CHAPTER 21

Application of Derivatives

1. The maximum distance from origin of a point on the curve $x = a\sin t - b\sin\left(\frac{at}{b}\right)$, $y = a\cos t - b\cos\left(\frac{at}{b}\right)$, both $a, b > 0$ is **[2002]**

   (a) $a-b$ (b) $a+b$

   (c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2}$

2. If $2a+3b+6c=0$, $(a, b, c \in R)$ then the quadratic equation $ax^2+bx+c=0$ has **[2002]**

   (a) at least one root in [0, 1]

   (b) at least one root in [2, 3]

   (c) at least one root in [4, 5]

   (d) none of these

3. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals **[2003]**

   (a) $\frac{1}{2}$ (b) 3

   (c) 1 (d) 2

4. The real number x when added to its inverse gives the minimum value of the sum at x equal to **[2003]**

   (a) –2 (b) 2

   (c) 1 (d) –1

5. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is **[2004]**

   (a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$

   (c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

6. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2,1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is **[2004]**

   (a) $(x+1)^2$ (b) $(x-1)^3$

   (c) $(x+1)^3$ (d) $(x-1)^2$

7. The normal to the curve $x = a(1+\cos\theta)$, $y = a\sin\theta$ at 'θ' always passes through the fixed point **[2004]**

   (a) (a, a) (b) $(0, a)$

   (c) $(0, 0)$ (d) $(a, 0)$

8. If $2a+3b+6c=0$, then at least one root of the equation $ax^2+bx+c=0$ lies in the interval **[2004]**

   (a) $(1, 3)$ (b) $(1, 2)$

   (c) $(2, 3)$ (d) $(0, 1)$

9. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **[2005]**

   (a) $2ab$ (b) ab

   (c) $\sqrt{ab}$ (d) $\frac{a}{b}$

10. The normal to the curve **[2005]**

$x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that

(a) it passes through the origin

(b) it makes an angle $\frac{\pi}{2} + \theta$ with the x- axis

(c) it passes through $\left(a\frac{\pi}{2}, -a\right)$

(d) It is at a constant distance from the origin

11. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm,then the rate at which the thickness of ice decreases is **[2005]**

(a) $\frac{1}{36\pi}$ cm/min. (b) $\frac{1}{18\pi}$ cm/min.

(c) $\frac{1}{54\pi}$ cm/min. (d) $\frac{5}{6\pi}$ cm/min

12. If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$

$a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is **[2005]**

(a) greater than α

(b) smaller than α

(c) greater than or equal to α

(d) equal to α

13. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched? **[2005]**

	Interval	*Function*
(a)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(b)	$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(c)	$\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(d)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$

14. A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of $2\,cm/s^2$ and pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then the lizard will catch the insect after **[2005]**

(a) $20\,s$ (b) $1\,s$

(c) $21\,s$ (d) $24\,s$

15. Two points A and B move from rest along a straight line with constant acceleration f and f' respectively. If A takes m sec. more than B and describes 'n'units more than B in acquiring the same speed then **[2005]**

(a) $(f - f')m^2 = ff'n$

(b) $(f + f')m^2 = ff'n$

(c) $\frac{1}{2}(f + f')m = ff'n^2$

(d) $(f' - f)n = \frac{1}{2}ff'm^2$

16. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at **[2006]**

(a) $x = 2$ (b) $x = -2$

(c) $x = 0$ (d) $x = 1$

17. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is **[2006]**

(a) $\frac{3}{2}x^2$ (b) $\sqrt{\frac{x^3}{8}}$

(c) $\frac{1}{2}x^2$ (d) πx^2

18. value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is **[2007]**

(a) $\log_3 e$ (b) $\log_e 3$

(c) $2\log_3 e$ (d) $\frac{1}{2}\log_3 e$

19. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[2007]**

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

20. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is **[2007]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$ (d) 2.

21. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **[2008]**

(a) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

22. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[2008]**

(a) 7 (b) 1

(c) 3 (d) 5

23. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is **[2009]**

(a) $(x-2)y'^2 = 25 - (y-2)^2$

(b) $(y-2)y'^2 = 25 - (y-2)^2$

(c) $(y-2)^2 y'^2 = 25 - (y-2)^2$

(d) $(x-2)^2 y'^2 = 25 - (y-2)^2$

24. Let $f(x) = x\,|x|$ and $g(x) = \sin x$.

Statement-1 : gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement-2 : gof is twice differentiable at $x = 0$. **[2009]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is false.

(c) Statement-1 is false, Statement-2 is true.

