Module 1: Sets & Logic

Introduction to Set Theory :

Set:

A set is a well defined collection of objects called elements or members of the set.

Eg: * A collection of blue coloured birds.

* Collection of real numbers between 0 & 1.

* $A = \{\alpha, \beta, \gamma, \delta\}$

* $B = \{$Ram, Shyam, Rohan, Sohan$\}$

Subsets

If every element in a set A is present in the set B, then A is called as the subset of B.

We can also say that A is contained in B.

This relationship is denoted by

$$A \subseteq B \quad \text{OR}$$

$$B \supseteq A$$

If A is not a subset of B or at least one element of A does not belong to B, then we write:

$$A \nsubseteq B \quad \text{or} \quad B \nsupseteq A$$

Example:

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5\}$

$\because$ All the elements of set A is present in set B.

$\therefore$ Set A is a subset of set B

$\Rightarrow$ $A \subseteq B$ OR

$B \supseteq A$

Universal Set

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the universal set.

For example, in a plane geometry, the universal set consists of all the points in te plane.

It is usually denoted by U.

Empty Set

The Set with no elements is called as an Empty Set or Null Set and is denoted by ϕ

e.g :

A= {x: x is a +ve integer, x<1}

=> A has no elements

Note: (1) A is said to be equal to B iff A is a subset of B and B is a subset of A.
(2) For any set A, $\phi \subseteq A$

Cardinality of a Set :-

The cardinality of a finite set is "number of elements in the set"

It is denoted by $|A|$ for set A.

e.g. $A = \{1, 2, 3\}$

$\therefore |A| = 3$

Finite and Infinite Set :

A set A is called <u>finite</u> set if it has 'n' distinct elements, where $n \in \mathbb{N}$.

In this case, n is called the cardinality of A and is denoted by $|A|$.

A set that is not finite is called an infinite set.

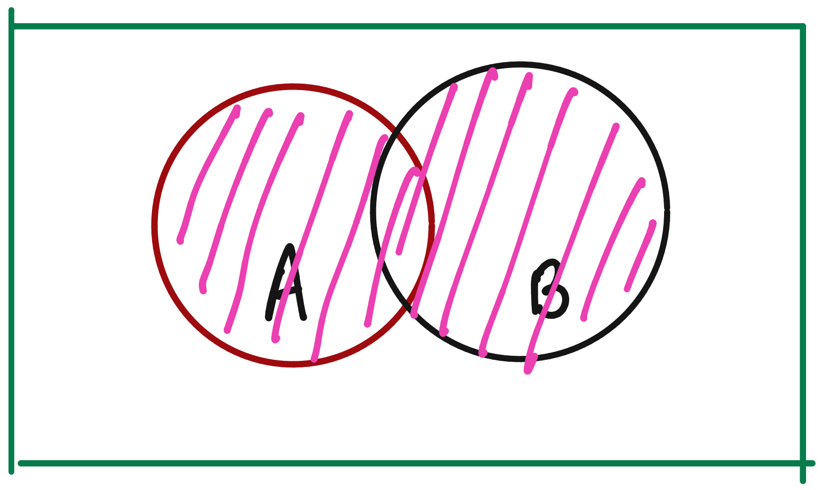
Operations on Sets & Venn Diagrams

1. Union of Sets

Let A and B be two non-empty sets, then the union of A and B is also a set

i.e the collection of all the elements of either A or B

$\therefore A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

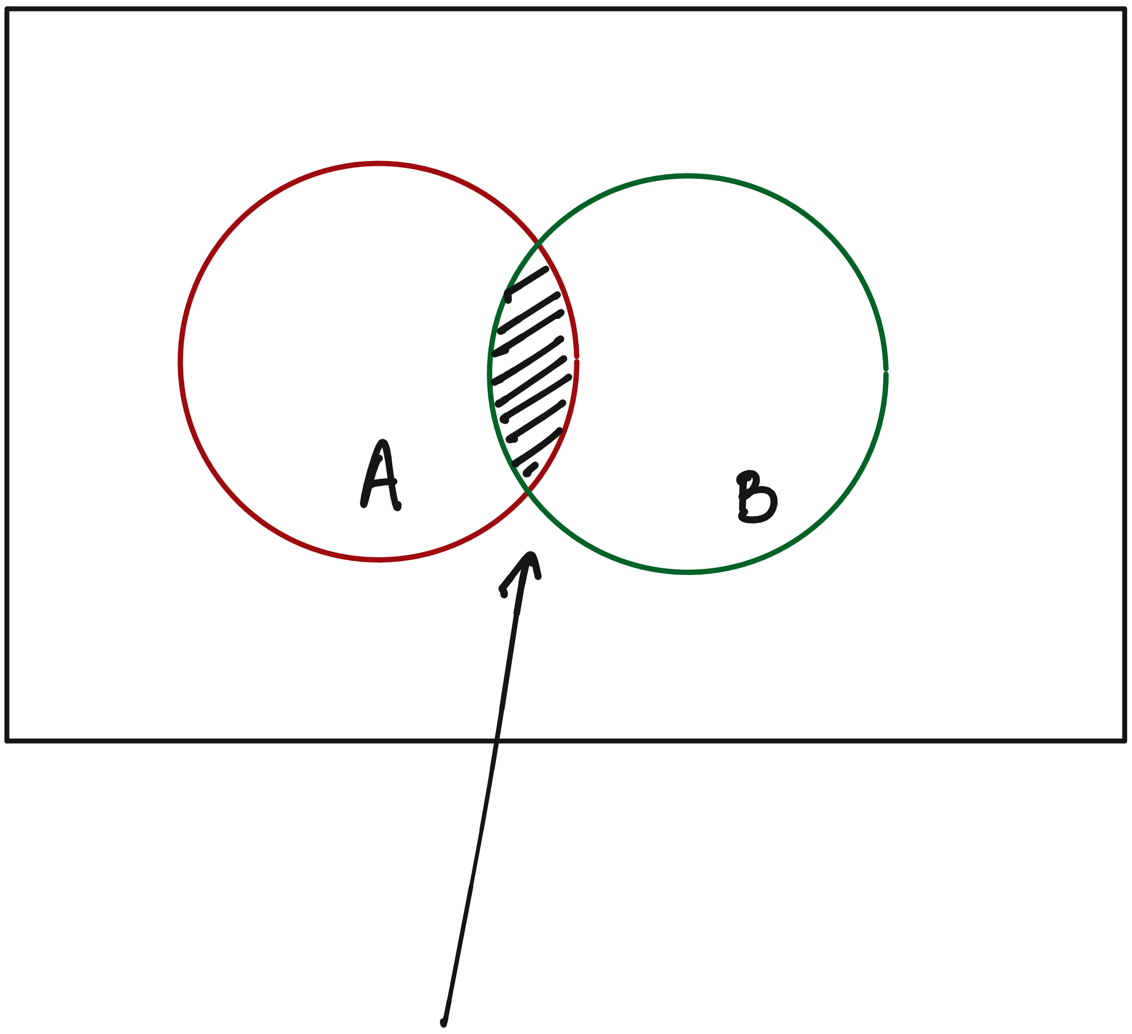


2. Intersection

The intersection of two non-empty sets A and B is denoted by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

i.e The collection of all the elements of both A and B.



