

Z - Transforms

1. Introduction

We have already studied Laplace Transforms and Fourier Transforms. Both these transforms are continuous functions. These transforms are not useful for studying discrete systems. Linear systems in which the input signals are in the form of discrete pulses of short duration are called 'Linear Time Invariant' (LTI) systems. For the analysis of such systems we need Z-transforms. In this chapter we shall first get acquainted with sequences, then study Z-transforms and then inverse Z-transforms. After studying Laplace Transforms and Z-transforms, you will find that Z-transform is the discrete analogue of Laplace Transform. For every operational rule and application of Laplace transform there corresponds an operational rule or application of Z-transform. For example, you will find Linearity Property, Shifting Theorem, Convolution Theorem etc. in both Laplace Transforms and Z-transforms.

2. Sequences

If objects are arranged according to a certain rule, this arrangement is called a sequence. We are particularly interested in sequences whose members are real or complex numbers. So we define a sequence as follows.

Definition : An ordered set of real or complex numbers is called a **sequence**.

We shall denote a sequence by $\{f(k)\}$ and k -th term of the sequence by $f(k)$. For example, we have a sequence

$$\{2^0, 2^1, 2^2, 2^3, \dots, 2^k, \dots\}$$

For $k = 0$, $f(k) = 2^0$; for $k = 1$, $f(k) = 2^1$ Thus, in a sequence we have to take into account, the order of a term k , and the term of k -th order $f(k)$. The set of all such ordered terms $\{f(0), f(1), \dots, f(k), \dots\}$ is called a sequence.

1. The most elementary way to denote a sequence is to list all the members of the sequence. For example,

$$\{f(k)\} = \{12, 10, 8, 5, 3, 6, 9\}$$

↑

The arrow ↑ indicates the element corresponding to $k = 0$. The elements on the left of the arrow correspond to $k = -1, -2, -3, \dots$ and those on the right correspond to $k = 1, 2, 3, \dots$

2. Another way of denoting a sequence is to give the general term in terms of k which varies from $-\infty$ to ∞ taking integral values.

For example $\{f(k)\} = 2^k$ (where k is an integer). This sequence is

$$\{\dots, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, 2^3, \dots\}$$

As illustrations we can have the following sequences and can have many more,

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As illustrations we can have the following sequences and can have many more,

$$\begin{aligned}
 & \{1, 1, 1, 1, \dots\} \\
 & \{1, 2, 3, \dots, k, \dots\} \\
 & \{1^2, 2^2, 3^2, \dots, k^2, \dots\} \\
 & \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots\right\} \\
 & \{1 \cdot 2^1, 2 \cdot 2^2, 3 \cdot 2^3, \dots, k \cdot 2^k, \dots\} \\
 & \{1 \cdot \alpha^1, 2 \cdot \alpha^2, 3 \cdot \alpha^3, \dots, k \cdot \alpha^k, \dots\} \\
 & \left\{\frac{\alpha^1}{1}, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots, \frac{\alpha^k}{k}, \dots\right\} \\
 & \left\{\frac{5^1}{1!}, \frac{5^2}{2!}, \frac{5^3}{3!}, \dots, \frac{5^k}{k!}, \dots\right\} \\
 & \{e^{1 \cdot \alpha}, e^{2 \cdot \alpha}, e^{3 \cdot \alpha}, \dots, e^{k \cdot \alpha}, \dots\}
 \end{aligned}$$

3. Basic Operations On Sequences

We shall see below some properties of sequences through examples.

1. Addition : The sum (or difference) of two sequences is obtained by adding (or subtracting) the corresponding terms of the two sequences.

For example, if $\{f(k)\} = 1^3, 2^3, 3^3, 4^3, \dots$

$\{g(k)\} = 1^2, 2^2, 3^2, 4^2, \dots$

$$\begin{aligned}
 \text{then } \{f(k)\} + \{g(k)\} &= \{(1^3 + 1^2), (2^3 + 2^2), (3^3 + 3^2), \dots, (k^3 + k^2), \dots\} \\
 &= \{1^2 \cdot 2, 2^2 \cdot 3, 3^2 \cdot 4, \dots, k^2(k+1), \dots\} \\
 \{f(k)\} - \{g(k)\} &= \{(1^3 - 1^2), (2^3 - 2^2), (3^3 - 3^2), \dots, (k^3 - k^2), \dots\} \\
 &= \{1^2 \cdot 0, 2^2 \cdot 1, 3^2 \cdot 2, \dots, k^2 \cdot (k-1), \dots\}
 \end{aligned}$$

2. Scalar Multiplication : If α is a scalar then from a given sequence $\{f(k)\}$ we can obtain another sequence $\alpha \{f(k)\}$ by multiplying each term $f(k)$ of the sequence $\{f(k)\}$ by α .

For example, if $\{f(k)\} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots$, then

$$3 \cdot \{f(k)\} = 3, \frac{3}{2}, \frac{3}{3}, \dots, \frac{3}{k}, \dots$$

If $\{f(k)\} = \sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{k}, \dots$, then

$$2 \cdot \{f(k)\} = 2\sqrt{1}, 2\sqrt{2}, 2\sqrt{3}, \dots, 2\sqrt{k}, \dots$$

3. Linearity : If α and β are two scalars then from two sequences $\{f(k)\}$ and $\{g(k)\}$, we can obtain another sequence by multiplying the terms of the two sequences by α and β as above and adding the corresponding terms.

$$\text{i.e. } \alpha \{f(k)\} + \beta \{g(k)\} = \{\alpha \cdot f(k) + \beta \cdot g(k)\}$$

For example, if $\{f(k)\} = 1, 2, 3, \dots, k, \dots$

$$\text{and } \{g(k)\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots$$

$$\text{then, } 2 \cdot \{f(k)\} + 3 \cdot \{g(k)\} = \{2 \cdot f(k) + 3 \cdot g(k)\}$$

$$= 2(1) + 3\left(\frac{1}{1}\right), 2 \cdot 2 + 3\left(\frac{1}{2}\right), 2(3) + 3\left(\frac{1}{3}\right), \dots, 2 \cdot k + 3\left(\frac{1}{k}\right) + \dots$$

$$= 2 + \frac{3}{1}, 4 + \frac{3}{2}, 6 + \frac{3}{3}, \dots, 2k + \frac{3}{k}, \dots$$

4. Convergence And Divergence : Consider the following sequence

$$\frac{1+1}{1}, \frac{2+1}{2}, \frac{3+1}{3}, \frac{4+1}{4}, \dots, \frac{k+1}{k}, \dots$$

$$\text{i.e. } \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, 1 + \frac{1}{k}, \dots \quad \text{i.e., } 2, 1.5, 1.33, 1.25, \dots$$

It is easy to see that as the number of terms become infinite the sequence goes on decreasing and ultimately takes the value 1. Such a sequence $\{f(k)\}$ is called a **convergent sequence**.

Definition : If $\{f(k)\}$ is a given sequence and if $f(k)$ tends to a (finite) real number L as k tends to infinity then $\{f(k)\}$ is called a **convergent sequence**.

The following sequences are convergent.

(i) $a, a, a, \dots, a, \dots, a, \dots$ converges to a

(ii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots$ converges to 0

(iii) $1 + \frac{1}{2^0}, 1 + \frac{1}{2^1}, 1 + \frac{1}{2^2}, \dots, 1 + \frac{1}{2^k}, \dots$ converges to 1.

Definition : A sequence which is not convergent i.e. which does not tend to a (finite) real number is called a **divergent sequence**.

The following are divergent sequences.

(i) $1, 2, 3, \dots, k, \dots$ diverges to ∞

(ii) $-1, -2, -3, \dots, -k, \dots$ diverges to $-\infty$

(iii) $1, 2, 1, 2, 1, 2, \dots$ oscillates between 1 and 2

(iv) $0, 1, 0, 1, 0, 1, \dots$ oscillates between 0 and 1.

EXERCISE - I

1. Write down the term corresponding to $k = 3$ of the following sequence

$$\{-6, -3, -1, 0, 2, 4, 6, 8, 10\}$$



[Ans. : 8]

2. Write down the term corresponding to $k = -3$ of the following sequence.

$$\{-12, -10, -9, -7, -5, -3, 1, 4, 6, 10\}$$



[Ans. : -10]

3. Write down the sequence if k -th term is 3^k for $-2 \leq k \leq 4$.

[Ans. : $\frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81$]

4. Write down the sequence whose k -th term is 2^k for $-\infty < k < \infty$.

[Ans. : $\left\{ \dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \dots \right\}$]

5. Write down the sequence whose k -th term = $\begin{cases} 4^k, & k < 0 \\ 3^k, & k \geq 0 \end{cases}$

[Ans. : $\left\{ \dots, \frac{1}{64}, \frac{1}{16}, \frac{1}{4}, 1, 3, 9, 27, \dots \right\}$]

6. Write down the sequence whose k -th term = $\begin{cases} a^k, & k < 0 \\ b^k, & k \geq 0 \end{cases}$

[Ans. : $\left\{ \dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, b, b^2, b^3, \dots \right\}$]

4. Z - transforms

We shall now define Z-transform of a sequence.

Definition : Let $\{f(k)\} = \{\dots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \dots\}$ be a sequence of terms where k varies from $-\infty$ to ∞ .

Let $z = x + iy$ be a complex number then

$$\begin{aligned} Z\{f(k)\} &= \dots + f(-3)z^3 + f(-2)z^2 + f(-1)z^{-1} + f(0)z^0 + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots \\ &= \sum_{k=-\infty}^{\infty} f(k)z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k} \end{aligned}$$

is called the **Z-transform of the sequence $\{f(k)\}$** .

In words, the sum of the product of k -th term of the sequence $f(k)$ with z^{-k} taken from $-\infty$ to ∞ is called the **Z-transform of the sequence $\{f(k)\}$** .

Thus,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

Notation : Unfortunately there is no unanimity in the notations used in the case of Z-transform. Some use x_k, y_k, \dots to denote sequences and $x_0, x_1, \dots, y_0, y_1, \dots$ to denote the terms of the sequences. We shall denote the sequences by $\{f(k)\}, \{g(k)\}, \dots$ the terms of the sequences by $f(1), f(2), \dots, g(1), g(2), \dots$ and Z-transforms by $Z\{f(k)\}, Z\{g(k)\}, \dots$ or by $F(z), G(z), \dots$ etc

Notes

- It is necessary to know which is the zeroth term, first term, second term minus first term minus second term i.e. we must know the order of each term. To obtain Z-transform of sequence we multiply each term by **negative power of z of the order of that term** and take the sum.
- $Z\{f(k)\}$ is a function of a complex variable z and is defined only if the sum is finite i.e. if the infinite series $\sum f(k)z^{-k}$ is **absolutely convergent**. We shall denote the Z-transform of the sequence $\{f(k)\}$ by $Z\{f(k)\}$ or by $F(z)$.
- Wherever necessary we shall denote the sequences by $\{f(k)\}, \{g(k)\}$ etc.
- If Z-transform of $\{f(k)\}$ is $F(z)$ we call $\{f(k)\}$ the **inverse Z transform of $F(z)$** and denote it $Z^{-1}[F(z)]$.

Example 1 : If $\{f(k)\} = \{-6, -3, 0, 2, 4\}$,

find $Z\{f(k)\}$ where \uparrow denotes the element corresponding to $k=0$.

Sol. : To obtain $Z\{f(k)\}$, we multiply each term $f(k)$ of the sequence by z^{-k} , and take the sum.

Multiply the 0th term 0 by z^0 , the first term 2 by z^{-1} , the second term 4 by z^{-2} , (-1)st term (-3) by $(z^{-1})^{-1}$ i.e., z , (-2)nd term (-6) by $(z^{-1})^{-2}$ i.e., z^2 and take the sum of these products.

$$\therefore Z\{f(k)\} = f(-2)(z^{-1})^{-2} + f(-1)(z^{-1})^{-1} + f(0)z^0 + f(1)z^{-1} + f(2)z^{-2}$$

where, $f(-2) = -6, f(-1) = -3, f(0) = 0, f(1) = 2, f(2) = 4$.

$$\begin{aligned}\therefore Z\{f(k)\} &= \sum_{k=-2}^2 f(k) z^{-k} = (-6)z^2 + (-3)z^1 + 0 \cdot z^0 + 2z^{-1} + 4z^{-2} \\ &= -6z^2 + 3z + 0 + \frac{2}{z} + \frac{4}{z^2}\end{aligned}$$

Example 2 : If $\{f(k)\} = \{9, 6, 3, 0, -3, -6, -9\}$, find $Z\{f(k)\}$.

Sol. : Since 3 is the term corresponding to $k=0$. We have

$$f(-2) = 9, f(-1) = 6, f(0) = 3, f(1) = 0, f(2) = -3, f(3) = -6, f(4) = -9.$$

$$\begin{aligned}\therefore Z\{f(k)\} &= \sum_{k=-2}^4 f(k) z^{-k} = 9z^2 + 6z^1 + 3z^0 + 0z^{-1} - 3z^{-2} - 6z^{-3} - 9z^{-4} \\ &= 9z^2 + 6z + 3 + 0 - \frac{3}{z^2} - \frac{6}{z^3} - \frac{9}{z^4}.\end{aligned}$$

Example 3 : If $\{f(k)\} = \{2^0, 2^1, 2^2, 2^3, \dots\}$ find $Z\{f(k)\}$.

Sol. : Z-transform of the sequence is

$$\begin{aligned}\therefore Z\{f(k)\} &= \sum_{k=0}^{k=\infty} f(k) z^{-k} = 2^0 z^0 + 2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \dots \\ &= 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots \\ &= \frac{1}{1-(2/z)} = \frac{z}{z-2} \quad \text{if } \left|\frac{2}{z}\right| < 1 \quad \left(\because s_\infty = a + ar + ar^2 + \dots = \frac{a}{1-r}, \text{ if } |r| < 1\right)\end{aligned}$$

Example 4 : If $\{f(k)\} = \begin{cases} 4^k, & \text{for } k < 0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$, find $Z\{f(k)\}$.

Sol. : The sequence is $\{f(k)\} = \{\dots, 4^{-4}, 4^{-3}, 4^{-2}, 4^{-1}, 3^0, 3^1, 3^2, 3^3, \dots\}$

And Z-transform of $f(k)$ is

$$\therefore Z\{f(k)\} = \{\dots, 4^{-4}z^4 + 4^{-3}z^3 + 4^{-2}z^2 + 4^{-1}z + 3^0z^0 + 3^1z^{-1} + 3^2z^{-2} + 3^3z^{-3} + \dots\}$$

We write positive powers of z in reverse order.

$$\begin{aligned}\therefore Z\{f(k)\} &= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \dots \right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots \right] \\ &= \frac{z}{4} \left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \dots \right] + \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \dots \right]\end{aligned}$$

$$\begin{aligned}\therefore Z\{f(k)\} &= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(3/z)} \quad \text{if } \left| \frac{z}{4} \right| < 1, \left| \frac{3}{z} \right| < 1 \\ &= \frac{z}{4} \cdot \frac{4}{4-z} + \frac{z}{z-3} = \frac{z}{4-z} + \frac{z}{z-3} \\ &= \frac{z}{(4-z)(z-3)} \quad \text{if } 3 < |z| < 4.\end{aligned}$$

Note

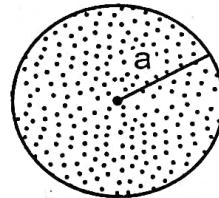
We shall require the following results in finding Z-transforms.