(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

25. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$: **[2009]**

(a) P(–1) is not minimum but P(1) is the maximum of P

(b) P(–1) is the minimum but P(1) is not the maximum of P

(c) Neither P(–1) is the minimum nor P(1) is the maximum of P

(d) P(–1) is the minimum and P(1) is the maximum of P

26. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[2010]**

(a) $y = 1$ (b) $y = 2$

(c) $y = 3$ (d) $y = 0$

27. Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} k - 2x, & if \quad x \le -1 \\ 2x + 3, & if \quad x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is **[2010]**

(a) 0 (b) $-\frac{1}{2}$

(c) –1 (d) 1

28. Let $f: R \to R$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$ **[2010]**

Statement -1 : $f(c) = \dfrac{1}{3}$, for some $c \in R$.

Statement -2 : $0 < f(x) \le \dfrac{1}{2\sqrt{2}}$, for all $x \in R$

(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

29. Let f be a function defined by - **[2011RS]**

$$f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

Statement - 1 : $x = 0$ is point of minima of f

Statement - 2 : $f'(0) = 0$.

(a) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.
(c) Statement-1 is true, statement-2 is false.
(d) Statement-1 is false, statement-2 is true.

30. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact is given by : **[2011RS]**

(a) $2y - 3x = 0$ (b) $y = \dfrac{6}{x}$

(c) $x^2 + y^2 = 13$ (d) $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2$

31. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is:

(a) $\dfrac{9}{7}$ (b) $\dfrac{7}{9}$ **[2012]**

(c) $\dfrac{2}{9}$ (d) $\dfrac{9}{2}$

32. Let $a, b \in R$ be such that the function f given by $f(x) = ln|x| + bx^2 + ax, x \ne 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1 : f has local maximum at $x = -1$ and *at* $x = 2$.

Statement-2 : $a = \dfrac{1}{2}$ and $b = \dfrac{-1}{4}$ **[2012]**

(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

33. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[2013]**

(a) lies between 1 and 2
(b) lies between 2 and 3
(c) lies between .1 and 0
(d) does not exist.

34. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then **[2014]**

(a) $\alpha = 2, \beta = -\dfrac{1}{2}$ (b) $\alpha = 2, \beta = \dfrac{1}{2}$

(c) $\alpha = -6, \beta = \dfrac{1}{2}$ (d) $\alpha = -6, \beta = -\dfrac{1}{2}$

35. The normal to the curve, $x^2+2xy-3y^2=0$, at (1, 1) **[2015]**

(a) meets the curve again in the third quadrant.
(b) meets the curve again in the fourth quadrant.
(c) does not meet the curve again.
(d) meets the curve again in the second quadrant.

36. Consider

$$f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right).$$

A normal to y = f(x) at $x=\frac{\pi}{6}$ a so passes through the point : **[2016]**

(a) $\left(\frac{\pi}{6},0\right)$ (b) $\left(\frac{\pi}{4},0\right)$
(c) (0, 0) (d) $\left(0,\frac{2\pi}{3}\right)$

37. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[2016]**

(a) x = 2r (b) 2x = r
(c) $2x=(\pi+4)r$ (d) $(4-\pi)x=\pi r$

38. The normal to the curve $y(x-2)(x-3)=x+6$ at the point where the curve intersects the y-axis passes through the point:

(a) $\left(\frac{1}{2},\frac{1}{3}\right)$ (b) $\left(-\frac{1}{2},-\frac{1}{2}\right)$
(c) $\left(\frac{1}{2},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},-\frac{1}{3}\right)$

39. Twenty metres of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is : **[2017]**

(a) 30 (b) 12.5
(c) 10 (d) 25

40. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is x = –4, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is : **[2017]**

(a) x + 2y = 4 (b) 2y – x = 2
(c) 4x – 2y = 1 (d) 4x + 2y = 7

Answer Key

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(b)	(a)	(d)	(c)	(a)	(b)	(d)	(d)	(a)	(d)	(b)	(b)	(c)	(c)	(d)
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
(a)	(c)	(c)	(d)	(c)	(a)	(b)	(c)	(b)	(a)	(c)	(c)	(d)	(b)	(b)
31	32	33	34	35	36	37	38	39	40					
(c)	(b)	(d)	(a)	(b)	(d)	(a)	(c)	(d)	(c)					

SOLUTIONS

1. **(b)** Distance of origin from $(x,y)=\sqrt{x^2+y^2}$

$$=\sqrt{a^2+b^2-2ab\cos\left(t-\frac{at}{b}\right)};$$

$$\le\sqrt{a^2+b^2+2ab}$$

$$\left[\left\{\cos\left(t-\frac{at}{b}\right)\right\}_{\min}=-1\right]$$

$=a+b$

$\therefore$ Maximum distance from origin $=a+b$

2. **(a)** Let $f(x)=\frac{ax^3}{3}+\frac{bx^2}{2}+cx$

$\Rightarrow f(0)=0$ and

$$f(1)=\frac{a}{3}+\frac{b}{2}+c=\frac{2a+3b+6c}{6}=0$$

Also $f(x)$ is continuous and differentiable in [0, 1] and [0, 1[. So by Rolle's theorem, $f'(x)=0$.

i.e $ax^2+bx+c=0$ has at least one root in [0, 1].

