MA 207 Autumn 2022
Tutorial Sheet 2

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1. Let $P_{58}(x)$ denote the Legendre polynomial which solves the Legendre's equation

$$(1-x^2)y'' - 2xy' + 58(59)y = 0 \quad \text{for } x \in (-1,+1).$$

The value of the integral

$$\int_{-1}^{+1} x^{59} P_{58}(x)\,dx$$

is __________ and the value of the integral

$$\int_{-1}^{+1} P_{58}(x)\,dx$$

is __________

Sol. We are given the following theorem

$$\int_{-1}^{+1} P_n(x)P_m(x)\,dx = 0 \quad \forall \quad m \neq n$$

We also know that x^m is some linear combination of the Legendre polynomials P_m, P_{m-2}, P_{m-4}...
We can therefore express x^59 as a linear combination as follows

$$x^{59} = \sum_{i=1,3,5,7..}^{59} \alpha_i P_i(x)$$

The sum skips over $P_{58}(x)$ and hence the integral will 0.

Similarly for the second part as $P_0(x)$=1 and the integral now becomes

$$\int_{-1}^{+1} P_0(x)\,P_{58}(x)\,dx$$

This integral is 0 by the theorem.

■

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that it has at least 7 distinct zeros. Using the fact that $e^{\frac{x}{2}} f(x)$ has the same number of zeros as $f(x)$, prove that $f + 4f' + 4f''$ has at least 5 distinct zeros.

Sol.

$$f + 4f' + 4f'' = f + 2f' + 2f' + 4f''$$

by inspection we can interpret $f + 2f'$ as

$$e^{-x/2}\frac{d}{dx}(2\,f\,e^{x/2})$$

similarly by inspection we can interpret $2f' + 4f''$ as

$$e^{-x/2}\frac{d}{dx}(4\,f'\,e^{x/2})$$

This gives us

$$e^{-x/2}\frac{d}{dx}(4\,f'\,e^{x/2}) + e^{-x/2}\frac{d}{dx}(2\,f\,e^{x/2}) = e^{-x/2}\frac{d}{dx}(4\,f'\,e^{x/2} + 2\,f\,e^{x/2})$$

Once again it can be seen that this is in fact the differential of the following function wrt x

$$4\,f'\,e^{x/2} + 2\,f\,e^{x/2} = \frac{d}{dx}(4\,f\,e^{x/2})$$

Which gives us the final relation

$$f + 4f' + 4f'' = e^{-x/2}\frac{d^2}{dx^2}(4\,f\,e^{x/2})$$

As f has atleast 7 roots, $f\,e^{x/2}$ also has atleast distinct 7 roots. Furthermore, by the mean value theorem we get that the double differential of this function has atleast distinct 5 roots. ■

3. The formula of Rodrigues for the Legendre polynomial of degree n is given by

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$

Considering the analogous expressions for the Legendre polynomials $P_{n+1}(x)$ and $P_{n-1}(x)$ followed by their differentiation, show that

$$P_n(x) = \frac{1}{2n+1}\Big(P'_{n+1}(x) - P'_{n-1}(x)\Big) \quad \text{for } n = 1, 2, \ldots$$

Sol. We will show that simplifying the RHS gives the LHS.

$$P_{n+1}(x) = \frac{1}{(n+1)!\,2^{n+1}}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}$$

$$P'_{n+1}(x) = \frac{1}{(n+1)!\,2^{n+1}}\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^{n+1}$$

$$P_{n-1}(x) = \frac{1}{(n-1)!\,2^{n-1}}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

$$P'_{n-1}(x) = \frac{1}{(n-1)!\,2^{n-1}}\frac{d^{n}}{dx^{n}}(x^2-1)^{n-1}$$

This gives us P'_{n+1} - P'_{n-1} as

$$= \frac{1}{(n-1)!\,2^{n-1}}\frac{d^n}{dx^n}[\frac{d^2}{dx^2}\frac{(x^2-1)^{n+1}}{n(n+1)4} - (x^2-1)^{n-1}]$$

$$= \frac{1}{(n-1)!\,2^{n-1}}\frac{d^n}{dx^n}[\frac{(x^2-1)^n}{2n} + x^2(x^2-1)^{n-1} - (x^2-1)^{n-1}]$$

$$= \frac{2n+1}{(n-1)!\,2^{n-1}\,2n}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= (2n+1)P_n(x)$$

■

4. Let $q(x)$ be a polynomial of degree $n \geq 1$ such that

$$\int_{-1}^{+1} x^k q(x)\,dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1.$$

Using the orthogonality property of the Legendre polynomials show that

$$q(x) = \alpha P_n(x)$$

for some constant α, where $P_n(x)$ is the Legendre polynomial of degree n.

