$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin sx \, dx}$$

Ex. 1 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin sx \cdot dx = \sqrt{\frac{2}{\pi}} \int_0^{1} 1 \cdot \sin sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[-\frac{\cos sx}{s}\right]_0^1 = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1-\cos s}{s}\right]$$

Ex. 2 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{a} 0 \sin xs \cdot dx + \int_a^{b} x \sin sx \cdot dx + \int_b^{\infty} 0 \sin sx \cdot dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_a^{b} x \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[x \frac{(-\cos sx)}{s} - \int -\frac{\cos sx}{s} \cdot 1 \cdot dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[-\frac{x \cos sx}{s} + \frac{\sin sx}{s^2}\right]_a^b$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{-b \cos bs + a \cos as}{s} + \frac{\sin bs - \sin as}{s^2}\right]$$

Ex. 3 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} \sin kx, & 0 \le x < a \\ 0, & x > a \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cdot \sin sx \, dx$$

$$\therefore \quad F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^a \sin kx \cdot \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^a \frac{1}{2} \{-[\cos(k+s)x + \cos(k-s)x]\} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[-\frac{\sin(k+s)x}{k+s} + \frac{\sin(k-s)x}{k-s} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{\sin(k-s)a}{k-s} - \frac{\sin(k+s)a}{k+s} \right]$$

Ex. 4 : Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cdot \sin sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{1}{x} \sin sx \cdot dx \qquad \text{.......... (1)}$$

Put $sx = t$ $\quad \therefore \; s\,dx = dt$, when $x = 0$, $t = 0$, when $x = \infty$, $t = \infty$.

$$\therefore \quad F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{s}{t} \sin t \cdot \frac{dt}{s}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{\sin t}{t} \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\pi}{2}\right) \qquad \text{[By result (iii) of Ex. 1 of § 6]}$$

$$= \sqrt{\frac{\pi}{2}}.$$

Alternatively : To evaluate (1), consider

$$\int_0^\infty e^{-\alpha x} \sin sx \, dx = \frac{1}{\alpha^2 + s^2} \left[e^{-\alpha x} (-\alpha \sin sx - s \cos sx) \right]_0^\infty$$

$$= \frac{s}{\alpha^2 + s^2}.$$

Integrate both sides w.r.t. α between the limits α_1 and α_2,

$$\therefore \quad \int_0^\infty \left[\int_{\alpha_1}^{\alpha_2} e^{-\alpha x} \, d\alpha \right] \sin sx \, dx = \int_{\alpha_1}^{\alpha_2} \frac{s}{\alpha^2 + s^2} \cdot d\alpha$$

$$\therefore \quad \int_0^\infty \left(\frac{e^{-\alpha_1 x} - e^{-\alpha_2 x}}{x} \right) \sin sx \, dx = \tan^{-1} \frac{\alpha_2}{s} - \tan^{-1} \frac{\alpha_1}{s}$$

Now, when $\alpha_1 \to 0$ and $\alpha_2 \to \infty$,

$$\int_0^\infty \frac{\sin sx}{x} \, dx = \frac{\pi}{2}$$

Hence, from (1), $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$.

For still another method see cor. of Ex. 4 of § 8.

from (2), $F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1} \infty = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$

$$\therefore \int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}.$$

8. Fourier Cosine Transform

The infinite **Fourier Cosine Transform** of $f(x)$, $0 < x < \infty$, denoted by $F_c(s)$ is defined by

$$\boxed{F_c(s) = \sqrt{\frac{2}{x}} \cdot \int_0^\infty f(x) \cos sx \, dx}$$

Ex. 1 : Find the Fourier cosine transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

Sol. : By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^1 1 \cdot \cos sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin sx}{s}\right]_0^1 = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin s}{s}.$$

Ex. 2 : Find the Fourier cosine transform of $f(x)$ if

$$f(x) = \begin{cases} \cos kx, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^a \cos kx \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a [\cos(k+s)x + \cos(k-s)x] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{\sin(k+s)x}{k+s} + \frac{\sin(k-s)x}{k-s}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{\sin(k+s)a}{k+s} + \frac{\sin(k-s)a}{k-s}\right]$$

Ex. 3 : Find the Fourier cosine transform of $f(x) = e^{-x^2}$.

