PH 223 TSC

Shashwat Chakraborty

14^{th} September 2023

Recap

Cauchy-Riemann conditions

Let f be a complex function with u and v as its real and imaginary parts, respectively. Then, f is differentiable *iff* the following holds:

$$u_x = v_y$$
$$u_y = -v_x$$

In a more compact form:

$$f_y = i f_x$$

Equivalent condition for differentiability

f is differentiable $\longleftrightarrow$ "Pure function of z"

Q.
Is $u(x,y) = xy + 3x^2y - y^3$ harmonic? If yes, find it's harmonic conjugate.

Ans.
Yes! $u_{xx} + u_{yy} = 0 \implies u$ is harmonic. Let the harmonic conjugate be $v(x,y)$. u and v satisfy the CR conditions:

$$
\begin{aligned}
v_y &= u_x = y + 6xy \\
v &= \frac{y^2}{2} + 3xy^2 + g(x) \\
v_x &= 3y^2 + g'(x) = -u_y = -x - 3x^2 + 3y^2 \\
g(x) &= -\frac{x^2}{2} - x^3
\end{aligned}
$$

$$v(x,y) = \frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3$$

Questions

Q.
Consider a function f defined on a simply connected domain Ω.
Statement: If $f'(z) = 0 \ \forall z \in \Omega$, then $f(z) = \text{constant in } \Omega$.
True or False?

Ans.
False! The keyword to note is *simply connected*. Thus, Ω can be a union of two (or more) simply connected islands:

$$\Omega = \Omega_1 \cup \Omega_2$$

f can take different values c_1 and c_2 on the two islands.

Recap

Analytic functions

In simple terms, an analytic function is: *a function with a Taylor expansion.*

Radius of convergence:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Cauchy condition for analytic functions

Let f be an analytic function and $\mathcal{C}$ be a closed contour in the complex plane $\mathbb{C}$. Then

$$\int_{\mathcal{C}} f(z)dz = 0$$

Questions

Q.
Find the radius of convergence of the following power series (about $z = 0$):

$$S = \sum_{n} \frac{(-1)^n}{n!} z^{n^2}$$

Ans.
Use ratio test (or root test)! $R = 1$

Recap

Cauchy Integral Formula (CIF)

If $f(z)$ is analytic on and inside a contour $\mathcal{C}$, then (for $z_0 \in \mathcal{C}$)

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Recap

Cauchy Residue Theorem

Let f be a complex function having n singular points $(z_1, z_2, \cdots, z_n)$ in the interior of a closed contour $\mathcal{C}$. Then

$$\int_{\mathcal{C}} f(z)dz = \sum_{i=1}^{n} \mathrm{Res}(z_i)$$

Recap

Laurent Series

Suppose a *non-analytic* function f has an isolated singularity at z_0. In that case, it can be expressed as a series (Laurent series) in an annular region around z_0 bounded by two closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$
\begin{aligned}
f(z) &= \sum_{n\geq 0} a_n (z - z_0)^n + \sum_{n\geq 1} b_n (z - z_0)^{-n} \\
a_n &= \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \\
b_n &= \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi
\end{aligned}
$$

Q.
Evaluate the following integral:

$$\int_0^{2\pi} \frac{\cos^2(3x)}{5 - 4\cos(2x)} dx$$

Ans.
Define $z = e^{i\theta}$ and substitute

$$\cos(n\theta) = \left(z^n + z^{-n}\right)/2$$

Important Definitions:

$|a\rangle, |b\rangle \in S_N \longleftarrow$ N-dimensional vector space

1. Scalar product: $\langle a|b\rangle$
   - $\langle a|b\rangle = \overline{\langle b|a\rangle}$
   - $\langle a|a\rangle \geq 0$
   - bi-linear
2. Metric distance: $P(|a\rangle, |b\rangle)$
   - $P(|a\rangle, |b\rangle) = P(|b\rangle, |a\rangle)$
   - $P(|a\rangle, |b\rangle) = 0$ iff $|a\rangle = |b\rangle$
   - triangle inequality

