PH107 Tutorial Solutions

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§ Photoelectric Effect

1. In a photoelectric effect experiment, excited hydrogen atoms are used as light source. The light emitted from this source is directed to a metal of work function ϕ. In this experiment, the following data on stopping potentials (Vs), for various Balmer lines of hydrogen, is obtained.

$$n = 4 \rightarrow 2, \text{ transition line : } V_S = 0.43V$$
$$n = 5 \rightarrow 2, \text{ transition line : } V_S = 0.75V$$
$$n = 6 \rightarrow 2, \text{ transition line : } V_S = 0.94V$$

1. What is the work function ϕ of the metal in eV?

First we find the energies corresponding to the given transitions. It is known that energy released when a transition two energy levels with principal quantum numbers n_1 and n_2 takes place is given by

$$E_{n_2 \rightarrow n_1} = 13.6 \times [\frac{1}{n_1^2} - \frac{1}{n_2^2}] \tag{1.1}$$

Which implies,

$$E_{4\rightarrow 2} = 13.6 \times [\frac{1}{4} - \frac{1}{16}] = 2.55 \text{ eV} \tag{1.2}$$
$$E_{5\rightarrow 2} = 13.6 \times [\frac{1}{4} - \frac{1}{25}] = 2.86 \text{ eV} \tag{1.3}$$
$$E_{6\rightarrow 2} = 13.6 \times [\frac{1}{4} - \frac{1}{36}] = 3.02 \text{ eV} \tag{1.4}$$

On subtracting the measured values of stopping potential from the obtained transition energies, we get three values of work functions corresponding to the three experiments: 2.12, 2.12 and 2.08. We take their mean.

$$\phi = \frac{2.12 + 2.12 + 2.08}{3} \approx 2.11 \tag{1.5}$$

b) What is the stopping potential (in Volts) for Balmer line of the shortest wavelength?

Shortest wavelength transition => largest energy transition, i.e., $E_{max} = E_{\infty \to 2} = 13.6 \times [\frac{1}{4} - 0] = 3.4$ eV.
Stopping potential $(V_s) = E_{max} - \phi = 3.4 - 2.11$ eV $= 1.29$ eV

2. In an experiment on photoelectric effect of a metal, the stopping potentials were found to be 4.62 V and 0.18 V for λ_1 = 1850 Å and λ_2 = 5460 Å, respectively. Find the value of Planck's constant, the threshold frequency and the work function of the metal.

Energy of light of wavelength λ:

$$E = \frac{hc}{\lambda} \tag{1.6}$$

Thus, we can easily write the following equations

$$4.62 = \frac{hc}{1850\text{Å}} - \phi \tag{1.7}$$

$$0.18 = \frac{hc}{5460\text{Å}} - \phi \tag{1.8}$$

Subtract equations (1.8) from (1.7) to get:

$$4.44 = h \times 3 \times 10^8 \times 3.57 \times 10^6 \tag{1.9}$$

$$h = 4.15 \times 10^{-15} \text{ eV Hz}^{-1} \tag{1.10}$$

Now, put the value of h in equation (1.7) (or (1.8)) to obtain

$$\phi = 6.72 - 4.62 \text{ eV} = 2.1 \text{ eV} \tag{1.11}$$

and

$$\nu_0 = 5.06 \times 10^{14} Hz \tag{1.12}$$

3. A monochromatic light of intensity 1.0 $\mu W/cm^2$ falls on a metal surface of area 1 cm^2 and work function 4.5 eV. Assume that only 3% of the incident light is absorbed by the metal (rest is reflected back) and that the photoemission efficiency is 100% (i.e. each absorbed photon produces one photoelectron). The measured saturation current is 2.4 nA.

