MA 207 TSC

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Power Series

Take $c \in \mathbb{R}$. A power series centered at c is given by

$$\sum_{n \geq 0} a_n (x - c)^n$$

where $a_0, a_1, \cdots$ are real numbers dependent on x.

Radius of Convergence

Given any power series

$$\sum_{n\geq 0} a_n(x-c)^n$$

$\exists\ R \in [0,\infty]$ (*note the brackets*), such that the series converges for all $x \in (c-R, c+R)$, and diverges elsewhere. One can find it using the Ratio test:

$$R = \limsup_{n\to\infty} \mid \frac{a_{n+1}}{a_n} \mid$$

Classical and Real Analytic Solution

Classical Solution

Consider the initial value problem for $x \in (a, b)$

$$y'' + P(x)y' + Q(x)y = R(x)$$
$$y(x_0) = y_0,\ y'(x_0) = y'_0 \text{ for some } \ x_0 \in (a, b)$$

If the functions $P(x), Q(x)$ and $R(x)$ are all continuous on (a, b), then

- $\exists$ a unique solution $y : (a, b) \longrightarrow \mathbb{R}$
- $y(x)$ is twice continuously differentiable in (a, b)

A point $x_0 \in (a, b)$ is called an ordinary point of the differential equation if $P(x)$ and $Q(x)$ are *real analytic* at x_0.

Classical and Real Analytic Solution

Real Analytic Solution

Let $x_0 \in (a, b)$ be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0,\ y'(x_0) = y_0' \text{ for some } \ x_0 \in (a, b)$$

Then

- $\exists$ a unique solution $y(x)$ real analytic at x_0.
- If the power series expansions (centered at x_0) of $P(x)$ and $Q(x)$ have radii R_1 and R_2, respectively, then the radius of convergence of $y(x)$ is $\min(R_1, R_2)$

Problems

1. RoC of $\sum_{n \text{ is prime}} x^n$
   Answer:1
2. RoC of $\sum_n n! x^n$
   Answer: 0
3. RoC of $\sum_n \frac{(-1)^n}{n!} x^{n^2}$
   Answer:1

Legendre Equation

The Legendre equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \qquad x \in (-1,1)$$

Use the power series method to solve this equation. We get the recurrence relation between the coefficients to be

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n \qquad n \geq 0$$

Odd and even coefficients don't interact.

Linearly Independent solutions

$$y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{p(p-2)\ldots(p-2(n-1))(p+1)(p+3)\ldots(p+2n-1)}{(2n)!} x^{2n}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{(p-1)(p-3)\ldots(p-(2n-1))((p+2)(p+4)\ldots(p+2n))}{(2n+1)!} x^{2n+1}$$

y_1 is even and y_2 is odd.
y_1 and y_2 are linearly independent
Questions?

The series we saw in the previous slides terminates for $p \in \mathbb{N}$.
The solutions obtained in this case are the Legendre Polynomials.
We define

Boundary condition on P_n

$$P_n(1) = 1 \qquad \forall n \in \mathbb{N}$$

Orthogonality

$$\int_{-1}^{1} P_m(x) P_n(x) dx \propto \delta_{mn}$$

Interlude: Inner Product Spaces

Definition: Inner Product Space

A vector space V is said to be an inner product space if $\exists F : V \times V \to \mathbb{R}$, such that

$$\begin{aligned} F(au + w, v) &= aF(u, v) + F(w, v) \\ F(u, aw + v) &= aF(u, w) + F(u, v) \\ F(u, v) &= F(v, u) \\ F(u, u) &\geq 0 \qquad F(u, u) = 0 \iff u = 0 \end{aligned}$$

Orthogonality

u and v are said to be orthogonal under F if

$$F(u, v) = 0 \qquad u, v \in V$$

Interlude: Inner Product Spaces

Let us define $\mathbb{P}(x)$ to be the space of all polynomials over a domain D.