1. $1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$ if $|r| < 1$.
2. $1 - x + x^2 - x^3 + \dots = (1+x)^{-1}$ if $|x| < 1$.
3. $1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ if $|x| < 1$.
4. $1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(x-2)}{3!} x^3 + \dots = (1+x)^n$
5. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$
6. If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$.
7. The set $|z| < a$. Since $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$.

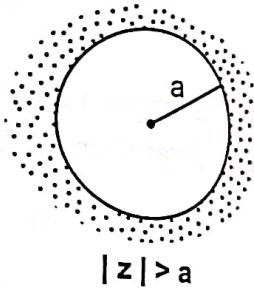
$\therefore |z| < a$ means $\sqrt{x^2 + y^2} = a$

i.e. $x^2 + y^2 < a^2$.

This means, $|z| < a$ is the set of points inside the circle of radius a and centre at the origin. By the same reasoning $|z| > a$ is the set of points outside the circle with radius a and centre at the origin.



$|z| < a$



$|z| > a$

EXERCISE - II

1. Write down the Z-transforms of the following sequences

(i) $\{f(k)\} = \{8, 6, 4, 2, 0, 1, 3, 5, 7\}$ (ii) $\{f(k)\} = \{-6, -4, -2, 1, 2, 4, 6\}$

↑

↑

[Ans. : (i) $8z^6 + 6z^5 + 4z^4 + 2z^3 + 0 + 1z + 3 + \frac{5}{z} + \frac{7}{z^2}$

(ii) $-6z^3 - 4z^2 - 2z + 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{6}{z^3}$]

2. Write down the Z-transform of the following sequences

(i) $\{f(k)\} = 3^k, k \geq 0$ (ii) $\{f(k)\} = 5^k, k \geq 0$

[Ans. : (i) $\frac{z}{z-3}, \left| \frac{3}{z} \right| < 1$, (ii) $\frac{z}{z-5}, \left| \frac{5}{z} \right| < 1$]

Inverse Z-Transform

Definition : If $F(z)$ is the Z-transform of the sequence $\{f(k)\}$ then the sequence $\{f(k)\}$ is called the inverse Z-transform of $F(z)$ and is denoted as

$$\{f(k)\} = Z^{-1}[F(z)]$$

Thus, we have if $Z\{f(k)\} = F(z)$, then $\{f(k)\} = Z^{-1}[f(z)]$ and vice versa.

5. Region of Convergence (ROC)

We shall try to understand this important concept in relation to Z-transforms through two examples.

Example 1 : Consider the sequence $f(k) = \begin{cases} 0 & \text{for } k < 0 \\ 4^k & \text{for } k \geq 0 \end{cases}$

i.e., the sequence $\{f(k)\} = \{4^0, 4^1, 4^2, 4^3, \dots, 4^k, \dots\}$.

Its Z-transform by definition is

$$\begin{aligned} Z\{f(k)\} &= \sum f(k) z^{-k} = \sum_{k=0}^{\infty} 4^k z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + 4^3 z^{-3} + \dots \\ &= 1 + \frac{4}{z} + \frac{16}{z^2} + \frac{64}{z^3} + \dots \end{aligned}$$

Notice that $Z\{f(k)\}$ is a Geometric Progression with common ratio $4/z$. We know that the sum

S of infinite terms of a G.P. with first term 1 and common ratio r is given by $S = \frac{1}{1-r}$ if $|r| < 1$.

The sum of the above series i.e. of the Z-transform is

$$Z\{f(k)\} = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \text{ if } \left|\frac{4}{z}\right| < 1$$

i.e. $4 < |z|$ i.e. $|z| > 4$.

Since, for the Z-transform to exist the corresponding series must be convergent. The above Z-transform is defined only if $|z| > 4$.

Note

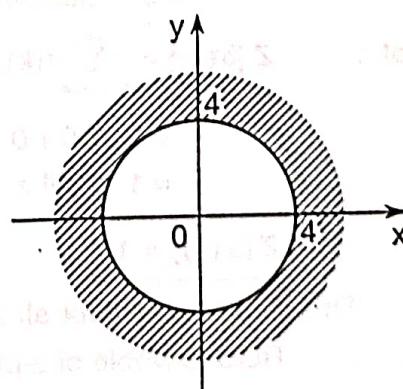
Note that a G.P. $a, ar, ar^2, \dots, ar^n, \dots$ is convergent if $|r| < 1$ and its sum

$$S = \frac{a}{1-r} \text{ where } |r| < 1.$$

But $|z| = 4$ is a circle with centre at the origin and radius $= 4$. Hence, the above Z-transform is defined if $z > 4$ i.e. if z is on the exterior of the circle $|z| = 4$.

In this case, we say that the region of convergence of this Z-transform is the exterior of the circle $|z| = 4$.

The region for which $\sum f(k) z^{-k}$ is convergent is called the region of convergence denoted in short by R.O.C.



Example 2 : Find the Z-transform and the region of convergence of $f(k) = \begin{cases} 5^k & \text{for } k < 0 \\ 3^k & \text{for } k \geq 0 \end{cases}$

Sol. : By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

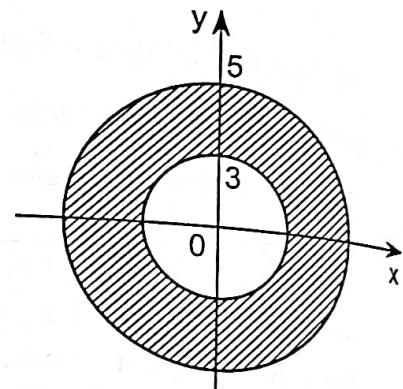
Putting $k = -n$ in the first series, we get

$$\begin{aligned} Z\{f(k)\} &= \sum_{n=1}^{\infty} 5^{-n} z^n + \sum_{k=0}^{\infty} 3^k z^{-k} \\ Z\{f(k)\} &= \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \dots \right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots \right] \\ &= \frac{z}{5} \left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \dots \right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots \right] \\ &= \frac{z}{5} \cdot \frac{1}{1-(z/5)} + \frac{1}{1-(3/z)} = \frac{z}{5-z} + \frac{z}{z-3} \\ &= \frac{2z}{(5-z)(z-3)} \end{aligned}$$

Now, $Z\{f(k)\}$ is the sum of two Geometric Progressions with the common ratios $(z/5)$ and $(3/z)$ respectively. The series will be convergent if $|z/5| < 1$ and $|3/z| < 1$. i.e. $|z| < 5$ and $3 < |z|$ i.e. $3 < |z| < 5$.

But $|z| = 3$ is a circle with centre at the origin and radius 3 and $|z| = 5$ is a circle with centre at the origin and radius 5. Hence, $Z\{f(k)\}$ is convergent if z lies between the annulus as shown in the figure. This is the region of convergence of $Z\{f(k)\}$ which is shown by shaded area.

\therefore ROC is $3 < |z| < 5$.



6. Z-Transforms of Some Standard Functions

Example 1 : Find the Z-transform of Unit Impulse function

$$\begin{aligned} \delta(k) &= 1 \quad \text{for } k = 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Sol. : $Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

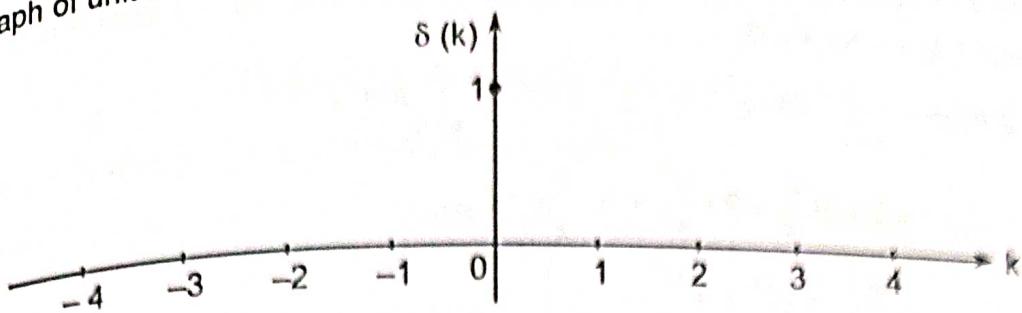
$$\begin{aligned} &= \{ \dots, 0 + 0 + 0 + 1 \cdot z^0 + 0 + 0 + 0, \dots \} \\ &= 1 \text{ for all } z \end{aligned}$$

$\therefore \boxed{Z\{\delta(k)\} = 1}$

This is convergent for all z .

\therefore ROC is whole of z -plane.

The graph of unit impulse function is,



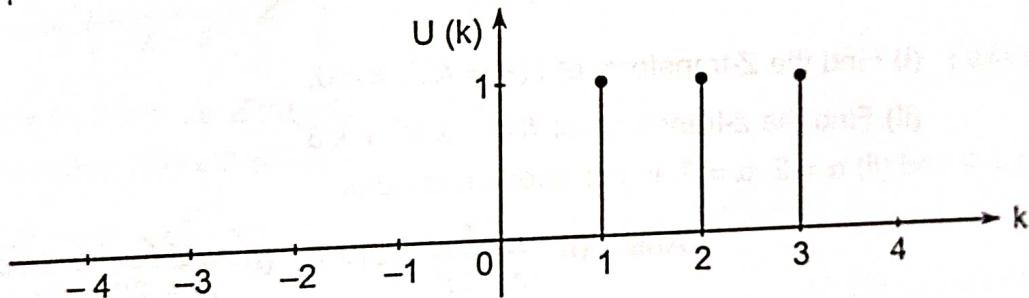
Example 2 : Find the Z-transform of Discrete Unit Step function

$$\begin{aligned} U(k) &= 1 \quad \text{for } k \geq 0 \\ &= 0 \quad \text{for } k < 0 \end{aligned}$$

$$\begin{aligned} \text{Sol. : } Z\{U(k)\} &= \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] = \frac{1}{1 - (1/z)} = \frac{z}{z-1} \end{aligned}$$

$$Z\{U(k)\} = \frac{z}{z-1}, \quad k \geq 0$$

The graph of discrete unit step function is,



This is convergent if $|1/z| < 1$ i.e. $1 < |z|$ i.e. $|z| > 1$

∴ ROC is $|z| > 1$.

Since, $Z\{U(k)\} = \frac{z}{z-1}$, $Z^{-1}\left[\frac{z}{z-1}\right] = \{U(k)\}$ where Z^{-1} denotes inverse Z-transform.

Example 3 : Find the z-transform of $f(k) = \alpha^k$, $\alpha > 0$, $k \geq 1$.

$$\begin{aligned} \text{Sol. : We have } Z\{f(k)\} &= \sum_{k=1}^{\infty} \alpha^k z^{-k} = \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \frac{\alpha^3}{z^3} + \dots \\ &= \frac{\alpha}{z} \left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + \dots \right) \quad [\text{G.P.}] \\ &= \frac{\alpha}{z} \cdot \frac{1}{1 - (\alpha/z)} = \frac{\alpha}{z - \alpha}, \quad |\alpha| < |z|. \end{aligned}$$

$$\therefore Z\{\alpha^k\} = \frac{\alpha}{z - \alpha}, \quad k \geq 1$$

$$\text{Cor. : Putting } \alpha = 1, \quad Z\{1\} = \frac{1}{z-1}, \quad k \geq 1$$

Example 4 : Find the Z-transform of $f(k) = k\alpha^k$, $k \geq 0$.

Sol. : Assuming $f(k) = 0$ for $k < 0$,

$$\begin{aligned} Z\{f(k)\} &= \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^k + \sum_{k=0}^{\infty} k \alpha^k z^{-k} \\ &= 0 + 1 \cdot \frac{\alpha}{z} + 2 \cdot \frac{\alpha^2}{z^2} + 3 \cdot \frac{\alpha^3}{z^3} + \dots \\ &= \frac{\alpha}{z} \left(1 + 2 \frac{\alpha}{z} + 3 \frac{\alpha^2}{z^2} + \dots \right) \end{aligned} \quad (1)$$

$$= \frac{\alpha}{z} \left(1 - \frac{\alpha}{z} \right)^{-2} = \frac{\alpha}{z} \cdot \frac{1}{[1 - (\alpha/z)]^2} = \frac{\alpha z}{(z - \alpha)^2}$$

∴

$$Z\{k\alpha^k\} = \frac{\alpha z}{(z - \alpha)^2}$$

Cor. : Putting $\alpha = 1$,

$$Z\{k\} = \frac{z}{(z - 1)^2}$$

Applying D'Alembert's ratio test to (1), we find that the series is convergent if $|\alpha/z| < 1$ i.e., $|\alpha| < |z|$ i.e., $|z| > |\alpha|$.

∴ ROC is $|z| > |\alpha|$

Particular Cases : (i) Find the Z-transform of $f(k) = k2^k$, $k \geq 0$.

(ii) Find the z-transform of $f(k) = k2^k + k3^k$.

Sol. : Put (i) $\alpha = 2$ and (ii) $\alpha = 3$, in the above example.

[Ans. : (i) $\frac{2z}{(z-2)^2}$, $|z| > 2$, (ii) $\frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}$, $|z| > 3$]

Example 5 : Find the Z-transform of $f(k) = \frac{\alpha^k}{k}$, $k \geq 1$.

Sol. : Assuming $f(k) = 0$ for $k \leq 0$,

$$\begin{aligned} Z\{f(k)\} &= \sum_{k=-\infty}^0 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} z^{-k} \\ &= \frac{\alpha}{z} + \frac{\alpha^2}{2z^2} + \frac{\alpha^3}{3z^3} + \frac{\alpha^4}{4z^4} + \dots \\ Z\{f(k)\} &= -\log\left(1 - \frac{\alpha}{z}\right) \end{aligned} \quad (1)$$

∴

$$Z\left(\frac{\alpha^k}{k}\right) = -\log\left(1 - \frac{\alpha}{z}\right)$$

Applying D'Alembert's Ratio Test to (1), we find that the series is convergent if $|\alpha/z| < 1$ i.e., $|\alpha| < |z|$ i.e., $|z| > |\alpha|$.

∴ ROC is $|z| > |\alpha|$.

- Particular cases :**
- Find Z-transform of $f(k) = \frac{1}{k}$, $k \geq 1$.
 - Find the Z-transform of $f(k) = \frac{2^k}{k}$, $k \geq 1$.

Sol. : Put (i) $\alpha = 1$, and (ii) $\alpha = 2$ in the above example.

$$[\text{Ans. : (i)} -\log\left(1-\frac{1}{z}\right), |z| > 1; \text{(ii)} -\log\left(1-\frac{2}{z}\right), |z| > 2.]$$

Example 6 : Find the Z-transform of $f(k) = a^k$, $k \geq 0$. (M.U. 2009)

Assuming that $f(k) = 0$ when $k < 0$

$$\begin{aligned} Z\{f(k)\} &= \sum_{k=-\infty}^{\infty} a^k z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k} \\ &= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots = \frac{1}{1-(a/z)} = \frac{z}{z-a} \end{aligned}$$

$$\therefore Z(a^k) = \boxed{\frac{z}{z-a}}$$

The series being G.P. is convergent if $1 > |a/z|$ i.e. $|z| > |a|$.