Intensity $I = 1.0 \times 10^{-6}\ W/cm^2$ and area $A = 1\ cm^2$ =>

$$\text{Incident power } P = 1 \times 10^{-6}\ W \tag{1.13}$$

$$\text{Saturation current } (I_s) = 2.4\ nA \tag{1.14}$$

$$\text{number of photo-electrons ejected per second} = \frac{I_s}{e} = 1.5 \times 10^{10} \tag{1.15}$$

Let the number of photons incident on the surface (per second) be n, then

$$0.03 \times n = 1.5 \times 10^{10} \tag{1.16}$$

$$n = 5 \times 10^{11} \text{ (answer to a-part)} \tag{1.17}$$

Energy of incident photon can be calculated by dividing the total incident power P (equation (1.13)) by the number of photons incident per second n (equation (1.17)).

$$E_p = \frac{P}{n} = 0.2 \times 10^{-17} = 20 \text{ eV} \tag{1.18}$$

5. Light of wavelength 2000 Å falls on a metal surface. If the work function of the metal is 4.2 eV, find the kinetic energy of the fastest and the slowest emitted photoelectrons. Also find the stopping potential and cutoff wavelength for the metal.

KE of fastest moving photo-electron is simply $12424/2000 - 4.2$ eV ≈ 2 eV (which is also the magnitude of stopping potential). Slowest photo-electrons are the the ones which barely come out of the metal surface, i.e., $KE = 0$.

Cutoff wavelength λ_0 is $hc/\phi = 12424/4.2$ Å $= 2958.1$ Å.

§Black-body Radiation

1. According to Planck, the spectral energy density $u(\lambda)$ of a black-body maintained at temperature T is given by

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

where λ denotes the wavelength of radiation emitted by the black-body.

a) Find an expression for λ_{max} at which $u(\lambda, T)$ attains its maximum value (at a fixed temperature T). λ_{max} should be in terms of T and fundamental constants h, c and k_B.

We just need to equate the partial derivative of the $u(\lambda, T)$ w.r.t λ to zero, and solve the transcendental equation so obtained.

$$\frac{\partial u}{\partial \lambda} = \frac{5}{\lambda^6(e^{\frac{hc}{\lambda k_B T}} - 1)} = \frac{hc \times e^{\frac{hc}{\lambda k_B T}}}{k_B T \lambda^7 (e^{\frac{hc}{\lambda k_B T}} - 1)^2} = 0 \tag{1.19}$$

Set $\frac{hc}{\lambda k_B T} = x$. Equation (1.19) becomes

$$\frac{xe^x}{e^x - 1} = 5 \tag{1.20}$$

Solve this using scientific calculator or google Newton-Raphson method if you wish to make a manual attempt!
Answer: $x = 4.9651$

Using this value of x we get $\lambda_{max} = \frac{hc}{xk_B T} = \frac{0.0029}{T}$ or $\lambda_{max} T = 0.0029 mK$. The above result is, in fact, the famous Wien's Law! (This is the solution to fourth question under this section too)

b) Expressing λ_{max} as $\frac{\alpha}{T}$, obtain an expression for $u_{max}(T)$ in terms of α, T and the fundamental constants.

Simply put $\lambda_{max} = \frac{\alpha}{T}$ in the Planck's formula to get

$$u_{max}(T) = \frac{8\pi hcT^5}{\alpha^5} \frac{1}{e^{\frac{hc}{\alpha k_B}} - 1} \tag{1.21}$$

2. The earth rotates in a circular orbit about the sun. The radius of the orbit is 140×10^6 km. The radius of the earth is 6000 km and the radius of the sun is 700,000 km. The surface temperature of the sun is 6000 K. Assuming that the sun and the earth are perfect black bodies, calculate the equilibrium temperature of the earth.

Power radiated by the sun $= \sigma 4\pi R^2_{\mathsf{Sun}} T^4_{\mathsf{Sun}}$
Intensity at a distance R_{SE} from the sun $= \sigma R^2_{\mathsf{Sun}} T^4_{\mathsf{Sun}} / R^2_{\mathsf{SE}}$
Since $R_{\mathsf{SE}} >> R_{\mathsf{Earth}}$, the radiation incident on the earth can be approximated to be a parallel beam.