Sol. Since

$$\int_{-1}^{+1} x^k q(x)\,dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1$$

We have

$$\int_{-1}^{+1} P_k(x) q(x)\,dx = 0 \quad \text{for } k = 0, 1, \ldots, n-1$$

This is because the $P_k(x)$ are made up of terms orthogonal to $q(x)$. Since q is a polynomial of order n, we can express it as a linear combination of the Legendre Polynomials. Since q is orthogonal to all Legendre polynomials of order less than n,it must be proportional to P_n. ■

5. Let $n \in \mathbb{N}$ and let $\mathbb{P}_n(x)$ denote the collection of all polynomials in x of degree at most n. Let $\mathbb{Q}_n(x)$ be a sub-collection defined as follows:

$$\mathbb{Q}_n(x) := \left\{ p(x) \in \mathbb{P}_n(x) \text{ such that } p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \text{ where } a_0, a_1, \ldots, a_{n-1} \in \mathbb{R} \right\}.$$

Find the value of

$$\min_{q(x) \in \mathbb{Q}_n(x)} \int_{-1}^{+1} |q(x)|^2 \,dx.$$

Find a candidate $\ell(x) \in \mathbb{Q}_n(x)$ which attains the above minimum value.
Furthermore, find the values of

$$\min_{q(x) \in \mathbb{Q}_n(x)} \int_{-1}^{+1} |q(x)|^2 \,dx$$

when $n = 1, 2, 3$.

Sol. We have

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2 - 1)^n$$

In this, the coefficient of x^n is $\binom{2n}{n}\frac{1}{2^n}$. Thus, it we can multiply by $\frac{2^n}{\binom{2n}{n}}$ to make the leading coefficient 1.
$\forall q(x) \in \mathbb{Q}_n$, we have

$$q = \frac{2^n}{\binom{2n}{n}} P_n + \sum_{k=0}^{n-1} c_k P_k$$

Thus, we get

$$\int_{-1}^{+1} |q(x)|^2 \,dx = \frac{2^{2n+1}}{(2n+1)\binom{2n}{n}^2} + \sum_{k=0}^{n-1} \frac{2|c_k|^2}{2k+1}$$

The minimum value of this is clearly $\frac{2^{2n+1}}{(2n+1)\binom{2n}{n}^2}$ occurring for $q(x) = \frac{2^n}{\binom{2n}{n}} P_n(x)$
We have the minimum values for the integral (denoted as I) to be

$$n = 1 : I = \frac{2}{3}$$
$$n = 2 : I = \frac{8}{45}$$
$$n = 3 : I = \frac{8}{175}$$

$\blacksquare$

6. Consider the functions $f_1, f_2, f_3 : [-1, 1] \to \mathbb{R}$ defined as follows:

$$
\begin{aligned}
f_1(x) &:= x \quad \text{for } x \in [-1,1] \\
f_2(x) &:= |x| \quad \text{for } x \in [-1,1]
\end{aligned}
$$

and

$$
f_3(x) := \begin{cases} 0 & \text{for } x \in [-1, 0] \\ x & \text{for } x \in (0, 1] \end{cases}
$$

Write down the series representation (first few (the word *few* is left to your interpretation) terms) for the above three functions in terms of the Legendre polynomials using the recipe given in the Legendre expansion theorem. Among the above three functions, who has the series representation with finite number of non-zero terms?

Sol. Let's list the first few Legendre polynomials.

