CHAPTER 7

Recurrence Relations

1. Introduction

We know that a sequence is an ordered set of objects *i.e.*, there is a first object, then the second object, then the third object and so on. If we have a sequence of numbers then the numbers are arranged in such a way that there is the first number, the second number, the third number and so on. For example, consider the following sequence

$$\{ 2^1, 2^2, 2^3, 2^4, \ldots \}$$

Here the first term is 2, the second is second power of 2, the third is third power of 2. Thus, the general term or the n th term can be stated as n th power of 2 *i.e.*, 2^n.

But the same sequence can be written in another form as follows :

$$\{ 2, 2 \cdot 2^1, 2 \cdot 2^2, 2 \cdot 2^3, \ldots, 2 \cdot 2^{n-1}, \ldots \}$$

The first term is 2, the second is double of the first, the third is double of the second. In general, any term is double of the previous term *i.e.*, $a_n = 2a_{n-1}$. Thus, the sequence is completely known if we write

$$a_n = 2a_{n-1}, \quad n \geq 2 \text{ and } a_1 = 2$$

Such a relation which expresses a term in terms of its previous term or terms with initial conditions is called a recurrence relation.

Definition : Given a sequence $\{a_n\}$, an equation which gives a relation between its n th term a_n with its previous terms $a_0, a_1, \ldots, a_{n-1}$ where $a_0, a_1, \ldots, a_{n-1}$ are given explicitly is called a **recursive relation**. The terms $a_0, a_1, \ldots, a_{n-1}$ which are given explicitly are called **initial conditions** or **boundary conditions**.

Example 1 : In the arithmetic progression $\{ 5, 9, 13, 17, \ldots \}$

We see that each term is greater by 4 than the previous term and the first term is 5.

Its recurrence relation is $a_n = a_{n-1} + 4, \ a_1 = 5, \ n \geq 2$.

Example 2 : In the geometric progression

$$\{ 5, 5^2, 5^3, \ldots \} \ i.e., \ \{ 5, 5 \cdot 5, 5 \cdot 5^2, 5 \cdot 5^3, \ldots \}$$

We see that each term is 5 times its previous term and the first term is 5.

Its recurrence relation is $a_n = 5a_{n-1}, \ a_1 = 5, \ n \geq 2$.

Example 3 : In Fibonacci sequence $\{ 1, 1, 2, 3, 5, 8, 13, \ldots \}$

We see that each term is not related to the previous term as above but is related to two previous terms. Each term is the sum of two preceding terms *i.e.*, a_n is equal to $a_{n-1} + a_{n-2}$ and the first two terms are 1 and 1.

Its recurrence relation is $a_n = a_{n-1} + a_{n-2}, \ a_1 = 1, \ a_2 = 1, \ n \geq 3$.

It may be noted that in the first two examples only one previous term is enough to give the general term while in the third example we need two previous terms. We may need three or even more terms to specify the general term. It may also be noted that with change in the initial condition or conditions, we get different sequences from the same relation. In the first example, if $a_1 = 4$ then we get the sequence { 4, 8, 12, 14, 18, ... }, if $a_1 = 6$ we get the sequence { 6, 10, 14, ... }. In the second example, if $a_1 = 3$, ($a_n = 3a_{n-1}$, $a_1 = 3$, $n \geq 2$) we get the sequence $\{3, 3 \cdot 3, 3 \cdot 3^2, 3 \cdot 3^3,$ .... } and if $a_1 = 7$, ($a_n = 7a_{n-1}$, $a_1 = 7$, $n \geq 2$), we get the sequence $\{7, 7 \cdot 7, 7 \cdot 7^2, 7 \cdot 7^3,$ .... }.

Example 4 : Find recursive relation for the factorial function $n\,!$.

Sol. : We know that when $n = 0$, $0! = 1$ by definition.

$\therefore \quad a_0 = 1$

$a_1 = 1\,! = 1$

$a_2 = 2\,! = 2 \times 1 = 2 \cdot a_1$

$a_3 = 3\,! = 3 \times 2\,! = 3 \cdot a_2$

$a_4 = 4\,! = 4 \times 3\,! = 4 \cdot a_3$

....

....

$\therefore$ The sequence {0 !, 1 !, 2 !, 3 !, ...} can be given as

$$a_n = n\, a_{n-1}, \quad n \geq 1, \; a_0 = 1.$$

2. To find the terms when a Recurrence Relation is given

We have seen above how to obtain the recurrence relation when a sequence is given. We shall now consider the reverse problem of finding the sequence *i.e.* the terms when the recurrence relation is given. To find the sequence, we put $n = 1, 2, 3,$ ..... successively in the recurrence relation and use the initial conditions.

Example : Find the sequences from the following recurrence relations

(i) $a_n = n\,(a_{n-1})^2, \; a_0 = 1, \; n \geq 1$ (ii) $a_{n+1} = 3a_n + n, \; a_1 = 1, \; n \geq 1$

(iii) $a_n = 2a_{n-1} + 3a_{n-2}, \; a_0 = 2, \; a_1 = 3, \; n \geq 2$

Sol. : (i) Putting $n = 1, 2, 3,$ ..... successively and using $a_0 = 1$

$a_1 = 1 \cdot a_0^{\,2} = 1$

$a_2 = 2 \cdot (a_1)^2 = 2 \cdot 1^2$

$a_3 = 3 \cdot (a_2)^2 = 3 \cdot 2^2$

$a_4 = 4 \cdot (a_3)^2 = 4 \cdot 3^2$

....

....

$\therefore$ The sequence is $\{1, \; 2 \cdot 1^2, \; 3 \cdot 2^2, \; 4 \cdot 3^2, \; .....\}$

(ii) Putting $n = 1, 2, 3,$ ..... successively and using $a_1 = 1$

$a_2 = 3a_1 + 1 = 3\,(1) + 1 = 4$

$a_3 = 3a_2 + 2 = 3\,(4) + 2 = 14$

$a_4 = 3a_3 + 3 = 3\,(14) + 3 = 45$

....

....

$\therefore$ The sequence is { 1, 4, 14, 45, ... }

(ii) Putting $n = 1, 2, 3, ...$ successively and using $a_0 = 2$, $a_1 = 3$.

$$a_2 = 2(a_1) + 3(a_0) = 2(3) + 3(2) = 12$$
$$a_3 = 2(a_2) + 3(a_1) = 2(12) + 3(3) = 33$$
$$a_4 = 2(a_3) + 3(a_2) = 2(33) + 3(12) = 102$$

$\therefore$ The sequence is $\{2, 3, 12, 33, 102, ...\}$.

3. Applications of Recursive Relations

There are many situations where we use recursive relations.

Example 1 (Compound Interest) : A person deposits ₹ 10,000 in a bank at the rate of interest 8% compounded annually. Define recursively the amount A_n (compounded) at the end of n years.

Sol. : Clearly A_0 = initial deposit = 10,000

If $n \geq 1$, we see that

$$A_n = [\text{ Amount at the end of } (n-1) \text{ years }] + [\text{ Interest during the } n \text{ th year }]$$
$$A_n = A_{n-1} + (0{\cdot}08)\, A_{n-1} = (1{\cdot}08)\, A_{n-1}$$

Thus, we have the recursive relation.

$$A_n = 1{\cdot}08\, A_{n-1}, \quad A_0 = 10{,}000.$$

For instance the amount at the end of three years is obtained by putting $n = 3$.

$$\therefore A_3 = 1{\cdot}08\, A_2$$

But $A_2 = 1{\cdot}08\, A_1$ and $A_1 = 1{\cdot}08\, A_0$.

Hence, $A_2 = (1{\cdot}08)^2 A_0$, $\quad A_3 = (1{\cdot}08)^2 A_0$

$$\therefore \quad A_3 = (1{\cdot}08)^3\, 10{,}000 = 1559{\cdot}71.$$

Example 2 : (Tower of Brahma) : According to a legend, there is a temple of Brahma in Varanasi in India in which there is a brass platform having three diamond pegs x, y, z and one of them is having 64 golden discs. God asked the priest of the temple to transfer the discs from the peg x to peg z with the help of the peg y under the following conditions.

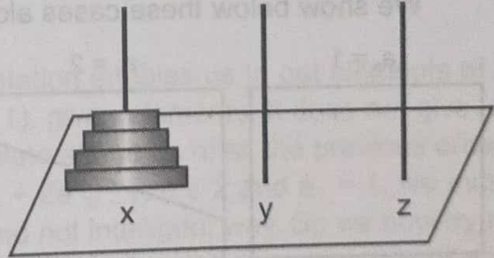

Fig. 7.1

(a) Only one disc is to be moved at a time.

(b) No disc can be placed on a smaller disc.

Suppose there are n discs on the peg x. Suppose b_n denotes the number of moves required to transfer n discs from the peg x to the peg z using the auxiliary peg y. Express b_n as a recursive relation.

Sol. : If there is only one disc, clearly we can transfer it to the peg z in one move and so $b_1 = 1$. If $n \geq 2$, by definition, we require b_{n-1} moves to transfer $(n-1)$ disc from x to z using y as auxiliary peg. This means the largest disc is on the peg x. We need one move to transfer this (largest disc) to the peg z which is empty.

Now, $(n-1)$ discs at the peg Y can be transferred to the peg Z using X as auxiliary peg in b_{n-1} moves. So the total number of moves required to transfer $(n-1)$ discs from the peg X to the peg Y is $b_{n-1} + 1 + b_{n-1} = 2b_{n-1} + 1$.

Thus, we have $\quad b_n = \begin{cases} 1, & \text{if } n = 1 \\ 2b_{n-1} + 1, & n \geq 2 \end{cases}$

Note ....

It is easy to see that to transfer 1 disc we need 1 move, to transfer 2 discs we need 2^1 moves and so on. Assuming that the priest knows this job well *i.e.* he does not commit any mistake and assuming that he transfers one disc in one second the time required to transfer all 64 disc is given by

$$\text{Total Time} = 1 + 2 + 2^2 + \ldots\ldots + 2^{64}$$

This is a geometric progression with first term unity and common ratio 2,

$$\therefore \quad \text{Sum} = \frac{2^{64} - 1}{2 - 1} = 2^{64} - 1 \text{ seconds}$$

This time is found to be much less than the life of the universe, calculated by modern scientists.