Function Spaces

Things to remember:

1. Infinite-dimensional basis
2. Parseval's equality condition → "completeness"
3. Metric Distance:

$$P(|f\rangle, |g\rangle) = \sqrt{\int_a^b w(x)|f(x) - g(x)|^2 dx}$$

4. Scalar product:

$$\langle f|g\rangle = \int_a^b \underbrace{w(x)}_{\text{weight}} \overline{f}(x) g(x) dx$$

Fourier Series

Let f be a function on $[-L, L]$,then it's Fouries series expansion is given by

$$FS(f(x)) = \frac{1}{\sqrt{2L}}\left[A_0 + \sum_{m\geq 1} A_m \cos(\frac{m\pi x}{L}) + \sum_{m\geq 1} B_m \sin(\frac{m\pi x}{L})\right]$$

$$A_0 = \frac{1}{\sqrt{2L}}\int_{-L}^{L} f(x)dx$$

$$A_m = \sqrt{\frac{2}{L}}\int_{-L}^{L} f(x)\cos(\frac{m\pi x}{L})dx$$

$$B_m = \sqrt{\frac{2}{L}}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})dx$$

Q. When is it equal to $f(x)$??

Q.
Consider the following 2π-periodic function:

$$f(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ x^2 & \text{for } x \in [0, \pi) \end{cases}$$

Note that the 2π-periodicity of the above function essentially says that

$$f(\pi) := f(-\pi) = 0.$$

Find the Fourier series of the above function and deduce the value of the following, pretty well-known, infinite sum:

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Ans.

Let us note the following two results before proceeding ($n \in \mathbb{N}$)

$$\int_0^\pi x^2 \sin(nx)dx = \frac{(-1)^{n+1}}{n}\pi^2 + 2\frac{1-(-1)^n}{n^3}$$

and

$$\int_0^\pi x^2 \cos(nx)dx = \frac{2\pi(-1)^n}{n^2}$$

Thus, we obtain the Fourier series for

$f(x) := \begin{cases} 0 & \text{for } x \in [-\pi, 0) \\ x^2 & \text{for } x \in [0, \pi) \end{cases}$ is

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\cos(nx) + \left[\frac{(-1)^{n+1}}{n}\pi + \frac{2}{\pi}\cdot\frac{1-(-1)^n}{n^3}\right]\sin(n$$

Plugging in $x = 0$, we get

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2} \implies 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \to S = ?$$

Fourier Transform

FT and inverse FT

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is related to it as

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

Generalized Distributions

Generalized distributions are limits of sequences of *good* functions.

Dirac Delta

Properties:

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

$$\delta(x) f(x) = f(0) \delta(x)$$

FT for Differential Equations

Some important results:

1. FT of derivatives:

$$FT\left(g^{(m)}(x)\right) = (ik)^m FT\left(g(x)\right)$$

2. Derivatives of FT:

$$\frac{d^n}{d(-ik)^n} FT\left(g(x)\right) = FT\left(x^n g(x)\right)$$

3. Convolution theorem:

$$f(x) \otimes g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) g(x-y) dy$$

$$FT(f(x) \otimes g(x)) = FT(f(x)) \times FT(g(x))$$

Questions

Q.
Let $a > 0$. Applying the Fourier transform in the x variable, show the solution $u(t,x)$ to the following first-order partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

is

$$u(t,x) = u_0(x - at) \qquad \text{for } (t,x) \in [0,\infty) \times \mathbb{R}.$$

Ans.

On taking the Fourier transform of the given differential equation, we get

$$\frac{d}{dt}\hat{u} + ika\hat{u} = 0$$

Let the Fourier transform of the initial condition be denoted as $\hat{u}_0(k)$, given by

$$\hat{u}_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x) e^{-ikx} dx$$

The solution of the differential equation is clearly

$$\hat{u}(t,k) = \hat{u}_0(k) e^{-ikat}$$

Taking its inverse Fourier transform gives

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(k) e^{ik(x-at)} dx \\ &= u_0(x-at) \end{aligned}$$