Sol. : By definition

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x^2} \cdot \cos sx \, dx \qquad \text{.......... (1)}$$

Differentiating w.r.t. s, we get,

$$\frac{d}{ds}[F_c(s)] = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty -e^{-x^2} \sin sx \cdot (x) \, dx$$

Integrating by parts,

$$= -\sqrt{\frac{2}{\pi}} \cdot \left[\sin sx \cdot \left(-\frac{e^{-x^2}}{2} \right) - \int \left(-\frac{e^{-x^2}}{2} \right) \cdot (\cos sx) \, s \, dx \right]_0^\infty$$

$$= 0 - \frac{s}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \cos sx \cdot e^{-x^2} dx$$

$$= -\frac{s}{2} \cdot F_c(s)$$

$$\therefore \quad \frac{d\,[F_c(s)]}{F_c(s)} = -\frac{s}{2} \, ds$$

By integration $\log [F_c(s)] = -\dfrac{s^2}{4} + \log c$ .... (2)

But from (1) when $s = 0$,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x^2} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}.$$

$$\therefore \quad \log c = \log \frac{1}{\sqrt{2}}$$

From (2), $\log [F_c(s)] = -\dfrac{s^2}{4} + \log \dfrac{1}{\sqrt{2}}$

$$\therefore \quad \log \left\{ \frac{F_c(s)}{1/\sqrt{2}} \right\} = -\frac{s^2}{4} \qquad \therefore \quad F_c(s) = \frac{1}{\sqrt{2}} e^{-s^2/4}.$$

9. Inverse Fourier Cosine Transform

If $F_c(s)$ is the Fourier cosine transform of $f(x)$ which satisfies Dirichlet's conditions in every finite interval $(0, l)$ and if $\int_0^\infty |f(x)| \, dx$ exists at every point of continuity of $f(x)$, then

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2} \cos sx \, ds$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \frac{\cos sx}{1+s^2} ds$$

$$\therefore \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{1+s^2} ds$$

Putting $x = m$, $\displaystyle\int_0^\infty \frac{\cos ms}{1+s^2} ds = \frac{\pi}{2} e^{-m}$

Since in definite integral the variable does not matter,

$$\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$$

Ex. 5 : Find the Fourier cosine transform of $f(x) = e^{-x} + e^{-2x}$, $(x > 0)$.

Sol. : By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty (e^{-x} + e^{-2x}) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\int_0^\infty e^{-x} \cos sx \, dx + \int_0^\infty e^{-2x} \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) + \frac{e^{-2x}}{4+s^2} (-2 \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\left(0 + \frac{1}{1+s^2}\right) + \left(0 + \frac{2}{4+s^2}\right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{1+s^2} + \frac{2}{4+s^2} \right] = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{6+3s^2}{4+5s^2+s^4} \right]$$

EXERCISE - IV

1. Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$ **(M.U. 2008)**

[**Ans. : (i)** $2\sqrt{\frac{2}{\pi}} \cdot \frac{\sin s}{s^2}(1 - \cos s)$, **(ii)** $2\sqrt{\frac{2}{\pi}} \cdot \frac{\cos s}{s^2}(1 - \cos s)$]

2. $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$\left[\textbf{Ans. :} \ \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right]; \right.$$

$$\left. -\frac{1}{\sqrt{2\pi}} \left[\frac{\cos(1+s)a}{1+s} + \frac{\cos(1-s)a}{1-s} - \frac{2}{1-s^2} \right] \right]$$

