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Tutorial (1)

(1)

(1) Find the laplace transform of

$$f(t) = \begin{cases} \cos t & , 0 < t < \pi \\ \sin t & , t > \pi \end{cases}$$

→

By definition,

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$L[f(t)] = \int_0^{\pi} e^{-st} \cos t\, dt + \int_{\pi}^{\infty} e^{-st} \sin t\, dt \quad \text{——— (1)}$$

Now

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\therefore \int_0^{\pi} e^{-st} \cos t\, dt = \left[\frac{e^{-st}}{s^2+1}\left[-s \cos t + \sin t\right]\right]_0^{\pi}$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(s) - \frac{e^0}{s^2+1}(-s)\right]$$

$$= \frac{s}{s^2+1}\left(e^{-\pi s} + 1\right)$$

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$$\int_{\pi}^{\infty} e^{-st}\sin t\,dt = \left[\frac{e^{-st}}{s^2+1}\left(-s\sin t - \cos t\right)\right]_{\pi}^{\infty}$$

$$= \left[0 - \frac{e^{-\pi s}}{s^2+1}\left(-s(0) - (-1)\right)\right]$$

$$= -\frac{e^{-\pi s}}{s^2+1}(1)$$

$$= -\frac{e^{-\pi s}}{s^2+1}$$

From ①.

$$L[f(t)] = \frac{s}{(s^2+1)}\left(e^{-\pi s}+1\right) + \left(-\frac{e^{-\pi s}}{s^2+1}\right).$$

$$= \cancel{\frac{e^{-\pi s}(s-1) - 1}{(s^2+1)}}$$

$$= \frac{se^{-\pi/s} + s - e^{-\pi/s}}{s^2+1}.$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-\pi/s}\right].$$

Q.2.) $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

$\longrightarrow$

$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$

$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots$

$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots$

$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[\frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots\right]$

Now $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$

$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma 7/2}{s^{7/2}} + \cdots$

$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1/2\,\Gamma 1/2}{2!\,s^{3/2}} + \frac{1}{4!}\frac{3/2\;1/2\;\Gamma 1/2}{s^{5/2}} - \frac{1}{6!}\frac{5/2\;3/2\;1/2\;\Gamma 1/2}{s^{7/2}} + \cdots$

$= \frac{\Gamma 1/2}{s^{1/2}}\left[1 - \frac{1/2}{2!\,s} + \frac{1}{4!}\frac{3/2\times 1/2}{s^2} - \frac{1}{6!}\left(\frac{5/2\times 3/2\times 1/2}{s^3}\right) + \cdots\right]$

$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{1!(4s)} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \cdots\right]$

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$$= \sqrt{\frac{\pi}{s}}\left(e^{-1/4s}\right)\left[\because e^{-x} = 1 - \frac{1}{x} + \frac{1}{2!x^2} - \frac{1}{3!x^3} + \cdots\right]$$

(5) Evaluate by Laplase Transform $\int_0^\infty e^{-3t}\sin^3 t\, dt$.

$\longrightarrow$

Now, $\int_0^\infty e^{-3t}\sin^3 t\, dt = L[\sin^3 t]_{s=3}$

$[\because$ By definition$]$.

~~$L[\sin^3 t]$~~ we know,

$$\sin^3 t = \frac{3\sin t - \sin 3t}{4}.$$

So, $L[\sin^3 t]_{s=3} = L\left[\frac{3\sin t - \sin 3t}{4}\right]_{s=3}$

$$= \frac{1}{4}\left(L[3\sin t]_{s=3} - L[\sin 3t]_{s=3}\right).$$

$$= \frac{1}{4}\left[\left(\frac{3}{s^2+1}\right)_{s=3} - \left(\frac{3}{s^2+9}\right)_{s=3}\right]$$

$$= \frac{1}{4}\left(\frac{3}{10} - \frac{3}{18}\right).$$

$$= \frac{3}{4}\left(\frac{18-10}{180}\right)$$

$$= \frac{3\times 8}{4\times 180} = \frac{2}{\cancel{4}\times 60} = \frac{1}{30}$$

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(4) If $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s}$. find $L[\sin(2\sqrt{t})]$

$\longrightarrow$

Its Given, $L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\ e^{-1/4s}$

$L[\sin(2\sqrt{t})] = L[\sin(\sqrt{4t})]$

By change of order by scalar

$$= \left(\frac{1}{2\frac{s}{4}}\sqrt{\frac{\pi}{s/4}}\ e^{-1/4\times s/4}\right) \times \frac{1}{4}$$

$$= \left(\frac{2\times 2\times}{s}\sqrt{\frac{\pi}{s}} \times e^{-1/s}\right) \times \frac{1}{4}$$

$$= \frac{1}{s}\sqrt{\frac{\pi}{s}} \times e^{-1/s}.$$

Here, $F(s) = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$.

So, By change of order by scalar.

$L[\sin(2\sqrt{t})] = \frac{1}{4} F\left(\frac{s}{4}\right)$.

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5. $L[e^{-3t}\cosh 4t \sin 3t]$

$\longrightarrow$

$L[\sin 3t] = \frac{3}{s^2+9}$ [By formula]

$L[e^{-3t}\cosh 4t \sin 3t]$

$= L\left[e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t\right]$

$= \frac{1}{2} L\left[\left(e^{t}+e^{-7t}\right)\sin 3t\right]$

$= \frac{1}{2}\left(L[e^{t}\sin 3t] + L[e^{-7t}\sin 3t]\right)$.

$= \cancel{\frac{1}{2}}$ By First shifting method.

$= \frac{1}{2}\left(\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right)$

$= \frac{1}{2}\left(\frac{3}{s^2-2s+10} + \frac{3}{s^2+14s+58}\right)$

~~$= \frac{1}{2}\Big(3s^2+14\times 3s+58\times 3$~~

$= \frac{3}{2}\left(\frac{s^2+14s+58+s^2-2s+10}{(s^2-2s+10)(s^2+14s+58)}\right)$

$= \frac{3}{2}\frac{(2s^2+12s+68)}{(s^2-2s+10)(s^2+14s+58)}$

$= \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$.

