$$2a^3 - 1 = 2b^3 - 1.$$

$\Rightarrow$ $a = b.$

$\therefore$ f is one to one

Now for $y = 2x^3 - 1.$

$$1 + y = 2x^3.$$

$$x^3 = \frac{1}{2} + \frac{y}{2}.$$

$\therefore$ $x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}.$

$\therefore$ for each $y \in B$. There is a unique x in A such that $f(x) = y$.

$\therefore$ f is onto.

$\therefore$ f is bijective function between A and B.

Similarly for $g : B \rightarrow A$ to be one to one and onto

$$g(a) = g(b) = \sqrt[3]{\frac{1}{2} + \frac{a}{2}} = \sqrt[3]{\frac{1}{2} + \frac{b}{2}}$$

$\therefore$ $\frac{1}{2} + \frac{a}{2} = \frac{1}{2} + \frac{b}{a}$

$\Rightarrow$ $a = b$

$\therefore$ g is one to one.

Also for $x = \sqrt[3]{\frac{1}{2} + \frac{y}{2}}$

$$x^3 = \frac{1}{2} + \frac{y}{2}$$

$$2x^3 = 1 + y$$

$$y = 2x^3 - 1$$

for each x in A. There is a corresponding y in B. Such that $g(y) = x$

$\therefore$ g is onto function

So g is bijective function between B and A.

Ex. 2.10.16 : Test whether the following function is one-to-one onto or both. $f : Z \rightarrow Z$, $f(x) = x^2 + x + 1$

MU - May 18

Soln. :

A function from 'A' to 'B' is one to one if no two elements of A have the same image.

Let, $X = -2, f(-2) = (-2)^2 + (-2) + 1$

$$= 4 - 2 + 1 = 3$$

Let $X = 1,$

$$f(1) = 1^2 + 1 + 1 = 3$$

Elements 1 and – 2 have same image so the function is not one to one.

A function from 'A' to 'B' is said to be an onto function of every element of B is image of one or more elements of A.

In the given function not every element is image of an element from A. For example, '2' is not an image of any elements of A. So it is not 'onto' function.

Ex. 2.10.17 : Let f : ROR, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x + 1)/2$ Find $(f \,O\, f^{-1})(x)$.

MU - May 19

Soln. :

$$(f \,O\, f^{-1})(x) = f(f^{-1}(x))$$

$$= f\left(\frac{(x+1)}{2}\right)$$

$$= \frac{2(x+1)}{2} - 1$$

$$= x + 1 - 1$$

$$= x$$

$$\therefore (f \circ f^{-1})(x) = x$$

2.11 Composition

2.11.1 Definition

Let f be a function from A to B (i.e. $f : A \rightarrow B$) and g be a function from B to C (i.e. $g : B \rightarrow C$). Then the composition of f and g denoted as $\boxed{gof}$ is a relation from A to C, where gof (a) = g (f (a)). $g \circ f : A \rightarrow C$ is also a function. This is because if there exists elements c, $d \in C$ such that g o f (a) = c and g of (a) = d, for some $a \in A$, this would imply that g (f (a)) = c and g (f (a)) = d. But f is a function, hence f (a) is unique. Then since g is also a function, it follows that c = d. Hence g o f is a function from A to C. Note that g o f is defined only when the range of f is a subset of the domain of g.

If f(a) = b and g (b) = c then g o f (a) = c The Fig. 2.11.1 depicts a composite function.The rule to compose two functions can be extended to a finite number of functions :

$f_1 : A, \rightarrow A_2$, $f_2 : A_2 \rightarrow A_3$, ... $f_n : A_n \rightarrow A_n + 1$, where range of f_i = domain of $f_i + 1$, for $1 \leq i \leq n$. Thus $f_n \circ f_n - 1 \circ \ldots \circ f_1$ (denoted usually as $f_n f_{n-1} \ldots f_1$) is a function from A_1 to A_{n+1}. In particular if $A_1 = A_2 = \ldots = A_{n+1} = A$ and $f_1 = f_2 = \ldots = f_n = f$ then f o f ...... o f (n times) denote as f^n is the composite function from A to A.

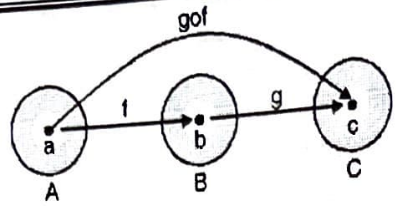

Fig. 2.11.1

Note : We read g (f (x)) as 'gof f of x' or "f followed by gof x".

2.11.2 Solved Examples on Composition

Ex. 2.11.1 : Let A = { 1, 2, 3 }, B = { a, b } and C = { 5, 6, 7 }.