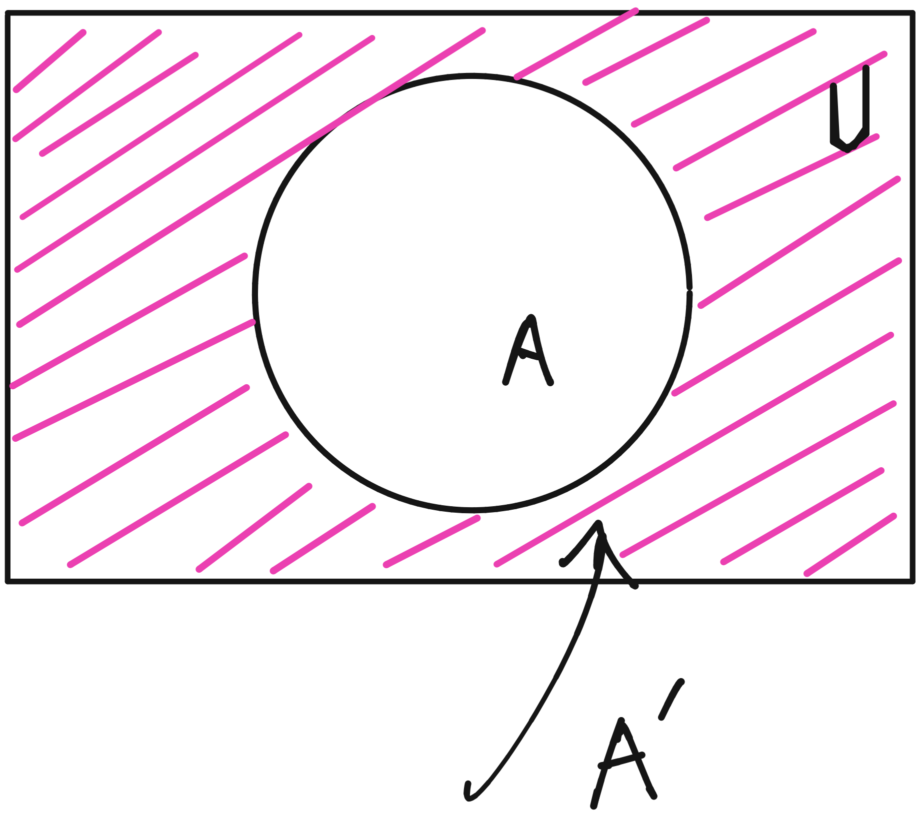
$A \cap B$

3. <u>Complement</u>

Let A be a non-empty subset of a given universal set U, then the complement of A is a set such that it is the collection of all elements of U but not A.

It is denoted by A' or $\bar{A}$ or A^c

$$A' = \{x \mid x \in U \text{ and } x \notin U\}$$



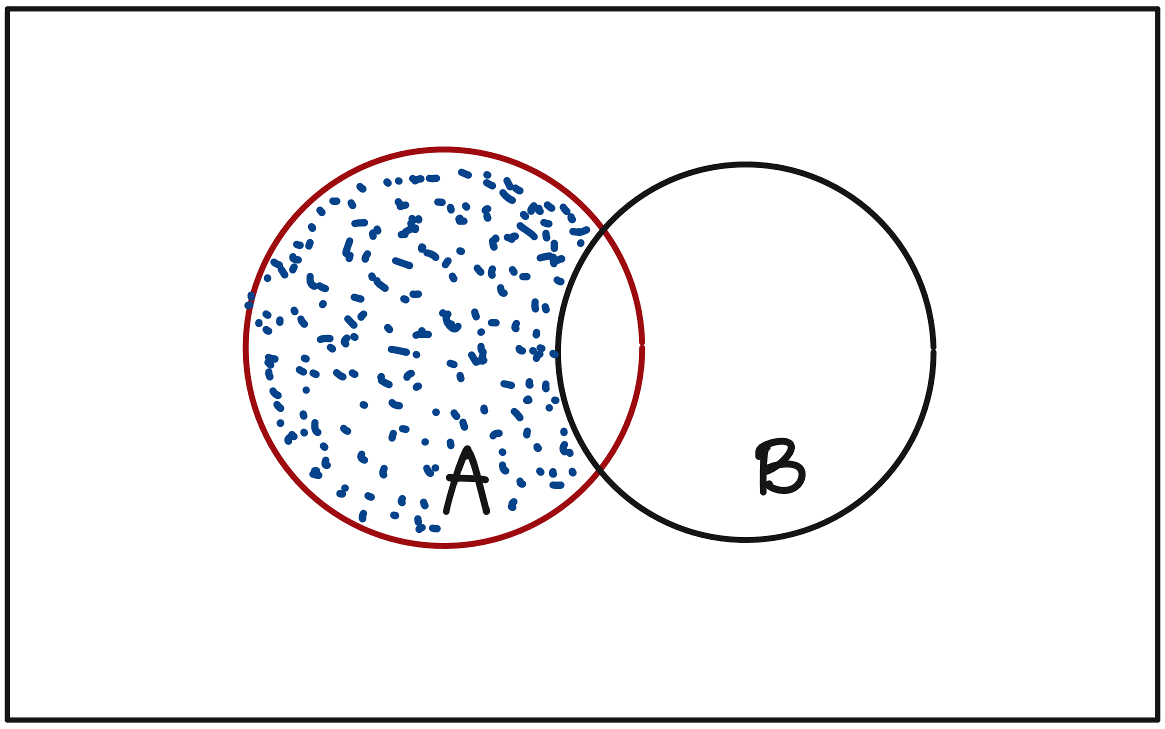
4. Set Difference

Let A and B be 2 non-empty sets, then the difference of A and B is a set

i.e the collection of all the elements of A but not the elements of B.

$$\therefore A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Also denoted as $A - B$.