3. **(d)** $f(x)=2x^3-9ax^2+12a^2x+1$

$f'(x)=6x^2-18ax+12a^2;\ f''(x)=12x-18a$

For max. or min.

$6x^2-18ax+12a^2=0 \Rightarrow x^2-3ax+2a^2=0$

$\Rightarrow x=a$ or $x=2a$. At $x=a$ max.

and at $x=2a$ min

$\therefore\ p=a$ and $q=2a$

As per question $p^2=q$

$\therefore a^2=2a \Rightarrow a=2$ or $a=0$

but $a>0$, therefore, $a=2$.

4. **(c)** $y=x+\dfrac{1}{x}$ or $\dfrac{dy}{dx}=1-\dfrac{1}{x^2}$

For max. or min., $1-\dfrac{1}{x^2}=0 \Rightarrow x=\pm 1$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=2}=2$$

(+ve minima)

$\therefore x=1$

5. **(a)** $y^2=18x \Rightarrow 2y\dfrac{dy}{dx}=18 \Rightarrow \dfrac{dy}{dx}=\dfrac{9}{y}$

Given $\dfrac{dy}{dx}=2 \Rightarrow \dfrac{9}{y}=2 \Rightarrow y=\dfrac{9}{2}$

Putting in $y^2=18x \Rightarrow x=\dfrac{9}{8}$

$\therefore$ Required point is $\left(\dfrac{9}{8},\dfrac{9}{2}\right)$

6. **(b)** $f''(x)=6(x-1)$. Inegrating, we get

$f'(x)=3x^2-6x+c$

Slope at (2, 1) $= f'(2)=c=3$

[$\because$ slope of tangent at (2,1) is 3]

$\therefore f'(x)=3x^2-6x+3=3(x-1)^2$

Inegrating again, we get

$f(x)=(x-1)^3+D$

The curve passes through (2, 1)

$\Rightarrow 1=(2-1)^3+D \Rightarrow D=0$

$\therefore f(x)=(x-1)^3$

7. **(d)** $\dfrac{dx}{d\theta}=-a\sin\theta$ and $\dfrac{dy}{d\theta}=a\cos\theta$

$\therefore \dfrac{dy}{dx}=-\cot\theta.$

$\therefore$ The slope of the normal at $\theta=\tan\theta$

$\therefore$ The equation of the normal at θ is

$y-a\sin\theta=\tan\theta(x-a-a\cos\theta)$

$\Rightarrow y\cos\theta-a\sin\theta\cos\theta=x\sin\theta-a\sin\theta$
$-a\sin\theta\cos\theta$

$\Rightarrow x\sin\theta-y\cos\theta=a\sin\theta$

$\Rightarrow y=(x-a)\tan\theta$

which always passes through $(a, 0)$

8. **(d)** Let us define a function

$f(x)=\dfrac{ax^3}{3}+\dfrac{bx^2}{2}+cx$

Being polynomial, it is continuous and differentiable, also,

$f(0)=0$ and $f(1)=\dfrac{a}{3}+\dfrac{b}{2}+c$

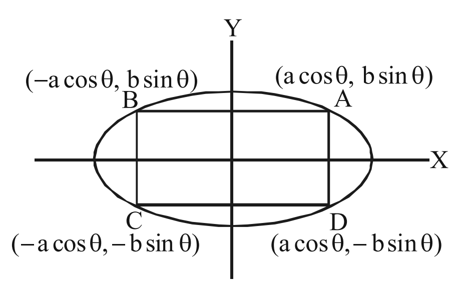
$\Rightarrow f(1)=\dfrac{2a+3b+6c}{6}=0$ (given)

$\therefore f(0)=f(1)$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem therefore $f'(x)=0$ has a root in (0, 1)

i.e. $ax^2+bx+c=0$ has at lease one root in (0, 1)

9. **(a)** Area of rectangle $ABCD=2a\cos\theta$ $(2b\sin\theta)=2ab\sin 2\theta$



$\Rightarrow$ Area of greatest rectangle is equal to $2ab$

When $\sin 2\theta=1$.