∴ ROC is $|z| > |a|$.

$$\text{Since, } Z\{a^k\} = \frac{z}{z-a}, \quad Z^{-1}\left[\frac{z}{z-a}\right] = a^k, \quad k \geq 0.$$

Example 7 : Find the Z-transform of $f(k) = b^k$, $k < 0$. (M.U. 2008, 09)

Sol. : Assuming that $f(k) = 0$ when $k \geq 0$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{n=1}^{\infty} b^{-n} z^n \text{ where } n = -k$$

(Note the substitution $n = -k$)

$$\begin{aligned} &= \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \dots = \frac{z}{b} \left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \dots\right) \\ &= \frac{z}{b} \frac{1}{1-(z/b)} = \frac{z}{b-z} \end{aligned}$$

$$\therefore Z(b^k) = \boxed{\frac{z}{b-z}}$$

The series being G.P. is convergent if $1 > |z/b|$ i.e. $|b| > |z|$.

∴ ROC is $|z| < |b|$.

$$\text{Since, } Z\{b^k\} = \frac{z}{b-z}, \quad Z^{-1}\left[\frac{z}{b-z}\right] = b^k, \quad k < 0.$$

Example 8 : Find Z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k, & k \geq 0 \end{cases}$

Sol. : By Examples 7 and 6, we get

$$Z\{f(k)\} = \frac{z}{b-z} + \frac{z}{z-a} = \frac{z^2 - az + bz - z^2}{(z-a)(b-z)}$$

$$\therefore Z\{f(k)\} = \frac{-z(a-b)}{-(z-a)(z-b)} = \frac{(a-b)z}{(z-a)(z-b)}$$

if $|z| > a$ and $|z| < b$ i.e., $a < |z| < b$.

Note

In general the Z-transform of $\{f(k)\} = b^k$, for $k < 0$ and $\{f(k)\} = a^k$ for $k \geq 0$ is $\frac{(a-b)z}{(z-a)(z-b)}$ and $a < |z| < b$. This Z-transform exists only if $a < b$.

Example 9 : Find the Z-transform of $f(k) = {}^n C_k$, $0 \leq k \leq n$.

$$\begin{aligned}\text{Sol. : } Z\{f(k)\} &= \sum_{k=-\infty}^{\infty} {}^n C_k z^{-k} = \sum_{k=0}^n {}^n C_k z^{-k} \\ &= {}^n C_0 + {}^n C_1 \frac{1}{z} + {}^n C_2 \frac{1}{z^2} + \dots + {}^n C_n \frac{1}{z^n} = \left(1 + \frac{1}{z}\right)^n\end{aligned}$$

The series being finite is obviously convergent if $z \neq 0$.

$$\therefore Z({}^n C_k) = \left(1 + \frac{1}{z}\right)^n$$

\therefore ROC is all of z-plane except the origin.

Example 10 : Find the Z-transform of $f(k) = {}^k C_n$, $k \geq n$.

$$\text{Sol. : } Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^k C_n z^{-k} = \sum_{k=n}^{\infty} {}^k C_n z^{-k}$$

To find the sum we change the dummy index k by $k = n+r$.

$$\begin{aligned}\therefore Z\{f(k)\} &= \sum_{r=0}^{\infty} {}^{n+r} C_n z^{-(n+r)} = \sum_{r=0}^{\infty} {}^{n+r} C_n z^{-n} z^{-r} \\ &= z^{-n} \sum_{r=0}^{\infty} {}^{n+r} C_r z^{-r} \quad [\because {}^n C_r = {}^n C_{n-r}, \text{ we get } {}^{n+r} C_n = {}^{n+r} C_{n+r-n} = {}^{n+r} C_r] \\ &= z^{-n} \left[1 + {}^{n+1} C_1 z^{-1} + {}^{n+1} C_2 z^{-2} + \dots \right] = z^{-n} \left(1 - \frac{1}{z}\right)^{-(n+1)}\end{aligned}$$

$$\therefore Z({}^k C_n) = z^{-n} \left(1 - \frac{1}{z}\right)^{-(n+1)}$$

\therefore ROC is $|z| > 1$.

Example 11 : Find the Z-transform of $f(k) = {}^{k+n} C_n$.

$$\text{Sol. : } Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^{k+n} C_n z^{-k}$$

But ${}^{k+n} C_n = 0$ if $k+n < n$ i.e. if $k < 0$.

$$\begin{aligned}\therefore Z\{f(k)\} &= \sum_{k=0}^{\infty} {}^{k+n} C_n z^{-k} = \sum_{k=0}^{\infty} {}^{k+n} C_k z^{-k} \quad (\text{as in the previous example}) \\ &= 1 + {}^{n+1} C_1 z^{-1} + {}^{n+2} C_2 z^{-2} + \dots\end{aligned}$$

$$\therefore Z\{f(k)\} = \left(1 - \frac{1}{z}\right)^{-(n+1)}$$

$$\therefore Z(k^n C_n) = \left(1 - \frac{1}{z}\right)^{-(n+1)}$$

As above ROC is $|z| > 1$.

Example 12 : Find the Z-transform of $f(k) = \frac{a^k}{k!}$, $k \geq 0$. (M.U. 2009)

$$\begin{aligned} \text{Sol. : } Z\{f(k)\} &= \sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{(a/z)^k}{k!} \\ &= 1 + \frac{a}{z} + \frac{1}{2!} \left(\frac{a}{z}\right)^2 + \frac{1}{3!} \left(\frac{a}{z}\right)^3 + \dots \\ &= e^{a/z} \text{ ROC is all of } z\text{-plane.} \end{aligned}$$

Remark

You are advised to memorise the Z-transforms of these standard functions.

EXERCISE - III

Find the Z-transform and its ROC of each of the following sequences $\{f(k)\}$ where $f(k)$ is given by

- | | | |
|--|--|---|
| 1. $f(k) = 3^k$, $k \geq 0$ | 2. $f(k) = 4^k$, $k \geq 0$ | 3. $f(k) = (1/6)^k$, $k \geq 0$ |
| 4. $f(k) = 2$, $k \geq 0$ | 5. $f(k) = 4$, $k \geq 0$ | 6. $f(k) = 5$, $k \geq 0$ |
| 7. $f(k) = 2^k$, $k < 0$ | 8. $f(k) = 4^k$, $k < 0$ | 9. $f(k) = (1/3)^k$, $k < 0$ |
| 10. $f(k) = 3^k$, $k < 0$ $= 2^k$, $k \geq 0$ | 11. $f(k) = 4^k$, $k < 0$ $= 3^k$, $k \geq 0$ | 12. $f(k) = a^k$, $k < 0$ $= b^k$, $k \geq 0$ ($a, b > 0$, $a > b$) |
| 13. $f(k) = k3^k$, $k \geq 0$ | 14. $f(k) = k5^k$, $k \geq 0$ | 15. $f(k) = ka^k$, $k \geq 0$ ($a > 0$) |
| 16. $f(k) = \frac{3^k}{k}$, $k > 1$ | 17. $f(k) = \frac{2^k}{k}$, $k > 1$ | 18. $f(k) = \frac{a^k}{k}$, $k > 1$, $a > 0$ |
| 19. $f(k) = (1/2)^{ k }$, for all k | | 20. $f(k) = (1/4)^{ k }$, for all k |
| 21. $f(k) = a^k$ for all k ($0 < a < 1$) | | 22. $f(k) = (3^k / k!)$, $k \geq 0$ |
| 23. $f(k) = (5^k / k!)$, $k \geq 0$ | | 24. $f(k) = e^{k\alpha}$, $k \geq 0$ |
| 25. $f(k) = \begin{cases} 2^k, & k \leq -1, \dots \\ (1/2)^k, & k = 0, 2, 4, \dots \\ (1/3)^k, & k = 1, 3, 5, \dots \end{cases}$ | | |

$$\text{Ans. : (1) } \frac{1}{1-(3/z)} ; |z| > 3, \quad (2) \frac{1}{1-(4/z)} ; |z| > 4, \quad (3) \frac{1}{1-(1/6z)} ; |z| > \frac{1}{6},$$

$$(4) 2 \cdot \frac{1}{1-(1/z)} ; |z| > 1, \quad (5) 4 \cdot \frac{1}{1-(1/z)} ; |z| > 1, \quad (6) 5 \cdot \frac{1}{1-(1/z)} ; |z| > 1,$$

$$(7) \frac{z}{2} \cdot \frac{1}{1-(z/2)} ; |z| < 2, \quad (8) \frac{z}{4} \cdot \frac{1}{1-(z/4)} ; |z| < 4, \quad (9) 3z \frac{1}{1-3z} ; |z| < \frac{1}{3},$$

$$(10) \frac{2z}{(3-z)(z-2)} ; 2 < |z| < 3, \quad (11) \frac{3z}{(4-z)(z-3)} ; 3 < |z| < 4,$$

$$(12) \frac{(a-b)z}{(a-z)(z-b)} ; b < |z| < a, \quad (13) \frac{3z}{(z-3)^2} ; |z| > 3,$$

$$(14) \frac{5z}{(z-5)^2} ; |z| > 5, \quad (15) \frac{az}{(z-a)^2} ; |z| > a, \quad (16) -\log\left(1 - \frac{3}{z}\right) ; |z| > 3,$$

$$(17) -\log\left(1 - \frac{2}{z}\right) ; |z| > 2, \quad (18) -\log\left(1 - \frac{a}{z}\right) ; |z| > a,$$

$$(19) \frac{1}{2} \cdot \frac{z}{1-(z/2)} + \frac{1}{1-2z} ; \frac{1}{2} < |z| < 2,$$

$$(20) \frac{1}{4} \cdot \frac{z}{1-(z/4)} + \frac{1}{1-(1/4z)} ; \frac{1}{4} < |z| < 4,$$

$$(21) \frac{az}{1-az} + \frac{1}{1-(a/z)} ; |a| < |z| < \frac{1}{|a|},$$

$$(22) e^{3/z}, \text{ ROC } z \text{ plane,}$$

$$(23) e^{5/z}, \text{ ROC } z \text{ plane,}$$

$$(24) \left(1 - \frac{e^\alpha}{z}\right)^{-1} ; |z| > |e^\alpha|,$$

$$(25) Z\{f(k)\} = \sum_{k=1}^{-\infty} 2^k z^{-k} + \sum_{k=0}^{2n} \left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=0}^{2n-1} \left(\frac{1}{3}\right)^k z^{-k} \text{ as } n \rightarrow \infty$$

$$= \sum_1^{\infty} \left(\frac{z}{2}\right)^k + \sum_{k=0}^{2n} \left(\frac{1}{2z}\right)^k + \sum_{k=1}^{2n-1} \left(\frac{1}{3z}\right)^k$$

$$= \frac{z}{2-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1} ; \frac{1}{2} < |z| < 2]$$

7. Properties of Z-transforms

As in the Laplace transforms we have the following properties of Z-transforms. We shall prove below these properties and use them to solve some problems.

(1) Linearity

If a and b are constants and $\{f(k)\}$ and $\{g(k)\}$ are two sequences which can be added then,

$$Z\{a f(k) + b g(k)\} = a Z\{f(k)\} + b Z\{g(k)\}$$

Proof : We have by definition,

$$\begin{aligned} Z\{a f(k) + b g(k)\} &= \sum_{k=-\infty}^{\infty} [a f(k) + b g(k)] z^{-k} \\ &= \sum [a f(k) z^{-k} + b g(k) z^{-k}] = a \sum f(k) z^{-k} + b \sum g(k) z^{-k} \\ &= a Z\{f(k)\} + b Z\{g(k)\} \end{aligned}$$

Corollary : If $a = b = 1$, we have $Z\{f(k) + g(k)\} = Z\{f(k)\} + Z\{g(k)\}$ where, $\{f(k)\}$ and $\{g(k)\}$ can be added.

In words, this means, the Z-transform of the sum (or difference when $b = -1$) of two sequences which can be added (or subtracted) is equal to the sum (or difference) of the Z-transforms of the two sequences.

We shall now use this property to solve some problems.

✓ Example 1 : Find $Z\{a^{|k|}\}$.

Sol. : We have

$$\begin{aligned}
 Z\{a^{|k|}\} &= \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k} \\
 &= [.... + a^3 z^3 + a^2 z^2 + az] + [1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} +] \\
 &= az(1 + az + az^2 + a^3 z^3 +) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} +\right) \\
 &= az \cdot \frac{1}{1 - az} + 1 \cdot \frac{1}{1 - (a/z)} = \frac{az}{1 - az} + \frac{z}{z - a} \\
 &= \frac{az^2 - a^2 z + z - az^2}{(1 - az)(z - a)} = \frac{z(1 - a^2)}{(1 - az)(z - a)}.
 \end{aligned}$$

$$Z\{a^{|k|}\} = \frac{z(1 - a^2)}{(1 - az)(z - a)}$$

The series in G.P. are convergent if $1 > |az|$ and $|z| > a$ i.e. $\frac{1}{a} > |z|$ and $|z| > a$.
∴ The ROC is $(1/a) > |z| > a$.

✓ Example 2 : Find the Z-transform of $\left\{ \left(\frac{1}{3}\right)^{|k|} \right\}$.

(M.U. 2014)

Sol. : We have

$$\begin{aligned}
 Z\left\{ \left(\frac{1}{3}\right)^{|k|} \right\} &= \sum \left(\frac{1}{3}\right)^{|k|} \cdot z^{-k} = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=\infty}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} \\
 &= \left[.... + \left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right) z\right] + \left[1 + \frac{1}{3} \cdot z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} +\right] \\
 &= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} +\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} +\right] \\
 &= \frac{z}{3} \left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 +\right] \left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^2 +\right] \\
 &= \frac{z}{3} \cdot \frac{1}{1 - (z/3)} + 1 \cdot \frac{1}{1 - [1/(3z)]}, \quad \left|\frac{z}{3}\right| < 1, \quad \left|\frac{1}{3z}\right| < 1 \\
 &= \frac{z}{3} \cdot \frac{3}{3 - z} + \frac{3z}{3z - 1} = \frac{z}{3 - z} + \frac{3z}{3z - 1}, \quad |z| < 3, \quad \frac{1}{3} < |z| \\
 &= \frac{3z^2 - z + 9z - 3z^2}{(3 - z)(3z - 1)} = \frac{8z}{(3 - z)(3z - 1)}, \quad \frac{1}{3} < |z| < 3.
 \end{aligned}$$

Remark

The above Ex. 2 is a particular case of Ex. 1 where $a = 1/3$.

Example 3 : Find the Z-transform $\{3^{1/k}\}$.

Sol. : Putting $a = 3$ in Ex. 1 or proceeding independently, we get

$$Z\{3^{|k|}\} = \frac{-8z}{(1-3z)(z-3)}$$

Example 4 : Find the Z-transform of $f(k) = c^k \cos \alpha k$, $k \geq 0$, where α is real.

Sol. : Assuming $f(k) = 0$ for $k < 0$

$$\begin{aligned}
 Z\{f(k)\} &= \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos \alpha k z^{-k} \\
 &= \sum_{k=0}^{\infty} c^k \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2} \right] z^{-k} \\
 &= \sum_{k=0}^{\infty} c^k \cdot \frac{e^{i\alpha k}}{2} z^{-k} + \sum_{k=0}^{\infty} c^k \cdot e^{-i\alpha k} z^{-k} \\
 &= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{i\alpha}}{z} \right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{-i\alpha}}{z} \right)^k \\
 &= \frac{1}{2} \cdot \left[\frac{1}{1 - (ce^{i\alpha}/z)} \right] + \frac{1}{2} \left[\frac{1}{1 - (ce^{-i\alpha}/z)} \right]
 \end{aligned}$$

[See note (3), page 10-6]

$$\therefore Z(c^k \cos \alpha k) = \frac{z(z - c \cos \alpha)}{z^2 - 2zc \cos \alpha + c^2}$$

From (1), we find that the series being in G.P. are convergent if $|z| > |ce^{i\alpha}|$ and $|z| > |ce^{-i\alpha}|$ i.e. if $|z| > |c(\cos \alpha \pm i \sin \alpha)|$ i.e. if $|z| > |c|$.