Let f : A → B be

defined by f(1) = a. f(2) = a. f(3) = b.

i.e. f = {(1, a), (2, a), (3, b)}.

Let g : B → C be defined by

g(a) = 5.

g(b) = 7.

i.e. g = {(a, 5), (b, 7)}.

Find composition of f and g i.e. (g o f)

Soln. :

If f(1) = a and g(a) = 5 then g o f (1) = 5

If f(2) = a and g(a) = 5 then g o f (2) = 5

If f(3) = b and g(b) = 7 then g o f (3) = 7

i.e. (g o f) (1) = 5

(g o f) (2) = 5

(g o f) (3) = 7

g o f = {(1, 5), (2, 5), (3, 7)}.

Ex. 2.11.2 : A = { a_1, a_2, a_3, a_4 }, B = { b_1, b_2, b_3, b_4 }, C = {c_1, c_2, c_3}, D = { d_1, d_2, d_3 }.

(i) For the function f and g, determine g o f.

(ii) For the function f, g and h, determine h o (g o f) and (h o g) o f

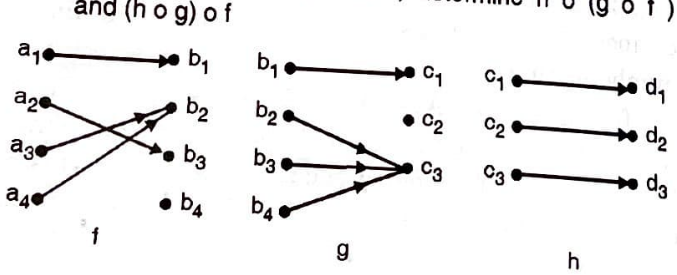

Fig. Ex. 2.11.2(a)

Soln. :

(i) f : A → B i.e. and g : B → C i.e.

$f(a_1) = b_1$, $g(b_1) = c_1$,

$f(a_2) = b_3$, $g(b_2) = c_3$,

$f(a_3) = b_2$, $g(b_3) = c_3$,

$f(a_4) = b_2$. $g(b_4) = c_3$

$$g \circ f = g(f(a_1)) = g(b_1) = c_1$$
$$= g(f(a_2)) = g(b_3) = c_3$$
$$= g(f(a_3)) = g(b_2) = c_3$$
$$= g(f(a_4)) = g(b_2) = c_3$$

∴ $g \circ f(a_1) = c_1$

$g \circ f(a_2) = c_3$

$g \circ f(a_3) = c_3$

$g \circ f(a_4) = c_3$

Graphical representation of 'g o f' is shown in Fig. Ex. 2.11.2(b).

(ii) $h \circ (g \circ f) = h(g \circ f(a_1))$
$= h(c_1)$
$= d_1$.
$= h(g \circ f(a_2))$
$= h(c_3)$
$= d_3$.
$= h(g \circ f(a_3))$
$= h(c_3)$
$= d_3$.
$= h(g \circ f(a_4))$
$= h(c_3)$
$= d_3$.

(h o g) o f $h \circ g = h(g(b_1))$
$= h(c_1)$
$= d_1$.
$= h(g(b_2))$
$= h(c_3)$
$= d_3$.
$= h(g(b_3))$
$= h(c_3)$
$= d_3$.
$= h(g(b_4))$
$= h(c_3)$
$= d_3$.

∴ $h \circ g(b_1) = d_1$.

$h \circ g(b_2) = d_3$.

$$h \circ g\,(b_3) = d_3.$$

$$h \circ g\,(b_4) = d_3.$$

$$(h \circ g) \circ f = (h \circ g)\,(f\,(a_1))$$

$$= hog\,(b_1)$$

$$= d_1.$$

$$= (h \circ g)\,(f\,(a_2))$$

$$= hog\,(b_3)$$

$$= d_3.$$

$$= (h \circ g)\,(f\,(a_3))$$

$$= hog\,(b_2)$$

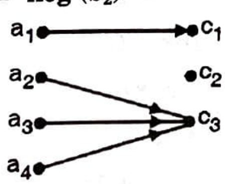

Fig. Ex. 2.11.2(b)

$$= d_3.$$

$$= (h \circ g)\,(f\,(a_4))$$

$$= hog\,(b_2)$$

$$= d_3.$$

Ex. 2.11.3 : Let A = B = Z and C be the set of even integers. Let f : A → B and g : B → C be defined by, f (a) = a + 1. g (b) = 2b. Find g o f.

Soln. : $(g \circ f)\,(a) = g\,(f\,(a)) = g\,(a + 1).$

$$= 2\,(a + 1) = 2a + 2.$$

Ex. 2.11.4 : Let f : Z → Z. where f(x) = 3x. and g : Z → Z. where $g(x) = x^2$. What are g o f (2) and f o g (2) ? Are there any values of x for which g o f (x) = f o g (x) ?

