MA 207 Autumn 2022
Tutorial Sheet 3

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1. Consider the differential equation:

$$xy'' + (1-x)y' + py = 0 \qquad \text{for } x > 0.$$

Here $p > 0$ is a real number.

(1a) Show that the point $x = 0$ is a regular singular point of the above differential equation.

(1b) Write down the indicial equation associated with the above differential equation and find its root(s).

(1c) Derive an expression for the guaranteed Frobenius series solution to the above differential equation.

(1d) Deduce that the expression for the solution obtained in Question (1c) reduces to a polynomial when p is a positive integer.

(1e) Write down the polynomial solutions of the above differential equation when $p = 1$, when $p = 2$ and when $p = 3$.

Sol.

(1a) Write the differential equation in the standard form

$$y'' + \frac{(1-x)}{x}y' + \frac{p}{x}y = 0$$

Here, $P(x)$ and $Q(x)$ are $(1-x)/x$ and p/x, respectively. Clearly $xP(x) = 1-x$ and $x^2Q(x) = px$ are both analytic at $x = 0$. Thus, $x = 0$ is a regular singular point.

(1b) Here $p(x) = 1 - x$ and $q(x) = px$. The indicial equation looks like

$$f(m) = m(m-1) + 1 \times m + 0 = m^2 = 0$$

The roots are $m_1 = 0 = m_2$.

(1c) The guaranteed Frobenius solution can be obtained in the following way: We know that the recurrence relation for the larger root m_1 (in our case both m_1 and m_2 are equal) is

$$a_n = \frac{-\sum_{i=0}^{n-1} p_{n-i}(i+m_1)a_i - \sum_{i=0}^{n-1} q_{n-i}a_i}{f(n+m_1)}$$

In our case, $p_0 = 1$, $p_1 = -1$ and $q_1 = p$ are the only non-zero terms. This gives the following recurrence relation

$$\begin{aligned} a_n &= \frac{(n-1-0)a_{n-1} - pa_{n-1}}{f(n+0)} \\ &= \frac{n-1-p}{n^2}a_{n-1} \end{aligned}$$

This recurrence relation results in the following closed form

$$a_n = \frac{(n-1-p)(n-2-p)\cdots(1-p)(0-p)}{(n!)^2}$$

The Frobenius solution would look like

$$y(x) = \sum_{n\geq 0} \frac{(n-1-p)(n-2-p)\cdots(1-p)(0-p)}{(n!)^2} x^n$$

(1d) Clearly, when p is a positive integer, all coefficients a_n for $n \geq p+1$ vanish, leaving us with a polynomial solution.

(1e) When $p = 1$, we have $a_1 = \frac{0-1}{(1!)^2} a_0 = -a_0$. Thus, the polynomial solution is linear, $y(x) = a_0 - a_0 x$. Similarly, when $p = 2$, $a_1 = -2a_0$ and $a_2 = \frac{(1-2)(0-2)}{(2!)^2} a_0 = a_0/2 \implies y(x) = a_0 - 2a_0 x + \frac{a_0}{2} x^2$.

■

2. Consider the following differential equation:

$$x^\alpha y'' + \sin(x) y = 0 \qquad \text{for } x > 0.$$

Here α is a positive parameter taking values from the set

$$\mathcal{S} := \{1, 2, 3, 4, 5\}.$$

For what values of $\alpha \in \mathcal{S}$, the point $x = 0$ is

- an ordinary point of the above differential equation?
- a regular singular point of the above differential equation?
- an irregular singular point of the above differential equation?

Sol. Rewrite the given differential equation as

$$y'' + \frac{\sin x}{x^\alpha} y = 0$$

So, $P(x) = 0$ and $Q(x) = \sin x / x^\alpha$.

- For $\alpha = 1$, $Q(x)$ is analytic at $x = 0$, hence it is an ordinary point in this case. One can easily check that for any other value of $\alpha \in \mathcal{S}$, $Q(x)$ is not analytic.
- Now, we demand $x = 0$ to be a regular singular point $\implies$ $Q(x)$ is not analytic but $x^2 \times Q(x)$ is. Clearly, $\alpha = 2, 3$ satisfy this condition.
- For the remaining values of $\alpha \in \mathcal{S}$, i.e., $4, 5$, $x = 0$ is is an irregular singular point.