Example 3 : Find the recurrence relation of the number of regions into which a plane is decomposed by n straight lines, no two of which are parallel and no three of which are concurrent.

Sol. : Let n lines no two of which are parallel and no three of which are concurrent divide a plane into a_n number of regions.

For small values of n, by drawing the lines (under the given conditions) we can count the number of regions a_n.

If no line is drawn *i.e.* $n = 0$, there is only one region, the region of the plane

$\therefore \quad a_n = 1.$

If one line is drawn, the plane is divided into two regions *i.e.*, if $n = 1$, $a_n = 2$. (The lines are numbered in the following figures.)

If two lines are drawn the plane is divided into four regions *i.e.*, if $n = 2$, $a_n = 4$.

If three lines are drawn the plane is divided into seven regions *i.e.*, if $n = 3$, $a_n = 7$.

If four lines are drawn *i.e.*, if $n = 4$, $a_n = 11$.

We show below these cases along with the lines in the following diagrams.

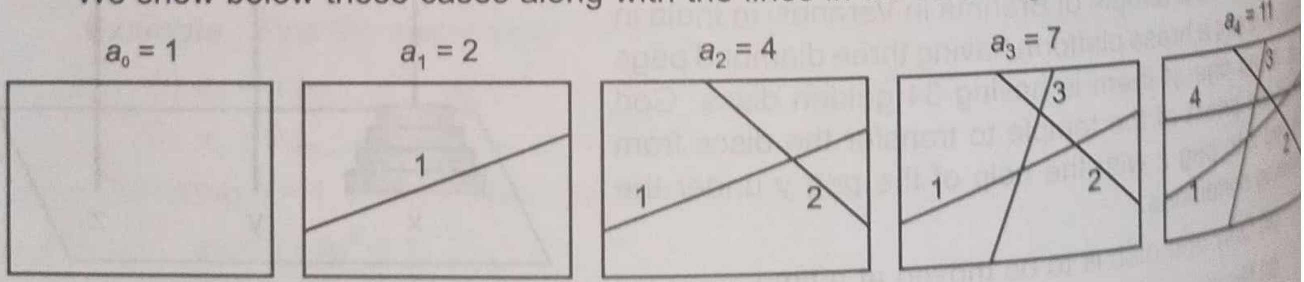

Fig. 7.2

If we observe these values carefully, we may see the following pattern.

$$a_0 = 1$$
$$a_1 = 2 = 1 + 1 = a_0 + 1$$
$$a_2 = 4 = 2 + 2 = a_1 + 2$$
$$a_3 = 7 = 4 + 3 = a_2 + 3$$
$$a_4 = 11 = 7 + 4 = a_3 + 4$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\therefore \quad a_n = \ldots\ldots = a_{n-1} + n$$

with initial conditions $a_0 = 1$ and $a_1 = 2$.

EXERCISE - I

Compute four more terms of the sequence defined by each of the following recursive relation satisfying the initial conditions.

1. $a_n = a_{n-1}^2, \quad n \geq 2$
   $a_1 = 2$
2. $a_n = n\,(a_{n-1})^2, \quad n \geq 2$
   $a_1 = 1$
3. $a_n = 2a_{n-1} + 3, \quad n \geq 2$
   $a_1 = 1$
4. $a_n = a_{n-1} + n, \quad n \geq 2$
   $a_1 = 1$
5. $a_n = 2^n a_{n-1}, \quad n \geq 2$
   $a_1 = 1$
6. $a_n = \left(\frac{n}{n+1}\right) a_{n-1}, \quad n \geq 2$
   $a_1 = 1$
7. $a_n = 2a_{n-1} + 3a_{n-2}, \quad n \geq 2$
   $a_0 = 1, \; a_1 = 1$
8. $a_n = n\,a_{n-1} + n^2 a_{n-2}, \quad n \geq 2$
   $a_0 = 1, \; a_1 = 1$
9. $a_n = a_{n-1} + a_{n-2} + a_{n-3}, \quad n \geq 4$
   $a_1 = 1, \; a_2 = 2, \; a_3 = 3$
10. $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}, \quad n \geq 4$
    $a_1 = 1, \; a_2 = 1, \; a_3 = 1$

[**Ans.:** **(1)** $2, 2^2, 4^2, 16^2, 256^2$; **(2)** 1, 2, 12, 576, $5\,(576)^2$; **(3)** 1, 5, 13, 29, 61;
(4) 1, 3, 6, 10, 15; **(5)** $1, 2^2, 2^5, 2^9, 2^{14}$; **(6)** $1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}$;
(7) 1, 1, 5, 13, 41, 121; **(8)** 1, 1, 6, 27, 204, 1695; **(9)** 1, 2, 3, 6, 11, 20, 37;
(10) 1, 1, 1, 6, 11, 26, 66.]

4. Solving Recurrence Relations

As is clear from the above discussion, a recurrence relation enables us to get elements of a sequence successively from the first, two or three, ... $(n-1)$, given elements. It does not give us the element of any desired order. For this, we need to calculate elements of all the previous order. For example, if we want 100th element of the sequence $a_n + 2a^2_{n-1}$, $n \geq 2$ and $a_1 = 1$, we must calculate all the previous 99 terms. This is clearly tedious and not intelligent way. So we now try to obtain a formula for a_n in terms of n which will enable us to get an element of a sequence of any order. This is called **solving a recurrence relation**.

Definition : To obtain an explicit formula for a_n in terms of n. From a recurrence relation is called **solving a recurrence relation.**

There are no standard rules to solve a recurrence relation. But the following three methods are given below.

(1) Iteration, **(2) Characteristic Roots,** **(3) Generating Functions.**

Iterative Method

The iterative method of solving a recurrence relation involves the following two steps.

1. Apply the recurrence relation repeatedly and see if you find a pattern for explicit formula.
2. Use the method of induction to verify that the formula gives all the terms of the sequence correctly.

Example 1 : Solve the recurrence relation $a_n = a_{n-1} + 2,\ n \geq 2,\ a_1 = 3.$ • (M.U. 2013)

Sol. : We backtrack the value of a_n by substituting in the definition the values of $a_{n-1}, a_{n-2}, \ldots$ and so on. At some place, pattern can be clear as illustrated below.

$$\begin{aligned} a_n &= a_{n-1} + 2 & \therefore\ a_{n-1} &= (a_{n-2} + 2) + 2 \\ &= a_{n-2} + 2 \cdot 2 & \therefore\ a_{n-2} &= (a_{n-3} + 2 \cdot 2) + 2 \\ &= a_{n-3} + 3 \cdot 2 \end{aligned}$$

Ultimately, we will get

$$\begin{aligned} a_n &= a_{n-(n-1)} + (n-1) \cdot 2 \\ &= a_1 + (n-1)\, 2 \\ &= a_1 + 2n - 2. \qquad \text{But } a_1 = 3 \\ &= 2n + 1 \end{aligned}$$

This is the required formula.

Alternatively, we may argue as follows.

$$\begin{aligned} a_n &= a_{n-1} + 2 \\ a_{n-1} &= a_{n-2} + 2 \\ a_{n-2} &= a_{n-3} + 2 \\ &\cdots\cdots\cdots \\ a_4 &= a_3 + 2 \\ a_3 &= a_2 + 2 \\ a_2 &= a_1 + 2 \\ a_1 &= 3 \end{aligned}$$

Adding vertically, (since 2 are added $(n-1)$ times)

$$\begin{aligned} a_n &= 3 + (2 + 2 + \ldots + 2)\ (n-1) \text{ times} \\ &= 3 + 2\,(n-1) \\ &= 2n + 1 \end{aligned}$$

Example 2 : Solve the following recursive relation,

$$a_n = a_{n-1} + n, \qquad n \geq 2,\ a_1 = 1$$

Sol. : We write a few terms and see if there is any pattern.

$$\begin{aligned} a_n &= a_{n-1} + n, \\ a_{n-1} &= a_{n-2} + (n-1) \\ a_{n-2} &= a_{n-3} + (n-2) \\ &\cdots\cdots\cdots \\ a_3 &= a_2 + 3 \\ a_2 &= a_1 + 2 \end{aligned}$$

Adding vertically, we find that

$$\begin{aligned} a_n &= a_1 + 2 + 3 + 4 + \ldots + (n-1) + n \\ &= 1 + 2 + 4 + \ldots + n \qquad [\because\ a_1 = 1] \\ &= \frac{n}{2}(n+1) \end{aligned}$$

Example 3 : Solve the following recursive relation

$$a_n = 2a_{n-1} + 1,\ n \geq 2,\ a_1 = 3.$$

Sol. : We use iteration and write a few terms.

$$\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
&= 2\,[2a_{n-2} + 1] + 1 \\
&= 2 \cdot 2 \cdot a_{n-2} + 2 + 1 \\
&= 2 \cdot 2 \cdot [\,2a_{n-3} + 1\,] + 2 + 1 \\
&= 2 \cdot 2 \cdot 2\, a_{n-3} + 2 \cdot 2 + 2 + 1 \\
&= 2^3\, a_{n-3} + 2^2 + 2^1 + 1
\end{aligned}$$

We have to continue this process upto a_1 which is given. Carefully observe the pattern that is revealing.

Before we generalise we write one more term

$$a_n = 2^4\, a_{n-4} + 2^3 + 2^2 + 2^1 + 1$$

We want $n - 4 = 1$ *i.e.* $n - (n - 1) = 1$

Hence, replacing 4 by $n - 1$,

$$\begin{aligned}
a_n &= 2^{n-1} \cdot a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \ldots\ldots + 2^3 + 2^2 + 2 + 1 \\
&= 2^{n-1} \cdot a_1 + [\,2^{n-2} + 2^{n-3} + \ldots\ldots + 2^3 + 2^2 + 2^1 + 1\,]
\end{aligned}$$

But the bracketed terms form a Geometric Series with first term 1, common ration 2 and the number of terms $(n - 1)$.

$$\text{Its sum} = a\frac{(1-r^n)}{(1-r)} = 1 \cdot \frac{(1-2^{n-1})}{1-2} = 2^{n-1} - 1$$

$$\begin{aligned}
\therefore \quad a_n &= 2^{n-1} \cdot a_1 + 2^{n-1} - 1 \qquad [\text{ But } a_1 = 3\,] \\
&= 2^{n-1} \cdot 3 + 2^{n-1} - 1 \\
&= 2^{n-1}\,[3 + 1] - 1 \\
&= 2^{n-1} \cdot 2^2 - 1 = 2^{n-1} - 1
\end{aligned}$$

[Referring again to the problem of Tower of Brahma if there are 64 discs and that is what the legend goes then the number of transfers required to transfer all the discs from the peg x to th peg $z = 2^{64} - 1$.

