Corollary : Taking the inverse Fourier transforms of both sides,

$$F^{-1}\left[F[f(x)*g(x)]\right]=F^{-1}[F(s)G(s)]$$

$$\therefore \quad f(x)*g(x)=F^{-1}[F(s)G(s)]$$

$$\therefore \quad F^{-1}[F(s)]\,F^{-1}[G(s)]=F^{-1}\left[F(s)*G(s)\right]$$

Relationship Between Fourier Transform And Laplace Transform

Let us consider the function,

$$f(t)=\begin{cases} e^{-xt}g(t), & \text{for } t>0 \\ 0, & \text{for } t<0 \end{cases}$$

Now, the Fourier transform of $f(t)$ is given by

$$F[f(t)]=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} f(t)e^{ist}\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\int_{0}^{\infty} f(t)\cdot e^{ist}\,dt \qquad [\because f(t)=0 \text{ for } t<0]$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\int_{0}^{\infty} e^{-xt}\,g(t)\cdot e^{ist}\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\int_{0}^{\infty} e^{(is-x)t}\cdot g(t)\,dt$$

Put $x-is=p$,

$$\therefore \quad F[f(t)]=\frac{1}{\sqrt{2\pi}}\cdot\int_{0}^{\infty} e^{-pt}\,g(t)\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\cdot L[g(t)]$$

$\therefore$ Fourier transform of $f(t)$

$$=\frac{1}{\sqrt{2\pi}} \text{ Laplace transform of } g(t)$$

where $g(t)$ is defined as above.

11. Finite Fourier Sine Transform and its Inverse

(i) If $f(x)$ is a sectionally continuous function of x over a finite interval $(0, l)$ then the **finite Fourier sine transform of $f(x)$ on $(0, l)$** is defined as

$$\boxed{F_s(s)=\int_{0}^{l} f(x)\cdot\sin\frac{s\pi x}{l}\cdot dx}$$

where s is positive integer ($s=1, 2, 3, \ldots\ldots$)

If by proper choice of origin and scale the interval becomes $(0, \pi)$, then **the finite Fourier sine transform of $f(x)$ on $(0, \pi)$ is defined by**

$$\boxed{F_s(s) = \int_0^{\pi} f(x) \cdot \sin sx \cdot dx}$$

where s is an integer.

Note ....

If for finite Fourier sine transform the interval is not given, we assume it to be $(0, \pi)$.

(ii) Inverse Finite Fourier Sine Transform

If $F_s(s)$ is the finite Fourier sine transform of $f(x)$ over the interval $(0, l)$ then the inverse of $F_s(s)$ *i.e.* $f(x)$ is given by

$$\boxed{f(x) = \frac{2}{l} \sum_{s=1}^{\infty} F_s(s) \sin \frac{s\pi x}{l}}$$

If the interval is $(0, \pi)$, then

$$\boxed{f(x) = \frac{2}{\pi} \cdot \sum_{s=1}^{\infty} F_s(s) \sin sx}$$

These formulae are called **Inverse Finite Fourier Sine transforms** of $F_s(s)$.

These formulae can be proved by using Fourier Series.

Ex. 1 : Find the finite Fourier sine transform of $\sin kx$, if k is not an integer and $0 < x < \pi$.

Sol. : By definition

$$F_s(s) = \int_0^{\pi} f(x) \sin sx \, dx$$

$$= \int_0^{\pi} \sin kx \cdot \sin sx \cdot dx$$

$$= -\frac{1}{2} \int_0^{\pi} [\cos(k+s)x - \cos(k-s)x] \, dx$$

$$= -\frac{1}{2} \left[\frac{\sin(k+s)x}{k+s} - \frac{\sin(k-s)x}{k-s} \right]_0^{\pi}$$

$$= -\frac{1}{2} \cdot \left[\frac{\sin(k+s)\pi}{k+s} - \frac{\sin(k-s)x}{k-s} \right]$$

Ex. 2 : Find the finite Fourier sine transform of $\sin kx$, if k is an integer and $0 < x < \pi$.

