CHAPTER

# 5 Fourier Transforms

## 1. Introduction

In this chapter we shall first state the Fourier Integral Theorem and then consider Fourier Transform. Fourier transforms transform a non-periodic function $f(t)$ in time-domain into a function $F(\lambda)$ in frequency domain. Fourier transform are highly useful in the study of conduction of heat, wave propagation, communication, etc.

## 2. Fourier Integral Theorem

If $f(x)$ satisfies Dirichlet's conditions (stated under Fourier Series) in each finite interval $-l \le x \le l$ and if $f(x)$ is integrable in $-\infty$ to $\infty$ then Fourier Integral Theorem states that

$$\boxed{f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos\omega(s-x)\,d\omega\,ds} \quad \text{..............} (1)$$

We assume this result without proof.

## 3. Fourier Sine and Cosine Integrals

The above integral can be written as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\{\cos\omega s\cos\omega x\}\,d\omega\,ds + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\{\sin\omega s\sin\omega x\}\,d\omega\,ds$$

*i.e.* $$f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\omega x\int_{-\infty}^{\infty} f(s)\cos\omega s\,d\omega\,ds + \frac{1}{\pi}\int_0^{\infty}\sin\omega x\int_{-\infty}^{\infty} f(s)\sin\omega s\,d\omega\,ds$$

### (a) Fourier Cosine Integral

**When $f(x)$ is an even function,** $f(s)$ will be even but $f(s)\sin\omega s$ will be **odd function** and $f(s)\cos\omega s$ will be **even function. Hence, the second integral will be zero and we will get**

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\omega x\,d\omega\int_0^{\infty} f(s)\cos\omega s\,ds} \quad \text{..............} (2)$$

This is called **Fourier Cosine Integral.**

### (b) Fourier Sine Integral

**When $f(x)$ is an odd function,** $f(s)$ will be odd but $f(s)\sin\omega s$ will be **even** and $f(s)\cos\omega s$ will be odd. Hence, the first integral will be zero and we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\omega x\,d\omega\int_0^{\infty} f(s)\sin\omega s\,ds} \quad \text{..............} (3)$$

This is called **Fourier Sine Integral.**

**Note ....**

If the given function $f(x)$ is even then we can use (2) and obtain the Fourier Cosine Integral representation of $f(x)$ (See Ex. 6, page 5-9). If the given function $f(x)$ is odd then we can use (3) and obtain the Fourier Sine Integral representation of $f(x)$ (See Ex. 1, page 5-5).

## Type I : Fourier Integral Representation

**Example 1 :** Express the function $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as Fourier Integral.

1 **(M.U. 1997, 99, 2002, 03)**

cos

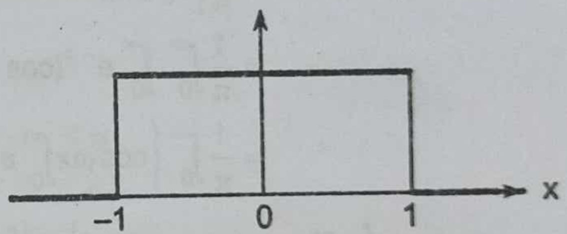
Hence, evaluate $\int_0^{\infty} \frac{\sin\omega \cdot \sin\omega x}{\omega} d\omega$. **(M.U. 1994, 95, 2003, 11, 12)**

**Sol. :** The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\cos\omega(s-x)\,d\omega\,ds$$

[ By data $f(s) = 0$ from $-\infty$ to $-1$, $f(s) = 1$ from $-1$ to 1 and $f(s) = 0$ from 1 to $\infty$. ]

The graph of $f(x)$ is shown in adjoining figure.




Hence, $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} 1 \cdot \cos\omega(s-x)\,d\omega\,ds$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\omega(s-x)}{\omega}\right]_{-1}^{1} d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x) - \sin\omega(-1-x)}{\omega}\,d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x) + \sin\omega(1+x)}{\omega}\,d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\,d\omega$$

This is the Fourier Integral representation of $f(x)$ given in the example.

$$\therefore \int_0^{\infty}\frac{\sin\omega \cdot \cos\omega x}{\omega}\,d\omega = \frac{\pi}{2}\cdot f(x)$$

$$= \begin{cases} \frac{\pi}{2} \text{ for } f(x) = 1 \text{ when } |x| < 1 \\ 0 \text{ for } f(x) = 0 \text{ when } |x| > 1 \end{cases} \qquad \text{.............. (1)}$$

At $|x| = 1$ *i.e.* $x = \pm 1$, $f(x)$ is discontinuous and the integral

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left\{\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right\} = \frac{\pi}{4}[1+0] = \frac{\pi}{4}. \qquad \text{.............. (2)}$$

From (1) and (2), we get

$$\int_0^{\infty}\frac{\sin\omega\cdot\cos\omega x}{\omega}\,d\omega = \begin{cases} \pi/2 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1 \\ \pi/4 & \text{when } |x| = 1 \end{cases}$$

**Cor. :** Putting $x = 0$ in the above result, we get

$$\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}.$$

**Note ....**

Unfortunately there is no uniformity in the notation of Fourier Integral and Fourier transforms. Some authors use $\lambda$ or $\alpha$ in place of $\omega$ and $t$ in place of $s$.

