TARGET : JEE (Advanced) 2015

Course : VIJETA & VIJAY (ADP & ADR) **Date : 01-05-2015**

MATHEMATICS DPP
DAILY PRACTICE PROBLEMS

DPP NO. 08

TEST INFORMATION

DATE : 03.05.2015 **OPEN TEST (OT-02) ADVANCED**

Syllabus : Full Syllabus

REVISION DPP OF

DEFINITE INTEGRATION & ITS APPLICATION AND INDEFINITE INTEGRATION

Total Marks : 149		**Max. Time : 105.5 min.**
Single choice Objective (–1 negative marking) Q. 1 to 14	**(3 marks 2.5 min.)**	**[42, 35]**
Multiple choice objective (–1 negative marking) Q. 15 to 31	**(4 marks, 3 min.)**	**[64, 48]**
Comprehension (–1 negative marking) Q.32 to 33 & Q.34 to Q.36	**(3 marks 2.5 min.)**	**[15, 12.5]**
Single digit type (no negative marking) Q. 37 to 39	**(4 marks 2.5 min.)**	**[12, 7.5]**
Double digit type (no negative marking) Q. 40	**(4 marks 2.5 min.)**	**[16, 2.5]**

1. If A = $\int_0^{505\pi} |\cos x|\,dx$ and B = $\int_{505\pi}^{1007\pi} |\sin x|\,dx$, then A + B is equal to

(A) 2013 (B) 2014

(C) 2015 (D) 2016

2. The least integer greater than $\int_0^{100} \left\{\sqrt{x}\right\} dx$ is (where {.} is fractional part function)

(A) 50 (B) 51

(C) 52 (D) 53

3. $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}\,dx$ =

(A) $e^x \cdot \sqrt{\frac{1-x}{1+x}} + c$ (B) $e^x\sqrt{\frac{1+x}{1-x}} + c$

(C) $\frac{e^x}{\sqrt{1-x}} + c$ (D) $\frac{e^x}{\sqrt{1+x}} + c$

4. Let $f(x) = \int x^2\cos^2 x\left(2x+6\tan x-2x\tan^2 x\right)dx$ and f(x) passes through the point (π, 0), then the number of solutions of the equation $f(x) = x^3$ in [0, 2π] is

(A) 1 (B) 2 (C) 3 (D) 4

5. Let f(x) is a continuous function symmetric about the lines x = 1 and x = 2. If $\int_0^2 f(x)dx = 3$ and $\int_0^{50} f(x)\ dx = I$, then $\left[\sqrt{I}\right]$ is equal to (where [.] is G.I.F.)

(A) 5 (B) 8 (C) 7 (D) 6

6. $\int\left(x^6+x^4+x^2\right)\sqrt{2x^4+3x^2+6}\ \ dx$ is equal to

(A) $\frac{\left(3x^6+2x^4+6x^2\right)^{3/2}}{18}+C$

(B) $\frac{\left(2x^6+3x^4+6x^2\right)^{3/2}}{24}+C$

(C) $\frac{\left(2x^6+3x^4+6x^2\right)^{3/2}}{18}+C$

(D) None of these

7. For each positive integer n > 1, let S_n represents the area of the region bounded by $\frac{x^2}{n^2}+y^2\le 1$ and $x^2+\frac{y^2}{n^2}\le 1$, then $\lim\limits_{n\to\infty} S_n$ is equal to

(A) 4 (B) 1 (C) 2 (D) 3

8. $\int\frac{8x^{43}+13x^{38}}{\left(x^{13}+x^5+1\right)^4}dx =$

(A) $\frac{x^{39}}{3\left(x^{13}+x^5+1\right)^3}$ + c

(B) $\frac{x^{39}}{\left(x^{13}+x^5+1\right)^3}$ + c

(C) $\frac{x^{39}}{5\left(x^{13}+x^5+1\right)^5}$ + c

(D) $\frac{x^{52}}{3\left(x^{13}+x^5+1\right)}$ + c

9. Let $\int e^x(f(x)-f'(x))dx = \phi(x)$, then $\int e^x f(x)dx$ is equal to

(A) $\phi(x) + e^x f(x) + c$

(B) $\phi(x) - e^x f(x) + c$

(C) $\frac{1}{2}\{\phi(x) + e^x f(x)\} + c$

(D) $\frac{1}{2}(\phi(x) + e^x f'(x)) + c$

10. Suppose f(x) is a real valued differentiable function defined on $[1, \infty)$ with f(1) = 1. Further let f(x) satisfy $f'(x) = \frac{1}{x^2 + f^2(x)}$, then the range of values of f(x) is

(A) $[1, \infty)$
(B) $[1, 1 + \pi/4)$
(C) $[1, \pi/4)$
(D) $[1 - \pi/4, 1]$

11. The value of $\lim\limits_{x\to 0} \frac{\int_0^x x e^{t^2} dt}{1 + x - e^x}$ is equal to

(A) 1
(B) 2
(C) –1
(D) –2

12. Let f(x) be a differentiable function such that f(0) = 0 and $\int_0^2 f'(2t)\, e^{f(2t)} dt = 5$, then the value of f (4) equals

(A) $2\,\ell n3$
(B) $\ell n10$
(C) $\ell n11$
(D) $3\,\ell n2$

13. The area enclosed by the curve $y \le \sqrt{4 - x^2}$, $y \ge \sqrt{2}\ \sin\left(\frac{\pi x}{2\sqrt{2}}\right)$ and the x-axis is divided by y-axis in the ratio