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Tutorial ②

Laplace Transform

① Evaluate $\int_0^\infty \frac{t^2 \sin t}{e^{2t}}\, dt$

$\rightarrow$

$$= \int_0^\infty e^{-2t}\, t^2 \sin t \, dt$$

$$= L\left[t^2 \sin t\right]_{s=2}$$

Now, $L[\sin t] = \frac{1}{s^2+1}$ [By definition]

$\therefore$ By Multiplication of order of power of t

$$L[t^2 \sin t]_{s=2} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{1+s^2}\right)$$

$$= 1 \times \frac{d}{ds}\left(-\frac{2s}{(1+s^2)^2}\right)$$

$$= -1 \times \left(\frac{(1+s^2)^2(2) - 2(1+s^2)(2s)(2s)}{(1+s^2)^4}\right)$$

$$= -\left(\frac{2+2s^2 - 8s^2}{(1+s^2)^3}\right)$$

Learn to write SD

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$$= -\left(\frac{2+2s^2-8s^2}{(1+s^2)^3}\right)$$

$$= -\left(\frac{2-6s^2}{(1+s^2)^3}\right)$$

$$= \cancel{\frac{8s^4+6s^2-2}{(1+s^2)^4}} \quad \frac{6s^2-2}{(1+s^2)^3}$$

Putting $s=2$, $= \frac{6\times 2^2 - 2}{(1+4)^3}$

$$= \cancel{\frac{8\times 2^4 + 6\times 2^2 - 2}{(1+2^2)^4}} \quad \frac{24-2}{125}$$

$$= \frac{22}{125}$$

Q.2) Find $L\left[e^{-2t}\, t^4 \sinh(4t)\right]$

$\longrightarrow$ We know, $\sinh(4t) = \dfrac{e^{4t} - e^{-4t}}{2}$

$$L\left[e^{-2t}\, t^4 \sinh(4t)\right] = L\left[t^4 e^{-2t}\left(\frac{e^{4t} - e^{-4t}}{2}\right)\right]$$

$$= \frac{1}{2} L\left[t^4\left(e^{2t} - e^{-6t}\right)\right]$$

$$L[t^4] = \frac{\Gamma 5}{s^5} = \frac{4!}{s^5} \quad \text{[By definition]}$$

By first shifting them.

$$= \frac{1}{2}\left[\frac{4!}{(s-2)^5} - \frac{4!}{(s+6)^5}\right]$$

$$= \frac{4!}{2}\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right]$$

$$= 12\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right]$$

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Q.3 c)

→ Now, $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

By scalar property

$$L[J_0(at)] = \frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}} \times \frac{1}{a}$$

$$= \frac{a}{\sqrt{a^2+s^2}} \times \frac{1}{a} = \frac{1}{\sqrt{s^2+a^2}}.$$

By Multiplication by power of t

$$L[t\,J_0(at)] = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right).$$

$$= (-1)\left(\frac{-1}{2} \times \frac{2s}{(s^2+a^2)^{3/2}}\right)$$

$$= \frac{s}{(s^2+a^2)^{3/2}}.$$

4) Prove: $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{\pi}{8}$

$\longrightarrow$

$$\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt$$

$$= \int_0^\infty \frac{e^{-\sqrt{2}t}}{t}\left(\frac{e^t - e^{-t}}{2}\right) \sin t \, dt$$

$$= \int_0^\infty \left(e^{(-\sqrt{2}+1)t} \frac{\sin t}{2t} - e^{(-\sqrt{2}-1)t} \frac{\sin t}{2t}\right) dt$$

$$= L\left[\frac{\sin t}{2t}\right]_{s=\sqrt{2}-1} - L\left[\frac{\sin t}{2t}\right]_{s=\sqrt{2}+1} \quad \text{———} (1)$$

for $L\left[\frac{\sin t}{2t}\right]$,

$$L[\sin t] = \frac{1}{1+s^2}$$

$$\therefore L\left[\frac{\sin t}{2t}\right] = \frac{1}{2}\int_s^\infty \frac{1}{1+s^2} ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s)\right]$$

$\therefore$ Eq (1) will be by putting $s = \sqrt{2}-1$ & $s = \sqrt{2}+1$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(\sqrt{2}-1) - \frac{\pi}{2} + \tan^{-1}(\sqrt{2}+1)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \pi/8.$$

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5c) $L\left[\int_0^t u e^{-3u} \sin 4u\, du\right]$

→ for

$L\left[u e^{-3u} \sin 4u\right]$

Now,

$L[\sin 4u] = \frac{4}{s^2+16}$. [By definition].

So,

$L[e^{-3u} \sin 4u] = \frac{4}{(s+3)^2+16}$

— [By first shifting theorem].

$L[e^{-3u} \sin 4u] = \frac{4}{s^2+6s+25}$

So,

$L[u e^{-3u} \sin 4u] = (-1)\frac{d}{ds}\left(\frac{4}{s^2+6s+25}\right)$ [By Multiplication by power of t]

$= -1 \times \frac{(-4(2s+6))}{(s^2+6s+25)^2}$

$= \left[\frac{8s+24}{(s^2+6s+25)^2}\right]$

By Laplace of integration

$L\left[\int_0^t u e^{-3u} \sin 4u\, du\right] = \frac{1}{s}\left[\frac{8s+24}{(s^2+6s+25)^2}\right]$

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Tutorial 3: Inverse Laplase Transform & convolution theorem

COMPS.