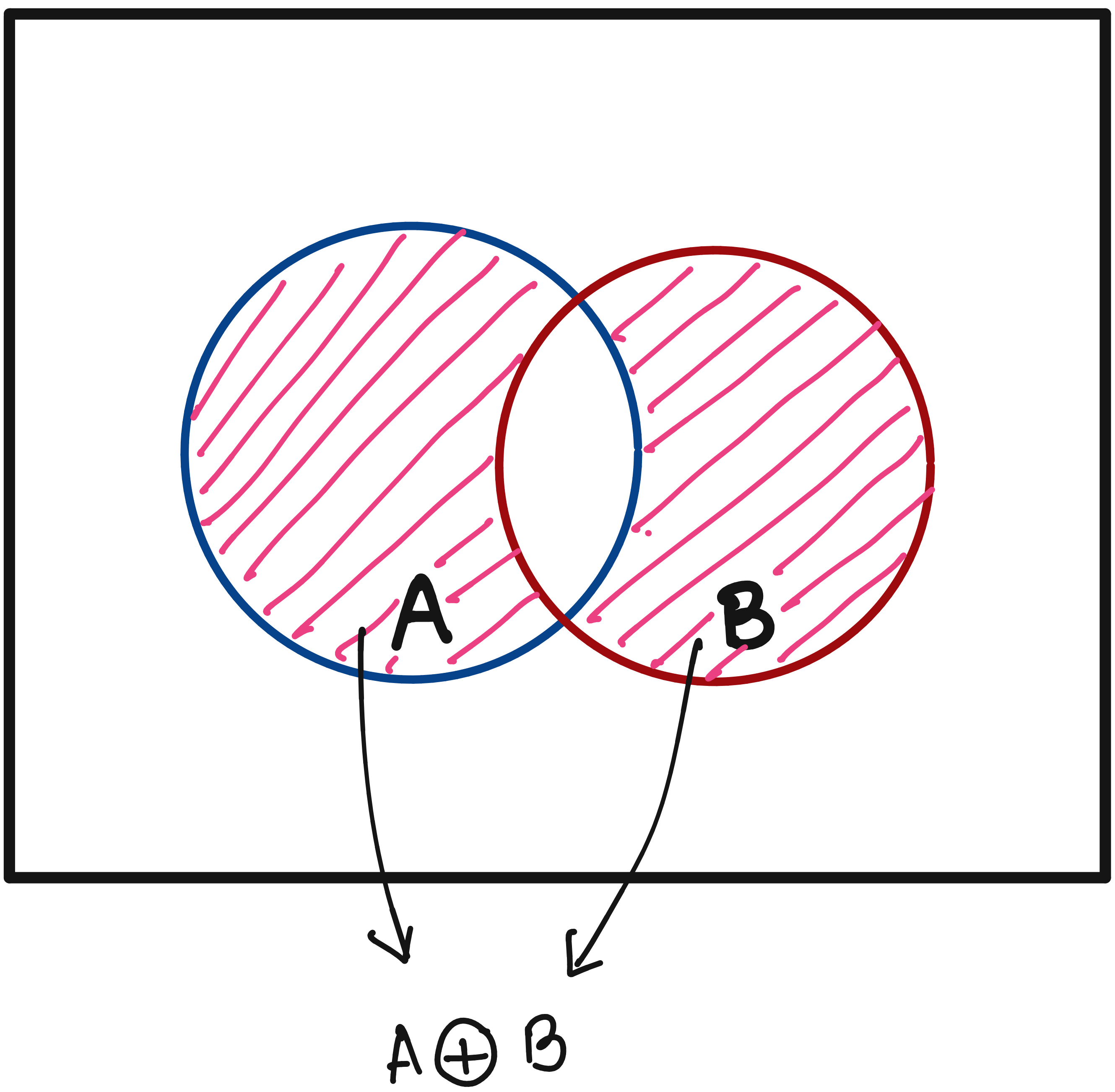
$A \setminus B$

$A - B$

5. Ring Sum ($\oplus$)

Let A and B be 2 non empty sets, then the ring sum of A and B is the set of all the elements which are either in A or in B but not in both.

$$A \oplus B = \{ x \mid x \in A \cup x \in B \text{ but } x \notin A \cap B \}$$



Algebraic Properties of Sets:

1. Commutative Property

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Associative Property

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributive Property

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. <u>Idempotent Property</u>

$$A \cup A = A$$
$$A \cap A = A$$

5. <u>Properties of Complement</u>

$$(\bar{\bar{A}}) = A$$
$$A \cup \bar{A} = U$$
$$A \cap \bar{A} = \phi$$
$$\bar{\phi} = U$$
$$\bar{U} = \phi$$

$$\left.\begin{aligned} \overline{A \cup B} &= \bar{A} \cap \bar{B} \\ \overline{A \cap B} &= \bar{A} \cup \bar{B} \end{aligned}\right\} \text{De. Morgan's Laws}$$

6. <u>Misc.</u>

$$\phi \cap A = A \cap \phi = \phi$$
$$U \cap A = A \cap U = A$$

Some Solved Examples

1. Let A, B, C be the subsets of universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer.

Sol: B can be expressed as

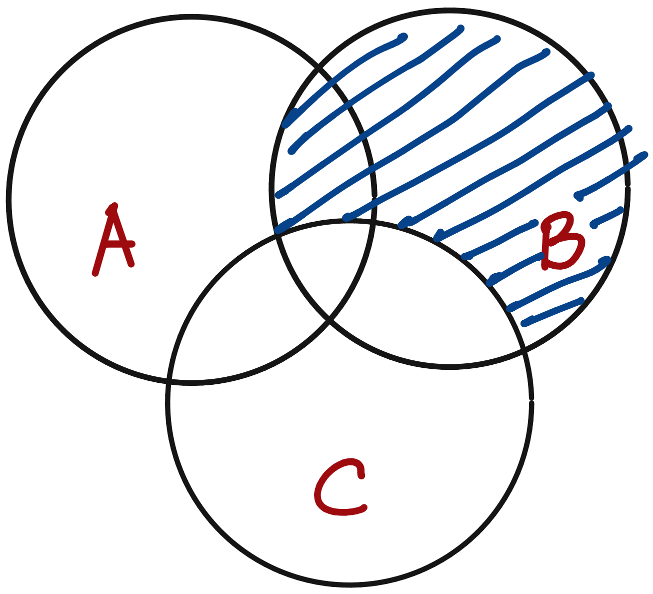
$$\begin{aligned}
B &= B \cap U \\
&= B \cap (A \cup \bar{A}) && \because A \cup \bar{A} = U \\
&= (B \cap A) \cup (B \cap \bar{A}) && \because \text{By DL} \\
&= (A \cap B) \cup (\bar{A} \cap B) && \because \text{By CL} \\
&= (A \cap C) \cup (\bar{A} \cap C) && \because \text{Given} \\
&= (A \cup \bar{A}) \cap C && \because \text{By D.L} \\
&= U \cap C && \because \text{Universal Set} \\
&= C
\end{aligned}$$

$\therefore B = C$ Hence Proved.

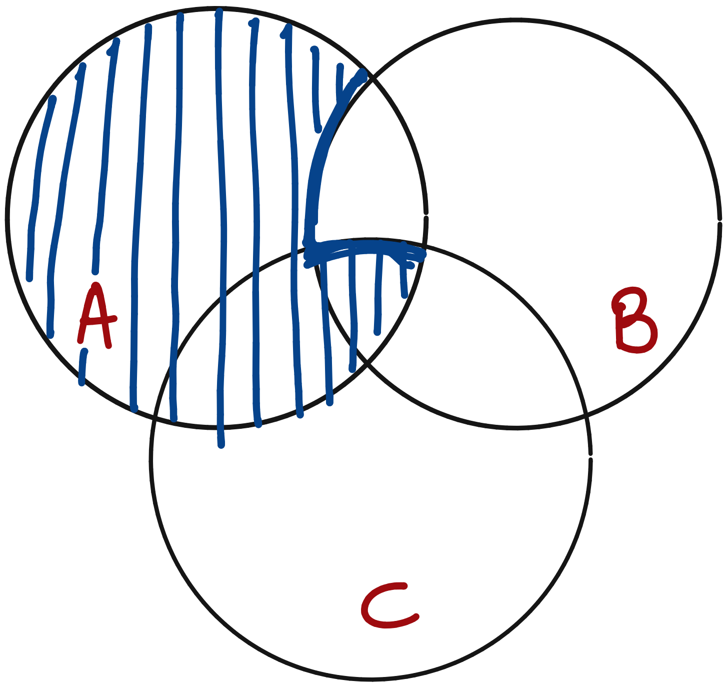
2. Using Venn Diagrams, prove that

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap B \cap C)$$

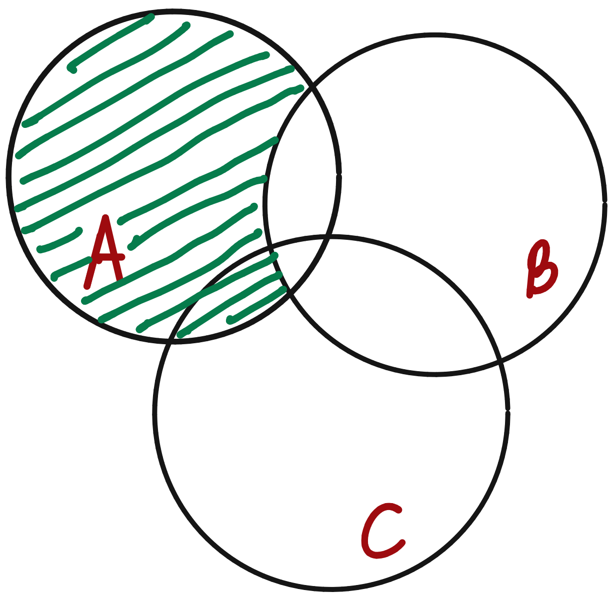
Sol:

a) 