Soln. :

$$g \circ f\,(2) = g\,(f\,(2)) = g\,(6). = 36$$

$$f \circ g\,(2) = f\,(g\,(2)) = f\,(4). = 12$$

$$g \circ f\,(x) = g\,(f\,(x)) = g\,(3x). = (3x)^2$$

$$f \circ g\,(x) = f\,(g\,(x)) = f\,(x^2). = 3x^2$$

Now if $f\,(g\,(x)) = g\,(f\,(x))$

then $3\,(x)^2 = (3x)^2$ which is true only if x = 0 and not otherwise.

Ex. 2.11.5 : Consider the functions f (x) = 2x – 3 and $g(x) = x^2 + 3x + 5$. Find a formula for the composition function (i) gof and (ii) fog.

Soln. :

(i)

$$g \circ f\,(x) = g\,(f\,(x))$$

$$= g\,(2x - 3)$$

$$= (2x - 3)^2 + 3\,(2x - 3) + 5$$

$$= 4x^2 - 12x + 9 + 6x - 9 + 5$$

$$= 4x^2 - 6x + 5$$

(ii)

$$(f \circ g)\,(x) = f\,(g\,(x))$$

$$= f\,(x^2 + 3x + 5\,)$$

$$= 2\,(x^2 + 3x + 5) - 3$$

$$= 2x^2 + 6x + 10 - 3$$

$$= 2x^2 + 6x + 7$$

Ex. 2.11.6 : Consider the above function f (x) = 2x – 3. Find a formula for the composition functions (i) $f^2 = f \circ f$ and (ii) $f^3 = f \circ f \circ f$. **MU - May 17**

Soln. :

(i)

$$(f \circ f)\,(x) = f\,(f\,(x))$$

$$= f\,(2x - 3)$$

$$= 2\,(2x - 3) - 3$$

$$= 4x - 6 - 3$$

$$= 4x - 9$$

(ii)

$$(f \circ f \circ f)\,(x) = f\,(f\,(f\,(x)))$$

$$= f\,(f\,(2x - 3)) = f\,(4x - 9)$$

$$= 2\,(4x - 9) - 3$$

$$= 8x - 18 - 3$$

$$= 8x - 21$$

Ex. 2.11.7 : If A = B = C = R where R is set of real number and f : A → B, g : B → C are functions defined by f(x) = x + 1, $g(x) = x^2 + 2$, then find (g o f) (x) and (f o g) (2).

Soln. :

Given : $f(x) = x + 1,\ g(x) = x^2 + 2$

(i)

$$(g \circ f)\,(x) = g\,(\,f\,(x))$$

$$= g\,(x + 1) = (x + 1)^2 + 2$$

$$= x^2 + 2x + 3$$

(ii)

$$(f \circ g)\,(2) = f\,(g\,(2))$$

$$= f\,(6) = 7$$

Ex. 2.11.8 : Following examples (i) to (iv) refer to maps f : A → B, g : B → A, h : C → B, F : B → C, and G : A → C which are pictured in the diagram of maps in Fig. Ex. 2.11.8.

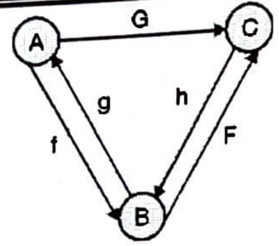

Fig. Ex. 2.11.8

(i) Is g o f defined? If so, what is its domain and codomain.

(ii) Is h o f defined? If so, what is its domain and codomain.

(iii) Is F o h o G defined. If so, what is its domain and codomain?

(iv) Is G o F o h defined? If so, what is its domain and codomain?

Soln. :

(i) Since f goes from A to B (i.e. $f : A \to B$) and g goes from B to A (i. e. $g : B \to A$), (g o f) is defined and A is its domain and co-domain.

(ii) Note that h does not "follow" f in the diagram, i. e. the codomain B of f is not the domain of h. Hence h o f is not defined.

(iii) The arrows representing G, h and F do follow each other. Refer obove fig and go from A to C to B to C. Thus F o h o g is defined with domain A and codomain C. (We note that compositions are "read" from right to left)

(iv) F follows h in the Fig. 4.17, but G does not follow F, i.e; the codomain C of F is not the domain of G. Hence G o F o h is not defined.

