MA 207 Autumn 2022
Tutorial Sheet 1

Rishabh Ravi & Anurag Pendse & Shashwat Chakraborty

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1. We are given that there are non-zero real coefficients $a_0, a_1, a_2, \dots$ such that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{for all } x \in \mathbb{R}.$$

If we are told that $a_0 = 2$, determine the coefficients a_n for $n \geq 1$. Furthermore, do you recognise the function represented by the above power series?

Sol. The RHS of the equation can be rewritten as

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Since these are two equal power series, their coefficients **must** be equal.
Equating coefficients we get the following recursive relations

$$a_n = (n+1) a_{n+1} \qquad \text{for all } x \in \mathbb{R}$$

Which on further simplification gives

$$a_n = \frac{a_0}{n!}$$

The series thus becomes

$$\sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 e^x$$

Finally the initial condition that a_0=2 in the question gets us $2e^x$. ■

2. In the lectures, we constructed a series solution centered at the origin for the Airy's equation

$$y'' - xy = 0.$$

Note that the coefficients in the Airy's equation are real-analytic everywhere on $\mathbb{R}$. The objective of this question is to build power series solution centered at 1. More precisely, we look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

(2a) Write a deduce a series representation for $y''(x)$ whose generic term is $(x-1)^n$.

(2b) Rewriting x as $1 + (x-1)$, deduce a series representation for $xy(x)$ whose generic term is $(x-1)^n$.

(2c) Substituting your expressions from (2a) and (2b) in the Airy's equation, arrive at a recurrence relation for the coefficients a_n and hence determine these coefficients.

(2d) Using the coefficients a_n determined in (2c), write down the expression for your series solution centered at 1. Further, show that it has the structure of a linear combination of two linearly independent solutions.

(2e) Using ratio test, show that the series representations of your two linearly independent solutions converge for all $x \in \mathbb{R}$.

(2f) Recall that the series solutions (centered at the origin) for the Airy's equation that we obtained in lectures were

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1) \cdot (3n)}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n) \cdot (3n+1)}.$$

Show, using ratio test, that both these series converge for all $x \in \mathbb{R}$.

Sol.

(2a) We need solutions of the form

$$y = \sum_{n \geq 0} a_n (x-1)^n$$

The second derivative looks like

$$y'' = \sum_{n \geq 2} n(n-1) a_n (x-1)^{n-2} = \sum_{n \geq 0} (n+2)(n+1) a_{n+2} (x-1)^n$$

(2b) Let's rewrite the given differential equation as $y'' = y + (x-1)y$. Now we plug in the power series (define $a_{-1} \equiv 0$)

$$\sum_{n \geq 2} n(n-1) a_n (x-1)^{n-2} = \sum_{n \geq 0} a_n (x-1)^n + \sum_{n \geq 0} a_n (x-1)^{n+1}$$

$$\sum_{n \geq 0} (n+2)(n+1) a_{n+2} (x-1)^n = \sum_{n \geq 0} a_n (x-1)^n + \sum_{n \geq 0} a_{n-1} (x-1)^n$$

(2c) Recurrence relation:

$$a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)}$$

The first few coefficients are

$$a_2 = \frac{a_0}{2 \times 1}$$
$$a_3 = \frac{a_1 + a_0}{3 \times 2}$$
$$a_4 = \frac{a_2 + a_1}{4 \times 3}$$

and so on. We have two degrees of freedom here (a_0 and a_1) that are fixed by the initial conditions. Let's take $a_1 = 0$ and $a_0 \neq 0$.

$$a_2 = \frac{a_0}{2!}$$
$$a_3 = \frac{a_0}{3!}$$
$$a_4 = \frac{a_0}{4!}$$
$$a_5 = \frac{4a_0}{5!}$$
$$a_6 = \frac{5a_0}{6!}$$
$$a_7 = \frac{9a_0}{7!}$$
$$a_8 = \frac{29a_0}{8!}$$

and so on.