$\Rightarrow$ $B \setminus C$ or $B - C$

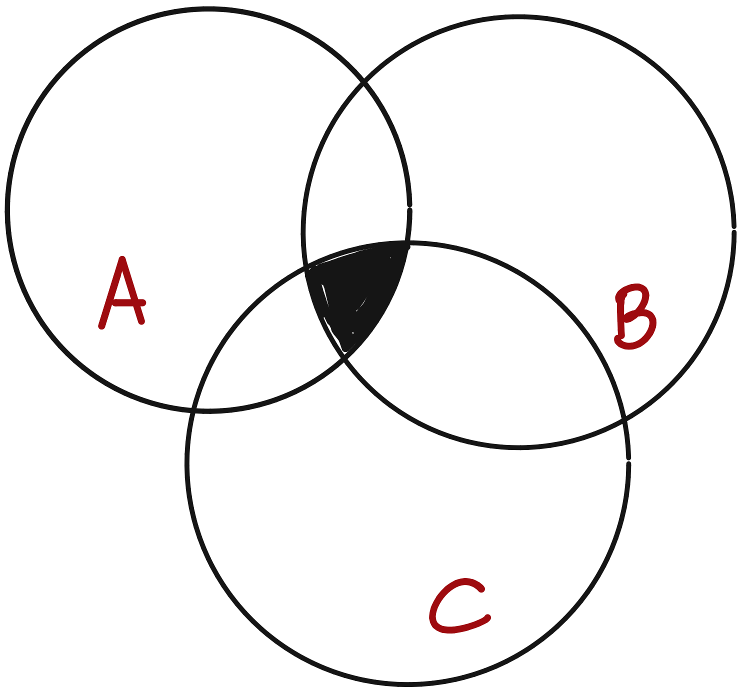
b) 

$\Rightarrow$ $A \setminus (B \setminus C)$

c) 

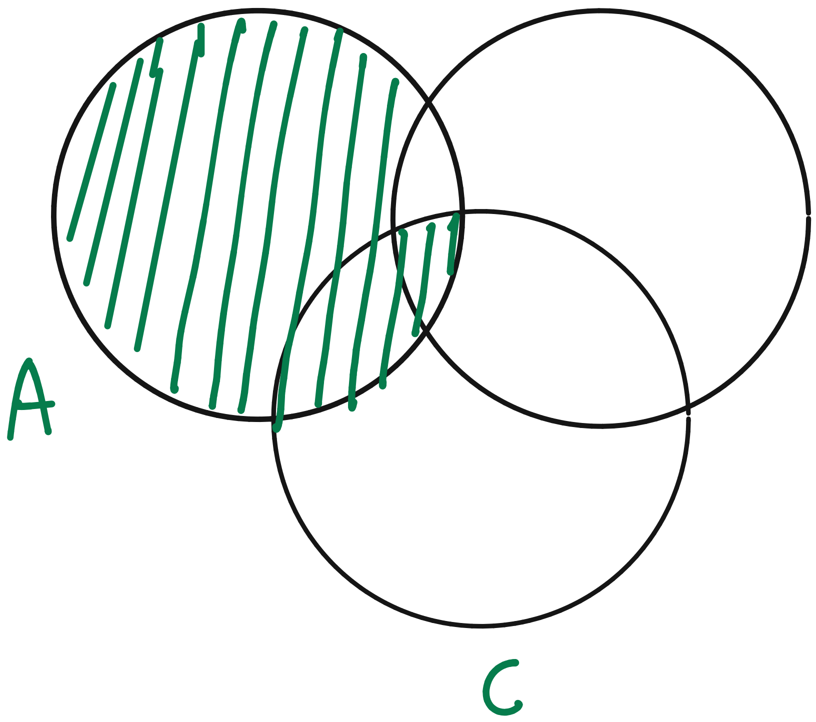
$\Rightarrow$ $A \setminus B$

d)



$\Rightarrow \quad A \cap B \cap C$

e)



$\Rightarrow \quad (A \setminus B) \cup (A \cap B \cap C)$

If we compare a & e, we find both the Venn Diagrams to be same, hence L.H.S = R.H.S.

Practice Problems.

1. Using Venn Diagram show

   a) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

   b) $A \cap B \cap C = A \setminus [(A \setminus B) \cup (A \setminus C)]$

2. Using Venn Diagram, prove that

   $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   $A = \{1, 2, 4, 6, 8\}$

   $B = \{2, 4, 5, 9\}$

   $C = \{x \mid x \text{ is a +ve integer and } x^2 < 16\} = \{1, 2, 3\}$

   $D = \{7, 8\}$

Compute

a) $A \cup B$

b) $A \cup C$

c) $A \cup D$

d) $B \cup C$

e) $A \cap C$

f) $C \cap D$

g) $A \cap D$

h) $B \cap C$

i) $A \setminus B$

j) $B \setminus A$

k) $C \setminus D$

l) $\overline{C}$

m) $\overline{A}$

n) $A \oplus B$

o) $C \oplus D$

p) $B \oplus C$

Compute

a) $A \cup B$ $= \{1,2,4,5,6,8,9\}$

b) $A \cup C$ $= \{1,2,3,4,6,8\}$

c) $A \cup D$ $= \{1,2,4,6,7,8\}$

d) $B \cup C$ $= \{1,2,3,4,5,9\}$

e) $A \cap C$ $= \{1,2\}$

f) $C \cap D$ $= \phi$

g) $A \cap D$ $= \{8\}$

h) $B \cap C$ $= \{2\}$

i) $A \setminus B$ $= \{1,6,8\}$

j) $B \setminus A$ $= \{5,9\}$

k) $C \setminus D$ $= \{1,2,3\}$

l) $\overline{C}$ $= \{4,5,6,7,8,9\}$

m) $\overline{A}$ $= \{3,5,7,9\}$

n) $A \oplus B$ $= \{1,5,6,8,9\}$

o) $C \oplus D$ $= \{1,2,3,7,8\}$

p) $B \oplus C$ $= \{1,2,4,5,9\}$

Compute:

a) $A \cup B \cup C =$

b) $A \cap B \cap C =$

c) $A \cap (B \cup C) =$

d) $(A \cup B) \cap C =$

e) $\overline{A \cup B} =$

f) $\overline{A \cap B} =$

g) $B \cup C \cup D =$

h) $A \cup A =$

i) $A \cap \overline{A} =$

j) $A \cup \overline{A} =$

k) $A \cap (\overline{C} \cup D) =$

Compute :

