CHAPTER 12

Limits and Derivatives

1. $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}x}$ is **[2002]**

(a) 1 (b) –1

(c) zero (d) does not exist

2. $\operatorname{Lim}\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$ **[2002]**

(a) e^4 (b) e^2

(c) e^3 (d) 1

3. Let $f(x) = 4$ and $f'(x) = 4$. Then $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}$ is given by **[2002]**

(a) 2 (b) –2

(c) –4 (d) 3

4. $\lim\limits_{n\to\infty}\dfrac{1^p+2^p+3^p+.....+n^p}{n^{p+1}}$ is **[2002]**

(a) $\dfrac{1}{p+1}$ (b) $\dfrac{1}{1-p}$

(c) $\dfrac{1}{p}-\dfrac{1}{p-1}$ (d) $\dfrac{1}{p+2}$

5. $\lim\limits_{x\to 0}\dfrac{\log x^n-[x]}{[x]}, n\in N$, ($[x]$ denotes greatest integer less than or equal to x) **[2002]**

(a) has value -1 (b) has value 0

(c) has value 1 (d) does not exist

6. If $f(1)=1, f'(1)=2$, then $\lim\limits_{x\to 1}\dfrac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is **[2002]**

(a) 2 (b) 4

(c) 1 (d) 1/2

7. If $\lim\limits_{x\to 0}\dfrac{\log(3+x)-\log(3-x)}{x}=k$, the value of k is **[2003]**

(a) $-\dfrac{2}{3}$ (b) 0

(c) $-\dfrac{1}{3}$ (d) $\dfrac{2}{3}$

8. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is **[2003]**

(a) ∞ (b) $\dfrac{1}{8}$

(c) 0 (d) $\dfrac{1}{32}$

9. If $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x}=e^2$, then the values of a and b, are **[2004]**

(a) $a=1$ and $b=2$ (b) $a=1, b\in R$

(c) $a\in R, b=2$ (d) $a\in R, b\in R$

10. Let α and β be the distinct roots of $ax^2+bx+c=0$, then $\lim\limits_{x\to\alpha}\dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to **[2005]**

(a) $\dfrac{a^2}{2}(\alpha-\beta)^2$ (b) 0

(c) $\dfrac{-a^2}{2}(\alpha-\beta)^2$ (d) $\dfrac{1}{2}(\alpha-\beta)^2$

11. Let f : R → R be a positive increasing function with $\lim\limits_{x\to\infty}\dfrac{f(3x)}{f(x)}=1$ then $\lim\limits_{x\to\infty}\dfrac{f(2x)}{f(x)}=$ **[2010]**

(a) $\dfrac{2}{3}$ (b) $\dfrac{3}{2}$

(c) 3 (d) 1

12. $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[2011]**

(a) equals $\sqrt{2}$ (b) equals $-\sqrt{2}$

(c) equals $\dfrac{1}{\sqrt{2}}$ (d) does not exist

13. Let $f : R \to [0,\infty)$ be such that $\lim\limits_{x\to 5} f(x)$ exists and $\lim\limits_{x\to 5}\dfrac{(f(x))^2-9}{\sqrt{|x-5|}}=0$ **[2011RS]**

Then $\lim\limits_{x\to 5} f(x)$ equals :

(a) 0 (b) 1

(c) 2 (d) 3

14. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[2013]**

(a) $-\dfrac{1}{4}$ (b) $\dfrac{1}{2}$

(c) 1 (d) 2

15. $\lim\limits_{x\to 0}\dfrac{\sin\left(\pi\cos^2 x\right)}{x^2}$ is equal to: **[2014]**

(a) $-\pi$ (b) π

(c) $\dfrac{\pi}{2}$ (d) 1

16. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$ equals : **[2017]**

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{24}$

(c) $\dfrac{1}{16}$ (d) $\dfrac{1}{8}$

Answer Key

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(d)	(a)	(c)	(a)	(d)	(a)	(d)	(d)	(b)	(a)	(d)	(d)	(d)	(d)	(b)
16														
(c)														

