Cosmology and Dark Matter

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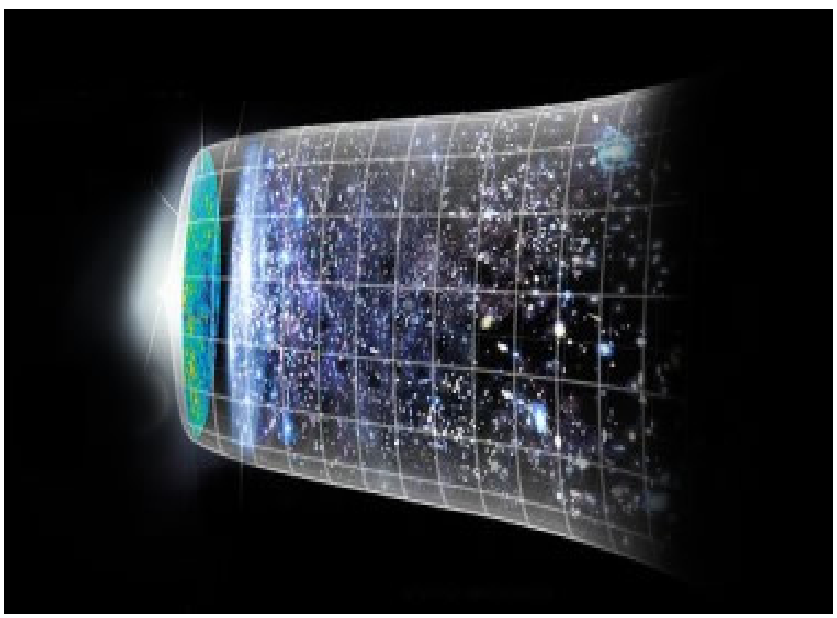


Mentor: Parth Sastry

Contents

1 Preface

This is my Summer of Science final report on the topic "Cosmology and Dark Matter" under the guidance of my mentor Parth Sastry. For preparing this report I did the following:

1. Read the topics "Vectors and Tensors", "Tensor algebra", "Covariant Differentiation" and "Gradient, Curl and Divergence", "Cosmological Principle", "Red Shift" and "Steady State Cosmology" from the book "Gravitation and Cosmology" by Steven Weinberg.

2. Read the chapter "Curvature" from the book "Spacetime and Geometry" by Sean Carroll.

3. Read about "Dark Matter", "Dark Energy" and "Density Parameter" from COSMOS- The SAO Encyclopedia of Astronomy - `https://astronomy.swin.edu.au/cosmos/`

4. Watched first 9 lectures of MIT OCW youtube lecture-series on General relativity (instructor: Scott Hughes) - link: `https://youtube.com/playlist?list=PLUl4u3cNGP629n_3fX7HmKKgin_rqGzbx`.

5. Watched all ten lectures of youtube lecture-series on Cosmology by Prof. Leonard Susskind (Stanford University) - link: `https://youtube.com/playlist?list=PLpGHT1n4-mAuVGJ2E1uF9GSwLsx7p1xtm`.

2 Introduction to Tensors and Tensor Algebra

We are all familiar with Scalars and Vectors. Now, I shall introduce **Tensors**. But before that it is important to explore the formal definitions of **Vectors** and **Dual Vectors** (also known as One-Forms).

2.1 Vectors and Dual Vectors

We know what a vector means and we are quite familiar with three-vectors, but here vector refers to four-vector (4-vector). These four-vectors help to probe the structure of the **Minkowski Space**.

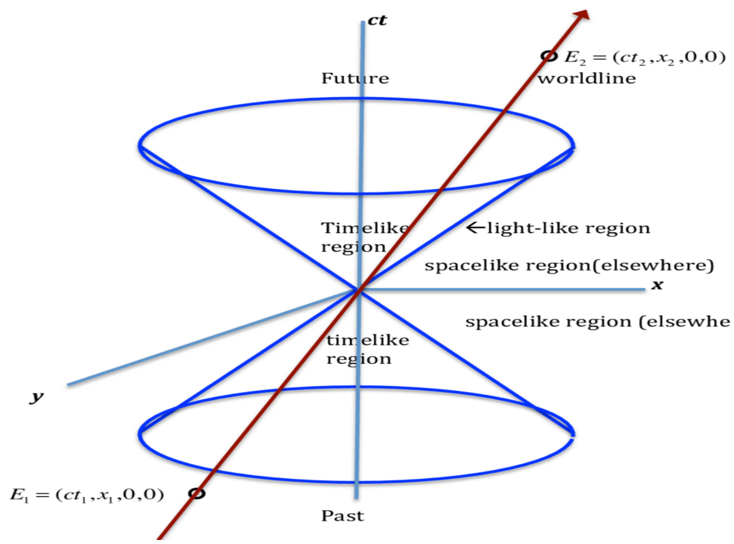


Figure 1: Space-time diagram

2.1.1 Minkowski Space

Minkowski Space or Spacetime is basically a combination of three spatial dimensions and one time dimension. Thinking of it as Euclidean Space with an additional time axis is a good way to comprehend it. Spacetime is a **four-dimensional manifold**. Manifold, in layman terms, is a set of points with well-understood connectedness properties defined by a **metric**. **Metric** associates a notion of distance with a manifold. This shall be discussed in more detail in upcoming sections. This simple idea of manifold would suffice and as far as our discussion is concerned.

2.1.2 Four-Vectors

Just the way three-vectors represent a point in Euclidean space, a four-vector represents a point in spacetime. A typical four-vector is represented as:

$$\mathbf{A} = (A^0, A^1, A^2, A^3)$$

The up-stair indices imply **contravariance**. There are two very important terminologies, **contravariance** and **covariance**. These terms are related to coordinate transformations. **Contravariant vectors** are those which "contra-vary" with a change of basis to compensate, i.e., the matrix which transforms components of the vector must be inverse of the matrix which transforms the basis vectors. Examples include position 4-vector, velocity 4-vector etc. On the other hand, **covariant vectors** are those which co-vary with a change of basis, i.e., vector components and basis vectors are both transformed by the same matrix. Examples of such vectors appear when gradient of a function is taken. Following is a mathematical demonstration of the contravariance property. For that basis vectors $\boldsymbol{e_i}$ need to be defined. **Basis vectors** form a set of linearly independent vectors which span the entire space, i.e., any vector in the space can be expressed as linear combination of the basis vectors with a unique tuple of coefficients. So

$$\mathbf{A} = \Sigma A^{\alpha}\mathbf{e}_{\alpha} \tag{1}$$

Suppose, a contravariant four-vector **A** is linearly transformed to **A'** (components measured in a different frame) using a transformation matrix Λ i.e.,

$$A^{\mu\prime} = \Sigma \Lambda^{\mu\prime}{}_{\alpha} A^{\alpha} \tag{2}$$

The summation sign shall be skipped from now. This act is in accordance with **Einstein's Summation Convention** which states that lined up indices imply summation over it. In the above case α is dummy index or running index whereas μ' is fixed index. Transformation of basis vectors is shown next.

$$\mathbf{A} = A^{\alpha}\mathbf{e}_{\alpha} = A^{\mu'}\mathbf{e}'_{\mu}$$

This is because a four-vector is a geometric object in space-time which is independent of representation in terms of components and basis vectors.

$$\mathbf{A} = A^{\alpha}\mathbf{e}_{\alpha} = \Lambda^{\mu'}_{\alpha} A^{\alpha}\mathbf{e}'_{\mu}$$

Using elementary properties of summation, we get:

$$A^{\alpha}\mathbf{e}_{\alpha} = A^{\alpha}\Lambda^{\mu'}_{\alpha}\mathbf{e}'_{\mu}$$

From here, by simple manipulations we can reach:

$$\mathbf{e}_{\alpha} = \Lambda^{\mu'}_{\alpha}\mathbf{e}'_{\mu} \tag{3}$$

It can be clearly seen that the transformation matrix for linear transformation of basis vectors is inverse of the transformation matrix used for linearly transforming the components of **A**.