Similarly, take $a_0 = 0$ and $a_1 \neq 0$.

$$\begin{aligned}
a_2 &= 0 \\
a_3 &= \frac{a_1}{3!} \\
a_4 &= \frac{2a_1}{4!} \\
a_5 &= \frac{a_1}{5!} \\
a_6 &= \frac{3a_1}{6!} \\
a_7 &= \frac{11a_0}{7!}
\end{aligned}$$

(2d) The general solution can be written as

$$y = a_0(1 + \frac{a_0}{2!}(x-1)^2 + \frac{a_0}{3!}(x-1)^3 + \cdots) + a_1(x - 1 + \frac{a_1}{3!}(x-1)^3 + \frac{2a_1}{4!}(x-1)^4 + \cdots)$$

Clearly, it has the stucture of a linear combination of two independent solutions, say y_1 and y_2, where

$$\begin{aligned}
y_1 &= 1 + \frac{a_0}{2!}(x-1)^2 + \frac{a_0}{3!}(x-1)^3 + \cdots \\
y_2 &= x - 1 + \frac{a_1}{3!}(x-1)^3 + \frac{2a_1}{4!}(x-1)^4 + \cdots
\end{aligned}$$

(2e) Here we cannot obtain a closed form for the coefficients, so applying the ratio test is not possible. But observe that $x = 1$ is an ordinary point! Thus, real analytic solutions exist (using the theorem on the existence of real analytic solutions discussed in lecture 1).

(2f) Series solutions (centered at origin) for the Airy's equation are lectures were

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1) \cdot (3n)}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n) \cdot (3n+1)}.$$

Ratio test:

y_1:

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{x^{3n+3}}{2 \cdot 3 \cdots (3n-1) \cdot (3n) \cdot (3n+2) \cdot (3n+3)} \middle/ \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1) \cdot (3n)} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^3}{(3n+2) \cdot (3n+3)} \right| \\
&= 0 \quad \text{for all } x \in \mathbb{R}
\end{aligned}$$

y_2:

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{x^{3n+4}}{3 \cdot 4 \cdots (3n) \cdot (3n+1) \cdot (3n+3) \cdot (3n+4)} \middle/ \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n) \cdot (3n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^3}{(3n+3) \cdot (3n+4)} \right| \\
&= 0 \quad \text{for all } x \in \mathbb{R}
\end{aligned}$$

Therefore both the solutions y_1 and y_2 converge $\forall x \in \mathbb{R}$.

3. We are given that the radius of convergence of the following power series is $R \in (0, \infty)$:

$$\sum_{n=0}^{\infty} a_n x^n$$

(3a) Show that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^{2n}$$

is $\sqrt{R}$.

(3b) Show that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n^2 x^n$$

is R^2.

While answering (3b), you may recall the root test for convergence of an infinite series

$$\sum_{n=0}^{\infty} \alpha_n$$

which says that the series converges if the following limit (when it exists) is strictly less than one and that the series diverges if the following limit (when it exists) is strictly larger than one

$$\lim_{n\to\infty} |\alpha_n|^{\frac{1}{n}}$$

Applying this root test to a general power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

we find that it converges if

$$|x-c| < \frac{1}{\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

Disclaimer: Root test is usually given in terms of $\limsup\limits_{n\to\infty}$. *But, we have chosen not to consider this subtle point.*

Sol.

(3a) We will directly use the following formula for the radius of convergence.

$$R = \frac{1}{\text{limsup}|a_n|^{\frac{1}{n}}}$$

Now, in this series, let us call the coefficients b_n. We have

$$b_n = \begin{cases} a_m, & \text{if } n = 2m \\ 0, & \text{otherwise} \end{cases}$$

Applying the formula here, we get

$$\begin{aligned}
R' &= \frac{1}{\limsup |a_n|^{\frac{1}{2n}}} \\
&= \left(\frac{1}{\limsup |a_n|^{\frac{1}{n}}}\right)^{\frac{1}{2}} \\
&= \sqrt{R}
\end{aligned}$$

Thus, the radius of convergence of the given series is $\sqrt{R}$.

(3b) This part follows trivially from the formula for radius of convergence.