10. (d) $x = a(\cos\theta + \theta\sin\theta)$

$\Rightarrow \quad \dfrac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$\Rightarrow \quad \dfrac{dx}{d\theta} = a\theta\cos\theta$ .....(1)

$y = a(\sin\theta - \theta\cos\theta)$

$\dfrac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta]$

$\Rightarrow \dfrac{dy}{d\theta} = a\theta\sin\theta$ .....(2)

From equations (1) and (2) we get

$\dfrac{dy}{dx} = \tan\theta \Rightarrow$ Slope of normal $= -\cot\theta$

Equation of normal at 'θ' is $y - a(\sin\theta$
$-\ \theta\cos\theta)$
$= -\cot\theta\ (x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow \quad y\sin\theta - a\sin^2\theta + a\ \theta\cos\theta\sin\theta$

$= -x\cos\theta + a\cos^2\theta + a\,\theta\sin\theta\cos\theta$

$\Rightarrow \quad x\cos\theta + y\sin\theta = a$

Clearly this is an equation of straight line which is at a constant distance 'a' from origin.

11. (b) Given that

$\dfrac{dv}{dt} = 50\ \text{cm}^3/\text{min} \Rightarrow \dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right) = 50$

$\Rightarrow 4\pi r^2 \dfrac{dr}{dt} = 50$

$\Rightarrow \dfrac{dr}{dt} = \dfrac{50}{4\pi(15)^2} = \dfrac{1}{18\pi}$ cm/min

(here $r = 10+5$)

12. (b) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1 x = 0$

The other given equation,

$na_n x^{n-1} + (n-1)\,a_{n-1}x^{n-2} + \ldots. + a_1 = 0 = f'(x)$

Given $a_1 \neq 0 \Rightarrow f(0) = 0$

Again $f(x)$ has root α, $\Rightarrow f(\alpha) = 0$

$\therefore \quad f(0) = f(\alpha)$

$\therefore \quad$ By Rolle's theorem $f'(x) = 0$ has root between $(0, \alpha)$

Hence $f'(x)$ has a positive root smaller than α.

13. (c) Clearly function $f(x) = 3x^2 - 2x + 1$ is increasing when $f'(x) = 6x - 2 \geq 0$

$\Rightarrow \quad x \in [1/3, \infty)$

$\therefore f(x)$ is incorrectly matched with

$\left(-\infty, \dfrac{1}{3}\right]$

14. (c) Let the lizard catches the insect after time t then distance covered by lizard = 21cm + distance covered by insect

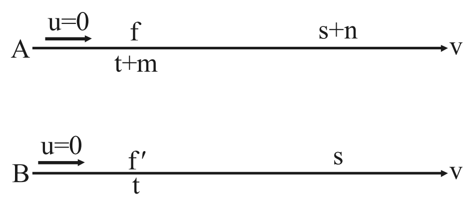
$\Rightarrow \dfrac{1}{2}ft^2 = 4 \times t + 21$

$\Rightarrow \dfrac{1}{2} \times 2 \times t^2 = 20 \times t + 21$

$\Rightarrow t^2 - 20t - 21 = 0$

$\Rightarrow t = 21$ sec

15. (d)



As per question if point B moves s distance in t time then point A moves $(s + n)$ distance in time $(t + m)$ after which both have same velocity v.

Then using equation $v = u + at$ we get

$v = f(t+m) = f't \Rightarrow t = \dfrac{f\,m}{f' - f}$ ....(1)

Using equation $v^2 = u^2 + 2$, as we get

$v^2 = 2f(s+n) = 2f's \quad \Rightarrow s = \dfrac{f\,n}{f' - f}$

....(2)

Also for point B using the eqn

$s = ut + \dfrac{1}{2}at^2$, we get

$s = \dfrac{1}{2}f't^2$

Substituting values of t and s from equations (1) and (2) in the above relation, we get

$$\frac{f\,n}{f'-f}=\frac{1}{2}f'\frac{f^2m^2}{(f'-f)^2}$$

$$\Rightarrow (f'-f)n=\frac{1}{2}ff'm^2$$

16. (a) $\frac{x}{2}+\frac{2}{x}$ is of the form $y+\frac{1}{y}$ where

$y+\frac{1}{y}\ge 2$ and equality holds for $y=1$

$\therefore$ Min value of function occurs at $\frac{x}{2}=1$

i.e.,
at $x=2$

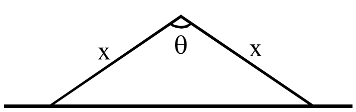
ALTERNATE SOLUTION

$$f(x)=\frac{x}{2}+\frac{2}{x}\Rightarrow f'(x)=\frac{1}{2}-\frac{2}{x^2}=0$$

$$\Rightarrow x^2=4 \text{ or } x=2,-2; \qquad f''(x)=\frac{4}{x^3}$$

$f''(x)]_{x=2}=+ve\Rightarrow f(x)$ has local min at $x=2$.