**Example 2 :** Find the Fourier Integral representation of

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

**(M.U. 2010, 11)**

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**Sol. :** The Fourier Integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(s) \cos \omega (s-x)\, d\omega\, ds\right]$$

$$= \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^0 0\, d\omega\, ds + \int_0^\infty \int_0^\infty e^{-s} \cos \omega (s-x)\, d\omega\, ds\right]$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-s} (\cos \omega s \cos \omega x + \sin \omega s \sin \omega x)\, d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\cos \omega x \int_0^\infty e^{-s} \cos \omega s\, ds + \sin \omega x \int_0^\infty e^{-s} \sin \omega s\, ds\right\} d\omega$$

But $\displaystyle\int e^{ax} \cos bx\, dx = \frac{1}{a^2+b^2}\left[e^{ax}(a \cos bx + b \sin bx)\right]$

and $\displaystyle\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2}\left[e^{ax}(a \sin bx - b \cos bx)\right]$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \left\{\cos \omega x \left[\frac{1}{1+\omega^2} e^{-s}(-\cos \omega s + \omega \sin \omega s)\right]_0^\infty\right.$$

$$\left. + \sin \omega x \left[\frac{1}{1+\omega^2} e^{-s}(-\sin \omega s - \omega \cos \omega s)\right]_0^\infty\right\} d\omega$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{\cos \omega x}{1+\omega^2} + \frac{\omega \sin \omega x}{1+\omega^2}\right) d\omega = \frac{1}{\pi}\int_0^\infty \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2}\, d\omega$$

And when $x = 0$,

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\, d\omega = \frac{1}{\pi}\left[\tan^{-1} \omega\right]_0^\infty = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}.$$

Hence, the Fourier Integral representation of $f(x)$ is

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2}\, d\omega, & x > 0 \end{cases}$$

**Example 3 :** Express the function $f(x)=\begin{cases}-e^{kx} & \text{for } x<0\\ e^{-kx} & \text{for } x>0\end{cases}$

as Fourier Integral and hence, prove that $\int_0^\infty \frac{\omega \sin \omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}e^{-kx}$ if $x>0,\ k>0.$ **(M.U. 2010)**

**Sol. :** The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds$$

But since the given function $f(x)$ is an odd function we use (3) of § 3, page 5-1.

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty e^{-ks}\sin\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\left[\frac{1}{k^2+\omega^2}e^{-ks}(-k\sin\omega s-\omega\cos\omega s)\right]_0^\infty d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{\omega}{k^2+\omega^2}\,d\omega$$

This is the Fourier integral representation of $f(x)$.

$$\therefore\quad \int_0^\infty \frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-kx}\quad \text{if } x>0.$$

**Example 4 :** Express the function $f(x)=\begin{cases}\sin x, & 0<x\le\pi\\ 0, & x<0,\ x>\pi\end{cases}$

as Fourier Integral and prove that $f(x)=\frac{1}{\pi}\int_0^\infty \frac{\sin\omega x+\cos[\omega(\pi-x)]}{1-\omega^2}\,d\omega$. **(M.U. 2001, 06)**

Hence, deduce that $\int_0^\infty \frac{\cos(\omega\pi/2)}{1-\omega^2}\,d\omega=\frac{\pi}{2}$.

**Sol. :** The Fourier integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi \sin s\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\int_0^\pi 2\sin s\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\int_0^\pi[\sin(s+\omega s-\omega x)+\sin(s-\omega s+\omega x)]\,d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\left[-\frac{\cos(s+\omega s-\omega x)}{1+\omega}-\frac{\cos(s-\omega s+\omega x)}{1-\omega}\right]_0^\pi d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[-\frac{\cos(\pi+\pi\omega-\omega x)}{1+\omega}-\frac{\cos(\pi-\pi\omega+\omega x)}{1-\omega}+\frac{\cos\omega x}{1+\omega}+\frac{\cos\omega x}{1-\omega}\right]d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[\frac{\cos(\pi\omega-\omega x)}{1+\omega}+\frac{\cos(-\pi\omega+\omega x)}{1-\omega}+\frac{\cos\omega x}{1+\omega}+\frac{\cos\omega x}{1-\omega}\right]d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[\left(\frac{1}{1+\omega}\right)\{\cos\omega x+\cos\omega(\pi-x)\}+\left(\frac{1}{1-\omega}\right)\{\cos\omega x+\cos\omega(\pi-x)\}\right]d\omega$$

$$\therefore \quad f(x) = \frac{1}{2\pi}\int_0^\infty \left(\frac{1}{1+\omega} + \frac{1}{1-\omega}\right)\{\cos\omega x + \cos\omega(\pi - x)\}\, d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{2}{(1-\omega^2)}\{\cos\omega x + \cos\omega(\pi - x)\}\, d\omega$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\omega x + \cos\omega(\pi - x)}{(1-\omega^2)}\right] d\omega$$

This is the Fourier integral representation of $f(x)$.

Putting $x = \pi/2$, we get,

$$\sin\frac{\pi}{2} = \frac{1}{\pi}\int_0^\infty \frac{\left[\cos\frac{\pi\omega}{2} + \cos\frac{\pi\omega}{2}\right]}{(1-\omega^2)} d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\pi\omega/2}{1-\omega^2} d\omega$$

$$\therefore \quad 1 = \frac{2}{\pi}\int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2} d\omega \qquad \therefore \quad \frac{\pi}{2} = \int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2} d\omega$$

**Example 5 :** Find the Fourier integral representation of $f(x) = e^{-|x|}, \quad -\infty < x < \infty.$

**(M.U. 2010**

**Sol. :** The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds\right]$$

Since $f(x)$ is an even function, we have by (2) of § 2, page 5-1

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\, d\omega\, ds = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s}\cos\omega s\, ds\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{e^{-s}}{1+\omega^2}(-\cos\omega s + \omega\sin\omega s)\right]_0^\infty d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left[0 + \frac{1}{1+\omega^2}\right] d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2} d\omega.$$