■

3. Find the indicial equation and its roots for each of the following differential equations:

$$x^3 y'' + \Big(\cos(2x) - 1\Big) y' + 2xy = 0 \qquad \text{for } x > 0.$$

$$4x^2 y'' + \Big(2x^4 - 5x\Big) y' + \Big(3x^2 + 2\Big) y = 0 \qquad \text{for } x > 0.$$

Sol. Rewrite the first differential equation as

$$x^3 y'' + \Big(\cos(2x) - 1\Big) y' + 2xy = 0 \qquad \text{for } x > 0.$$

as

$$x^2y'' + \left(x\frac{\cos(2x)-1}{x^2}\right)y' + 2y = 0 \qquad \text{for } x > 0.$$

$p(x) = (\cos(2x)-1)/x^2$ and $q(x) = 2$. Taylor expanding the numerator of $p(x)$ gives

$$p(x) = \frac{-2^2}{2!} + \frac{2^4x^2}{4!} - \cdots$$

This implies that $p_0 = -2$ and $q_0 = 2 \implies$ the indicial equation is as follows:

$$f(m) = m(m-1) - 2m + 2 = m^2 - 3m + 2 = 0$$

Its roots are $m_1 = 2$ and $m_2 = 1$.

Similarly, write the second differential equation as

$$x^2y'' + x\left(\frac{2x^3-5}{4}\right)y' + \left(\frac{3x^2+2}{4}\right)y = 0 \qquad \text{for } x > 0.$$

$p_0 = -5/4$ and $q_0 = 1/2 \implies$

$$f(m) = m(m-1) - \frac{5}{4}m + \frac{1}{2} = m^2 - \frac{9}{4}m + \frac{1}{2} = 0$$

The roots of the indicial equation are $m_1 = 2$ and $m_2 = 1/4$. ■

4. Find two linearly independent Frobenius series solutions to the following equation:

$$xy'' + 2y' + xy = 0 \qquad \text{for } x > 0.$$

Sol. Let's write the given differential equation as

$$x^2y'' + x \times 2y' + x^2y = 0$$

$p(x) = 2$ and $q(x) = x^2$. The indicial equation would look like $f(m) = m(m-1) + 2m + 0 = m(m+1) = 0$. The roots are $m_1 = 0$ and $m_2 = -1$. Let's find out the guaranteed Frobenius solution, which corresponds to $m_1 = 0$.

$$a_n = \frac{-\sum_{i=0}^{n-1} p_{n-i}(i+m_1)a_i - \sum_{i=0}^{n-1} q_{n-i}a_i}{f(n+m_1)}$$

In our case, only $p_0 = 2$ and $q_2 = 1$ are the non-zero terms. Thus, the recurrence relation becomes

$$a_n = \frac{-q_2a_{n-2}}{f(n+0)} = -\frac{a_{n-2}}{n(n+1)}$$

The above recurrence relation gives two linearly independent closed forms:

$$a_{2k} = \frac{(-1)^k}{(2k+1)!}a_0$$

and

$$a_{2k+1} = \frac{(-1)^k \times 2}{(2k+2)!}a_1$$

Therefore the two linearly independent solutions are

$$y_1(x) = \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!}x^{2k}$$

and

$$y_2(x) = \sum_{k\geq 0} \frac{(-1)^k \times 2}{(2k+2)!}x^{2k+1}$$

■

5. Here is the expression for the Bessel function of order $p \geq 0$:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

(5a) Employing the ratio test, deduce that the above infinite series converges and that its radius of convergence is infinity.

(5b) Find the values of $J_0(0)$ and $J_p(0)$ for $p > 0$.

Sol.