Sol. : By definition

$$F_s(s) = \int_0^{\pi} f(x) \sin sx \, dx = \int_0^{\pi} \sin kx \cdot \sin sx \, dx.$$

Then as above

$$= -\frac{1}{2}\cdot\left[\frac{\sin(k+s)\pi}{k+s} - \frac{\sin(k-s)\pi}{k-s}\right]$$

Since s is a positive integer and k is given to be an integer, $k+s$ and $k-s$ are both integers. If $k \neq s$ and since $\sin m\pi = 0$ whenever m is an integer,

$\therefore \; F_s(s) = 0$

If $k = s$, then we have

$$F_s(s) = \int_0^{\pi} \sin^2 sx\, dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2sx)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2sx}{2s}\right]_0^{\pi}$$

$$= \frac{\pi}{2} \quad [\because \; s \text{ is an integer}]$$

Ex. 3 : Find the finite Fourier sine transform of $f(x) = 2x,\ 0 < x < 4$.

Sol. : By definition,

$$F_s(s) = \int_0^{l} f(x)\cdot\frac{\sin \pi x}{l}\cdot dx$$

$$= \int_0^{4} 2x\cdot\sin\frac{s\pi x}{4}\cdot dx \quad [\because l = 4]$$

$$= \left[2x\left(-\cos\left(\frac{s\pi x}{4}\right)\right)\cdot\frac{4}{s\pi}\right]_0^4 - \int_0^4 -\cos\left(\frac{s\pi x}{4}\right)\cdot\frac{4}{s\pi}\cdot 2\cdot dx$$

$$= \left[-\frac{8x}{s\pi}\cdot\cos\left(\frac{s\pi x}{4}\right)\right]_0^4 + \left[\frac{8}{s\pi}\cdot\sin\left(\frac{s\pi x}{4}\right)\cdot\frac{4}{s\pi}\right]_0^4$$

$$= -\frac{32}{s\pi}\cos s\pi \qquad [\because \; s \text{ is an integer}]$$

12. Finite Fourier Cosine Transform and Its Inverse

(i) If $f(x)$ is sectionally continuous function of x over a finite interval $(0, l)$ then the **finite Fourier cosine transform of $f(x)$ on $(0, l)$** is defined as

$$\boxed{F_c(s) = \int_0^{l} f(x)\cdot\cos\frac{s\pi x}{l}\cdot dx}$$

where, s is a positive integer or zero ($s = 0, 1, 2, 3, \ldots$)

If as before by proper choice of origin and scale the interval becomes $(0, \pi)$ then the **finite Fourier cosine transform of $f(x)$ on $(0, \pi)$** is defined as

$$\boxed{F_c(s) = \int_0^{\pi} f(x)\cdot\cos sx\cdot dx}$$

where s is an integer.

> **Note ....**
>
> If for finite Fourier cosine transform interval is not given we assume it to be $(0, \pi)$.

(ii) **Inverse Finite Cosine Transform**

If $F_c(s)$ is the finite Fourier cosine transform of $f(x)$ over the interval $(0, l)$ then the inverse of $F_c(s)$ i.e. $f(x)$ is given by

$$\boxed{f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} F_c(s) \cos \frac{s\pi x}{l}} \quad \text{where } F_c(0) = \int_0^l f(x)\, dx$$

If the interval is $(0, \pi)$, then

$$\boxed{f(x) = \frac{1}{\pi} F_c(0) + \frac{2}{\pi} \sum_{s=1}^{\infty} F_c(s) \cos sx} \quad \text{where } F_c(0) = \int_0^{\pi} f(x)\, dx$$

These formulae are called **inverse finite Fourier transforms** of $F_s(s)$.

These formulae can be proved by using Fourier Series.

Ex. 1 : Find the finite Fourier cosine transform for $\sin kx$, $0 < x < \pi$.

Sol. : By definition,

$$F_c(s) = \int_0^{\pi} f(x) \cos sx\, dx = \int_0^{\pi} \sin kx \cdot \cos sx\, dx$$

$$= \frac{1}{2} \int_0^{\pi} [\sin(k+s)x + \sin(k-s)x]\, dx$$

$$= \frac{1}{2} \left[-\frac{\cos(k+s)x}{k+s} - \frac{\cos(k-s)x}{k-s} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[-\frac{\cos(k+s)\pi}{k+s} - \frac{\cos(k-s)\pi}{k-s} + \frac{1}{k+s} + \frac{1}{k-s} \right]$$

(If k is not an integer)

If k is an integer, we see that since s is an integer.