## Type II : Fourier Sine Integral Representation

**Example 1 :** Express the function $f(x) = \begin{cases} \sin x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

as Fourier sine integral and evaluate $\displaystyle\int_0^\infty \frac{\sin\omega x \cdot \sin\pi\omega}{1-\omega^2} d\omega.$

**Sol. :** The Fourier Sine Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\pi \sin s \sin\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \left(-\frac{1}{2}\right)\int_0^\pi [\cos s(1+\omega) - \cos s(1-\omega)]\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left(-\frac{1}{2}\right)\left[\frac{\sin s(1+\omega)}{1+\omega} - \frac{\sin s(1-\omega)}{1-\omega}\right]_0^\pi d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \left(-\frac{1}{2}\right)\left[-\frac{2\sin\pi\omega}{1-\omega^2}\right] d\omega$$

[ $\because \sin(\pi+\theta) = -\sin\theta$ and $\sin(\pi-\theta) = \sin\theta$ ]

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x \sin\pi\omega}{1-\omega^2} dx$$

$$\therefore \quad \int_0^\infty \frac{\sin\omega x \cdot \sin\pi\omega}{1-\omega^2} dx = \frac{\pi}{2} f(x) = \frac{\pi}{2}\begin{cases}\sin x, & |x| < \pi \\ 0, & |x| > \pi\end{cases}$$

**Example 2 :** Express the function $f(x) = \begin{cases}\pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi\end{cases}$

as Fourier sine Integral and show that $\int_0^\infty \frac{1-\cos\pi\omega}{\omega} \sin\omega x\, d\omega = \frac{\pi}{2}$ when $0 < x < \pi$. **(M.U. 1998)**

**Sol. :** Fourier sine integral is,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\pi \frac{\pi}{2} \cdot \sin\omega s\, d\omega\, ds = \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \frac{\pi}{2}\left[-\frac{\cos\omega s}{\omega}\right]_0^\pi d\omega$$

$$\therefore \quad f(x) = \int_0^\infty \sin\omega x\left[\frac{-\cos\pi\omega + 1}{\omega}\right] d\omega = \int_0^\infty \frac{1-\cos\pi\omega}{\omega} \cdot \sin\omega x\, d\omega$$

$$\therefore \quad \int_0^\infty \frac{1-\cos\pi\omega}{\omega} \cdot \sin\omega x \cdot d\omega = f(x) = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

**Example 3 :** Find Fourier Sine integral representation for $f(x) = \frac{e^{-ax}}{x}$. **(M.U. 2004, 09, 16)**

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**Sol. :** By (3) Fourier Sine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s) \sin\omega s\, d\omega\, ds = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty \frac{e^{-as}}{s} \sin\omega s\, ds\, d\omega$$

To evaluate $\int_0^\infty \frac{e^{-as}}{s} \sin\omega s\, ds$ we use the rule of differentiation under the integral sign.

Let $\quad I = \int_0^\infty \frac{e^{-as}}{s} \cdot \sin\omega s\, ds$

$$\therefore \quad \frac{dI}{d\omega} = \int_0^\infty \frac{\partial}{\partial\omega}\left(\frac{e^{-as}}{s} \cdot \sin\omega s\right) ds = \int_0^\infty \frac{e^{-as}}{s} \cdot (\cos\omega s) \cdot s\, ds$$

$$= \int_0^\infty e^{-as} \cos\omega s\, ds = \frac{1}{a^2+\omega^2}\left[e^{-as}(-a\cos\omega s + \omega\sin\omega s)\right]_0^\infty$$

$$\therefore \quad \frac{dI}{d\omega} = \frac{1}{a^2+\omega^2}(a) = \frac{a}{a^2+\omega^2}$$

Integrating w.r.t. $\omega$, $\quad I = a \cdot \frac{1}{a}\tan^{-1}\frac{\omega}{a} + C$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s} \cdot \sin\omega s\, ds = \tan^{-1}\frac{\omega}{a} + C$$

To find $C$, we put $\omega = 0$. $\therefore \; 0 = 0 + C \quad \therefore \; C = 0.$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s} \cdot \sin \omega s \, ds = \tan^{-1} \frac{\omega}{a}$$

Hence, $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \sin \omega x \cdot \tan^{-1} \frac{\omega}{a} \, d\omega$.

**Example 4 :** Find Fourier Sine integral of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$. **(M.U. 1999)**

**Sol. :** Fourier Sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \int_0^\infty f(s) \sin \omega s \cdot d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \sin \omega x \left[ \int_0^1 s \sin \omega s \, ds + \int_1^2 (2 - s) \sin \omega s \, ds + \int_2^\infty 0 \cdot \sin \omega s \, ds \right] d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \sin \omega x \left\{ \left[ s \left( -\frac{\cos \omega s}{\omega} \right) - \left( -\frac{\sin \omega s}{\omega^2} \right)(1) \right]_0^1 + \left[ (2 - s) \left( -\frac{\cos \omega s}{\omega} \right) - \left( -\frac{\sin \omega s}{\omega^2} \right)(-1) \right]_1^2 \right\} d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \sin \omega x \left\{ \left[ -\frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2} \right] + \left[ 0 - \frac{\sin 2\omega}{\omega^2} + \frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2} \right] \right\} d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \cdot \frac{(2 \sin \omega - \sin 2\omega)}{\omega^2} \, d\omega$$