Example 5 : Find the Z-transform of

(i) $f(k) = \cos \alpha k$, $k > 0$ where α is real. (ii) $f(k) = \cos \frac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example

$$Z\{\cos \alpha k\} = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}; |z| > 1$$

$$\text{Putting } \alpha = \frac{\pi}{3}, \quad Z \left\{ \cos \frac{k \pi}{3} \right\} = \frac{z(z - 1/2)}{z^2 - z + 1}; \quad |z| > 1$$

Example 6 : Find the Z-transform of $f(k) = c^k \sin \alpha k$, $k \geq 0$.

Sol. : Following the above lines we find that

$$Z\{c^k \sin \alpha k\} = \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}, |z| > |c|$$

Example 7 : Find the Z-transform of

(i) $f(k) = \sin \alpha k$, $k \geq 0$ where α is real. (ii) $f(k) = \sin \frac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}; |z| > 1$$

$$\text{Putting } \alpha = \frac{\pi}{3}, Z\left\{\sin \frac{k\pi}{3}\right\} = \frac{\sqrt{3}z/2}{z^2 - z + 1}.$$

Example 8 : Find the Z-transform of $f(k) = c^k \cos h \alpha k$, $k \geq 0$.

Sol. : By definition,

$$\begin{aligned} Z\{f(k)\} &= \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos h \alpha k \cdot z^{-k} \\ &= \sum_{k=0}^{\infty} c^k \left(\frac{e^{\alpha k} + e^{-\alpha k}}{2} \right) \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{c^k e^{\alpha k}}{2} \cdot z^{-k} + \sum_{k=0}^{\infty} \frac{c^k e^{-\alpha k}}{2} \cdot z^{-k} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{\alpha}}{z} \right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{ce^{-\alpha}}{z} \right)^k \\ &= \frac{1}{2} \left[\frac{1}{1 - (ce^{\alpha}/z)} \right] + \frac{1}{2} \left[\frac{1}{1 - (ce^{-\alpha}/z)} \right] \quad [\text{By note (3), page 10-6}] \\ &= \frac{1}{2} \left[\frac{z}{z - ce^{\alpha}} + \frac{z}{z - ce^{-\alpha}} \right] = \frac{z}{2} \left[\frac{1}{z - ce^{\alpha}} + \frac{1}{z - ce^{-\alpha}} \right] \\ &= \frac{z}{2} \left[\frac{z - ce^{-\alpha} + z - ce^{\alpha}}{z^2 - cz(e^{\alpha} + e^{-\alpha}) + c^2} \right] \\ &= \frac{z}{2} \left[\frac{2z - 2c \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right)}{z^2 - 2cz \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right) + c^2} \right] = \frac{z}{2} \cdot 2 \left[\frac{z - c \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right)}{z^2 - 2cz \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right) + c^2} \right] \\ &= \frac{z(z - cc \cos h \alpha)}{z^2 - 2cz \cos h \alpha + c^2} \quad |z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|) \end{aligned}$$

Corollary : Putting $c = 1$, we get if $f(k) = \cos h \alpha k$, then

$$Z\{f(k)\} = \frac{z(z - \cos h \alpha)}{z^2 - 2z \cos h \alpha + c^2} \quad |z| > \max(|e^{\alpha}|, |e^{-\alpha}|)$$

Example 9 : Find the Z-transform of $f(k) = c^k \sin h\alpha k$, $k \geq 0$.

Sol. : Following the above lines, we find that,

$$Z\{f(k)\} = \frac{cz \sin h\alpha}{z^2 - 2cz \cos h\alpha + c^2} \quad |z| > \max(|c e^{\alpha}|, |c e^{-\alpha}|)$$

Corollary : Putting $c = 1$, we get, if $f(k) = \sin h\alpha k$, then

$$Z\{f(k)\} = \frac{z \sin h\alpha}{z^2 - 2z \cos h\alpha + 1} \quad |z| > \max(|e^\alpha|, |e^{-\alpha}|)$$

Example 10 : Find the Z-transform of $\left\{ \sin \left(\frac{k\pi}{3} + \alpha \right) \right\}$, $k \geq 0$.

Sol. : We have

$$\begin{aligned} Z\{f(k)\} &= Z\left\{ \sin \left(\frac{k\pi}{3} + \alpha \right) \right\} = Z\left\{ \sin \frac{k\pi}{3} \cos \alpha + \cos \frac{k\pi}{3} \sin \alpha \right\} \\ &= \cos \alpha \cdot Z\left\{ \sin \frac{k\pi}{3} \right\} + \sin \alpha \cdot Z\left\{ \cos \frac{k\pi}{3} \right\} \\ &= \cos \alpha \cdot \frac{z \sin(\pi/3)}{z^2 - 2z \cos(\pi/3) + 1} + \sin \alpha \cdot \frac{z^2 - z \cos(\pi/3)}{z^2 - 2z \cos(\pi/3) + 1} \end{aligned}$$

[By Ex. 7 and 5 above]

$$\begin{aligned} &= \frac{z\{z \sin \alpha - \sin \alpha \cos(\pi/3) + \cos \alpha \sin(\pi/3)\}}{z^2 - 2z \cos(\pi/3) + 1} \\ &= \frac{z\{z \sin \alpha + \sin[(\pi/3) - \alpha]\}}{z^2 - 2z \cos(\pi/3) + 1} = \frac{z[z \sin \alpha + \sin(\pi/3 - \alpha)]}{z^2 - z + 1}. \end{aligned}$$

Example 11 : Find the Z-transform of $\left\{ \cos \left(\frac{k\pi}{3} + \alpha \right) \right\}$, $k \geq 0$. (M.U. 2009)

Sol. : We have

$$\begin{aligned} Z\{f(k)\} &= Z\left\{ \cos \left(\frac{k\pi}{3} + \alpha \right) \right\} = Z\left\{ \cos \frac{k\pi}{3} \cdot \cos \alpha - \sin \frac{k\pi}{3} \cdot \sin \alpha \right\} \\ &= \cos \alpha \cdot Z\left\{ \cos \frac{k\pi}{3} \right\} - \sin \alpha \cdot Z\left\{ \sin \frac{k\pi}{3} \right\} \\ \therefore F(z) &= \cos \alpha \cdot \frac{z^2 - z \cos(\pi/3)}{z^2 - 2z \cos(\pi/3) + 1} - \sin \alpha \cdot \frac{z \sin(\pi/3)}{z^2 - 2z \cos(\pi/3) + 1} \\ &= \frac{z\{z \cos \alpha - [\cos(\pi/3) \cdot \cos \alpha + \sin(\pi/3) \sin \alpha]\}}{z^2 - 2 \cos(\pi/3) \cdot z + 1} \\ &= \frac{z\{z \cos \alpha - \cos[(\pi/3) - \alpha]\}}{z^2 - 2 \cos(\pi/3) \cdot z + 1} = \frac{z[z \cos \alpha - \cos(\pi/3 - \alpha)]}{z^2 - z + 1} \end{aligned}$$

Example 12 : Find the Z-transform of $\{\sin(ak + b)\}$, $k \geq 0$.

Sol. : We have $\sin(ak + b) = \sin ak \cos b + \cos ak \sin b$

$$\begin{aligned} \therefore Z\{\sin(ak + b)\} &= \cos b \cdot Z\{\sin ak\} + \sin b \cdot Z\{\cos ak\} \\ &= \cos b \cdot \frac{z \sin a}{z^2 - 2z \cos a + 1} + \sin b \cdot \frac{z(z - \cos a)}{z^2 - 2z \cos a + 1} \end{aligned}$$

(M.U. 2008)

$$\therefore Z\{\sin(ak+b)\} = \frac{z[\sin a \cos b - \cos a \sin b + z \sin b]}{z^2 - 2z \cos a + 1}$$

$$= \frac{z[\sin(a-b) + z \sin b]}{z^2 - 2z \cos a + 1}.$$

Example 13 : Find the Z-transform of $\{\cos(ak+b)\}$, $k \geq 0$.

Sol. : We have $\cos(ak+b) = \cos ak \cos b - \sin ak \sin b$

$$\therefore Z\{\cos(ak+b)\} = \cos b \cdot Z\{\cos ak\} - \sin b \cdot Z\{\sin ak\}$$

$$= \cos b \cdot \frac{z(z - \cos a)}{z^2 - 2z \cos a + 1} - \sin b \cdot \frac{z \sin a}{z^2 - 2z \cos a + 1}$$

$$= \frac{z[z \cos b - (\cos a \cos b + \sin a \sin b)]}{z^2 - 2z \cos a + 1}$$

$$= \frac{z[z \cos b - \cos(a-b)]}{z^2 - 2z \cos a + 1}.$$

EXERCISE - IV

Find the Z-transforms of $\{f(k)\}$ where $f(k)$ is given by ($k \geq 0$).

$$1. 2^{|k|}$$

$$2. \left(\frac{1}{2}\right)^k$$

$$3. \cos k$$

$$4. \cos 2k$$

$$5. \sin k$$

$$6. \sin 2k$$

$$7. \cos hk$$

$$8. \cos h2k$$

$$9. \sin hk$$

$$10. \sin h2k$$

$$11. \sin(k+1)$$

$$12. 2^k \cos k$$

$$13. \sin(3k+2)$$

$$14. \cos(3k+2)$$

$$15. \sin\left(\alpha k + \frac{\pi}{2}\right)$$

$$16. \cos\left(\alpha k + \frac{\pi}{2}\right)$$

$$17. \sin\left(\frac{k\pi}{4} + a\right)$$
 (M.U. 2008)

[Ans. : (1) $-\frac{3z}{(1-2z)(z-2)}$, (2) $\frac{3z}{(2-z)(2z-1)}$, (3) $\frac{z(z-\cos 1)}{z^2-2z \cos 1+1}$, (4) $\frac{z(z-\cos 2)}{z^2-2z \cos 2+1}$,
 (5) $\frac{z \sin 1}{z^2-2 \cos 1+1}$, (6) $\frac{z \sin 2}{z^2-2 \cos 2+1}$, (7) $\frac{z(z-\cos h1)}{z^2-2z \cos h1+1}$, (8) $\frac{z(z-\cos h2)}{z^2-2z \cos h2+1}$,

$$(9) \frac{\sin h1}{z^2-2z \cos h1+1},$$

$$(10) \frac{\sinh 2}{z^2-2z \cosh 2+1},$$

$$(11) \frac{z^2 \sin 1}{z^2-2z \cos 1+1},$$

$$(12) \frac{z^2-2 \cos 1 \cdot z}{z^2-4 \cos 1 \cdot z+4},$$

$$(13) \frac{z(\sin 1+z \sin 2)}{z^2-2z \cos 3+1},$$

$$(14) \frac{z(z \cos 2-\cos 1)}{z^2-2z \cos 3+1},$$

$$(15) \frac{z(z-\cos \alpha)}{z^2-2z \cos \alpha+1},$$

$$(16) \frac{-z \cos \alpha}{z^2-2z \cos \alpha+1},$$

$$(17) \frac{z \sin[(\pi/4)-a]+z^2 \sin a}{z^2-\sqrt{2} \cdot z+1}.$$

(2) Change of Scale

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F(z/a)$ and if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ is $|a| R_1 < |z| < |a| R_2$.

Proof : By definition, $F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

Replacing z by z/a , we get, $F\left(\frac{z}{a}\right) = \sum f(k) \left(\frac{z}{a}\right)^{-k}$ (1)

But by definition again, $Z\{a^k f(k)\} = \sum_{k=-\infty}^{\infty} a^k f(k) z^{-k} = \sum f(z) \left(\frac{z}{a}\right)^{-k}$ (2)

From (1) and (2), we get $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

Further, if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ from (2) will be $R_1 < |z/a| < R_2$ i.e. $|a|R_1 < |z| < |a|R_2$.

Example 1 : Obtain $Z\{1\}$ and hence deduce $Z\{a^k\}$, $k \geq 0$.

Sol. : By definition

$$\begin{aligned} Z\{1\} &= \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\ &= \frac{1}{1-(1/z)} = \frac{z}{z-1}. \end{aligned}$$

Now, $a^k = a^k \cdot 1$. Hence, by change of scale property,

$$Z\{a^k\} = Z\{a^k \cdot 1\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}.$$

Example 2 : Find $Z\{c^k \sin \alpha k\}$ from $Z\{\sin \alpha k\}$

Sol. : We know that, $Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}$

By change of scale property,

$$Z\{c^k \sin \alpha k\} = \frac{(z/c) \sin \alpha}{(z/c)^2 - 2(z/c) \cos \alpha + 1} = \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}.$$

Example 3 : Find $Z\{c^k \cos \alpha k\}$ from $Z\{\cos \alpha k\}$.

Sol. : We know that $Z\{\cos \alpha k\} = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}$

By change of scale property,

$$Z\{c^k \cos \alpha k\} = \frac{\frac{z}{c} \left\{ \frac{z}{c} - \cos \alpha \right\}}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right) \cos \alpha + 1} = \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}.$$

(3) Shifting Property

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{f(k+n)\} = z^n F(z)$ and $Z\{f(k-n)\} = z^{-n} F(z)$.

Proof : We have $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$

$$\begin{aligned} \therefore Z\{f(k+n)\} &= \sum f(k+n) z^{-k} = \sum f(k+n) z^{-(k+n)} \cdot z^n \\ &= z^n \sum_{k=-\infty}^{\infty} f(k+n) z^{-k} \end{aligned}$$

If we put $k+n=m$ when $k=-\infty$, $m=-\infty$ and when $k=+\infty$, $m=+\infty$.

$$\therefore Z\{f(k+n)\} = z^n \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^n F(z)$$

Changing the sign of n or proceeding as above $Z\{f(k-n)\} = z^{-n} F(z)$.

Example : Find the Z-transform of $\frac{1}{k+1}$, $k \geq 1$. Indicate the region of convergence.

(M.U. 2009)

Sol. : By Ex. 5, page 10-10, putting $\alpha = 1$, we have

$$Z\left(\frac{1}{k}\right) = -\log\left(1 - \frac{1}{z}\right), \quad |z| > 1$$

By Shifting property, $Z\{f(k-n)\} = z^{-n} F(z)$.

$$\text{Hence, } Z\left[\frac{1}{k+1}\right] = z \cdot \left\{-\log\left(1 - \frac{1}{z}\right)\right\} = -z \log\left(1 - \frac{1}{z}\right), \quad |z| > 1. \quad [\text{Put } n = -1]$$

R.O.C. is $|z| > 1$.

Unilateral or one sided or causal sequence : If a sequence $\{f(k)\}$ is defined for the right side only i.e. if $\{f(k)\}$ extends to infinity on the right only i.e. if k varies from 0 to $+\infty$, the sequence is called unilateral or one sided or causal sequence.

For example, $f(k) = \begin{cases} 0, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$ is a causal sequences.