Case (i) : if $k - s$ is even, then $k + s$ is also even,

$$\therefore \quad F_c(s) = \frac{1}{2} \left[-\frac{1}{k+s} - \frac{1}{k+s} + \frac{1}{k+s} + \frac{1}{k+s} \right] = 0$$

Case (ii) : if $k - s$ is odd, then $k + s$ is also odd,

$$\therefore \quad F_c(s) = \frac{1}{2} \left[\frac{1}{k+s} + \frac{1}{k-s} + \frac{1}{k+s} + \frac{1}{k-s} \right] = \frac{2k}{k^2 - s^2}.$$

Ex. 2 : Find the finite Fourier cosine transform for

$$f(x) = 2x, \qquad 0 < x < 4.$$

Sol. : By definition,

$$F_c(s) = \int_0^l f(x)\cdot \cos\frac{s\pi x}{l}\,dx$$

$$= \int_0^4 2x\cdot\cos\frac{s\pi x}{4}\cdot dx \qquad [\because l = 4]$$

$$= \left[2x\sin\left(\frac{s\pi x}{4}\right)\cdot\frac{4}{s\pi}\right]_0^4 - \int_0^4 \sin\left(\frac{s\pi x}{4}\right)\cdot\frac{4}{s\pi}\cdot 2\,dx$$

$$= \left[\frac{8x}{s\pi}\cdot\sin\left(\frac{s\pi x}{4}\right)\right]_0^4 + \frac{8}{s\pi}\left[\cos\left(\frac{s\pi x}{4}\right)\cdot\frac{4}{s\pi}\right]_0^4$$

$$= \frac{32}{s^2\pi^2}[\cos s\pi - 1]$$

$$= \frac{32}{s^2\pi^2}\left[(-1)^s - 1\right] \qquad [\because s \text{ is an integer}]$$

If $s = 0$, $\quad F_c(s) = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16.$

EXERCISE - V

1. Find the finite Fourier sine and cosine transform of

   1. $f(x) = 1$ [**Ans. :** (i) $\frac{1-(-1)^s}{s}$, (ii) 0 and π if $s = 0$]

   2. $f(x) = x$ [**Ans. :** (i) $-\pi(-1)^s$, (ii) $\frac{(-1)^s - 1}{s^2}$ and $\frac{\pi^2}{2}$ if $s = 0$]

   3. $f(x) = x^2$ [**Ans. :** (i) $\frac{2}{s^3}(\cos s\pi - 1) - \frac{\pi^2}{s}\cos s\pi$

      (ii) $\frac{2\pi}{s^3}\cos s\pi$ and $\frac{\pi^3}{3}$ if $s = 0$]

   4. $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$

      [**Ans. :** (i) $\frac{\pi}{6s}[(-1)^s + 2] + \frac{1}{\pi s^3}[(-1)^s - 1]$, (ii) $\frac{1}{s^2}$, if $s = 0$]

2. Find the finite Fourier sine transform of

   1. $f(x) = \frac{x}{\pi}$ [**Ans. :** $-\frac{1}{s}(-1)^s$]

   2. $f(x) = 1 - \frac{x}{\pi}$ [**Ans. :** $\frac{1}{s}$]

   3. $f(x) = \frac{x}{4\pi}$ [**Ans. :** $(-1)^{s+1}\cdot\frac{1}{4s}$]

   4. $f(x) = x^3$ [**Ans. :** $(-1)^s\cdot\pi\left(\frac{6}{s^3} - \frac{\pi^2}{s}\right)$]

5. $f(x) = e^{ax}$ [Ans. : $\dfrac{s}{a^2+s^2}[1-(-1)^s]e^{a\pi}$]

6. $f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & (\pi/2) \le x < \pi \end{cases}$ [Ans. : $\dfrac{2}{s^2}\sin\dfrac{s\pi}{2}$]

3. Find the finite Fourier cosine transform of

1. $f(x) = 1 - \dfrac{x}{\pi}$ [Ans. : $\dfrac{1}{\pi s^2}[1-(-1)^s]$ and $\dfrac{\pi}{2}$, if $s = 0$]

2. $f(x) = \dfrac{x}{4\pi}$ [Ans. : $\dfrac{1}{4\pi s^2}[(-1)^s - 1]$ and $\dfrac{\pi}{8}$, if $s = 0$]

3. $f(x) = \left(1 - \dfrac{x}{\pi}\right)^2$ [Ans. : $\dfrac{2}{\pi s^2}$ and $\dfrac{\pi}{3}$ if $s = 0$]