a) $A \cup B \cup C = \{1,2,3,4,5,6,8,9\}$

b) $A \cap B \cap C = \{2\}$

c) $A \cap (B \cup C) = \{1,2,4\}$

d) $(A \cup B) \cap C = \{8\}$

e) $\overline{A \cup B} = \{3,7\}$

f) $\overline{A \cap B} = \{1,3,5,6,7,8,9\}$

g) $B \cup C \cup D = \{1,2,3,4,5,7,8,9\}$

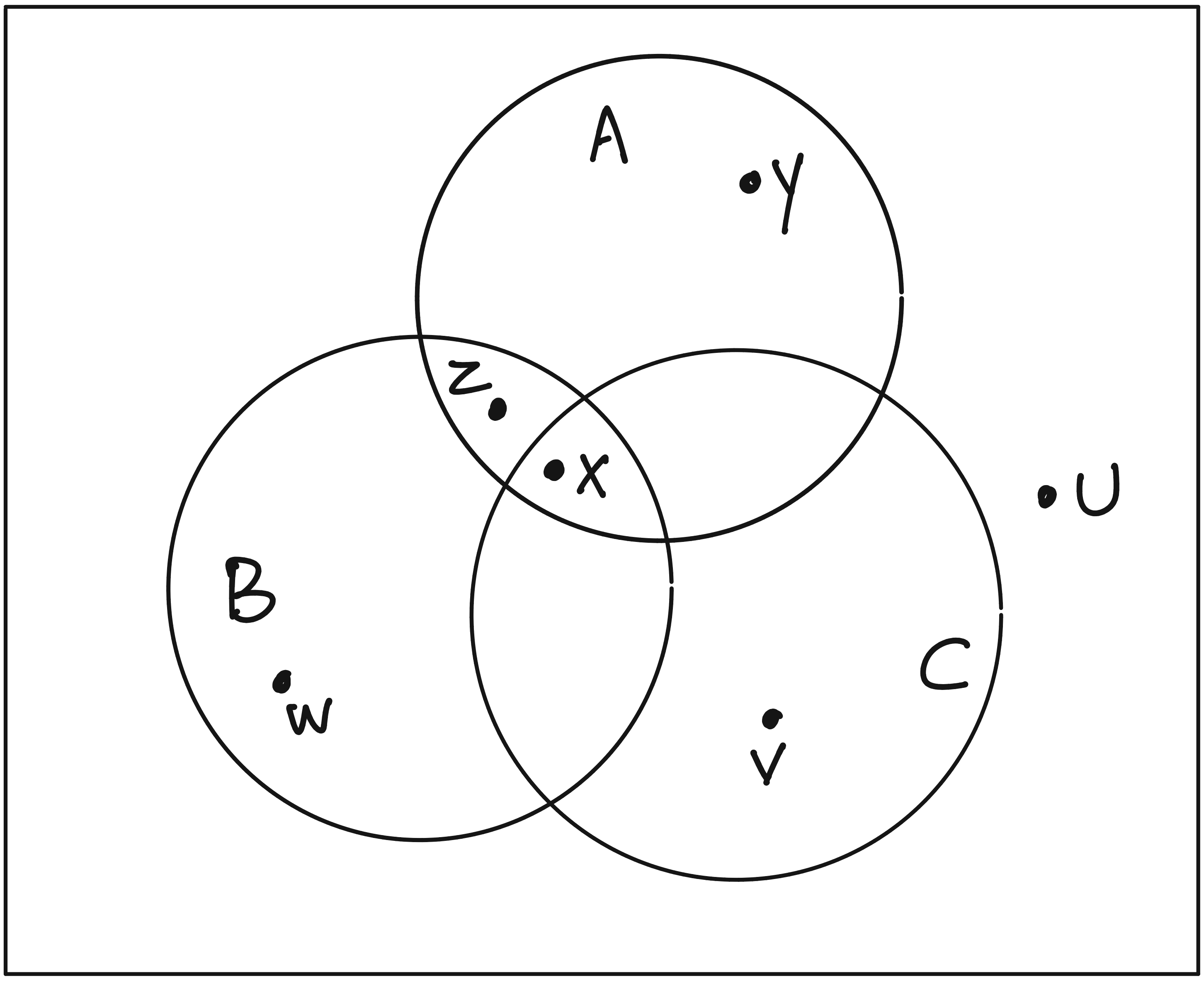
h) $A \cup A = A$

i) $A \cap \bar{A} = \phi$

j) $A \cup \bar{A} = U$

k) $A \cap (\bar{C} \cup D) = \{4,6,8\}$

4.



Identify the following as true or false

a) $y \in A \cap B$

b) $x \in B \cup C$

c) $w \in B \cap C$

d) $u \notin C$

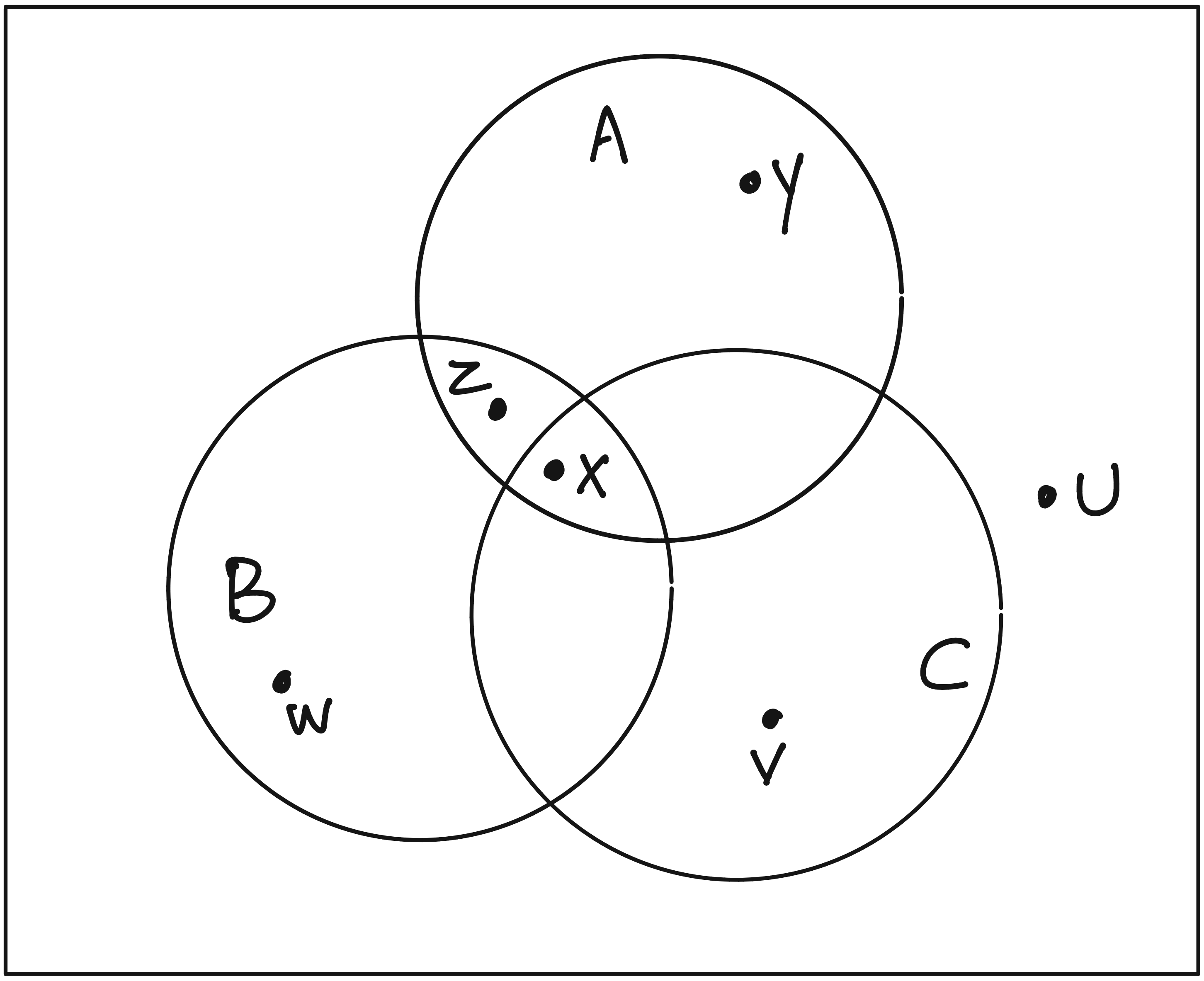
e) $x \in A \cap B \cap C$

f) $y \in A \cup B \cup C$

g) $z \in A \cap C$

h) $v \in B \cap C$

4.