Assuming that the priest commits no mistake and assuming that he takes one second for one transfer,

Time taken $= 2^{64} - 1$ seconds $= 1{\cdot}84 \times 10^{11}$ seconds

Dividing this number by the number of seconds $365 \times 24 \times 60 \times 60$ in a year, we get

Time taken = 600 billion years approximately.

According to some estimates the life of the universe is just 18 billion years.]

Example 4 : Solve the following recursive relation

$$a_n = a_{n-1} + (n-1), \qquad n \geq 2,\ a_1 = 0$$

Sol. : We use iteration and try to find a pattern

$$\begin{aligned}
a_n &= a_{n-1} + (n-1) \\
&= a_{n-2} + (n-2) + (n-1) \\
&= a_{n-3} + (n-3) + (n-2) + (n-1)
\end{aligned}$$

We want to continue this process upto a_1 which is given, we want $n-3=1$

$$\therefore \quad a_n = a_1 + 1 + 2 + 3 + \ldots + (n-2) + (n-1)$$
$$= 0 + 1 + 2 + 3 + \ldots + (n-1) \qquad [\because a_1 = 0]$$

But this is the sum of $(n-1)$ natural numbers.

$$\therefore \quad a_n = \frac{(n-1)[(n-1)+1]}{2} = \frac{n(n-1)}{2}.$$

(Or you may use the alternative method shown in Ex. 1. Try it.)

Example 5 : Solve the following recursive relation

$$\therefore \quad a_n = a_{n-1} + \frac{n(n+1)}{2}, \quad n \geq 1, \; a_0 = 0$$

Sol. : We use iteration and try to find a pattern.

$$a_n = a_{n-1} + \frac{n(n+1)}{2}$$
$$= a_{n-2} + \frac{(n-1)[(n-1)+1]}{2} + \frac{n(n+1)}{2}$$
$$= a_{n-2} + \frac{(n-1)(n)}{2} + \frac{n(n+1)}{2}$$
$$\therefore \quad a_n = a_{n-3} + \frac{(n-2)(n-1)}{2} + \frac{(n-1)(n)}{2} + \frac{(n)(n+1)}{2}$$

We want to continue this process upto a_1 which is given. As in the previous example we want $a_{n-3} = a_1$

$$\therefore \quad a_n = a_0 + \frac{(1)(2)}{2} + \frac{(2)(3)}{2} + \frac{(3)(4)}{2} + \ldots + \frac{n(n+1)}{2}$$
$$= \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \ldots + \frac{n(n+1)}{2} \qquad [\because a_0 = 0]$$
$$= \frac{1}{2}\sum_{r=1}^{n} r(r+1) = \frac{1}{2}\left[\sum_{1}^{n} r^2 + \sum_{1}^{n} r\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$
$$= \frac{1}{2}\left[\frac{n(n+1)}{6}[2n+1+3]\right] = \frac{1}{2} \cdot \frac{n(n+1)}{6}(2n+4)$$
$$= \frac{n(n+1)(n+2)}{6}, \quad n \geq 0$$

EXERCISE - II

Solve the following recurrence relations.

1. $a_n = 3a_{n-1}, \quad n \geq 1, \; a_0 = 1$ [**Ans. :** $a_n = 3^n, n \geq 1$]
2. $a_n = a_{n-1} + n, \quad n \geq 1, \; a_0 = 1.$ [**Ans. :** $a_n = \frac{n}{2}(n+1), n \geq 1$]
3. $a_n = a_{n-1} + 4n, \quad n \geq 1, \; a_0 = 0$ [**Ans. :** $a_n = 3n(n+1), n \geq 0$]
4. $a_n = a_{n-1} + n^2, \quad n \geq 2, \; a_1 = 1$ [**Ans. :** $a_n = \frac{n}{6}(n+1)(2n+1), n \geq 1$]
5. $a_n = a_{n-1} + n^3, \quad n \geq 2, \; a_1 = 1.$ [**Ans. :** $a_n = \frac{n^2}{4}(n+1)^2, n \geq 1$]

5. Linear Homogeneous Recurrence Relations with Constant Coefficients

(M.U. 2008, 09)

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ where $c_1, c_2, ..., c_k$ are real constants, $c_k \neq 0$ is called a **k th order linear homogeneous recurrence relation with constant coefficients.** (LHRRWCCs)

For example, $a_n = 2a_{n-1} + 3a_{n-2}$

$$a_n = a_{n-1} + 2a_{n-2} + 5a_{n-3}$$

are respectively second and third order LHRRWCCs. Since $a_{n-1}, a_{n-2}, ..., a_{n-k}$ appear in the first degree, the relation is called **linear**. Because **all** $a_{n-1}, a_{n-2}, ..., a_{n-k}$ appear in the same degree (one) the relation is called **homogeneous.** Alternatively we may say that, if each term is some constant multiple of a_i or if we get $a_n = 0$ for $a_{n-1} = 0, a_{n-2} = 0, ..., a_{n-k} = 0$ then the relation is called homogeneous. All the coefficients $c_1, c_2, ..., c_k$ are real constants.

Thus, $a_n = 2a_{n-1} + 3$, $a_n = 2a_{n-1} + 3a_{n-2} + 4$ are not homogeneous because of the terms 3, 4. $a_n = a_n^2 + 2a_{n-1}$ also is not linear because of the term a_n^2 which is of the second degree.

6. Method of Characteristic Roots

The basic approach for solving homogeneous recurrence relation $f(n) = 0$ is to look for the solutions of the form $a_n = r^n$, which is obtained as follows.

Let the given relation be $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

$a_n = r^n$ will be a solution of this relation if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + ... + c_k r^{n-k}.$$

Dividing by r^{n-k}, we get

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + ... + c_k r^0$$

$$\therefore \quad r^k - c_1 r^{k-1} - c_2 r^{k-2} - ..... - c_k = 0$$

This is called the **characteristic equation** of the recurrence relation. The roots of this equation are called **characteristic roots**.

(a) **Real Distinct Roots**

If the roots are **distinct** say $r_1, r_2, ..., r_k$ then .... (A)

$$a_n = b_1 r_1^r + b_2 r_2^r + ... + b_k r_k^r$$

can be a solution where $b_1, b_2, ..., b_k$ are constants which satisfy initial conditions.

Example 1 : Solve the following recurrence relation

$$a_n = 5a_{n-1}, \qquad n \geq 0, \ a_0 = 1.$$

Sol. : Let $a_n = r^n$ be a solution. Then the characteristic equation is

$$r^n = 5r^{n-1} \qquad \therefore \ r = 5. \qquad ....(1)$$

Hence, the general solution is $\quad a_n = b_1 r^n = b_1 5^n$

But by data, when $n = 0$, $a_0 = 1 \quad \therefore \ 1 = b_1 \cdot 5^0 \quad \therefore \ b_1 = 1$

Hence, the solution from (1) is $\quad a_n = b_1 r^n = 1 \cdot 5^n$.

Example 2 : Solve $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 2$, $a_0 = 0$, $a_1 = 1$.

Sol. : The given recurrence relation is

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{i.e.,} \quad a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \text{.......... (1)}$$

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^n - 5r^{n-1} + 6r^{n-2} = 0 \qquad \therefore \; r^{n-2}(r^2 - 5r + 6) = 0$$

$$\therefore \; r^2 - 5r + 6 = 0 \qquad \therefore \; (r-3)(r-2) = 0 \qquad \therefore \; r = 2, 3.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1\, 2^n + b_2\, 3^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$ and $a_0 = 0$, $\quad b_1 + b_2 = 0$.

Putting $n = 1$ and $a_1 = 1$, $\quad 2b_1 + 3b_2 = 1$.

Solving the two equations, we get $b_1 = -1$, $b_2 = 1$.

Hence, the explicit solution of the given recurrence relation from (2) is

$$a_n = (-1)\,2^n + 3^n, \; n \geq 2, \; a_0 = 0, \; a_1 = 1.$$

Example 3 : Solve $a_{r+2} + 2a_{r-1} - 3a_r = 0$ that satisfies $a_0 = 1$, $a_1 = 2$. **(M.U. 2002, 16)**

Sol. : The given recurrence relation is

$$a_n + 2a_{n-1} - 3a_{n-2} = 0 \quad \text{.......... (1)}$$

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

$$\therefore \; r^n + 2r^{n-1} - 3r^{n-2} = 0 \qquad \therefore \; r^{n-2}(r^2 + 2r - 3) = 0$$

The characteristic equation of (1) is

$$r^2 + 2r - 3 = 0 \qquad \therefore \; (r+3)(r-1) = 0 \qquad \therefore \; r = 1, -3.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot 1^n + b_2(-3)^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = 1$, $\quad b_1 + b_2 = 1$

Putting $n = 1$, $a_1 = 2$, $\quad b_1 - 3b_2 = 2$

Solving the two equations, we get $b_1 = \dfrac{5}{4}$ and $b_2 = -\dfrac{1}{4}$.

Hence, the explicit solution of the given recurrence relation is

$$a_n = \frac{5}{4} - \frac{1}{4}(-3)^n$$

Example 4 : Find the sequence whose recurrence relation is $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_1 = 5$, $a_2 = 3$. **(M.U. 2008)**

Sol. : The given recurrence relation is $a_n - 3a_{n-1} + 2a_{n-2} = 0$ .... (1)

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - 3r + 2 = 0 \qquad \therefore \; (r-2)(r-1) = 0 \qquad \therefore \; r = 1, 2.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot 1^n + b_2 \cdot 2^n$ .... (2)

We now use initial conditions to find b_1 and b_2.

Putting $n = 1$, $a_1 = b_1 + 2b_2 = 5$. Putting $n = 2$, $a_2 = b_1 + 4b_2 = 3$.

Solving these equations, we get $b_2 = -1$ and $b_1 = 7$.