Theorem : If $\{f(k)\}$ is one sided and if $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$,

$$\text{then } Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$$

$$\text{and } Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^n f(-r) z^{-n+r}$$

Proof : Since $\{f(k)\}$ is one sided sequence, by data,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} \quad \dots \dots \dots (1)$$

$$\begin{aligned} \therefore Z\{f(k+n)\} &= \sum_{k=0}^{\infty} f(k+n) z^{-k} = \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n \\ &= z^n \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \end{aligned}$$

Put $k+n = m$. When $k = 0$, $n = m$; when $k = \infty$, $m = \infty$.

$$\therefore Z\{f(k+n)\} = z^n \sum_{m=n}^{\infty} f(m) z^{-m}$$

The interval from n to ∞ can be changed to the interval from 0 to ∞ by subtracting from it, the terms in the interval 0 to $n-1$.

$$\begin{aligned} \therefore Z\{f(k+n)\} &= z^n \sum_{m=0}^{\infty} f(m) z^{-m} - z^n \sum_{m=0}^{n-1} f(m) z^{-m} \\ &= z^n \cdot F(z) - z^n \sum_{m=0}^{n-1} f(m) z^{-m} \end{aligned}$$

Taking z^n inside the summation,

$$\therefore Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}.$$

Further, from (1) again,

$$\begin{aligned} Z\{f(k-n)\} &= \sum_{k=0}^{\infty} f(k-n) z^{-k} = \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)} \cdot z^{-n} \\ &= z^{-n} \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)} \end{aligned}$$

Put $k-n = m$. When $k=0$, $m=-n$ and when $k=\infty$, $m=\infty$.

$$\therefore Z\{f(k-n)\} = z^{-n} \sum_{m=-n}^{\infty} f(m) z^{-m}$$

As before we split the interval from $-n$ to ∞ into two intervals, $-n$ to -1 and 0 to ∞ .

$$\begin{aligned} \therefore Z\{f(k-n)\} &= z^{-n} \sum_{m=-n}^{-1} f(m) z^{-m} + z^{-n} \sum_{m=0}^{\infty} f(m) z^{-m} \\ &= z^{-n} F(z) + z^{-n} \sum_{m=-n}^{-1} f(m) z^{-m} \end{aligned}$$

Taking z^{-n} inside the summation,

$$\therefore Z\{f(k-n)\} = z^{-n} F(z) + \sum_{m=-n}^{-1} f(m) z^{-(m+n)}$$

Putting $m=-r$, when $m=-1$, $r=1$ and when $m=-n$, $r=n$.

$$\therefore Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^n f(-r) z^{-n+r}.$$

Corollary 1 : For one sided sequence ($k \geq 0$), we have

$$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^n f(-r) z^{-n+r}$$

Since for causal sequence $f(-1) = f(-2) = \dots = f(-n) = 0$, the second term is zero.

$$\therefore Z\{f(k-n)\} = z^{-n} F(z).$$

Corollary 2 : Since for one sided sequence ($k \geq 0$), we have

$$Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$$

Putting $n=1$,

$$Z\{f(k+1)\} = z^1 F(z) - \sum_{m=0}^0 f(0) z^{1-0} = z F(z) - z f(0) \quad \dots \dots \dots (4)$$

Putting $n=2$,

$$\begin{aligned} Z\{f(k+2)\} &= z^2 F(z) - \sum_{m=0}^{2-1} f(m) z^{2-m} = z^2 F(z) - [f(0) z^{2-0} + f(1) z^{2-1}] \\ &= z^2 F(z) - z^2 f(0) - z f(1). \end{aligned}$$

(4) Multiplication by k

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{k f(k)\} = -z \frac{d}{dz} F(z)$.

Proof: We have, by definition,

$$\begin{aligned} Z\{kf(k)\} &= \sum_{k=-\infty}^{\infty} kf(k)z^{-k} = \sum k f(k) z^{-k-1} \cdot z \\ &= -z \sum -kf(k)z^{-k-1} = -z \sum f(k) \frac{d}{dz}(z^{-k}) \\ &= -z \cdot \frac{d}{dz} \sum f(k)z^{-k} = -z \frac{d}{dz} F(z) \end{aligned}$$

$$\text{In general, } Z\{k^n f(k)\} = \left(-z \frac{d}{dz}\right)^n F(z)$$

Note that $\left(-z \frac{d}{dz}\right)^2 \neq z^2 \frac{d^2}{dz^2}$, but it is equal to repeated operations $\left(-z \frac{d}{dz}\right) \left(-z \frac{d}{dz}\right)$.

$$\text{Corollary 1: } Z\{k\} = \frac{z}{(z-1)^2}, \quad |z| > 1$$

Proof: By definition,

$$Z\{1\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + z^{-1} + z^{-2} + \dots$$

$$\therefore Z\{1\} = \frac{1}{1-z^{-1}} = \frac{1}{1-(1/z)}, \quad |z^{-1}| < 1$$

$$= \frac{z}{z-1}, \quad |z| > 1$$

Now, by the above theorem,

$$\begin{aligned} Z\{k\} &= Z\{k \cdot 1\} = \left(-z \frac{d}{dz}\right)[Z\{1\}] \\ &= \left(-z \frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = -z \left[\frac{(z-1)1 - z \cdot 1}{(z-1)^2} \right] \\ &= -z \left[\frac{-1}{(z-1)^2} \right] = \frac{z}{(z-1)^2}, \quad |z| > 1. \end{aligned}$$

$$\text{Corollary 2: } Z\{k^2\} = \frac{z(z+1)}{(z-1)^3}.$$

Proof: We have proved above that $Z\{1\} = \frac{z}{z-1}$.

By the above theorem,

$$\begin{aligned} Z\{k^2\} &= Z\{k^2 \cdot 1\} = \left(-z \frac{d}{dz}\right)^2 Z\{1\} \\ &= \left(-z \frac{d}{dz}\right) \left(-z \frac{d}{dz}\right) \left(\frac{z}{z-1}\right) = \left(-z \frac{d}{dz}\right) \left[\frac{z}{(z-1)^2}\right] \quad [\text{As above}] \\ &= -z \left[\frac{(z-1)^2 \cdot 1 - z \cdot 2(z-1) \cdot 1}{(z-1)^4} \right] \\ &= -z \left[\frac{(z-1) - 2z}{(z-1)^3} \right] = \frac{z(z+1)}{(z-1)^3}. \end{aligned}$$

(5) Initial Value

Theorem : If $Z\{f(k)\} = F(z)$, $k \geq 0$, then $f(0) = \lim_{z \rightarrow \infty} F(z)$.

Proof : By definition $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = F(z)$

$\therefore f(0)z^0 + f(1)z^{-1} + f(2)z^{-2} + \dots = F(z)$

\therefore Taking the limit as $z \rightarrow \infty$ of both sides of

$$f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots = F(z) \quad \therefore f(0) = \lim_{z \rightarrow \infty} F(z).$$

(6) Final Value

Theorem : $\lim_{k \rightarrow \infty} f(k) = \lim_{z \rightarrow 1^-} (z-1)F(z)$

Proof : By definition,

$$\begin{aligned} Z\{f(k+1) - f(k)\} &= \sum_{k=0}^{\infty} [f(k+1) - f(k)]z^{-k} \\ \therefore Z\{f(k+1)\} - Z\{f(k)\} &= \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1) - f(k)]z^{-k} \\ \therefore zF(z) - zf(0) - F(z) &= \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1) - f(k)]z^{-k} \quad [\text{By (A), page 10-22}] \\ \therefore (z-1)F(z) &= zf(0) + \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1) - f(k)]z^{-k} \end{aligned}$$

Taking the limits of both sides as $z \rightarrow 1^-$,

$$\therefore \lim_{z \rightarrow 1^-} (z-1)F(z) = f(0) + \lim_{z \rightarrow 1^-} \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1) - f(k)]z^{-k}$$

Changing the order of limits,

$$\begin{aligned} \lim_{z \rightarrow 1^-} (z-1)F(z) &= f(0) + \lim_{n \rightarrow \infty} \cdot \lim_{z \rightarrow 1^-} \sum_{k=0}^n [f(k+1) - f(k)] \cdot z^{-k} \\ &= f(0) + \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(k+1) - f(k)] \left[\lim_{z \rightarrow 1^-} z^{-k} \right] = 1 \end{aligned}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} [f(0) + f(1) - f(0) + f(2) - f(1) + \dots + f(n+1) - f(n)] \\ &= \lim_{n \rightarrow \infty} f(n+1) = \lim_{n \rightarrow \infty} f(n) \end{aligned}$$

$$\begin{aligned} &\therefore \lim_{z \rightarrow 1^-} (z-1)F(z) = f(0) \end{aligned}$$

(7) Convolution

If $\{f(k)\}$ and $\{g(k)\}$ are two sequences then their convolution $\{f(k)\} * \{g(k)\}$ is defined by

$$h(k) = \{f(k)\} * \{g(k)\}$$

$$\text{where, } \{h(k)\} = \sum_{m=-\infty}^{\infty} f(m)g(k-m) = \{g(k)\} * \{f(k)\}$$

Theorem : If $\{h(k)\}$ is the convolution of two sequences $\{f(k)\}$ and $\{g(k)\}$ then

$$Z\{h(k)\} = Z\{f(k)\} Z\{g(k)\} \text{ i.e. } H(z) = F(z) G(z).$$

Proof : By definition, $H(z) = Z\{h(k)\} = Z[\{f(k)\} * \{g(k)\}]$

$\therefore H(z) = Z\left[\sum_{m=-\infty}^{\infty} f(m)g(k-m)\right]$

Since the power series converges absolutely, it converges uniformly within ROC. Hence we can interchange the order of summation.

$$H(z) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (f(m)g(k-m))z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} (f(m)z^{-m}) \cdot \sum_{k=-\infty}^{\infty} g(k-m)z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} (f(m)z^{-m}) \cdot \sum_{p=k-m}^{\infty} g(p)z^{-p}$$

$$= \sum_{m=-\infty}^{\infty} f(m)z^{-m} \cdot \sum_{p=-\infty}^{\infty} g(p)z^{-p}$$

(When $k = -\infty$, $p = -\infty$ and when $k = \infty$, $p = \infty$)

$\therefore H(z) = F(z) G(z)$.

Example 1 : If $f(k) = U(k)$ and $g(k) = 2^k U(k)$, find Z-transform of $\{f(k) * g(k)\}$. (M.U. 2008)

$$\therefore Z\{f(k)\} = \sum_{k=0}^{\infty} z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$\therefore F(z) = \frac{1}{1-(1/z)} = \frac{z}{z-1}, \left| \frac{1}{z} \right| < 1.$$

By the change of scale property (page 10-19),

$$Z\{g(k)\} = Z\{2^k U(k)\} = \frac{z/2}{(z/2)-1}$$

$$G(z) = \frac{z}{z-2}, \left| \frac{2}{z} \right| < 1$$

By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) G(z) = \frac{z}{z-1} \cdot \frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)}, |z| > 2.$$

Example 2 : If $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$, then find the Z-transform of $\{f(k) * g(k)\}$. (M.U. 2009, 14)

Sol. : As above, $\{f(k)\} = \{4^0, 4^1, 4^2, \dots\}$

$$\therefore Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + \dots$$

$$\therefore Z\{f(k)\} = 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \dots = \frac{1}{1-(4/z)} = \frac{z}{z-4}, \left| \frac{4}{z} \right| < 1$$

$$\therefore \{g(k)\} = \{5^0, 5^1, 5^2, \dots\}$$

$$\begin{aligned} \therefore Z\{g(k)\} &= \sum_{k=0}^{\infty} g(k)z^{-k} = 5^0 z^0 + 5z^{-1} + 5^2 z^{-2} + \dots \\ &= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + \dots = \frac{1}{1-(5/z)} = \frac{z}{z-5}, \left| \frac{5}{z} \right| < 1. \end{aligned}$$

\therefore By convolution theorem

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{(z-4)} \cdot \frac{z}{(z-5)} = \frac{z^2}{(z-4)(z-5)}, \quad |z| > 5.$$

Alternatively : If $\{h(k)\} = U(k) = \{1, 1, 1, \dots\}$, then

$$Z\{h(k)\} = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \dots = \frac{1}{1 - (1/z)} = \frac{z}{z-1}$$

$$\therefore F(z) = \frac{z}{z-1}, \quad \left| \frac{1}{z} \right| < 1.$$

By change of scale property,

$$Z\{f(k)\} = Z\{4^k U(k)\} \quad \therefore F(z) = \frac{z^{1/4}}{(z/4)-1} = \frac{z}{z-4}$$

$$Z\{g(k)\} = Z\{5^k U(k)\} \quad \therefore G(z) = \frac{z^{1/5}}{(z/5)-1} = \frac{z}{z-5}$$

\therefore By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \frac{z}{z-4} \cdot \frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}.$$

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \frac{1}{2^k} * \frac{1}{3^k}$.

$$\text{Sol. : } Z\left\{\frac{1}{2^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(2z)^k} = 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \dots \\ = \frac{1}{1-[1/(2z)]} = \frac{2z}{2z-1}, \quad |2z| > 1 \text{ i.e. } |z| > \frac{1}{2}.$$

Similarly, $Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}, \quad |z| > \frac{1}{3}$.

By convolution theorem,

$$Z\{f(k)\} = \left(\frac{2z}{2z-1} \right) \left(\frac{3z}{3z-1} \right), \quad |z| > \frac{1}{2}.$$

Example 4 : Find $Z\{f(k) * g(k)\}$ if $f(k) = \frac{1}{3^k}$ and $g(k) = \frac{1}{5^k}$.

(M.U. 2015)

Sol. : We have $Z\left\{\frac{1}{3^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{3^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(3z)^k}$

$$= \frac{1}{1-\frac{1}{3z}} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \dots$$

$$= \frac{1}{1-(1/3z)} \quad \left[\because S_{\infty} = \frac{a}{1-r} \text{ and } |3z| > 1 \right]$$

$$= \frac{3z}{3z-1} \quad \left[|z| > \frac{1}{3} \right] \quad [\text{Or by Ex. 6, page 10-11, } Z\left\{\frac{1}{3^k}\right\} = \frac{z}{z-(1/3)}]$$

Similarly, $Z\left\{\frac{1}{5^k}\right\} = \frac{5z}{5z-1} \quad \left[|5z| > 1 \text{ i.e., } |z| > \frac{1}{5} \right]$

Example 5 : Find $Z\{f(k) * g(k)\}$ if $f(k) = \frac{1}{5^k}$ and $g(k) = \frac{1}{7^k}$.

$$\text{Sol. : We have } Z\left\{\frac{1}{5^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{5^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$$

$$= \frac{1}{1+\frac{1}{5z}} + \frac{1}{(5z)^2} + \frac{1}{(5z)^3} + \dots$$

$$= \frac{1}{1-(1/5z)} \quad \left[\because S_{\infty} = \frac{a}{1-r} \text{ and } |5z| > 1 \right]$$

$$\text{Similarly, } Z\left\{\frac{1}{7^k}\right\} = \frac{7z}{7z-1} \quad \left[|7z| > 1 \text{ i.e., } |z| > \frac{1}{7} \right]$$

\therefore By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z) = \left(\frac{5z}{5z-1} \right) \left(\frac{7z}{7z-1} \right), \quad |z| > \frac{1}{5}.$$

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{e^{-ak} f(k)\} = F(e^a z)$.

Proof : By definition,

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore Z\{e^{-ak} f(k)\} = \sum_{k=-\infty}^{\infty} e^{-ak} f(k) z^{-k} = \sum_{k=-\infty}^{\infty} f(k) (e^a z)^{-k} = F(e^a z).$$

Example 1 : Find the Z-transform of $\{k e^{-ak}\}$, $k \geq 0$.

Sol. : We know that if $U(k) = 1$, for $k > 0$, then $Z\{U(k)\} = \frac{z}{z-1}$.

By the above theorem, $Z\{e^{-ak} U(k)\} = Z(e^{-ak}) = \frac{e^a z}{e^a z - 1}$.