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Q.1.) $L^{-1}\left[\frac{3s+5}{9s^2-25}\right]$, find this

$\longrightarrow$

so,

$$L^{-1}\left[\frac{3s+5}{9s^2-25}\right] = L^{-1}\left[\frac{3s}{9s^2-25}\right] + L^{-1}\left[\frac{5}{9s^2-25}\right]$$

$$= 3L^{-1}\left[\frac{s}{9(s^2-25/9)}\right] + 5L^{-1}\left[\frac{1}{9(s^2-25/9)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{s}{(s^2-(5/3)^2)}\right] + \frac{5}{9}L^{-1}\left[\frac{1}{(s^2-(5/3)^2)}\right]$$

We know,

$$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh(at)}{a}, \quad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh(at)$$

$$\therefore L^{-1}\left[\frac{3s+5}{9s^2-25}\right]$$

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$$= \frac{1}{3}\cosh(5/3t) + \frac{5}{9}\sinh(5/3t) \times \frac{1}{5/3}$$

$$= \frac{1}{3}\cosh\left(\frac{5t}{3}\right) + \frac{1}{3}\sinh\left(\frac{5t}{3}\right)$$

$$L^{-1}\left[\frac{3s+5}{9s^2-25}\right] = \frac{1}{3}\left[\cosh\left(\frac{5t}{3}\right) + \sinh\left(\frac{5t}{3}\right)\right]. = \frac{1}{3}e^{5/3t}$$

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Q. 20) find $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$

$\longrightarrow$

SQ)

$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+3}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+(\sqrt{3})^2}\right]$$

By first shifting therom.

$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = e^{-2t} L^{-1}\left[\frac{s}{s^2+(\sqrt{3})^2}\right]$$

Now

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$$

$$\boxed{L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = e^{-2t}\cos(\sqrt{3}t)}$$ $\left[\because \text{Here } a = \sqrt{3}\right]$.

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Q.3) find $L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$

$\longrightarrow$

So, using parital function.

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1}$$

$\therefore 5s^2+8s-1 = A(s^2+1) + (Bs+C)(s+3)$

Put $s=-3$

$20 = 10(A) \Rightarrow \boxed{A=2}$

Put $s=0$

$-1 = A + C(3)$

$3C = -3 \Rightarrow \boxed{C=-1}$

Put $s=1$

$12 = 2(A) + (B+C)(4)$

$12 = 4 + (B-1)4$

$4(B-1) = 8 \Rightarrow B-1=2 \Rightarrow \boxed{B=3}$

$$\therefore \frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{2}{s+3} + \frac{3s-1}{s^2+1}.$$

So,

$$L^{-1}\left[\frac{2}{s+3} + \frac{3s-1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{2}{s+3}\right] + 3L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+1}\right]$$

—— [By Linearity property]

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Now

$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$, $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$, $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \frac{\sin(at)}{a}$

$\therefore$

$$L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

$$\therefore \boxed{L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = 2e^{-3t} + 3\cos t - \sin t}$$

Q. 4] find $L^{-1}\left[\cot^{-1}(s+1)\right]$.

we know

$$L^{-1}\left[F(s)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}F(s)\right].$$

$$\therefore L^{-1}\left[\cot^{-1}(s+1)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}(s+1)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\frac{\pi}{2} - \tan^{-1}(s+1)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$$

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$$= \frac{1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

By first shifting theorem.

$$= \frac{1}{t} e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

we know, $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$$= \frac{1}{t} e^{-t} \sin t$$

$$\therefore \boxed{L^{-1}\left[\cot^{-1}(s+1)\right] = \frac{e^{-t}\sin t}{t}}$$

(5) Find using Convolution theorem.

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right]$$

$\longrightarrow$

$$L^{-1}\left[\frac{s}{(s^2+4)} \times \frac{1}{s^2+1}\right]$$

Here

$F(s) = \frac{s}{s^2+4}$ & $G(s) = \frac{1}{s^2+1}$

so, $f(t) = \cos(2t)$ & $g(t) = \sin t$

$\therefore$ By Convolution theorem.

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$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = f(t) * g(t) \quad \text{——— (1)}$$

where

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$\therefore \cos(2t) * \sin(t) = \int_0^t \cos(2u)\sin(t-u)\, du$$

$$= \int_0^t \cos(2u)\left[\sin t \cos u - \cos t \sin u\right] du$$

$$= \int_0^t \cos 2u \cos u \sin t \, du$$

$$- \int_0^t \cos 2u \sin u \cos t \, du$$

$$= \int_0^t \left(\frac{\cos 3u + \cos u}{2}\right) \sin t \, du$$

$$- \int_0^t \left(\frac{\sin 3u - \sin u}{2}\right) \cos t \, du$$

$$= \frac{\sin t}{2}\left[\frac{\sin 3x}{3} + \frac{\sin x}{1}\right]_0^t$$

$$- \frac{\cos t}{2}\left[-\frac{\cos 3u}{3} + \frac{\cos x}{1}\right]_0^t$$

$$= \frac{\sin t}{2}\left[\frac{\sin 3t}{3} + \frac{\sin t}{1}\right]$$

$$- \frac{\cos t}{2}\left[\left(\cos t - \frac{\cos 3t}{3}\right) - \frac{2}{3}\right]$$

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$$= \frac{\sin t \sin 3t}{2 \times 3} + \frac{\sin^2 t}{2} - \frac{(\cos t)^2}{2}$$

$$+ \frac{\cos t \cos 3t}{6} + \frac{\cos t}{3}$$

$$= \frac{\cos 2t}{6} + \frac{\cos t}{3} - \frac{\cos 2t}{2}$$

$$= \cancel{\frac{1}{3}\left(\frac{\cos 2t}{2} + \cos t - \frac{3\cos 2t}{2}\right)}$$

$$= \frac{\cos t}{3} - \frac{\cos 2t}{3}$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}(\cos t - \cos 2t)$$