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}(3x^2 - 1)
\end{aligned}
$$

The Legendre expansion theorem states that if $f(x)$ and $f'(x)$ are piecewise continuous functions then we can write $f(x)$ as

$$
f(x) = \sum_{n \geq 0} c_n P_n
$$

where the coefficients are given by

$$
c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)
$$

Clearly all three given functions are continuous and have piecewise continuous derivatives. Thus, we can apply the Legendre expansion theorem here.

$$
f_1(x) = x = \sum_{n \geq 0} c_n P_n(x)
$$

Let's find the first three coefficients.

$$
\begin{aligned}
c_0 &= \frac{1}{2} \int_{-1}^{1} x dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{3}{2} \int_{-1}^{1} x^2 dx \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{5}{2} \int_{-1}^{1} x \frac{1}{2}((3x^2 - 1) dx \\
&= 0
\end{aligned}
$$

Now let's look at $f_2(x)$.

$$
f_2(x) = |x| = \sum_{n \geq 0} c_n P_n(x)
$$

The first three coefficients are as follows:

$$\begin{aligned} c_0 &= \frac{1}{2}[-\int_{-1}^{0} x dx + \int_{0}^{1} x dx] \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{3}{2}[-\int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx] \\ &= 0 \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{5}{2}[-\int_{-1}^{0} x\frac{1}{2}((3x^2-1)dx + \int_{0}^{1} x\frac{1}{2}((3x^2-1)dx] \\ &= \frac{5}{8} \end{aligned}$$

Similarly, in case of $f_3(x)$, we have

$$f_1(x) = x = \sum_{n\geq 0} c_n P_n(x)$$

The first three coefficients are as follows:

$$\begin{aligned} c_0 &= \frac{1}{2}\int_{0}^{1} x dx \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{3}{2}\int_{0}^{1} x^2 dx \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{5}{2}\int_{0}^{1} x\frac{1}{2}((3x^2-1)dx \\ &= \frac{5}{16} \end{aligned}$$

It can be easily seen that $f_1(x)$ will have finite number of terms in its Legendre expansion ($a_1 = 1$ and rest all coefficients are zero. This is because we know that

$$\int_{-1}^{1} xP_m(x)dx = 0 \quad m \geq 2$$

using the result that the inner product of any polynomical of degree less than m with P_m is zero! ■

7. Let $P_k(x)$ denote the Legendre polynomial of degree k.

(7a) Using the formula of Rodrigues (see Question 3), show that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \qquad \text{for } n = 0, 1, 2, \ldots$$

(7b) Using parity properties of the Legendre polynomials, deduce that

$$\begin{aligned} P_{2n+1}(0) &= 0 \qquad \text{for } n = 0, 1, 2, \ldots \\ P'_{2n}(0) &= 0 \qquad \text{for } n = 0, 1, 2, \ldots \end{aligned}$$

(7c) Using the formula of Rodrigues (see Question 3), show that

$$P_n'(1) = \frac{n(n+1)}{2} \qquad \text{for } n = 0, 1, 2, \ldots$$

Sol.

7a We just need to find the coefficient of x^n in $(x^2-1)^n$. If n is odd, then that coefficient is 0. If $n = 2m$,

$$P_{2m}(0) = \frac{(-1)^m}{2^{2m}}\binom{2m}{m}$$

7b The results follow directly from the fact that P_{2m+1} is an odd function. Same reasoning for P_{2m}'.

7c We have

$$(x^2-1)^n = (x+1)^n(x-1)^n$$

Now, we can look at terms in the derivative which will give a non-zero contribution to $P_n(1)$. Thus, we will have

$$P_n'(x) = \frac{1}{n!2^n}\left(n!\binom{n+1}{2}(x+1)^n + \ldots\right)$$

Thus, we have

$$P_n'(1) = \frac{n(n+1)}{2}$$

■

8. Let $P_k(x)$ denote the Legendre polynomial of degree k. Let m, n be two distinct non-negative integers such that $m \leq n$ and the difference $n - m$ is divisible by 2.

(8a) Using integration by parts and the formula of Rodrigues, show that

$$\int_{-1}^{+1} P_n'(x)P_m'(x)\,dx = m(m+1).$$

(8b) Using integration by parts, show that

$$\int_{-1}^{+1} x^m P_n'(x)\,dx = 0.$$

(8c) Furthermore, if $m < n$ then show that

$$\int_0^1 P_n(x)P_m(x)\,dx = 0.$$

Sol.

8a This integral can evaluated using Integration by parts to

$$[P_n(x)P_m'(x)]_{-1}^{+1} - \int_{-1}^{+1} P_n(x)P_m''(x)\,dx$$

The second term is the product of a polynomial of order m-2 which is strictly less than n, hence its representation as a linear combination of Legendre polynomials won't include P_n. Thus the term evaluates to 0. Whereas the the first term is trivial to evaluate and it gives m(m+1).

