CHAPTER 16

Relations and Functions

1. The period of $\sin^2\theta$ is **[2002]**
(a) π^2 (b) π
(c) 2π (d) $\pi/2$

2. Which one is not periodic? **[2002]**
(a) $|\sin 3x| + \sin^2 x$
(b) $\cos\sqrt{x} + \cos^2 x$
(c) $\cos 4x + \tan^2 x$
(d) $\cos 2x + \sin x$

3. The function $f(x) = \log\left(x + \sqrt{x^2+1}\right)$, is **[2003]**
(a) neither an even nor an odd function
(b) an even function
(c) an odd function
(d) a periodic function.

4. A function f from the set of natural numbers to integers defined by **[2003]**

$$f(n) = \begin{cases} \dfrac{n-1}{2}, \text{when n is odd} \\ -\dfrac{n}{2}, \text{when n is even} \end{cases} \quad \text{is}$$

(a) neither one -one nor onto
(b) one-one but not onto
(c) onto but not one-one
(d) one-one and onto both.

5. If $f: R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is **[2004]**
(a) $[-1, 3]$ (b) $[-1, 1]$
(c) $[0, 1]$ (d) $[0, 3]$

6. Let R = $\{(1,3),(4,2),(2,4),(2,3),(3,1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$.. The relation R is **[2004]**
(a) reflexive (b) transitive
(c) not symmetric (d) a function

7. Let f: (– 1, 1) $\to$ B, be a function defined by $f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$, then f is both one - one and onto when B is the interval **[2005]**
(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left[0, \dfrac{\pi}{2}\right)$
(c) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ (d) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

8. A real valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ where a is a given constant and $f(0) = 1$, $f(2a-x)$ is equal to **[2005]**
(a) $-f(x)$ (b) $f(x)$
(c) $f(a) + f(a-x)$ (d) $f(-x)$

9. Let $R = \{(3,3),(6,6),(9,9),(12,12),(6,12),(3,9),(3,12),(3,6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is **[2005]**
(a) reflexive and transitive only
(b) reflexive only
(c) an equivalence relation
(d) reflexive and symmetric only

10. Let W denote the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W|$ the words x and y have at least one letter in common.$\}$ Then R is **[2006]**

(a) not reflexive, symmetric and transitive
(b) relexive, symmetric and not transitive
(c) reflexive, symmetric and transitive
(d) reflexive, not symmetric and transitive

11. Let $f: N \to Y$ be a function defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$. Show that f is invertible and its inverse is **[2008]**

(a) $g(y) = \frac{3y+4}{3}$ (b) $g(y) = 4 + \frac{y+3}{4}$

(c) $g(y) = \frac{y+3}{4}$ (d) $g(y) = \frac{y-3}{4}$

12. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y) : y = x + 1$ and $0 < x < 2\}$
$T = \{(x, y) : x - y$ is an integer$\}$,
Which one of the following is true? **[2008]**

(a) Neither S nor T is an equivalence relation on R
(b) Both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

13. **DIRECTIONS :** This question contains two statements:
Statement-1 (Assertion) and Statement-2 (Reason).
This question also has four alternative choices, only one of which is the correct answer. You have to select the correct choice.

Let $f(x) = (x+1)^2 - 1,\ x \geq -1$

Statement -1 : The set $\{x : f(x) = f^{-1}(x) = \{0, -1\}$
Statement-2 : f is a bijection. **[2009]**

(a) Statement-1 is true, Statement-2 is true. Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true. Statement-2 is not a correct explanation for Statement-1.

