Module 1: Sets & Logic

Introduction to Set Theory :

Set:

A set is a well defined collection of objects called elements or members of the set.

Eg: * A collection of blue coloured birds.

* Collection of real numbers between 0 & 1.

* $A = \{\alpha, \beta, \gamma, \delta\}$

* $B = \{Ram, Shyam, Rohan, Sohan\}$

Subsets

If every element in a set A is present in the set B, then A is called as the subset of B.

We can also say that A is contained in B.

This relationship is denoted by

$$A \subseteq B \quad \text{OR}$$

$$B \supseteq A$$

If A is not a subset of B or at least one element of A does not belong to B, then we write:

$$A \nsubseteq B \quad \text{or} \quad B \nsupseteq A$$

Example:

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5\}$

$\because$ All the elements of set A is present in set B.

$\therefore$ Set A is a subset of set B

$\Rightarrow$ $A \subseteq B$ OR

$B \supseteq A$

Universal Set

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set.

For example, in a plane geometry, the universal set consists of all the points in te plane.

It is usually denoted by U.

Empty Set

The Set with no elements is called as an Empty Set or Null Set and is denoted by ϕ

e.g :

A= {x: x is a +ve integer, x<1}

=> A has no elements

Note: (1) A is said to be equal to B iff A is a subset of B and B is a subset of A.
(2) For any set A, $\phi \subseteq A$

Cardinality of a Set :-

The cardinality of a finite set is "number of elements in the set"

It is denoted by $|A|$ for set A.

e.g. $A = \{1, 2, 3\}$

$\therefore \quad |A| = 3$

Finite and Infinite Set:

A set A is called finite set if it has 'n' distinct elements, where $n \in N$.

In this case, n is called the cardinality of A and is denoted by $|A|$.

A set that is not finite is called an infinite set.