17. (c) Area $=\frac{1}{2}x^2\sin\theta$



Maximum value of $\sin\theta$ is 1 at $\theta=\frac{\pi}{2}$

$$A_{max}=\frac{1}{2}x^2$$

18. (c) Using Lagrange's Mean Value Theorem
Let $f(x)$ be a function defined on $[a, b]$

then, $f'(c)=\frac{f(b)-f(a)}{b-a}$ ....(i)

$c\in[a, b]$

$\therefore$ Given $f(x)=\log_e x \quad \therefore f'(x)=\frac{1}{x}$

$\therefore$ equation (i) become

$$\frac{1}{c}=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \frac{1}{c}=\frac{\log_e 3-\log_e 1}{2}=\frac{\log_e 3}{2}$$

$$\Rightarrow c=\frac{2}{\log_e 3}\Rightarrow c=2\log_3 e$$

19. (d) Given $f(x)=\tan^{-1}(\sin x+\cos x)$

$$f'(x)=\frac{1}{1+(\sin x+\cos x)^2}.(\cos x-\sin x)$$

$$=\frac{\sqrt{2}.\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)}{1+(\sin x+\cos x)^2}$$

$$=\frac{\left(\cos\frac{\pi}{4}.\cos x-\sin\frac{\pi}{4}.\sin x\right)}{1+(\sin x+\cos x)^2}$$

$$\therefore \quad f'(x)=\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

if $f'(x)>0$ then $f(x)$ is increasing function.
Hence $f(x)$ is increasing, if

$$-\frac{\pi}{2}<x+\frac{\pi}{4}<\frac{\pi}{2}$$

$$\Rightarrow -\frac{3\pi}{4}<x<\frac{\pi}{4}$$

Hence, $f(x)$ is increasing when $n\in\left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

20. (c) Given that $p^2+q^2=1$

$\therefore \; p=\cos\theta$ and $q=\sin\theta$

Then $p+q=\cos\theta+\sin\theta$
We know that

$$-\sqrt{a^2+b^2}\le a\cos\theta+b\sin\theta\le\sqrt{a^2+b^2}$$

$$\therefore -\sqrt{2}\le\cos\theta+\sin\theta\le\sqrt{2}$$

Hence max. value of $p+q$ is $\sqrt{2}$

ALTERNATE SOLUTION

Since, p and q are positive real numbers
$p^2+q^2=1$ (Given)
Using AM $\ge$ GM

$$\therefore \left(\frac{p+q}{2}\right)^2\ge\sqrt{(pq)^2}$$

$$=\frac{p^2+q^2+2pq}{4}\ge pq$$

$\frac{1+2pq}{4} \geq pq$ or $1+2pq \geq 4pq$

$1 \geq 2pq$ or, $2pq \leq 1$

$pq \leq \frac{1}{2}$ or, $pq \leq \frac{1}{2}$

Now, $(p+q)^2 = p^2+q^2+2pq$

$\Rightarrow (p+q)^2 \leq 1+2\times\frac{1}{2} \Rightarrow p+q \leq \sqrt{2}$

21. (a) Let $y = x^3 - px + q \Rightarrow \frac{dy}{dx} = 3x^2 - p$

For $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - p = 0$

$\Rightarrow x = \pm\sqrt{\frac{p}{3}}$

$\frac{d^2y}{dx^2} = 6x$

$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{\frac{p}{3}}} = +ve$ and $\left.\frac{d^2y}{dx^2}\right|_{x=-\sqrt{\frac{p}{3}}} = -ve$

$\therefore$ y has minima at $x = \sqrt{\frac{p}{3}}$ and maxima at $x = -\sqrt{\frac{p}{3}}$

22. (b) Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$\Rightarrow f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall\ x \in R$

$\Rightarrow f$ is an increasing function on R

Also $\lim_{x\to\infty} f(x) = \infty$ and

$\lim_{x\to-\infty} f(x) = -\infty$

$\Rightarrow$ The curve $y = f(x)$ crosses x-axis only once.

$\therefore f(x) = 0$ has exactly one real root.

23. (c) Let the centre of the circle be $(h, 2)$

$\therefore$ Equation of circle is

$(x-h)^2 + (y-2)^2 = 25$ …(1)

Differentiating with respect to x, we get

$2(x-h) + 2(y-2)\frac{dy}{dx} = 0$

$\Rightarrow x - h = -(y-2)\frac{dy}{dx}$

Substituting in equation (1) we get

$(y-2)^2\left(\frac{dy}{dx}\right)^2 + (y-2)^2 = 25$

$\Rightarrow (y-2)^2 (y')^2 = 25 - (y-2)^2$

24. (b) Given that $f(x) = x|x|$ and $g(x) = \sin x$

So that

$go\,f(x) = g(f(x)) = g(x|x|) = \sin x|x|$

$= \begin{cases} \sin(-x^2), & \text{if } x < 0 \\ \sin(x^2), & \text{if } x \geq 0 \end{cases}$