Q.5) Find using Convolution theorem

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$$

$\rightarrow$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$$

Here $F(s) = \frac{1}{(s-2)^4}$, $G(s) = \frac{1}{s+3}$

So, $f(t) = e^{2t} L^{-1}\left[\frac{1}{s^4}\right]$, $g(t) = e^{-3t}$

[By first shifting theorem]

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i.e. $f(t) = e^{2t} \frac{t^3}{6}$, $g(t) = e^{-3t}$

So, By Convolution theorem

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = f(t) * g(t)$$

where

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

$$\therefore \left(e^{2t}\frac{t^3}{6}\right) * \left(e^{-3t}\right) = \int_0^t e^{2u}\frac{u^3}{6} \times e^{-3(t-u)}\, du$$

$$= \int_0^t e^{2u}\frac{u^3}{6}\left(e^{-3t}e^{3u}\right) du$$

$$= e^{-3t}\int_0^t \frac{u^3}{6} e^{5u}\, du$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\left(\frac{e^{5u}}{5}\right)\right]_0^t - \int_0^t \frac{3u^2}{6}\left(\frac{e^{5u}}{5}\right) du\right\}$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\left(\frac{e^{5u}}{5}\right)\right]_0^t - \left(\left[\frac{1}{2}u^2\frac{e^{5u}}{25}\right]_0^t + \int_0^t u\frac{e^{5u}}{25}\, du\right)\right\}$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\times\frac{e^{5u}}{5}\right]_0^t - \left[\frac{u^2}{2}\frac{e^{5u}}{25}\right]_0^t\right.$$

$$\left. + \left[u\frac{e^{5u}}{125}\right]_0^t - \left[\frac{e^{5u}}{625}\right]_0^t\right\}$$

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$$= e^{-3t}\left[\frac{t^3 e^{5t}}{30} - \frac{t^2 e^{5t}}{50} + \frac{t e^{5t}}{125} - \frac{e^{5t}}{625} + \frac{1}{625}\right]$$

$$= \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right]$$

$\therefore$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$$

$$= \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right].$$

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Tutorial 4: Application, Heavy side & direct delta.

Q.1.) $\longrightarrow$

$(D^2-D-2)y = 20\sin 2t$, $y(0)=1$, $y'(0)=2$

Taking Laplace Transform on both side.

$L[(D^2-D-2)y] = L[20\sin 2t]$

$L[D^2y - Dy - 2y] = L[20\sin 2t]$

$L[D^2y] - L[Dy] - 2L[y] = \frac{20\times 2}{s^2+4}$

$s^2 L[y(t)] - s\,y(0) - y'(0) - sL[y(t)] + y(0) - 2L[y(t)]$

$= \frac{40}{s^2+4}$

$L[y(t)][s^2-s-2] - s - 2 + 1 = \frac{40}{s^2+4}$

$L[y(t)](s^2-s-2) = \frac{40}{s^2+4} + (s+1)$

$L[y(t)] = \frac{40}{(s^2+4)(s^2-s-2)} + \frac{(s+1)}{(s^2-s-2)}$

$= \frac{40+(s+1)(s^2+4)}{(s^2+4)(s^2-s-2)}$

25 8 24/2

$= \frac{40+s^3+4s+s^2+4}{(s^2+4)(s^2-s-2)}$

$= \frac{s^3+s^2+4s+44}{(s^2+4)(s-2)(s+1)}$

Applying partial function.

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$$\frac{s^3+s^2+4s+44}{(s+1)(s-2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$$

$$s^3+s^2+4s+44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2)$$

put $s=2$,

$$8+4+8+44 = B(3)(8) \Rightarrow B = \frac{64}{24} = \frac{8}{3} \Rightarrow \boxed{B = \frac{8}{3}}$$

Put $s=-1$

$$-1+1-4+44 = A(-3)(5) \Rightarrow A = \frac{40}{5\times(-3)} \Rightarrow \boxed{A = \frac{-8}{3}}$$

Put $s=0$

$$44 = A(-2)(4) + B(4) + D(-2)$$

$$44 = \frac{-8}{3}\times -8 + \frac{8}{3}\times 4 - 2D$$

$$12 = -2D \Rightarrow \boxed{D=-6}$$

Put $s=1$

$$50 = A(-1)(5) + B(2)(5) + (C+D)(2)(-1)$$

$$50 = -5A + 10B + (C-6)(-2)$$

$$50 = -\frac{5\times -8}{3} + \frac{10\times 8}{3} + (-2C) + 12.$$

$$-2 = -2C \Rightarrow \boxed{C=1}$$

$$\therefore L[y(t)] = \frac{-8}{3(s+1)} + \frac{8}{3(s-2)} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace Transform

$$y(t) = \frac{-8}{3} L^{-1}\left[\frac{1}{s+1}\right] + \frac{8}{3} L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{s-6}{s^2+4}\right]$$

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$$y(t) = \frac{-8}{3} e^{-t} + \frac{8}{3} e^{2t} + \cos 2t - 3\sin 2t$$

$$\left[\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at}, \right.$$

$$\left. L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at,\ L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a} \right]$$

Q.2)

$\rightarrow$

$2x' + y' = 5e^{t}$, $x(0) = 0$, $y(0) = 0$

$y' - 3x' = 5$

Solving both equation simultaneously

Taking Laplace transform on both side on both equation

$L[2x' + y'] = L[5e^{t}]$, $L[-3x' + y'] = L[5]$ — ①

$2L[x'] + L[y'] = 5L[e^{t}]$

$2[sL[x(t)] - x(0)] + sL[y(t)] - y(0) = \frac{5}{s-1}$ $\left[\because L[e^{at}] = \frac{1}{s-a}\right]$

$2sL[x(t)] + sL[y(t)] = \frac{5}{s-1}$

Let $L[x(t)]$ be X & $L[y(t)]$ be Y.