(A) $\frac{\pi^2 - 8}{\pi^2 + 8}$
(B) $\frac{\pi^2 - 4}{\pi^2 + 4}$
(C) $\frac{\pi - 3}{\pi + 4}$
(D) $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

14. For any $t \in R$ and f being a continuous function.

Let $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f\big(x(2 - x)\big) dx$

$I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f\big(x(2 - x)\big) dx$, then

(A) $I_1 = I_2$
(B) $I_1 = 2I_2$
(C) $2I_1 = I_2$
(D) $I_1 + I_2 = 0$

15. If $\int_2^x g(t) dt = \frac{x^2}{2} + \int_x^2 t^2 g(t) dt$, then equation g(x) = λ has

(A) 2 solution if $|\lambda| < \frac{1}{2}$
(B) 2 solution if $|\lambda| < \frac{1}{2}$ & $\lambda \ne 0$
(C) 1 solution if $\lambda = -\frac{1}{2}$
(D) No solution if $|\lambda| > \frac{1}{2}$

16. If $g(x) = \{x\}^{[x]}$, where {.} and [.] represents fractional part and greatest integer function respectively and $f(k) = \int_{k}^{k+1} g(x)dx$ $(k \in N)$, then

(A) f(1), f(2), f(3), ... are in H.P.

(B) $\sum_{r=1}^{\infty} (-1)^{r+1} f(r) = 1 - \ell n\, 2$

(C) $\sum_{r=1}^{\infty} (-1)^{r} f(r) = \ell n\left(\frac{2}{e}\right)$

(D) $\sum_{r=0}^{n} f\left(\frac{1}{r}\right) = \frac{n(n+1)}{2}$

17. If f(x) is a differentiable function such that $f(x + y) = f(x)\, f(y)\ \forall\, x, y \in R$, $f(0) \neq 0$ and $g(x) = \frac{f(x)}{1+(f(x))^2}$, then

(A) $\int_{-2014}^{2015} g(x)dx = \int_{0}^{2015} g(x)dx$

(B) $\int_{-2014}^{2015} g(x)dx - \int_{0}^{2014} g(x)dx = \int_{0}^{2015} g(x)dx$

(C) $\int_{-2014}^{2015} g(x)dx = 0$

(D) $\int_{-2014}^{2014} 2g(-x) - g(x)\, dx = 2\int_{0}^{2014} g(x)dx$

18. Let $I = \int_{-1}^{2}\left(\cot^{-1}\frac{1}{x} + \cot^{-1}x\right)dx$ and $J = \int_{-2\pi}^{7\pi} \frac{\sin x}{|\sin x|}dx$, then which of the following is/are correct ?

(A) $2I + J = 6\pi$

(B) $2I - J = 3\pi$

(C) $4I^2 + J^2 = 26\pi^2$

(D) $\frac{I}{J} = \frac{5}{2}$

19. If $I_n = \int_{0}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx$, where $n \in w$, then

(A) $I_{n+2} = I_n$

(B) $\sum_{m=1}^{10} I_m = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m-1} = 10\pi$

(D) $I_{n+1} = I_n$

20. If $\int_0^1 \frac{x^4\left(1+x^{10065}\right)}{\left(1+x^5\right)^{2015}}\,dx = \frac{1}{p}$, then

(A) Number of ways in which p can be expressed as a product of two relatively prime factors is 8.

(B) Number of ways in which p can be expressed as a product of two relatively prime factors is 4.

(C) Number of ways in which p can be expressed as a product of two factors is 8.

(D) Number of ways in which p can be expressed as a product of two factors is 4.

21. If $T_n = \sum_{r=2n}^{3n-1} \frac{r}{r^2+n^2}$ and $S_n = \sum_{r=2n+1}^{3n} \frac{r}{r^2+n^2}$ $\forall\, n \in \{1, 2, 3, \ldots\}$, then

(A) $T_n > \ell n\sqrt{2}$

(B) $S_n < \ell n\sqrt{2}$

(C) $T_n < \ell n\sqrt{2}$

(D) $S_n > \ell n\sqrt{2}$

22. If $I_1 = \int_0^1 \frac{1+x^8}{1+x^4}dx$ and $I_2 = \int_0^1 \frac{1+x^9}{1+x^3}dx$, then

(A) $I_2 < I_1 < \pi/4$

(B) $\pi/4 < I_2 < I_1$

(C) $1 < I_1 < I_2$

(D) $I_2 < I_1 < 1$

23. Consider a continuous function 'f' where $x^4 - 4x^2 \le f(x) \le 2x^2 - x^3$ such that the area bounded by $y = f(x)$, $g(x) = x^4 - 4x^2$, the y-axis and the line $x = t$ $(0 \le t \le 2)$ is twice of the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, y-axis and the line $x = t$ $(0 \le t \le 2)$ then

(A) $f(2) = 0$

(B) $f(1) = 1/3$

(C) $f'(1) = -2/3$

(D) $f(x)$ has two points of extrema

24. The value of the definite integral $\int_2^4 \left(\sqrt{x(3-x)(4+x)(6-x)(10-x)+\sin x}\right)dx$ equals

(A) cos 2 + cos 4

(B) cos 2 – cos 4

(C) 2cos1 cos3

(D) 2sin1 sin 3

25. The value of the definite integral $\int_{-\infty}^{a} \frac{(\sin^{-1}e^x + \sec^{-1}e^{-x})dx}{(\cot^{-1}e^a + \tan^{-1}e^x)(e^x+e^{-x})}$ $(a \in R)$ is

(A) Independent of a

(B) dependent on a

(C) $\frac{\pi}{2}\ell n 2$

(D) $-\frac{\pi}{2}\ell n\left(\frac{2}{\pi}\tan^{-1}e^{-a}\right)$

26. Let I = $\int_{k\pi}^{(k+1)\pi} \frac{|\sin 2x|dx}{|\sin x|+|\cos x|}$, (k $\in$ N) and J = $\int_{0}^{\pi/4} \frac{dx}{\sin x+\cos x}$, then which of the following hold(s) good ?