SOLUTIONS

1. (d)

$$\lim\frac{\sqrt{1-\cos 2x}}{\sqrt{2}x}\Rightarrow\lim\frac{\sqrt{1-(1-2\sin^2 x)}}{\sqrt{2}x};$$

$$\lim_{x\to 0}\frac{\sqrt{2\sin^2 x}}{\sqrt{2}x}\Rightarrow\lim_{x\to 0}\frac{|\sin x|}{x}$$

The limit of above does not exist as LHS $=-1\neq$ RHL $=1$

2. (a) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$

$$=\lim_{x\to\infty}\left(1+\frac{4x+1}{x^2+x+2}\right)^x$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{\frac{x^2+x+2}{4x+1}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$$

$$= e^{\lim_{x\to\infty}\frac{4x^2+x}{x^2+x+2}} \left[\because \lim_{x\to\infty}(1+\lambda x)^{\frac{1}{x}} = e^{\lambda}\right]$$

$$= e^{\lim_{x\to\infty}\frac{4+\frac{1}{x}}{1+\frac{1}{x}+\frac{2}{x^2}}} = e^4$$

3. **(c)** Apply L H Rule

We have, $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 2} f(2)-2f'(x) = f(2)-2f'(2)$$

$= 4-2\times 4 = -4.$

4. **(a)** We have $\lim_{n\to\infty}\frac{1^p+2^p+....+n^p}{n^{p+1}}$;

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^p}{n^p\cdot n} = \int_0^1 x^p dx = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{1}{p+1}$$

5. **(d)** Since $\lim_{x\to 0}[x]$ does not exist, hence the required limit does not exist.

6. **(a)** $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ $\left(\frac{0}{0}\right)$ form using L' Hospital's rule $= \lim_{x\to 1}\frac{\frac{1}{2\sqrt{f(x)}}f'(x)}{1/2\sqrt{x}}$

$$= \frac{f'(1)}{\sqrt{f(1)}} = \frac{2}{1} = 2.$$

7. **(d)** $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x} = k$

(by L'Hospital rule)

$$\Rightarrow \lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{-1}{3-x}}{1} = k \quad \therefore \frac{2}{3} = k$$

8. **(d)** $\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right).(1-\sin x)}{(\pi-2x)^3}$

Let $x = \frac{\pi}{2}+y;\, y\to 0$

$$= \lim_{y\to 0}\frac{\tan\left(-\frac{y}{2}\right).(1-\cos y)}{(-2y)^3}$$

$$= \lim_{y\to 0}\frac{-\tan\frac{y}{2}2\sin^2\frac{y}{2}}{(-8).\frac{y^3}{8}.8}$$

$$= \lim_{y\to 0}\frac{1}{32}\frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\frac{\sin y/2}{y/2}\right]^2 = \frac{1}{32}$$

9. **(b)** We know that $\lim_{x\to\infty}(1+x)^{\frac{1}{x}} = e$

$$\therefore \lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$$

$$\Rightarrow \lim_{x\to\infty}\left[\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{\left(\frac{1}{\frac{a}{x}+\frac{b}{x^2}}\right)}\right]^{2x\left(\frac{a}{x}+\frac{b}{x^2}\right)} = e^2$$

$$\Rightarrow e^{\lim_{x\to\infty}2\left[a+\frac{b}{x}\right]} = e^2 \Rightarrow e^{2a} = e^2$$

