Single Value Decomposition :- (SVD)

[factorization of the matrix]

There are several ways of decomposing matrices.

Decomposition is done to convert the original matrix into new form which is easy to work with.

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2 or more matrices.

SVD — matrix factorization technique

SVD takes original Data matrix and converts or rather decomposes it into 3 matrices.

$$\underset{m\times n}{A} = U \Sigma V^T$$

$\underset{m\times m}{\Sigma}$ → is a diagonal matrix (i.e only diagonal elements)

diagonal elts are sorted from Large to Smaller values.

(square root of eigen value)

Singular value matrix.

$\underset{m\times n}{U}$ → left singular value matrix

→ $A \cdot A^T$ (we find eigenvectors)

V^T → right singular value matrix.

~~m x~~ n x n → $A^T \cdot A$. (we find eigenvectors)

Example :-

Find SVD of $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Solution :-

Computing 'U' —— part I

a) $U = A \cdot A^T = \underset{2\times3}{\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}}$

b.) computing eigenvalues (λ) and eigen vectors of $A \cdot A^T$.

$$Av = \lambda v$$

here $A \longrightarrow A \cdot A^T$ say $P. = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$

$\therefore \quad P - \lambda I = 0$

$\therefore \quad \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$

$\therefore \quad \begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix} = 0$

$\therefore \quad (11-\lambda)(11-\lambda) - (1)(1) = 0$

$\therefore \quad 121 - 11\lambda - 11\lambda + \lambda^2 - 1 = 0$

$\therefore \quad \lambda^2 - 22\lambda + 120 = 0$

$\therefore \quad \lambda_1 = 10$ or $\lambda_2 = 12.$

$\longrightarrow$ eigenvalues.

eigenvectors of $\lambda_1 = 10$ & $\lambda_2 = 12$.

$\boxed{\text{when } \lambda_1 = 10}$ let, $V = \begin{bmatrix} x \\ y \end{bmatrix}$ $A = P$.

$$AV = \lambda V$$

$$\therefore \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 10 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11x + y \\ x + 11y \end{bmatrix} - \begin{bmatrix} 10x \\ 10y \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} x + y \\ x + y \end{bmatrix} = 0$$

We get, $x + y = 0$ —— (1)

$x + y = 0$ —— (2)

$\therefore x + y = 0$

$x = -y$. i.e if $x = 1$, $y = -1$

$x + y = 0$.

Hence $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ — vector for λ_1.

Normalizing the vector:-

unit length of vector $= \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$.

$$\therefore N \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\boxed{\text{when } \lambda_2 = 12}$

$$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 12 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11x + y \\ x + 11y \end{bmatrix} - \begin{bmatrix} 12x \\ 12y \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} y - x \\ +x \mp y \end{bmatrix} = 0$$

$$-x + y = 0 \quad \text{—— (3)}$$

$$+x - y = 0 \quad \text{—— (4)}$$

$\therefore$ $x - y = 0$ ie if $x = y = 1$

$\therefore$ $x = y$ $x - y = 0$.

Hence, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Normalizing the above vector.

unit length $= \sqrt{(1)^2 + (1)^2} = \sqrt{2}$.

$$\therefore N \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

[Greater the λ value — eigen vectors of that λ would be written first].

Hence, $$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Computing V^T — part II.

a.) $V = A^T \cdot A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

b.) Computing eigen values & eigenvectors of $A^T \cdot A$.

Since, the matrix is 3-D.
writing characteristic eq. of $A^T A$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of diagonal elts
$= 10 + 10 + 2 = 22$

S_2 = sum of minors of diagonal elts

$$= \begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}$$

$= 4 + 16 + 100 = 120$

S_3 = determinant of $A^T A$.

$$\begin{vmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{vmatrix} = 10(20-16) - 0 + 2(0-20)$$
$= 40 - 40$
$= \underline{\underline{0}}$

$\therefore \quad \lambda^3 - 22\lambda^2 + 120\lambda - 0 = 0$

$\therefore \quad \lambda_1 = 12, \ \lambda_2 = 10, \ \lambda_3 = 0.$

eigen vectors of $\lambda_1 = \cancel{0}12$, $\lambda_2 = 10$ & $\lambda_3 = 0$

when $\lambda_1 = 12$.

$$AV = \lambda V$$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 12 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} 10x + 0y + 2z \\ 0x + 10y + 4z \\ 2x + 4y + 2z \end{bmatrix} - \begin{bmatrix} 12x \\ 12y \\ 12z \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -2x + 0y + 2z \\ 0x - 2y + 4z \\ 2x + 4y - 10z \end{bmatrix} = 0$$

$$-2x + 0y + 2z = 0 \quad \text{—— (1)}$$
$$0x - 2y + 4z = 0 \quad \text{—— (2)}$$
$$2x + 4y - 10z = 0 \quad \text{—— (3)}$$

Consider eq. (1) and eq (2).

$$\frac{x}{\begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{-y}{-8} = \frac{z}{4}$$

$$\therefore \quad v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Normalizing vector.

$$\cancel{\sqrt{(4)^2+(-8)^2+(4)^2} = \sqrt{16+64+16}}$$

$$\sqrt{(1)^2+(-2)^2+(1)^2} = \sqrt{6}$$

$$v_1 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

when $\lambda = 10$.

$$Av = \lambda v$$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 10 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore \begin{bmatrix} 10x + 0y + 2z - 10x \\ 0x + 10y + 4z - 10y \\ 2x + 4y + 2z - 10z \end{bmatrix} = 0$$

$$\therefore$$

$$0x + 0y + 2z = 0 \quad \text{--- (1)}$$

$$0x + 0y + 4z = 0 \quad \text{--- (2)}$$

$$2x + 4y - 8z = 0 \quad \text{--- (3)}$$

Consider (2) & (3).

$$\frac{x}{\begin{vmatrix} 0 & 4 \\ 4 & -8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 0 & 4 \\ 2 & -8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix}}$$

$$\frac{x}{-16} = \frac{-y}{-8} = \frac{z}{0}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{0}$$

$$\therefore V_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Normalizing vector.

Unit length $= \sqrt{(2)^2 + (-1)^2 + (0)^2} = \sqrt{5}$

$$V_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

When $\lambda = 0$.

$$AV = \lambda V$$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\therefore$

1)— $10x + 0y + 2z = 0$

2) $0x + 10y + 4z = 0$

$2x + 4y + 2z = 0$

(1) & (2)

$$\frac{x}{\begin{vmatrix} 0 & 2 \\ 10 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 10 & 2 \\ 0 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{40} = \frac{z}{100}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{5}$$

$$v_3 = \begin{bmatrix} 1 \\ +2 \\ -5 \end{bmatrix}$$

Normalizing vector:-

$$\sqrt{1^2 + 2^2 + (-5)^2} = \sqrt{30}$$

$$\therefore \quad v_3 = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$$

$$\therefore \quad V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

Hence,

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

Computing Σ —— part III.

eigen values = diagonal elts in decreasing order.

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & \sqrt{0} \end{bmatrix}$$

Hence, decomposed matrix A is

$$\begin{array}{ccc} U & \Sigma & V^T \end{array}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ & \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$