From your JEE knowledge, it must be obvious that in this we have to multiply the projected area, i.e., πR_{Earth}^2 with the incident intensity to obtain the power incident on the Earth. Condition for equilibrium:

$$\text{Power incident on Earth} = \text{Power radiated by Earth}$$

$$\frac{\sigma R_{\text{Sun}}^2 T_{\text{Sun}}^4}{R_{\text{SE}}^2} \times \pi R_{\text{Earth}}^2 = \sigma 4\pi R_{\text{Earth}}^2 T_{\text{Earth}}^2$$

After a little simplification we get $T_{\text{Earth}} = \frac{1}{4^{\frac{1}{4}}}(\frac{R_{\text{Sun}}}{R_{\text{SE}}})^{\frac{1}{2}} = 300$ K

3. a) Given Planck's formula for the energy density, obtain an expression for the Rayleigh-Jeans formula for $u(\nu, T)$.

Planck's formula: $u_{\text{Planck}}(\nu, T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{k_BT}}-1}$

Rayleigh-Jeans Formula: $u_{\text{R-J}}(\nu, T) = \frac{8\pi\nu^2}{c^3}k_BT$
The graphical results suggest that the predictions of the Planck's law and Rayleigh-Jeans law are appreciably close in the "low frequency" domain. Thus, the limit of the Planck's formula as $\nu \to 0$ should be the Rayleigh-Jeans formula.

$$\lim_{\nu\to 0}\frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{k_BT}}-1} = \frac{8\pi\nu^2}{c^3}k_BT\lim_{\nu\to 0}\frac{\frac{h\nu}{k_BT}}{e^{\frac{h\nu}{k_BT}}-1} = \frac{8\pi\nu^2}{c^3}k_BT \tag{1.22}$$

Hence, $\lim_{\nu\to 0} u_{\text{Planck}}(\nu, T) = u_{\text{R-J}}(\nu, T)$!

§Compton Scattering

1. A photon of energy $h\nu$ is scattered through 90° by an electron initially at rest. The scattered photon has a wavelength twice that of the incident photon. Find the frequency of the incident photon and the recoil angle of the electron.

Compton's formula: $\lambda' = \lambda + \lambda_C(1 - cos\theta)$, where $\lambda_C = \frac{h}{m_0c}$. We are given $\lambda' = 2\lambda$, and $\theta = 90°$. This renders

$$2\lambda = \lambda + \lambda_C(1 - cos(90°)) \tag{1.23}$$

$$\lambda = \lambda_C = \frac{h}{m_0c} \tag{1.24}$$

We know that $\nu = c/\lambda$ =>

$$\nu = \frac{c \times m_0c}{h} = \frac{m_0c^2}{h} \tag{1.25}$$

To find the recoil angle of the electron, we use momentum conservation. Let the recoil angle be ϕ and the final momentum of the electron be p_e. The component along the initial direction photon: $p_e cos\phi$ and that along the final direction of photon (in this case orthogonal to the initial direction): $p_e sin\phi$.

$$\frac{h}{\lambda} + 0 = 0 + p_e cos\phi \tag{1.26}$$

$$\frac{h}{2\lambda} + 0 = 0 + p_e sin\phi \tag{1.27}$$

Take the ratio of equations (1.27) and (1.26) to obtain

$$tan\phi = \frac{1}{2} \tag{1.28}$$

$$\phi = tan^{-1}(\frac{1}{2}) \approx 26.56° \tag{1.29}$$

2. Find the energy of the incident x-ray if the maximum kinetic energy of the Compton electron is $m_0c^2/2.5$.

Using energy conservation, we can see that the kinetic energy of the electron is the difference between initial and final energy of the photon. We also know that photon energy $\propto \frac{1}{\lambda}$. Thus for maximising kinetic energy of the electron we need to minimise the final energy of the photon, which is equivalent to maximising the final wavelength of the photon. Just put $\theta = 180°$ in the Compton's formula.