$= \begin{cases} -\sin x^2, & \text{if } x < 0 \\ \sin x^2, & \text{if } x \geq 0 \end{cases}$

$\therefore (go\,f)'(x) = \begin{cases} -2x\cos x^2, & \text{if } x < 0 \\ 2x\cos x^2, & \text{if } x \geq 0 \end{cases}$

Here we observe

$L(go\,f)'(0) = 0 = R(go\,f)'(0)$

$\Rightarrow$ $go\,f$ is differentiable at $x = 0$

and $(go\,f)'$ is continuous at $x = 0$

Now $(go\,f)''(x)$

$= \begin{cases} -2\cos x^2 + 4x^2\sin x^2, & x < 0 \\ 2\cos x^2 - 4x^2\sin x^2, & x \geq 0 \end{cases}$

Here

$L(go\,f)''(0) = -2$ and $R(go\,f)''(0) = 2$

$\because$ $L(go\,f)''(0) \neq R(go\,f)''(0)$

$\Rightarrow$ $go\,f(x)$ is not twice differentiable at $x = 0$.

$\therefore$ Statement - 1 is true but statement -2 is false.

25. (a) We have $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow$ $P'(x) = 4x^3 + 3ax^2 + 2bx + c$

But $P'(0) = 0 \Rightarrow c = 0$

$\therefore$ $P(x) = x^4 + ax^3 + bx^2 + d$

As given that $P(-1) < P(a)$

$\Rightarrow$ $1 - a + b + d < 1 + a + b + d$

$\Rightarrow$ $a > 0$

Now $P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

As $P'(x) = 0$, there is only one solution $x = 0$, therefore $4x^2 + 3ax + 2b = 0$ should not have any real roots i.e. $D < 0$

$\Rightarrow$ $9a^2 - 32b < 0 \Rightarrow b > \frac{9a^2}{32} > 0$

Hence $a, b > 0 \Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx > 0$

$\forall x > 0$

$\therefore$ $P(x)$ is an increasing function on (0,1)

$\therefore$ $P(0) < P(a)$

Similarly we can prove $P(x)$ is decreasing on $(-1, 0)$

$\therefore$ $P(-1) > P(0)$

So we can conclude that

Max $P(x) = P(1)$ and Min $P(x) = P(0)$

$\Rightarrow$ $P(-1)$ is not minimum but $P(1)$ is the maximum of P.

26. **(c)** Since tangent is parallel to x-axis,

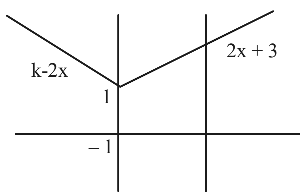
$$\therefore \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x = 2 \Rightarrow y = 3$$

Equation of tangent is $y - 3 = 0\,(x-2)$

$\Rightarrow y = 3$

27. **(c)**

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$



This is true where k = – 1

28. **(d)** $f(x) = \dfrac{1}{e^x + 2e^{-x}} = \dfrac{e^x}{e^{2x} + 2}$

$$f'(x) = \frac{(e^{2x}+2)e^x - 2e^{2x}.e^x}{(e^{2x}+2)^2}$$

$f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

maximum $f(x) = \dfrac{\sqrt{2}}{4} = \dfrac{1}{2\sqrt{2}}$

$0 < f(x) \le \dfrac{1}{2\sqrt{2}} \quad \forall x \in R$

Since $0 < \dfrac{1}{3} < \dfrac{1}{2\sqrt{2}} \Rightarrow$ for some $c \in R$

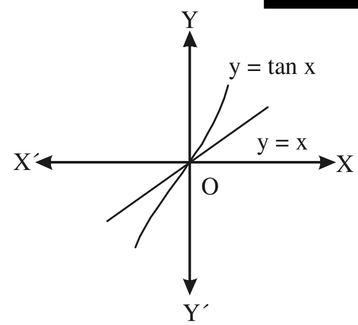
$f(c) = \dfrac{1}{3}$

29. **(b)** $f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

In right neighbourhood of '0'

$\tan x > x$

$\dfrac{\tan x}{x} > 1$



In left neighbourhood of '0'

$\tan x < x$

$\dfrac{\tan x}{x} > 1$ as $(\tan x < 0)$

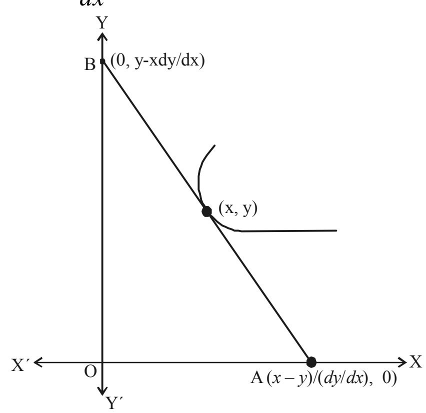
at $x = 0$, $f(x) = 1$

$\Rightarrow$ $x = 0$ is the point of minima

So Statement 1 is true.