## Type III : Fourier Cosine Integral Representation

**Example 1 :** Find the Fourier Cosine integral representation of the function

$$f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$$

**Sol. :** Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^a s^2 \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left[ s^2 \left( \frac{\sin \omega s}{\omega} \right) - \left( -\frac{\cos \omega s}{\omega^2} \right)(2s) + \left( -\frac{\sin \omega s}{\omega^n} \right)(2) \right]_0^a d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left[ \frac{a^2 \sin a\omega}{\omega} + \frac{2a \cos a\omega}{\omega^2} - \frac{2 \sin a\omega}{\omega^3} \right] d\omega$$

**Example 2 :** Find the Fourier cosine integral representation of the function $f(x) = e^{-ax}, x > 0$ and hence, show that

$$\int_0^\infty \frac{\cos \omega s}{1 + \omega^2} \, d\omega = \frac{\pi}{2} e^{-x}, \quad x \geq 0.$$

**Sol. :** Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-ax}\cos\omega x \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + s\sin\omega s)\right]_0^\infty d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{a}{a^2+\omega^2}\right] d\omega = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2} d\omega$$

For deduction put $a = 1$,

$$\therefore \; e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2} d\omega \qquad \therefore \; \int_0^\infty \frac{\cos\omega x}{1+\omega^2} d\omega = \frac{\pi}{2}e^{-x}.$$

**Example 3 :** Find Fourier cosine integral representation of the function

$$f(x) = \begin{cases} \cos x, & |x| < (\pi/2) \\ 0, & |x| > (\pi/2) \end{cases}$$

**Sol. :** Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^{\pi/2} \cos s \cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\int_0^{\pi/2}[\cos(1+\omega)s + \cos(1-\omega)s] \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\left[\frac{\sin(1+\omega)s}{1+\omega} + \frac{\sin(1-\omega)s}{1-\omega}\right]_0^{\pi/2} ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\left[\frac{\sin\pi(1+\omega)/2}{1+\omega} + \frac{\sin\pi(1-\omega)/2}{1-\omega}\right] d\omega$$

But $\sin\left(\frac{\pi}{2}+\frac{\pi\omega}{2}\right) = \cos\frac{\pi\omega}{2}$ and $\sin\left(\frac{\pi}{2}-\frac{\pi\omega}{2}\right) = \cos\frac{\pi\omega}{2}$.

$$\therefore \; f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \cdot \left(\frac{1}{2}\right)\cdot\left[\frac{\cos(\pi\omega/2)}{1+\omega} + \frac{\cos(\pi\omega/2)}{1-\omega}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\cdot\frac{2\cdot\cos(\pi\omega/2)}{1-\omega^2} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\cdot\cos(\pi\omega/2)}{1-\omega^2} d\omega.$$

**Example 4 :** Using Fourier Cosine Integral prove that

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{(\omega^4+4)}\cdot\cos\omega x \, d\omega$$ **(M.U. 2002, 05, 07)**

**Sol. :** By Fourier cosine integral formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s \, d\omega \, ds$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s} \cos s \cdot \cos \omega s \, d\omega \, ds$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s}\left[\cos(\omega+1)s + \cos(\omega-1)s\right] d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x \left[\frac{1}{1+(\omega+1)^2}\cdot e^{-s}\{-\cos(\omega+1)s + (\omega+1)\sin(\omega+1)s\}_0^\infty \right.$$

$$\left. + \frac{1}{1+(\omega-1)^2}\cdot e^{-s}\{-\cos(\omega-1)s + (\omega-1)\sin(\omega-1)s\}_0^\infty\right]$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x \left[\frac{1}{1+(\omega+1)^2} + \frac{1}{1+(\omega-1)^2}\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x \left[\frac{\omega^2 - 2\omega + 2 + \omega^2 + 2\omega + 2}{\{(\omega^2+2)+2\omega\}\{(\omega^2+2)+2\omega\}}\right] d\omega$$

$$\therefore \quad f(x) = \frac{1}{\pi}\int_0^\infty \cos\omega x \cdot \frac{2(\omega^2+2)}{(\omega^2+2)^2 - 4\omega^2}\cdot d\omega = \frac{1}{\pi}\int_0^\infty \cos\omega x \cdot \frac{(\omega^2+2)}{(\omega^4+4)}\, d\omega.$$

**Example 5 :** Find Fourier cosine integral for $f(x) = \begin{cases} 1-x^2, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$.

Hence, evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\cdot dx$. **(M.U. 2003)**

**Sol. :** Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty f(t)\cos\omega t\, dt\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty (1-t^2)\cos\omega t\, dt\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\left[(1-t^2)\frac{\sin\omega t}{\omega} - (-2t)\left(-\frac{\cos\omega t}{\omega^2}\right) + (-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right]_0^1\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{-2\cdot\frac{\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right\} d\omega$$

$$1 - x^2 = \frac{4}{\pi}\int_0^\infty \cos\omega x\left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right) d\omega$$

Now put $x = \frac{1}{2}$, $\therefore \quad \frac{3\pi}{16} = \int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\cdot d\omega$

**Example 6 :** Find Fourier Integral representation for

$$f(x) = \begin{cases} 1-x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

**(M.U. 1998, 2003, 08, 12)**

**Sol. :** By data $f(s) = 0$ from $-\infty$ to $-1$, $f(s) = 1 - s^2$ from $-1$ to $1$ and $f(s) = 0$ from $1$ to $\infty$.

Also $f(-s) = 1-(-s)^2 = 1-s^2$

$= f(s)$ from $-1$ to $1$

Hence, $f(s)$ is an even function and we use (2) and obtain Fourier cosine integral representation of $f(x)$.