Hence, the explicit solution of the recurrence relation is

$$a_n = 7 \cdot 1^n - 2^n \quad i.e., \quad a_n = 7 - 2^n.$$

Example 5 : Sole the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with initial conditions $a_0 = 1$, $a_1 = 6$. **(M.U. 2016)**

Sol. : The given recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$ .... (1)

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$$r^2 - 7r + 10 = 0 \quad \therefore \quad (r-2)(r-5) = 0 \quad \therefore \quad r = 2, 5.$$

The roots are real rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot 2^n + b_2 \cdot 5^n$ .... (2)

We now, use initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = b_1 + b_2 = 1$.

Putting $n = 1$, $a_1 = 2b_1 + 5b_2 = 6$

Solving these equations, we get $b_1 = -\frac{1}{3}$ and $b_2 = \frac{4}{3}$.

Hence, the explicit solution of the recurrence relation is

$$a_n = \left(-\frac{1}{3}\right)2^n + \left(\frac{4}{3}\right)5^n$$

Example 6 : Solve the recurrence relation $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 2$ and $a_2 = 6$. **(M.U. 2011)**

Sol. : The given recurrence relation is $a_n - 4a_{n-1} - 5a_{n-2} = 0$ .... (1)

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$$r^2 - 4r - 5 = 0 \quad \therefore \quad (r+1)(r-5) = 0 \quad \therefore \quad r = -1, 5.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot (-1)^n + b_2 \cdot 5^n$ .... (2)

We now, use initial conditions to find b_1 and b_2.

Putting $n = 1$, $a_1 = -b_1 + 5b_2 = 2$.

Putting $n = 2$, $a_2 = b_1 + 25b_2 = 6$

Solving these equations, we get $b_1 = -\frac{2}{3}$ and $b_2 = \frac{4}{15}$.

Hence, the explicit solution of the recurrence relation is

$$a_n = \left(-\frac{2}{3}\right) \cdot (-1)^n + \left(\frac{4}{15}\right) \cdot 5^n$$

Example 7 : Solve the recurrent relation $a_n - 10a_{n-1} + 9a_{n-2} = 0$ with initial conditions $a_0 = 3$, $a_1 = 19$.

Sol. : The given recurrence relation is $a_n - 10a_{n-1} + 9a_{n-2} = 0$ ...... (1)

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$$r^2 - 10r + 9 = 0 \quad \therefore \; (r-1)(r-9) = 0 \quad \therefore \; r = 1, 9.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot 1^n + b_2 \cdot 9^n$ ...... (2)

We now, use initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = b_1 + b_2 = 3$.

Putting $n = 1$, $a_1 = b_1 + 9b_2 = 19$

Solving these equations, we get $b_1 = 1$ and $b_2 = 2$.

Hence, the explicit solution of the recurrence relation is

$$a_n = 1 \cdot (1)^n + 2 \cdot (9)^n.$$

Example 8 : Find the solution of the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$. **(M.U. 2003, 05, 15)**

Sol. : The given recurrence relation is $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$ ...... (1)

This is a third order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^3 - 6r^2 + 11r - 6 = 0 \quad \therefore \; (r-1)(r-2)(r-3) = 0 \quad \therefore \; r = 1, 2, 3.$$

The roots are real, rational and distinct.

Hence, let the general solution be $a_n = b_1 \cdot 1^n + b_2 \cdot 2^n + b_3 \cdot 3^n$

We now use the initial conditions to find b_1, b_2 and b_3.

Putting $n = 0$, $a_0 = 2$, $b_1 + b_2 + b_3 = 2$.

Putting $n = 1$, $a_1 = 5$, $b_1 + 2b_2 + 3b_3 = 5$

Putting $n = 2$, $a_2 = 15$, $b_1 + 4b_2 + 9b_3 = 15$

Solving the three equations, we get $b_1 = 1$, $b_2 = -1$, $b_3 = 2$.

Hence, the explicit solution of the given recurrence relation is

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

Example 9 : Solve $a_n = 2a_{n-1} + a_{n-2}$, for $n \geq 2$, $a_0 = 0$, $a_1 = 1$.

Sol. : The given recurrence relation is

$$a_n = 2a_{n-1} + a_{n-2} \qquad i.e., \qquad a_n - 2a_{n-1} + a_{n-2} = 0 \quad \text{...... (1)}$$

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation is

$$r^2 - 2r - 1 = 0 \quad \therefore \; r = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

(The roots of $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$)

The roots are real, irrational and distinct.

Hence, let the general solution be $a_n = b_1(1+\sqrt{2})^n + b_2(1-\sqrt{2})^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = 0$, $b_1 + b_2 = 0$.

Putting $n = 1$ and $a_1 = 1$, $b_1(1+\sqrt{2}) + b_2(1-\sqrt{2}) = 1$

Solving the two equations, we get $b_2 = -\dfrac{1}{2\sqrt{2}}$ and $b_1 = \dfrac{1}{2\sqrt{2}}$.

Hence, the explicit solution of the given recurrence relation from (2) is

$$a_n = \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n]$$

Example 10 : Solve the Fibonacci sequence relation

$a_n = a_{n-1} + a_{n-2}$, for $n \geq 2$, $a_0 = 0$, $a_2 = 1$. **(M.U. 2002)**

Sol. : The given recurrence relation is

$a_n = a_{n-1} + a_{n-2}$ *i.e.,* $a_n - a_{n-1} - a_{n-2} = 0$ .... (1)

This is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ be a solution of (1).

The characteristic equation is

$$r^2 - r - 1 = 0 \qquad \therefore \quad r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

The roots are real, irrational and distinct.

Hence, let the general solution be

$$a_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_n\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad \text{............... (2)}$$

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = 0$,

$b_1 + b_2 = 1$.

Putting $n = 1$, $a_1 = 1$,

$$b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solving the two equations, we get $b_2 = -\dfrac{1}{\sqrt{5}}$ and $b_1 = \dfrac{1}{\sqrt{5}}$.

Hence, the explicit solution of the given Fibonacci recurrence relation from (2) is,

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(b) Complex Roots

Example 1 : Solve the recurrence relation $a_{n+2} + 9a_n = 0$.

Sol. : The given recurrence relation is $a_{n+2} + 9a_n = 0$. ....(1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is $r^2 + 9 = 0$ $\therefore$ $r = 3i, -3i$.

The roots are imaginary.

Hence, its general solution is $a_n = b_1 (3i)^n + b_2 (-3i)^n$.

Example 2 : Solve the recurrence relation $a_{n+2} - a_{n+1} + a_n = 0$.

Sol. : The given recurrence relation is $a_{n+2} - a_{n+1} + a_n = 0$. ....(1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - r + 1 = 0 \qquad \therefore \quad r = \frac{1 \pm \sqrt{3} \cdot i}{2}$$

The roots are imaginary.

Hence, its general solution is $a_n = b_1 \left(\dfrac{1+\sqrt{3}\,i}{2}\right)^n + b_2 \left(\dfrac{1-\sqrt{3}\,i}{2}\right)^n$

Example 3 : Solve the recurrence relation $a_{n+3} + a_n = 0$.

Sol. : The given recurrence relation is $a_{n+3} + a_n = 0$. ....(1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^3 + 1 = 0 \qquad \therefore \quad (r+1)(r^2 - r + 1) = 0 \qquad \therefore \quad r = -1, \frac{1 \pm \sqrt{3} \cdot i}{2}.$$

One root is real and the remaining two roots are complex.

Hence, its general solution is

$$a_n = b_1 (-1)^n + b_2 \left(\frac{1+\sqrt{3}\,i}{2}\right)^n + b_3 \left(\frac{1-\sqrt{3}\,i}{2}\right)^n$$

(c) Real Repeated Roots (Not Distinct Roots)

If a root α, say, is repeated then the above formula (A) given on page 7-9 needs to be modified.

If α is repeated twice then we assume $a_n = b_1 \alpha^n + b_2 n \alpha^n$.

If α is repeated thrice then we assume $a_n = b_1 \alpha^n + b_2 n \alpha^n + b_3 n^2 \alpha^n$

If α is repeated r times, we generalise the above expression.

Here after we use the earlier procedure. This is illustrated below.

Example 1 : Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 0$.

Sol. : The given recurrence relation is $a_{n+2} - 6a_{n+1} + 9a_n = 0$. ....(1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - 6r + 9 = 0 \qquad \therefore \quad (r-3)^2 = 0 \qquad \therefore \quad r = 3, 3.$$

The roots are real, rational and repeated.

Hence, its general solution is $a_n = (b_1 + n\, b_2) \cdot 3^n$.

Example 2 : Solve the recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ with the initial conditions $a_0 = 8$, $a_1 = 20$.

Sol. : The given recurrence relation is $a_n - 8a_{n-1} + 16a_{n-2} = 0$. .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - 8r + 16 = 0 \qquad \therefore \ (r-4)^2 = 0 \qquad \therefore \ r = 4, 4.$$

The roots are real, rational and repeated.

Hence, its general solution is $a_n = (b_1 + n\, b_2) \cdot 4^n$. .... (2)

We now, use initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = b_1 = 8$ $\therefore \ b_1 = 8$

Putting $n = 1$, $a_1 = (8 + b_2) \cdot 4 = 20$ $\therefore \ b_2 + 8 = 5$ $\therefore \ b_2 = -3$.

Hence, the solution of the given recurrence relation from (2) is

$$a_n = (8 - 3n) \cdot 4^n.$$

Example 3 : Solve the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$ with initial conditions $d_1 = 3/2$, $d_2 = 3$. **(M.U. 2008, 09)**

Sol. : The given recurrence relation is $a_n - 2a_{n-1} + a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - 2r + 1 = 0 \qquad \therefore \ (r-1)^2 = 0 \qquad \therefore \ r = 1, 1.$$

The roots are real, rational but repeated.

Hence, let the general solution be $a_n = (b_1 + nb_2) \cdot 1^n$ .... (2)

Putting $n = 1$, $a_1 = \dfrac{3}{2}$, $b_1 + b_2 = \dfrac{3}{2}$

Putting $n = 2$, $a_2 = 3$, $b_1 + 2b_2 = 3$

Solving the two equations, we get $b_1 = 0$ and $b_2 = \dfrac{3}{2}$.

Hence, the solution of the given recurrence relation is $a_n = \dfrac{3n}{2}$.