14. For real x, let $f(x) = x^3 + 5x + 1$, then **[2009]**

(a) f is onto R but not one-one
(b) f is one-one and onto R
(c) f is neither one-one nor onto R
(d) f is one-one but not onto R

15. Consider the following relations:
R = $\{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$;

$S = \{\left(\frac{m}{n}, \frac{p}{q}\right) \mid m, n, p$ and q are integers such that $n, q \neq 0$ and $qm = pn\}$. Then **[2010]**

(a) Neither R nor S is an equivalence relation
(b) S is an equivalence relation but R is not an equivalence relation
(c) R and S both are equivalence relations
(d) R is an equivalence relation but S is not an equivalence relation

16. Let R be the set of real numbers. **[2011]**
Statement-1: $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.
Statement-2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

17. Let f be a function defined by

$f(x) = (x-1)^2 + 1, (x \geq 1)$. **[2011RS]**

Statement - 1 :

The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$.

Statement - 2:

f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is NOT a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

18. If g is the inverse of a function f and $f'(x)=\dfrac{1}{1+x^5}$, then $g'(x)$ is equal to: **[2014]**

(a) $\dfrac{1}{1+\{g(x)\}^5}$ (b) $1+\{g(x)\}^5$

(c) $1+x^5$ (d) $5x^4$

19. The function $f: R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x)=\dfrac{x}{1+x^2}$, is: **[2017]**

(a) neither injective nor surjective
(b) invertible
(c) injective but not surjective
(d) surjective but not injective

Answer Key

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(b)	(b)	(c)	(d)	(a)	(c)	(d)	(a)	(a)	(b)	(d)	(d)	(b)	(b)	(b)
16	17	18	19											
(a)	(a)	(b)	(d)											

SOLUTIONS

1. **(b)** $\sin^2\theta=\dfrac{1-\cos 2\theta}{2}$; Period $=\dfrac{2\pi}{2}=\pi$

2. **(b)** $\because \cos\sqrt{x}$ is non periodic

$\therefore \cos\sqrt{x}+\cos^2 x$ can not be periodic.

3. **(c)** $f(x)=\log(x+\sqrt{x^2+1})$

$f(-x)=\log\left\{-x+\sqrt{x^2+1}\right\}$

$=\log\left\{\dfrac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right\}$

$=-\log(x+\sqrt{x^2+1})=-f(x)$

$\Rightarrow f(x)$ is an odd function.

4. **(d)** We have $f: N \to I$

If x and y are two even natural numbers,

then $f(x)=f(y)\Rightarrow \dfrac{-x}{2}=\dfrac{-y}{2} \Rightarrow x=y$

Again if x and y are two odd natural numbers then

$f(x)=f(y)\Rightarrow \dfrac{x-1}{2}=\dfrac{y-1}{2} \Rightarrow x=y$

$\therefore f$ is onto.

Also each negative integer is an image of even natural number and each positive integer is an image of odd natural number.

$\therefore f$ is onto.

Hence f is one one and onto both.

5. **(a)** $f(x)$ is onto $\therefore S=$ range of $f(x)$

Now $f(x)=\sin x-\sqrt{3}\cos x+1$

$=2\sin\left(x-\dfrac{\pi}{3}\right)+1$

$\because -1\le \sin\left(x-\dfrac{\pi}{3}\right)\le 1$

$-1\le 2\sin\left(x-\dfrac{\pi}{3}\right)+1\le 3$

$\therefore f(x)\in[-1, 3]=S$

ALTERNATE SOLUTION

We know that

$-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$

$\therefore -2\le \sin x-\sqrt{3}\cos x\le 2$

$\Rightarrow -1 \le \sin x - \sqrt{3}\cos x + 1 \le 3$

$\therefore f(x) \in [-1, 3]$

6. **(c)** $\because$ (1, 1) $\notin R \Rightarrow R$ is not reflexive (2,3) $\in R$ but (3, 2) $\notin R$
$\therefore$ R is not symmetric

7. **(d)** Given $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$

for $x \in (-1, 1)$

If $x \in (-1, 1) \Rightarrow \tan^{-1} x \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow 2\tan^{-1} x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Clearly, range of $f(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