(A) I = 2 $\int_{0}^{\pi/2} \frac{\sin 2x\, dx}{\sin x+\cos x}$

(B) I = 4 – 4J

(C) I = 4 – 2J

(D) I = 2 – 2J

27. If f(x) = $\int \frac{x^8+4}{x^4-2x^2+2}$ dx and f(0) = 0, then

(A) f(x) is an odd function

(B) f(x) has range R

(C) f(x) = 0 has at least one real root

(D) f(x) is a monotonic function

28. If f(x) = $\int_{0}^{\pi/2} \frac{\ell n\left(1+x\sin^2\theta\right)}{\sin^2\theta}$ dθ, x $\geq$ 0, then

(A) f(x) = $\pi\left(\sqrt{x+1}-1\right)$

(B) f'(3) = $\frac{\pi}{4}$

(C) f(x) cannot be determined

(D) f'(0) = $\frac{\pi}{2}$

29. If f : R $\to$ R be a continuous function such that f(x) = $\int_{1}^{x} 2tf(t)dt$, then which of the following does not hold(s) good?

(A) f(π) = e^{π^2}

(B) f(1) = e

(C) f(0) = 1

(D) f(2) = 2

30. If $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\left(\frac{3r}{n}\right)^2+2\right)\frac{3}{n}=\int_{0}^{b} f(x)dx$, then

(A) b = 1

(B) f(x) = $9x^2$ + 6

(C) $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\left(\frac{3r}{n}\right)^2+2\right)\frac{3}{n}$ = 9

(D) $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\left(\frac{3r}{n}\right)^2+2\right)\frac{3}{n}$ = 15

31. A real valued function f(x) : $R^+ \to R^+$ satisfies $\int_{0}^{1} f(tx)dt = nf(x)$. If $\lim_{n\to\infty}$ f(x) = g(x), g(1) = 2 and area bounded by y = g(x) with x-axis from x = 3 to x = 7 is S, then

(A) S $\in$ $\left(2, \frac{8}{3}\right)$

(B) S $\in$ $\left(\frac{8}{7}, \frac{8}{3}\right)$

(C) S < $\frac{40}{21}$

(D) S > ℓn4

Comprehension # 1 (For Q. No. 32 to 33)

Consider the integral $I = \int_0^{10\pi} \frac{\cos 4x \cos 5x \cos 6x \cos 7x}{1+e^{2\sin 2x}} dx$

32. If $I = k . \int_0^{\pi/2} \cos 4x \cos 5x \cos 6x \cos 7x \, dx$, then 'k' is equal to

(A) 5 (B) 10 (C) 1 (D) 20

33. If $I = \lambda . \int_0^{\pi/4} \cos 2x \cos 4x \cos 6x \, dx$, then 'λ' is equal to

(A) 5 (B) 20 (C) 10 (D) 5/2

Comprehension # 2 (For Q. No. 34 to 36)

For i = 0, 1, 2, ... , n, let S_i denotes the area of region bounded by the curve $y = e^{-2x} \sin x$ with x-axis from $x = i\pi$ to $x = (i + 1)\pi$.

34. The value of S_0 is

(A) $\frac{1+e^{2\pi}}{5}$ (B) $\frac{1-e^{-2\pi}}{5}$

(C) $\frac{1+e^{-2\pi}}{5}$ (D) $\frac{1+e^{-\pi}}{5}$

35. The ratio $\frac{S_{2014}}{S_{2015}}$ is equal to

(A) $e^{-2\pi}$ (B) $e^{2\pi}$ (C) $2e^{\pi}$ (D) $e^{-\pi}$

36. The value of $\sum_{i=0}^{\infty} S_i$ is equal to

(A) $\frac{e^{\pi}\left(1+e^{\pi}\right)}{5\left(e^{\pi}-1\right)}$ (B) $\frac{e^{2\pi}\left(e^{2\pi}+1\right)}{5\left(e^{2\pi}-1\right)}$

(C) $\frac{e^{2\pi}+1}{5\left(e^{2\pi}-1\right)}$ (D) $\frac{e^{2\pi}+1}{e^{2\pi}-1}$

37. Let f(x) be differentiable function satisfying the condition $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ $\forall$ x, y $\in$ R – {0} and f(x) $\neq$ 0, $f'(1) = 2$. If the area enclosed by $y \geq f(x)$ and $x^2 + y^2 \leq 2$ is A, then find [2A], where [.] represents G.I.F.

38. The value of the definite integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{(\sin x + \cos x + 2\sqrt{\sin x \cos x})\sqrt{\sin x \cos x}}$ equals

39. A continuous real function 'f' satisfies f(2x) = 3 f(x) $\forall$ x $\in$ R. If $\int_0^1 f(x)dx = 1$, then compute the value of definite integral $\int_1^2 f(x)dx$

40. If $2^{2010} \frac{\int_0^1 x^{1004}(1-x)^{1004}dx}{\int_0^1 x^{1004}\left(1-x^{2010}\right)^{1004}dx} = \lambda$, then find the highest prime factor of λ.