Example 4 : Determine whether the sequence $\{ a_n \}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ where $a_n = 3n$ for every non-negative integer n. Answer the same question for $n = 5n$. **(M.U. 2003)**

Sol. : The given recurrence relation is $a_n - 2a_{n-1} + a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation is

$$r^2 - 2r + 1 = 0 \qquad \therefore \ (r-1)^2 = 0 \qquad \therefore \ r = 1, 1.$$ .... (2)

The roots are real, rational and repeated.

Hence, let the general solution be $a_n = (b_1 + n\, b_2) \cdot 1^n$

Putting $n = 2$, $a_2 = b_1 + 2b_2$

But $a_n = 3n$ $\therefore$ $a_2 = 6$ $\therefore$ $b_1 + 2b_2 = 6$

Putting $n = 3$, $a_3 = b_1 + 3b_2$

But $a_n = 3n$ $\therefore$ $a_3 = 9$ $\therefore$ $b_1 + 3b_2 = 9$

Solving these equations, we get $b_2 = 3$ and $b_1 = 0$.

Hence, from (2), we get $a_n = 3n$ as the solution of (1).

If $a_n = 5n$, putting $n = 2$ in (2), we get $a_2 = b_1 + 2b_2$.

But $a_n = 5n$ $\therefore$ $a_2 = 10$ $\therefore$ $b_1 + 2b_2 = 10$

Putting $n = 3$ in (2), we get $a_3 = b_1 + 3b_2$.

But $a_n = 5n$ $\therefore$ $a_3 = 15$ $\therefore$ $b_1 + 3b_2 = 15$

Solving these equation, we get $b_2 = 5$ and $b_1 = 0$.

Hence, from (2), we get $a_n = 5n$ as the solution of (1).

Example 5 : Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$ and $a_0 = 2$, $a_1 = 3$.

Sol. : The given recurrence relation is $a_n - 6\, a_{n-1} + 9\, a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation is

$$r^2 - 6r + 9 = 0 \quad \therefore \quad (r-3)^2 = 0 \quad \therefore \quad r = 3, 3.$$

The roots are real but repeated twice.

Hence, let the general solution be $a_n = (b_1 + n\, b_2) \cdot 3^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = 2$, we get $b_1 = 2$.

Putting $n = 1$, $a_1 = 3$, we get, $3 = b_1\, 3 + b_2\, 3$

But $b_1 = 2$ $\therefore$ $3b_2 = -3$ $\therefore$ $b_2 = -1$

Hence, the solution of the given recurrence relation from (2), is

$$a_n = 2 \cdot 3^n - n\, 3^n, \; n \geq 0.$$

Example 6 : Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$. **(M.U. 2004)**

Sol. : The given recurrence relation is $a_n - 6a_{n-1} + 9a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^2 - 6r + 9 = 0 \quad \therefore \quad (r-3)^2 = 0 \quad \therefore \quad r = 3, 3.$$

The roots are real, rational and repeated twice.

Hence, let the general solution be $a_n = (b_1 + n\, b_2) \cdot 3^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$ in (2), $a_0 = b_1 = 1$

Putting $n = 1$ in (2), $a_1 = (b_1 + b_2)\, 3 = 6$

$\therefore$ $3\,(1 + b_2) = 6$ $\therefore$ $3b_2 = 3$ $\therefore$ $b_2 = 1$.

Hence, the solution of the given recurrence relation from (2), is

$$a_n = (1 + n) \cdot 3^n.$$

Example 7 : Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ subject to the conditions $a_0 = 1 = a_1$. **(M.U. 2005, 06, 07)**

Sol. : The given recurrence relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$r^2 - 4r + 4 = 0 \quad \therefore \ (r-2)^2 = 0 \quad \therefore \ r = 2, 2.$

The roots are real, rational and repeated twice.

Hence, let the general solution be $a_n = (b_1 + n\, b_2) \cdot 2^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$ in (2), $a_0 = b_1 \cdot 2^0 = 1 \quad \therefore \ b_1 = 1$

Putting $n = 1$ in (2), $a_1 = (b_1 + b_2) \cdot 2^1 = 1 \quad \therefore \ 2(1 + b_2) = 1$

$\therefore \ 2 + 2b_2 = 1 \quad \therefore \ 2b_2 = -1 \quad \therefore \ b_2 = -1/2$

Hence, the solution of the given recurrence relation from (2), is

$$a_n = \left(1 - \frac{n}{2}\right) \cdot 2^n$$

Example 8 : Solve the recurrence relation $a_n = 4a_{n-2} - 4a_{n-2}$, $n \geq 2$, $a_0 = 2$, $a_1 = 2$.

Sol. : The given recurrence relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$r^2 - 4r + 4 = 0 \quad \therefore \ (r-2)^2 = 0 \quad \therefore \ r = 2, 2.$

The roots are real, rational and repeated twice.

Hence, let the general solution be $a_n = (b_1 + n\, b_2)\, 2^n$ .... (2)

We now use the initial conditions to find b_1 and b_2.

Putting $n = 0$, $a_0 = 2$, we get $2 = b_1$.

Putting $n = 1$, $a_1 = 2$, we get $2 = b_1 \cdot 2 + b_2 \cdot 1 \cdot 2$

But $b_1 = 2$, $\quad \therefore \ b_2 = -1$.

Hence, the solution of the given recurrence relation from (2) is

$$a_n = 2 \cdot 2^n - n\, 2^n = (2 - n)\, 2^n.$$

Example 9 : Solve the recurrence relation

$a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3}$, $\quad n \geq 3$, $a_0 = 0$, $a_1 = 2$, $a_2 = 13$.

Sol. : The given recurrence relation is $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ .... (1)

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$r^3 - 8r^2 + 21r - 18 = 0 \quad \therefore \ r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 = 0$

$\therefore \ (r-2)(r^2 - 6r + 9) = 0 \quad \therefore \ (r-2)(r-3)^2 = 0 \quad \therefore \ r = 2, 3, 3.$ .... (2)

The roots are real, rational and repeated.

Hence, let the general solution be $a_n = b_1\, 2^n + (b_2 + n\, b_3)\, 3^n$ .... (3)

We now use initial conditions to find b_1, b_2, b_3.

Putting $n = 0$, $a_0 = 0$, we get $0 = b_1 + b_2$

Putting $n = 1$, $a_1 = 2$, we get $\quad 2 = 2b_1 + 3b_2 + 3b_3$ .... (4)

Putting $n = 2$, $a_2 = 13$, we get $\quad 13 = 4b_1 + 9b_2 + 18b_3$ .... (5)

Multiply (4) by 6 and subtract from (5), $\quad \therefore\ 1 = -8b_1 - 9b_2$

From (3), $0 = 8b_1 + 8b_2$ $\quad \therefore\ b_2 = -1.$ Hence, $b_1 = 1$.

Now, from (5), $\quad 13 = 4 - 9 + 18b_3$ $\quad \therefore\ b_3 = 1.$

Putting these values in (2), the solution of the given recurrence relation is

$$a_n = 2^n - 3^n + n\,3^n.$$

Example 10 : Solve the recurrence relation $a_n - 6a_{n-1} + 12a_{n-2} = 8$.

Sol. : The given recurrence relation is

$$a_n - 6a_{n-1} + 12a_{n-2} - 8 = 0. \quad \text{..................... (1)}$$

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is

$$r^3 - 6r^2 + 12r - 8 = 0 \quad \therefore\ (r-2)^3 = 0 \quad \therefore\ r = 2, 2, 2.$$

The roots are real, rational and repeated.

Hence, its general solution is $\quad a_n = (b_1 + n\,b_2 + n^2\,b_3)\cdot 2^n.$

Example 11 : Solve the recurrence relation $a_n = -3\,(a_{n-1} + a_{n-2}) - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, $a_2 = 15$. **(M.U. 2014)**

Sol. : The given recurrence relation is

$$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0 \quad \text{..................... (1)}$$

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$$r^3 + 3r^2 + 3r + 1 = 0 \quad \therefore\ (r+1)^3 = 0 \quad \therefore\ r = -1, -1, -1.$$

The roots are real, rational and repeated thrice.

Hence, let the general solution be

$$a_n = b_1\,\alpha^n + b_2\,n\,\alpha^n + b_3\,n^2\,\alpha^n$$

Here, $\alpha = -1$

$$\therefore\ a_n = b_1\,(-1)^n + b_2\,n\,(-1)^n + b_3\,n^2\,(-1)^n \quad \text{..................... (2)}$$

We now use initial conditions to find b_1, b_2, b_3.

Putting $n = 0$, $a_0 = 5$ in (2), we get $\quad 5 = b_1$

Putting $n = 1$, $a_1 = -9$ in (2), we get .... (3)

$$-9 = -5 - b_2 - b_3 \quad \therefore\ b_2 + b_3 = 4$$

Putting $n = 2$, $a_2 = 15$ in (2), we get

$$15 = 5 + 2b_2 + 4b_3 \quad \therefore\ 2b_2 + 4b_3 = 10$$

But from (3), $\quad 2b_2 + 3b_3 = 8$

By subtraction, we get $\quad 2b_3 = 2 \quad \therefore\ b_3 = 1$

Now, from (3), we get $\quad b_2 = 3$

Hence, the required recurrence relation from (2) is

$$a_n = 5\,(-1)^n + 3n\,(-1)^n + n^2\,(-1)^n.$$

Example 12 : Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$. **(M.U. 2003)**

Sol. : The given recurrence relation is

$$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0 \quad \text{.............. (1)}$$

Let $a_n = r^n$ be a solution of (1).

The characteristic equation of (1) is,

$$r^3 + 3r^2 + 3r + 18 = 0 \qquad \therefore \; (r+1)^3 = 0$$

$\therefore \; r = -1, -1, -1.$

The roots are real, rational and repeated thrice.