Identify the following as true or false

a) $y \in A \cap B$ False

b) $x \in B \cup C$ True

c) $w \in B \cap C$ False

d) $u \notin C$ True

e) $x \in A \cap B \cap C$ True

f) $y \in A \cup B \cup C$ True

g) $z \in A \cap C$ False

h) $v \in B \cap C$ False

Principle of Inclusion & Exclusion

(Addition Principle)

1. Let A and B be two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Also called as <u>disjoint union</u>

2. Let A, B and C be finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Problems based on Addition Principle

1. Let $A = \{a, b, c, d, e\}$, $B = \{a, b, e, g, h\}$ and $C = \{b, d, e, g, h, k, m, n\}$

We have,

$$A \cup B \cup C = \{a, b, c, d, e, g, h, k, m, n\}$$

$$A \cap B = \{a, b, e\}$$

$$B \cap C = \{b, e, g, h\}$$

$$A \cap C = \{b, d, e\}$$

$$A \cap B \cap C = \{b, e\}$$

Now, $|A \cup B \cup C| = 10$

$|A| = 5 \quad |B| = 5 \quad |C| = 8$

$|A \cap B| = 3 \quad |B \cap C| = 4$

$|A \cap C| = 3$

$|A \cap B \cap C| = 2$

Thus

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 5 + 5 + 8 - 3 - 3 - 4 + 2$$

$$= 10$$

$$= |A \cup B \cup C|$$

Hence addition principle is verified

Q.2. $|A| = 6$, $|B| = 8$, $|C| = 6$

$|A \cup B \cup C| = 11$, $|A \cap B| = 3$

$|A \cap C| = 2$, $|B \cap C| = 5$

Find $|A \cap B \cap C|$

3. In a survey of 260 college students, the data were obtained

64 had taken mathematics
94 " " Computer Science
58 " " business
28 had taken both Mathematics & business
26 had taken both Mathematics & computer science
22 had taken both computer science & business
14 had taken all three courses.

a) How many students had taken none of the three types of courses?

b) How many had taken only a computer science course?

Sol: $|M \cup C \cup B|$ = The set of students who had taken at least one of the courses.

$|M| = 64$, $|C| = 94$, $|B| = 58$

$|M \cap B| = 28$, $|M \cap C| = 26$

$|C \cap B| = 22$, $|M \cap B \cap C| = 14$

By Addition Principle

$$|M \cup C \cup B| = |M| + |C| + |B| - |M \cap C| - |M \cap B| - |C \cap B| + |M \cap B \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$

$$= 154$$

a) Now, number of students who had taken none of the course

= (Total No. of students) − (No. of students with atleast one course)

$= 260 - 154 = 106$

---

b) No. of students who had taken only computer science course

$= |C| - |M \cap C| - |C \cap B| + |M \cap B \cap C|$

$= 94 - 26 - 22 + 14$

$= 60$

④ Among the integers from 1 to 300,

i) How many of them are divisible by 3, 5 or 7 and are not divisible by 3 nor 5 nor by 7.

ii) How many of them are divisible by 3 but not by 5 nor by 7.

<u>Sol:</u> Let,

A be a set of integers among 1 and 300 divisible by 3

B be a set of integers among 1 and 300 divisible by 5.

C be a set of integers among 1 and 300 divisible by 7.

$$\therefore \quad |A| = \frac{300}{3} = 100$$

$$|B| = \frac{300}{5} = 60$$

$$|C| = \frac{300}{7} = 42$$

No. of integers divisible by 3 & 5 are

$$|A \cap B| = \frac{300}{3 \times 5} = 20$$

Similarly, for 3 & 7, $|A \cap C| = \frac{300}{3 \times 7} = 14$

& for 5 & 7, $|B \cap C| = \frac{300}{5 \times 7} = 8$

(i)

Also, the no. of integers which are divisible by at least one of them, i.e 3, 5 or 7.

$$= |A \cup B \cup C|$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 100 + 60 + 42 - 20 - 14 - 8 + 2$$

$$= 162$$

$\therefore$ No. of integers which are not divisible by 3, nor by 5 nor by 7

$= 300 - 162$

$= 138$

ii) No. of integers divisible by 3 but not by 5 nor by 7

$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 100 - 20 - 14 + 2$

$= 68$

Practice Problems

1. A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of the three popular options, air conditioned A, radio R, and power windows W, were already installed. The survey found

15 had A

12 had R

11 had W

5 had A & W

9 had A & R

4 had R & W

5 had all three

Find the number of cars with

a) Only Power Windows

b) Only Air Conditioning

c) Only Radio

d) Radio & Power Window but no air conditioner.

e) air conditioning & radio but no power window.

f) Only one of the options

g) At least one option

h) None of the options

(2) Determine the number of integers between 1 and 250 that are divisible by 2 or 3 or 5 or 7.

Cartesian Product

The cartesian product of two non-empty sets A and B is defined as

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

eg. $A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$$\therefore A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$$

Q.1. Given $A = \{2, 3, 4, 5\}$ and $B = \{4, 16, 23\}$, $a \in A$ and $b \in B$, find the set of ordered pairs such that $a^2 < b$?

Sol. As $2^2 < 16, 23$

$3^2 < 16, 23$

$4^2 < 23$,

$\therefore$ we have the set of ordered pairs such that $a^2 < b$ is

$\{\{2, 16\}, \{2, 23\}, \{3, 16\}, \{3, 23\}, \{4, 23\}\}$

Q.2. If $A = \{9, 10\}$ & $B = \{3, 4, 6\}$.

Find $A \times B$ & $|A \times B|$.

Sol.

$A \times B = \{(9,3), (9,4), (9,6), (10,3),$

$(10,4), (10,6)\}$

$|A \times B| = |A| * |B|$

$= 2 \times 3$

$= 6$

Q.3. Given $A \times B$ has 15 ordered pairs and A has 5 elements, Find the number of elements in B.