3. $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$\left[\textbf{Ans. :} -\frac{1}{\sqrt{2\pi}} \left[\frac{\cos(1+s)a}{1+s} - \frac{\cos(1-s)a}{1-s} - \frac{2}{1-s^2} \right]; \right.$$

$$\left. \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s} \right] \right]$$

4. $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & x > a \end{cases}$

$$\left[\textbf{Ans. : (i)} \ k \cdot \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1-\cos sa}{s} \right], \ \textbf{(ii)} \ k \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\sin sa}{s} \right) \right]$$

5. Find Fourier Cosine Transform of $e^{-2x} + 4e^{-3x}$.

$$\left[\textbf{Ans. :} \ 2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{s^2+4} + \frac{6}{s^2+9} \right] \right]$$

6. Find the Fourier Sine Transform of e^{-ax}, $(a > 0)$.

$$\left[\textbf{Ans. :} \ \sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2} \right]$$

7. Find the Fourier sine transform of $e^{-|x|}$.

$$\left[\textbf{Ans. :} \ \sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2} \right]$$

8. Find the Fourier Sine Transform of $2e^{-5x} + 5e^{-2x}$.

$$\left[\textbf{Ans. :} \ s \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{2}{s^2+25} + \frac{5}{s^2+4} \right) \right]$$

9. Find Fourier sine and cosine transforms of e^{-x} and use the inversion formulae to recover the original function in both cases.

[**Ans. :** $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2}$, $F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2}$

To recover the function use Fourier integral. For $f(x) = e^{-x}$ use Fourier sine integral and Fourier Cosine Integral.]

10. Properties of Fourier Transform

1. Linearity Property : If $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$, then the Fourier transform of the sum of a $f(x)$ and $bg(x)$ is given by

$$F[af(x) + bg(x)] = aF(s) + bG(s)$$

where, a and b are constants.

Proof : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

and $$G(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

$$\therefore \quad F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{isx}\, dx$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx + b \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= aF(s) + bG(s)$$

2. Change of Scale Property : If $F(s)$ is the complex Fourier transform of $f(x)$, then $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$.

Proof : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

where $F(s)$ denotes the Fourier transform of $f(x)$.

$$\therefore \quad F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

Now put $ax = t$, $x = \frac{t}{a}$, $dx = \frac{dt}{a}$

$$\therefore \quad F[f(ax)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{ist/a} \cdot \frac{dt}{a}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{i(s/a)t}\, dt$$

$$\therefore \quad F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right).$$

3. Shifting Property : If $F(s)$ is the complex Fourier transform of $f(x)$ then $F[f(x-a)] = e^{isa} F(s)$.

Proof : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

where $F(s)$ denotes the Fourier transform of $f(x)$.

$$\therefore \quad F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x-a) e^{isx} \, dx$$

Now, put $x - a = t$, $x = a + t$, $dx = dt$.

$$\therefore \quad F(x-a) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) e^{is(a+t)} \, dt$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) e^{ist} \, dt$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = e^{isa} F(s).$$

4. Convolution Theorem : We know that the convolution of two functions $f(x)$ and $g(x)$ denoted by $f(x) * g(x)$ is defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x) g(x-u) \, du$$

Theorem : The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$.

In symbols, $F[f(x) * g(x)] = F(s)\, G(s)$

where, $F(s)$ and $G(s)$ denote the Fourier transforms of $f(x)$ and $g(x)$ respectively

i.e. $F(s) = F[f(x)]$ and $G(s) = F[g(x)]$ **(M.U. 2008)**

Proof : By the definition of convolution, the convolution of $f(x)$ and $g(x)$ denoted by $f(x) * g(x)$ is given by

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(u) g(x-u) \, du$$

(Constant is adjusted)

Taking Fourier transforms of both sides,

$$F[f(x) * g(x)] = F\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u) \, du\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u) \, du\right] e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \, du \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-u) \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \, du\, F[g(x-u)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \, du\, e^{ius} F[g(x)] \quad \text{[By shifting property]}$$

$$= G(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{ius} \, ds$$

$$= G(s) F(s)$$