8b Using Integration by parts we get the reduced integral as

$$-m\int_{-1}^{+1} x^{m-1} P_n(x)\, dx$$

Now either m-1 is odd and n is even or vice versa, which gives us the situation in Q1 a). Hence this integral is also 0

8c As both m - n is divisible by 2, they're both either odd or even.

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0$$

But also substituting x as -x we get the same integral, as the profuct of the two functions is even. Hence,

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 2\int_{0}^{1} P_n(x)P_m(x)\, dx$$

hence this integral is 0.

■

9. The formula of Rodrigues for the Legendre polynomial of degree n is given by

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$

Using the binomial expansion for $(x^2-1)^n$, arrive at the following alternate expression for the Legendre polynomial

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, x^{n-2k}.$$

Here $[\frac{n}{2}]$ denotes the greatest integer less than or equal to $\frac{n}{2}$.
Sol. The solution follows directly from binomial expansion of $(x^2-1)^n$ and differentiating it n times. ■

10. Given a sequence of numbers $a_0, a_1, a_2, \ldots$ the generating function of that sequence is the formal power series wherein a_n appear as coefficients, i.e.

$$g(t) := \sum_{n=0}^{\infty} a_n t^n.$$

Even if we have got a family of functions $f_0(x), f_1(x), f_2(x), \ldots$ we can consider the generating function of that family as follows

$$g(x,t) := \sum_{n=0}^{\infty} f_n(x) t^n.$$

Note that the coefficients in the above formal power series are depending on the x variable whereas the power series is in the t variable.
For our family $P_0(x), P_1(x), P_2(x), \ldots$ of Legendre polynomials, there is an associated generating function. More precisely we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{1}$$

The objective of this question is to prove the equality in (1).

(10a) Rewriting $1-2xt+t^2$ as $1-t(2x-t)$, write down the binomial series for

$$\frac{1}{\sqrt{1-2xt+t^2}}$$

(10b) Gather all the like terms (involving powers of t) in the series expansion from Question (10a) and identify the coefficient of t^n.

(10c) Show that the coefficient of t^n obtained in Question (10b) coincides with the alternate expression for the degree n Legendre polynomial obtained in Question 9.

Sol.

10a We have

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(2k-1)}{2^n n!} x^n$$

Thus, we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(2k-1)}{2^n n!} t^n(2x-t)^n$$

10b Thus, the coefficient of t^n is $\sum_{m=[\frac{n}{2}]}^{n} \frac{\prod_{k=1}^{n}(2k-1)}{2^m m!}\binom{m}{n-m}(-1)^{m-n}(2x)^{2m-n}$.

10c This matches the expression obtained in question 9

■

11. By manipulating the generating relation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

arrive at the following recurrence relations involving the Legendre polynomials:

(11a) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ for $n = 1, 2, \ldots$

(11b) $xP'_n(x) - P'_{n-1}(x) - nP_n(x) = 0$ for $n = 1, 2, \ldots$

(11c) $P'_{n+1}(x) - P'_{n-1}(x) - (2n+1)P_n(x) = 0$ for $n = 1, 2, \ldots$

(11d) $P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0$ for $n = 1, 2, \ldots$

Sol.

11a Let

$$\frac{1}{\sqrt{1-2xt+t^2}} = f(x,t)$$

Then $(n+1)P_{n+1}(x)$ is the coefficient of t^n in $f'(x,t)$. Similarly $nP_{n-1}(x)$ is the coefficient of t^n in $t\,(tf(x,t))'$.

$$t\,(tf(x,t))' = t^2f' + tf$$

Furthermore, we obtain $-(2n+1)xP_n(x)$ as $-2xtf' - xf$ if we compare coefficients.
This gives us

$$-2xtf' - xf + t^2f' + tf + f' = f'(1-2xt+t^2) + f(t-x)$$

Substituting f(x,t) as taken above into the equation gives us 0.

11b These are the coefficients of t^n as

$$xP'_n(x) = x\frac{\partial f}{\partial x}$$

$$P'_{n-1}(x) = t\frac{\partial f}{\partial x}$$

as P_{n-1} is the coefficient of t^n when the first derivative is taken.

$$nP_n'(x) = t\frac{\partial f}{\partial t}$$

as P_n is the coefficient of t^n when the first derivative is taken and then multiplied by t. Lastly substituting f as taken above and taking the partial derivatives gives 0.

11c Similar to solution of Q3

11d Performing 11c-11b gives this result.

■