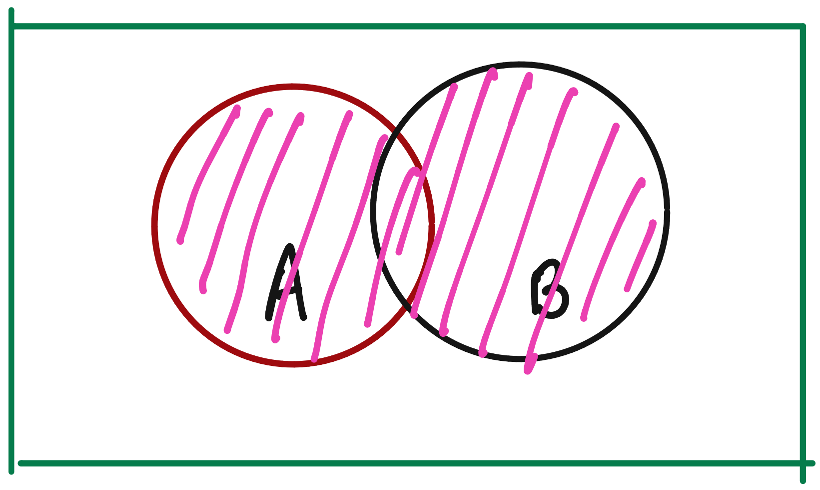
Operations on Sets & Venn Diagrams

1. Union of Sets

Let A and B be two non-empty sets, then the union of A and B is also a set

i.e the collection of all the elements of either A or B

$$\therefore A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

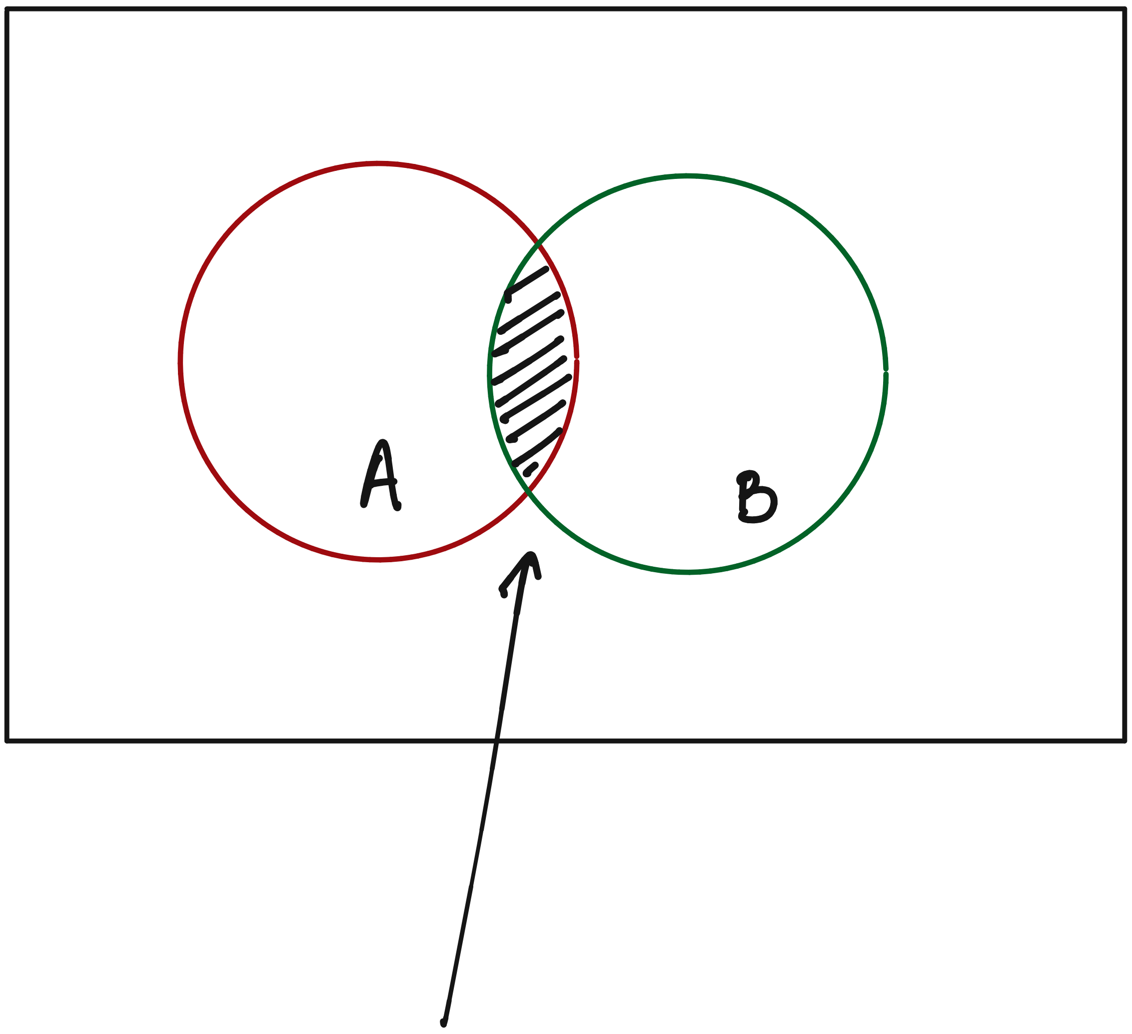


2. Intersection

The intersection of two non-empty sets A and B is denoted by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

i.e The collection of all the elements of both A and B.



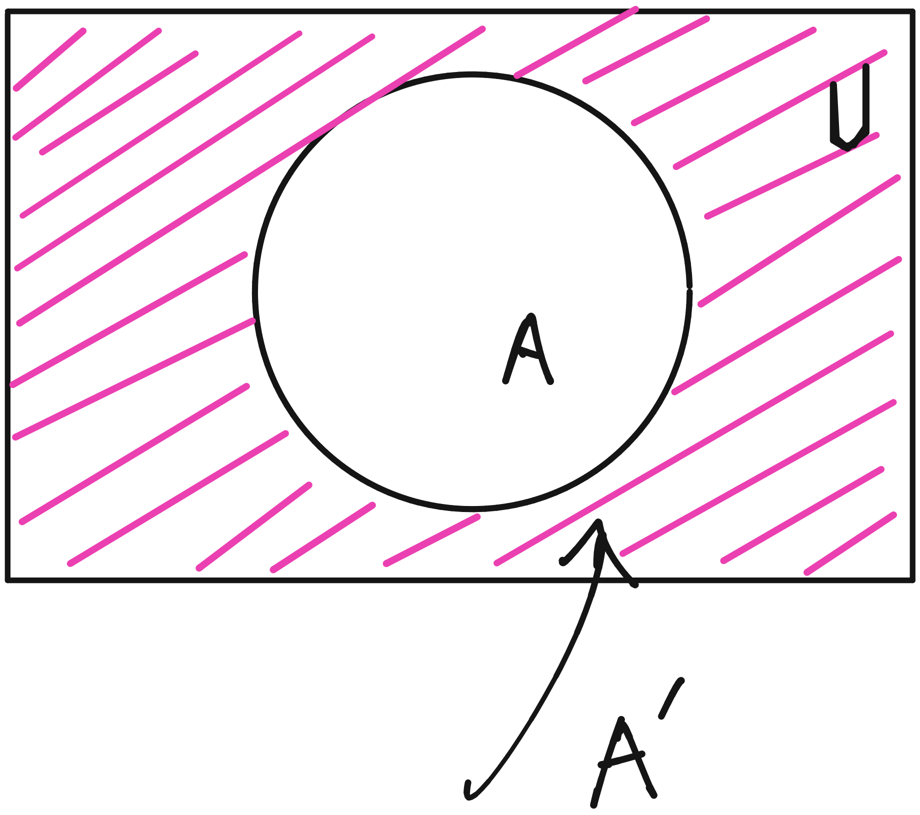
$A \cap B$

3. <u>Complement</u>

Let A be a non-empty subset of a given universal set U, then the complement of A is a set such that it is the collection of all elements of U but not A.

It is denoted by A' or $\bar{A}$ or A^c

$$A' = \{x \mid x \in U \text{ and } x \notin U\}$$



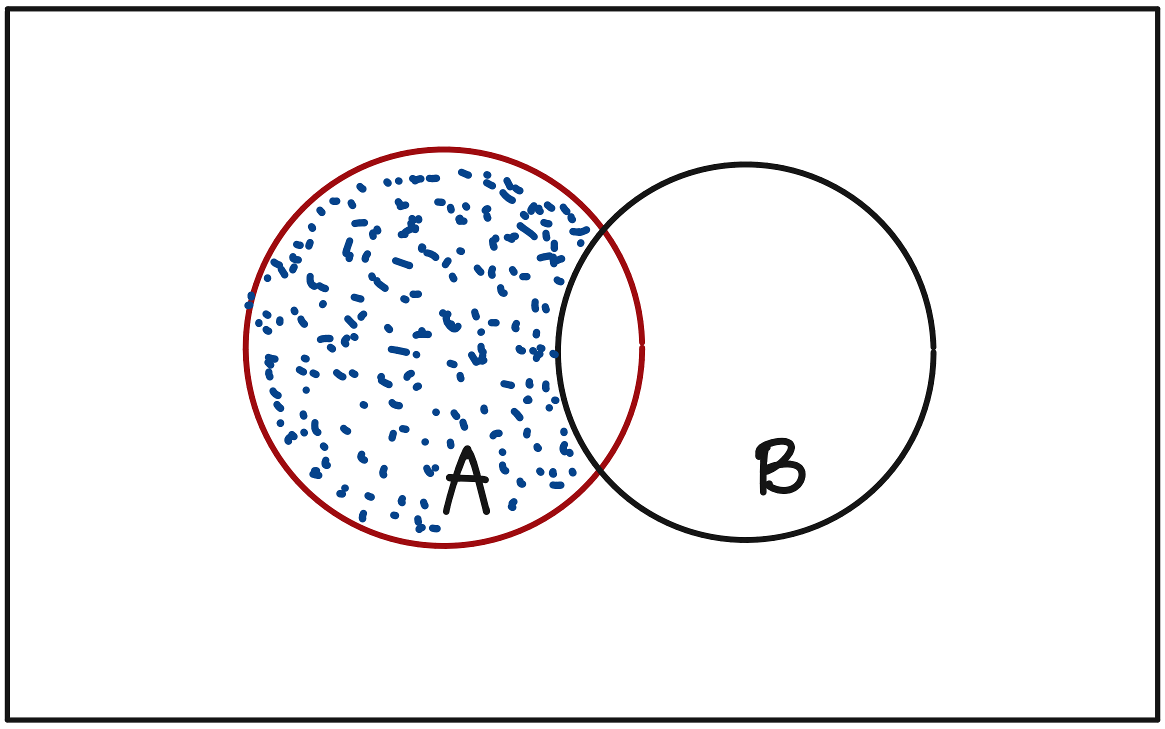
4. Set Difference

Let A and B be 2 non-empty sets, then the difference of A and B is a set

i.e the collection of all the elements of A but not the elements of B.

$$\therefore A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Also denoted as $A - B$.