$\therefore$ $2sX + sY = \frac{5}{s-1}$ — ②

from ①,

$L[-3x' + y'] = L[5]$

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$$-3L[x'] + L[y'] = 5\,L[1]$$

$$-3\left[sL[x(t)] - x(0)\right] + \left[sL[y(t)] - y(0)\right] = \frac{5}{s}$$

$$-3sL[x(t)] + sL[y(t)] = \frac{5}{s}$$

Let $L[x(t)]$, $L[y(t)]$ be X, Y respectively

$$-3s(X) + s(Y) = \frac{5}{s} \quad \text{— ③}$$

Subtracting ② & ③

$$X(2s) + Y(s) = 5/(s-1)$$
$$X(-3s) + Y(s) = 5/s$$
$$- \qquad - \qquad -$$

$$X(5s) = \frac{5}{s-1} - \frac{5}{s}$$

Now, $X = L[x(t)]$

$$L[x(t)](5s) = \frac{5}{s-1} - \frac{5}{s}$$

$$L[x(t)] = \frac{1}{s(s-1)} - \frac{1}{s^2}$$

$$L[x(t)] = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \quad \text{— ④}$$

Taking Inverse Laplace transform

$$x(t) = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right]$$

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$x(t) = e^{-t} - 1 - t$

i.e. $x(t) = e^{-t} - t - 1$

from ②,

$$Y(s) = \frac{5}{s-1} - X(2s)$$

Putting Y, X as L[y(t)] & L[x(t)] resp.

$$(s) L[y(t)] = \frac{5}{s-1} - L[x(t)](2s)$$

As $L[x(t)] = \frac{1}{s^2(s-1)}$

$$(s) L[y(t)] = \frac{5}{s-1} - \frac{1}{s^2(s-1)}(2s)$$

$$(s) L[y(t)] = \frac{5}{s-1} - \frac{2}{s(s-1)}$$

$$(s) L[y(t)] = \frac{5}{s-1} - 2\left[\frac{1}{s-1} - \frac{1}{s}\right]$$

~~Taking Inverse~~

$$L[y(t)] = \frac{3}{s(s-1)} + \frac{2}{s^2}$$

$$L[y(t)] = 3\left[\frac{1}{s-1} - \frac{1}{s}\right] + \frac{2}{s^2}$$

Taking inverse Laplace Transform

$$y(t) = 3L^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = 3e^{t} - 3 + 2t$$

$$y(t) = 3e^{t} + 2t - 3$$

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$\therefore x(t) = e^{t} - t - 1$

& $y(t) = 3e^{t} + 2t - 3$

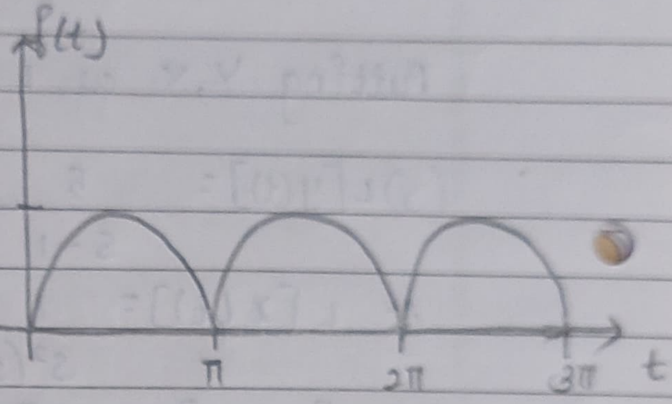
Q-3)

$f(t) = |\sin pt|, \ t \geq 0$

here $|\sin pt|$ is periodic function

$f(t + \pi/p) = f(t), \ \forall t$

$\therefore$ period $(T) = \pi/p$



for periodic function

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-s\pi/p}} \int_0^{\pi/p} e^{-st} \sin pt \, dt$$

$$= \frac{1}{1 - e^{-s\pi/p}} \left[\frac{e^{-st}}{s^2 + p^2} \left(-s \sin pt - p \cos pt \right) \right]_0^{\pi/p}$$

$$= \frac{1}{1 - e^{-s\pi/p}} \left[\frac{e^{-\pi s/p}}{s^2 + p^2} [p] - \frac{(-p)}{s^2 + p^2} \right]$$

$$= \frac{1}{1 - e^{-s\pi/p}} \times \frac{p}{s^2 + p^2} \times \left[e^{-\pi s/p} + 1 \right]$$

$$= \frac{p}{s^2 + p^2} \frac{(1 + e^{-\pi s/p})}{(1 - e^{-\pi s/p})}$$

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$$= \frac{p}{s^2+p^2}\left(\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2}\cosh\left(\frac{\pi s}{2p}\right).$$

4.)

$\longrightarrow$

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)\,dt$$

By definition

$$L\left[(1+2t-t^2+t^3)H(t-1)\right]_{s=1}$$

Now, $L[f(t)H(t-a)] = e^{-as}L[f(t+a)]$

$\therefore$

$$L[(1+2t-t^2+t^3)H(t-1)]$$

$$= e^{-s}L\left[1+2(t+1)-(t+1)^2+(t+1)^3\right]$$

$$= e^{-s}L\left[1+2t+2-t^2-2t-\cancel{1}+t^3+1+3t^2+3t\right]$$

$$= e^{-s}L\left[t^3+2t^2+3t+3\right]$$

$$= e^{-s}\left[\frac{3!}{s^4}+\frac{2\cdot 2!}{s^3}+\frac{3\times 1!}{s^2}+\frac{3}{s}\right]$$

Now putting $s=1$

$$= e^{-1}\left[3!+4+3+3\right] = e^{-1}(16) = 16/e$$

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$$\therefore \int_0^\infty e^{-t}\left(t^3 - t^2 + 2t + 1\right) H(t-1)\, dt = \frac{16}{e}$$