Ex. 2.11.9 : Let $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$ find (i) f o (g o f) (ii) g o (f o g). **MU - Dec. 12**

Soln. :

Given $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$

(i) $$f \circ (g \circ f) = f(g(f(x)))$$
$$= f(g(x^2 - 1)) = f(4(x^2 - 1)^2 + 2)$$
$$= f(4(x^4 - 2x^2 + 1) + 2)$$
$$= f(4x^4 - 8x^2 + 4 + 2)$$
$$= f(4x^4 - 8x^2 + 6)$$
$$= (4x^4 - 8x^2 + 6)^2 - 1$$

(ii) $$g \circ (f \circ g) = g(f(g(x)))$$
$$= g(f(4x^2 + 2)) = g((4x^2 + 2)^2 - 1)$$
$$= g(16x^4 + 16x^2 + 4)$$
$$= 4(16x^4 + 16x^2 + 4)^2 + 2$$

Ex. 2.11.10 : Let the functions f, g and h defined as follows : $f : R \to R$, $f(x) = 4x - 3$. $g : R \to R$, $g(x) = x^2 + 1$.

$$h : R \to R,\ h(x) = \begin{cases} 1 \text{ if } x \geq 0. \\ 0 \text{ if } x < 0. \end{cases}$$

Find rules for the following functions

(i) f o f (ii) f o g (iii) g o f

(iv) f o h (v) h o f (vi) g o h

(vii) h o g

Soln. :

(i) $$(f \circ f)(x) = f(f(x))$$
$$= f(4x - 3) = 4(4x - 3) - 3$$
$$= 16x - 12 - 3$$
$$= 16x - 15$$

(ii) $$(f \circ g)(x) = f(g(x))$$
$$= f(x^2 + 1)$$
$$= 4(x^2 + 1) - 3$$
$$= 4x^2 + 4 - 3$$
$$= 4x^2 + 1$$

(iii) $$(g \circ f)(x) = g(f(x))$$
$$= g(4x - 3)$$
$$= (4x - 3)^2 + 1$$
$$= 16x^2 - 24x - 9 + 1$$
$$= 16x^2 - 24x - 8$$

(iv) $$(f \circ h)(x) = f(h(x))$$
$$= \begin{cases} f(1) \text{ if } x \geq 0 \\ f(0) \text{ if } x < 0 \end{cases}$$
$$= \begin{cases} 1 \text{ if } x \geq 0 \\ -3 \text{ if } x < 0 \end{cases}$$

$\therefore$ $$(f \circ h)(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ -3 \text{ if } x < 0 \end{cases}$$

(v) $$(h \circ f)(x) = h(f(x))$$
$$= h(4x - 3)$$
$$= \begin{cases} 1 \text{ if } x \geq 3/4 \\ 0 \text{ if } x < 3/4 \end{cases}$$

$\therefore$ $$(h \circ f)(x) = \begin{cases} 1 \text{ if } x \geq 3/4 \\ 0 \text{ if } x < 3/4 \end{cases}$$

(vi) $$(g \circ h)(x) = g(h(x))$$
$$= \begin{cases} g(1) \text{ if } x \geq 0 \\ g(0) \text{ if } x < 0 \end{cases}$$
$$= \begin{cases} 2 \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{cases}$$

$$\therefore \quad (g \circ h)(x) = \begin{cases} 2 \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{cases}$$

(vii) $(h \circ g)(x) = h(g(x))$

$= h(x^2 + 1)$

$= 1$

Ex. 2.11.11 : Let A = B = C = IR and Let $f : A \rightarrow B$, $g : B \rightarrow C$ be defined by $f(a) = a + 1$ and $g(b) = b^2 + 2$. Find :

(i) $(g \circ f)(-2)$ (ii) $(f \circ g)(-2)$

(iii) $(g \circ f)(x)$ (iv) $(f \circ g)(x)$

(v) $(f \circ f)(y)$ (vi) $(g \circ g)(y)$

Soln. :

(i) $(g \circ f)(-2) = g(f(-2)) = g(-2+1)$

$= g(-1) = (-1)^2 + 2$

$= 1 + 2$

$= 3$

(ii) $(f \circ g)(-2) = f(g(-2)) = f((-2)^2 + 2)$

$= f(4 + 2)$

$= f(6)$

$= 6 + 1$

$= 7$

(iii) $(g \circ f)(x) = g(f(x))$

$= g(x + 1)$

$= (x + 1)^2 + 2$

$= x^2 + 2x + 3$

(iv) $(f \circ g)(x) = f(g(x))$

$= f(x^2 + 2)$

$= (x^2 + 2) + 1$

$= x^2 + 3$

(v) $(f \circ f)(y) = f(f(y))$

$= f(y + 1)$

$= (y + 1) + 1$

$= y + 2$

(vi) $(g \circ g)(y) = g(g(y))$

$= g(y^2 + 2)$

$= (y^2 + 2)^2 + 2$

Ex. 2.11.12 : Let A = { 1, 2, 3, 4 },

B = { a, b, c } and

C = { x, y, z } and let $f : A \rightarrow B$ and $g : B \rightarrow C$. Find $(g \circ f)$.