ANSWER KEY
DPP # 7

REVISION DPP OF
VECTORS AND THREE DIMENSIONAL GEOMETRY

1.	(C)	**2.**	(C)	**3.**	(B)	**4.**	(B)	**5.**	(A)	**6.**	(A)	**7.**	(A)
8.	(C)	**9.**	(A)	**10.**	(C)	**11.**	(B)	**12.**	(C)	**13.**	(C)	**14.**	(B)
15.	(C)	**16.**	(B)	**17.**	(A)	**18.**	(A,D)	**19.**	(B,D)	**20.**	(B,D)		
21.	(B,C,D)	**22.**	(B,C)	**23.**	(B,C)	**24.**	(A,B,C)	**25.**	(A,B,C,D)	**26.**	(A,B,D)		
27.	(A,C,D)	**28.**	(A,C,D)	**29.**	(A,B)	**30.**	(A,C,D)	**31.**	(A,C,D)	**32.**	(A,B,D)		
33.	(A,C,D)	**34.**	(C,D)	**35.**	(B,D)	**36.**	(A,B,D)	**37.**	(A,D)	**38.**	(D)	**39.**	(C)
40.	(B)												

MATHEMATICS

1. $A = \int_0^{505\pi} |\cos x|\,dx = 505\int_0^{\pi}|\cos x|dx = 1010$ $\quad B = \int_{505\pi}^{1007\pi} |\sin x| = (1007-505)\int_0^{\pi}|\sin x|\,dx = 1004$

2. $I = \int_0^{100} \sqrt{x} - \left[\sqrt{x}\right]dx = \left(\frac{x^{3/2}}{3/2}\right)_0^{100} - \int_0^{100}\left[\sqrt{x}\right]dx$

$= \frac{2000}{3} - \left[\int_1^4 1.dx + \int_4^9 2.dx + \int_9^{16} 3.dx + \int_{16}^{25} 4.dx + \int_{25}^{36} 5.dx + \int_{36}^{49} 6.dx + \int_{49}^{64} 7.dx + \int_{64}^{81} 8.dx + \int_{81}^{100} 9.dx\right]$

$= \frac{2000}{3} - [3 + 10 + 21 + 36 + 55 + 78 + 105 + 136 + 171] = \frac{2000}{3} - 615 = \frac{155}{3}$

3. $I = \int e^x.\frac{2-x^2}{(1-x)\left(\sqrt{1-x^2}\right)}dx = \int e^x\left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1-x^2}{(1-x)\sqrt{1-x^2}}\right)dx = \int e^x\left(\frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}}\right)dx$

$= e^x.\sqrt{\frac{1+x}{1-x}} + c$

4. $f(x) = \int 2x^3.\cos^2 x + 6x^2 \sin x \cos x - 2x^3 \sin^2 x dx$

$= \int 2x^3.\cos 2x dx + \int \underbrace{3x^2}_{\text{II}}\underbrace{\sin 2x dx}_{\text{I}} = \int x^3.2\cos 2x\, dx + \sin 2x.x^3 - \int x^3.2\cos 2x\, dx$

$\Rightarrow \quad f(x) = x^3 \sin 2x + c \qquad \Rightarrow \quad f(x) = x^3 \sin 2x$

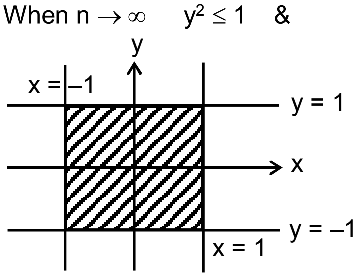
5. $f(x) = f(2-x)$ & $f(x) = f(4-x) \quad\Rightarrow\quad f(x) = f(x+2) \quad\Rightarrow\quad f(x)$ is periodic with period 2

Now $\quad I = \int_0^{50} f(x)\ dx = 25\int_0^2 f(x)\ dx = 75$

6. $I = \int\left(x^6 + x^4 + x^2\right)\sqrt{2x^4 + 3x^2 + 6}\,dx = \int\left(x^5 + x^3 + x\right)\sqrt{2x^6 + 3x^4 + 6x^2}\,dx$

Put $2x^6 + 3x^4 + 6x^2 = t \qquad \therefore \qquad I = \int\sqrt{t}.\frac{dt}{12} = \frac{t^{3/2}}{18} + c$

7. When $n \to \infty \quad y^2 \le 1$ & $x^2 \le 1 \quad\Rightarrow\quad -1 \le y \le 1$ & $-1 \le x \le 1$



$\lim_{n\to\infty} S_n = 4$

8. $I = \int\frac{8x^{43} + 13x^{38}}{\left(x^{13} + x^5 + 1\right)^4}dx = \int\frac{8x^{-9} + 13x^{-14}}{\left(1 + x^{-8} + x^{-13}\right)^4}dx \quad$ Put $\quad 1 + x^{-8} + x^{-13} = t \qquad \therefore \quad I = \int\frac{-dt}{t^4} = \frac{1}{3t^3} + c$

9. $\int e^x (f(x) - f'(x))dx = \phi(x)$ ...(i)

and $\int e^x (f(x) + f'(x))dx = e^x f(x)$...(ii)

equation (i) & (ii)

$2\int e^x f(x)dx = \phi(x) + e^x f(x)$

10. $f'(x) > 0 \Rightarrow f(x)\uparrow \Rightarrow f(x) \geq 1 \ \forall\ x \geq 1$

$\Rightarrow f'(x) \leq \dfrac{1}{1+x^2}\ \forall\ x \geq 1 \Rightarrow \displaystyle\int_1^x f'(x)dx \leq \int_1^x \frac{1}{1+x^2}dx$