Statement 2 obvious.

30. **(b)** $Y - y = \dfrac{dy}{dx}(X - x)$



X-intercept $= x - \dfrac{y}{dy/dx}$

Y-intercept $= y - \dfrac{xdy}{dx}$

According to given statement

$x - \dfrac{y}{dy} = 2x$ and $y - \dfrac{xdy}{dx} = 2y$

$\dfrac{-y}{\frac{dy}{dx}} = x$ and $\dfrac{-xdy}{dx} = y$

$\Rightarrow \dfrac{dx}{x} + \dfrac{dy}{y} = 0$

$\ell ny = -\ell nc + \ell nc$

$y = \dfrac{c}{x}$

Since the above line passes through the point (2, 3).

$\therefore \quad c = 6$

Hence $y = \dfrac{6}{x}$ is the required equation.

31. (c) Volume of spherical balloon

$= V = \dfrac{4}{3}\pi r^3$

$\Rightarrow \quad 4500\,\pi = \dfrac{4\pi r^3}{3}$

$(\because$ Given, volume $= 4500\pi m^3)$

Differentiate both the side, w.r.t 't' we get,

$$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$$

Now, it is given that $\dfrac{dV}{dt} = 72\pi$

$\therefore$ After 49 min,

Volume $= (4500 - 49 \times 72)\pi$

$= (4500 - 3528)\pi = 972\,\pi\, m^3$

$\Rightarrow \quad V = 972\,\pi\, m^3$

$\therefore \; 972\pi = \dfrac{4}{3}\pi r^3$

$\Rightarrow \quad r^3 = 3 \times 243 = 3 \times 3^5 = 3^6 = (3^2)^3$

$\Rightarrow \quad r = 9$

Also, we have $\dfrac{dV}{dt} = 72\pi$

$\therefore \quad 72\pi = 4\pi \times 9 \times 9\left(\dfrac{dr}{dt}\right)$

$\Rightarrow \dfrac{dr}{dt} = \left(\dfrac{2}{9}\right)$

32. (b) Given, $f(x) = ln|x| + bx^2 + ax$

$\therefore \quad f'(x) = \dfrac{1}{x} + 2bx + a$

At $x = -1$, $f'(x) = -1 - 2b + a = 0$

$\Rightarrow \quad a - 2b = 1 \qquad$ ...(i)

At $x = 2$, $\quad f'(x) = \dfrac{1}{2} + 4b + a = 0$

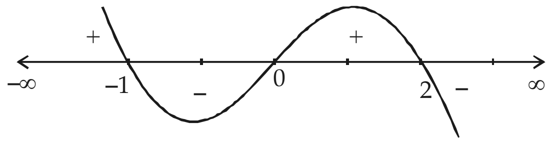
$\Rightarrow \quad a + 4b = -\dfrac{1}{2} \qquad$ ...(ii)

On solving (i) and (ii) we get $a = \dfrac{1}{2}, b = -\dfrac{1}{4}$

Thus, $f'(x) = \dfrac{1}{x} - \dfrac{x}{2} + \dfrac{1}{2} = \dfrac{2 - x^2 + x}{2x}$

$= \dfrac{-x^2 + x + 2}{2x} = \dfrac{-\left(x^2 - x - 2\right)}{2x}$

$= \dfrac{-(x+1)(x-2)}{2x}$



So maxima at $x = -1, 2$

33. (d) $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \;\; \forall\, x \in R \quad (\because \;\; x^2 > 0)$

$\Rightarrow \quad f(x)$ is strictly increasing function

$\Rightarrow \quad f(x) = 0$ has only one real root, so two roots are not possible.

34. (a) Let $f(x) = \alpha \log|x| + \beta x^2 + x$

Differentiate both side,

$$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$$

Since $x = -1$ and $x = 2$ are extreme points therefore $f'(x) = 0$ at these points.