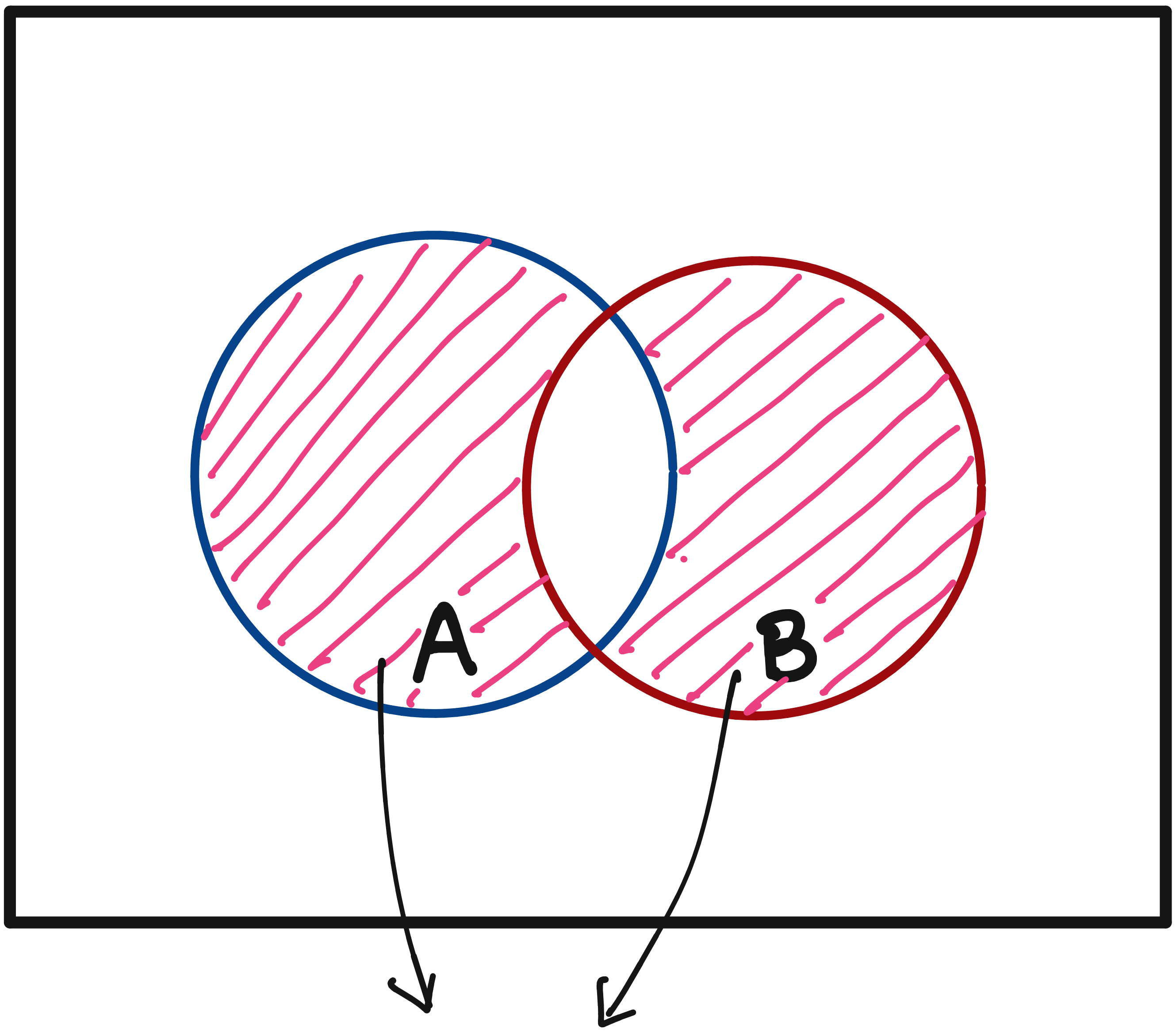
$A \setminus B$

$A - B$

5. <u>Ring Sum (⊕)</u>

Let A and B be 2 non empty sets, then the ring sum of A and B is the set of all the elements which are either in A or in B but not in both.

$$A \oplus B = \{ x \mid x \in A \cup x \in B \text{ but } x \notin A \cap B \}$$



$A \oplus B$

Algebraic Properties of Sets:

1. Commutative Property

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

2. Associative Property

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributive Property

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. <u>Idempotent Property</u>

$$A \cup A = A$$

$$A \cap A = A$$

5. <u>Properties of Complement</u>

$$(\overline{\overline{A}}) = A$$

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \phi$$

$$\bar{\phi} = U$$

$$\bar{U} = \phi$$

$$\left.\begin{aligned} \overline{A \cup B} &= \bar{A} \cap \bar{B} \\ \overline{A \cap B} &= \bar{A} \cup \bar{B} \end{aligned}\right\} \text{De-Morgan's Laws}$$