(5a) Ratio test: For the series to converge, we need

$$\lim_{n\to\infty} \left| \frac{n!\Gamma(n+p+1)}{(n+1)!\Gamma(n+p+2)}\left(\frac{x}{2}\right)^2 \right| < 1$$

Using the property $\Gamma(n+1) = n\Gamma(n)$, it can be easily shown that the limit is 0 $\forall x \in \mathbb{R}$. Therefore, the radius of convergence is infinite!

(5b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \implies J_0(0) = \frac{(-1)^0}{(0!)^2} = 1$. Similarly, we have

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n}$$

Therefore, $J_p(0) = 0$ for $p > 0$.

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6. Consider the differential equation:

$$y'' + P(x)y' + Q(x)y = 0 \qquad \text{for } x > 0.$$

Here $P(x)$ and $Q(x)$ are continuous functions. Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the above equation. While answering the following questions, it may be useful to recall that the Wronskian associated with two linearly independent solutions of a differential equation never vanishes.

(6a) Does there exist a point $x^* > 0$ such that $y_1(x^*) = y_2(x^*) = 0$? Justify your answer.

(6b) Let x_1 and x_2 be two consecutive zeros of $y_2(x)$. Does there exist a point $x^{**} \in (x_1, x_2)$ such that $y_1(x^{**}) = 0$? Justify your answer.

(6c) How many such points $x^{**} \in (x_1, x_2)$ (found in Question (1b)) exist?

Sol.

(6a) As y_1 and y_2 are linearly independent solutions, their Wronskian cannot be zero. If they have a common root, the Wronskian would become zero at that point and thus, would be zero everywhere which cannot be the case due to linear independence.(We have assumed analyticity of y_1 and y_2)

(6b) If possible, let y_1 not have a zero in the interval (x_1, x_2). This implies that $\frac{y_2}{y_1}$ is a well defined continuous function on the interval. Furthermore, it is differentiable with

$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1^2}$$

Where

$$W = y_1 y_2' - y_1' y_2$$

Now, since $\frac{y_1}{y_2}$ is continuous and differentiable on the interval, we can use Rolle's theorem. Notice that $\frac{y_2}{y_1}$ is 0 at x_1 and x_2. Thus, we have a $x_3 \in (x_1, x_2)$ with

$$\left(\frac{y_2}{y_1}\right)' = 0$$

Thus, we get that $W(x_3) = 0$. This is a contradiction since W can never be 0. Thus, we can conclude that y_1 has atleast one zero in the interval (x_1, x_2).

(6c) We can easily conclude that y_1 has only one zero in the interval (x_1, x_2).(If y_1 had multiple zeroes, with the same logic as 6b, we could find a zero of y_2 between them which contradicts the fact that x_1 and x_2 are consecutive zeroes of y_2.)

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7. Let $y(x)$ be a solution to the Bessel's equation of order p:

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \qquad \text{for } x > 0.$$

Introduce a new unknown $u : (0, \infty) \to \mathbb{R}$ as follows:

$$u(x) := \sqrt{x}\, y(x) \quad \text{for } x > 0.$$

Show that $u(x)$ satisfies the differential equation:

$$u''(x) + \left(1 + \frac{1 - 4p^2}{4x^2}\right) u(x) = 0 \qquad \text{for } x > 0.$$

Sol.
We can rewrite y(x) as $u(x)/\sqrt{x}$ and follow chain rule of differentiation to obtain y"(x) and y'(x). This gives us the following

$$y'(x) = \frac{u'}{\sqrt{x}} - \frac{2u}{x^{3/2}}$$

$$y''(x) = \frac{u''}{\sqrt{x}} - \frac{u'}{x^{3/2}} + \frac{3u}{4x^{5/2}}$$

Substituting the same into the initial DE gives us the required DE. ■

8. Here is the expression for the Bessel function for order $p \geq 0$:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p + n + 1)} \left(\frac{x}{2}\right)^{2n+p}$$

Mimicking the calculations performed in the lectures, show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}} \cos(x).$$