2.1.3 Dual Vectors

A dual vector or one-form is a linear mapping from a vector space to scalars.Similarly, **dual spaces** can be defined. If **V** is any vector space then its dual space **V*** is defined as the set of all the linear transformations ϕ: $\mathbf{V} \rightarrow \Re$. Basis dual vectors are generally represented as $\boldsymbol{\omega^{\alpha}}$ where α varies from 0 to 4. These are linearly independent dual vectors using which any dual vector in the manifold can be expressed as a linear combination. Suppose **p** is a dual vector. Then it can be expressed in term of its components and basis dual vectors as follows:

$$\mathbf{p} = p_{\alpha}\boldsymbol{\omega^{\alpha}}$$

A dual vector takes a vector as its argument and returns a scalar. When the argument is $\boldsymbol{e_{\alpha}}$, some nice result is obtained.

$$\mathbf{p}(\mathbf{e}_{\alpha}) = p_{\beta}\boldsymbol{\omega^{\beta}}(\mathbf{e}_{\alpha})$$

The term $\boldsymbol{\omega^{\beta}}(\boldsymbol{e^{\alpha}})$ for any particular coordinate system is equal to Kroneckar delta (δ_{α}^{β}). This leads to a the inference: plugging in $\boldsymbol{e_{\alpha}}$ as an argument gives out the α^{th} component of the dual vector.

$$\mathbf{p}(\mathbf{e}_{\alpha}) = p_{\alpha} \tag{4}$$

So, in general, if any vector **A** is plugged in as an argument to **p**, the following is observed:

$$\mathbf{p}(\mathbf{A}) = \mathbf{p}(A^{\alpha}\boldsymbol{e_{\alpha}}) = A^{\alpha}p_{\alpha}$$

This is a very simple example of **contraction** (discussed in the subsequent parts). Using this result along with the fact that scalars are transformation invariant, linearly transformed components of a dual vector (using the same transformation matrix Λ as before) can be written down easily.

$$p_{\alpha'} = \Lambda^{\mu}_{\alpha'}p_{\mu} \tag{5}$$

2.2 Tensors

A **tensor** is a geometric object that describes a multilinear relationship between sets of algebraic objects related to a vector space. Objects that tensors may map between include scalars, vectors and even other tensors. A tensor has many covariant and/or contravariant indices with corresponding linear transformation properties. For example, the transformation $T^{\gamma}_{\alpha\beta} \rightarrow T'^{\gamma}_{\alpha\beta}$ would look like the following:

$$T'^{\gamma}_{\alpha\beta} = \Lambda^{\gamma}_{\delta}\Lambda^{\epsilon}_{\alpha}\Lambda^{\zeta}_{\beta}T^{\delta}_{\epsilon\zeta}$$

The various algebraic operations on tensors are as follows:

1. **Linear Combination**: A linear combination of tensors with same upper and lower indices is a tensor itself with the same indices. For instance, say R^{α}_{β} and S^{α}_{β} are two tensors, then $T^{\alpha}_{\beta} = aR^{\alpha}_{\beta} + bS^{\alpha}_{\beta}$ is also a tensor (with the same indices), because $T'^{\alpha}_{\beta} = aR'^{\alpha}_{\beta} + bS'^{\alpha}_{\beta}$ upon straight-forward mathematical manipulations would give $T'^{\alpha}_{\beta} = \Lambda^{\alpha}_{\rho}\Lambda^{\sigma}_{\beta}T^{\rho}_{\sigma}$

2. **Direct Products**: The product of components of two tensors yields a tensor whose upper and lower indices consist of all the upper and lower indices of the two original tensors. For example, say $A^{\alpha\beta}_{\gamma}$ and B^{μ}_{ν} are tensors, then $T^{\alpha\beta\mu}_{\gamma\nu} = A^{\alpha\beta}_{\gamma}B^{\mu}_{\nu}$ is a tensor.

3. **Contraction**: Setting an upper and lower index equal and summing over its values 0,1,2,3 would yield a tensor with these indices absent. For example, let $A^{\alpha\gamma\delta}_{\beta}$ be a tensor then $A^{\alpha\gamma\beta}_{\beta}$ (note that according to Einstein summation convention this notation implies summation over β) is equal to a tensor $T^{\alpha\gamma}$ with β and δ indices absent. Linear transformation would look like the following:

$$\begin{aligned} T'^{\alpha\gamma} &= A'^{\alpha\gamma\beta}_{\beta} \\ &= \Lambda^{\alpha}_{\kappa}\Lambda^{\zeta}_{\beta}\Lambda^{\gamma}_{\eta}\Lambda^{\beta}_{\epsilon}A^{\kappa\eta\epsilon}_{\zeta} \\ &= \Lambda^{\alpha}_{\kappa}\Lambda^{\gamma}_{\eta}\delta^{\zeta}_{\epsilon}A^{\kappa\eta\epsilon}_{\zeta} \\ &= \Lambda^{\alpha}_{\kappa}\Lambda^{\gamma}_{\eta}T^{\kappa\eta} \end{aligned}$$

A very important tensor, which will be useful for us in the subsequent parts, is the **metric tensor**. Roughly speaking, metric tensor $g_{\alpha\beta}$ is a function which tells how to compute distance between 2 points in the given manifold.

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$$

It is easy to see that in Euclidean Space, $g_{\alpha\beta} = \delta_{\alpha\beta}$ (Kroneckar delta) because in this case distance between two points is simply given by Pythagoras Theorem. When there is no curvature in space-time, metric tensor has a special name, i.e., **Minkowski Tensor** (denoted by $\eta_{\alpha\beta}$). Let x^μ represent a general coordinate system and ζ^α represent locally inertial coordinate system, then $g_{\mu\nu} \equiv \eta_{\alpha\beta}\frac{\partial\zeta^\alpha}{\partial x^\mu}\frac{\partial\zeta^\beta}{\partial x^\nu}$. Just the way $g_{\mu\nu}$ is the covariant metric tensor, $g^{\mu\nu}$ is the contravariant metric tensor.

3 Tensor Calculus

In the previous section basic definition of tensors and the algebraic operations associated them were discussed. Tensor calculus will be discussed next.

3.1 Gradient

Gradient is the **covariant derivative** of a scalar.

$$S_{;\lambda} = \frac{\partial S}{\partial x^\lambda}$$

To understand this, **affine connection** and **covariant derivative** are needed.

Affine connection is a very important non-tensor defined as $\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial\zeta^\alpha}\frac{\partial^2\zeta^\alpha}{\partial x^\mu\partial x^\nu}$. Detailed discussion about affine connection is beyond the scope of this project. Although, discussion on Christoffel symbol (in a later section) also requires familiarity with affine connection.

Covariant Derivative of a contravariant vector is defined as the following:

$$V^\mu_{;\lambda} \equiv \frac{\partial V^\mu}{\partial x^\lambda} + V^\kappa\Gamma^\mu_{\kappa\lambda} \tag{6}$$

By the properties of affine connection, this comes out to be a tensor. Similarly, we can define covariant derivative of a dual vector.

$$p_{\mu;\lambda} \equiv \frac{\partial p_\mu}{\partial x^\lambda} - \Gamma^\kappa_{\mu\lambda}p_\kappa \tag{7}$$

This definition can be extended to any general tensor. For example,

$$T^{\mu\sigma}_{\lambda;\rho} = \frac{\partial T^{\mu\sigma}_{\lambda}}{\partial x^\rho} + \Gamma^\mu_{\rho\nu}T^{\nu\sigma}{}_\lambda + \Gamma^\sigma_{\rho\nu}T^{\mu\nu}{}_\lambda - \Gamma^\kappa_{\lambda\rho}T^{\mu\sigma}{}_\kappa$$

and this also turns out to be a tensor itself (by the properties of affine connection).

3.2 Curl

Covariant curl is just like the simple curl we know. It is define as the following:

$$V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}$$

This happens because the affine connection $\Gamma^\kappa_{\mu\nu}$ is symmetric in μ and ν which is why it gets cancelled out. There is another operation called **covariant divergence** which will not be discussed in detail because it involves a lot of advanced linear algebra which is beyond the scope of this project. But, for the sake of completeness the result is stated and associated terms are briefly described.