Inner Product of polynomials

$$F(p(x), q(x)) = \int_D p(x)q(x)dx \qquad p(x)q(x) \in \mathbb{P}(x)$$

This is used to observe orthogonality of the Legendre Polynomials

Legendre Polynomials

Rodrigues Formula

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n \qquad n \in \mathbb{N}$$

Orthogonality of Legendre polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

Here, $m, n \in \mathbb{N}$

Legendre Expansion Theorem

Let $\mathbb{P}_n$ denote the space of all polynomials with degree atmost n.We have

$$span(P_0, \dots P_n) = \mathbb{P}_n$$

Approximation Property

Let q(x) be a polynomial of degree atmost n. Then

$$q(x) = \sum_{k=0}^{n} \eta_k P_k(x)$$

$$\eta_k = \frac{2k+1}{2} \int_{-1}^{1} q(x) P_k(x) dx$$

Legendre Expansion Theorem

The same formula we saw in the last slide applies to all functions in $C^1[-1,1]$. We also get

Legendre Expansion Theorem

The series we saw in the last slide can be defined for all function in $C^1[-1,1]$ by

$$f(x) = \sum_{n=0}^{\infty} \eta_n P_n(x)$$

This converges to $\frac{1}{2}(f^+(x) + f^-(x))$ at all points in $(-1,1)$. It converges to $f^+(-1)$ and $f^-(1)$ at -1 and 1 respectively.

We want to minimize $\int_{-1}^{1} |f_n(x) - f(x)|^2 dx$.

Least Squares approximation

The best approximation to n^{th} degree to a function(by least squares) is given by

$$f_n(x) = \sum_{k=0}^{n} \eta_k P_k(x)$$

$$\eta_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

Questions?

Zeroes of Legendre Polynomials

Application of Rolle's Theorem

If $f : \mathbb{R} \to \mathbb{R}$ has zeroes of order n at a and b, then $f^{(n)}$ has n zeroes in (a, b)

Zeroes of Legendre Polynomials

P_n has n zeroes in $(-1, 1)$

Singular Points

A point $x_0 \in (a, b)$ is called a singular point of the differential equation $y'' + P(x)y' + Q(x)y = 0$, if one/both of the coeffecient functions $P(x)$ and $Q(x)$ is/are not real analytic at x_0. There are two types of singular points:

- Regular
- Irregular

Regular Singular Point

A singular point $x_0 \in (a, b)$ is classified as regular if the functions $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are real analytic at x_0.

A singular point that is *NOT* regular is called irregular.

Cauchy-Euler Equation

Consider the differential equation

$$y'' + \frac{p}{x}y + \frac{q}{x^2}y = 0 \text{ for } \ x > 0$$

Here

$$P(x) = \frac{p}{x}$$
$$Q(x) = \frac{q}{x^2}$$

Make the change of variables $z = \ln(x)$. The Cauchy-Euler equation becomes

$$\frac{d^2}{dz^2}y + (p-1)\frac{d}{dz}y + qy = 0$$

Since it is a differential equation with constant coefficients, we seek soutions of the form $y = e^{\lambda z}$. This gives two solutions

- $e^{\lambda_1 z} = x^{\lambda_1}$ and $e^{\lambda_2 z} = x^{\lambda_2}$ when $\lambda_1 \neq \lambda_2$
- $e^{\lambda_1 z} = x^{\lambda_1}$ and $ze^{\lambda_1 z} = x^{\lambda_1}\ln(x)$

where λ_1 and λ_2 are the roots of the auxiliary equation.

Frobenius Solution

When $x_0 \in (a, b)$ is a ergular singular point, $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ can be expanded as power series. Let the series expansions be

$$(x - x_0)P(x) = \sum_{n \geq 0} p_n (x - x_0)^n$$

$$(x - x_0)^2 Q(x) = \sum_{n \geq 0} q_n (x - x_0)^n$$

Look for solutions of the form

$$y = (x - x_0)^m \sum_{n \geq 0} a_n (x - x_0)^n$$

Here $a_0 \neq 0$.

Indicial equation: $f(m) = m(m - 1) + p_0 m + q_0 = 0 \implies$ zeroes: m_1 and m_2 $(\leq m_1)$.