$\Rightarrow f(x) - f(1) \leq \tan^{-1}x - \tan^{-1}1 \Rightarrow f(x) \leq 1 - \dfrac{\pi}{4} + \tan^{-1}x < 1 + \dfrac{\pi}{4}$

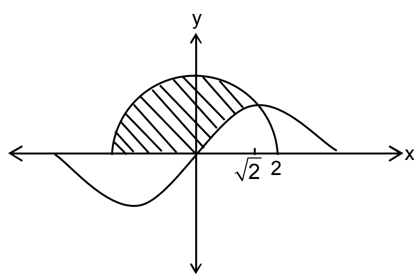
11. Let $\ell = \displaystyle\lim_{x\to 0} \frac{x\int_0^x e^{t^2}dt}{-(e^x - x - 1)}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{x\int_0^x e^{t^2}dt}{-x^2\left(\frac{e^x - x - 1}{x^2}\right)} = -2\lim_{x\to 0} \frac{\int_0^x e^{t^2}dt}{x}\left(\frac{0}{0}\right) = -2\lim_{x\to 0} \frac{e^{x^2}}{1} = -2$

12. We have $\displaystyle\int_0^2 f'(2t)\, e^{f(2t)}dt = 5$ Put $e^{f(2t)} = y \Rightarrow 2f'(2t)\, e^{f(2t)}\, dt = dy$

Now $\displaystyle\frac{1}{2}\int_{e^{f(0)}}^{e^{f(4)}} e^y dy = 5 \Rightarrow \int_{e^{f(0)}}^{e^{f(4)}} e^y dy = 10 \Rightarrow e^{f(4)} - e^{f(0)} = 10 \Rightarrow e^{f(4)} = 10 + e^0 = 11$

Hence $f(4) = \ell n\ 11$

13.



Area to the left of y-axis = π

Area to the right of y-axis = $\displaystyle\int_0^{\sqrt{2}} \left(\sqrt{4 - x^2} - \sqrt{2}\sin\left(\frac{\pi x}{2\sqrt{2}}\right)\right)dx$

$= \left(\dfrac{x\sqrt{4-x^2}}{2} + \dfrac{4}{2}\sin^{-1}\dfrac{x}{2}\right)_0^{\sqrt{2}} + \left(\dfrac{4}{\pi}\cos\dfrac{\pi x}{2\sqrt{2}}\right)_0^{\sqrt{2}} = 1 + \pi/2 - 4/\pi$

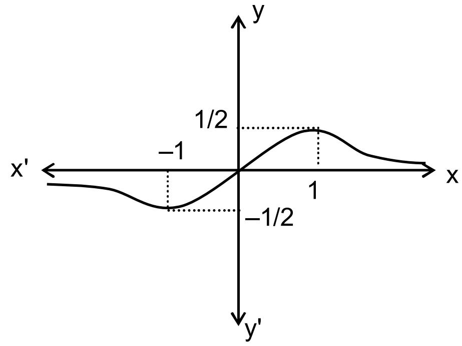
14. $I_1 = \displaystyle\int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx$

$= \displaystyle\int_{\sin^2 t}^{1+\cos^2 t} (1 + \cos^2 t + \sin^2 t - x) f\left\{(1 + \cos^2 t + \sin^2 t - x)(2 - (1 + \cos^2 t + \sin^2 t - x))\right\}$ **[P-5]**

$= 2\displaystyle\int_{\sin^2 t}^{1+\cos^2 t} f\{(2-x)x\}dx - \int_{\sin^2 t}^{1+\cos^2 t} xf\{(2-x)x\}dx \Rightarrow I_1 = 2I_2 - I_1 \Rightarrow 2I_1 = 2I_2 \Rightarrow \dfrac{I_1}{I_2} = 1$

15. Differentiating both sides $g(x) = x - x^2\, g(x)$ $\Rightarrow$ $g(x) = \dfrac{x}{1+x^2}$

Now, graph of y = g(x) is



16. $f(k) = \int_k^{k+1} (x-k)^k\, dx = \left(\dfrac{(x-k)^{k+1}}{k+1}\right)_k^{k+1}$ $\Rightarrow$ $f(k) = \dfrac{1}{k+1}$

Now $\sum_{r=1}^{\infty} (-1)^{r+1} f(r) = f(1) - f(2) + f(3) + \ldots\ldots.$

$= \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{5} + \ldots\ldots = 1 - \ell n\, 2$

17. $f(x + y) = f(x)\, f(y)$ $\Rightarrow$ $f(x) = e^x$

$\therefore$ $g(x) = \dfrac{e^x}{1+e^{2x}} = \dfrac{1}{e^x + e^{-x}}$ $\Rightarrow$ g(x) is an even function

18. $I = \int_{-1}^{0} \dfrac{3\pi}{2} dx + \int_{0}^{2} \dfrac{\pi}{2} dx = \dfrac{5\pi}{2}$ $J = \int_{-2\pi}^{6\pi} \dfrac{\sin x}{|\sin x|} dx + \int_{6\pi}^{7\pi} \dfrac{\sin x}{|\sin x|} dx = 0 + \pi = \pi$

19. $I_{n+1} - I_n = \int_0^{\pi} \dfrac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} dx$

$\Rightarrow$ $I_{n+1} - I_n = \int_0^{\pi} \dfrac{2\cos(n+1)x \quad \sin\frac{x}{2}}{\sin\frac{x}{2}}\, dx$