■

4. Consider the differential equation

$$x^2y'' - (1+x)y = 0.$$

Determine the coefficients a_n in the following power series supposing that the power series solves the above differential equation

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Sol. Double differentiating the power series and multiplying x^2 we obtain

$$x^2y'' = 1 \cdot 2a_2x^2 + 2 \cdot 3a_3x^3 + 3 \cdot 4a_4x^4 ... = \sum_{n=2}^{\infty} n \cdot (n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} n \cdot (n-1)a_n x^n$$

Similarly we get

$$xy = a_0x + a_1x^2 + a^2x^3 ... = \sum_{n=1}^{\infty} a_{n-1}x^n$$

Putting this in the equation

$$x^2y'' - (1+x)y = 0.$$

We get the following reccursive relation between the coefficients

$$\sum_{n=1}^{\infty}[n \cdot (n-1)a_n - a_n - a_{n-1}]x^n - a_0x^0 = 0$$

Notice that the series is written from n=1, and the terms for n=0 are/is included separately. Now by equating coefficients for each power of x we get $a_0 = 0$ as it is the only term with x^0. Hence for all n¿0 we get

$$n \cdot (n-1)a_n - a_n - a_{n-1} = 0 \qquad \text{for all } n \in \mathbb{N}$$

This gives a_n=0 for all n∈ ℕ. We thus see that x=0 is the only solution to this differential equation. ■

5. Consider the differential equation

$$(1-x^2)y'' - xy' + p^2y = 0,$$

where p is a constant.

(5a) Find two linearly independent series solutions valid for $|x| < 1$

(5b) Show that if $p = n$ where n is a non-negative integer, then there is a polynomial solution of degree n.

(5c) Determine the polynomial solutions from (5b) for $n = 0, 1, 2, 3, 4$.

Sol.

(5a) We seek solutions of the form

$$y = \sum_{n \geq 0} a_n x^n$$

The derivatives of y look like

$$y' = \sum_{n \geq 1} n a_n x^{n-1}$$

and

$$y'' = \sum_{n \geq 2} n(n-1)a_n x^{n-2}$$

Let's write down the three terms of the given differential equation in power series form.

$$\begin{aligned}(1-x^2)y'' &= \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n - \sum_{n\geq 0} n(n-1)a_n x^n \\ xy' &= \sum_{n\geq 0} na_n x^n \\ p^2 y &= p^2 \sum_{n\geq 0} a_n x^n\end{aligned}$$

Substituting the above expressions in the differential equation gives

$$\sum_{n\geq 0}((n+2)(n+1)a_{n+2}x^n - n(n-1)a_n x^n - na_n x^n + p^2 a_n x^n) = 0$$

Recurrence relation:

$$a_{n+2} = \frac{n^2 - p^2}{(n+2)(n+1)}a_n$$

Clearly, we have two degrees of freedom, a_0 and a_1. To find the two linearly independent solutions, let's first put $a_1 = 0 \implies a_{2k+1} = 0\ \forall k \in \mathbb{N}$. The even numbered coefficients are:

$$\begin{aligned}a_2 &= \frac{0-p^2}{2\times 1}a_0 \\ a_4 &= \frac{4-p^2}{4\times 3}.\frac{0-p^2}{2\times 1}a_0\end{aligned}$$

and so on $\implies$

$$a_{2k} = \frac{(4(k-1)^2-p^2)(4(k-2)^2-p^2)...(0-p^2)}{(2k)!}a_0$$

Similarly, if we take $a_0 = 0$ and $a_1 \neq 0$, we get

$$a_{2k+1} = \frac{(2k-1)^2-p^2)((2k-3)^2-p^2)...(1-p^2)}{(2k+1)!}a_1$$

Therefore, the general solution can be written as

$$y = a_0 \sum_{k\geq 0}\frac{(4(k-1)^2-p^2)...(0-p^2)}{(2k)!}x^{2k} + a_1 \sum_{k\geq 0}\frac{(2k-1)^2-p^2)...(1-p^2)}{(2k+1)!}x^{2k+1}$$

(5b) $p = n \in \mathbb{N}$. Let's take a look at the recurrence relation

$$a_{k+2} = \frac{k^2-n^2}{(k+2)(k+1)}a_k$$

It is easy to observe that all coefficients of one of the two independent solutions (either odd or even) vanish beyond $k = n$. Thus, we get a a polynomial solution of degree $= n$.