Some Solved Examples

1. Let A, B, C be the subsets of universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer.

Sol: B can be expressed as

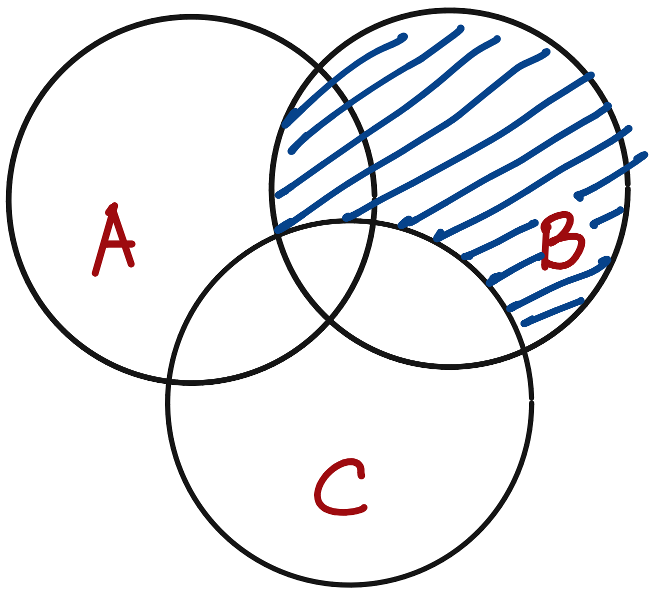
$$
\begin{aligned}
B &= B \cap U \\
&= B \cap (A \cup \bar{A}) && \because A \cup \bar{A} = U \\
&= (B \cap A) \cup (B \cap \bar{A}) && \because \text{By DL} \\
&= (A \cap B) \cup (\bar{A} \cap B) && \because \text{By CL} \\
&= (A \cap C) \cup (\bar{A} \cap C) && \because \text{Given} \\
&= (A \cup \bar{A}) \cap C && \because \text{By D.L} \\
&= U \cap C && \because \text{Universal Set} \\
&= C
\end{aligned}
$$

$\therefore B = C$ Hence Proved.

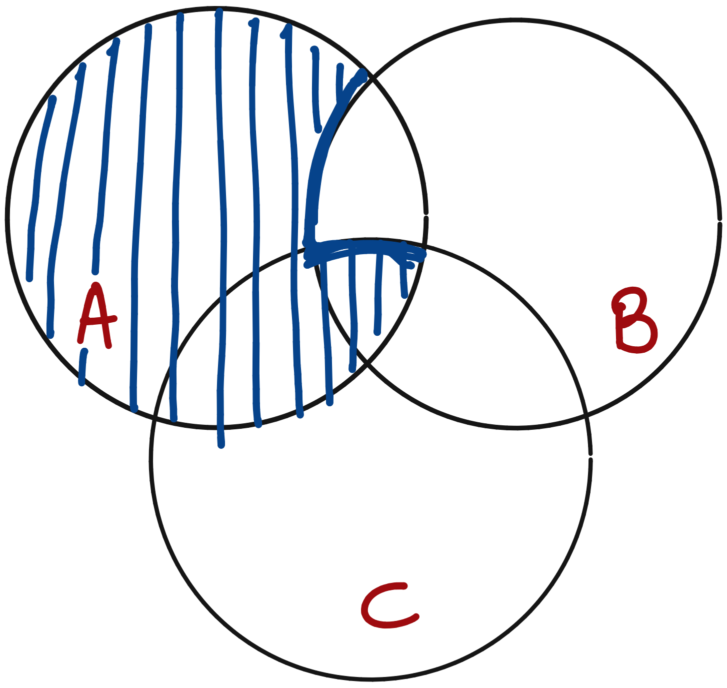
2. Using Venn Diagrams, prove that

$$A\setminus(B\setminus C) = (A\setminus B)\cup(A\cap B\cap C)$$

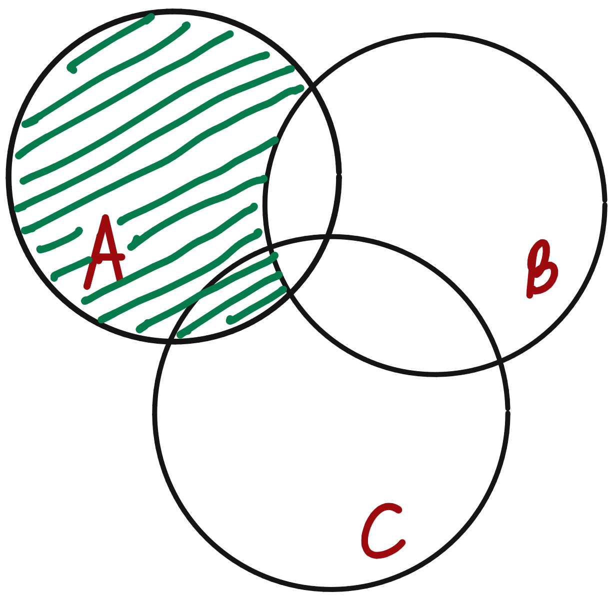
Sol:

a) 

$\Rightarrow$ $B\setminus C$ or $B - C$

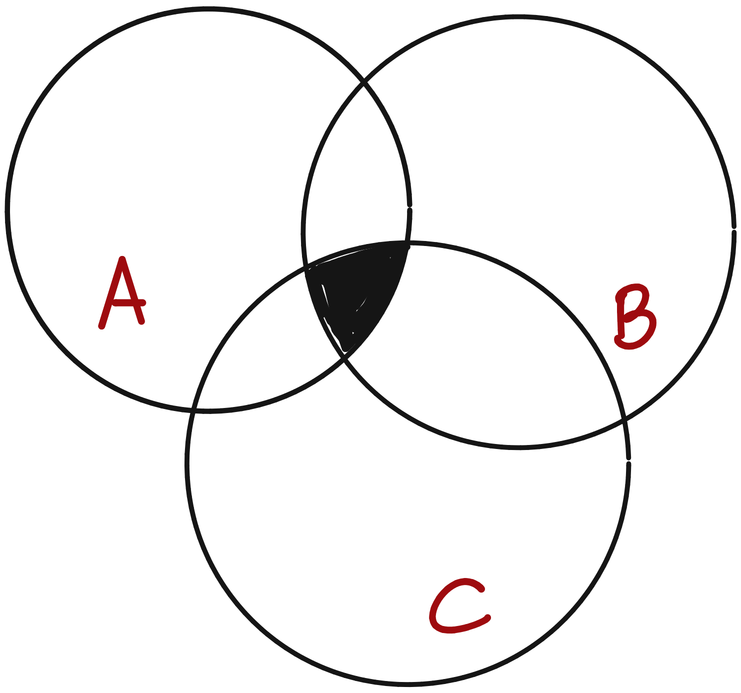
b) 