3.3 Covariant Divergence

The metric tensor $g_{\rho\sigma}$ can be expressed as a matrix. Let determinant be g. The covariant divergence of a contravariant vector (V^μ) is given by

$$V^\mu_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} V^\mu$$

The upcoming section discusses one of the most amazing theories in the history of physics, the **The General Theory of Relativity**.

4 General Theory of Relativity

This theory was published by Albert Einstein in the year 1915. The most beautiful thing about this theory is that it provides a geometric interpretation of gravitation, i.e., it relates gravity with the geometry of space-time. The first thing to be understood is the principle of equivalence of gravitation and inertia.

4.1 The Principle of Equivalence

The principle may be stated as "gravitation and inertia are equivalent". Einstein observed that the gravitational force experienced "locally" while standing on a massive body (like the Earth) is same as the fictitious (pseudo) force experienced in a non-inertial frame of reference. The reason why emphasis is given over the word "locally" is that the region of interest should be taken to be small enough so that variation in gravitational effect is not much since, in practice, there are no perfectly homogeneous gravitational field (just think about Earth's gravitational field and what is the effect of Sun's gravitational field on it). The forces which arise due to these inhomogeneities are known as **tidal forces** (as the name suggests, they are named after the process of tide formation, which we are all aware of).

4.2 Christoffel Symbols

Affine connection was discussed under section 2.1 of this project. The most important thing was how it was used to define covariant derivative and this is all we need to know about affine connections to accomplish what this project intends to. The reason why I am talking about affine connections under this heading is that affine connections share the same relation with Christoffel symbols as vectors with their components! That is, given an affine connection and a frame, one can determine the Christoffel symbols just the way given a vector and a basis, one can determine the components. We can also obtain a nice relation between metric tensor and the symbols.

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}) \tag{8}$$

4.3 Curvature

Manifolds have already been talked about but now it's time to go a little deeper into curved manifolds. Here, I shall emphasise upon parallelly transporting a vector along a curve in curved manifold. We know what happens when transport a vector parallelly in flat manifold. The vector does not change when you bring it back to the same point after having transported it along some curve in flat manifold. But the same does not apply to curved manifolds. The length still remains same but the angle is not the same as before. Thinking of a curve on 2 two sphere might help in visualisation. If we try to move a vector along a curve on a 2-sphere trying to keep it as parallel as possible, it is clear that when the vector comes back to the initial position it is inclined to the initial orientation of the vector (and hence, not exactly the same). This is a basic notion

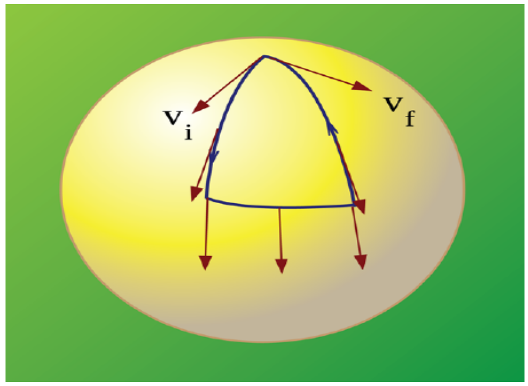


Figure 2: Parallel transport of a vector in curved space

of what curvature is. Now, suppose a "straight" line on a curved surface is demanded. Of course this doesn't make sense but what can be done is making a line as straight as possible. For accomplishing this one needs to parallelly transport the tangent vector and the lines so obtained are called **geodesics**. Mathematically, equation describing a geodesic can be obtained in the following way. Suppose we have a curve in the manifold $\vec{x}$ and let the parameter of the curve be λ. Now suppose we have a vector field $\mathbf{V}$ which needs to be parallelly transported along the curve (having tangent vector field U. The parallel transport can be defined as

$$\frac{d\mathbf{V}}{d\lambda} = U^{\beta} V^{\alpha}_{;\beta} = 0 \tag{9}$$

There is alternative way of representing the same, i.e., $\nabla_{\mathbf{U}}\mathbf{V} = 0$. Now we want the tangent vector to be parallelly transported. This implies $\nabla_{\mathbf{U}}\mathbf{U} = 0$ which renders

$$\frac{d}{d\lambda}(\frac{dx^{\alpha}}{d\lambda}) + \Gamma^{\alpha}_{\beta\mu}\frac{dx^{\mu}}{d\lambda} = 0 \tag{10}$$

Definition of **commutator** is discussed next. Commutator([A,B]) of 2 operators A and B is basically the difference of their products taken in different orders, i.e. $[A, B] = AB - BA$. If you think of A and B as numbers then you will find that their commutator will always be zero, but not everything is commutative, so the commutator can be non-zero (which is, in-fact, the scenario in the cases we are going to consider). Suppose, two coordinates of a manifold are x^{μ} and x^{ν}. Now we consider an infinitesimal parallelogram ABCD such that |AB|=|CD|=dx^{μ} and |BC|=|AD|=dx^{ν}. Now, we want to parallelly transport a vector V along the path A $\rightarrow B \rightarrow C \rightarrow D \rightarrow A$.

$$V_B - V_A = dx^\mu \nabla_\mu V$$
$$(V_C - V_D) - (V_B - V_A) = dx^\nu dx^\mu \nabla_\nu \nabla_\mu V$$
$$V_C - V_B = dx^\nu \nabla_\nu V$$
$$(V_D - V'_A) - (V_C - V_B) = dx^\mu dx^\nu \nabla_\mu \nabla_\nu V$$

Observe that in the last equation I have written V'_A instead of V_A. This is because vector won't remain the same as before at the starting point after the entire process. Now, we take the difference of differences:

$$dV = V'_A - V_A = dx^\mu dx^\nu (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) V \tag{11}$$

It can be clearly seen that the term inside brackets on the RHS is the commutator $[\nabla_\nu, \nabla_\mu]$. So we may write the last equation as the following:

$$dV = dx^\mu dx^\nu [\nabla_\nu, \nabla_\mu] V \tag{12}$$

The definition of covariant derivative (equation 6) leads us to expressing $[\nabla_\nu, \nabla_\mu]$ in terms of partial derivative terms and Christoffel symbols.

$$\nabla_\nu = \partial_\nu + \Gamma_\nu$$

$$\nabla_\mu = \partial_\nu + \Gamma_\mu$$

Here ∂_ν is a shorthand notation for $\frac{\partial}{\partial x^\nu}$. Proceeding further:

$$[\nabla_\nu, \nabla_\mu] = (\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu) - (\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu)$$
$$[\nabla_\nu, \nabla_\mu] = -[\partial_\mu, \Gamma_\nu] + [\partial_\nu, \Gamma_\mu] + [\Gamma_\nu, \Gamma_\mu] \tag{13}$$

It can be observed that equation (13) renders the commutator $[\nabla_\nu, \nabla_\mu]$ in terms of three more elementary commutators. This is called the **Riemann tensor** and for all our purposes this is the **Ricci tensor**. Defining it formally, Riemann tensor is a (1,3) tensor (this notation means that the tensor has 1 contravariant index and 3 covariant indices) $R^\alpha_{\beta\mu\nu}$ is related to Christoffel symbols and its derivatives as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\kappa\mu} \Gamma^\kappa_{\beta\nu} - \Gamma^\alpha_{\kappa\nu} \Gamma^\kappa_{\beta\mu} \tag{14}$$

The Riemann Tensor upon contraction yields Ricci tensor. Using the Ricci tensor, the **curvature scalar** can be defined. The curvature scalar R is the trace of the Ricci tensor with respect to the metric associated with manifold of interest.

$$R = g^{\mu\nu} R_{\mu\nu} \tag{15}$$

The curvature scalar associates a real number to a point in the manifold depending upon the intrinsic geometry.

4.4 Position four-vector

The position 4-vector provides information about the position in space-time of the concerned event.

$$\mathbf{X} = \{x^0, x^1, x^2, x^3\}$$

Conventionally, x^0 is taken to be the time coordinate and x^1, x^2 and x^3 are taken to be the spatial coordinates. The proper-time derivative of the position 4-vector gives the velocity 4-vector. Some concepts from special theory of relativity are required for finding the derivative. Discussing all those details is beyond the scope of this project, so in the next section 4-velocity is discussed without going very deep into special relativity.