For f to be onto, codomain = range
$\therefore$ Co-domain of function = B =

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

8. **(a)** $f(2a-x) = f(a-(x-a))$
$= f(a)f(x-a) - f(0)f(x) = f(a)f(x-a) - f(x)$
$= -f(x)$

$[\because x = 0, y = 0,\ f(0) = f^2(0) - f^2(a)$

$\Rightarrow f^2(a) = 0 \Rightarrow f(a) = 0$

$\Rightarrow f(2a-x) = -f(x)$

9. **(a)** Reflexive and transitive only.
e.g. (3, 3), (6, 6), (9, 9), (12, 12) [Reflexive]
(3, 6), (6, 12), (3, 12) [Transitive].
(3, 6) $\in R$ but (6, 3) $\notin R$ [non symmetric]

10. **(b)** Clearly $(x, x) \in R \,\forall x \in W$. So R is reflexive.

Let $(x, y) \in R$, then $(y, x) \in R$ as x and y have at least one letter in common. So, R is symmetric.
But R is not transitive for example
Let x = INDIA, y = BOMBAY and z = JOKER
then $(x, y) \in R$ (A is common) and
$(y, z) \in R$ (O is common) but $(x, z) \notin R$. (as no letter is common)

11. **(d)** Clearly f is one one and onto, so invertible

Also $f(x) = 4x + 3 = y$

$\Rightarrow x = \frac{y-3}{4} \qquad \therefore\ g(y) = \frac{y-3}{4}$

12. **(d)** Given $S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$\because x \ne x + 1$ for any $x \in (0, 2)$
$\Rightarrow (x, x) \notin$ S
$\therefore$ S is not reflexive.
Hence S in not an equivalence relation.
Also $T = \{x, y) : x - y$ is an integer$\}$
$\because x - x = 0$ is an integer $\forall x \in R$
$\therefore$ T is reflexive.
If $x - y$ is an integer then $y - x$ is also an integer $\quad\therefore$ T is symmetric
If $x - y$ is an integer and $y - z$ is an integer then
$(x - y) + (y - z) = x - z$ is also an integer.
$\therefore$ T is transitive

13. **(b)** Given that $f(x) = (x+1)^2 - 1,\ x \ge -1$
Clearly $D_f = [-1, \infty)$ but co-domain is not given. Therefore $f(x)$ need not be necessarily onto.
But if $f(x)$ is onto then as $f(x)$ is one one also, $(x + 1)$ being something +ve,
$f^{-1}(x)$ will exist where
$(x+1)^2 - 1 = y$

$\Rightarrow x + 1 = \sqrt{y+1}$

(+ve square root as $x + 1 \ge 0$)

$\Rightarrow x = -1 + \sqrt{y+1}$

$\Rightarrow f^{-1}(x) = \sqrt{x+1} - 1$
Then $\quad f(x) = f^{-1}(x)$
$\Rightarrow (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow (x+1)^2 = \sqrt{x+1} \Rightarrow (x+1)^4 = (x+1)$
$\Rightarrow (x+1)[(x+1)^3 - 1] = 0 \Rightarrow x = -1, 0$
$\therefore$ The statement-1 is correct but statement-2 is false.

14. **(b)** Given that $f(x) = x^3 + 5x + 1$
$\therefore f'(x) = 3x^2 + 5 > 0,$
$\forall x \in R$
$\Rightarrow f(x)$ is strictly increasing on R
$\Rightarrow f(x)$ is one one
$\therefore$ Being a polynomial $f(x)$ is continuous and increasing.

on R with $\lim_{x \to \infty} f(x) = -\infty$

and $\lim_{x \to \infty} f(x) = \infty$

$\therefore$ Range of $f = (-\infty, \infty) = R$

Hence f is onto also. So, f is one one and onto R.