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$$L\left[\cos t\left[H(t-\pi/2) - H(t-3\pi/2)\right]\right]$$

$$= L\left[\cos t\left(H(t-\pi/2)\right)\right] - L\left[\cos t\left(H(t-3\pi/2)\right)\right]$$

Now $L[f(t)H(t-a)] = e^{-as} F(s+a)$

$$= e^{-s\pi/2} L\left[\cos(t+\pi/2)\right] - e^{-3\pi s/2} L\left[\cos(t+3\pi/2)\right]$$

$$= e^{-s\pi/2} L[-\sin t] - e^{-3\pi s/2} L[\sin t]$$

$$= -e^{-s\pi/2}\left(\frac{1}{s^2+1}\right) - e^{-3\pi s/2}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{-1}{s^2+1}\left[e^{\pi s/2} + e^{3\pi s/2}\right]$$

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$$L\left[t^2 H(t-2) - \cosh t\ \delta(t-4)\right]$$

Now, $L[f(t)H(t-a)] = e^{-as} L[f(t+a)]$

$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$

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⑨

$\therefore$

$$L\left[t^2 H(t-2)\right] - L\left[\cosh t\, \delta(t-4)\right]$$

$$= e^{-2s} L\left[(t+2)^2\right] - e^{-4s} \cosh 4$$

$$= e^{-2s} L\left[t^2 + 2t + 4\right] - e^{-4s} \cosh 4$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{2\times 2}{s^2} + \frac{4}{s}\right] - e^{-4s} \cosh 4$$

$\therefore$

$$L\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right]$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s} \cosh 4.$$

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Tutorial 5

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Fourier series

Q.1) find fourier series for $f(x)$ in $(0, 2\pi)$

$$f(x) = \begin{cases} x, & 0 < x \leq \pi \\ 2\pi - x, & \pi \leq x < 2\pi \end{cases}$$

$\longrightarrow$

$$f(x) = \begin{cases} x, & 0 < x \leq \pi \\ 2\pi - x, & \pi \leq x < 2\pi \end{cases}$$

Here fourier series in $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \left\{ \int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right\}$$

$$= \frac{1}{\pi} \left[\left[\frac{x^2}{2} \right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2} \right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2} - \left(2\pi^2 - \frac{\pi^2}{2} \right) \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \times \pi^2$$

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$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi} \int_{\pi}^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx \right]$$

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$$a_n = \frac{1}{\pi}\left\{\left[\frac{x \sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\,dx + \left[\frac{(2\pi - x)\sin nx}{n}\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{0 + \left[\frac{\cos nx}{n^2}\right]_0^{\pi} + 0 - \left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2} - \left[\frac{\cos(2n\pi)}{n^2} - \frac{\cos(n\pi)}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left[2\frac{\cos(n\pi)}{n^2} - \frac{\cos(2n\pi)}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n - 2}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{-4}{\pi n^2}, & n \text{ is add} \\ 0, & n \text{ is even.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx - \left[\frac{(2\pi - x)\cos nx}{n}\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\frac{\cos nx}{n}\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi\cos(n\pi)}{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi} + \frac{\pi\cos n\pi}{\pi} - \left[\frac{\sin nx}{n^2}\right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi}\left[0\right] = 0$$

$\therefore$ fourier series of $f(x)$ is

$$f(x) = \cancel{\frac{\pi}{2}}\ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$f(x) = \frac{\pi}{2} + \left[\frac{-4}{\pi}\cos x - \frac{4}{\pi 9}\cos 3x - \frac{4}{\pi 25}\cos 5x - \dots \right]$$

(If n is odd)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

~~[If n is odd]~~

~~(If n is even)~~ As for n = even

~~$f(x) = \frac{\pi}{2} \cancel{4}0$~~ $$\sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx = 0.$$

~~$f(x) = \frac{\pi}{2}$.~~

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

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Q.2) obtain fourier series

$$f(x) = \begin{cases} \pi/2 + x, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases}$$

deduce in

(1) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

(2) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

→

$$f(x) = \begin{cases} \pi/2 + x, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases} \qquad \text{———— (1)}$$

$$f(-x) = \begin{cases} \pi/2 - x, & \pi > x > 0 \\ \pi/2 + x & 0 > x > -\pi \end{cases}$$

Now Here

$f(x) = f(-x)$

$\therefore$ $f(x)$ is even in $(-\pi, \pi)$

so, fourier series for $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

but for even function, $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

$$\therefore \boxed{a_0 = 0}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} (-1)\frac{\sin nx}{n}\, dx\right\}$$

$$= \frac{2}{\pi}\left[0 - \left[\frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\right]$$

$$= -\frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 4/\pi n^2, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$\therefore$ fourier series for $f(x)$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \cos nx$$

[as for n = even $\sum_{n=1}^{\infty} a_n \cos nx = 0$]

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$$f(x) = \frac{4}{\pi}\cos x + \frac{4}{\pi 3^2}\cos 3x + \frac{4}{\pi 5^2}\cos 5x + \dots \quad \text{——— } (2)$$

Put $x = 0$

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}.$$

$$\therefore \frac{\pi}{2} = \frac{4}{\pi} + \frac{4}{\pi 3^2} + \frac{4}{\pi 5^2} + \dots \quad [\text{from } (2)]$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence proved

(3) By Parseval's identity for even function

$$\frac{2}{\pi}\int_0^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)^2 dx = \frac{16}{\pi^2} + \frac{16}{\pi^2 3^4} + \frac{16}{\pi^2 5^4} + \dots$$

$$\cancel{\frac{2}{\pi}\left[\frac{\pi^2 x}{4} + \frac{x^3}{3}\right]}$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{4} + x^2 - \pi x\right) dx = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

$$\frac{\pi}{8}\left[\frac{\pi^3}{4}+\frac{\pi^3}{3}-\frac{\pi^3}{2}\right] = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