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s)\cos \omega s \, d\omega \, ds = \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\int_0^1 (1-s^2)\cos \omega s \, ds\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[(1-s^2)\left(\frac{\sin \omega s}{\omega}\right) - \left(-\frac{\cos \omega s}{\omega^2}\right)(-2s) + \left(-\frac{\sin \omega s}{\omega^3}\right)(-2)\right]_0^1 d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[0 - \frac{2\cos \omega}{\omega^2} + \frac{2\sin \omega}{\omega^3}\right] d\omega$$

$$\therefore \quad f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin \omega - \omega \cos \omega}{\omega^3} \cdot \cos \omega x \, d\omega .$$

**Example 7 :** Find Fourier integral representation of

$$f(x) = \begin{cases} e^{ax} & x \le 0,\ a > 0 \\ e^{-ax} & x \ge 0,\ a > 0 \end{cases}$$ **(M.U. 1996, 97, 2002, 09)**

Hence, show that $\int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}, \ x > 0,\ a > 0.$

**Sol. :** Since, $f(x)$ is an even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s)\cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty e^{-as}\cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{1}{a^2+\omega^2} \cdot e^{-as}(-a\cos \omega s + \omega \sin \omega s)\right]_0^\infty d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \cdot \frac{a}{a^2+\omega^2} d\omega$$

$$\therefore \quad \int_0^\infty \frac{\cos \omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax}, \ x > 0,\ a > 0$$

**Example 8 :** Find Fourier Integral representation of

$$f(x) = x, \quad 0 < x < a$$
$$= 0, \quad x > a$$
$$f(-x) = f(x)$$ **(M.U. 1995)**

**Sol. :** Since, $f(x)$ is even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s)\cos \omega s \, d\omega \, ds = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^a s\cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} - \int \frac{\sin \omega s}{\omega}(1) \cdot ds\right]_0^a d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} + \frac{\cos \omega s}{\omega^2}\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left[\frac{a \sin a\omega}{\omega} + \frac{\cos a\omega}{\omega^2} - \frac{1}{\omega^2}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left(\frac{a\omega \sin a\omega + \cos a\omega - 1}{\omega^2}\right) d\omega$$

**Example 9 :** Find the Fourier cosine and sine integrals of the following function.

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$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$$ **(M.U. 2011)**

**Sol. :** (i) Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos \omega x \int_0^{\infty} f(s) \cos \omega s \; d\omega \; ds$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left[\int_0^1 s \cos \omega s \; ds + \int_1^2 (2 - s) \cos \omega s \; ds\right] d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left\{\left[s \cdot \frac{\sin \omega s}{\omega} + \frac{\cos \omega s}{\omega^2} \cdot 1\right]_0^1 \right.$$

$$\left. + \left[(2 - s) \cdot \frac{\sin \omega s}{\omega} + \frac{\cos \omega s}{\omega^2} \cdot (-1)\right]_1^2 \right\} d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left\{\left[\frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2} - \frac{1}{\omega^2}\right] + \left[-\frac{\cos 2\omega}{\omega^2} - \frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2}\right]\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left[\frac{2 \cos \omega}{\omega^2} - \frac{(1 + \cos 2\omega)}{\omega^2}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \left[\frac{2 \cos \omega - 2 \cos^2 \omega}{\omega^2}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos \omega x \cdot \frac{2 \cos \omega (1 - \cos \omega)}{\omega^2} d\omega$$

$$= \frac{4}{\pi}\int_0^{\infty} \cos \omega x \cdot \frac{\cos \omega (1 - \cos \omega)}{\omega^2} d\omega$$

**(ii)** For Fourier sine integral see Ex. 4, page 5-7.

## EXERCISE - I

**(A)** 1. Express $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$ as a Fourier integral and show that

$$\int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1 + \omega^2} d\omega = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$

(**Hint :** For the second part put $x = 0$ in the integral, then

$$f(0) = \int_0^\infty \frac{1}{1+\omega^2}\, d\omega = \frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty = \frac{1}{2}.\ )$$

2. Express $f(x) = \begin{cases} \cos x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$ as a Fourier integral and show that

$$\int_0^\infty \frac{\omega \sin \pi\omega \cos \omega x}{1-\omega^2}\, d\omega = \begin{cases} \frac{\pi}{2}\cos x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$$

(B) 1. Express the function $f(x) = e^{-ax} - e^{-bx},\ x \geq 0;\ a, b > 0$ as Fourier sine Integral and evaluate

$$\int_0^\infty \frac{\omega \sin \omega x}{(1+\omega^2)(4+\omega^2)}\, d\omega. \qquad \left[\textbf{Ans. :}\ \frac{\pi}{6}(e^{-x} - e^{-2x})\right]$$

2. Express the function $f(x) = e^{-x}$ as Fourier sine integral $(x \geq 0)$ and show that

$$\int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} \cdot e^{-x}.$$ **(M.U. 2006)**

3. Express $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$ as Fourier sine integral and show that

$$\int_0^\infty \frac{\omega^3 \sin \omega x}{\omega^4 + 4}\, d\omega = \frac{\pi}{2} e^{-x} \cos x.$$ **(M.U. 2002)**

(C) 1. Express the function $f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x > 1 \end{cases}$

as a Fourier cosine Integral and hence, show that

$$\int_0^\infty \frac{\sin \omega \cdot \cos \omega x}{\omega}\, d\omega = \frac{\pi}{2} \text{ if } 0 \leq x < 1.$$

(**Hint :** $f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^1 1 \cdot \cos \omega s\, d\omega\, ds = \frac{2}{\pi}\int_0^\infty \frac{\cos \omega x \cdot \sin \omega}{\omega}\, d\omega$)