Hence, let the general solution be

$$a_n = b_1(-1)^n + b_2 n(-1)^n + b_3 n^2(-1)^n \quad \text{.............. (2)}$$

Putting $n = 0$ in (2), $\quad b_1 + 0 + 0 = a_0 = 1 \qquad \therefore \; b_1 = 1$

Putting $n = 1$ in (2), $\quad -b_1 - b_2 - b_3 = a_1 = -2$

$\therefore \; -b_2 - b_3 = -1 \qquad \therefore \; b_2 + b_3 = 1$

Putting $n = 2$ in (2), $\quad b_1 + 2b_2 + 4b_3 = a_2 = -1$

$\therefore \; 2b_2 + 4b_3 = -2 \qquad \therefore \; b_2 + 2b_3 = -1$

Solving these equations, we get

$b_3 = -2$ and $b_2 = -1 - 2b_3 \qquad \therefore \; b_2 = 3$

Hence, the required recurrence relation from (2) is

$$a_n = (-1)^n + 3n(-1)^n - 2n^2(-1)^n.$$

(M.U. 2009)

7. Non-homogeneous Recurrence Relation

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n) \quad \text{.............. (1)}$$

where $c_1, c_2, \ldots, c_k$ are real constants $c_k \neq 0$ and $f(n)$ is not identically zero is called ***k* th order linear non-homogeneous recurrence relation with constant coefficient** (LNHRRWCCs).

The solution of equation (1) depends upon the solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \quad \text{.............. (2)}$$

which is called associated linear homogeneous recurrence relation with constant coefficients (ALHRRWCCs).

We have learned how to solve the equation (2).

To Solve Non-homogeneous Recurrence Relation

The general solution of (1) consists of two parts.

First part is the solution of equation (2) and the second part is some solution of (1) which is called **particular solution**. The **general solution** of (1) is the sum of these two.

Suppose the solution of (2) is of the exponential form $a_n^{(h)}$ and the known solution of (1) is of the exponential from $a_n^{(p)}$ then **general solution** of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$.

We know how to solve (2) and obtain $a_n^{(h)}$. The part $a_n^{(p)}$ depends upon the characteristic roots (α) and the nature of $f(n)$. We give below the list of forms of $a_n^{(p)}$ along with the nature of $f(n)$.

$f(n)$	$a_n^{(p)}$
If $f(n)$ is a constant and $\alpha \neq 1$	A, a constant
If $f(n)$ is a constant and $\alpha = 1$ of multiplicity m	$A n^m$
If $f(n)$ is of the form $an + b$ where a, b are constants.	$An + B$ where A, B are constants.
If $f(n)$ is of the form an^2 where a is a constant	$An^2 + Bn + c$
If $f(n)$ is of the form ea^n.	$A a^n$
If $f(n)$ is of the form $a_n e^n$.	$(An + B) e^n$

Example 1 : Find the particular solution of $a_n - 3a_{n-1} - 18a_{n-2} = 20$.

Sol. : Since $f(n)$ is a constant, we assume $a_n^{(p)} = A$, a constant.

Putting the solution in the given equation, we get

$$A - 3A - 18A = 20 \quad \therefore \; -20A = 20 \quad \therefore \; A = -1$$

$\therefore$ Particular solution, $a_n^{(p)} = -1$.

Example 2 : Find the particular solution of $a_n - 2a_{n-1} + a_{n-2} = 6$.

Sol. : Since $f(n)$ is a constant but the characteristic equation $n^2 - 2n + 1$ *i.e.*, $(n-1)^2$ has repeated roots of order 2, we assume $a_n^{(p)} = An^2$.

Putting this solution in the given equation, we get

$$An^2 - 2A(n-1)^2 + A(n-2)^2 = 6$$

$$\therefore \; An^2 - 2A(n^2 - 2n + 1) + A(n^2 - 4n + 4) = 6$$

$$\therefore \; 2A = 6 \quad \therefore \; A = 3.$$

$\therefore$ Particular solution is $a_n^{(p)} = 3n^2$.

Example 3 : Find the particular solution of $a_n - 4a_{n-1} + 4a_{n-2} = 3n$.

Sol. : Since $f(n)$ is of the form $a \cdot n + b$ where a, b are constant, we assume

$$a_n^{(p)} = An + B$$

Putting this solution in the given equation, we get,

$$(An + B) - 4[A(n-1) + B] + 4[A(n-2) + B] = 3n$$

$$\therefore \; An + B - 4An + 4A - 4B + 4An - 8A + 4B = 3n$$

$$\therefore \; An - 4A + B = 3n$$

$$\therefore \; A = 3 \text{ and } B - 4A = 0 \quad \therefore \; B = 12.$$

Hence, the particular solution is

$$a_n^{(p)} = 3n + 12.$$

Example 4 : Find the particular solution of $a_n - 7a_{n-1} + 10a_{n-2} = 2 + 8n$.

Sol. : Since $f(n)$ is of the form $an + b$ where a, b are constants, we assume

$$a_n^{(p)} = An + B$$

Putting this solution in the given equation, we get,

$$(An + B) - 7[A(n-1) + B] + 10[A(n-2) + B] = 2 + 8n$$

$$\therefore \; An + B - 7An + 7A - 7B + 10An - 20A + 10B = 2 + 8n$$

$\therefore \quad 4An - 13A + 4B = 2 + 8n \qquad \therefore \quad 4A = 8$ and $4B - 13A = 2$

$\therefore \quad A = 2$ and $4B = 28 \qquad \therefore \quad B = 7.$

Hence, the particular solution is

$$a_n^{(p)} = 2n + 7.$$

Example 5 : Find the complete solution of $a_n + 2a_{n-1} = n + 3$ for $n \geq 1$ with $a_0 = 3$. **(M.U. 2007, 17)**

Sol. : The characteristic equation is $r + 2 = 0 \quad \therefore \quad r = -2$

Hence, the solution is $a_n^{(h)} = B \cdot (-2)^n$

Let the particular solution be $a_n^{(p)} = an + b$

Putting this value of a_n in the given equation

$$(an + b) + 2\,[\,a\,(n-1) + b\,] = n + 3$$

$\therefore \quad 3an + (3b - 2a) = n + 3 \qquad \therefore \quad 3a = 1 \qquad \therefore \quad a = 1/3.$

And $\quad 3b - 2a = 3 \qquad \therefore \quad 3b = 2a + 3 = 11/3.$

Hence, $a_n^{(p)} = \dfrac{n}{3} + \dfrac{11}{9}.$

The total solution is $a_n = B \cdot (-2)^n + \dfrac{n}{3} + \dfrac{11}{9}.$

We find the constant B by using the given condition $a_0 = 3$ when $n = 0$

$\therefore \quad 3 = B + \dfrac{11}{9} \qquad \therefore \quad B = \dfrac{16}{9}$

$\therefore$ The required solution is

$$a_n = \frac{16}{9} \cdot (-2)^n + \frac{n}{3} + \frac{11}{9}.$$

Example 6 : Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$ with $a_0 = 13$ and $a_1 = 29$. **(M.U. 2018)**

Sol. : Since this is a non-homogeneous equation, its solution consists of two parts : (i) the solution of the corresponding homogeneous equation, and (ii) the particular solution.

We shall first obtain the solution of the corresponding homogeneous equation

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

Let $a_n = r^n$, then the characteristic equation is

$$r^2 - 7r + 10 = 0 \qquad \therefore \quad (r-2)(r-5) = 0 \qquad \therefore \quad r = 2, 5.$$

The roots are real, rational and distinct.

Let the solution of the corresponding homogeneous equation be .... (1)

$$a_n^{(h)} = A\,2^n + B\,5^n$$

Since $f(n) = 6 + 8n$ is linear, we assume the particular solution to be $a_n = an + b$.

Putting this in the given recurrence relation, we have,

$$(an + b) - 7\,[\,a\,(n-1) + b\,] + 10\,[\,a\,(n-2) + b\,] - 6 - 8n = 0$$

$\therefore \quad an + b - 7an + 7a - 7b + 10an - 20a + 10b - 6 - 8n = 0$

$\therefore \quad 4an - 13a + 4b - 6 - 8n = 0$

$\therefore \quad (4a - 8)\,n + (-13a + 4b - 6) = 0$

Since this is identically zero,

$4a - 8 = 0 \quad \therefore \; a = 2$

and $\; -13a + 4b - 6 = 0 \quad \therefore \; -26 + 4b - 6 = 0 \quad \therefore \; 4b = 32 \quad \therefore \; b = 8$

$\therefore$ The particular solution is

$$a_n^{(p)} = 2n + 8$$

$\therefore$ The general solution of the given equation is .... (2)

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\therefore \quad a_n = A\,(2)^n + B\,(5)^n + 2n + 8 \qquad \text{............... (3)}$$

To find A and B, we use the initial conditions $a_0 = 13$, $a_1 = 29$.

Putting $n = 0$, in (3), we get,

$$a_0 = A + B + 8 = 13 \qquad \therefore \; A + B = 5$$

Putting $n = 1$, in (3), we get,

$$a_1 = 2A + 5B + 2 + 8 = 29 \qquad \therefore \; 2A + 5B = 19$$

Solving these equations, we get $B = 3$ and $A = 2$.

Hence, the solution is from (3),

$$a_n = 2\,(2)^n + 3\,(5)^n + 2n + 8.$$

Example 7 : Solve the recurrence relation $a_{n+2} - a_{n+1} - 6a_n = 4$. **(M.U. 2014)**

Sol. : Since this is a non-homogeneous equation, its solution consists of two parts : (i) the solution of the corresponding homogeneous equation, and (ii) the particular solution,.

We shall first obtain the solution of the corresponding homogeneous equation.

$$a_{n+2} - a_{n+1} - 6a_n = 0.$$

Let $a_n = r^n$, then the characteristic equation is

$$r^2 - r - 6 = 0 \qquad \therefore \; (r-3)(r+2) = 0 \qquad \therefore \; r = -2, 3.$$

The roots are real, rational and distinct.

Let the solution of the corresponding homogeneous equation be

$$a_n^{(h)} = A\,(-2)^n + B\,(3)^n$$

Since $f(n)$ = a constant, we assume the particular solution to be a constant *i.e.*, we assume

$$a_n^{(p)} = c, \text{ a constant.} \qquad \therefore \; a_{n+1} = a_{n+2} = c.$$

Putting these values in the given recurrence relation

$$c - c - 6c = 4 \qquad \therefore \; c = -\frac{2}{3}$$

$\therefore$ The particular solution is $a_n^{(p)} = -\dfrac{2}{3}$.

Hence, the solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore \; a_n = A\,(-2)^n + B\,(3)^n - \frac{2}{3}.$$

Example 8 : Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial conditions $a_0 = 1$, $a_1 = -1$. **(M.U. 2009)**

Sol. : Since this is a non-homogeneous equation, its solution consists of two parts : (i) the solution of the corresponding homogeneous equation, and (ii) the particular solution.

