pretty straight forward in this case. After a slight manipulation of the coefficients one can obtain

$$y(x) = Ae^{\sqrt{\kappa}x} + Be^{-\sqrt{\kappa}x}$$

(1b) Let's look at what happens in each case:

(i) $y(0) = 0$ and $y(1) = 0$: These boundary conditions give us two the following two equations

$$\begin{aligned} A + B &= 0 \\ Ae^{\sqrt{\kappa}} + Be^{-\sqrt{\kappa}} &= 0 \end{aligned}$$

This renders $A = B = 0 \implies$ trivial solution.

(ii) $y'(0) = 0$ and $y'(1) = 0$: These conditions give

$$\begin{aligned} \sqrt{\kappa}A - \sqrt{\kappa}B &= 0 \\ \sqrt{\kappa}Ae^{\sqrt{\kappa}} - \sqrt{\kappa}Be^{-\sqrt{\kappa}} &= 0 \end{aligned}$$

Once again $A = B = 0 \implies$ trivial solution.

(iii) $y'(0) = 0$ and $y(1) = 0$: In this case we have

$$\begin{aligned} \sqrt{\kappa}A - \sqrt{\kappa}B &= 0 \\ Ae^{\sqrt{\kappa}} + Be^{-\sqrt{\kappa}} &= 0 \end{aligned}$$

Once again, we get $A = B = 0 \implies$ trivial solution.

(iv) $y(0) = 0$ and $y'(1) = 0$: In this case we have

$$A + B = 0$$
$$\sqrt{\kappa} A e^{\sqrt{\kappa}} - \sqrt{\kappa} B e^{-\sqrt{\kappa}} = 0$$

Again, we get $A = B = 0 \implies$ trivial solution.

■

2. A constant $\lambda \in \mathbb{R}$ and a function $y : [-\pi, \pi] \to \mathbb{R}$ are said to be an eigenvalue and an eigenfunction, respectively, of the following Sturm-Liouville problem if

$$\begin{cases} y''(x) = \lambda y(x) & \text{for } x \in (-\pi, \pi) \\ y(-\pi) = y(\pi), \; y'(-\pi) = y'(\pi) \end{cases}$$

(2a) Can $\lambda = 0$ be an eigenvalue of the above Sturm-Liouville problem? If your answer is YES, then find the corresponding eigenfunction. On the other hand, if your answer is NO, justify your answer.

(2b) Does there exist an eigenvalue $\lambda < 0$ for the above Sturm-Liouville problem? If your answer is YES, then find the corresponding eigenfunction. On the other hand, if your answer is NO, justify your answer.

(2c) Find all possible eigenvalues of the above Sturm-Liouville problem and their corresponding eigenfunctions.

Sol.

(2a) Let's take $\lambda = 0$ and see if there exists a solution that satisfies the boundary conditions. So, $\lambda = 0$ gives $y(x) = Ax + B$. The boundary conditions give

$$-A\pi + B = A\pi + B$$
$$A = A$$

These equations render $A = 0 \implies y(x) \equiv B$ (constant function).

(2b) Let's take $\lambda = -k^2$. We can rewrite our differential equation as

$$y''(x) = -k^2 y(x)$$

The general solution can be written as

$$y(x) = A\sin(kx) + B\cos(kx)$$

Under the periodic boundary conditions we have

$$-A\sin(k\pi) + B\cos(k\pi) = A\sin(k\pi) + B\cos(k\pi)$$
$$kA\cos(k\pi) + kB\sin(k\pi) = kA\cos(k\pi) - kB\sin(k\pi)$$

These equations can be simplified to give

$$A\sin(k\pi) = 0$$
$$B\sin(k\pi) = 0$$

Thus we need $k = n \in \mathbb{N}$ for satisfying the equations. Thus, the Sturm-Liouville problem has a non-trivial solution if $\lambda = -n^2$ where $n \in \mathbb{N}$. The corresponding family of eigenfunctions comprises of all the functions of the form

$$y(x) = A\sin(nx) + B\cos(nx)$$

(2c) Done in part b.