$$\lambda'_{max} = \lambda + \lambda_C(1 - cos(180°)) \tag{1.30}$$

$$\lambda'_{max} = \lambda + 2\lambda_C \tag{1.31}$$

Maximum kinetic energy of the electron is given by

$$KE_{max} = hc[\frac{1}{\lambda} - \frac{1}{\lambda'_{max}}] = \frac{2}{5}m_0c^2 \text{ (given)} \tag{1.32}$$

$$\frac{2}{5}m_0c^2 = hc[\frac{1}{\lambda} - \frac{1}{\lambda + 2\lambda_C}] \tag{1.33}$$

$$\lambda(\lambda + 2\lambda_C) = 5\lambda_C^2 \tag{1.34}$$

$$\lambda^2 + 2\lambda\lambda_C - 5\lambda_C^2 = 0 \tag{1.35}$$

$$\lambda = \frac{-2\lambda_C + \sqrt{4\lambda_C^2 + 4 \times 5\lambda_C^2}}{2} = (\sqrt{6} - 1)\lambda_C \tag{1.36}$$

We know that $\lambda_C \approx 0.0243$ Å => $\lambda = 0.035$ Å. Thus, the initial energy of the photon E is

$$E = \frac{hc}{\lambda} = \frac{12424}{0.035} \text{ eV} \approx 0.35 \text{ MeV} \tag{1.37}$$

3. Show that a free electron cannot absorb a photon so that a photo-electron requires bound electron. However, the electron can be free in Compton Effect. Why?

Lets assume that it is possible for a free electron to absorb a photon. We will now write the energy and momentum conservation equations for this scenario. A little bit of special relativity is required for this purpose. The relativistic energy and momentum of a particle moving at a speed v are given by

$$E = \gamma_v m_0 c^2 \tag{1.38}$$

$$p = \gamma_v m_0 v \tag{1.39}$$

where m_0 is the rest mass of the particle, and $\gamma_v = 1/\sqrt{1 - v^2/c^2}$. Multiply equation (1.39) by c and then square and add both the equations to get

$$E = \sqrt{p^2c^2 + m_0^2c^4} \tag{1.40}$$

Conservation of energy and momentum equations are as follows

$$m_0c^2 + \frac{hc}{\lambda} = \sqrt{p^2 + m_0^2c^4} \tag{1.41}$$

$$\frac{h}{\lambda} = p \tag{1.42}$$

Equations (1.41) and (1.42) may be clubbed together as

$$m_0c^2 + pc = \sqrt{p^2c^2 + m_0^2c^4} \tag{1.43}$$

$$m_0^2c^4 + p^2c^2 + 2m_0pc^3 = p^2c^2 + m_0^2c^4 \tag{1.44}$$

CONTRADICTION!!!

Hence, our initial assumption is incorrect!

3. Two Compton scattering experiments were performed using x-rays (incident energies E_1 and $E_2 = E_1/2$). In the first experiment, the increase in wavelength of the scattered x-ray, when measured at an angle $\theta = 45°$, is 7×10^{-14}

m. In the second experiment, the wavelength of the scattered x-ray, when measured at an angle $\theta = 60°$, is 9.9×10^{-12} m.

a) Calculate the Compton wavelength and the mass (m) of the scatterer.

$$\Delta\lambda^{(1)} = \lambda_C(1 - cos\theta^{(1)}) \tag{1.45}$$
$$7 \times 10^{-14} = \lambda_C(1 - \frac{1}{\sqrt{2}}) \tag{1.46}$$
$$\lambda_C = \frac{h}{mc} = 23.9 \times 10^{-14} \text{ m} \tag{1.47}$$
$$m = \frac{h}{\lambda_C c} = 9.24 \times 10^{-30} \text{ Kg} \tag{1.48}$$

b) Find the wavelengths of the incident x-rays in the two experiments.

$$\lambda'^{(2)} = \lambda^{(2)} + \lambda_C(1 - cos\theta^{(2)}) \tag{1.49}$$
$$\lambda^{(2)} = \lambda'^{(2)} - \lambda_C(1 - cos\theta^{(2)}) \tag{1.50}$$
$$\lambda^{(2)} = 9.9 \times 10^{-12} - 23.9 \times 10^{-14}(1 - \frac{1}{2}) = 9.78 \times 10^{-12} \text{ m} \tag{1.51}$$
$$\lambda^{(1)} = 0.5 \times \lambda^{(2)} = 4.89 \times 10^{-12} \text{ m} \quad (\text{Since } E_2 = E_1/2\ ) \tag{1.52}$$

5. Find the smallest energy that a photon can have and still transfer 50% of its energy to an electron initially at rest.