We shall first obtain the solution of the corresponding homogeneous equation.

$$a_{n+2} - 5a_{n+1} + 6a_n = 0$$

Let $a_n = r^n$, then the characteristic equation is

$$r^2 - 5r + 6 = 0 \qquad \therefore \ (r-2)(r-3) = 0 \qquad \therefore \ r = 2, 3.$$

The roots are real, rational and distinct.

Let the solution of the corresponding homogeneous equation be

$$a_n^{(h)} = A(2)^n + B(3)^n$$

Since $f(n)$ = a constant, we assume the particular solution to be a constant *i.e.*, we assume

$$a_n^{(p)} = c, \text{ constant.} \qquad \therefore \ a_{n+1} = a_{n+2} = c.$$

Putting these values in the given recurrence relation,

$$c - 5c + 6c = 2 \qquad \therefore \ c = 1$$

$\therefore$ The particular solution is $a_n^{(p)} = 1$.

Hence, the solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore \ a_n = A(2)^n + B(3)^n + 1$$

To find A and B, we use the given conditions.

When $n = 0$, $\quad a_0 = A(2)^0 + B(3)^0 + 1 = 1 \qquad \therefore \ A + B = 0$

When $n = 1$, $\quad a_1 = A(2)^1 + B(3)^1 + 1 = -1 \qquad \therefore \ 2A + 3B = -2$

Solving these equations, we get, $B = -2 \qquad \therefore \ A = -B = -2$

$\therefore$ The solution is $\quad a_n = 2(2)^n - 2(3)^n + 1$

Example 9 : Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 8n^2$, with $a_0 = 45$, $a_1 = 7$.

Sol. : Since this a non-homogeneous equation, its solution consists of two parts (i) the solution of the corresponding homogeneous equation $a_n^{(h)}$, and (ii) the particular solution $a_n^{(p)}$.

We shall first obtain the solution of the corresponding homogeneous equation

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

Let $a_n = r^n$, then the characteristic equation is

$$r^2 - 5r + 6 = 0 \qquad \therefore \ (r-3)(r-2) = 0 \qquad \therefore \ r = 2, 3.$$

The roots are real, rational and distinct.

Let the solution of the corresponding homogeneous equation be

$$a_n^{(h)} = A \cdot 2^n + B \cdot 3^n.$$

Since $f(n) = n^2$ (quadratic), we assume the particular solution to be

$$a_n^{(p)} = an^2 + bn + c.$$

Putting this in the given recurrence relation, we have,

$$[an^2 + bn + c] - 5[a(n-1)^2 + b(n-1) + c] + 6[a(n-2)^2 + b(n-2) + c] - 8n^2 = 0$$

$$\therefore \ an^2 + bn + c - 5[an^2 + (-2a+b)n + (a-b+c)]$$
$$+ 6[an^2 + (-4a+b)n + (4a-2b+c)] - 8n^2 = 0$$

$$\therefore \ (2a-8)n^2 + (-14a+2b)n + (19a-7b+2c) = 0$$

Since this is identically zero, we get

$$2a = 8 \qquad \therefore \ a = 4$$

$$2b = 14a \qquad \therefore \ b = 7a = 28$$

$$2c = -19a + 7b = -76 + 196 = 120 \qquad \therefore \ c = 60$$

$\therefore$ The particular solution is $a_n^{(p)} = 4n^2 + 28n + 60.$

$\therefore$ The general solution of the given equation is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore\ a_n = A \cdot 2^n + B \cdot 3^n + 4n^2 + 28n + 60$$

To find A and B, we use initial conditions $a_0 = 4,\ a_1 = 7.$

Putting $n = 0$, and $a_0 = 4$, $\quad A + B + 60 = 4.$

Putting $n = 1$ and $a_1 = 7$, $\quad 2A + 3B + 92 = 7$

Solving these equations, we get $A = -83$, $B = 27$.

Hence, the desired solution is

$$a_n = (-83) \cdot 2^n + 27 \cdot 3^n + 4n^2 + 28n + 60, \quad n \geq 0.$$

Example 10 : Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

(M.U. 2006, 12, 13, 15)

Sol. : As before the solution of the corresponding homogeneous equation is

$$a_n^{(h)} = A \cdot 2^n + B \cdot 3^n$$

Since $f(n) = 7^n$, we assume the particular solution to be $a_n = C \cdot 7^n$.

Putting this in the given equation

$$C \cdot 7^n - 5C \cdot 7^{n-1} + 6C \cdot 7^{n-2} = 7^n$$

$$\therefore\ C \cdot 49 \cdot 7^{n-2} - 5 \cdot C \cdot 7 \cdot 7^{n-2} + 6C \cdot 7^{n-2} = 49 \cdot 7^{n-2}$$

$$\therefore\ \frac{(49 - 35 + 6)}{49} C = 1 \quad \therefore\ C = \frac{49}{20}$$

Hence, the desired solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore\ a_n = A \cdot 2^n + B \cdot 3^n + \frac{49}{20} \cdot 7^n$$

Example 11 : Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 4^n$ with the initial conditions $a_0 = -1$, $a_1 = 1$.

Sol. : As in the above two examples, the solution of the corresponding homogeneous equation is

$$a_n^{(h)} = A \cdot 2^n + B \cdot 3^n.$$

Since $f(n) = 4^n$, we assume the particular solution to be $a_n = C \cdot 4^n$.

Putting this in the given equation,

$$C \cdot 4^n - 5C \cdot 4^{n-1} + 6C \cdot 4^{n-2} = 4^n$$

$$\therefore\ C \cdot 4^2 \cdot 4^{n-2} - 5C \cdot 4 \cdot 4^{n-2} + 6C \cdot 4^{n-2} = 4^2 \cdot 4^{n-2}$$

$$\therefore\ 16C - 20C + 6C = 16 \quad \therefore\ 2C = 16 \quad \therefore\ C = 8$$

$$\therefore\ a_n = 8 \cdot 4^n.$$

$\therefore$ The required solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore\ a_n = A \cdot 2^n + B \cdot 3^n + 8 \cdot 4^n.$$

Example 12 : Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 6 \cdot 5^n$ with $a_0 = 4$, $a_1 = 7$.

Sol. : As before the solution of the corresponding homogeneous equation is

$$a_n^{(h)} = A \cdot 2^n + B \cdot 3^n.$$

Since $f(n) = 6 \cdot 5^n$ we assume the particular solution to be $a_n = C \cdot 5^n$.

Putting this in the given equation

$$C \cdot 5^n - 5 \cdot C \cdot 5^{n-1} + 6 \cdot C \cdot 5^{n-2} = 6 \cdot 5^n$$

Cancelling out 5^{n-2}, from both sides

$$25C - 25C + 6C = 150 \qquad \therefore \quad C = 25$$

$\therefore$ The particular solution is $a_n^{(p)} = 25 \cdot 5^n$.

$\therefore$ The general solution of the given equation is

$$a_n = a_n^{(h)} + a_n^{(p)} = A \cdot 2^n + B \cdot 3^n + 25 \cdot 5^n.$$

To find A and B, we use initial conditions $a_0 = 4$, $a_1 = 7$.

Putting $n = 0$, $a_0 = 4$, $\quad A + B + 25 = 4$.

Putting $n = 1$, $a_1 = 7$, $\quad 2A + 3B + 125 = 7$

Solving the two equations, we get $A = 55$ and $B = -76$.

Hence, the desired solution is

$$a_n = 55 \cdot 2^n - 76 \cdot 3^n + 25 \cdot 5^n.$$

Example 13 : Find the general solution of $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$. **(M.U. 2006)**

Sol. : We have $a_n + 5a_{n-1} + 6a_{n-2} = 42 \cdot 4^n$ .... (1)

Let $a_n = r^n$ then the characteristic equation is

$$r^2 + 5r + 6 = 0 \qquad \therefore \quad (r+2)(r+3) = 0 \qquad \therefore \quad r = -2, -3.$$

Since the roots are distinct, let the solution of the corresponding homogeneous equation be

$$a_n^{(h)} = A(-2)^n + B(-3)^n.$$

Since $f(n) = 42 \cdot 4^n$ we assume the particular solution to be $a_n = C \cdot 4^n$.

Putting this in the given equation (1),

$$C \cdot 4^n + 5C \cdot 4^{n-1} + 6C \cdot 4^{n-2} = 42 \cdot 4^n$$

Cancelling out 4^{n-2} from both sides,

$$16C + 20C + 6C = 42 \cdot 16$$

$$\therefore \quad 42C = 42 \cdot 16 \qquad \therefore \quad C = 16$$

$\therefore$ The particular solution is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$.

Hence, the desired solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \qquad \therefore \quad a_n = A(-2)^n + B(-3)^n + 4^{n+2}.$$

Example 14 : Solve the equation $\quad a_n + a_{n-1} = 3n \cdot 2^n$, $a_1 = 8/3$.

Sol. : The characteristic equation is $r + 1 = 0 \qquad \therefore \quad r = -1$.

$\therefore$ The solution is $a_n^{(h)} = A(-1)^n$.

Let the particular solution be $\quad a_n^{(p)} = (Bn + C)\, 2^n$

Putting this value in the given equation, we get

$$(Bn + C) \cdot 2^n + [B(n-1) + C]\, 2^{n-1} = 3n \cdot 2^n$$

Cancelling out 2^{n-1}, we have

$$(Bn + C)\, 2 + (Bn - B + C) = 3n \cdot 2 \qquad \therefore \quad 3Bn + (3C - B) = 6n$$

Equating the coefficients of like powers of n,

$$3B = 6 \qquad \therefore \quad B = 2$$

$$\therefore \quad 3C - B = 0 \qquad \therefore \quad C = 2/3.$$

Hence, the solution is $a_n = A(-1)^n + \left(2n + \frac{2}{3}\right) \cdot 2^n$.

To find A we use the initial condition that when $n = 0$, $a_1 = 8/3$.

$\therefore \quad \frac{8}{3} = A + \frac{2}{3} \quad \therefore \quad A = \frac{6}{3} = 2$

$\therefore$ Hence, the desired solution is

$$a_n = 2(-1)^n + \left(2n + \frac{2}{3}\right) \cdot 2^n.$$

Example 15 : Solve the equation $a_r + a_{r-1} = 3r \cdot 2^r$, with $a_0 = 11/3$.