■

3. Consider the eigenvalue problem:

$$(p(x)y'(x))' + q(x)y(x) = \lambda y(x) \qquad \text{for } x \in (a,b).$$

Here $p(x)$ and $q(x)$ are coefficients which are at least once continuously differentiable on the interval (a,b). A boundary condition for the above Sturm-Liouville problem is said to be *symmetric* if

$$\left(p(x)\left(u'(x)v(x) - v'(x)u(x)\right)\right)\Big|_a^b = 0$$

for all functions $u(x)$ and $v(x)$ satisfying the given boundary condition. Who among the following boundary conditions are symmetric?

* Dirichlet: $y(a) = 0$ and $y(b) = 0$
* Neumann: $y'(a) = 0$ and $y'(b) = 0$
* Mixed-I: $y'(a) = 0$ and $y(b) = 0$
* Mixed-II: $y(a) = 0$ and $y'(b) = 0$
* Periodic: $y(a) = y(b)$ and $y'(a) = y'(b)$

Sol.

* $y(a) = 0$ and $y(b) = 0$: Using the fact that u and v satisfy the boundary conditions, we get that $p(x)(u'(x)v(x) - v'(x)u(x))$ is zero at both the boundary points $x = a$ and b. Thus, the Dirichlet boundary condition is symmetric.
* Similarly, the Neumann boundary condition is also symmetric.
* $y'(a) = 0$ and $y(b) = 0$: At $x = a$, we have

  $$p(a)(u'(a)v(a) - v'(a)u(a)) = p(a)(0 \times v(a) - 0 \times u(a)) = 0$$

  At $x = b$

  $$p(b)(u'(b)v(b) - v'(b)u(b)) = p(b)(u'(b) \times 0 - v'(b) \times 0) = 0$$

  Therefore, $\left(p(x)\left(u'(x)v(x) - v'(x)u(x)\right)\right)\Big|_a^b = 0 \implies$ the boundary condition is symmetric.
* $y(a) = 0$ and $y'(b) = 0$: Proceeding similar to the previous part, we get this boundary condition is also symmetric.
* $y(a) = y(b)$ and $y'(a) = y'(b)$: In this case we have

  $$\begin{aligned}\left(p(x)\left(u'(x)v(x) - v'(x)u(x)\right)\right)\Big|_a^b &= p(b)(u'(b)v(b) - v'(b)u(b)) - p(a)(u'(a)v(a) - v'(a)u(a)) \\ &= (p(b) - p(a))\left(u'(a)v(a) - v'(a)u(a)\right) \\ &\neq 0\end{aligned}$$

■

4. Let λ be an eigenvalue and let $y(x)$ be the associated twice continuously differentiable eigenfunction on the interval $[a,b]$ satisfying the eigenvalue problem:

$$\begin{cases} y''(x) = \lambda y(x) & \text{for } x \in (a,b) \\ y(a) = y(b) = 0. \end{cases}$$

(4a) Show that the following equality holds:

$$\lambda \int_a^b (y(x))^2 \, dx = -\int_a^b (y'(x))^2 \, dx.$$

(4b) Justify the following claim: For the eigenfunction $y(x)$, we have

$$\int_a^b (y'(x))^2 \, dx > 0.$$

(4c) Deduce from the equality proved in Question (4a) and the positivity of the integral established in Question (4b) that the eigenvalue λ should be strictly negative.

Sol.

(4a) The required result ca n be easily obtained by multiplying both sides by $y(x)$ and then using the method of integration by parts.

(4b) Since $y(x)$ is a real function, so is its derivative $\implies$ $(y'(x))^2 > 0$, assuming that $y(x)$ is a non-trivial solution, thus, $y'(x)$ can't be zero on the entire interval. The limits of the integral are such that $b > a$. Since, the integrand is positive and the upper limit is larger than the lower limit, the integral has to be. positive!