$\Rightarrow$ $I_{n+1} - I_n = 0$

20. Put $x^5 = t$

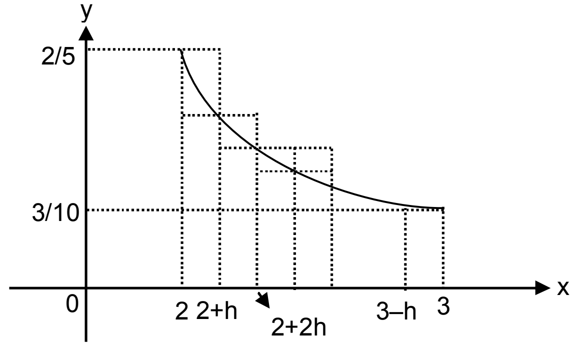
$I = \dfrac{1}{5} \int_0^1 \dfrac{1+t^{2013}}{(1+t)^{2015}}\, dt$

$= \dfrac{1}{5} \int_0^1 \dfrac{1}{(1+t)^{2015}}\, dt + \dfrac{1}{5} \int_0^1 \dfrac{t^{-2}}{\left(t^{-1}+1\right)^{2015}}\, dt$

$= \dfrac{1}{5} \times \dfrac{1}{2014}$

$\therefore$ $p = 5 \times 2014 = 2 \times 5 \times 19 \times 53$

21. Consider $f(x) = \dfrac{x}{1+x^2}$



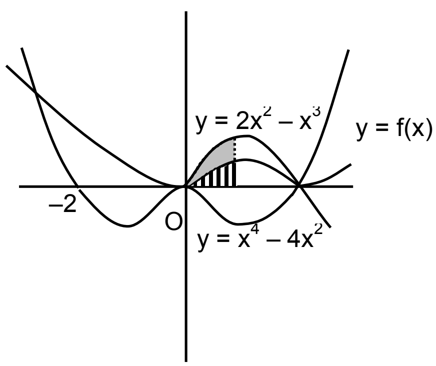
Area bounded by f(x) with x-axis $\int_2^3 \dfrac{x}{x^2+1} = \ell n\sqrt{2}$

Clearly, $h[f(2)+ f(2 + h) + \ldots + f(3 - h)] > \ell n\sqrt{2} > h[f(2 + h) + f(2 + 2h) + \ldots + f(3)]$

22. For all $x \in (0, 1)$

$\Rightarrow \quad \dfrac{1}{1+x^2} < \dfrac{1+x^9}{1+x^3} < \dfrac{1+x^8}{1+x^4} < 1 \qquad \therefore \int_0^1 \dfrac{1}{1+x^2}dx < I_2 < I_1 < \int_0^1 1 \; dx \qquad \therefore \quad \pi/4 < I_2 < I_1 < 1$

23.



$\int_0^t \left[f(x) - (x^4 - 4x^2)\right] \; dx = 2\int_0^t \left[\left(2x^2 - x^3\right) - f(x)\right]dx$

on differentiating with respect to t.

$\Rightarrow \quad f(t) - (t^4 - 4t^2) = 2(2t^2 - t^3 - f(t)) \qquad \Rightarrow \quad f(t) = \dfrac{1}{3}(t^4 - 2t^3)$

24. We have $I = \int_2^4 \left(x(3-x)(4+x)(6-x)(10-x) + \sin x \right)dx$ ....(1)

Now $I = \int_2^4 \left((6-x)\left(3-(6-x)\right)\left(4+(6-x)\right)\left(6-(6-x)\right)\left(10-(6-x)\right) + \sin(6-x) \right)dx$

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \qquad I = \int_2^4 \left((6-x)(x-3)(10-x)x(4+x) + \sin(6-x) \right)dx$ ....(2)

$\therefore$ On adding (1) and (2), we get

$2I = \int_2^4 \left(\sin x + \sin(6-x)\right)dx = \left(-\cos x + \cos(6-x)\right)_2^4 = -\cos 4 + \cos 2 + \cos 2 - \cos 4$

$= 2(\cos 2 - \cos 4)$ Hence $I = \cos 2 - \cos 4$ **Ans.**

25. We have $I = \int\limits_{-\infty}^{a}\left(\frac{\sin^{-1}e^x+\cos^{-1}e^x}{\cot^{-1}e^a+\tan^{-1}e^x}\right)\left(\frac{e^x}{e^{2x}+1}\right)dx = \frac{\pi}{2}\int\limits_{-\infty}^{a}\frac{1}{(\cot^{-1}e^a+\tan^{-1}e^x)}\left(\frac{e^x}{(e^{2x}+1)}\right)dx$

Put $\tan^{-1}e^x = t$ $\Rightarrow$ $\frac{e^x}{e^{2x}+1}dx = dt$

$I = \frac{\pi}{2}\int\limits_{0}^{\tan^{-1}e^a}\frac{dt}{(t+\cot^{-1}e^a)} = \frac{\pi}{2}\Big[\ell n(t+\cot^{-1}e^a)\Big]_0^{\tan^{-1}e^a} = \frac{\pi}{2}\left[\ell n\left(\frac{\pi}{2}\right)-\ell n\left(\cot^{-1}e^a\right)\right] = -\frac{\pi}{2}\ell n\left(\frac{2}{\pi}\tan^{-1}e^{-a}\right)$

26. We have $I = \int\limits_{k\pi}^{(k+1)\pi}\frac{|\sin 2x|dx}{|\sin x|+|\cos x|}$; put $x = k\pi + t \Rightarrow dx = dt$