Sol. : The characteristic equation is $r + 1 = 0 \quad \therefore \quad r = -1$

Hence, the solution is $a_n^{(h)} = A(-1)^n$

Let the particular solution be $a_n^{(p)} = (an + b) \, 2^n$ .... (1)

Putting this value in the given equation, we get .... (2)

$$(an + b) \, 2^n + [\, a(n-1) + b \,] \, 2^{n-1} = 3n \cdot 2^n$$

Cancelling out 2^{n-1},

$\therefore \quad (an + b) \, 2 + (an - a + b) = 3n \cdot 2$

$\therefore \quad 2an + 2b + an - a + b = 6n \qquad \therefore \quad 3an + (-a + 3b) = 6n$

Equating the coefficients of n on both sides,

$3a = 6 \qquad \therefore \quad a = 2$

and $\quad -a + 3b = 0 \quad \therefore \quad 3b = a = 2 \quad \therefore \quad b = 2/3$

$\therefore \quad a_n^{(p)} = \left(2n + \frac{2}{3}\right) \cdot 2^n$ .... (3)

$\therefore$ The solution is $a_n = A(-1)^n + \left(2n + \frac{2}{3}\right) \cdot 2^n$ .... (4)

We find the constant A from the initial condition $a_0 = \frac{11}{3}$.

Putting $n = 0$ in (4), $\quad a_0 = A + \frac{2}{3} \cdot 1 = \frac{11}{3} \qquad \therefore \quad A = \frac{11}{3} - \frac{2}{3} = 3 \qquad \therefore \quad A = 3.$

Hence, the required solution is $a_n = 3(-1)^n + \left(2n + \frac{2}{3}\right) \cdot 2^n$.

Example 16 : Solve the equation $a_n - 4a_{n-1} + 4a_{n-2} = n^2 (n + 1) \, 2^n$, $a_0 = 2$, $a_1 = 37/3$.

Sol. : The characteristic equation is

$r^2 - 4r + 4 = 0 \qquad \therefore \quad (r - 2)^2 = 0 \qquad \therefore \quad r = 2, 2.$

Hence, the solution is $a_n^{(h)} = (An + B) \, 2^n$.

Let the particular solution be $a_n^{(p)} = n^2 (an + b) \cdot 2^n$.

Putting this value in the given equation

$$n^2 (an + b) \, 2^n - 4 \, [(n-1)^2 \{a(n-1) + b\} \,] \, 2^{n-1} + 4 \, [\, (n-2)^2 \{a(n-2) + b\} \,] \, 2^{n-2} = (n + 1) \, 2^n.$$

$$\therefore \quad n^2 (an + b) \, 2^n - [(n-1)^2 (an - a + b)] \, 2^{n+1} + [(n-2)^2 (an - 2a + b)] \, 2^n = (n + 1) \, 2^n$$

Cancelling out 2^n, we have,

$$n^2(an+b)-2(n^2-2n+1)(an-a+b)+(n^2-4n+4)(an-2a+b)=n+1$$

$$\therefore \quad an^3+bn^2-2an^3+4an^2-2an+2an^2-4an+2a-2n^2b+4bn-2b$$
$$+an^3-4an^2+4an-2an^2+8an-8a+bn^2-4bn+4b=n+1$$

$\therefore \quad 6an+(2b-6a)=n+1 \qquad \therefore \; 6a=1 \qquad \therefore \; a=1/6$

and $\quad 2b-6a=1 \qquad \therefore \; 2b-1=1 \qquad \therefore \; 2b=2 \qquad \therefore \; b=1$

$$\therefore \quad a_n^{(p)}=\left(\frac{1}{6}\cdot n+1\right)n^2 2^n$$

$\therefore$ The total solution is $\quad a_n=(An+B)2^n+\left(\frac{1}{6}\cdot n+1\right)n^2\,2^n$

We find A and B by using given conditions $a_0=2$ and $a_1=37/3$, we get

Putting $n=0$, $\qquad 2=B$

Putting $n=1$, $\qquad \frac{37}{3}=(A+2)2+\left(\frac{1}{6}+1\right)1\cdot 2$

$$\therefore \quad \frac{37}{3}=2A+4+\frac{7}{3} \qquad \therefore \; 2A=6 \qquad \therefore \; A=3$$

$\therefore$ The required solution is

$$a_n=(3n+2)\cdot 2^n+\left(\frac{n}{6}+1\right)\cdot n^2\cdot 2^n$$

EXERCISE - III

(A) State whether each of the following recurrence relation is a linear homogeneous recurrence relation with constant coefficients (LHRRWCCs).

1. $a_n=2a_{n-1}+3a_{n-2}$
2. $a_n=a_{n-1}+2^n$
3. $a_n=3a_{n-1}+4a_{n-2}+a_{n-3}$
4. $a_n=a_{n-1}+3n^2+1$
5. $a_n=2a_{n-1}+3^n-n$
6. $a_n=a_{n-1}+3a_{n-2}$
7. $a_n=2a_{n-1}$
8. $a_n=2a_{n-1}+n$

[**Ans. :** (1) Yes, (2) No, (3) Yes, (4) No, (5) No, (6) Yes, (7) Yes, (8) No.]

(B) Solve each of the following linear homogeneous recurring relation with constant coefficients.

1. $a_n=2a_{n-1},\; n\geq 0,\; a_0=1$
2. $a_n=3a_{n-1}-2a_{n-2},\; a_0=1,\, a_1=3,\; n\geq 2$
3. $a_n=a_{n-1}+2a_{n-2},\; a_0=3,\, a_1=0,\, n\geq 2.$
4. $a_n=a_{n-1}+6a_{n-2},\; a_0=3,\, a_1=4.$
5. $a_n=8a_{n-1}-16a_{n-2},\; a_0=1,\, a_1=4.$
6. $a_n=10a_{n-1}-25a_{n-2},\; a_0=3,\, a_1=5.$
7. $a_n=7a_{n-1}-16a_{n-2}+12a_{n-3},\; a_0=0,\, a_1=3,\, a_2=13,\, n\geq 3.$
8. $a_n=8a_{n-1}-21a_{n-2}+18a_{n-3},\; a_0=3,\, a_1=2,\, a_2=13,\, n\geq 3.$

[**Ans.:** (1) $a_n=2^n$ (2) $a_n=(-1)+2^n$ (3) $a_n=2(-1)^n+2^n$
(4) $a_n=2\cdot 3^n+1\cdot(-2)^n$ (5) $a_n=2\cdot 4^n-n\cdot 4^4$ (6) $a_n=3\cdot 5^n-2\cdot n\cdot 5^n$
(7) $a_n=3^n-2^n+n\cdot 2^n$ (8) $a_n=2^n-3^n+n\cdot 3^n$]

(C) Find the particular solution of

1. $a_n - 2a_{n-1} - 8a_{n-2} = 9$
2. $a_n - 2a_{n-1} + a_{n-2} = 4$
3. $a_n - 6a_{n-1} + 9a_{n-2} = 4n$
4. $a_n - 7a_{n-1} + 10a_{n-2} = 8n + 6$
5. $a_n - 7a_{n-1} + 12a_{n-2} = 6n^2 + 2n - 1$
6. $a_n - 4a_{n-1} + 4a_{n-2} = 4^n$
7. $a_n - 4a_{n-1} + 4a_{n-2} = (n + 1)\, 2^n$

[**Ans. :** (1) $a_n^{(p)} = -1$ (2) $a_n^{(p)} = 2n^2$ (3) $a_n^{(p)} = n + 3$ (4) $a_n^{(p)} = 2n + 8$

(5) $a_n^{(p)} = 2n + 8$ (6) $a_n^{(p)} = n^2 + 6n + 10$ (7) $a_n^{(p)} = 4^{n+1}$ (8) $a_n^{(p)} = n^2\left(\frac{n}{6}+1\right)\cdot 2^n$]

(D) Solve the following non-homogeneous recurrence relations.

1. $a_n - 2a_{n-1} = 1, \quad a_0 = 1$
2. $a_n - 2a_{n-1} - 8a_{n-2} = 6, \ a_0 = 1, \ a_1 = 3$
3. $a_n - 5a_{n-1} + 6a_{n-2} = 2, \ a_0 = 1, \ a_1 = -1$
4. $a_n - 7a_{n-1} + 10a_{n-2} = 8n + 6, \ a_0 = 1, \ a_1 = 5.$
5. $a_n - 5a_{n-1} + 6a_{n-2} = 8n^2, \ a_0 = 4, \ a_1 = 7.$
6. $a_n - 7a_{n-1} + 12a_{n-2} = 3^n, \ a_0 = 0, \ a_1 = 2.$
7. $a_n - 5a_{n-1} + 6a_{n-2} = 3 \cdot 5^n, \ a_0 = 4, \ a_1 = 7.$
8. $a_n + a_{n-1} = 3n\, 2^n, \ a_0 = 5/3.$
9. $a_n - 6a_{n-1} + 9a_{n-2} = 4\,(n + 1) \cdot 3^n, \ a_0 = 2, \ a_1 = 3.$

[**Ans. :** (1) $a_n = 2^n - 1, \ n \ge 0$ (2) $a_n = \frac{3}{2} \cdot 2^n + \frac{1}{6} \cdot 4^n - \frac{2}{3}, \ n \ge 0$

(3) $a_n = -2 \cdot 3^n + 2 \cdot 2^n + 1, \ n \ge 0$ (4) $a_n = -10 \cdot 2^n + 3 \cdot 5^n + 2n + 8, \ n \ge 0$

(5) $a_n = -83 \cdot 2^n + 27 \cdot 3^n + 4n^2 + 28n + 60, \ n \ge 0$

(6) $a_n = 11 \cdot 4^n - 11 \cdot 3^n - n \cdot 3^{n+1}, \ n \ge 0$

(7) $a_n = 30 \cdot 2^n - (77/2) \cdot 3^n + (25/2) \cdot 5^n, \ n \ge 0$

(8) $a_n = (-1)^n + \left(2n + \frac{2}{3}\right) \cdot 2^n, \ n \ge 0$

(9) $a_n = (6 - 19n) \cdot 3^{n-1} + 2n^2\,(n + 6) \cdot 3^{n-1}, \ n \ge 0$]

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