4.5 Velocity four-vector

As discussed above, 4-velocity can be obtained by differentiating 4-position with respect to "proper time". Proper time (represented by τ) is the time which all observers irrespective of their frames can agree on. The mathematically rigorous definition of proper time will not be discussed here.

$$\mathbf{V} = \frac{d\mathbf{X}}{d\tau}$$

Looking at one component at a time gives:

$$V^\mu = \frac{dX^\mu}{d\tau}$$

The energy-momentum 4-vector is defined using the velocity 4-vector.

4.6 Energy-momentum four-vector

We know that momentum p=mv, so in an analogous manner, the energy-momentum 4-vector ($\mathbf{p}$) is defined as

$$\mathbf{p} = m\mathbf{V}$$

Looking at one component at a time gives:

$$p^\mu = mV^\mu$$

The time-like component of the energy-momentum 4-vector (p^0) gives the energy (this is because for ease of calculations in special relativity, c (light speed) is taken to be equal to 1).

4.7 Energy-Momentum Tensor

The energy momentum tensor or the stress-energy tensor or the stress-energy-momentum tensor ($T^{\alpha\beta}$) is a tensor of order 2 which gives the flux ofth component of momentum vector across a surface with constant x^β co-ordinate. For all our purposes, it is a symmetric tensor, i.e.,

$$T^{\alpha\beta} = T^{\beta\alpha}$$

In Layman's terms, it gives information about the energy-like aspects of a system such as energy density, pressure, stress, etc.

Perfect fluid is the one which has no viscosity or heat conduction. It can be completely characterized by its density and pressure. **Dust** refers to a collection of particles at rest with respect to each other. A velocity field ($U^\mu(\mathbf{x})$)may be associated with it such that all components are same at each point. In case of dust, the energy-momentum tensor can be written as

$$T^{\mu\nu}{}_{dust} = \rho U^\mu U^\nu \tag{16}$$

In case of a general perfect fluid, the relationship is not as straightforward. A perfect fluid, by definition, is **isotropic** in rest frame. Using this definition it can be inferred that the matrix representing the energy momentum tensor will be a diagonal matrix. Furthermore, the non-zero indexed diagonal elements will be equal. $T^0 0$ is assigned to energy density (ρ) and the other diagonal elements are assigned to pressure (P). The matrix looks like:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{17}$$

Therefore, the energy-momentum of any general perfect fluid can be represented as

$$T^{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} + P\eta^{\mu\nu} \tag{18}$$

4.8 Einstein Field Equations

One of the most famous set equations in physics is "Einstein Field Equations". The various terms which make up the EFL's (abbreviated form of Einstein Field Equations) have been discussed in the previous parts, namely, the metric tensor, the Ricci tensor, the Curvature Scalar and the energy-momentum tensor. The equations are derived by suitably guessing a relationship similar to the Poisson's equation ($\nabla^2\phi = 4\pi G\rho$, where G is the gravitational constant and ρ is the mass density). The EFL's look like:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{19}$$

The left side consists of terms dealing with geometry and curvature of space-time whereas, the right side consists of energy-momentum term. Thus, in simple words it can be said "*Curvature of space-time tells the mass how to move, and mass tells space-time how to warp and curve*".

5 Introduction to Cosmology

The branch of astronomy which involves study of the universe is called cosmology. It has 2 sub-branches:

1. Cosmogony: This sub-branch deals with the "origin of universe"

2. Cosmography: This sub-branch involves recording and analysing features of the universe.

5.1 Cosmological Principle

Cosmological principle is the hypothesis that at the scale of the largest structures of the universe, it is spatially **homogeneous** and **isotropic**. The two words mean:

1. Homogeneous: The universe is uniformly filled on an average.
2. Isotropic: The universe is similar in all directions.

5.2 Hubble's Law

Einstein's field equations led many mathematicians and physicists to theoretically infer that the universe is expanding. In the early 20^{th} century, scientists observed that the distant galaxies seemed to be moving away due to which they could find a red-shift in the incoming light which can be explained using Doppler's effect. The red-shift (z) is calculated using the following formula:

$$z = \frac{\Delta\lambda}{\lambda} \tag{20}$$

Here λ is wavelength of emitted light and $\Delta\lambda$ is difference between observed wavelength and emitted wavelength. Red-shift can be expressed in terms of relative velocity as:

$$z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1 \tag{21}$$

In the year 1929, Edwin Hubble made a number of observations about the velocity at which galaxies were moving away and finally led to the conclusion that this velocity was proportional to the distance between them. Mathematically, the Hubble's law can be stated as follows:

$$v = H_0 D \tag{22}$$

Here H_0 is the proportionality constant known as Hubble's constant which corresponds to the value of time-dependent Hubble's parameter **H**, v is the recessional velocity and D is the proper distance (which varies with time.

Consider that the galaxies are arranged in a grid like fashion with the galaxies on at the lattice points (which is not a bad thing to do considering the assumptions in the cosmological principle). Now, considering the universe is expanding, we associate a scale factor (a(t)) with the smallest unit of distance in reference to the grid. It can be easily understood that the distance between two galaxies (say A and B) in the grid is

$$D_{AB} = a(t)\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

Here Δx, Δy and Δz are the differences in there x,y and z co-moving coordinates respectively. The scale factor appears in the equation because the grid itself is expanding such that the galaxies are always at the lattice points or we can say that the smallest unit of length itself is getting scaled with time by the factor $a(t)$. For simplicity of calculation, the two galaxies under consideration are assumed to be placed at two points on the x-axis such that there coordinate differs by Δx. So the distance between them is at time 't' is

$$D_{AB} = a(t)\Delta x \tag{23}$$

Taking it's time derivative yields

$$v_{AB} = \dot{a}(t)\Delta x \tag{24}$$

Here $\dot{a}(t)$ represents the time derivative of $a(t)$. Dividing equation (23) by equation (22) renders an interesting result:

$$\frac{v_{AB}}{D_{AB}} = \frac{\dot{a}(t)}{a(t)} = \mathbf{H}(t) \tag{25}$$

Here $\mathbf{H}(t)$ is the Hubble's parameter (discussed above).

6 Math of the Universe

Finding the dependence of scale factor (a(t)) on time is a very important and rigorous task. It can be used to calculate the value of Hubble's parameter. The observations combined with mathematical ideas give rise to mathematical models of the universe. These models can be used to understand a lot about the present state of the universe, how it used to be in the distant past and how it will be in the future.

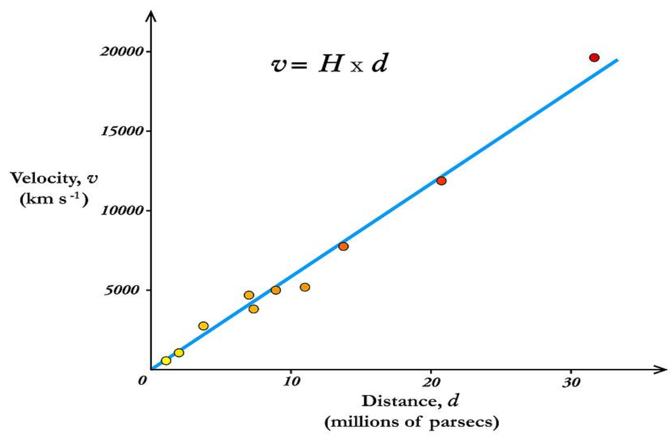


Figure 3: Plot of recessional velocity against distance

6.1 Newtonian Approach

Consider a cube in this grid having dimensions Δx, Δy and Δz. The volume of this cube (V) is $a^3(t)\Delta x\Delta y\Delta z$. Since, the grid itself is expanding (i.e, the unit of length itself is changing), the mass content inside any volume will remain time-invariant. Let the mass inside a cube with $\Delta x = \Delta y = \Delta z = 1$ time-varying unit be ν. Using this the density(ρ) can be found out in the following way:

$$\rho = \frac{Mass}{Volume} = \frac{\nu\Delta x\Delta y\Delta z}{a^3(t)\Delta x\Delta y\Delta z} = \frac{\nu}{a^3(t)}$$