Now, by (4) (property of multiplication by k), page 10-22

$$\begin{aligned} Z[k e^{-ak}] &= -z \frac{d}{dz} \left(\frac{e^a z}{e^a z - 1} \right) \\ &= -z e^a \cdot \left[\frac{(e^a \cdot z - 1) - z(e^a)}{(e^a \cdot z - 1)^2} \right] \\ &= -z \cdot e^a \cdot \frac{(-1)}{(e^a \cdot z - 1)^2} = \frac{e^a \cdot z}{(e^a z - 1)^2}. \end{aligned}$$

Example 2 : Find the Z-transform of $\{k^2 e^{-ak}\}$, $k \geq 0$.

Sol. : As in Ex. 1 above $Z[k e^{-ak}] = \frac{e^a \cdot z}{(e^a z - 1)^2}$

(M.U. 2009)

Now, by (4) (property of multiplication by k), page 10-22

$$\begin{aligned} Z[k^2 e^{-ak}] &= \left(-z \frac{d}{dz} \right) F(z) \\ &= -z \frac{d}{dz} \left[\frac{e^a z}{(e^a z - 1)^2} \right] = -z \cdot e^a \cdot \frac{d}{dz} \left[\frac{z}{(e^a \cdot z - 1)^2} \right] \\ &= -z e^a \cdot \left[\frac{(e^a \cdot z - 1)^2 \cdot 1 - z \cdot 2(e^a \cdot z - 1) \cdot e^a}{(e^a \cdot z - 1)^4} \right] \\ &= -z \cdot e^a \left[\frac{e^a \cdot z - 1 - 2 \cdot z \cdot e^a}{(e^a \cdot z - 1)^3} \right] = z \cdot e^a \frac{(z \cdot e^a + 1)}{(z \cdot e^a - 1)^3}. \end{aligned}$$

Example 3 : Find $Z\{e^{-ak} \cos bk\}$.

Sol. : We have already obtained (Ex. 5, page 10-16)

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1}$$

Now, by the above result,

$$Z\{e^{-ak} \cos bk\} = \frac{e^a z (e^a z - \cos b)}{(e^a \cdot z)^2 - 2(e^a z) \cos b + 1}$$

Multiply in the numerator and denominator by e^{-2a} .

$$Z\{e^{-ak} \cos bk\} = \frac{z(z - e^{-a} \cos b)}{z^2 - 2e^{-a} z \cos b + e^{-2a}}.$$

Example 4 : Find $Z\{e^{-ak} \sin bk\}$.

Sol. : We have already proved that (Ex. 7, page 10-17)

$$Z\{\sin bk\} = \frac{z \sin b}{z^2 - 2z \cos b + 1}$$

∴ By the above property,

$$\begin{aligned} Z\{e^{-ak} \sin bk\} &= \frac{(e^a z) \sin b}{(e^a z)^2 - 2(e^a z) \cos b + 1} \\ &= \frac{e^{-a} z \cdot \sin b}{z^2 - 2e^{-a} z \cos b + e^{-2a}}. \end{aligned}$$

We give below the list of Z-transforms obtained above.

Table of Z-transforms

- | | |
|--|--|
| 1. $Z[\delta(k)] = 1$, for all z | 2. $Z[U(k)] = \frac{z}{z-1}$, $ z > 1$ |
| 3. $Z[1] = \frac{z}{z-1}$, $ z > 1$ | 4. $Z\{k\} = \frac{z}{(z-1)^2}$, $ z > 1$ |
| 5. $Z[a^k] = \frac{z}{z-a}$, $k \geq 0$, $ z > a $. | 6. $Z[a^k] = \frac{z}{a-z}$, $k < 0$, $ z < a $. |
| 7. $Z\{ka^k\} = \frac{az}{(z-a)^2}$, $k \geq 0$, $ z > a $. | 8. $Z\{n!C_k\} = \left(1 + \frac{1}{z}\right)^n$, $0 \leq k \leq n$, $ z > 0$ |

Example 1 : Find Z-transform of $\{a \cos k\alpha + b \sin k\alpha\}$, $k \geq 0$.

Sol. : $Z\{a \cos k\alpha + b \sin k\alpha\} = aZ\{\cos k\alpha\} + bZ\{\sin k\alpha\}$ [By linearity property]

$$\begin{aligned} &= a \cdot \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} + b \cdot \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \quad |z| > 1 \\ &= \frac{az^2 + z(b \sin \alpha - a \cos \alpha)}{z^2 - 2z \cos \alpha + 1}. \end{aligned}$$

Example 2 : Find $Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\}$, $k \geq 0$.

(M.U. 2008)

$$\begin{aligned} \text{Sol. :} \quad \text{Since } \sin\left(\frac{k\pi}{4} + a\right) &= \sin\frac{k\pi}{4} \cos a + \cos\frac{k\pi}{4} \sin a \\ Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\} &= Z\left\{\sin\frac{k\pi}{4} \cos a + \cos\frac{k\pi}{4} \sin a\right\} \\ &= \cos a Z\left\{\sin\left(\frac{\pi}{4} k\right)\right\} + \sin a Z\left\{\cos\left(\frac{\pi}{4} k\right)\right\} \\ &= \cos a \frac{z \sin(\pi/4)}{z^2 - 2z \cos(\pi/4) + 1} + \sin a \frac{z[z - \cos(\pi/4)]}{z^2 - 2z \cos(\pi/4) + 1} \\ &= \frac{\cos a \cdot (z/\sqrt{2})}{z^2 - (2z/\sqrt{2}) + 1} + \frac{\sin a z \cdot [z - (1/\sqrt{2})]}{z^2 - (2z/\sqrt{2}) + 1} \\ &= \frac{z}{\sqrt{2}} \cdot \frac{[\cos a + \sin a \cdot (\sqrt{2}z - 1)]}{z^2 - \sqrt{2}z + 1}. \end{aligned}$$

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$, $k \geq 0$.

Sol. : We have $\cos\left(\frac{k\pi}{4} + a\right) = \cos\frac{k\pi}{4} \cos a - \sin\frac{k\pi}{4} \sin a$

$$\begin{aligned}
 \text{Now, by (4) (property of multiplication by } k\text{), page 10-22} \\
 Z[k^2 e^{-ak}] &= \left(-z \frac{d}{dz} \right) F(z) \\
 &= -z \frac{d}{dz} \left[\frac{e^a z}{(e^a z - 1)^2} \right] = -z \cdot e^a \cdot \frac{d}{dz} \left[\frac{z}{(e^a z - 1)^2} \right] \\
 &= -z e^a \cdot \left[\frac{(e^a \cdot z - 1)^2 \cdot 1 - z \cdot 2(e^a \cdot z - 1) \cdot e^a}{(e^a \cdot z - 1)^4} \right] \\
 &= -z \cdot e^a \left[\frac{e^a \cdot z - 1 - 2 \cdot z \cdot e^a}{(e^a \cdot z - 1)^3} \right] = z \cdot e^a \frac{(z \cdot e^a + 1)}{(z \cdot e^a - 1)^3}.
 \end{aligned}$$

Example 3 : Find $Z\{e^{-ak} \cos bk\}$.

Sol. : We have already obtained (Ex. 5, page 10-16)

$$Z\{\cos(bk)\} = \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1}$$

Now, by the above result,

$$Z\{e^{-ak} \cos bk\} = \frac{e^a z (e^a z - \cos b)}{(e^a \cdot z)^2 - 2(e^a z) \cos b + 1}$$

Multiply in the numerator and denominator by e^{-2a} .

$$Z\{e^{-ak} \cos bk\} = \frac{z(z - e^{-a} \cos b)}{z^2 - 2e^{-a} z \cos b + e^{-2a}}.$$

Example 4 : Find $Z\{e^{-ak} \sin bk\}$.

Sol. : We have already proved that (Ex. 7, page 10-17)

$$Z\{\sin(bk)\} = \frac{z \sin b}{z^2 - 2z \cos b + 1}$$

∴ By the above property,

$$\begin{aligned}
 Z\{e^{-ak} \sin bk\} &= \frac{(e^a z) \sin b}{(e^a z)^2 - 2(e^a z) \cos b + 1} \\
 &= \frac{e^{-a} z \cdot \sin b}{z^2 - 2e^{-a} z \cos b + e^{-2a}}.
 \end{aligned}$$

We give below the list of Z-transforms obtained above.

Table of Z-transforms

1. $Z[\delta(k)] = 1$, for all z
2. $Z[U(k)] = \frac{z}{z-1}$, $|z| > 1$
3. $Z[1] = \frac{z}{z-1}$, $|z| > 1$
4. $Z[k] = \frac{z}{(z-1)^2}$, $|z| > 1$
5. $Z[a^k] = \frac{z}{z-a}$, $k \geq 0$, $|z| > |a|$.
6. $Z[a^k] = \frac{z}{a-z}$, $k < 0$, $|z| < |a|$.
7. $Z\{ka^k\} = \frac{az}{(z-a)^2}$, $k \geq 0$, $|z| > |a|$.
8. $Z\{^n C_k\} = \left(1 + \frac{1}{z}\right)^n$, $0 \leq k \leq n$, $|z| > 0$.

$$\begin{aligned}
 9. Z\{^k C_n\} &= z^{-n} \left(1 - \frac{1}{z}\right)^{-(n+1)}, \quad |z| > 1 \quad 10. Z\left[\frac{a^k}{k!}\right] = e^{az}, \quad k \geq 0 \text{ for all } z. \\
 11. Z\{a^{|k|}\} &= \frac{az}{1-az} + \frac{z}{z-a}, \quad |a| < |z| < \frac{1}{|a|} \\
 12. Z\{c^k \cos \alpha k\} &= \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c| \\
 13. Z\{c^k \sin \alpha k\} &= \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|. \\
 14. Z\{c^k \cosh \alpha k\} &= \frac{z(z - c \cosh \alpha)}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|) \\
 15. Z\{c^k \sinh \alpha k\} &= \frac{cz \sinh \alpha}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|).
 \end{aligned}$$

Miscellaneous Examples

Example 1 : Find Z-transform of $\{a \cos k \alpha + b \sin k \alpha\}$, $k \geq 0$.

$$\begin{aligned}
 \text{Sol. :} \quad Z\{a \cos k \alpha + b \sin k \alpha\} &= aZ\{\cos k \alpha\} + bZ\{\sin k \alpha\} \quad [\text{By linearity property}] \\
 &= a \cdot \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} + b \cdot \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \quad |z| > 1 \\
 &= \frac{az^2 + z(b \sin \alpha - a \cos \alpha)}{z^2 - 2z \cos \alpha + 1}.
 \end{aligned}$$

Example 2 : Find $Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\}$, $k \geq 0$.

(M.U. 2008)

$$\begin{aligned}
 Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\} &= Z\left\{\sin\frac{k\pi}{4} \cos a + \cos\frac{k\pi}{4} \sin a\right\} \\
 &= \cos a Z\left\{\sin\left(\frac{\pi}{4} k\right)\right\} + \sin a Z\left\{\cos\left(\frac{\pi}{4} k\right)\right\} \\
 &= \cos a \frac{z \sin(\pi/4)}{z^2 - 2z \cos(\pi/4) + 1} + \sin a \frac{z[z - \cos(\pi/4)]}{z^2 - 2z \cos(\pi/4) + 1} \\
 &= \frac{\cos a \cdot (z/\sqrt{2})}{z^2 - (2z/\sqrt{2}) + 1} + \frac{\sin a \cdot [z - (1/\sqrt{2})]}{z^2 - (2z/\sqrt{2}) + 1} \\
 &= \frac{z}{\sqrt{2}} \cdot \frac{[\cos a + \sin a \cdot (\sqrt{2}z - 1)]}{z^2 - \sqrt{2} \cdot z + 1}.
 \end{aligned}$$

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$, $k \geq 0$.

$$\text{Sol. :} \quad \text{We have } \cos\left(\frac{k\pi}{4} + a\right) = \cos\frac{k\pi}{4} \cos a - \sin\frac{k\pi}{4} \sin a$$

Now, by (4) (property of multiplication by k), page 10-22

$$\begin{aligned} Z[k^2 e^{-ak}] &= \left(-z \frac{d}{dz} \right) F(z) \\ &= -z \frac{d}{dz} \left[\frac{e^a z}{(e^a z - 1)^2} \right] = -z \cdot e^a \cdot \frac{d}{dz} \left[\frac{z}{(e^a \cdot z - 1)^2} \right] \\ &= -z e^a \cdot \left[\frac{(e^a \cdot z - 1)^2 \cdot 1 - z \cdot 2 \cdot (e^a \cdot z - 1) \cdot e^a}{(e^a \cdot z - 1)^4} \right] \\ &= -z \cdot e^a \left[\frac{e^a \cdot z - 1 - 2 \cdot z \cdot e^a}{(e^a \cdot z - 1)^3} \right] = z \cdot e^a \frac{(z \cdot e^a + 1)}{(z \cdot e^a - 1)^3}. \end{aligned}$$

Example 3 : Find $Z\{e^{-ak} \cos bk\}$.

Sol. : We have already obtained (Ex. 5, page 10-16)

$$Z[\cos bk] = \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1}$$

Now, by the above result,

$$Z\{e^{-ak} \cos bk\} = \frac{e^a z (e^a z - \cos b)}{(e^a \cdot z)^2 - 2(e^a z) \cos b + 1}$$

Multiply in the numerator and denominator by e^{-2a}

$$Z\{e^{-ak} \cos bk\} = \frac{z(z - e^{-a} \cos b)}{z^2 - 2e^{-a} z \cos b + e^{-2a}}.$$

Example 4 : Find $Z\{e^{-ak} \sin bk\}$.

Sol. : We have already proved that (Ex. 7, page 10-17)

$$Z[\sin bk] = \frac{z \sin b}{z^2 - 2z \cos b + 1}$$

\therefore By the above property,

$$\begin{aligned} Z\{e^{-ak} \sin bk\} &= \frac{(e^a z) \sin b}{(e^a z)^2 - 2(e^a z) \cos b + 1} \\ &= \frac{e^{-a} z \cdot \sin b}{z^2 - 2e^{-a} z \cos b + e^{-2a}}. \end{aligned}$$

We give below the list of Z-transforms obtained above.

Table of Z-transforms

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| 1. $Z[\delta(k)] = 1$, for all z | 2. $Z[U(k)] = \frac{z}{z-1}$, $ z > 1$ |
| 3. $Z\{1\} = \frac{z}{z-1}$, | $ z > 1$ |
| 4. $Z\{k\} = \frac{z}{(z-1)^2}$, | $ z > 1$ |
| 5. $Z\{a^k\} = \frac{z}{z-a}$, | $k \geq 0$, $ z > a $. |
| 6. $Z\{a^k\} = \frac{z}{a-z}$, | $k < 0$, $ z < a $. |
| 7. $Z\{ka^k\} = \frac{az}{(z-a)^2}$, | $k \geq 0$, $ z > a $. |
| 8. $Z\{n!C_k\} = \left(\frac{1}{z}\right)^n$, | $0 \leq k \leq n$, $ z > 0$ |

Miscellaneous Examples

Example 1 : Find Z-transform of $\{a \cos k\alpha + b \sin k\alpha\}$, $k \geq 0$.

$$\begin{aligned} \text{Sol. : } Z\{a \cos k\alpha + b \sin k\alpha\} &= aZ[\cos k\alpha] + bZ[\sin k\alpha] \\ &= a \cdot \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} + b \cdot \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \quad |z| > 1 \\ &= \frac{az^2 + z(b \sin \alpha - a \cos \alpha)}{z^2 - 2z \cos \alpha + 1}. \end{aligned}$$

Example 2 : Find $Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\}$, $k \geq 0$.