Let the initial wavelength be $\lambda \implies$ the final wavelength will be 2λ. We need the smallest wavelength for which this is possible. The limiting case will be the one in which maximum energy transfer ($\theta = 180°$) will be equal to half of the initial energy.

$$2\lambda = \lambda + \lambda_C(1 - cos(180°)) = \lambda + 2\lambda_C \tag{1.53}$$
$$\lambda = 2\lambda_C \tag{1.54}$$
$$E = \frac{hc}{2\lambda_C} = \frac{1}{2}m_0c^2 = \frac{1}{2} \times 0.51 \text{ MeV} = 0.255 \text{ MeV} \tag{1.55}$$

6. γ-rays are scattered from electrons initially at rest. Assume the it is back-scattered and its energy is much larger than the electron's rest-mass energy, $E \gg m_ec^2$.

a) Calculate the wavelength shift.

Given, $\theta = 180° \implies$

$$\Delta\lambda = \lambda_C(1 - cos(180°)) \tag{1.56}$$
$$\Delta\lambda = 2\lambda_C = 2 \times 0.0243 \text{ Å} \approx 0.05 \text{ Å} \tag{1.57}$$

b) Show that the energy of the scattered beam is half the rest mass energy of the electron, regardless of the energy of the incident beam.

We are given that $E \gg m_e c^2 \implies$

$$\frac{hc}{\lambda} \gg m_e c^2 \tag{1.58}$$
$$\lambda \ll \lambda_C \tag{1.59}$$

Using the results (1.57) and (1.59), we can infer the following

$$E' = \frac{hc}{\lambda'} = \frac{hc}{\lambda + 2\lambda_C} = \frac{hc}{2\lambda_C} \times \frac{1}{1 + \frac{\lambda}{2\lambda_C}} \approx \frac{hc}{2\lambda_C} = \frac{1}{2} m_e c^2 \tag{1.60}$$

c) Calculate the electron's recoil kinetic energy if the energy of the incident radiation is 150 MeV.

$$\text{Initial Energy} = \text{Final Energy} \tag{1.61}$$
$$E = E' + KE_e \tag{1.62}$$
$$150 \text{ MeV} = \frac{1}{2} \times 0.51 \text{ MeV} + KE_e \tag{1.63}$$
$$KE_e = 149.745 \text{ MeV} \tag{1.64}$$

7. In Compton Scattering, Show that if the angle of scattering θ increases beyond a certain value θ_0, the scattered photon will never have energy larger than $2m_0c^2$, irrespective of the energy of the incident photon. Find the value of θ_0.

$$\lambda' = \lambda + \lambda_C(1 - cos\theta) \tag{1.65}$$
$$E' = \frac{hc}{\lambda'} = \frac{hc}{\lambda + \lambda_C(1 - cos\theta)} \tag{1.66}$$

We need $E < 2m_0c^2 \ \forall \ \lambda > 0$ if $\theta > \theta_0$.

$$E = \frac{hc}{\lambda_C} \times \frac{1}{1 - cos\theta + \frac{\lambda}{\lambda_C}} \tag{1.67}$$

$$E = \frac{m_0c^2}{1 - cos\theta + \frac{m_0c^2}{E}} < \frac{m_0c^2}{1 - cos\theta_0 + \frac{m_0c^2}{E}} < \frac{m_0c^2}{1 - cos\theta_0} = 2m_0c^2 \tag{1.68}$$

$$cos\theta_0 = \frac{1}{2} \tag{1.69}$$

Therefore, $\theta_0 = 60°$