4. $f(x) = \begin{cases} 1, & 0 < x < \dfrac{\pi}{2} \\ -1, & \dfrac{\pi}{2} < x < \pi \end{cases}$ [Ans. : $\dfrac{2}{s}\sin\dfrac{s\pi}{2}$, π if $s = 0$]

5. $f(x) = 3x^2$ [Ans. : $6\pi\dfrac{(-1)^s}{s^2}$, π^3 if $s = 0$]

6. $f(x) = \dfrac{x^2}{2\pi} - \dfrac{\pi}{6}$ [Ans. : $\dfrac{1}{s^2}(-1)^s$, 0 if $s = 0$]

7. $f(x) = \cos kx$ if k is not an integer and $0 < x < \pi$.

[Ans. : $\dfrac{1}{2}\left[\dfrac{\sin(k+s)\pi}{k+s} + \dfrac{\sin(k-s)\pi}{k-s}\right]$

4. 1. Find the finite Fourier sine transform of $f(x) = 2x^2$, $0 < x < c$.

[Ans. : $2\left[-\dfrac{c^3}{s\pi}\cos s\pi + \dfrac{2c^3}{s^3\pi^3}\cos s\pi - \dfrac{2c^3}{s^3\pi^3}\right]$

2. Find the finite Fourier sine transform of $f(x) = 2x$, $0 < x < 4$.

[Ans. : $-\dfrac{32}{s\pi}\cos s\pi$]

Examples On Inverse Fourier Finite Sine And Cosine Transform

We have already noted the formulae for inverse Fourier finite sine and cosine transforms. We shall see below how to apply them to find $f(x)$ when $f_s(s)$ or $f_c(s)$ is given.

Ex. 1 : Find $f(x)$ if its Fourier finite co... sfor... is given by

$$F_c(s) = \frac{\left(\sin\frac{s\pi}{2} - \cos s\pi\right)}{(2 \ldots}$$

$$= \frac{2}{\ldots}$$

Sol. : By definition, we have

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} f_c(s) \cos \frac{s\pi x}{l}$$

Putting $l = 6$ in the above formula,

$$f(x) = \frac{1}{6} \cdot \frac{2}{\pi} + \frac{2}{6} \sum_{s=1}^{\infty} \frac{\left(\sin \frac{s\pi}{2} - \cos s\pi\right)}{(2s+1)\pi} \cos \frac{s\pi x}{6}$$

$$= \frac{1}{3\pi} + \frac{1}{3} \sum_{s=1}^{\infty} \frac{\left(\sin \frac{s\pi}{2} - \cos s\pi\right)}{(2s+1)\pi} \cos \frac{s\pi x}{6}.$$

Ex. 2 : Find $f(x)$ if its Fourier finite sine transform is given by

$$F_s(s) = \frac{2\pi(-1)^{s-1}}{s^2}, \quad s = 1, 2, 3, \ldots \text{ where } 0 < x < \pi.$$

Sol. : By definition, we have,

$$F_s(s) = \frac{2}{\pi} \sum_{s=1}^{\infty} F_s(s) \sin sx$$

$$\therefore \quad F_s(s) = \frac{2}{\pi} \sum_{s=1}^{\infty} \frac{2\pi(-1)^{s-1}}{s^2} \sin sx$$

EXERCISE - VI

1. Find $f(x)$ if its Fourier finite sine transform is

$$F_s(s) = \frac{1 - \cos s\pi}{s^2 \pi^2}, \quad 0 < x < \pi$$

[**Ans. :** $f(x) = \frac{2}{\pi} \sum_{s=1}^{\infty} \left(\frac{1 - \cos s\pi}{s^2 \pi^2}\right) \sin sx.$]

2. Find $f(x)$ if its Fourier finite cosine transform is

$$F_c(s) = \frac{\sin(s\pi/2)}{2s}, \quad s = 1, 2, 3, \ldots$$

$$= \frac{\pi}{4}, \quad s = 0 \quad \text{where } 0 < x < 2\pi.$$

[**Ans. :** $\frac{1}{8} + \frac{1}{\pi} \sum \frac{\sin(s\pi/2)}{2s} \cos \frac{sx}{2}$]

EXERCISE - VII

Theory

1. Define the following terms :
   1. Fourier Integral
   2. Fourier Cosine Integral
   3. Fourier Sine Integral
   4. Fourier Transform