15. **(b)** x Ry need not implies yRx

$S : \frac{m}{n} s \frac{p}{q}$

Given $qm = pn \Rightarrow \frac{p}{q} = \frac{m}{n}$

$\therefore \frac{m}{n} s \frac{m}{n}$ reflexive $\frac{m}{n} s \frac{p}{q} \Rightarrow \frac{p}{q} s \frac{m}{n}$ symmetric

$\frac{m}{n} s \frac{p}{q}, \frac{p}{q} s \frac{r}{s} \Rightarrow qm = pn, ps = rq$

$\Rightarrow \frac{p}{q} = \frac{m}{n} = \frac{r}{s} \Rightarrow$ ms = rn transitive.

S is an equivalence relation.

16. **(a)** Let for statement 1: $xRy = x - y \in I$. As xRx is an integer and yRx as well as xRz (for xRy and yRz) is also an integer.
Hence equivalence.
Similarly as $x = \alpha y$ hence $\alpha = 1$ for reflexive and $\frac{1}{\alpha}$ being a rational for symmetric for some non zero α and product of rationals also being rational $\Rightarrow$ equivalence
But not symmetric because of $\alpha = 0$ case
Both relations are equivalence but not the correct explanation.

17. **(a)** $f(x) = (x-1)^2 + 1, x \geq 1$

Since f is a bijective function

$\therefore \quad f : [1, \infty) \to [1, \infty)$

$\Rightarrow \quad y = (x-1)^2 + 1 \Rightarrow (x-1)^2 = y - 1$

$\Rightarrow x = 1 \pm \sqrt{y-1} \Rightarrow f^{-1}(y) = 1 \pm \sqrt{y-1}$

$\Rightarrow \quad f^{-1}(x) = 1 + \sqrt{x-1} \; \{\because x \geq 1\}$

Hence statement-2 is correct

Now $f(x) = f^{-1}(x)$

$\Rightarrow f(x) = x \Rightarrow (x-1)^2 + 1 = x$

$\Rightarrow \quad x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

Hence statement-1 is correct

18. **(b)** Since $f(x)$ and $g(x)$ are inverse of each other

$\therefore \quad g'(f(x)) = \frac{1}{f'(x)}$

$\Rightarrow \quad g'(f(x)) = 1 + x^5$

$\left(\because \quad f'(x) = \frac{1}{1+x^5}\right)$

Here $x = g(y)$

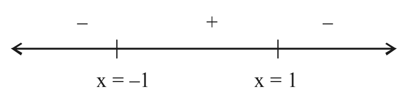
$\therefore \quad g'(y) = 1 + [g(y)]^5$

$\Rightarrow g'(x) = 1 + (g(x))^5$

19. **(d)** we have $\mathrm{f} : \mathrm{R} \to \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$\mathrm{f}(\mathrm{x}) = \frac{x}{1+x^2} \; \forall x \in R$

$\Rightarrow \mathrm{f}'(\mathrm{x}) = \frac{(1+\mathrm{x}^2).1 - \mathrm{x}.2\mathrm{x}}{(1+\mathrm{x}^2)^2} = \frac{-(\mathrm{x}+1)(\mathrm{x}-1)}{(1+\mathrm{x}^2)^2}$



sign of f′ (x)
$\Rightarrow$ f′ (x) changes sign in different intervals.
$\therefore$ Not injective

Now $\mathrm{y} = \frac{\mathrm{x}}{1+\mathrm{x}^2}$
$\Rightarrow \mathrm{y} + \mathrm{yx}^2 = \mathrm{x}$
$\Rightarrow \mathrm{yx}^2 - \mathrm{x} + \mathrm{y} = 0$
For $\mathrm{y} \neq 0$, $\mathrm{D} = 1 - 4\mathrm{y}^2 \geq 0$

$\Rightarrow \mathrm{y} \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$

For $\mathrm{y} = 0 \Rightarrow \mathrm{x} = 0$

$\therefore$ Range is $\left[\frac{-1}{2}, \frac{1}{2}\right]$

$\Rightarrow$ Surjective but not injective