$\therefore$ $I = \int\limits_{0}^{\pi}\frac{|\sin 2x|\,dx}{|\sin x|+|\cos x|} = 2\int\limits_{0}^{\pi/2}\frac{\sin 2x\,dx}{\sin x+\cos x} = 2\int\limits_{0}^{\pi/2}\frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}dx$

$= 2\int\limits_{0}^{\pi/2}(\sin x+\cos x)dx - 2\int\limits_{0}^{\pi/2}\frac{dx}{\sin x+\cos x} = 4-4\int\limits_{0}^{\pi/4}\frac{dx}{\sin x+\cos x} = 4-4J$

27. $f(x) = \int\frac{x^8+4+4x^4-4x^4}{x^4-2x^2+2}\,dx = \int\frac{(x^4+2)^2-4x^4}{x^4-2x^2+2}dx = \int\frac{(x^4+2x^2+2)(x^4-2x^2+2)}{(x^4-2x^2+2)}\,dx$

$\Rightarrow$ $f(x) = \frac{x^5}{5} + \frac{2x^3}{3} + 2x$

28. $f(x) = \int\limits_{0}^{\pi/2}\frac{\ell n(1+x\sin^2\theta)}{\sin^2\theta}d\theta$; $x \ge 0$ $\Rightarrow$ $f'(x) = \int\limits_{0}^{\pi/2}\frac{1}{1+x\sin^2\theta}\,d\theta$

$\Rightarrow$ $f'(x) = \int\limits_{0}^{\pi/2}\frac{\sec^2\theta\ \ dq}{1+(1+x)\tan^2\theta}$ put $\tan\theta = t$

$\Rightarrow$ $f'(x) = \int\limits_{0}^{\infty}\frac{dt}{1+\left\{\left(\sqrt{1+x}\right)t\right\}^2}$ $\Rightarrow$ $f'(x) = \frac{1}{\sqrt{1+x}}\left(\tan^{-1}\left(\sqrt{1+x}\times t\right)\right)_0^{\infty}$

$\Rightarrow$ $f'(x) = \frac{\pi}{2}.\frac{1}{\sqrt{1+x}}$ $\Rightarrow$ $f(x) = \pi\ .\ \sqrt{1+x}+c$ put $x = 0$

$\pi + c = f(0)$ $\Rightarrow$ $c = -\pi$ $\therefore$ $f(x) = \pi\left(\sqrt{1+x}-1\right)$

29. $f(x) = \int\limits_{0}^{x}2t\ \ f(t)dt$ $\Rightarrow$ $f'(x) = 2xf(x)$ $\Rightarrow$ $\frac{f'(x)}{f(x)} = 2x$ $\Rightarrow$ $\ell n f(x) = x^2 + \ell n c$

$\Rightarrow$ $f(x) = c.e^{x^2}$ put $x = 1$ $c.e = f(1) = 0$ $\Rightarrow$ $c = 0$ $\therefore$ $f(x) = 0$

30. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left(\left(\frac{3r}{n}\right)^2+2\right)\frac{3}{n} = \int\limits_{0}^{1}\left(9x^2+2\right).3dx$

31. $tx = y$ $\Rightarrow$ $\int\limits_{0}^{x}f(y)\,dy = xn\,f(x)$ $\Rightarrow$ $f(x) = n[f(x) + xf'(x)]$ $\Rightarrow$ $f(x)(1-n) = nx\,f'(x)$

$\Rightarrow$ $\frac{f'(x)}{f(x)} = \left(\frac{1-n}{n}\right).\frac{1}{x}$ $\Rightarrow$ $\ell n(f(x)) = \left(\frac{1-n}{n}\right)\ell n x + \ell n c$

$\Rightarrow$ $f(x) = c\,x^{\frac{1-n}{n}}$ as $n \to \infty$ $f(x) = cx^{-1} = \frac{c}{x}$ $\Rightarrow$ $g(x) = \frac{2}{x}$

(32 to 33)

$$I = \int_0^{10\pi} \frac{\cos 4x \cos 5x \cos 6x \cos 7x}{1+e^{2\sin 2x}} dx$$

$$I = \int_0^{10\pi} \frac{\cos 4x \cos 5x \cos 6x \cos 7x}{1+e^{-2\sin 2x}} dx \qquad \text{(from p-5)}$$

$$2I = \int_0^{10\pi} \cos 4x \cos 5x \cos 6x \cos 7x \, dx$$

$$2I = 10\int_0^{\pi} \cos 4x \cos 5x \cos 6x \cos 7x \, dx \qquad \text{(from p-7)}$$

$$2I = 20\int_0^{\pi/2} \cos 4x \cos 5x \cos 6x \cos 7x \, dx \qquad \text{(from p-6)}$$

$$I = 10\int_0^{\pi/2} \cos 4x \cos 5x \cos 6x \cos 7x \, dx \qquad \therefore \quad k = 10$$

Further,

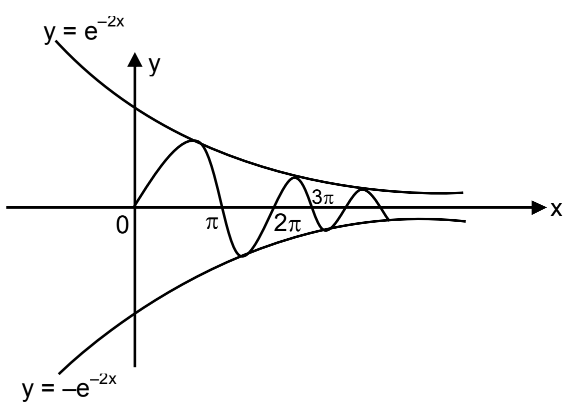
$$I = 5\int_0^{\pi/2} \cos 4x . \cos 6x . (\cos 12x + \cos 2x)$$

$$I = 5\left(\int_0^{\pi/2} \cos 4x \cos 6x \cos 12x \, dx + \int_0^{\pi/2} \cos 2x \cos 4x \cos 6x \, dx\right)$$

$$I = 5\left(0 + 2\int_0^{\pi/4} \cos 2x \cos 4x \cos 6x \, dx\right) \qquad \text{(from p-6)}$$

$$I = 10\int_0^{\pi/4} \cos 2x \cos 4x \cos 6x \, dx \qquad \therefore \quad \lambda = 10$$