$\Rightarrow$ $A\setminus(B\setminus C)$

c) 

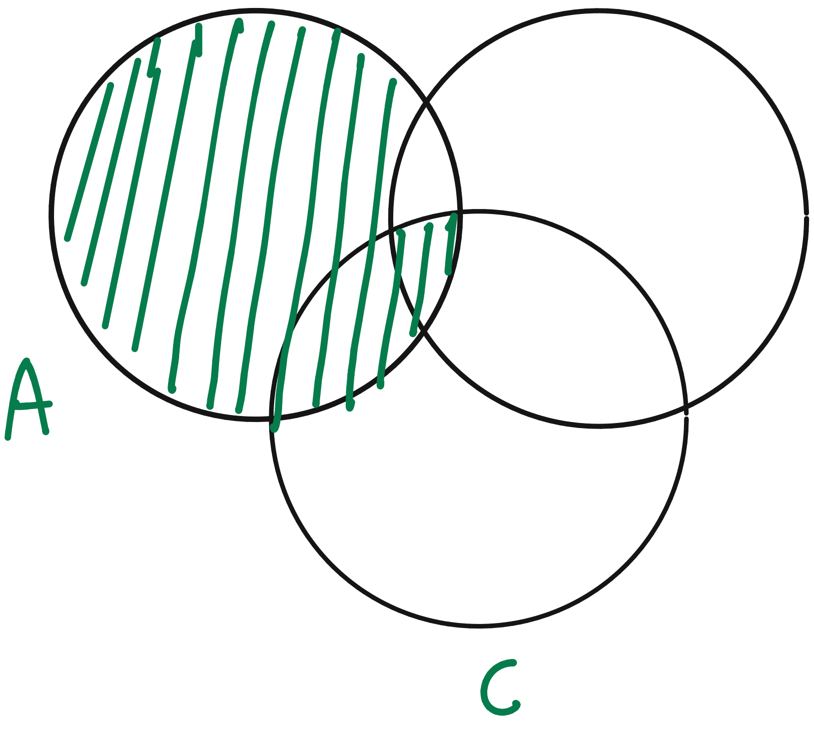
$\Rightarrow$ $A\setminus B$

d)



$\Rightarrow \quad A \cap B \cap C$

e)



$\Rightarrow \quad (A \setminus B) \cup (A \cap B \cap C)$

If we compare a & e , we find both the Venn Diagrams to be same, hence L·H·S = R·H·S.

Practice Problems.

1. Using Venn Diagram show

   a) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

   b) $A \cap B \cap C = A \setminus [(A \setminus B) \cup (A \setminus C)]$

2. Using Venn Diagram, prove that

   $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

3. If $U = \{1,2,3,4,5,6,7,8,9\}$

   $A = \{1,2,4,6,8\}$

   $B = \{2,4,5,9\}$

   $C = \{x \mid x$ is a +ve integer and $x^2 < 16\} = \{1,2,3\}$

   $D = \{7,8\}$

Compute

a) $A \cup B$

b) $A \cup C$

c) $A \cup D$

d) $B \cup C$

e) $A \cap C$

f) $C \cap D$

g) $A \cap D$

h) $B \cap C$

i) $A \setminus B$

j) $B \setminus A$

k) $C \setminus D$

l) $\overline{C}$

m) $\overline{A}$

n) $A \oplus B$

o) $C \oplus D$

p) $B \oplus C$

Compute

a) $A \cup B = \{1,2,4,5,6,8,9\}$

b) $A \cup C = \{1,2,3,4,6,8\}$

c) $A \cup D = \{1,2,4,6,7,8\}$

d) $B \cup C = \{1,2,3,4,5,9\}$

e) $A \cap C = \{1,2\}$

f) $C \cap D = \phi$

g) $A \cap D = \{8\}$

h) $B \cap C = \{2\}$

i) $A \setminus B = \{1,6,8\}$

j) $B \setminus A = \{5,9\}$

k) $C \setminus D = \{1,2,3\}$

l) $\overline{C} = \{4,5,6,7,8,9\}$

m) $\overline{A} = \{3,5,7,9\}$

n) $A \oplus B = \{1,5,6,8,9\}$

o) $C \oplus D = \{1,2,3,7,8\}$

p) $B \oplus C = \{1,2,4,5,9\}$

Compute:

a) $A \cup B \cup C =$

b) $A \cap B \cap C =$

c) $A \cap (B \cup C) =$

d) $(A \cup B) \cap C =$

e) $\overline{A \cup B} =$

f) $\overline{A \cap B} =$

g) $B \cup C \cup D =$

h) $A \cup A =$

i) $A \cap \overline{A} =$

j) $A \cup \overline{A} =$

k) $A \cap (\overline{C} \cup D) =$

Compute :

a) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8, 9\}$

b) $A \cap B \cap C = \{2\}$

c) $A \cap (B \cup C) = \{1, 2, 4\}$

d) $(A \cup B) \cap C = \{8\}$

e) $\overline{A \cup B} = \{3, 7\}$

f) $\overline{A \cap B} = \{1, 3, 5, 6, 7, 8, 9\}$

g) $B \cup C \cup D = \{1, 2, 3, 4, 5, 7, 8, 9\}$

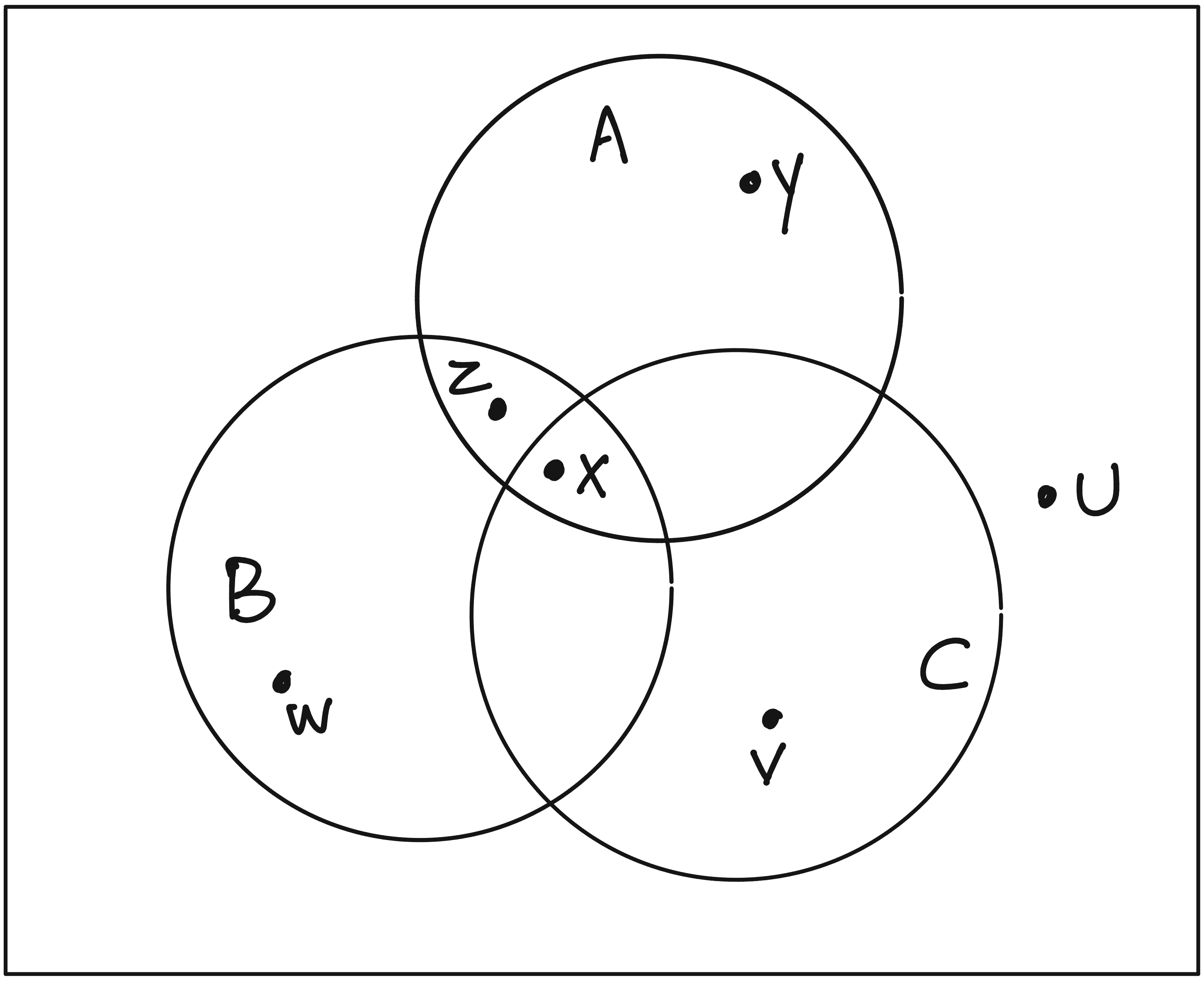
h) $A \cup A = A$

i) $A \cap \bar{A} = \phi$

j) $A \cup \bar{A} = U$

k) $A \cap (\bar{C} \cup D) = \{4, 6, 8\}$

4.