Hence deduced

Q.4) Obtain half range ~~sine~~ sine series to represent

$$f(x) = \begin{cases} 2x/3 , & 0 \le x \le \pi/3 \\ \pi - x/3 ; & \pi/3 \le x \le \pi \end{cases}$$

→

$$f(x) = \begin{cases} 2x/3 , & 0 \le x \le \pi/3 \\ \pi - x/3 & \pi/3 \le x \le \pi \end{cases}$$

Half Range sine series in $(0,\pi)$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{—— (1)}$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/3} \frac{2x}{3}\sin(nx)\,dx + \int_{\pi/3}^{\pi}\left(\pi - \frac{x}{3}\right)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{x\cos nx}{n}\right]_0^{\pi/3} + \int_0^{\pi/3}\frac{\cos nx}{n}\right.$$

$$\left. + \frac{1}{3}\left[\frac{2\pi}{3n}\cos\frac{n\pi}{3} - \left[\frac{\sin nx}{n^2}\right]_{\pi/3}^{\pi}\right]\right]$$

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$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{-\pi}{3n}\cos\left(\frac{n\pi}{3}\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi/3}\right] + \frac{1}{3}\left[\frac{2\pi}{3n}\cos\frac{n\pi}{3} - \left[\frac{\sin nx}{n^2}\right]_{\pi/3}^{\pi}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{-\pi}{3n}\cos\left(\frac{n\pi}{3}\right) + \frac{\sin n\pi/3}{n^2}\right] + \frac{1}{3}\left[\frac{2\pi}{3n}\cos\frac{n\pi}{3} + \frac{\sin n\pi/3}{n^2}\right]\right]$$

$$b_n = \frac{2}{\pi}\left[\left[\frac{-2\pi}{9n}\cos\left(\frac{n\pi}{3}\right) + \frac{2}{3n^2}\sin n\pi/3\right] + \frac{2\pi}{9n}\cos\left(\frac{n\pi}{3}\right) + \frac{\sin n\pi/3}{3n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{\pi n}{3}\right)\right]$$

$\therefore$ Half range sine series is

$$F(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{3}\right) \sin(nx) \quad [\text{from } ①]$$

$$F(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)\sin(nx)$$

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Q.5) find half range cosine series for $f(x) = e^x$, $0 < x < 1$

$\longrightarrow$

$$f(x) = e^x$$

Half Range cosine series in $(0, 1)$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cancel{dx} \quad \text{, Here } L = 1$$

where,

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = \left[2e^x\right]_0^1$$

$$\boxed{a_0 = 2e - 2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_0^1 e^x \cos(n\pi x)\,dx$$

$$= 2\left[\frac{e^x}{1+(n\pi)^2}\left[\cos(n\pi x) + n\pi\sin(n\pi x)\right]\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]$$

$$\therefore F(x) = (e-1) + \sum_{n=1}^{\infty} \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]\cos(n\pi x)$$

$$= (e-1) + 2\sum_{n=1}^{\infty} \frac{\left[(-1)^n e - 1\right]}{1+n^2\pi^2}\cos(n\pi x)$$

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Q.32 Find fourier expansion of $f(x) = 4 - x^2$ in $(0, 2)$. Also state the values for $x = 0, 1, 2, 10, 11$. Hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$\longrightarrow$

here $2l = 2$, $l = 1$

Now

fourier series in $(0, 2)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

where,

$$a_0 = \frac{1}{2l} \int_0^{2l} f(x)\,dx = \frac{1}{2} \int_0^2 4 - x^2\,dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[8 - \frac{8}{3}\right] = \frac{8}{3}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= 1 \int_0^2 (4 - x^2) \cos(n\pi x)\,dx$$

$$= \left[(4 - x^2)\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^2 - \int_0^2 \left(-2x \sin(n\pi x)\right)\right]$$

$$= (4 - x^2) \times 0 - \left[(-2x)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^2$$

$$+ \left[(-2)\frac{(-\sin n\pi x)}{(n\pi)^2}\right]_0^2$$

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$$= \left[\frac{-2x \cos n\pi x}{n^2\pi^2}\right]_0^2 = \frac{-2}{n^2\pi^2}(2)$$

$$= \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^2 (4-x^2)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \left[(4-x)^2\left(-\frac{\cos n\pi x}{n\pi}\right)\right]_0^2 - \int_0^2 -2x\,\frac{(-\cos n\pi x)}{n\pi}\,dx$$

$$= \frac{4}{n\pi} - \left[(-2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$+ (-2)\left[\frac{\cos n\pi x}{(n\pi)^3}\right]_0^2$$

$$b_n = \frac{4}{n\pi}$$

from (1)

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos(n\pi x) + \frac{4}{n\pi}\sin(n\pi x)$$

$$\cancel{= \frac{8}{3} + \left[\frac{-4}{\pi^2}\cos\pi x + \frac{4}{n\pi}\sin\pi x\right.}$$

$$f(x) = \frac{8}{3} + \left(\frac{-4}{\pi^2}\right)\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \cdots\right]$$

$$+ \frac{4}{\pi}\left[1(\sin\pi x) + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \cdots\right]$$

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Now $f(x) = 4 - x^2$

$$4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[1\cos \pi x + \frac{1}{4}\cos 2\pi x + \cdots\right]$$

$$+ \frac{4}{\pi}\left[1\sin \pi x + \frac{1}{2}\sin 2\pi x + \cdots\right]$$

Put $x = 0$

$$4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{-1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{—— } (2)$$

$3 - \frac{8}{3}$ Put $x = 1$

$$\frac{1}{3} = \frac{-4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{-\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Put $x = 2$

$$0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad (3)$$

Put $x = 10$

$$4 - 100 - 8/3 = -4/\pi^2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{-246}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

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Put $x = \Pi$

$$4 - 121 - \frac{8}{3} = \frac{-4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right]$$

$$\frac{359}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots\right]$$

Add eq ② & eq ③

$$\frac{-1}{3} + \frac{2}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

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Tutorial 6

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COMPS [B3]