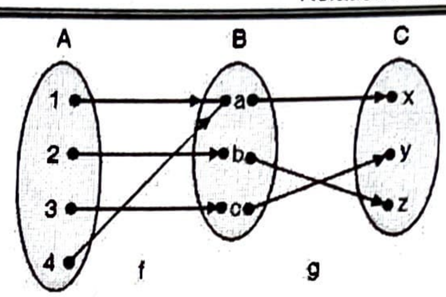

Fig. Ex. 2.11.12

Soln.:

$f(1) = a.$ and $g(a) = x.$

$f(2) = b.$ $g(b) = z.$

$f(3) = c.$ $g(c) = y.$

$f(4) = a.$

$(g \circ f)(1) = g(f(1)) = g(a) = x.$

$(g \circ f)(2) = g(f(2)) = g(b) = z.$

$(g \circ f)(3) = g(f(3)) = g(c) = y.$

$(g \circ f)(4) = g(f(4)) = g(a) = x.$

$\therefore \quad (g \circ f)(1) = x.$

$(g \circ f)(2) = z.$

$(g \circ f)(3) = y.$

$(g \circ f)(4) = x.$

$\therefore \quad (g \circ f) = \{(1, x), (2, z), (3, y), (4, x)\}.$

Ex. 2.11.13 : Define injective, surjective and bijective functions. If $f : R \rightarrow R$ and $g : R \rightarrow R$ are defined by the formulas $f(x) = x + 2$ and $g(x) = x^2$ Find (i) f . g . f ; (ii) g . f. g. **MU - May 13, Dec. 15**

Soln. :

Let $f(x) = x + 2$ and $g(x) = x^2$

(i) $f \circ g \circ f = f(g(f(x)))$

$= f(g(x + 2))$

$= f((x + 2)^2)$

$= (x + 2)^2 + 2$

$= x^2 + 4x + 6$

(ii) $g(f(g(x)) = g(f(x^2))$

$= g(x^2 + 2) = (x^2 + 2)^2$

$= (x^2 + 2)\ (x^2 + 2)$

$= x^4 + 2x^2 + 2x^2 + 4$

$= x^4 + 4x^2 + 4]$

Ex. 2.11.14 : Functions f, g, h are defined on a set,
X = { 1, 2, 3 } as
f = {(1, 2), (2, 3), (3, 1)}.
g = {(1, 2), (2, 1), (3, 3)}.
h = {(1, 1), (2, 2), (3, 1)}.
(i) Find f o g, g o f, Are they equal ?
(ii) Find f o g o h and f o h o g.

MU - May 16, Dec. 16

Soln. :

We may depict f, g , h graphically as shown in Fig. Ex. 2.11.14(a).

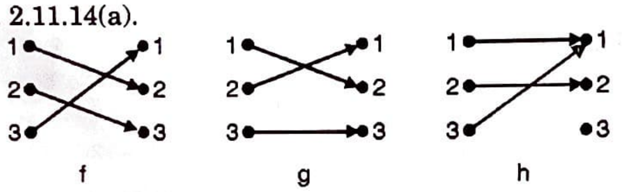

Fig. Ex. 2.11.14(a)

(i) $f(1) = 2,$ $\quad g(1) = 2,$

$f(2) = 3,$ $\quad g(2) = 1,$

$f(3) = 1.$ $\quad g(3) = -3.$

$$f \circ g(1) = f(g(1)) = f(2) = 3.$$
$$f \circ g(2) = f(g(2)) = f(1) = 2.$$
$$f \circ g(3) = f(g(3)) = f(3) = 1.$$
$$\therefore \quad f \circ g = \{(1, 3), (1, 2), (1, 1)\}.$$

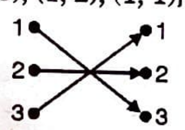

Fig. Ex. 2.11.14(b)

f o g is depicted as shown in Fig. Ex. 2.11.14(b)

$$g \circ f(1) = g(f(1)) = g(2) = 1$$
$$g \circ f(2) = g(f(2)) = g(3) = 3$$
$$g \circ f(3) = g(f(3)) = g(1) = 2$$
$$\therefore \quad g \circ f = \{(1, 1), (2, 3), (3, 2)\}.$$

g o f is depicted as shown in Fig. Ex. 2.11.14(c).