2. Express the function $f(x) = e^{-x}\ (x \geq 0)$ as a Fourier cosine integral and show that

$$\int_0^\infty \frac{\cos \omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x}.$$

3. Express $f(x) = e^{-kx} (k > 0)$ as Fourier sine and cosine Integral and show respectively that

(i) $\int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2} e^{-kx}$ (ii) $\int_0^\infty \frac{\cos \omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2k} e^{-kx}$ **(M.U. 2003)**

## 4. Fourier Transform or Complex Fourier Transform

**Definition :** If a function $f(x)$ is defined on $(-\infty, \infty)$, is piecewise continuous in each finite interval and is absolutely integrable in $(-\infty, \infty)$ then the integral $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$ is called the Fourier Transform of $f(x)$ and is denoted by $F\{f(x)\}$ or $F(s)$. Thus,

$$\boxed{F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx}$$

**Example 1 :** Find the Fourier Transform of $f(x)$, if $f(x)=\begin{cases} e^{i\omega x}, & a<x<b \\ 0, & x<a,\ x>b \end{cases}$.

**Sol. :** By definition,

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\cdot e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\omega+s)x}\,dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(\omega+s)x}}{i(\omega+s)}\right]_a^b$$

$$=\frac{1}{\sqrt{2\pi}\,i(\omega+s)}\left(e^{i(\omega+s)b}-e^{i(\omega+s)a}\right).$$

**Example 2 :** Find the Fourier Transform of $f(x)=e^{-x^2/2}$.

**Sol. :** By definition

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\cdot e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(x-is)^2}\cdot e^{-s^2/2}\,dx=\frac{e^{-s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(x-is)^2}\,dx$$

Now, put $\frac{1}{\sqrt{2}}(x-is)=y \quad \therefore\ dx=\sqrt{2}\cdot dy$

$$\therefore\quad F(s)=\frac{e^{-s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}\cdot\sqrt{2}\cdot dy$$

$$\therefore\quad F(s)=\frac{e^{-s^2/2}}{\sqrt{\pi}}\cdot\sqrt{\pi}=e^{-s^2/2}. \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2}\,dy=\sqrt{\pi}\right]$$

**Example 3 :** Find the Fourier Transform of $f(x)=\begin{cases} 1/2\epsilon, & |x|\le\epsilon \\ 0, & |x|>\epsilon \end{cases}$

**Sol. :** By definition

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\epsilon}^{\epsilon}\frac{1}{2\epsilon}\cdot e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}\cdot 2\epsilon}\cdot\left[\frac{e^{isx}}{is}\right]_{-\epsilon}^{\epsilon}=\frac{1}{\sqrt{2\pi}\cdot s\epsilon}\left[\frac{e^{is\epsilon}-e^{-is\epsilon}}{2}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{\sin s\epsilon}{s\epsilon}.$$

**Example 4 :** Find the Fourier Transform of $f(x)=e^{-|x|}$.

**Sol. :** By definition

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot\cos sx\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot\sin sx\,dx$$

Since the first integral is even and the second is odd (and hence zero.)

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-|x|} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx \quad [\because |x| = x \text{ when } x > 0]$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[ 0 + \frac{1}{1+s^2} \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2}.$$

**Example 5 :** Find the Fourier Transform of $f(x) = \begin{cases} k, & |x| < a \\ 0, & |x| > a \end{cases}$

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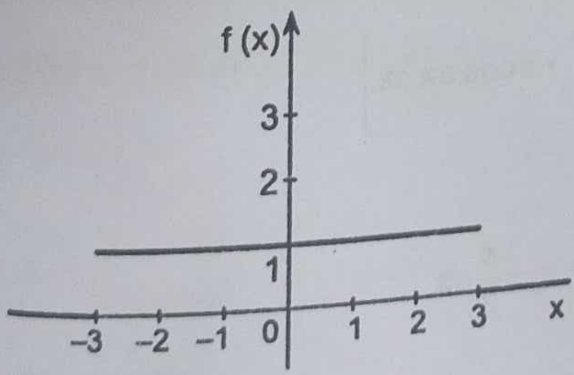
**Sol. :** By definition

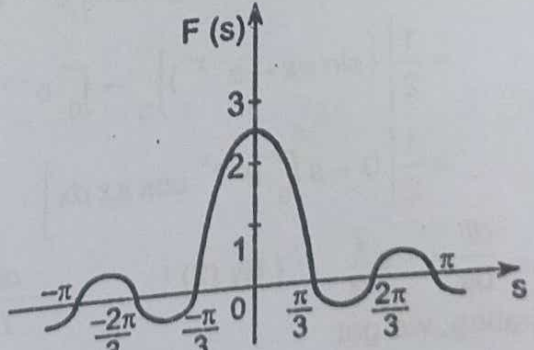
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} k \cdot e^{isx} \, dx$$

$$= \frac{k}{\sqrt{2\pi}} \left[ \frac{e^{isx}}{is} \right]_{-a}^{a} = \frac{k}{\sqrt{2\pi} \cdot s} \left[ \frac{e^{isa} - e^{-isa}}{2i} \right] \times 2 \sqrt{2}$$

$$= \frac{k \cdot 2}{\sqrt{2\pi} \cdot s} \left[ \frac{e^{isa} - e^{-isa}}{2i} \right] = \frac{k}{s} \cdot \sqrt{\frac{2}{\pi}} \cdot \sin sa.$$

We show below the graph of $f(x)$ and $F(s)$ at $k = 1$ and $a = 3$.