Example 3 : Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$, $k \geq 0$.

$$\begin{aligned} \text{Sol. : Since } \sin\left(\frac{k\pi}{4} + a\right) &= \sin\frac{k\pi}{4} \cos a + \cos\frac{k\pi}{4} \sin a \\ Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\} &= Z\left\{\sin\frac{k\pi}{4} \cos a + \cos\frac{k\pi}{4} \sin a\right\} \\ &= \cos a Z\left\{\sin\left(\frac{\pi}{4} k\right)\right\} + \sin a Z\left\{\cos\left(\frac{\pi}{4} k\right)\right\} \\ &= \cos a \frac{z \sin(\pi/4)}{z^2 - 2z \cos(\pi/4) + 1} + \sin a \frac{z[z - \cos(\pi/4)]}{z^2 - 2z \cos(\pi/4) + 1} \\ &= \frac{\cos a \cdot (z/\sqrt{2})}{z^2 - (2z/\sqrt{2}) + 1} + \frac{\sin a \cdot [z - (1/\sqrt{2})]}{z^2 - (2z/\sqrt{2}) + 1} \\ &= \frac{z}{\sqrt{2}} \cdot \frac{[\cos a + \sin a \cdot (\sqrt{2}z - 1)]}{z^2 - \sqrt{2} \cdot z + 1}. \end{aligned}$$

$$\begin{aligned} 9. \quad Z\{^k C_n\} &= z^{-n} \left(1 - \frac{1}{z}\right)^{-(n+1)}, \quad |z| > 1 \\ 10. \quad Z\left[\frac{a^k}{k!}\right] &= e^{az/z}, \quad k \geq 0 \text{ for all } z. \end{aligned}$$

$$\begin{aligned} 11. \quad Z\{a^{|k|}\} &= \frac{az}{1-az} + \frac{z}{z-a}, \quad |a| < |z| < \frac{1}{|a|} \\ 12. \quad Z\{c^k \cos ak\} &= \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|. \\ 13. \quad Z\{c^k \sin ak\} &= \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|. \\ 14. \quad Z\{c^k \cos h \alpha k\} &= \frac{z(z - c \cosh \alpha)}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|) \\ 15. \quad Z\{c^k \sinh \alpha k\} &= \frac{cz \sinh \alpha}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|). \end{aligned}$$

$$\begin{aligned} \text{Ex. 3: } & \text{We have } \cos\left(\frac{k\pi}{4} + a\right) = \cos\frac{k\pi}{4} \cos a - \sin\frac{k\pi}{4} \sin a \end{aligned}$$

$$\begin{aligned} 9. \quad Z\{^k C_n\} &= z^{-n} \left(1 - \frac{1}{z}\right)^{-(n+1)}, \quad |z| > 1 \\ 10. \quad Z\left[\frac{a^k}{k!}\right] &= e^{az/z}, \quad k \geq 0 \text{ for all } z. \end{aligned}$$

$$\begin{aligned} 11. \quad Z\{a^{|k|}\} &= \frac{az}{1-az} + \frac{z}{z-a}, \quad |a| < |z| < \frac{1}{|a|} \\ 12. \quad Z\{c^k \cos ak\} &= \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|. \\ 13. \quad Z\{c^k \sin ak\} &= \frac{cz \sin \alpha}{z^2 - 2cz \cos \alpha + c^2}, \quad |z| > |c|. \\ 14. \quad Z\{c^k \cos h \alpha k\} &= \frac{z(z - c \cosh \alpha)}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|) \\ 15. \quad Z\{c^k \sinh \alpha k\} &= \frac{cz \sinh \alpha}{z^2 - 2cz \cosh \alpha + c^2}, \quad k \geq 0 \quad |z| > \max(|ce^\alpha|, |ce^{-\alpha}|). \end{aligned}$$

$$\begin{aligned} \text{Ex. 3: } & \text{We have } \cos\left(\frac{k\pi}{4} + a\right) = \cos\frac{k\pi}{4} \cos a - \sin\frac{k\pi}{4} \sin a \end{aligned}$$

$$\begin{aligned} \therefore Z\left\{\cos\left(\frac{k\pi}{4} + a\right)\right\} &= Z\left[\cos\frac{k\pi}{4}\cos a - \sin\frac{k\pi}{4}\sin a\right] \\ &= \cos a \cdot Z\left\{\cos\frac{k\pi}{4}\right\} - \sin a \cdot Z\left\{\sin\frac{k\pi}{4}\right\} \\ &= \cos a \cdot \frac{z[z - \cos(\pi/4)]}{z^2 - 2z\cos(\pi/4) + 1} + \frac{\sin a \cdot z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1} \\ &= \frac{\cos a \cdot z[z - (1/\sqrt{2})]}{z^2 - \sqrt{2} \cdot z + 1} - \frac{\sin a \cdot z \cdot 1/\sqrt{2}}{z^2 - \sqrt{2} \cdot z + 1} \\ &= \frac{z}{\sqrt{2}} \cdot \left[\frac{\cos a(\sqrt{2}z - 1) - \sin a}{z^2 - \sqrt{2} \cdot z + 1} \right]. \end{aligned}$$

Example 4 : Find $Z\{2^k \cos(3k+2)\}, k \geq 0$.

Sol. : We have $\cos(3k+2) = \cos 3k \cos 2 - \sin 3k \sin 2$

$$\begin{aligned} \therefore Z\{\cos(3k+2)\} &= \cos 2 \cdot Z\{\cos 3k\} - \sin 2 \cdot Z\{\sin 3k\} \\ &= \cos 2 \cdot \frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} - \frac{\sin 2 \cdot z\sin 3}{z^2 - 2z\cos 3 + 1} \\ &= \frac{z[z\cos 2 - (\cos 3\cos 2 + \sin 3\sin 2)]}{z^2 - 2z\cos 3 + 1} \\ &= \frac{z \cdot [z\cos 2 - \cos 1]}{z^2 - 2z\cos 3 + 1} \end{aligned}$$

By change of scale property,

$$\text{If } Z\{f(k)\} = F(z), \text{ then } Z\{a^k f(k)\} = F\left(\frac{z}{a}\right).$$

$$\begin{aligned} \therefore Z\{2^k \cos(3k+2)\} &= \frac{z}{2} \left[\frac{z}{2} \cdot \cos 2 - \cos 1 \right] = \frac{z[z\cos 2 - 2\cos 1]}{z^2 - 4z\cos 3 + 4}. \\ &= \left(\frac{z}{2} \right)^2 - 2\left(\frac{z}{2} \right) \cos 3 + 1 \end{aligned}$$

(M.U. 2009)

Example 5 : Find $Z\{2^k \sin(3k+2)\}, k \geq 0$.

Sol. : We have $\sin(3k+2) = \sin 3k \cos 2 + \cos 3k \sin 2$

$$\begin{aligned} \therefore Z\{\sin(3k+2)\} &= \cos 2 \cdot Z\{\sin 3k\} + \sin 2 \cdot Z\{\cos 3k\} \\ &= \cos 2 \cdot \frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \frac{\sin 2 \cdot z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} \\ &= \frac{z[\sin 3\cos 2 - \cos 3\sin 2]}{z^2 - 2z\cos 3 + 1} \\ &= \frac{z[\sin(3-2) + z\sin 2]}{z^2 - 2z\cos 3 + 1} = \frac{z[\sin 1 + z\sin 2]}{z^2 - 2z\cos 3 + 1}. \end{aligned}$$

Now, by change of scale property as above,

$$\begin{aligned} \therefore Z\{2^k \sin(3k+2)\} &= \frac{z}{2} \left[\sin 1 + \frac{z}{2} \sin 2 \right] = \frac{z[2\sin 1 + z\sin 2]}{z^2 - 4z\cos 3 + 4}. \end{aligned}$$

Example 6 : Find $Z\{3^k \sin h \alpha k\}, k \geq 0$.

Sol. : We have $Z\{\sin h \alpha k\} = \frac{z(z - \cosh \alpha)}{z^2 - 2z\cosh \alpha + 1}$

By change of scale property,

$$\begin{aligned} \therefore Z\{3^k \sin h \alpha k\} &= \frac{\frac{z}{3}(z - \cosh \alpha)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cosh \alpha + 1} = \frac{z(z - 3\cosh \alpha)}{z^2 - 6z\cosh \alpha + 9}. \end{aligned}$$

Example 7 : Find $Z\{3^k \cos h \alpha k\}, k \geq 0$.

Sol. : We have $Z\{\cosh h \alpha k\} = \frac{z(z - \cosh \alpha)}{z^2 - 2z\cosh \alpha + 1}$

By change of scale property,

$$\begin{aligned} \therefore Z\{3^k \cosh h \alpha k\} &= \frac{\frac{z}{3}(z - \cosh \alpha)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cosh \alpha + 1} = \frac{z(z - 3\cosh \alpha)}{z^2 - 6z\cosh \alpha + 9}. \end{aligned}$$

Example 8 : Find $Z\{(k+1)a^k\}, k \geq 0$.

Sol. : We have $Z\{(k+1)a^k\} = Z\{ka^k\} + Z\{a^k\}$

But we know that $Z\{a^k\} = \frac{z}{z-a}$.

Now, by (4) (property of multiplication by k), page 10-22

$$\begin{aligned} \therefore Z\{ka^k\} &= -z \frac{d}{dz} \left\{ \frac{z}{z-a} \right\} = -z \left[\frac{(z-a)1-z \cdot 1}{(z-a)^2} \right] = -z \cdot \frac{(-a)}{(z-a)^2} = \frac{az}{(z-a)^2} \\ \therefore Z\{(k+1)a^k\} &= \frac{az}{(z-a)^2} + \frac{z}{(z-a)} = \frac{az+z^2-az}{(z-a)^2} = \frac{z^2}{(z-a)^2}. \end{aligned}$$

Example 9 : Find Z-transform of $\{k^2 a^{k-1}\}, k \geq 0$.

Sol. : We know that, $Z\{f(k-\eta)\} = z^{-\eta} \cdot F(z)$

$$\therefore Z\{a^{k-1}\} = z^{-1} F(z) \text{ where } F(z) = Z\{a^k\} = \frac{z}{z-a}$$

By the property of multiplication by k ,

$$\therefore Z\{k \cdot a^{k-1}\} = -z \frac{d}{dz} \left(\frac{1}{z-a} \right) = -z \cdot \frac{(-1)}{(z-a)^2} = \frac{z}{(z-a)^2}.$$

By the property of multiplication by k ,

$$\begin{aligned} \therefore Z\{k^2 \cdot a^{k-1}\} &= Z\{k \cdot (ka^{k-1})\} = -z \frac{d}{dz} \left[\frac{z}{(z-a)^2} \right] = -z \left[\frac{(z-a)^2 \cdot 1 - z \cdot 2(z-a) \cdot 1}{(z-a)^4} \right] \\ &= -z \left[\frac{(z-a)-2z}{(z-a)^3} \right] = \frac{z(z+a)}{(z-a)^3}, |z| > |a|. \end{aligned}$$

Example 11 : Find $Z\{k^2 a^{k-1} U(k-1)\}$.

Sol.: We know that, $Z\{U(k)\} = \frac{z}{z-1}$.

By change of scale property,

$$Z\{a^k U(k)\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}$$

$$\therefore Z\{f(k-1)\} = z^{-1} \cdot Z\{a^k U(k)\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}.$$

$$\therefore Z\{k^2 \cdot a^2 U(k-1)\} = \left(-z \frac{d}{dz}\right)^2 \left(\frac{1}{z-a}\right) = \frac{z(z+a)}{(z-a)^3} \text{ as above.}$$

EXERCISE - V

Find the Z-transforms of the following sequences.

1. $\{3^k + 5^k\}, k < 0$
2. $\{\alpha^k + \beta^k\}, k < 0$
3. $\{3^k + 5^k\}, k \geq 0$
4. $\{\alpha^k + \beta^k\}, k \geq 0$
5. $\{k2^k + k3^k\}, k \geq 0$
6. $\{k\alpha^k + k\beta^k\}, k \geq 0$
7. $\left\{\frac{2^k}{k} + \frac{3^k}{k}\right\}, k \geq 1$
8. $\left\{\frac{\alpha^k}{k} + \frac{\beta^k}{k}\right\}, k \geq 1$
9. $\left\{3^k + \frac{1}{3^k}\right\}, k \geq 0$
10. $\left\{\alpha^k + \frac{1}{\alpha^k}\right\}, k \geq 0$
11. Find $Z\{f(k) * g(k)\}$ where $f(k) = 4^k$ and $g(k) = 7^k$.

12. Find $Z\{f(k) * g(k)\}$ where $f(k) = 5^k$ and $g(k) = 7^k$.
13. Find $Z\{f(k) * g(k)\}$ where $f(k) = \frac{1}{4^k}$ and $g(k) = \frac{1}{6^k}$.

14. Find $Z\{f(k) * g(k)\}$ where $f(k) = \frac{1}{3^k}$ and $g(k) = \frac{1}{7^k}$.

15. $\{\sin 5k\}, k \geq 0$
16. $\{\cos 5k\}, k \geq 0$
17. $\left\{\sin\left(\frac{k\pi}{2} + a\right)\right\}, k \geq 0$

18. $\left\{\cos\left(\frac{k\pi}{2} + a\right)\right\}, k \geq 0$
19. $\left\{\sin\left(\frac{k\pi}{3} + a\right)\right\}, k \geq 0$
20. $\left\{\cos\left(\frac{k\pi}{3} + a\right)\right\}, k \geq 0$
21. $\left\{\sin\left(\frac{k\pi}{6} + a\right)\right\}, k \geq 0$
22. $\left\{\cos\left(\frac{k\pi}{6} + a\right)\right\}, k \geq 0$
23. $\left\{3^k \cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\}, k \geq 0$
24. $\left\{3^k \sin\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\}, k \geq 0$
25. $\{e^{-2k} \cos 3k\}, k \geq 0$
26. $\{e^{-3k} \sin 2k\}, k \geq 0$

3. Inverse Z-transforms

We shall now consider the reverse problem i.e. given the Z-transform $Z\{f(k)\} = F(z)$ of a sequence to find the original sequence denoted by $f(k)$ or $Z^{-1}[F(z)]$. We shall consider Z-transforms which are rational functions of z i.e. of the form $F(z) = \frac{P(z)}{Q(z)}$ and $P(z)$ and $Q(z)$ are algebraic polynomials in z . It should be noted that to find the inverse Z-transform we should know its region of convergence i.e. ROC. We shall consider the following three methods.

- (a) Direct Division,
- (b) Binomial Expansion,
- (c) Partial Fraction.