Sol.
Taking p as -1/2 in the expression for Bessel function gives us

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + 1/2)} \left(\frac{x}{2}\right)^{2n-1/2}$$

$$= \left(\frac{2}{x}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1/2)} \left(\frac{x}{2}\right)^{2n}$$

$$= \left(\frac{2}{x}\right)^{1/2} \left[\frac{1}{0!\,\Gamma(1/2)} \left(\frac{x}{2}\right)^{0} + \frac{-1}{1!\,\Gamma(1+1/2)} \left(\frac{x}{2}\right)^{2} + \frac{1}{2!\,\Gamma(2+1/2)} \left(\frac{x}{2}\right)^{4} + \frac{1}{3!\,\Gamma(3+1/2)} \left(\frac{x}{2}\right)^{6} + \ldots\right]$$

$$= \left(\frac{2}{x}\right)^{1/2} \left[\frac{1}{0!\,\Gamma(1/2)} \left(\frac{x}{2}\right)^{0} + \frac{-1}{1!\,\frac{\Gamma(1/2)}{2}} \left(\frac{x}{2}\right)^{2} + \frac{1}{2!\,\frac{1x3\Gamma(1/2)}{4}} \left(\frac{x}{2}\right)^{4} + \frac{1}{3!\,\frac{1x3x5\Gamma(1/2)}{8}} \left(\frac{x}{2}\right)^{6} + \ldots\right]$$

We know that $\Gamma(1/2) = \sqrt{\pi}$. Moreover,

$$n!\,2^n = \prod_{r=1}^{n} 2r$$

Therefore we get

$$\prod_{r=1}^{n} 2r \prod_{m=1}^{n} (2m-1) = 2n!$$

Thus the final reult now becomes $\sqrt{\frac{2}{x\pi}}\,\cos(x)$ ■

9. Consider the following functions for $p \in \mathbb{R}$:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}.$$

Mimicking the calculations performed in lectures, establish the following six identities:

$$\frac{d}{dx}\Big(x^p J_p(x)\Big) = x^p J_{p-1}(x)$$
$$\frac{d}{dx}\Big(x^{-p} J_p(x)\Big) = -x^{-p} J_{p+1}(x)$$
$$\frac{d}{dx} J_p(x) + \frac{p}{x} J_p = J_{p-1}(x)$$
$$\frac{d}{dx} J_p(x) - \frac{p}{x} J_p = -J_{p+1}(x)$$
$$\frac{d}{dx} J_p(x) = \frac{1}{2}\Big(J_{p-1}(x) - J_{p+1}(x)\Big)$$
$$\frac{p}{x} J_p = \frac{1}{2}\Big(J_{p-1}(x) + J_{p+1}(x)\Big)$$

Sol. Using the theorem $\Gamma(z+1) = z\Gamma(z)$, chain rule of differentiation and a bit of manipulation it is fairly simple to obtain these results.

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10. Recall that the Gamma function is defined as

$$\Gamma(z) := \int_0^{\infty} t^{z-1}\, e^{-t}\, dt \qquad \text{for } z > 0.$$

Recall further that the Gamma function satisfies the functional relation:

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{for } z > 0.$$

Define a function $\psi : (0,\infty) \to \mathbb{R}$ as follows

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{for } z > 0,$$

where $\Gamma'(z)$ denotes the derivative.

(10a) Does there exist a point $z^* > 0$ such that $\Gamma(z^*) = 0$? Justify your answer.

(10b) Show that the function ψ defined above satisfies the equation:

$$\psi(z+1) = \frac{1}{z} + \psi(z) \qquad \text{for } z > 0.$$

Sol.

10a If there exists such a $z*$, then there exists a whole sequence of numbers where $\Gamma(z)$ becomes 0. However, for all $z > 0$, the integrand is always positive. Furthermore, there exists a subset of the domain of integration where the integrand is strictly positive. Thus, we can conclude that the gamma function cannot take the value 0.

10b we have

$$\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z)$$

Substituting this expreession into thte one for $\Psi(z+1)$, we get

$$\psi(z+1) = \frac{1}{z} + \psi(z) \qquad \text{for } z > 0$$

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11. Using the first two identities from Question 9 above and employing the Rolle's theorem, show that

(11a) There exists precisely one zero of J_{p-1} between any two consecutive zeros of J_p.