**Example 6 :** Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

**Sol. :** By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)[\cos sx + i \sin sx] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx + \frac{i}{\sqrt{2}} \int_{-1}^{1} (1-|x|) \sin sx \, dx$$

But the first integral is even and the second is odd hence zero.

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_0^1 (1-|x|) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ (1-x) \frac{\sin sx}{s} - \left( -\frac{\cos sx}{s^2} \right)(-1) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[ \left( -\frac{\cos s}{s^2} \right) + \frac{1}{s^2} \right] = \sqrt{\frac{2}{\pi}} \cdot \left( \frac{1-\cos s}{s^2} \right)$$

**Example 7 :** Find the Fourier transform of $f(x) = e^{-x^2}$.

**Sol. :** By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} (\cos sx + i \sin sx) \, dx$$

Now, the first integral is even and the second is odd and hence zero.

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2} \cos sx \, ds \qquad \text{.............. (1)}$$

Let $\quad I = \int_0^{\infty} e^{-x^2} \cos sx \, dx \qquad$ .............. (2)

Differentiating under the integral sign with respect to $s$,

$$\frac{dI}{ds} = \int_0^{\infty} e^{-x^2} (-x) \sin sx \, dx = \frac{1}{2} \int_0^{\infty} \{ e^{-x^2} (-2x) \} \cdot \sin sx \, dx$$

Integrating by parts and noting that $\int e^{-x^2} (-2x) \, dx = e^{-x^2}$

$$= \frac{1}{2} \left[ \left\{ \sin sx \cdot (e^{-x^2}) \right\}_0^{\infty} - \int_0^{\infty} e^{-x^2} \cdot s \cos sx \, dx \right]$$

$$= \frac{1}{2} \left[ 0 - s \int_0^{\infty} e^{-x^2} \cos sx \, dx \right]$$

$$\therefore \quad \frac{dI}{ds} = -s \frac{I}{2} \qquad [\text{ By (2) }] \qquad \therefore \quad \frac{dI}{I} = -\frac{s}{2} ds$$

Integrating, we get

$$\log I = -\frac{s^2}{4} + \log c \qquad \therefore \quad \log \frac{I}{c} = -\frac{s^2}{4} \qquad \therefore \quad \frac{I}{c} = e^{-s^2/4}$$

Putting $s = 0$, $I(0) = c$.

But from (2), $\quad I(0) = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \qquad$ [ By Gamma function ]

$$\therefore \quad c = \frac{\sqrt{\pi}}{2} \qquad \therefore \quad I = \frac{\sqrt{\pi}}{2} \cdot e^{-s^2/4}$$

Hence, from (1),

$$F(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-s^2/4} = \frac{1}{\sqrt{2}} \cdot e^{-s^2/4} = \frac{e^{-s^2/4}}{\sqrt{2}}.$$

**Example 8 :** Find the Fourier transform of

$$f(x) = \begin{cases} 1+\dfrac{x}{a}, & -a < x < 0 \\ 1-\dfrac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

**Sol. :** By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-a}^{0} \left(1+\frac{x}{a}\right) e^{isx} dx + \int_{0}^{a} \left(1-\frac{x}{a}\right) e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \left\{ \left(1+\frac{x}{a}\right) \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2} \cdot \left(\frac{1}{a}\right) \right\}_{-a}^{0} + \left\{ \left(1-\frac{x}{a}\right) \cdot \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2} \left(-\frac{1}{a}\right) \right\}_{0}^{a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \left\{ \frac{1}{is} + \frac{1}{as^2} - 0 - \frac{e^{-isa}}{as^2} \right\} + \left\{ 0 - \frac{e^{isa}}{as^2} - \frac{1}{is} + \frac{1}{as^2} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{2}{as^2} - \frac{1}{as^2} (e^{isa} + e^{-isa}) \right] = \frac{1}{\sqrt{2\pi}} \left[ \frac{2}{as^2} - \frac{2\cos as}{as^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{as^2} (1 - \cos as)$$

## EXERCISE - II

Find the Fourier Transform of the following :

1. $f(x) = \begin{cases} 1, & a < x < b \\ 0, & x < a,\ x > b \end{cases}$ $\left[\textbf{Ans. :}\ \dfrac{1}{\sqrt{2\pi} \cdot is} (e^{isb} - e^{isa})\right]$

2. $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ $\left[\textbf{Ans. :}\ \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{1}{s} \cdot \sin sa\right]$

3. $f(x) = e^{-a|x|}$ $\left[\textbf{Ans. :}\ \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{a}{s^2 + a^2}\right]$

(**Hint :** $f(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{0} e^{(a+is)x} dx + \dfrac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a-is)x}$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[ \frac{1}{a+is} + \frac{1}{a-is} \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + s^2})$$

4. $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ $\left[\textbf{Ans. :}\ \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{1}{s^3} [(a^2 s^2 - 2) \sin as + 2as \cos as]\right]$

6. $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ $\left[\textbf{Ans. :}\ \dfrac{2i}{s^2} \cdot \dfrac{1}{\sqrt{2\pi}} [\sin sa - as \cos sa]\right]$

7. $f(x) = \begin{cases} \dfrac{\pi}{2} \cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ $\left[\textbf{Ans. :}\ \pi \cdot \dfrac{s \sin s\pi}{1 - s^2}\right]$

## 5. Inverse Fourier Transform or Complex Fourier Transform

If $F(s)$ is the Fourier transform of $f(x)$ and if $f(x)$ satisfies certain conditions (i.e. Dirichlets conditions) in every finite interval $(-l, l)$ and if $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent then at every point of continuity of $f(x)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds}$$

$f(x)$ is called the **Inverse Fourier Transform** of $F(s)$.