(4c) In part (b) we have shown

$$\int_a^b (y'(x))^2 \, dx > 0.$$

Thus, the right hand side of the equality in part (a) is negative $\implies$

$$\lambda \int_a^b (y(x))^2 \, dx < 0$$

Using arguments similar to part (b), one can show that

$$\int_a^b (y(x))^2 \, dx > 0$$

Therefore, $\lambda < 0$. Hence proved!

■

5. Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\begin{cases} y''(x) + 2y'(x) + y(x) = -\lambda y(x) & \text{for } x \in (0, \pi) \\ y(0) = y(\pi) = 0. \end{cases}$$

Sol. The general solution to

$$y''(x) + 2y'(x) + y(x) = -\lambda y(x)$$

is

$$y(x) = c_1 e^{\kappa_+ x} + c_2 e^{\kappa_- x} \qquad \kappa_\pm = -1 \pm \sqrt{-\lambda}$$

This solution can only satisfy the given boundary conditions when $\lambda > 0$. In that case, the solution we get which satisfies $y(0) = y(\pi) = 0$ is

$$y(x) = ce^{-x}\sin(\sqrt{\lambda}x) \qquad \lambda = n^2 \,;\, n \in \mathbb{N}$$

■

6. Recall that the Gamma function is defined as follows:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt \qquad \text{for } z > 0.$$

(6a) Show that the change of variable $t = s^2$ leads to

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-s^2}\, ds.$$

(6b) Since s is a dummy variable in Question (6a), we can write

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

By changing the above double integral to polar coordinates, show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Sol.

6a We have

$$\Gamma(\frac{1}{2}) = \int_\infty^0 t^{-\frac{1}{2}} e^{-t} dt$$

Upon making the substitution $t = s^2$, we will get

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{s} e^{-s^2} 2s ds$$

Thus, we get

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-s^2} ds$$

6b Solving this integral is a simple exercise using the substitution suggested in the problem.

$$\left(\Gamma(\frac{1}{2}\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

Thus,

$$\begin{aligned}\left(\Gamma(\frac{1}{2}\right)^2 &= 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta dr \\ &= 2\pi \int_0^\infty r e^{-r^2} dr \\ &= \pi\end{aligned}$$

This gives us

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

■

7. The Bessel function of first kind of order $p \geq 0$ is given by

$$J_p(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

Similarly, we have the functions

$$J_{-p}(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

for $p > 0$.

(7a) In the lectures, we have shown that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}}\sin(x).$$

Using similar computations, show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}}\cos(x).$$

(7b) Using the identity

$$\frac{p}{x}J_p = \frac{1}{2}\Big(J_{p-1}(x) + J_{p+1}(x)\Big),$$

derive expressions for $J_{\frac{5}{2}}(x)$ and $J_{-\frac{5}{2}}(x)$.

Sol.

7a We will use the following identity which we derived in Problem sheet 3

$$\frac{d}{dx}J_p(x) + \frac{p}{x}J_p = J_{p-1}(x)$$

Now, set $p = \frac{1}{2}$. This will give us

$$J_{-\frac{1}{2}} = \frac{d}{dx}\left(\sqrt{\frac{2}{x\pi}}\sin(x)\right) + \frac{1}{2x}\sqrt{\frac{2}{x\pi}}\sin(x)$$

Carrying out this computation gives

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}}\cos(x)$$

This result can also be derived as it was in the slides. That method however is long and unnecessary given that we already know $J_{\frac{1}{2}}$

7b Using the identity

$$\frac{p}{x}J_p = \frac{1}{2}\Big(J_{p-1}(x) + J_{p+1}(x)\Big)$$

We get

$$J_{\frac{3}{2}} = \sqrt{\frac{2}{x\pi}}\left(\frac{\sin(x)}{x} - \cos(x)\right)$$

and

$$J_{-\frac{3}{2}} = -\sqrt{\frac{2}{x\pi}}\left(\frac{\cos(x)}{x} - \sin(x)\right)$$