<u>Sol.</u> We know that:-

$$|A \times B| = |A| * |B|$$

$$\therefore 15 = 5 * |B|$$

Therefore, B has $15/5 = 3$ elements

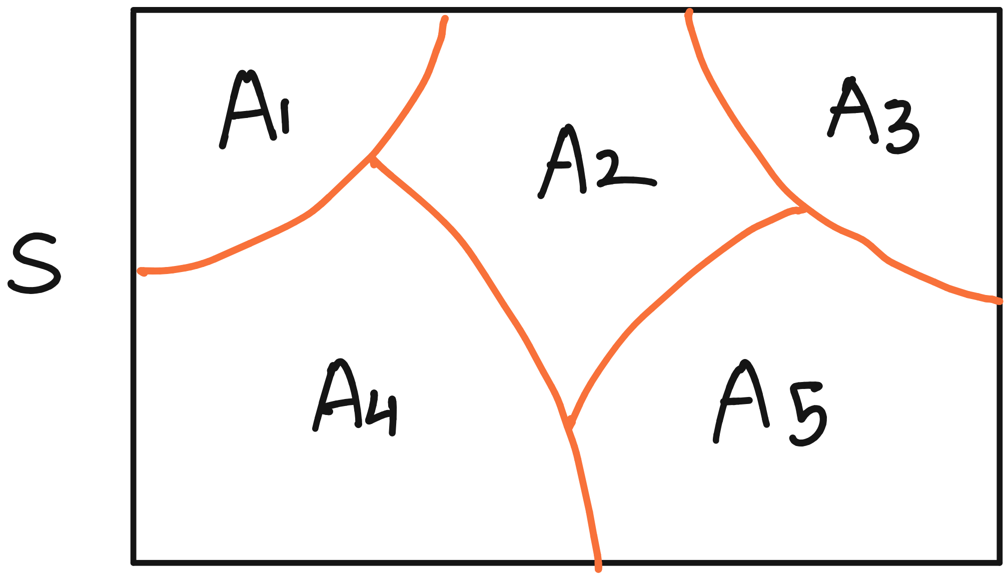
Partition

A partition of S is a collection $\{A_i\}$ of non empty subsets of S such that

i) Each a in S belongs to one of A_i

ii) The sets of $\{A_i\}$ are mutually disjoint. i.e $A_i \neq A_j$ then $A_i \cap A_j = \phi$

The subsets in partitions are called cells.



$S = \{A_1, A_2, A_3, A_4, A_5\}$ is a partition.

Q.1. Let $X = \{1, 2, 3, \ldots\ldots 8, 9\}$.

Determine whether or not each of the following is a partition.

a) $[\{1,3,6\}, \{2,8\}, \{5,7,9\}]$

b) $[\{2,4,5,8\}, \{1,9\}, \{3,6,7\}]$

c) $[\{1, 5, 7\}, \{2,4,8,9\}, \{3,5,6\}]$

d) $[\{1,2,7\}, \{3,5\}, \{4,6,8,9\}, \{3,5\}]$

Sol: a) $4 \in X$ but 4 does not belong to any cell.

$\therefore$ It is not a partition of X.

b) $\because$ $\{2,4,5,8\} \cup \{1,9\} \cup \{3,6,7\} = S$

& $\{2,4,5,8\} \cap \{1,9\} \cap \{3,6,7\} = \phi$

i.e mutually disjoint

Hence it is a PARTITION of X.

c) $\because \{1,5,7\} \cap \{3,5,6\} = 5$

$\therefore$ It is not a partition of X.

d) $\because$ 2nd & 4th cells are identical.

$\therefore$ It is not a partition of X.

Practice Problems

1. Let $S = \{1,2,3,4,5,6\}$. Determine whether or not the following are partitions of S.

a) $P_1 = [\{1,2,3\}, \{1,4,5,6\}]$

b) $P_2 = [\{1,2\}, \{3,5,6\}]$

c) $P_3 = [\{1,3,5\}, \{2,4\}, \{6\}]$

d) $P_4 = [\{1,3,5\}, \{2,4,6,7\}]$

2. Determine whether or not each of the following is a partition of a set N of +ve integers.

a) $[\{n : n > 5\}, \{n : n < 5\}]$

b) $[\{n : n > 5\}, \{0\}, \{1, 2, 3, 4, 5\}]$

c) $[\{n : n^2 > 11\}, \{n : n^2 < 11\}]$

Power Set

Let A be a given set, then the set of all possible subsets of A is called power set of A and is denoted by P(A).

eg. $A = \{1, 2, 3\}$

$$|A| = 3$$

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$\therefore \ |P(A)| = 2^{|A|} = 2^3$$

$$= 8$$

1. If $A = \{2^x \mid x^2 - 5x + 6 = 0\}$

$B = \{2^x \mid x^3 - 6x^2 + 11x - 6 = 0\}$

Verify that $P(A) \subseteq P(B)$

<u>Sol</u>:

$x^2 - 5x + 6 = 0$

$\Rightarrow \underline{x = 2, 3}$

$x^3 - 6x^2 + 11x - 6 = 0$

$\Rightarrow (x-1)(x^2 - 5x + 6) = 0$

$\Rightarrow (x-1)(x-3)(x-2) = 0$

$\therefore \underline{x = 1, 2, 3}$

$\therefore A = \{2^2, 2^3\} = \{4, 8\}$

$\therefore B = \{2^1, 2^2, 2^3\} = \{2, 4, 8\}$

$P(A) = \{\phi, \{4\}, \{8\}, \{4, 8\}\}$

$P(B) = \{\phi, \{1\}, \{4\}, \{8\}, \{1,4\},$
$\{1, 8\}, \{4, 8\}, \{1, 4, 8\}\}$

$\therefore$ Every element of $P(A)$ belongs to set $P(B)$

$\therefore \quad P(A) \subseteq P(B)$

Q.2. Let set $A = \{a, b, c, d\}$, determine the power set A.

Sol. $P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{d\},$
$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\},$
$\{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\},$
$\{b,c,d\}, \{a,b,c,d\}\}$

$\therefore$ $P(A)$ has $2^4 = 16$ elements

Q.3. If $A\{\phi, a\}$, then construct the sets $A \cup P(A)$, $A \cap P(A)$.

Sol. $P(A) = \{\phi, \{a\}, \{\phi\}, \{\phi, a\}\}$

$A \cup P(A) = \{\phi, a, \{\phi\}, \{a\}, \{\phi, a\}\}$

$A \cap P(A) = \{\phi\}$

Introduction to Propositional logic

Propositions & Logical Operations

Proposition or statement form is a declarative sentence which can be true or false but not both.

eg. i) Paris is in France (True)

ii) $1+2 = 2$ (False)

Compound Proposition consists of subpropositions and also the way in which they are connected.

eg. i) Roses are Red & Violets are blue.

Basic Logical Operation

1. Conjunction ($p \wedge q$) AND

Any two propositions when combined with the word 'AND' to form a compound proposition, it is called a conjunction.

$p \wedge q$, p & q , $p \cdot q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction, $p \vee q$ (or).

Any two propositions, when combined with the word 'OR' to form a compound proposition. It is called as disjunction.

$p \vee q$, $p + q$, p OR q.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3. Negation

Given any proposition p, the negation of p can be formed by inserting the word 'NOT' in p

p	$\sim p$
T	F
F	T

Logical Equivalence

Two compound propositions P & Q are said to be logically equivalent or simply equivalent or equal, if they have identical truth tables.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$$\therefore \sim(p \wedge q) \equiv \sim p \vee \sim q$$

Logical Implications

1) Conditional Statement:

If p and q are statements, the compound statement, "if p then q" denoted by $\boxed{p \Rightarrow q}$ is called conditional statement or implication.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:

$$\boxed{p \rightarrow q \equiv \sim p \vee q}$$

2) Biconditional Statement

If p and q are statements, the compound statement "p if and only if q" denoted by $p \Leftrightarrow q$, is called an equivalence or biconditional.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Laws of Logic

1. **Idempotent Law :-**

   1) $p \vee p \equiv p$

   2) $p \wedge p \equiv p$

2. **Associative Law :-**

   1) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

   2) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. Commutative Laws

1. $p \vee q \equiv q \vee p$
2. $p \wedge q \equiv q \wedge p$

4. Distributive Laws

1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. Identity Laws.