**Note ....**

Some authors define Fourier Transforms in different ways

1. $F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$
2. $F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$
3. $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}dx$ and $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$

Even some authors use $p$ in place of $s$, and $\bar{f}(p)$ in place of $F(s)$, some use $\lambda$ in place of $s$. We advise students to use the notation used in this book as it is more common and more convenient.

**Example 1 :** Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases}$ and hence, evaluate

(i) $\int_{-\infty}^{\infty} \frac{\sin sk \cos sx}{s}\,ds$ (ii) $\int_{-\infty}^{\infty} \frac{\sin ks}{s}\,ds$ **(M.U. 2009)** (iii) $\int_{-\infty}^{\infty} \frac{\sin s}{s}\,ds$

**Sol. :** By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} 1\cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-k}^{k} = \frac{1}{\sqrt{2\pi}}\cdot\frac{2}{s}\left[\frac{e^{isk}-e^{-isk}}{2i}\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{s}\cdot\sin sk \text{ for } s \neq 0$$

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For $s = 0$, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} dx = \frac{1}{\sqrt{2\pi}}[k+k] = \frac{2k}{\sqrt{2\pi}}$

Now, we use inverse Fourier Transform. We know that if

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \quad \text{then,} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds$$

(i) $\therefore$ $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\cdot\frac{1}{s}\cdot\sin sk\cdot e^{-isx}\cdot ds$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\cos sx - i\sin sx)}{s}\sin sk\cdot ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx\cdot\sin sk}{s}\,ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin sx\sin sk}{s}\,ds$$

The second integral being odd is zero.

$$\therefore \quad f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} \cdot ds$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} ds = \begin{cases} \pi, & |x| < k \\ 0, & |x| > k \end{cases}$$

(ii) In the above result, if we put $x = 0$, we put

$$\int_{-\infty}^{\infty} \frac{\sin ks}{s} ds = \pi \qquad \therefore \ 2\int_{0}^{\infty} \frac{\sin ks}{s} ds = \pi \qquad \therefore \ \int_{0}^{\infty} \frac{\sin ks}{s} ds = \frac{\pi}{2}.$$

**Note ....**

From the result (ii) above, we get

$$\boxed{\int_{0}^{\infty} \frac{\sin kx}{x} dx = \frac{\pi}{2}}$$

This is an important integral and can be used as a standard result when required. You are advised to memories it and also the following result.

(iii) In the above result put $k = 1$,

$$\therefore \quad \boxed{\int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}}$$

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**Example 2 :** Find the Fourier transform of

$f(x) = \begin{cases} (1-x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ and hence evaluate $\int_{0}^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cdot \cos \frac{x}{2} dx.$ **(M.U. 2008, 09)**

**Sol. :** By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

Integrating by parts, the integral $I$ is given by

$$I = (1-x^2) \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is} (-2x)\, dx$$

$$= (1-x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is}\left[x \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is} \cdot 1 \cdot dx\right]$$

$$\therefore \quad I = (1-x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is} \cdot \left[x \cdot \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\right]$$

$$\therefore \quad F(s) = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{(1-x^2)\, e^{isx}}{is}\right)_{-1}^{+1} + \frac{2}{is}\left(\frac{x\, e^{isx}}{is}\right)_{-1}^{+1} + \frac{2}{is}\left(\frac{e^{isx}}{s^2}\right)_{-1}^{+1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \frac{4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{4}{s^2}\cos s + \frac{4}{s^3}\sin s\right] = -2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{s\cos s - \sin s}{s^3}\right)$$

Now, we use inverse Fourier Transform. We know that if

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx \text{ then, } f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds$$

→ (c − is)

$$\therefore \quad f(x)=\frac{1}{\sqrt{2\pi}}\left(-2\cdot\sqrt{\frac{2}{\pi}}\right)\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)e^{-isx}\,ds$$

$$=-\frac{2}{\pi}\int_{-\infty}^{\infty}\cos sx\left(\frac{s\cos s-\sin s}{s^3}\right)dx - i\frac{2}{\pi}\int_{-\infty}^{\infty}\sin sx\left(\frac{s\cos - \sin s}{s^3}\right)ds$$

Now, the second integral being odd is zero.

$$\therefore \quad f(x)=\begin{cases}(1-x^2), & |x|\le 1\\ 0, & |x|>1\end{cases}=-\frac{2}{\pi}\int_{-\infty}^{\infty}\cos sx\left(\frac{s\cos s-\sin s}{s^3}\right)ds$$

Now, we put $x=\frac{1}{2}$,

$$\therefore \quad \frac{3}{4}=-\frac{2}{\pi}\int_{-\infty}^{\infty}\cos\left(\frac{s}{2}\right)\left(\frac{s\cos s-\sin s}{s^3}\right)ds$$

$$\therefore \quad \int_{-\infty}^{\infty}\cos\left(\frac{s}{2}\right)\cdot\left(\frac{s\cos s-\sin s}{s^3}\right)ds=-\frac{3\pi}{8}$$

$$\therefore \quad \int_{0}^{\infty}\cos\left(\frac{x}{2}\right)\cdot\left(\frac{x\cos x-\sin x}{x^3}\right)dx=-\frac{3\pi}{16}.$$

## EXERCISE - III

Find the inverse Fourier Transform of $F(s)=e^{-|s|a}$.

$$\left[\textbf{Ans. :}\ \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2}\right]$$

( **Hint :** $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a-ix)s}\,ds+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+ix)s}\,ds$

For $|s|=-s$ if $s\le 0$ and $|s|=s$ if $s\ge 0$. )