Put $x = -1$ and $x = 2$ in $f'(x)$, we get

$-\alpha - 2\beta + 1 = 0 \Rightarrow \alpha + 2\beta = 1$ ...(i)

$\dfrac{\alpha}{2} + 4\beta + 1 = 0 \Rightarrow \alpha + 8\beta = -2$ ...(ii)

On solving (i) and (ii), we get

$6\beta = -3 \Rightarrow \beta = -\dfrac{1}{2}$

$\therefore \; \alpha = 2$

35. (b) Given curve is

$x^2 + 2xy - 3y^2 = 0 \qquad$ ...(1)

Differentiatew.r.t. x

$, \; 2x + 2x\dfrac{dy}{dx} + 2y - 6y\dfrac{dy}{dx} = 0$

$\left(\dfrac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at (1, 1) is

$y = 2 - x \qquad$ ...(2)

Solving eq. (1) and (2), we get

$x = 1, 3$

Point of intersection (1, 1), (3, –1)

Normal cuts the curve again in 4th quadrant.

36. (d) $f(x) = \tan^{-1}\left(\sqrt{\dfrac{1 + \sin x}{1 - \sin x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{x} - \cos\frac{x}{2}\right)^2}}\right) = \tan^{-1}\left(\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$\Rightarrow \; y = \frac{\pi}{4} + \frac{x}{2}$

$\Rightarrow \; \frac{dy}{dx} = \frac{1}{2}$

Slope of normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = -2$

At $\left(\frac{\pi}{6}, \frac{\pi}{4} + \frac{\pi}{12}\right)$

$y - \left(\frac{\pi}{4} + \frac{\pi}{12}\right) = -2\left(x - \frac{\pi}{6}\right)$

$y - \frac{4\pi}{12} = -2x + \frac{2\pi}{6}$

$y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$

$y = -2x + \frac{2\pi}{3}$

This equation is satisfied only by the point $\left(0, \frac{2\pi}{3}\right)$

37. **(a)** $4x + 2\pi r = 2 \quad \Rightarrow 2x + \pi r = 1$

$S = x^2 + \pi r^2$

$S = \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2$

$\frac{dS}{dr} = 2\left(\frac{1 - \pi r}{2}\right)\left(\frac{-\pi}{2}\right) + 2\pi r$

$\Rightarrow \frac{-\pi}{2} + \frac{\pi^2 r}{2} + 2\pi r = 0 \quad \Rightarrow r = \frac{1}{\pi + 4}$

$\Rightarrow x = \frac{2}{\pi + 4} \quad \Rightarrow x = 2r$

38. **(c)** We have $y = \frac{x + 6}{(x - 2)(x - 3)}$

At y-axis, $x = 0 \Rightarrow y = 1$

On differentiating, we get

$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)\,(1) - (x + 6)\,(2x - 5)}{(x^2 - 5x + 6)^2}$

$\frac{dy}{dx} = 1$ at point $(0, 1)$

$\therefore$ Slope of normal $= -1$

Now equation of normal is $y - 1 = -1\,(x - 0)$

$\Rightarrow y - 1 = -x$

$x + y = 1$

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$ satisfy it.

39. **(d)** We have

Total length $= r + r + r\theta = 20$

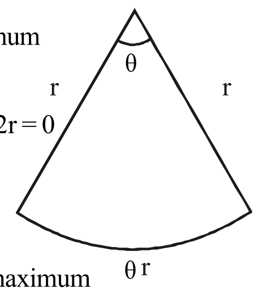
$\Rightarrow 2r + r\,\theta = 20$

$\Rightarrow \theta = \frac{20 - 2r}{r}$ ...(1)

$A = \text{Area} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \left(\frac{20 - 2r}{r}\right)$

$A = 10r - r^2$

For A to be maximum



$\frac{dA}{dr} = 0 \Rightarrow 10 - 2r = 0$

$\Rightarrow r = 5$

$\frac{d^2A}{dr^2} = -2 < 0$

$\therefore$ For $r = 5$ A is maximum

From (1)

$\theta = \frac{20 - 2(5)}{5} = \frac{10}{5} = 2$

$A = \frac{2}{2\pi} \times \pi (5)^2 = 25$ sq. m

40. **(c)** Eccentricity of ellipse $= \frac{1}{2}$

Now, $-\frac{a}{e} = -4 \Rightarrow a = 4 \times \frac{1}{2} = 2 \Rightarrow a = 2$

We have $b^2 = a^2\,(1 - e^2) = a^2\left(1 - \frac{1}{4}\right) = 4 \times \frac{3}{4} = 3$

$\therefore$ Equation of ellipse is

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Now differentiating, we get

$\Rightarrow \frac{x}{2} + \frac{2y}{3} \times y' = 0 \Rightarrow y' = -\frac{3x}{4y}$

$y'\big|_{(1, 3/2)} = -\frac{3}{4} \times \frac{2}{3} = -\frac{1}{2}$

Slope of normal $= 2$

$\therefore$ Equation of normal at $\left(1, \frac{3}{2}\right)$ is

$y - \frac{3}{2} = 2\,(x - 1) \Rightarrow 2y - 3 = 4x - 4$

$\therefore 4x - 2y = 1$