Identify the following as true or false

a) $y \in A \cap B$

b) $x \in B \cup C$

c) $w \in B \cap C$

d) $u \notin C$

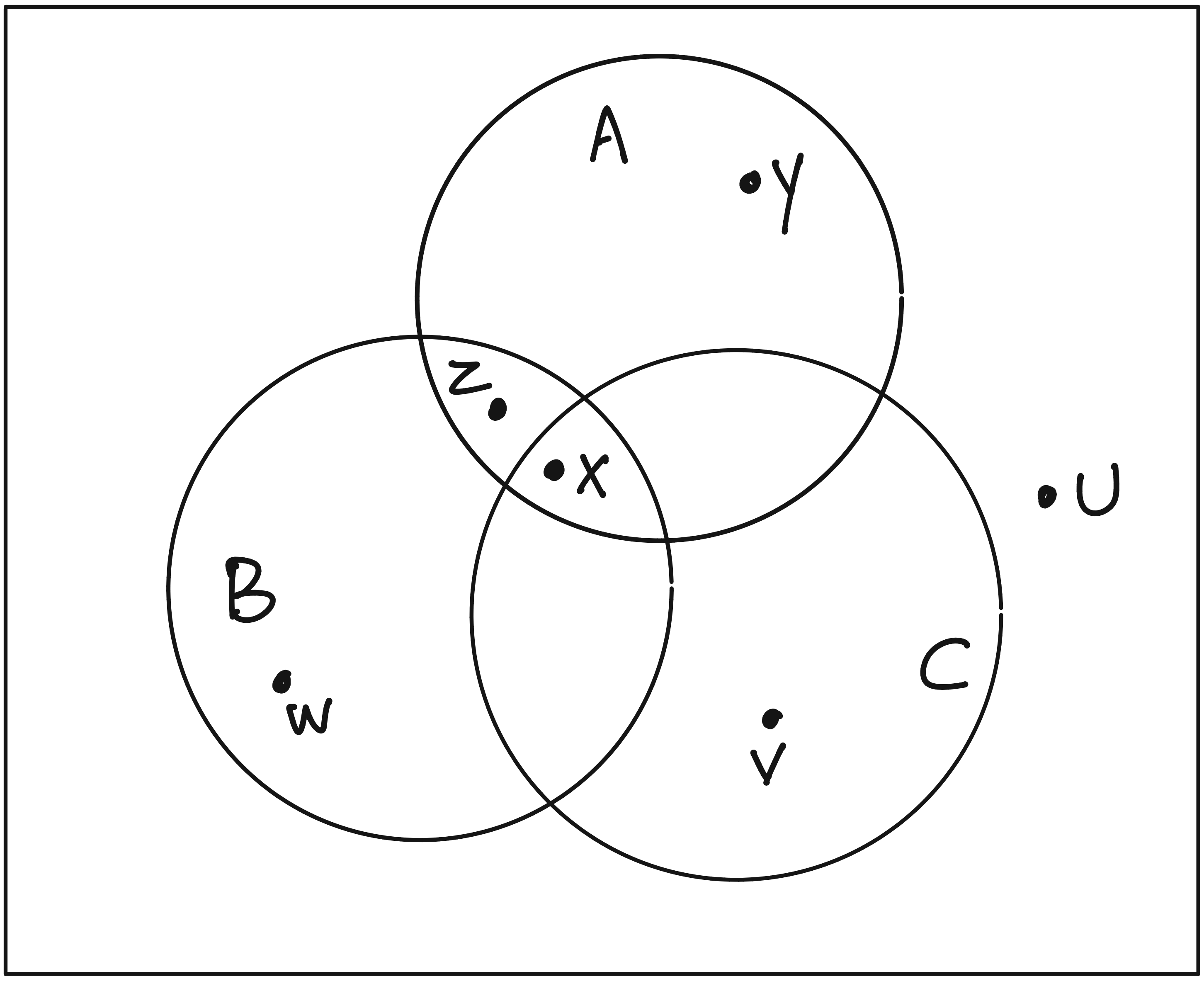
e) $x \in A \cap B \cap C$

f) $y \in A \cup B \cup C$

g) $z \in A \cap C$

h) $v \in B \cap C$

4.



Identify the following as true or false

a) $y \in A \cap B$ False

b) $x \in B \cup C$ True

c) $w \in B \cap C$ False

d) $u \notin C$ True

e) $x \in A \cap B \cap C$ True

f) $y \in A \cup B \cup C$ True

g) $z \in A \cap C$ False

h) $v \in B \cap C$ False

Principle of Inclusion & Exclusion

(Addition Principle)

1. Let A and B be two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Also called as <u>disjoint union</u>

2. Let A, B and C be finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Problems based on Addition Principle

1. Let $A = \{a, b, c, d, e\}$, $B = \{a, b, e, g, h\}$ and $C = \{b, d, e, g, h, k, m, n\}$

We have,

$A \cup B \cup C = \{a, b, c, d, e, g, h, k, m, n\}$

$A \cap B = \{a, b, e\}$

$B \cap C = \{b, e, g, h\}$

$A \cap C = \{b, d, e\}$

$A \cap B \cap C = \{b, e\}$

Now, $|A \cup B \cup C| = 10$

$|A| = 5 \quad |B| = 5 \quad |C| = 8$

$|A \cap B| = 3 \quad |B \cap C| = 4$

$|A \cap C| = 3$

$|A \cap B \cap C| = 2$

Thus

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 5 + 5 + 8 - 3 - 3 - 4 + 2$$

$$= 10$$

$$= |A \cup B \cup C|$$

Hence addition principle is verified

Q.2. $|A| = 6$, $|B| = 8$, $|C| = 6$

$|A \cup B \cup C| = 11$, $|A \cap B| = 3$

$|A \cap C| = 2$, $|B \cap C| = 5$

Find $|A \cap B \cap C|$

3. In a survey of 260 college students, the data were obtained

64 had taken mathematics

94 " " Computer Science

58 " " business

28 had taken both Mathematics & business

26 had taken both Mathematics & computer science

22 had taken both computer science & business

14 had taken all three courses.

a) How many students had taken none of the three types of courses?

b) How many had taken only a computer science course?

Sol: $|M \cup C \cup B|$ = The set of students who had taken at least one of the courses.

$|M| = 64$, $|C| = 94$, $|B| = 58$

$|M \cap B| = 28$, $|M \cap C| = 26$

$|C \cap B| = 22$, $|M \cap B \cap C| = 14$

By Addition Principle

$$|M \cup C \cup B| = |M| + |C| + |B| - |M \cap C| - |M \cap B| - |C \cap B| + |M \cap B \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$

$$= 154$$

a) Now, number of students who had taken none of the course

= (Total No. of students) − (No. of students with atleast one course)

$= 260 - 154 = 106$

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b) No. of students who had taken only computer science course

$= |C| - |M \cap C| - |C \cap B| + |M \cap B \cap C|$

$= 94 - 26 - 22 + 14$

$= 60$

④ Among the integers from 1 to 300,

i) How many of them are divisible by 3, 5 or 7 and are not divisible by 3 nor 5 nor by 7.

ii) How many of them are divisible by 3 but not by 5 nor by 7.

Sol: Let,

A be a set of integers among 1 and 300 divisible by 3

B be a set of integers among 1 and 300 divisible by 5.

C be a set of integers among 1 and 300 divisible by 7.

$$\therefore \quad |A| = \frac{300}{3} = 100$$

$$|B| = \frac{300}{5} = 60$$

$$|C| = \frac{300}{7} = 42$$

No. of integers divisible by 3 & 5 are

$$|A \cap B| = \frac{300}{3 \times 5} = 20$$

Similarly, for 3 & 7, $|A \cap C| = \frac{300}{3 \times 7} = 14$

& for 5 & 7, $|B \cap C| = \frac{300}{5 \times 7} = 8$

(i)

Also, the no. of integers which are divisible by at least one of them, i.e 3, 5 or 7.

$$= |A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 100 + 60 + 42 - 20 - 14 - 8 + 2$$

$$= 162$$

$\therefore$ No. of integers which are not divisible by 3, nor by 5 nor by 7

$= 300 - 162$

$= 138$

ii) No. of integers divisible by 3 but not by 5 nor by 7

$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 100 - 20 - 14 + 2$

$= 68$

Practice Problems

1. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of the three popular options, air conditioned A, radio R, and power windows W, were already installed. The survey found

15 had A

12 had R

11 had W

5 had A & W

9 had A & R

4 had R & W

5 had all three

Find the number of cars with

a) Only Power Windows

b) Only Air Conditioning

c) Only Radio

d) Radio & Power Window but no air conditioner.

e) air conditioning & radio but no power window.

f) Only one of the options

g) At least one option

h) None of the options

(2) Determine the number of integers between 1 and 250 that are divisible by 2 or 3 or 5 or 7.