$\Rightarrow a = 1$ and $b \in R$

ALTERNATE SOLUTION

$$\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$$

$$\Rightarrow e^{\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}-1\right)2x} = e^2$$

$$\Rightarrow \lim_{x\to\infty}\left(2a+\frac{2b}{x}\right) = 2$$

$\Rightarrow 2a+0 = 2, b\in \mathrm{R} \Rightarrow a = 1, b \in \mathrm{R}$

10. **(a)** Given limit =

$$\lim_{x\to\alpha}\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}$$

$$= \lim_{x\to\alpha}\frac{2\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{(x-\alpha)^2}$$

$$= \lim_{x\to\alpha} \frac{2}{(x-\alpha)^2} \times \frac{\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{\frac{a^2(x-\alpha)^2(x-\beta)^2}{4}} \times \frac{a^2(x-\alpha)^2(x-\beta)^2}{4}$$

$$= \frac{a^2(\alpha-\beta)^2}{2}.$$

11. (d) f(x) is a positive increasing function.

$\therefore\ 0 < f(x) < f(2x) < f(3x)$

$$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x\to\infty} 1 \le \lim_{x\to\infty} \frac{f(2x)}{f(x)} \le \lim_{x\to\infty} \frac{f(3x)}{f(x)}$$

By Sandwich Theorem.

$$\Rightarrow \lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1$$

12. (d) $\lim_{x\to 2} \frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}$

$$= \lim_{x\to 2} \frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$$

$$\underset{(\text{at } x=2)}{\text{L.H.L}} = -\lim_{x\to 2} \frac{\sqrt{2}\sin(x-2)}{(x-2)} = -1$$

$$\underset{(\text{at } x=2)}{\text{R.H.L}} = \lim_{x\to 2} \frac{\sqrt{2}\sin(x-2)}{(x-2)} = 1$$

Thus $\underset{(\text{at } x=2)}{\text{L.H.L}} \ne \underset{(\text{at } x=2)}{\text{R.H.L}}$

Hence, $\lim_{x\to 2} \frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}$ does not exist.

13. (d) $\lim_{x\to 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$

$\lim_{x\to 5} [(f(x))^2 - 9] = 0 \Rightarrow \lim_{x\to 5} f(x) = 3$

14. (d) Multiply and divide by x in the given expression, we get

$$\lim_{x\to 0} \frac{(1-\cos 2x)}{x^2} \frac{(3+\cos x)}{1} \cdot \frac{x}{\tan 4x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3+\cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2\lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x\to 0} 3+\cos x \cdot \lim_{x\to 0} \frac{x}{\tan 4x}$$

$$= 2.4\ \frac{1}{4} \lim_{x\to 0} \frac{4x}{\tan 4x} = 2.4.\frac{1}{4} = 2$$

15. (b) Consider $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$

$$= \lim_{x\to 0} \frac{\sin\left[\pi(1-\sin^2 x)\right]}{x^2}$$

$$= \lim_{x\to 0} \sin\frac{(\pi - \pi\sin^2 x)}{x^2}$$ $[\because \sin(\pi-\theta) = \sin\theta]$

$$= \lim_{x\to 0} \sin\frac{(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{\pi\sin^2 x}{x^2}$$

$$= \lim_{x\to 0} 1\times\pi\left(\frac{\sin x}{x}\right)^2 = \pi$$

16. (c) $\lim_{x\to\frac{\pi}{2}} \frac{\cot x(1-\sin x)}{-8\left(x-\frac{\pi}{2}\right)^3}$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cot x(1-\sin x)}{8\left(\frac{\pi}{2}-x\right)^3}$$

Put $\frac{\pi}{2} - x = t \Rightarrow$ as $x \to \frac{\pi}{2} \Rightarrow t \to 0$

$$= \lim_{t\to 0} \frac{\cot\left(\frac{\pi}{2}-t\right)\left(1-\sin\left(\frac{\pi}{2}-t\right)\right)}{8t^3}$$

$$= \lim_{t\to 0} \frac{\tan t(1-\cos t)}{8t^3}$$

$$= \lim_{t\to 0} \frac{\tan t}{8t} . \frac{1-\cos t}{t^2}$$

$$= \frac{1}{8}.1.\frac{1}{2} = \frac{1}{16}$$