①

Q. 1) Find the Complex form of fourier series of

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

$\longrightarrow$

In the range $x \in (c, c+2l)$, complex form of fourier integral is given as

$$f(x) = c_0 + \sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/L}, \text{ where } c_n = \frac{1}{2L}\int_c^{c+2L} f(x) e^{-in\pi x/L} dx \quad \text{— ①}$$

$$c_n = \frac{1}{2L}\int_c^{c+2L} f(x) e^{-in\pi x/L} dx \quad \text{here } L=1$$

$$= \frac{1}{2}\int_0^2 f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2}\left[\int_0^1 x^2 e^{-in\pi x} dx + \int_1^2 e^{-in\pi x} dx\right]$$

$$c_n = \frac{1}{2}\left[\left[\frac{x^2 e^{-in\pi x}}{-in\pi} - \frac{2x e^{-in\pi x}}{(-in\pi)^2} + \frac{2e^{-in\pi x}}{(-in\pi)^3}\right]_0^1 + \left[\frac{e^{-in\pi x}}{-in\pi}\right]_1^2\right]$$

$$c_n = \frac{1}{2}\left[-\frac{e^{-in\pi x}}{in\pi} + \frac{2e^{-in\pi}}{n^2\pi^2} - \frac{2e^{-in\pi}}{in^3\pi^3} + \frac{2}{in^3\pi^3} - \frac{e^{-i2n\pi}}{in\pi} + \frac{e^{-in\pi}}{(in\pi)}\right]$$

$$c_n = \frac{1}{2}\left[\frac{2e^{-in\pi}}{n^2\pi^2} + \frac{2ie^{-in\pi} - 2i}{n^3\pi^3} + \frac{ie^{-i2n\pi}}{n\pi}\right]$$

2H

Now, $e^{-in\pi} = (-1)^n$ & $e^{-2in\pi} = +1$

$$c_n = \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n - 1)}{n^3\pi^3} + \frac{i}{n\pi}\right]$$

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and $c_0 = \frac{1}{2L}\int_c^{c+2L} f(x)\,dx$

$= \frac{1}{2}\int_0^2 f(x)\,dx$

$= \frac{1}{2}\left[\int_0^1 x^2\,dx + \int_1^2 dx\right]$

$= \frac{1}{2}\left[\left[\frac{x^3}{3}\right]_0^1 + \left[x\right]_1^2\right]$

$= \frac{1}{2}\left[\frac{1}{3} + 1\right] = \frac{2}{3}$

substitutig c_0 & c_n in eq ①

$$f(x) = \frac{2}{3} + \sum_{n=-\infty}^{n=\infty} \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n - 1)}{n^3\pi^3} + \frac{i}{n\pi}\right] e^{in\pi x/L}$$

here $n \neq 0$ as numerator $\longrightarrow \infty$ at $n = 0$

$\therefore$

$$f(x) = \frac{2}{3} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{n=\infty} \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n - 1)}{n^3\pi^3} + \frac{i}{n\pi}\right] e^{in\pi x/L}.$$

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2) Show that set of functions $\phi_n(x) = \sin\left(\frac{\pi n x}{L}\right)$, $n = 1, 2, 3, \ldots$ is orthogonal set on interval $0 \le x \le L$ and find corresponding orthonormal set.

$\longrightarrow$

Let $f_1(x) = \sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$
& $f_2(x) = \sin(m\pi x/L)$, $m = 1, 2, 3, \ldots$

for the functions $\phi_n(x)$ to be orthogonal to each other in interval $0 \le x \le L$ the inner product i.e. $(f_1(x), f_2(x))$ should be zero

i.e. $(f_1(x), f_2(x)) = \int_0^L f_1(x) f_2(x)\, dx = 0.$

so,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx, \quad \text{Here } m \neq n$$

$$= \int_0^L \frac{1}{2} \times \left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)\pi x/L)}{\pi(m-n)/L} - \frac{\sin((m+n)\pi x/L)}{\pi(m+n)/L}\right]_0^L$$

$$= \frac{1}{2}(0 - 0)$$
$$= 0$$

Hence the given set of functions is orthogonal in $0 \le x \le L$ as inner product is zero.

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Now, their Corresponding orthonormal sets;

$$\left\| \sin\left(\frac{n\pi x}{L}\right) \right\|^2 = \int_0^L \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx$$

$$= \int_0^L \left[\frac{1 - \cos\left(\frac{2n\pi x}{L}\right)}{2}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2\pi n x/L)}{2n\pi/L}\right]_0^L$$

$$= \frac{1}{2}\left[L - 0\right]$$

$$= L/2$$

$\therefore$ corresponding orthonormal set is

$$\left\{ \sqrt{\frac{2}{L}}\, \phi_n x \right\}.$$

3) Find fourier integral representation of function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

→

fourier integral representation of f(x) is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(u) \cos\lambda(u-x)\, du\, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^0 0\, du + \int_0^\infty e^{-u} \cos\lambda(u-x)\, du + 0 \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ 0 + \int_0^\infty e^{-u} (\cos\lambda u \cos\lambda x + \sin\lambda u \sin\lambda x)\, du + 0 \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \int_0^\infty e^{-u} \cos\lambda u \cos\lambda x\, du + \int_0^\infty e^{-u} \sin\lambda u \sin\lambda x\, du. \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \left\{ \left[\frac{\cos\lambda x\, e^{-u} [-\cos\lambda u + \lambda \sin\lambda u]}{1+\lambda^2} \right]_0^\infty + \left[\frac{\sin\lambda x\, e^{-u} [-\sin\lambda u - \lambda\cos\lambda u]}{1+\lambda^2} \right]_0^\infty \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} + \frac{\sin\lambda x\, (\lambda)}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2}\, d\lambda$$

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