(5c) For $n = 0$: we get $y = a_0$ (constant function) as the polynomial solution, since the coefficients $a_2, a_4, ...$ all vanish.
Similarly, for $n = 1$: $y = a_1 x$; for $n = 2$: $y = a_0 - 2a_0 x^2$ (using the recurrence relation); for $n = 3$: $y = a_1 x - (4/3)a_1 x^3$, and for $n = 4$: $y = a_0 - 8a_0 x^2 + 8a_0 x^4$.

■

6. Consider the differential equation

$$y'' - 2xy' + 2py = 0,$$

where p is a constant.

(6a) Find two linearly independent series solutions valid for all x

(6b) Show that if $p = n$ where n is a non-negative integer, then there is a polynomial solution of degree n.

(6c) Determine the polynomial solutions from (6b) for $n = 0, 1, 2, 3, 4, 5$.

Sol.

(6a) Let us assume a solution of the form

$$y = \Sigma_{n=0}^{\infty} a_n x^n$$

Plugging this into the differential equation, we get

$$\Sigma_{n=0}^{\infty}((n+2)(n+1)a_{n+2} - 2(n-p)a_n)x^n = 0$$

Thus,

$$(n+2)(n+1)a_{n+2} - 2(n-p)a_n = 0, n \in \mathbb{N}$$

Thus, we obtain

$$a_{n+2} = \frac{2(n-p)}{(n+2)(n+1)} a_n$$

Thus, the two linearly independent solutions can be obtained by setting

$$a_0 = 1, a_1 = 0$$

and

$$a_0 = 0, a_1 = 1$$

For the first case, we get

$$a_{2n} = \frac{(-2)^n p(p-2)...(p-(2n-2))}{(2n)!} a_0$$

For the second, we get

$$a_{2n+1} = \frac{(-2)^n (p-1)(p-3)...(p-(2n-1))}{(2n+1)!} a_1$$

(6b) If $p \in \mathbb{N}$, we can find some $n = p$. Thus, $a_{n+2} = 0$ for that n. This means that the series terminates at some finite n meaning that the solution is a polynomial of degree n (Only considering the linearly independent part containing n, we get a polynomial of degree n. We disregard the other part)

(6c) (a) p=n=0 $a_2 = 0$. Thus,

$$y = a_0$$

(b) p=n=1:
$a_3 = 0$.Thus,

$$y = a_1 x$$

(c) p=n=2:
$a_4 = 0$. Thus,

$$y = a_0 - 2a_0 x^2$$

(d) p=n=3:
$a_5 = 0$. Thus,

$$y = a_1 x - \frac{2a_1}{3} x^3$$

(e) p=n=4:
$a_6 = 0$. Thus,

$$y = a_0 - 4a_0 x^2 + \frac{4a_0}{3} x^4$$

(f) p=n=5:
$a_7 = 0$. Thus,

$$y = a_1 x - \frac{4a_1}{3}x^3 + \frac{4a_1}{15}x^5$$

■

7. Arrive at a recurrence relation involving the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

when you plug it in the following differential equation

$$(1 - x^2)y' = y$$

Sol.
Taking a general expression of a power as a solution of the differential equation, we obtain the following

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = 1a_1x^0 + 2a_2x^1 + 3a_3x^2 ... = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$x^2y' = 1a_1x^2 + 2a_2x^3 + 3a_3x^4 ... = \sum_{n=0}^{\infty} n a_n x^{n+1} = \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n$$

Thus putting it in the equation

$$(1 - x^2)y' = y$$

we get

$$\sum_{n=1}^{\infty} [(n+1)a_{n+1} - (n-1)a_{n-1}]x^n + a_1x^0 = \sum_{n=0}^{\infty} a_n x^n$$

Equating coefficients we get the following relations

$$a_1 = a_0$$

$$(n+1)a_{n+1} - (n-1)a_{n-1} = a_n \qquad \text{for all } n \in \mathbb{N}$$

■