(11b) There exists precisely one zero of J_{p+1} between any two consecutive zeros of J_p.

Sol.

11a From the first identity and Rolle's theorem, we get that J_{p-1} needs to have atleast one zero between the two consecutive zeroes of J_p. Now, if we assume that there are more than 1 zeroes of J_{p-1} in the concerned region, we can obtain using the second identity and Rolle's Theorem that there exists a zero of J_p between the two zeroes of J_{p-1} in the concerned region. This contradicts the fact that the region in concern is between two consecutive zeroes of J_p. Thus, we can conclude that there exists precisely one zero of J_{p-1} between any two consecutive zeros of J_p.

11b Proceed the same way as done in 11a

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12. While answering the following questions, the sixth identity from Question 9 above may be useful. Let $J_p(x)$ denote the Bessel function of order $p \geq 0$.

(12a) Let $x^* > 0$ be such that $J_p(x^*) = 0$. Show that both $J_{p+1}(x^*)$ and $J_{p+2}(x^*)$ have the same sign.

(12b) Show that there is precisely one zero of J_{p+2} between any two consecutive positive zeros of J_p and vice versa.

Sol.

12a This follows directly from the sixth identity from Question 9

12b This follows immediately from the intermediate value theorem after looking at the sign of J_{p+2} at the zeroes of J_p. Same argument can be used to show the vice-versa.

13. While answering the following questions, the identities from Question 9 above may be useful.

(13a) Show that for all non-negative integers $p \geq 0$, we have

$$\frac{d}{dx}\Big((J_p(x))^2 + (J_{p+1}(x))^2 \Big) = \frac{p}{x}(J_p(x))^2 - \frac{p+1}{x}(J_{p+1}(x))^2$$

(13b) Summing the equality obtained in Question (13a) over all non-negative integers, show that

$$(J_0(x))^2 + 2\sum_{p=1}^{\infty}(J_p(x))^2 = 1.$$

(13c) Hence deduce that for all x,

$$|J_0(x)| \leq 1 \quad \text{and} \quad |J_p(x)| \leq \frac{1}{\sqrt{2}}.$$

Sol.

13a Follows directly from the identities

13b Summing over the equation, we get that

$$\frac{d}{dx}((J_0(x))^2 + 2\sum_{p=1}^{\infty}(J_p(x))^2) = 0$$

Thus, we get that $(J_0(x))^2 + 2\sum_{p=1}^{\infty}(J_p(x))^2$ must be a constant. Thus, looking at its value at $x = 0$, we get

$$(J_0(x))^2 + 2\sum_{p=1}^{\infty}(J_p(x))^2 = 1$$

13c This follows from 13b by imposing that each term must be less than 1.

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14. Consider the differential equation:

$$x^2y'' + (3x-1)y' + y = 0 \qquad \text{for } x > 0.$$

(14a) Show that $x = 0$ is an irregular singular point of the above differential equation.

(14b) Let us, nevertheless, attempt a solution of the form:

$$a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots \qquad \text{with } a_0 \neq 0.$$

Plugging the above expansion for the unknown in the above differential equation, show that the exponent m must be zero.

(14c) Determine the coefficients $a_1, a_2, \dots$ in terms of a_0.

(14d) Does the series solution, thus built, have a positive radius of convergence?

Sol.

14a Follows from the limit definition of irregular points.

14b Plugging the given form into the equation, we get

$$(m+k+1)^2 a_k - (m+k+1)a_{k+1} = 0$$

Since $m+k+1$ is not always 0, we can in general write

$$a_{k+1} = (m+k+1)a_k$$

This gives

$$a_k = \frac{\Gamma(m+k+1)}{\Gamma(m+1)} a_0$$

This assumes that m isn't a negative integer. Now, if we substitute the form of the solution in $x^2y'' + (3x-1)y' + y = 0$ and look at the coefficient of x^{m-1}, we get the coefficient to be $-a_0 m$. Thus, we must have $m = 0$.

14c We can now obtain

$$a_k = k! a_0$$

14d The radius of convergence is 0. However, it is well defined at $x = 0$ and is thus acceptable.

■