Frobenius Method

The guaranteed solution corresponds to $m = m_1$, i.e., the larger root of the indicial equation. We substitute

$$y = (x - x_0)^{m_1} \sum_{n \geq 0} a_n (x - x_0)^n$$

in the differential equation. The recurrence relation has an easy to remember form:

$$a_n = -\frac{\sum_{n=0}^{n-1} p_{n-i}(i + m_1)a_i + \sum_{n=0}^{n-1} q_{n-i}a_i}{f(n + m_1)}$$

For finding the second solution we need to first look at the difference between $m_1 - m_2$. If $m_1 - m_2 \notin \{0\} \cup \mathbb{Z}^+$, then the second solution can also be obtained using the above formula by changing m_1 to m_2!

Frobenius Method

When the difference between the roots of the indicial equations is an integer, then we have to build the second solution using the following recipe:

$$y_2(x) = v(x)y_1(x)$$

Let $m_1 - m_2 = k - 1$ for some $k \in \mathbb{N}$. $v(x)$ satisfies the following relation:

$$\begin{aligned}
v'(x) &= \frac{1}{y_1^2(x)} e^{-\int P(x)dx} \\
&= \frac{1}{y_1^2(x)} e^{-\int \frac{p_0}{x} + p_1 + p_2 x^2 + \cdots dx} \\
&= \frac{1}{x^{2m_1 + p_0}(a_0 + a_1 x + a_2 x^2 + \cdots)} e^{-(p_1 x + p_2 \frac{x^2}{2} + \cdots)} \\
&= \frac{1}{x^k} g(x)
\end{aligned}$$

Let the series expansion of $g(x)$ be $\sum_{n \geq 0} b_n x^n$. We obtain the second solution y_2 of the form

Bessel Equation

Bessels's equation of order p

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad x > 0$$

Using the Frobenius method gives us a solution of the form

$$y = x^p \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = -\frac{a_{n-2}}{n(n+2p)} \qquad n \geq 2$$

along with

$$a_1 = 0$$

This gives

$$a_{2k+1} = 0 \qquad k = 0, 1, \ldots$$

Thus, only even integer powers show up along with p

Interlude: Gamma Function

Definition

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad z > 0$$

Recurrence relation

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

This allows us to extend the definition of the Gamma function to $\mathbb{R} \backslash \mathbb{Z}^-$

Relation between Gamma function and factorial

$$\Gamma(n+1) = n! \qquad n \in \mathbb{N}$$

Gamma function at negative integers

We define

$$\frac{1}{\Gamma(z)} = 0 \qquad z \in \mathbb{Z}^-$$

Bessel Functions

Bessel Function of first kind of order p

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

J_{-p}

$$J_{-p} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

$p \in \mathbb{N}$

$$J_{-p}(x) = (-1)^p J_p(x)$$

Properties of Bessel Functions

Identities

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^p J_p(x)\Big) = x^p J_{p-1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-p} J_p(x)\Big) = -x^{-p} J_{p+1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} J_p(x) + \frac{p}{x} J_p = J_{p-1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} J_p(x) - \frac{p}{x} J_p = -J_{p+1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} J_p(x) = \frac{1}{2}\Big(J_{p-1}(x) - J_{p+1}(x)\Big)$$

$$\frac{p}{x} J_p = \frac{1}{2}\Big(J_{p-1}(x) + J_{p+1}(x)\Big)$$

Zeroes of $J_{p-1}, J_p and J_{p+1}$

Take $p \geq 0$.
There exists precisely one zero of J_{p+1} between two consecutive zeroes of J_p
There exists exactly one zero of J_{p-1} between two consecutive zeroes of J_p

Theorem: Finite zeroes in a closed interval

Let u(x) be a non-trivial solution of $u'' + qu = 0$ where $q(x) > 0 \forall x \in [a, b]$ is a continuous function.
Then, $u(x)$ has atmost finitely many zeroes in $[a, b]$.

Zeroes of Bessel Functions

Theorem: Infinite Zeroes

Let u(x) be a non-trivial solution of $u'' + qu = 0$ where $q(x)$ is a continuous function. If we have $q(x) > 0$ for $x > x*$ for some $x*$, and

$$\int_{x*}^{\infty} q(x)dx = +\infty$$

Then $u(x)$ has infinitely many zeroes in $(0, \infty)$.