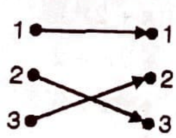

Fig. Ex. 2.11.14(c)

$$\therefore \quad f \circ g \neq g \circ f$$

(ii) $h(1) = 1, \quad h(2) = 2, \quad h(3) = 1$

$$f \circ g \circ h(1) = f(g(h(1))) = f(g(1)) = f(2) = 3.$$
$$f \circ g \circ h(2) = f(g(h(2))) = f(g(2)) = f(1) = 2.$$
$$f \circ g \circ h(3) = f(g(h(3))) = f(g(1)) = f(2) = 3.$$
$$\therefore \quad f \circ g \circ h = \{(1, 3), (2, 2), (3, 3)\}.$$

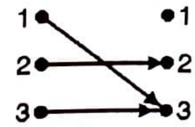

Fig. Ex. 2.11.14(d)

f o g o h can be depicted as shown in Fig. Ex. 2.11.14(d)

$$f \circ h \circ g(1) = f(h(g(1))) = f(h(2)) = f(2) = 3.$$
$$f \circ h \circ g(2) = f(h(g(2))) = f(h(1)) = f(1) = 2.$$
$$f \circ h \circ g(3) = f(h(g(3))) = f(h(3)) = f(1) = 2.$$
$$\therefore \quad f \circ h \circ g = \{(1, 3), (2, 2), (3, 2)\}.$$

f o h o g can be depicted as shown in Fig. Ex. 2.11.14(e).

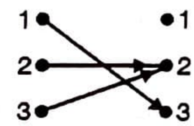

Fig. Ex. 2.11.14(e)

Ex. 2.11.15 : Let A = { a, b, c, d }, B = { s, t, u }, C = { *l*, m, n} obtain the composition of the following functions f : A → B, g : B → C

where f = {(a, s), (b, t), (c, u), (d, t)}.
g = {(s, m), (t, *l*), (u, n)}.

Soln. : $f(a) = s, \; f(b) = t, \; f(c) = u, \; f(d) = t.$

$g(s) = m, \; g(t) = l, \; g(u) = n.$

$$g \circ f(a) = g(f(a)) = g(s) = m.$$
$$g \circ f(b) = g(f(b)) = g(t) = l.$$
$$g \circ f(c) = g(f(c)) = g(u) = n.$$
$$g \circ f(d) = g(f(d)) = g(t) = l.$$

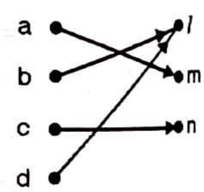

Fig. Ex. 2.11.15

$$\therefore \quad g \circ f = \{(a, m), (b, l), (c, n), (d, l)\}.$$

g o f can be depicted as shown in Fig. Ex. 2.11.15.

Ex. 2.11.16 : Let A = { 1, 2, 3, 4, 5 }, g : A → A is as shown in the Fig. Ex. 2.11.16(a). Find the composition g o g, g o (g o g). Determine each whether each is one to one or onto function.

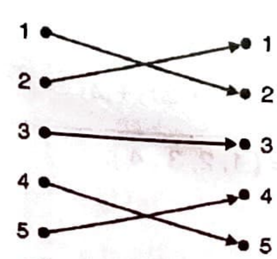

Fig. Ex. 2.11.16(a)

Soln. :

$$g(1) = 2,\ g(2) = 1,\ g(3) = 3,\ g(4) = 5,\ g(5) = 4$$

$\therefore$ $g \circ g(1) = g(g(1)) = g(2) = 1.$

$g \circ g(2) = g(g(2)) = g(1) = 2.$

$g \circ g(3) = g(g(3)) = g(3) = 3.$

$g \circ g(4) = g(g(4)) = g(5) = 4.$

$g \circ g(5) = g(g(5)) = g(4) = 5.$

$\therefore$ $g \circ g = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}.$

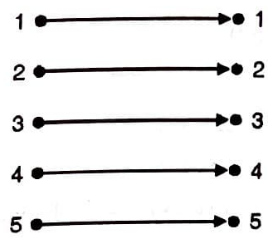

Fig. Ex. 2.11.16(b)

g o g can be depicted as shown in Fig. Ex. 2.11.16(b).

g o g is one to one and onto function

$(g \circ (g \circ g))(1) = g((g \circ g)(1)) = g(1) = 2.$

$(g \circ (g \circ g))(2) = g((g \circ g)(2)) = g(2) = 1.$

$(g \circ (g \circ g))(3) = g((g \circ g)(3)) = g(3) = 3.$

$(g \circ (g \circ g))(4) = g((g \circ g)(4)) = g(4) = 5.$

$(g \circ (g \circ g))(5) = g((g \circ g)(5)) = g(5) = 4.$

$\therefore$ $g \circ (g \circ g) = \{(1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\}.$

g o (g o g) can be depicted as as shown in Fig. Ex. 2.11.16(c).

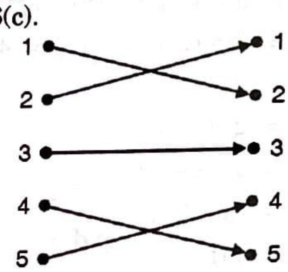

Fig. Ex. 2.11.16(c)

g o (g o g) is one to one and onto function.