(34 to 36)



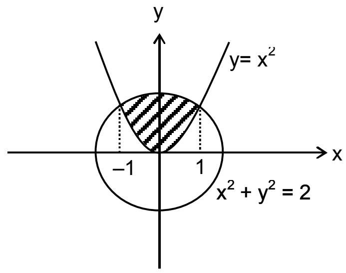
Now $S_i = \left|\int_{i\pi}^{(i+1)\pi} e^{-2x} \sin x \ \ dx\right| \Rightarrow \qquad S_i = \left|\left(\frac{e^{-2x}}{5}(-2\sin x - \cos x)\right)_{i\pi}^{(i+1)\pi}\right|$

$\Rightarrow \qquad S_i = \frac{1}{5}\left|e^{-2(i+1)\pi}\cos\ (i+1)\pi - e^{-2i\pi}\cos i\pi\right| \qquad \Rightarrow \qquad S_i = \frac{e^{-2i\pi}}{5}\left(1+e^{-2\pi}\right)$

(i) $\quad S_0 = \frac{1+e^{-2\pi}}{5}$ $\qquad$ (ii) $\frac{S_{2014}}{S_{2015}} = e^{2\pi}$ $\qquad$ (ii) $\sum_{i=0}^{\infty} S_i = \frac{\frac{1+e^{-2\pi}}{5}}{1-e^{-2\pi}} = \frac{e^{2\pi}+1}{5\left(e^{2\pi}-1\right)}$

37. $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ Differentiable both side w.r.t. y $f'\left(\frac{x}{y}\right).\left(\frac{-x}{y^2}\right)=\frac{-f(x)}{f^2(y)}.f'(y)$

Put y = 1 $\Rightarrow$ $f'(x).x = 2f(x)$ $\Rightarrow$ $f(x) = x^2$



$$A = 2\int_0^1\left(\sqrt{2-x^2}-x^2\right)\ dx=\frac{1}{3}+\frac{\pi}{2} \Rightarrow 2A = \frac{2}{3}+\pi \Rightarrow [2A] = 3$$

38. Let $I = \int_{\pi/4}^{\pi/2}\frac{dx}{\cos x(\tan x+1+2\sqrt{\tan x})\sqrt{\tan x\cos^2 x}}$ $\quad I = \int_{\pi/4}^{\pi/2}\frac{\sec^2 x\ dx}{(1+\sqrt{\tan x})^2\sqrt{\tan x}}$

Put $\tan x = t^2$ $\Rightarrow$ $\sec^2 x\ dx = 2t\ dt$ $\quad I = \int_1^{\infty}\frac{2t\,dt}{(t+1)^2\cdot t}$ $\quad I = -2\left[\frac{1}{t+1}\right]_1^{\infty} = -2\left[0-\frac{1}{2}\right] = 1$

39. We have $f(2x) = 3 f(x)$ ....(1) and $\int_0^1 f(x)dx = 1$ ....(2)

From (1) and (2), $\frac{1}{3}\int_0^1 f(2x)dx = 1$

Put 2x = t, $\frac{1}{6}\int_0^2 f(t)dt = 1$ $\Rightarrow$ $\int_0^2 f(t)dt = 6$ $\Rightarrow$ $\int_0^1 f(t)dt+\int_1^2 f(t)dt = 6$

Hence $\int_1^2 f(t)dt = 6-\int_0^1 f(t)dt = 6 - 1 = 5$

40. Consider $I_2 = \int_0^1 x^{1004}\left(1-x^{2010}\right)^{1004}dx$ Put $x^{1005} = t$ $\Rightarrow$ $1005x^{1004}dx = dt$

So $I_2 = \frac{1}{1005}\int_0^1\left(1-t^2\right)^{1004}dt$ …(i) Also $I_2 = \frac{1}{1005}\int_0^1\left[1-(1-t)^2\right]^{1004}dt$ …(ii)

$\Rightarrow$ $I_2 = \frac{1}{1005}\int_0^1\left(t(2-t)\right)^{1004}dt = \frac{1}{1005}\int_0^1 t^{1004}(2-t)^{1004}dx$ Put t = 2y $\Rightarrow$ dt = 2dy

So $I_2 = \frac{1}{1005}\int_0^{1/2}(2y)^{1004}(2-2y)^{1004}2dy = \frac{1}{1005}2.2^{1004}.2^{1004}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$

$I_2 = \frac{1}{1005}2^{2009}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$ …(iii)

Now $I_1 = \int_0^1 x^{1004}(1-x)^{1004}dx = 2\int_0^{1/2}x^{1004}(1-x)^{1004}dx$ …(iv)

$\therefore$ From (iii) and (iv) we get

$I_2 = \frac{1}{1005}2^{2010}\frac{I_1}{4}$ $\Rightarrow$ $2^{2010}\frac{I_1}{I_2} = 4020$