1. $p \vee F \equiv p$
2. $p \wedge F \equiv F$
3. $p \vee T \equiv T$
4. $p \wedge T \equiv p$

6. Complement Laws

1. $p \vee \sim p \equiv T$
2. $p \wedge \sim p \equiv F$
3. $\sim T \equiv F$
4. $\sim F \equiv T$

7. Involution Law

$\sim(\sim p) \equiv p$

8. De Morgan's Laws:

1. $\sim(p \vee q) \equiv \sim p \wedge \sim q$
2. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

9. Absorption Law

1. $p \wedge (p \vee q) \equiv p$

2. $p \vee (p \wedge q) \equiv p$

10. Implication Law

$p \rightarrow q \equiv \sim p \vee q$

11. Tautology:-

A statement which is true for all possible values of its propositional variables.

12. Contradiction:-

A statement which is always false is called contradiction.

Predicates:

Consider the following sentences:

i) "x is tall and handsome"

ii) "$x + 3 = 5$."

iii) "$x + y \geq 10$"

These sentences are not propositions, since they do not have any truth value.

However, if values are assigned to the variables, each of them becomes, which is either true or false.

For example, the above sentences can be converted to

a) He is tall and handsome

b) $2+3 = 5$

c) $2+5 \geq 10$

Definition :

An assertion that contains one or more variables is called a **predicate**.

A predicate P containing n variables $x_1, x_2 \dots\dots x_n$ is called an n-place predicate.

Examples a) and b) are 'one-place' predicate.

while example c) is a '2-place' predicate.

If we want to specify the variables in a predicate, we denote the predicate by $P(x_1, x_2 \cdots\cdots, x_n)$

Each variable X_i is also called as an argument.

<u>For ex:</u>

i) "<u>x is a city in India</u>" is denoted by $P(x)$.

ii) "<u>x is the father of y</u>" is denoted by $P(x, y)$

iii) "<u>$x + y \geq z$</u>" is denoted by $P(x, y, z)$

The values which the variables may assume constitute a collection or set called as the <u>universe of discourse</u>.

When we specify a value for a variable appearing in a predicate, we <u>bind</u> that variable.

A predicate becomes a proposition only when all its voriables are bound.

Consider the following examples:-

i) $P(x)$: $x + 3 = 5$

Let the universe of discourse be the set of all integers.

Binding x by putting $x = -1$, we get a false proposition.

Binding x by putting $x = 2$, we get a true proposition.

A second method of binding individual variables in a predicate is by Quantification of the variable.

The most common forms of quantification are Universal and Existential.

Universal Quantifier

If $P(x)$ is a predicate with the individual variable x as an argument, then the assertion

"For all values of x, the assertion $P(x)$ is true," is a statement in which the variable x is said to be Universally Quantified.

We denote the phrase "For all" by $\forall$

The meaning of $\forall$ is "for all" or "for every" or "for each".

If $P(x)$ is true for every possible value of x, then $\forall x\, P(x)$ is true; otherwise $\forall P(x)$ is false.

Example:

Let $P(x)$ be the predicate $x \geq 0$; where x is any positive integer.

Then the proposition $\forall x\, P(x)$ is true.

However, if x is any real number, then $\forall x\, P(x)$ is a false proposition.

Existential Quantifier

Suppose for the predicate $P(x)$, $\forall x P(x)$ is false, but there exists atleast one value of x for which $P(x)$ is true, then we say that in this proposition, x is bound by existential quantification.

We denote the words "there exists" by the symbol $\exists$.

Then the notation $\exists x\, P(x)$ means "there exists a value of x (in the universe of discourse) for which $P(x)$ is true".

Example

Let $P(x)$ be the predicate "$x+3=5$" and let the universe of discourse be the set of all integers. Then the proposition $\underline{\exists x\, P(x)}$ is true (by setting $x=2$) but $\underline{\forall x\, P(x)}$ is false.

Let $P(x,y)$ be a 2-phase predicate.

1. then $\underline{\exists x\, y\, P(x,y)}$ is the proposition.

"There exists a value of x such that for all values of y, $P(x,y)$ is true".

2. $\underline{\forall y\, \exists x\, P(x,y)}$ is the proposition

"For each value of y, there exists an x such that $P(x,y)$ is true.

③ $\underline{\exists x \exists y\, P(x,y)}$ is the proposition

"There exists a value of x and a value of y such that $P(x,y)$ is true".

④ $\underline{\forall x \forall y\, P(x,y)}$ is the proposition

"For all values of x and y, $P(x,y)$ is true".

Normal Forms

It is usually easier to find if a statement is a Tautology or a Contradiction, if we have limited or less number of variables with the help of truth table.

But when the statements consist of more variables or are of complex form, then the method of employing truth table is not efficient.

Thus, we will have to reduce the statement to a form called

"NORMAL FORM"

There are Two types of normal forms.

1. Disjunctive Normal Form (DNF)
2. Conjunctive Normal Form (CNF)

Disjunctive Normal Form

An expression is said to be in disjunctive normal form if it is a join of min terms.

For ex:

$$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

Alternatively, a DNF can also be defined as, A statement which consists of a disjunction of fundamental conjunctions is called a disjunctive normal form.

Examples

i) $(p \wedge q) \vee \sim q$

ii) $(\sim p \wedge q) \vee (p \wedge q) \vee q$

iii) $(p \wedge q \wedge r) \vee (p \wedge \sim r) \vee (q \wedge r)$

iv) $(p \wedge \sim q) \vee (p \wedge r)$

v) $(p \wedge q \wedge r) \vee \sim r$

Q. Obtain the DNF of the form

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Sol: $\equiv (p \wedge (\sim p \vee q)) \rightarrow q$

$$\equiv \sim (p \wedge (\sim p \vee q)) \vee q$$

$$\equiv \sim [\sim (p \wedge (\sim p \vee q)) \vee q]$$

$$\equiv p \wedge (\sim p \vee q) \vee \sim q$$

$$\equiv (p \wedge \sim p) \vee (p \wedge q) \vee \sim q$$

$\equiv F \vee (p \wedge q) \vee \sim q$

$\equiv (p \wedge q) \vee \sim q$

OR

$\sim q \vee (p \wedge q)$

Q. Obtain the DNF of the form

$p \wedge (p \rightarrow q)$

Sol: $p \wedge (p \rightarrow q)$

$\equiv p \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv F \vee (p \wedge q)$

$\equiv p \wedge q$

Practice

Q.1 $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

Q.2 $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

Conjunctive Normal Form

An expression is said to be in conjunctive normal form if it is a meet of its maxterms

For example:

$$x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_n$$

ex. i) $p \wedge q$

ii) $\sim p \wedge (p \vee q)$

iii) $(p \vee q \vee r) \wedge (\sim p \vee r)$

Alternatively, a CNF can also be defined as, A statement which consists of a conjunction of its fundamental disjunctions is called a <u>conjunctive normal form</u>.

Q.1. Obtain the CNF of the form:-

$$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$$

<u>Sol:</u>

$$\equiv (\sim p \rightarrow r) \wedge (p \leftrightarrow q)$$

$$\equiv (\sim p \rightarrow r) \wedge ((p \rightarrow q) \wedge (q \rightarrow p))$$

$$\equiv (\sim(\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p))$$

$$\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$$

2. Obtain the CNF for the following:-

$$(p \wedge q) \vee (\sim p \wedge q \wedge r)$$

2. Obtain the CNF for the following :-

$$(p \wedge q) \vee (\sim p \wedge q \wedge r)$$

$$\equiv (p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$$

$$\equiv (p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r) \wedge$$
$$(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)$$

$$\equiv T \wedge (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge$$
$$q \wedge (q \vee r)$$

$$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r)$$

3.