Ex. 2.11.17 : Let $f(x) = x + 2$, $g(x) = x - 2$, and $h(x) = 3x$ for $x \in R$, where R = set of real numbers. Find g o f, f o g, f o f, g o g, f o h, h o g, h o f, f o h o g.

Soln. :

$g \circ f(x) = g(f(x))$
$= g(x + 2)$
$= (x + 2) - 2$
$= x.$

$f \circ g(x) = f(g(x))$
$= f(x - 2)$
$= (x - 2) + 2$
$= x.$

$f \circ f(x) = f(f(x))$
$= f(x + 2)$
$= x + 2 + 2$
$= x + 4.$

$g \circ g(x) = g(g(x))$
$= g(x - 2)$
$= (x - 2) - 2$
$= x - 4.$

$f \circ h(x) = f(h(x))$
$= f(3x)$
$= 3x + 2.$

$h \circ g(x) = h(g(x))$
$= h(x - 2) = 3(x - 2)$
$= 3x - 6.$

$h \circ f(x) = h(f(x))$
$= h(x + 2)$
$= 3(x + 2)$
$= 3x + 6.$

$f \circ h \circ g(x) = f \circ h(g(x))$
$= f \circ h(x - 2)$
$= f(h(x - 2))$
$= f(3x - 6)$
$= 3x - 6 + 2$
$= 3x - 4.$

Ex. 2.11.18 : Let $f(x) = 2x + 3$, $g(x) = 3x + 4$, $h(x) = 4x$ for $x \in R$, where R = set of real numbers.
Find g o f, f o g, f o h, h o f, g o h.

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Soln. :

(i) $g \circ f(x) = g(f(x))$
$= g(2x + 3)$
$= 3(2x + 3) + 4$
$= 6x + 13$

(ii) $f \circ g(x) = f(g(x))$
$= f(3x + 4)$
$= 2(3x + 4) + 3$
$= 6x + 11$

(iii) $f \circ h(x) = f(h(x))$
$= f(4x)$
$= 2(4x) + 3$
$= 8x + 3$

(iv) $h \circ f(x) = h(2x + 3)$
$= 4(2x + 3)$
$= 8x + 12$

(v) $g \circ h(x) = g(4x)$
$= 3(4x) + 4$
$= 12x + 4$

Ex. 2.11.19 : If f (x) = $x^2 + 1$ and g (x) = x + 2 are functions from R to R, where R is the set of real numbers, find f o g and g o f.

Soln. :

(i) $$f \circ g(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$$

(ii) $$g \circ f(x) = g(f(x)) = g(x^2+1) = (x^2+1) + 2 = x^2 + 3$$

Ex. 2.11.20 : Let f (x) = ax + b and g(x) = cx + d where a, b, c, d are constants. Determine for which constants a, b, c, d it is true that f o g = g o f.

Soln. : $$f \circ g(x) = f(g(x)) = f(cx+d) = a(cx+d)+b. = acx + ad + b$$

$$g \circ f(x) = g(f(x)) = g(ax+b) = c(ax+b)+d = acx + cb + d.$$

$$f \circ g = g \circ f \Rightarrow acx + ad + b = acx + cb + d$$

$$ad + b = cb + d$$

$$ad - d = cb - b$$

$$d(a-1) = b(c-1).$$

i.e. $\frac{b}{d} = \frac{a-1}{c-1}$ is the relation between the constants if f o g = g o f.

Ex. 2.11.21 : f : R → R is defined as f (x) = x^3.g : R → R is defined as g (x) = $4x^2 + 1$.

h : R → R is defined as h (x) = 7x – 2.

Find the rule of defining as (h o g) of g o (h o f)

Soln. :

We have $$h \circ f = h[f(x)] = h(x^3) = 7(x^3) - 2 = 7x^3 - 2$$

$$\therefore \quad g \circ (h \circ f) = g[h \circ f] = g[7x^3 - 2] = 4(7x^3-2)^2 + 1$$

Now, $$h \circ g(x) = h[g(x)] = h[4x^2 + 1] = 7(4x^2+1) - 2 = 28x^2 + 5$$

Hence (h o g) of g o (h o f) (x) $= 28[4(7x^3-2)^2 + 1]^2 + 5$

Ex. 2.11.22 : If f and g be the functions from set of integers to set of integers defined by.

f (x) = 2 x + 3.

g (x) = 3 x + 2.

Find (f o g) and (g o f).