At this point, we consider the spherical polar coordinate system (r, θ, ϕ) for simplicity. Consider the observer to be at the origin of coordinates and that the universe is homogeneously and isotropically filled with galaxies (cosmological principle). A galaxy (say 'A') at the point (R, θ, ϕ) is taken into consideration. The distance (D) of the observer from the galaxy changes as $D = a(t)R$. Assuming that the only force which acts on the galaxies is gravitation. So the force experienced by the galaxy 'A' (assuming it's own mass to be m and the total mass contained within a sphere of radius R is M) is given by:

$$F = -\frac{GMm}{D^2} \tag{26}$$

Which means that the acceleration(A) of the galaxy is

$$A = -\frac{GM}{D^2} \tag{27}$$

Acceleration can also be found by double differentiating the equation $D = a(t)R$ with respect to time. This gives:

$$A = \ddot{D} = \ddot{a}(t)R \tag{28}$$

Equating the two different expressions of acceleration found in equations (27) and (28) gives:

$$\ddot{a}(t)R = -\frac{GM}{a^2(t)R^2} \tag{29}$$

Dividing both sides by $a(t)R$ gives an interesting result which is as follows:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{GM}{a^3(t)R^3} \tag{30}$$

This equation can be modified a little by writing M as ρV, i.e., $\frac{4}{3}\pi a^3(t)R^3\rho$.

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G\rho}{3} \tag{31}$$

Equation (31) shows that the ratio $\frac{\ddot{a}(t)}{a(t)}$ is independent of R. We know that $\rho = \frac{\nu}{a^3(t)}$ from equation (25).
Substituting ρ in equation (31) gives:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G\nu}{3a^3(t)} \tag{32}$$

An important inference is that $\ddot{a}(t)$ is negative which implies that the velocity of recession is retarding. So there are 3 possibilities:

1. The universe keeps on expanding indefinitely

2. The universe on the verge of contracting
3. The universe expands till a certain point of time and then contracts to ultimately collapse.

The cases are very similar to those which are possible when an object is thrown/fired up from the earth's surface. In the latter case, energy considerations helped in developing the math. So we write the potential and kinetic energies of the galaxy 'A'. The kinetic energy (K) would like:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{a}^2(t)R^2 \tag{33}$$

The potential energy (U) is purely gravitational. Thus the expression for U would look like:

$$U = -\frac{GMm}{D} = -\frac{GMm}{a(t)R} \tag{34}$$

Hence, the total energy (E) is given by:

$$E = K + U \tag{35}$$

Substituting values of K and U from equations (33) and (34) respectively gives:

$$E = \frac{1}{2}m\dot{a}^2(t)R^2 - \frac{GMm}{a(t)R} \tag{36}$$

The above mentioned case 1 corresponds to positive energy, case 2 corresponds to 0 energy and case 3 corresponds to negative energy. The case in which E=0 is easy to analyse. Equation (36) with E=0 looks like:

$$0 = \frac{1}{2}m\dot{a}^2(t)R^2 - \frac{GMm}{a(t)R} \tag{37}$$

This equation can be further modified by substituting M in terms V and ρ (which is $\frac{\nu}{a^3}$).

$$(\frac{\dot{a}(t)}{a(t)})^2 = \frac{8\pi G\rho}{3} = \frac{8\pi G\nu}{3a^3} \tag{38}$$

Solving this differential equation is straightforward and it can be concluded that $a(t)$ comes out to be varying with time as $t^{\frac{2}{3}}$. From now onwards $a(t)$ will be written simply as a. In order to solve the two other cases, we take

$2E/m$ to be C (a constant). Also, since R can be set arbitrarily, we set it to 1 for simplicity. Slight manipulation of equation (36) gives:

$$\dot{a}^2 - \frac{2GM}{a} = C \tag{39}$$

Dividing both sides by a^2 and using the relation $V = \frac{4\pi}{3}a^3$ gives:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\nu}{3a^3} + \frac{C}{a^2} \tag{40}$$

At this point, some approximations need to be made. When a is small enough, we have $a^2 >> a^3$ which implies $\frac{1}{a^2} << \frac{1}{a^3}$. So equation (39), to a great extent, reduces to equation (38). Thus in this case also the scale factor a varies with time as $t^{\frac{2}{3}}$. If C is taken to be positive then for large a, $\frac{1}{a^3}$ term will be greatly dominated by $\frac{1}{a^2}$ term. This approximates the RHS in equation (40) to $\frac{C}{a^2}$. The modified equation looks like

$$(\frac{\dot{a}}{a})^2 = \frac{C}{a^2} \tag{41}$$

On solving this differential equation, one can easily infer that a varies linearly with time, i.e., $a \propto t$. The above calculations also lead to the conclusion that in the case wherein C is negative, the value of a can't really become large enough so that the $\frac{1}{a^2}$ could outweigh the $\frac{1}{a^3}$ term. This sets an upper limit to a. Thus the variation of a with time would increase for a certain time interval and then start decreasing, ultimately returning back to 0, i.e., the universe re-collapses. But all the above calculations are valid only for "matter" dominated universe (i.e. energy of the universe largely comes from matter). In order to bring in contribution of radiations, the energy density needs to be found out. Consider a simple example, there is a box which is filled with photons which expands with time by a factor $a(t)$. As the box expands, the effective wavelength also increases by a factor of $a(t)$ which implies that the energy of the photon decreases by a factor of $a(t)$. Therefore the energy density ρ varies with $a(t)$ as $a^{-4}(t)$. Thus, in this case, the relation between ρ and a looks like:

$$\rho = \frac{\nu}{a^4} \tag{42}$$

Consider that the universe consisted of only radiation and no matter. In such a universe, the equivalent of equation (38) would look like:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} = \frac{8\pi G\nu}{3a^4} \tag{43}$$

Solving this differential equation gives $a \propto t^{\frac{1}{2}}$. Therefore it can be inferred that in a hypothetical universe consisting only of matter the scale factor $a(t)$ varies with time as $t^{\frac{2}{3}}$ and in another hypothetical universe consisting only of radiation it varies was $t^{\frac{1}{2}}$. Now the mixed case can be analysed, i.e., the universe which consists of both matter and radiation. In this case the square of Hubble's parameter will have contributions from both matter and radiation, i.e., the expression will contain both $\frac{1}{a^3}$ and $\frac{1}{a^4}$ terms.

$$\mathbf{H}^2(t) = (\frac{\dot{a}}{a})^2 = \frac{C_M}{a^3} + \frac{C_R}{a^4} \tag{44}$$

Here C_M and C_R are constants which indicate the weightage of each the terms. Clearly for low values of a, the radiation term dominates whereas for large values of a, the matter term dominates. Thus when the universe was newly born, it mainly consisted of radiations which means a varied with time as $t^{\frac{1}{2}}$ but as time passed, gradually the matter term started becoming significant and at sufficiently later times the matter term started dominating over the radiation term. Presently, the universe we see is matter dominated. Thus a varies with time approximately as $t^{\frac{2}{3}}$.

6.2 Relativistic Approach

Before getting into the differential equations, cosmological metric must be discussed.

6.2.1 Cosmological Metric

Geometry of space is defined by metric. The metric of a circular space (1-sphere) is denoted by Ω_1, that of a spherical space (2-sphere) is denoted by Ω_2 and so on. Thus the metric of flat space will be:

$$dF_n^2 = dr^2 + r^2 d\Omega_{n-1}^2 \tag{45}$$

Here F_n represents the metric of n-dimension flat space and Ω_{n-1} represents the metric of the sphere of one less dimension. Metric of a spherical space will be:

$$d\Omega_n^2 = dr^2 + sin^2 r d\Omega_{n-1}^2 \tag{46}$$

Another kind of spaces is hyperbolic space. It's metric is given by:

$$dH_n^2 = dr^2 + sinh^2 r d\Omega_{n-1}^2 \tag{47}$$

Usually curved spaces are represented using a method called stereographic projection. In this method each point of a sphere is mapped to a point on flat plane. Similarly for hyperbolic space. Figure 4 illustrates how the stereographic projection of a sphere looks.