Using these, we will get

$$J_{\frac{5}{2}} = \sqrt{\frac{2}{x\pi}}\left(\frac{3\sin(x)}{x^2} - \frac{3\cos(x)}{x} - \sin(x)\right)$$

and

$$J_{-\frac{5}{2}} = \sqrt{\frac{2}{x\pi}}\left(\frac{3\cos(x)}{x^2} + \frac{\sin(x)}{x} - \cos(x)\right)$$

■

8. Let $u(t,x) : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a solution to the heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) \qquad \text{for } (t,x) \in (0,\infty)\times\mathbb{R}.$$

Define a function $v(t,x):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ as follows

$$v(t,x):=u(9t,3x)\qquad \text{for } (t,x)\in(0,\infty)\times\mathbb{R}.$$

Show that the function v also solves the heat equation.

Sol. Checking this is trivial ■

9. Let $u(t,x):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)\qquad \text{for } (t,x)\in(0,\infty)\times\mathbb{R}.$$

Define a function $v(t,x):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ as follows

$$v(t,x):=u(3t,3x)\qquad \text{for } (t,x)\in(0,\infty)\times\mathbb{R}.$$

Show that the function v also solves the wave equation.

Sol. Checking this is trivial

■

10. Consider the initial value problem for the wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2} & \text{for } t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x) & \text{for } x\in\mathbb{R},\\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x) & \text{for } x\in\mathbb{R}. \end{cases}$$

Recall the formula of D'Alembert:

$$u(t,x)=\frac{1}{2}\Big(u_0(x-ct)+u_0(x+ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}u_1(s)\,ds.$$

(10a) Write down the solution to the above initial value problem when

$$u_0(x)=e^x\qquad \text{and}\qquad u_1(x)=\sin(x).$$

(10b) Write down the solution to the above initial value problem when

$$u_0(x)=\ln(1+x^2)\qquad \text{and}\qquad u_1(x)=4+x.$$

(10c) Suppose that the data $u_0(x)$ and $u_1(x)$ are odd functions of the x variable. Show that the solution $u(t,x)$ to the above initial value problem is an odd function in the x variable for all $t>0$.

Sol.

10a

$$u(t,x)=e^x\cosh(ct)+\frac{\sin(ct)\sin x}{c}$$

10b

$$u(t,x)=\frac{1}{2}\ln\left((1+(x-ct)^2)(1+(x+ct)^2)\right)+4t+2xt$$

10c

$$u(t,-x)=\frac{1}{2}\Big(u_0(-x-ct)+u_0(-x+ct)\Big)+\frac{1}{2c}\int_{-x-ct}^{-x+ct}u_1(s)\,ds.$$

As u(-x) is equal to -u(x),

$$u(t,-x)=-\frac{1}{2}\Big(u_0(x+ct)+u_0(x-ct)\Big)+\frac{1}{2c}\int_{-x-ct}^{-x+ct}u_1(s)\,ds.=-u(t,x)$$

The integral takes a negative sign upon choosing an appropriate variable

11. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } t > 0,\ x \in \mathbb{R}.$$

Let $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ be any two twice continuously differentiable functions. Show that the following function solves the above wave equation:

$$F(x - ct) + G(x + ct).$$

Sol. Checking this is trivial. ■

12. Consider the heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) \qquad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}.$$

(12a) Verify that the function

$$f(t, x) := 1 - x^2 - 2t$$

solves the heat equation.

(12b) Take $T > 0$. Find the locations of the maximum and the minimum of the function f in the closed rectangle

$$\Big\{(t, x) \in \mathbb{R}^2 \text{ such that } t \in [0, T] \text{ and } x \in [0, 1]\Big\}.$$

Sol.

12a Checking this is trivial.

12b As the function decraeses monotonically with respect to x as well as t i.e observing the function's behaviour by varying one variable and keeping the other as a constant. Hence the maxima and minima occur at the boundaries.

$$Maxima(T = 0, x = 0) = 1$$

$$Minima(T = T, x = 1) = -2T$$

■