Soln. :

We have $$f(x) = 2x + 3.$$
$$g(x) = 3x + 2.$$

(i) $$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x + 4 + 3 = 6x + 7.$$

(ii) $$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x + 9 + 2 = 6x + 11.$$

Ex. 2.11.23 : Let A = B = C = R and Let f : A → B and g : B → C be defined by f (a) = a – 1, g (b) = b^2. Find : (i) (f o g) (2) (ii) (g o f) (x) (iii) (f o f) (y) (iv) (g o f) (2) (v) (g o g) (y)

Soln. :

Given : $f(a) = a - 1$ and $g(b) = b^2$

(i) $$(f \circ g)(2) = f(g(2)) = f(4) = 4 - 1 = 3.$$

(ii) $$(g \circ f)(x) = g(f(x)) = g(x-1) = (x-1)^2.$$

(iii) $$(f \circ f)(y) = f(f(y)) = f(y-1) = (y-1)-1 = y - 2.$$

(iv) $$(g \circ f)(2) = g(f(2)) = g(2-1) = g(1) = 1.$$

(v) $$(g \circ g)(y) = g(g(y)) = g(y^2) = y^4.$$

Ex. 2.11.24 : Suppose that A is non empty set, and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f (y). Show that R is an equivalence relation on A.

Soln. :

Here $R = \{(x, y) \mid f(x) = f(y)\}$ ... $x \cdot y \in A$

For $x \in A$

For ordered pair (x, x) each $f(x) = f(y)$

$\therefore$ $x R x$ for all $x \in A$

$\therefore$ R is Reflexive

Also for ordered pair (x, y)

as $f(x) = f(y)$

$\Rightarrow$ $f(y) = f(x)$

$\Rightarrow$ $(y, x) \therefore y R x$

$\therefore$ R is symmetric

Now for (x, y) and (y , z)

$f(x) = f(y)$ and $f(y) = f(z)$

$\Rightarrow$ $f(x) = f(z)$

$\Rightarrow (x, z)$ $\therefore x R z$

$\therefore$ R is Transitive

$\therefore$ R is an equivalence Relation.

Ex. 2.11.25 : Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in R$. (R is the set of real number) Find (i) f o g o h (ii) h o g o f (iii) f o f o f **MU - May 14**

Soln. :

(i) $f \circ g \circ h(x) = f(g \circ (h(x)))$

$= f(g \circ (3x))$

$= f((3x) - 2)$

$= (3x - 2) + 2 = 3x$

(ii) $h \circ g \circ f(x) = h(g(f(x)))$

$= h(g(x + 2))$

$= h((x + 2) - 2)$

$= h(x)$

$= 3x$

(iii) $f \circ f \circ f = f(f(f(x)))$

$= f(f(x + 2))$

$= f((x + 2) + 2)$

$= f(x + 4) = (x + 4) + 2 = x + 6$

Ex. 2.11.26 : Let $f : R \to R$ defined as $f(x) = x^3$ and $g : R \to R$ defined as $g(x) = 4x^2 + 1$ Find out $g \circ f$, $f \circ g$, f^2, g^2. **MU - Dec. 17**

Soln. :

$g \circ f = g(f(x)) = g(x^3) = [4x^6+1]$

$f \circ g = f(g(x)) = f(4x^2 + 1) = (4x^2 + 1)^3$

$f \circ f = f(f(x)) = f(x^3) = x^9$

$g \circ g = g(g(x)) = g(4x^2 + 1) = 4(4x^2 + 1)^2 + 1$

2.12 Identity

Definition

Let A be a non-empty set. Then we can always define a function

$f : A \to A$ (i. e. B = A) as $f(a) = a$ for all $a \in A$

f is called the **Identify** function on A and is denoted by I_A.

$\therefore$ $I_A = \{(a,a) \mid a \in A\}$

Example

Let A = { 1, 2, 3 } and $f : A \to A$ is identify function since

$f(1) = 1$ $f(2) = 2$ $f(3) = 3$

2.13 Inverse Function

The concept of inverse of a function is a analogous to that of the converse of a relation.

2.13.1 Definition

Let $f : A \to B$, be function, then

$f^{-1} : B \to A$ is called the **inverse** mapping of f

f^{-1} is the set defined as

$f^{-1} = \{(b, a) \mid (a, b) \in f\}$.

Note : (i) In general the inverse f^{-1} of a function $f : A \to B$, need not be a function. It may be a relation.

Let $f : A \to A$. If there, exists a function $g : A \to A$ such that g o f = f o g = IA, then g is called the inverse of the function f and is denoted by f^{-1}, read as " f inverse".

Let $f : A \to A$ be such that f(a) = b. Then when it exists f^{-1} is a function from A to A. Such that $f^{-1}(b) = a$. Note that f^{-1} "undoes" what f does.

(ii) If $f : A \to B$ is a bijection and f(a) = b, then $a = f^{-1}(b)$ where $a \in A$ and $b \in B$

Example

Let A = { 1, 2, 3 } and f be the function defined on A such that f(1) = 2, f (2) = 3, f(3) = 1. Then $f^{-1} : A \to A$ is defined by

$f^{-1}(1) = \{3\}$.