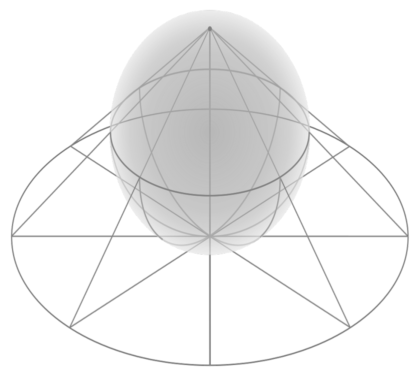


Figure 4: Stereographic projection of a sphere

Space along with the time coordinate makes space-time. The above discussed metrics are all space-only metrics. The metrics of flat, spherical and hyperbolic space-time are as follows. In case of n-dimensional flat space-time the metric looks like:

$$ds^2 = -dt^2 + a^2(t)dF_n^2$$

Notice that here the scale factor is multiplied with the spatial metric term. In case of n-sphere space-time the metric would look like:

$$ds^2 = -dt^2 + a^2(t)d\Omega_n \tag{48}$$

Similarly, in case of n-hyperbolic space-time the metric would look like:

$$ds^2 = -dt^2 + a^2(t)dH_n^2 \tag{49}$$

Non-Euclidean geometric spaces may have positive or negative curvature. In Layman's terms, the positively curved spaces are the ones in which the internal angle sum of a triangle exceeds π radians whereas in a negatively curved space the angle sum is less than pi radians. Thus it can be easily inferred that a spherical space is positively curved and a hyperbolic space is negatively curved. A scalar k is associated to curvature. k is 1 for positively curved spaces, 0 for flat space and -1 for negatively curved space.

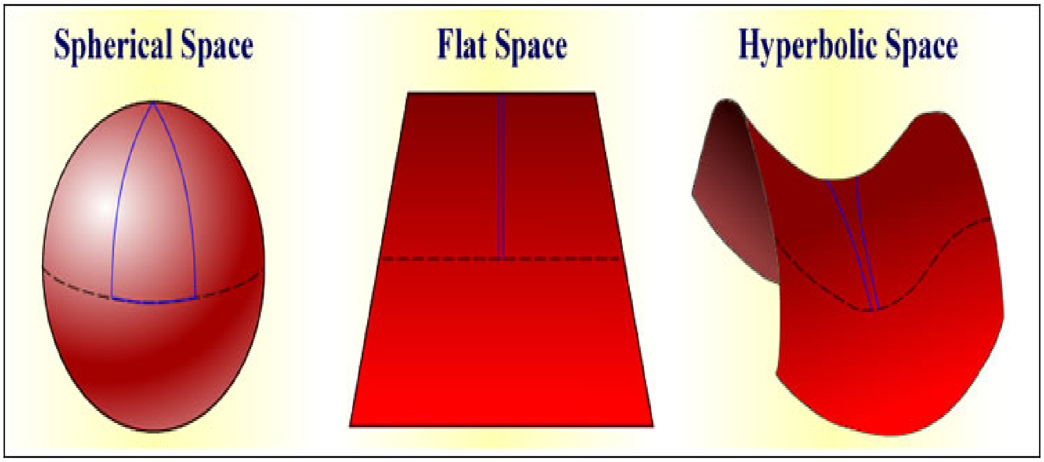


Figure 5: The possible geometries of universe

6.2.2 Friedmann Equation

It is a very important differential equation in cosmology which governs the expansion of the universe within the context of general relativity. The equation relates Hubble's parameter to curvature and energy density. On writing the Einstein's Field equations for such a universe it can be easily observed that the Einstein tensor $G_{\mu\nu}$ will have only one independent component (G_{00}), given by

$$G_{00} = 3(\frac{\dot{a}}{a})^2 + 3\frac{k}{a^2} \tag{50}$$

Substituting this in $G_{00} = 8\pi G T_{00}$ gives the Friedmann equation.

$$\mathbf{H}^2(t) = (\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{51}$$

6.3 Fluid Equation

The fluid equation links together density, its time derivative and pressure. The equation looks like:

$$\dot{\rho} + \frac{\dot{a}}{a}(\rho + \frac{P}{c^2}) = 0 \tag{52}$$

This along with the Friedmann equation yield what is called the **acceleration equation**.

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4G\pi}{3}(\rho + \frac{3P}{c^2}) \tag{53}$$

6.3.1 Equation of State

Equation of state of a perfect is a linear relationship between energy density ρ and pressure P. The equation is characterized by the slope w.

$$P = w\rho \tag{54}$$

Suppose the universe is modelled as an expanding box filled with some perfect fluid. It is known that $dE = -PdV$ and $E = \rho V$. Differentiating the latter gives:

$$dE = \rho dV + V d\rho \tag{55}$$

Equating the two expressions of dE, so obtained, gives:

$$V d\rho = -(P + \rho)dV \tag{56}$$

Substituting P using the relation $P = w\rho$ and solving the differential equation gives:

$$\rho \propto \frac{1}{a^{3(1+w)}} \tag{57}$$

The task is to find the value of w for matter, radiation and dark energy/vacuum energy. The case of matter is trivial as the energy consists of both rest energy and kinetic energy whereas pressure comes on from the collision of particles with the walls of the box. This means that the pressure is much smaller than the energy density. Thus w is practically 0 in this case. Solving the fluid equation for this case yields:

$$a(t) \propto t^{\frac{2}{3}} \tag{58}$$

Thus for matter dominated case we have:

$$\mathbf{H}(t) = \frac{2}{3t} \tag{59}$$

The case of radiation is discussed next. Consider a box of photons (all having same energy ϵ). Each photon has 3 components of velocity (v_x, v_y, v_z) such that $v_x^2 + v_y^2 + v_z^2 = c^2$=1 (natural units). On an average each direction is equally likely, so, $v_x = v_y = v_z = \frac{1}{\sqrt{3}}$. Let the length of the box be L. Motion along the x-axis is analysed next.

$$L = v_x \Delta t = \frac{\Delta t}{\sqrt{3}} \tag{60}$$

Number of particles moving towards a particular wall (N) can be easily calculated as shown below:

$$N = \frac{1}{2}\nu.Vol = \frac{1}{2}\nu L^3 \tag{61}$$

The momentum change (per photon) on hitting the wall is given by

$$\Delta p_x = 2p_x \tag{62}$$

It is known that for a photon energy ϵ is p (as is c is 1 in natural units). Momentum change along x-axis in terms of energy is $\frac{2\epsilon}{\sqrt{3}}$. Pressure is simply $\frac{\Delta p}{A\Delta t}$.

$$P = \frac{\frac{2\epsilon}{\sqrt{3}}\frac{L^3\nu}{2}}{L^2\Delta t} \tag{63}$$

This on simplification (using equation (56)) gives

$$P = \frac{1}{3}\rho \tag{64}$$

Hence the value of w for radiation is $\frac{1}{3}$. Using this to solve the fluid equation yields:

$$a(t) \propto t^{\frac{1}{2}} \tag{65}$$

Thus for radiation dominated case we have:

$$\mathbf{H(t)} = \frac{1}{2t} \tag{66}$$

In any volume vacuum energy is present and the associated energy density is constant universally. This can be derived using the **Quantum Field Theory** (QFT). Discussing QFT in detail is beyond the scope of this project. QFT states that every fundamental field must be quantized everywhere in space. Effects of vacuum energy can be seen in several experiments such as spontaneous emission, Casimir effect etc. Vacuum energy can be thought of in terms of virtual particles. A virtual particle is basically a short-spanned quantum fluctuation. The reason why it is called a "particle" is that it shares some properties with any ordinary particle but the Heisenberg's uncertainty principle limits it's existence. The perturbation theory gives rise to this concept as it expresses the various interactions between ordinary particles as exchanges of virtual-particles. In the year 1948, Richard

Feynman came up idea to represent abstract mathematical equations which describe the interactions of subatomic particles using simple diagrams. The scheme was named after him and hence called **Feynman diagrams**. Feynman diagrams are used to pictorially depict interactions between subatomic particles wherein the virtual particles are represented by internal lines. Figure 6 shows a typical Feynman diagram. In the figure, the sinusoidal line which starts and ends in the diagram represents a virtual particle (which in this case is a virtual photon).