Generating Functions

Formula of Schlomilch

$$exp\left(\frac{x}{2}(t-\frac{1}{t})\right) = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left(t^n + \frac{(-1)^n}{t^n}\right) \qquad t \in \mathbb{C}*$$

Another result

$$cos(xsin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)cos(2p\theta)$$

$$sin(xsin\theta) = 2\sum_{k=1}^{\infty} J_{2p+1}(x)sin((2p+1)\theta)$$

$$\frac{1}{\pi}\int_0^{\pi} (cos(xsin\theta)cos(n\theta) + sin(xsin\theta)sin(n\theta)) = J_n(x)$$

Orthogonality of Bessel Functions

Orthogonality of Scaled Bessel Functions

Let $\lambda_{p,i}$ be the zeroes of the Bessel function of order p. Then

$$\int_0^1 xJ_p(\lambda_{p,m}x)J_p(\lambda_{p,n}x)dx = \frac{1}{2}(J_{p+1}(\lambda_{p,n}))^2\delta_{mn}$$

Orthogonality if Bessel Functions

Least Squares Approximation

The squared error of f ($\int_0^1 |f|^2 dx$ is finite) is minimized if

$$f(x) = \sum_{k=0}^{m} a_k J_p(\lambda_{p,k} x)$$

$$a_k = \frac{2}{(J_{p+1}(\lambda_{p,k}))^2} \int_0^1 x f(x) J_p(\lambda_{p,k} x) dx$$

We get $f(x)$ in the limit as $m \to \infty$
The sum gives the value $\frac{1}{2}(f^+(x) + f^-(x))$ for $x \in (0,1)$. We have the sum to converge to 0 for $x = 1$, $f^+(0)$ for $x = 0, p = 0$ and 0 for $x = 0, p > 0$

The Heat Equation

The heat equation is of the form

$$u_t = u_{xx}$$

This can have Dirichlet or Neumann Boundary conditions.
Methods to solve it:
Separation of variables.
For non-homogeneous boundary conditions, add a function linear in x to the previously known solution with the time dependent boundary conditions and solve a heat equation with a source term.

Sturm-Liouville eigenvalue problem

Sturm Liouville Problem

These are eigenvalue problems of the form

$$(p(x)y'(x))' + q(x)y(x) = \lambda y(x)$$
$$k_1 y(a) + k_2 y'(a) = 0$$
$$k_3 y(b) + k_4 y'(b) = 0$$

The constants are such that k_1 and k_2 or k_3 and k_4 are not zero simultaneously.

The boundary condition is said to be symmetric if the following condition is met:

$$\big(p(x)(u'v - v'u)\big)\,\big|_a^b = 0$$

for all functions satisfying the boundary conditions.

Sturm-Liouville eigenvalue problem

Orthogonality

If y_1 and y_2 are two solutions associated with different eigenvalues (λ_1 and λ_2) of Sturm-Liouville problem with symmetric boundary conditions

$$\begin{gathered}
(p(x)y'(x))' + q(x)y(x) = \lambda y(x) \\
k_1 y(a) + k_2 y'(a) = 0 \\
k_3 y(b) + k_4 y'(b) = 0
\end{gathered}$$

then

$$\int_a^b y_1(x) y_2(x) dx = 0$$

Wave Equation

$$u_{tt} = c^2 u_{xx}$$

Variety of boundary conditions as before.
Methods of finding the solution:

- D'Alembert's method
- Separation of variables

Maximum Principle

Maximum Principle

Let $\mathcal{L}$ be a differential operator defined on $C^2[0,1]$ as $\mathcal{L}y(x) := -(p(x)y'(x))' + q(x)y(x)$. Let $u(x)$ be such that

$$\mathcal{L}u(x) \leq 0 \quad \text{for } x \in (0,1)$$

- If the coefficient $q(x) \equiv 0$, then

$$\max_{x \in (0,1)} u(x) = \max\{u(0), u(1)\}$$

- If $q(x) \geq 0$ for $x \in (0,1)$, then

$$\max_{x \in (0,1)} u(x) \leq \max\{u(0), u(1), 0\}$$

The minimum principle for u is the maximum principle for $-u$.

Thank You and All the Best!

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