Cartesian Product

The cartesian product of two non-empty sets A and B is defined as

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

eg. $A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$$\therefore A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$$

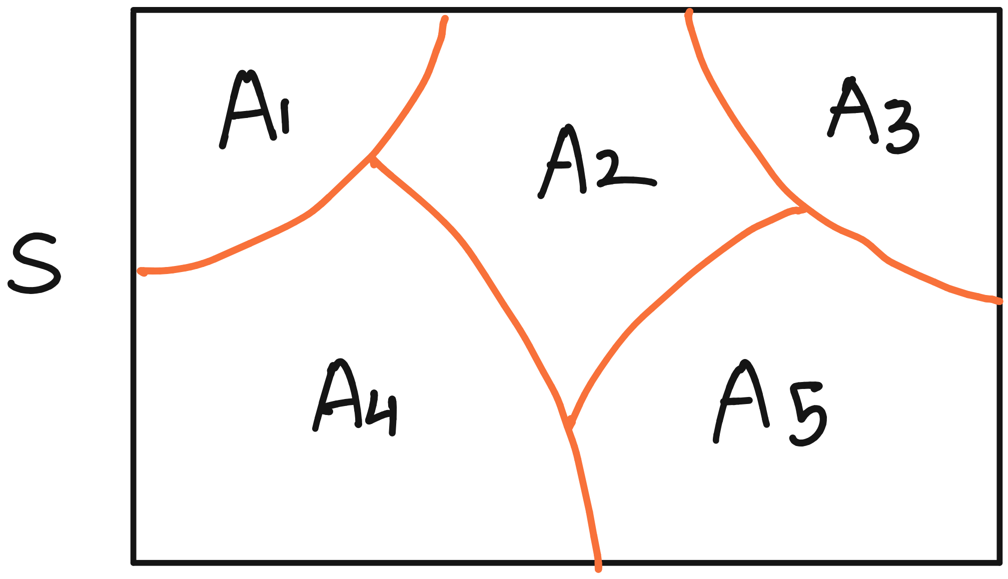
Partition

A partition of S is a collection $\{A_i\}$ of non empty subsets of S such that

i) Each a in S belongs to one of A_i

ii) The sets of $\{A_i\}$ are mutually disjoint . i.e $A_i \neq A_j$ then $A_i \cap A_j = \phi$

The subsets in partitions are called cells.



$S = \{A_1, A_2, A_3, A_4, A_5\}$ is a partition.

Q.1. Let $X = \{1, 2, 3, \ldots 8, 9\}$.

Determine whether or not each of the following is a partition.

a) $[\{1,3,6\}, \{2,8\}, \{5,7,9\}]$

b) $[\{2,4,5,8\}, \{1,9\}, \{3,6,7\}]$

c) $[\{1, 5, 7\}, \{2,4,8,9\}, \{3,5,6\}]$

d) $[\{1,2,7\}, \{3,5\}, \{4,6,8,9\}, \{3,5\}]$

Sol: a) $4 \in X$ but 4 does not belong to any cell.

$\therefore$ It is not a partition of X.

b) $\because$ $\{2,4,5,8\} \cup \{1,9\} \cup \{3,6,7\} = S$

& $\{2,4,5,8\} \cap \{1,9\} \cap \{3,6,7\} = \phi$

i.e mutually disjoint

Hence it is a PARTITION of X.

c) $\because \{1,5,7\} \cap \{3,5,6\} = 5$

$\therefore$ It is not a partition of X.

d) $\because$ 2nd & 4th cells are identical.

$\therefore$ It is not a partition of X.

Practice Problems

1. Let $S = \{1,2,3,4,5,6\}$. Determine whether or not the following are partitions of S.

a) $P_1 = [\{1,2,3\}, \{1,4,5,6\}]$

b) $P_2 = [\{1,2\}, \{3,5,6\}]$

c) $P_3 = [\{1,3,5\}, \{2,4\}, \{6\}]$

d) $P_4 = [\{1,3,5\}, \{2,4,6,7\}]$

2. Determine whether or not each of the following is a partition of a set N of +ve integers.

a) $[\{n : n > 5\}, \{n : n < 5\}]$

b) $[\{n : n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}]$

c) $[\{n : n^2 > 11\}, \{n : n^2 < 11\}]$

Power Set

Let A be a given set, then the set of all possible subsets of A is called power set of A and is denoted by P(A).

<u>eg.</u> $A = \{1, 2, 3\}$

$|A| = 3$

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$\therefore\ |P(A)| = 2^{|A|} = 2^3$$

$$= 8$$

1. If $A = \{2^x \mid x^2 - 5x + 6 = 0\}$

$B = \{2^x \mid x^3 - 6x^2 + 11x - 6 = 0\}$

Verify that $P(A) \subseteq P(B)$

Sol:

$x^2 - 5x + 6 = 0$

$\Rightarrow \quad x = 2, 3$

$x^3 - 6x^2 + 11x - 6 = 0$

$\Rightarrow (x-1)(x^2 - 5x + 6) = 0$

$\Rightarrow (x-1)(x-3)(x-2) = 0$

$\therefore \quad x = 1, 2, 3$

$\therefore \; A = \{2^2, 2^3\} = \{4, 8\}$

$\therefore \; B = \{2^1, 2^2, 2^3\} = \{2, 4, 8\}$

$$P(A) = \{\phi, \{4\}, \{8\}, \{4, 8\}\}$$

$$P(B) = \{\phi, \{1\}, \{4\}, \{8\}, \{1, 4\}, \{1, 8\}, \{4, 8\}, \{1, 4, 8\}\}$$

$\therefore$ Every element of $P(A)$ belongs to set $P(B)$

$$\therefore \quad P(A) \subseteq P(B)$$