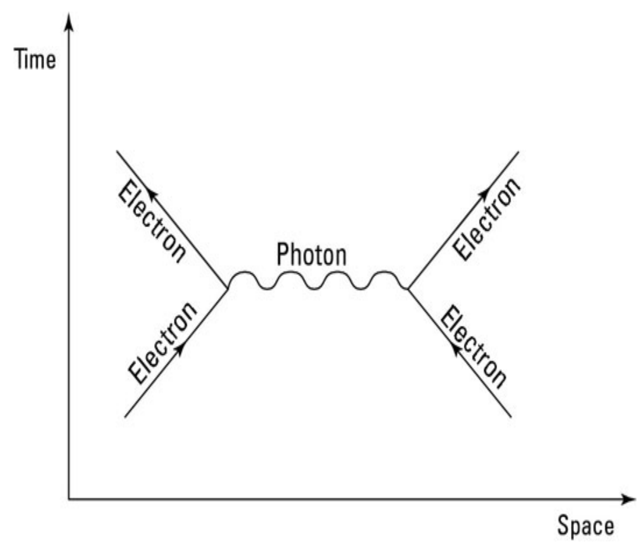


Figure 6: Feynman diagram representing electromagnetic force between 2 charges

Since it is known, from QFT, that the energy density of vacuum energy is constant universally, back calculation can be performed in order to obtain the value of w in this case. The following equation can derived using QFT.

$$8\pi G\rho_0 = \Lambda \tag{67}$$

Here ρ_0 is the vacuum energy density and Λ is the cosmological constant (discussed later). As ρ is constant ($=\rho_0$), we have:

$$\rho_0 dV = -PdV \tag{68}$$

Thus $P = -\rho_0$ implying that the value of w in this case is -1. Experimental evidences show that w actually comes close to -1 in various locations in the universe. Dark energy is anything that accelerates the expansion of the universe. Anything whose equation of state parameter w is $<-\frac{1}{3}$ is a subtle dark energy candidate. One of the simplest candidate for dark energy is the vacuum energy that is a perfect fluid having constant density.

6.3.2 Cosmological constant

The cosmological constant Λ was first introduced by Einstein as a modification to the original Field Equations. He did this because his equations implied that the universe expands and thus he wanted to put an extra term which could make it static. The modified Einstein's Field Equations look like:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{69}$$

Einstein called it his greatest mistake. But eventually physicists realised that incorporation of the cosmological actually helped the theoretical results to explain the experimental results. The Friedmann equation with the cosmological constant looks like:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{70}$$

The acceleration equation in this case would look like:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4G\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{71}$$

The observations shockingly reveal that the universe is accelerating. But the models discussed above have negative $\ddot{a}(t)$. This shows the importance of cosmological constants which acts as an accelerating agent for the universe.

6.4 Critical Density and Density Parameter

Critical density (ρ_C) is the average density of matter required for the universe to "just" halt its expansion after ∞ time. Alternatively, it can be said that the density that the universe would need to have to be flat. Critical density can be expressed as:

$$\rho_C = \frac{3\mathbf{H}^2}{8G\pi} \tag{72}$$

Here $\mathbf{H}$ is the Hubble's parameter. Another parameter of great cosmological importance is the density parameter (Ω_0). It is defined as the ratio of the average density of matter and energy in the universe to the critical density.

$$\Omega_0 = \frac{\rho}{\rho_C} \tag{73}$$

The present value of Ω_0 (found experimentally) is very close to 1. Theoretically, there are three possibilities:

1. $\Omega_0 < 1$: This implies the universe is open (hyperbolic geometry). Thus the universe will expand indefinitely.

2. $\Omega_0 = 1$: This implies the universe is flat. Thus it keeps on expanding to ultimately halt its expansion after infinite time.

3. $\Omega_0 > 1$: This implies the universe is closed (Spherical geometry). Thus it will eventually stop expanding and re-collapse.

The actual experimental result obtained has been discussed in the next section.

7 Experimental Results

This section covers how the values of various parameters are calculated and their experimental values. First of all, Hubble's parameter $\mathbf{H}(t)$ is discussed. First the red-shift experienced by the light coming from a particular galaxy is found. Using this the recessional velocity is calculated ($v = z.c$). The distance at which the galaxy is located is found using the formula $D = \frac{d}{\theta}$, where d is the size of galaxy and θ is the angular distance. Then the velocity is plotted against the distance of the galaxy. The curve obtained is approximately a line, the slope of which is the present value of Hubble's parameter (as $\mathbf{H}(t) = \frac{v}{D}$). The various methods used to determine the value of Hubble's parameter include ΛCDM (Lambda-Cold Dark Matter), WMAP (Wilkinson Microwave Anisotropy Probe) and ACT (Atacama Cosmology Telescope). The best estimate so far is 69.6 ± 0.7 km/s. Using this the age of universe is calculated to be approximately 13.8 billion years.

In this portion the experimental values of and conclusions drawn from density parameter have been discussed. The ρ in the numerator of RHS in equation (69) is the total energy density of the universe inclusive of all components. Thus the total density parameter is the sum of the density parameters of individual components.

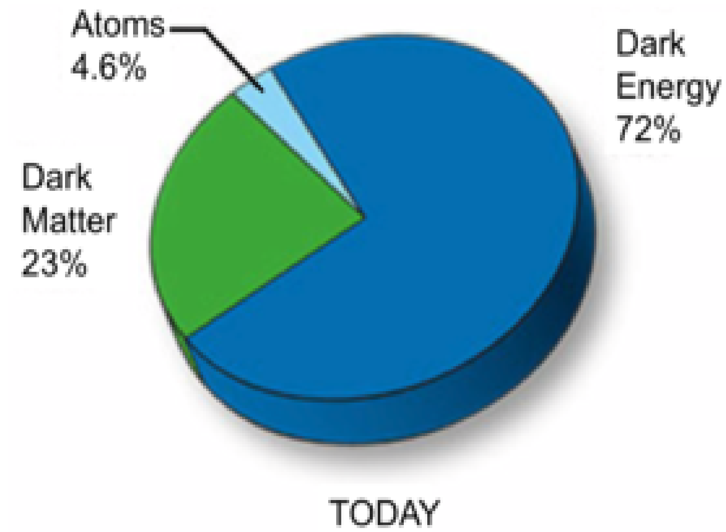


Figure 7: Schematic representation of energy density of components of the universe

The total density parameter of the universe can be expressed in the following fashion:

$$\Omega_0 = \Omega_B + \Omega_D + \Omega_\Lambda \tag{74}$$

Here Ω_B, Ω_D and Ω_Λ are the density parameter's of Baryonic matter, Dark matter and Dark energy respectively. Baryonic matter in Layman's terms refer to the matter built up of atom. Current observations suggest, $\Omega_\Lambda = 0.73$, $\Omega_D = 0.23$ and $\Omega_B = 0.04$. So their sum is 1. Thus the universe in the present times is flat. It is believed that the early universe was radiation dominated, then eventually matter component became dominant. Presently the most dominant component is dark energy. Another very mysterious component is dark matter which has been discussed in the next section.

8 Dark Matter

The name dark matter is somewhat misleading. What it actually refers to can be thought of as "invisible matter". It doesn't interact with light (or electromagnetic radiation in general) unlike all the visible matter in the universe. But they interact with the visible matter gravitationally. Physicists have found many evidences of its existence but still no one has the answer to what it is made up of. Though, physicists have some candidates in mind for dark matter.

8.1 Existence of Dark Matter

Provided that we don't know what dark matter is made up of, a common question could be what made the scientists to believe that such a thing even exists, i.e., what are the cosmological observations which necessitated the existence of dark matter. The observations and evidences which led to the concept of dark matter are discussed below.

8.1.1 Rotation of Galaxies

Galactic rotation curves are great evidences of existence of matter other visible matter. Figure 8 shows how a galaxy looks. It can be clearly seen that the visible matter is majorly concentrated near the core (where taking its density to be constant is a subtle approximation) and at larger radial distances the density of visible matter is quite small.
Rotation curves are variation of circular velocity (v_c) of galaxy with radial distance (r). Supposing that the galaxy is constituted of only matter. The circular velocity can be calculated by equating gravitational pull to the centripetal force.

$$\frac{mv_c^2}{r} = \frac{GM(r)m}{r^2} \tag{75}$$

Here $M(r)$ represents the mass inside a disc of uniform thickness with the core as it center and r as it radius. The above equation on simplification gives:

$$v_c = \sqrt{\frac{GM(r)}{r}} \tag{76}$$

Considering the density of matter (ρ) near the core to be constant, it can be inferred that $M(r)$ is proportional to r^2 for smaller radial distances.

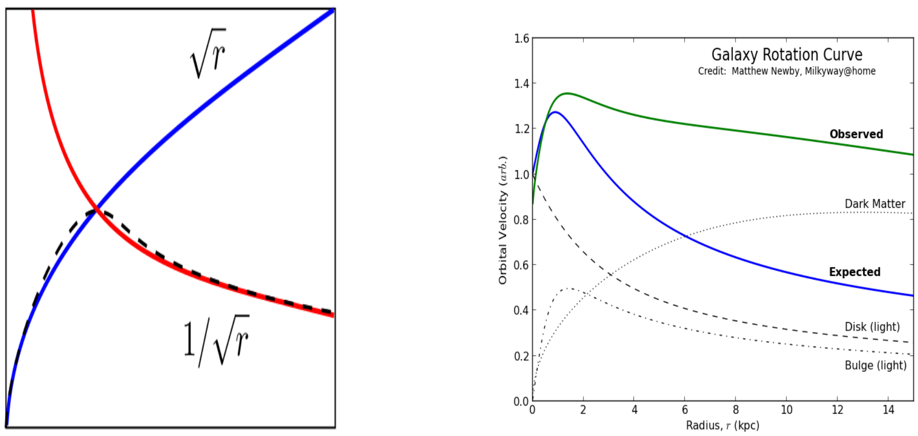
Figure 8: Structure of our galaxy-Milky Way

While, at larger distances the amount of matter is very less, thus, we can take the mass to be constant.

This means the circular velocity for small r is directly proportional to $\sqrt{r}$ and for large r it is inversely proportional to $\sqrt{r}$. Figure 9 (a) shows the expected . But the actual rotation curve (shown in Figure 9 (b)) obtained by experimentation is different. The actual curve becomes almost flat for large values of r. Explaining this behaviour was a challenge and thus it was necessary to think of some sort of matter which could not be detected using electromagnetic radiation but interacts gravitationally. This proved the existence of dark matter.

8.1.2 Galaxy Clusters

The amount of gas in the **Coma** cluster was estimated to be $M = 1.6 \times 10^{14} M_{\odot}$ ($M_{\odot}$ represents mass of the sun). Mass was then calculated using the virial theorem. The virial theorem provides a general equation that relates time-averaged total kinetic energy T of a stable system of discrete particles with the time-averaged total potential energy of the system.



(a) Expected Rotation Curve

(b) Actual Rotation Curve

Figure 9: The expected and actual rotation curves

Mathematically it can be shown as:

$$< T >= -\frac{1}{2}\sum_{k=1}^{N} < \vec{F}_k.\vec{r}_k > \tag{77}$$

Using this theorem, the mass of matter in the cluster was found to be $M = 1.9 \times 10^{15} M_{\odot}$, which is approximately 10-times the estimated mass of luminous matter. This led to the conclusion that more matter other than the luminous matter is present which cannot be detected using electromagnetic radiation. So this also necessitated the existence of dark matter.

8.2 Candidate Possibilities for Dark Matter

Dark matter is still a mystery for physicists thus at present only candidate possibilities can be thought of and nothing can be said with surety. Dark matter can be baryonic or non-baryonic. Non-baryonic dark matter is subdivided into two categories namely, Hot Dark Matter (HDM) and Cold Dark Matter (CDM).

8.2.1 Hot Dark Matter

Hot Dark Matter requires nearly mass-less particles. Thus neutrinos, axions and supersymmetric particles are prime examples. Neutrino is a Fermion (i.e. spin $\frac{1}{2}$ elementary particle) which interacts only via weak-interactions and gravity. Its rest mass is very small (in most practical cases negligible). The axion is a hypothetical elementary particle. Further discussion on axion requires the knowledge of topics like Peccei-Quinn Theory and Quantum Chromodynamics (QCD) which are beyond the scope of this project. From the special relativity theory it is known that particles with near-zero rest mass must move with speeds close to that of light. HDM does not fully account for the large scale structure of galaxies in the universe because of highly relativistic nature. The example of neutrino explains the previous line. Due to highly relativistic velocities they would tend to smooth-out the fluctuations in the matter density. Thus they will be good at forming large structures like super-clusters but not smaller ones like galaxies.

8.2.2 Cold Dark Matter

Cold Dark Matter requires objects of sufficient mass so that they move at sub-relativistic velocities. Observation of large scale structures and N-body simulations indicate that CDM is the major component of dark matter. A major candidate for CDM is WIMP (Weakly Interacting Massive Particle). Physicists aim for searching these particles by direct detection methods or production in particle accelerators. Several teams of physicists are trying to detect WIMP using ultra-sensitive detectors. The collision of WIMP with an atom would force the WIMP to change direction and slowdown, whilst the atom recoils. There are certain methods by which the ionisation charge released by the recoiling atom can be measured.

Many theories also suggest that some baryonic matter can also be dark matter candidates. For example, MACHOs (Massive Compact Halo objects) are dark matter candidates inclusive of black-holes, neutron stars, white dwarfs and non-luminous objects (planets, brown dwarfs etc.). They can be searched using gravitational lensing. Gravitational lens is a distribution of matter between a distant light source and an observer which is capable of bending light. This effect is known as gravitational lensing. Amount of baryonic dark matter can be inferred from **Big Bang nucle-**

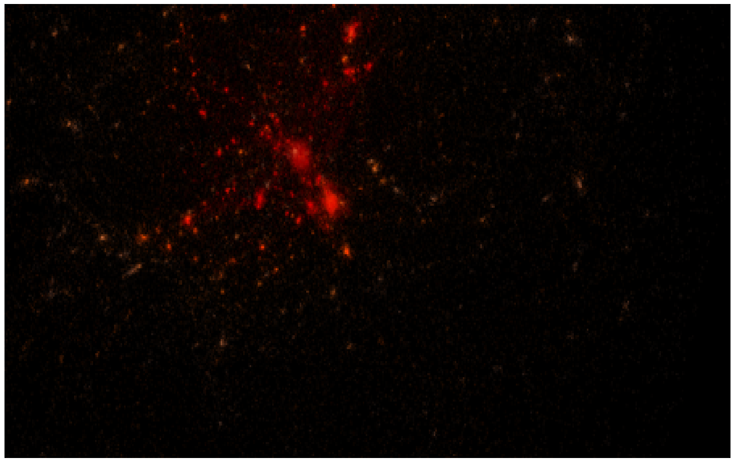


Figure 10: N-body simulation of formation of galaxy cluster

osynthesis and **Cosmic Microwave Background** radiation. The amount of baryonic dark matter estimated is much less as compared to the amount total dark matter. This clearly implies that baryonic dark matter can form only a small part of total dark matter present.

Big Bang nucleosynthesis refers to formation of nuclei other than H-1 (normal hydrogen) in the early phases of the universe. It is believed to be responsible for the formation of H-1 (hydrogen), H-2 (Deuterium), He-3 (Helium-3 isotope), He-4 (Helium) and Li-7 isotope. Twelve nuclear reactions are involved.

Cosmic Microwave Background is thought to be the leftover radiation from the Big Bang. According to the theory, the universe underwent rapid expansion and inflation when it was born. CMB represents the heat leftover from the Big Bang but it can't be seen or felt by human senses as it lies in the microwave region and is too cold (just 2.725 degrees above the absolute zero temperature).

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