In this method, we divide the numerator by the denominator and obtain a power series i.e. if

[Ans. : (1) $\frac{8z-2z^2}{(3-z)(5-z)}, |z| < 3$

(2) $\frac{(\alpha+\beta)z-2z^2}{(\alpha-z)(\beta-z)}, |z| < \min. \text{ of } |\alpha|, |\beta|$

(3) $\frac{2z^2-8z}{(z-3)(z-5)}, |z| > 5$

(4) $\frac{2z^2-(\alpha+\beta)z}{(z-\alpha)(z-\beta)}, |z| > \max. \text{ of } |\alpha|, |\beta|$

(5) $\frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}, |z| > 3$

(6) $\frac{\alpha z}{(z-\alpha)^2} + \frac{\beta z}{(z-\beta)^2}, |z| > \max. \text{ of } |\alpha|, |\beta|$

- (7) $-\log\left[\frac{(z-2)(z-3)}{z^2}\right], |z| > 3$
- (8) $-\log\left[\frac{(z-\alpha)(z-\beta)}{z^2}\right], |z| > \max. \text{ of } |\alpha|, |\beta|$
- (9) $\frac{z}{z-3} + \frac{3z}{3z-1}, |z| > 3$
- (10) $\frac{z}{z-\alpha} + \frac{\alpha z}{\alpha z-1}, |z| > |\alpha|$
- (11) $\left(\frac{z}{z-4}\right)\left(\frac{z}{z-7}\right), |z| > 7$
- (12) $\left(\frac{z}{z-5}\right)\left(\frac{z}{z-7}\right), |z| > 7$
- (13) $\left(\frac{4z}{4z-1}\right)\left(\frac{6z}{6z-1}\right), |z| > \frac{1}{4}$
- (14) $\left(\frac{3z}{3z-1}\right)\left(\frac{7z}{7z-1}\right), |z| > \frac{1}{3}$
- (15) $\frac{z \sin 5}{z^2 - 2z \cos 5 + 1}$
- (16) $\frac{z(z-\cos 5)}{z^2 - 2z \cos 5 + 1}$
- (17) $\frac{z^2 \sin a + z c \cos a}{z^2 + 1}, |z| > 1$
- (18) $\frac{z^2 \cos a - z s \sin a}{z^2 + 1}, |z| > 1$
- (19) $\frac{z}{2} \left[\frac{\cos a \cdot \sqrt{3} + \sin a (2z-1)}{z^2 - z + 1} \right]$
- (20) $\frac{z}{2} \left[\frac{\cos a (2z-1) - \sin a \cdot \sqrt{3}}{z^2 - z + 1} \right]$
- (21) $\frac{z}{2} \left[\frac{\cos a \cdot 1 + \sin a (2z - \sqrt{3})}{z^2 - \sqrt{3} \cdot z + 1} \right]$
- (22) $\frac{z}{2} \left[\frac{\cos a (2z - \sqrt{3}) - \sin a \cdot 1}{z^2 - \sqrt{3} \cdot z + 1} \right]$
- (23) $\frac{1}{\sqrt{2}} \cdot \frac{z^2 - 3z}{z^2 + 9}, |z| > 3$
- (24) $\frac{1}{\sqrt{2}} \cdot \frac{z^2 + 3z}{z^2 + 9}, |z| > 3$
- (25) $\frac{z(z - e^{-2} \cos 3)}{z^2 - 2e^{-2} z \cos 3 + e^{-4}}$
- (26) $\frac{e^{-3} z \sin 2}{z^2 - 2e^{-3} z \cos 2 + e^{-6}}$

Example 2 : Find the inverse Z-transform of $\frac{z}{z-a}$, $|z| > a$.

$$\text{Sol. : We have } F(z) = \frac{z}{z-a} = \frac{z}{z[1-(a/z)]} = \left(1-\frac{a}{z}\right)^{-1}$$

$$\begin{aligned} &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots + \left(\frac{a}{z}\right)^k + \dots \\ &\therefore \text{ Coefficient of } z^{-k} = a^k. \\ &\therefore Z^{-1}[F(z)] = \{a^k\}; k \geq 0, |z| > a \\ &\therefore Z^{-1}[F(z)] = \left\{ \frac{-k+1}{a^{k+2}} \right\}, k \leq 0 \end{aligned}$$

Example 3 : Find the inverse Z-transform of $\frac{z}{z+a}$, $|z| > a$.

Sol. : Changing the sign of a or proceeding as above we get

$$Z^{-1}F(z) = \{(-a)^k\}$$

Note ...

The following inverse Z-transforms may be remembered.

- (i) $Z^{-1}\left[\frac{1}{z-a}\right] = \{a^{k-1}\}, k \geq 1, |z| > a$
- (ii) $Z^{-1}\left[\frac{1}{z+a}\right] = \{(-a)^{k-1}\}, k \geq 1, |z| > a$
- (iii) $Z^{-1}\left[\frac{z}{z-a}\right] = \{a^k\}, k \geq 0, |z| > a$
- (iv) $Z^{-1}\left[\frac{z}{z+a}\right] = \{(-a)^k\}, k \geq 0, |z| > a$

Example 4 : Find the inverse Z-transform of $\frac{1}{(z-a)^2}$ (i) $|z| < a$, (ii) $|z| > a$.

Sol. : (i) If $|z| < a$,

$$\begin{aligned} F(z) &= \frac{1}{a^2[1-(z/a)]^2} = \frac{1}{a^2}\left[1-\frac{z}{a}\right]^{-2} \\ &= \frac{1}{a^2}\left[1+2\cdot\frac{z}{a}+3\cdot\frac{z^2}{a^2}+\dots+(n+1)\frac{z^n}{a^n}+\dots\right] \\ &= \frac{1}{a^2} + 2\cdot\frac{z}{a^3} + 3\cdot\frac{z^2}{a^4} + \dots + (n+1)\frac{z^n}{a^{n+2}} + \dots \end{aligned}$$

$$\text{Coefficient of } z^n = \frac{n+1}{a^{n+2}}, n \geq 0$$

Coefficient of $z^{-k} = \frac{-k+1}{a^{-k+2}}, -k \geq 0$ i.e. $k \leq 0$

$$\therefore Z^{-1}[F(z)] = \left\{ \frac{-k+1}{a^{-k+2}} \right\}, k \leq 0$$

$$\text{If } |z| > a$$

$$\begin{aligned} F(z) &= \frac{1}{z^2[1-(a/z)]^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2} \\ &= \frac{1}{z^2}\left[1+2\cdot\frac{a}{z}+3\cdot\frac{a^2}{z^2}+\dots+(n-1)\frac{a^{n-2}}{z^{n-2}}+\dots\right] \\ &= \frac{1}{z^2} + 2\cdot\frac{a}{z^3} + 3\cdot\frac{a^2}{z^4} + \dots + (n-1)\frac{a^{n-2}}{z^n} + \dots \end{aligned}$$

$$\begin{aligned} &\text{Coefficient of } z^{-k} = (k-1)a^{k-2} \\ &\therefore Z^{-1}[F(z)] = \{(k-1)a^{k-2}\}, k \geq 2. \end{aligned}$$

Example 5 : Find the inverse Z-transform of $\frac{1}{(z-5)^3}$, $|z| > 5$. (M.U. 2009)

Sol. : Since, $|z| > 5$, $\left|\frac{z}{5}\right| > 1$ i.e. $\frac{5}{|z|} < 1$ hence, we take out z^3 and write.

$$\begin{aligned} F(z) &= \frac{1}{(z-5)^3} = \frac{1}{z^3} \cdot \frac{1}{[1-(5/z)]^3} = \frac{1}{z^3}\left(1-\frac{5}{z}\right)^{-3} \\ &= \frac{1}{z^3}\left[1-\left(-\frac{3}{z}\right)\frac{5}{z} + \frac{(-3)(-4)}{2!} \cdot \left(\frac{5}{z}\right)^2 - \frac{(-3)(-4)(-5)}{3!} \cdot \left(\frac{5}{z}\right)^3 + \dots\right] \\ &= \frac{1}{z^3}\left[1+3\cdot\frac{5z^{-1}}{z}+6\cdot\frac{5^2z^{-2}}{z^2}+10\cdot\frac{5^3z^{-3}}{z^3}+\dots+\frac{(n+1)(n+2)}{2}\cdot\frac{5^n}{z^n}\cdot\frac{z^{-n}}{z}+\dots\right] \\ &= z^{-3} + 3\cdot 5 \cdot z^{-4} + 6 \cdot 5^2 \cdot z^{-5} + 10 \cdot 5^3 \cdot z^{-6} + \dots + \frac{(n+1)(n+2)}{2} \cdot 5^n \cdot z^{-n-3} \end{aligned}$$

$$\text{Coefficient of } z^{-n-3} = \frac{(n+1)(n+2)}{2} \cdot 5^n \quad [\text{Put } n+3=k]$$

$$\therefore \text{Coefficient of } z^{-k} = \frac{(k-3+1)(k-3+2)}{2} \cdot 5^{k-3}$$

$$= \frac{(k-2)(k-1)}{2} \cdot 5^{k-3}$$

$$Z^{-1}[F(z)] = \left\{ \frac{(k-2)(k-1)}{2} \cdot 5^{k-3} \right\}, k \geq 3.$$

Example 6 : Find the Z-transform of $\frac{1}{(z-5)^3}$ if $|z| < 5$.

Since $|z| < 5$ we take out 5 and write

$$\begin{aligned} F(z) &= \frac{1}{5^3[(z/5)-1]^3} = -\frac{1}{5^3}\left(1-\frac{z}{5}\right)^{-3} \\ &= -\frac{1}{5^3}\left[1+3\cdot\frac{z}{5}+6\cdot\frac{z^2}{5^2}+\dots+\frac{(n+1)(n+2)}{2}\cdot\frac{z^n}{5^n}+\dots\right] \end{aligned}$$

$$\begin{aligned} \text{Coefficient of } z^n &= -\frac{(n+1)(n+2)}{2} \cdot \frac{1}{5^{n+3}}, n \geq 0 \quad [\text{Put } n = -k] \\ \text{Coefficient of } z^{-k} &= -\frac{(-k+1)(-k+2)}{2} \cdot \frac{1}{5^{-k+3}}, k \leq 0 \\ \therefore Z^{-1}[F(z)] &= \left\{ \begin{array}{l} \frac{(-k+1)(-k+2)}{2} \cdot \frac{1}{5^{-k+3}}, k \leq 0 \\ \end{array} \right. \end{aligned}$$

(c) **Method of Partial Fractions**

If $F(z)$ can be factorised into partial fractions, linear, quadratic or repeated we express

$F(z) = \frac{P(z)}{Q(z)}$ as the sum such factors, find the constants and then use the method of Binomial Expansion. This is illustrated in the following problems. If the degree of $P(z)$ is greater than or equal to that of $Q(z)$ we write $\frac{F(z)}{z} = \frac{P(z)}{Q(z)}$ as in Ex. 3, page 10-43. We now discuss the three cases separately.

(i) **Linear non-repeated factors**

Let the linear non-repeated factor be $\frac{z}{z-a}$. Then

$$\begin{aligned} Z^{-1}\left(\frac{z}{z-a}\right) &= Z^{-1}\left[\frac{1}{1-(a/z)}\right] = Z^{-1}\left[1-\left(\frac{a}{z}\right)\right]^{-1}, \quad \text{if } |z| > |a| \\ &= Z^{-1}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\dots\right] \\ &= Z^{-1}\left[1+\frac{a}{z}+\left(\frac{a^2}{z^2}\right)+\left(\frac{a^3}{z^3}\right)+\dots\right] \\ &= Z^{-1}\left[1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\dots+a^kz^{-k}+\dots\right] \\ &= \{a^k\} \end{aligned}$$

Also, $Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{z/a}{(za)-1}\right] = -Z^{-1}\left[\frac{z/a}{1-(za)}\right], \quad \text{if } |a| > |z|$

$$\begin{aligned} &= -Z^{-1}\left[\left(\frac{z}{a}\right)\left(1-\frac{z}{a}\right)^{-1}\right] = -Z^{-1}\left[\frac{z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\dots\right)\right] \\ &= -Z^{-1}\left[\left(\frac{z}{a}\right)+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+\dots\right] = -Z^{-1}\left\{\frac{z^k}{a^k}\right\}, \quad k \geq 0 \\ &= -Z^{-1}\left[a^{-k}z^k\right], \quad k \geq 0 \\ &= -Z^{-1}\left\{a^kz^{-k}\right\}, \quad k \leq 0 \\ \therefore Z^{-1}\left(\frac{z}{z-a}\right) &= -\{a^k\}, \quad k \leq 0 \end{aligned}$$

$$\checkmark \quad \text{Example 1 : Find inverse Z-transform of } F(z) = \frac{1}{(z-1)(z-2)}, |z| > 2.$$

Sol.: We have $F(z) = \frac{1}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$

Since $|z| > 2$ clearly $|z| > 1$.

$\therefore |z/2| > 1$ and $|z| > 1$.

$\therefore |z/2| < 1$ and $|1/z| < 1$. \therefore We take z common.

$$\begin{aligned} \therefore F(z) &= \frac{2}{z[1-(2/z)]} - \frac{1}{z[1-(1/z)]} = \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\ &= \frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\dots+\frac{2^{k-1}}{z^{k-1}}+\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots+\frac{1}{z^{k-1}}+\dots\right) \\ &= \left(\frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots + \frac{2^k}{z^k} + \dots\right) - \left(\frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^k} + \dots\right) \end{aligned}$$

Coefficient of $z^{-k} = 2^k - 1$, $k \geq 1$

$$Z^{-1}[F(z)] = \{2^k - 1\}, \quad k \geq 1.$$

Example 2 : Find the inverse Z-transform of $F(z) = \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)}$, $3 < z < 4$.

Sol.: We have (by partial fractions) $F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$

Since $z > 3$, $z > 2$, hence we take z out from the first two terms. Since $4 > z$, ... take out 4 from the last bracket.

$$\begin{aligned} \therefore F(z) &= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{4}\left(1-\frac{4}{z}\right)^{-1} \\ \therefore F(z) &= \frac{1}{z}\left(1+\frac{2}{z}+\dots+\frac{2^{k-1}}{z^{k-1}}+\dots\right) + \frac{1}{z}\left(1+\frac{3}{z}+\dots+\frac{3^{k-1}}{z^{k-1}}+\dots\right) \end{aligned}$$

$$\begin{aligned} -\frac{1}{4}\left(1+\frac{2}{4}+\dots+\frac{2^k}{4^k}+\dots\right) \\ = -\frac{1}{4}\left(1+\frac{2}{4}+\dots+\frac{2^k}{4^k}+\dots\right) \end{aligned}$$

From the first series the coefficient of $z^{-k} = 2^{k-1}$, $k \geq 0$.

From the second series the coefficient of $z^{-k} = 3^{k-1}$, $k \geq 0$.

From the third series the coefficient of $z^k = -\frac{1}{4^{k+1}}$, $k \geq 0$.

\therefore The coefficient of $z^{-k} = -4^{k-1}$, $k \leq 0$

$$\begin{aligned} \therefore Z^{-1}[F(z)] &= \{2^{k-1} + 3^{k-1}\}, \quad k \geq 0 \\ &= \{-4^{k+1}\}, \quad k \leq 0 \end{aligned}$$

- Example 3 :** Find the inverse Z-transform of $F(z) = \frac{1}{(z-3)(z-2)}$ if ROC is (i) $|z| < 2$,
(ii) $2 < |z| < 3$, (iii) $|z| > 3$.

$$\text{Sol. : We have } F(z) = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

If $|z| < 2$, clearly $|z| < 3 \quad \therefore \left| \frac{z}{2} \right| < 1 \text{ and } \left| \frac{z}{3} \right| < 1$

Hence, we take out 3 and 2 from the fractions and write.

$$F(z) = \frac{1}{3[(z/3)-1]} - \frac{1}{2[(z/2)-1]} = -\frac{1}{3[1-(z/3)]} + \frac{1}{2[1-(z/2)]}$$

$$\begin{aligned} &= -\frac{1}{3} \left(1 - \frac{z}{3} \right)^{-1} + \frac{1}{2} \left(1 - \frac{z}{2} \right)^{-1} \\ &= -\frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \dots + \frac{z^k}{3^k} + \dots \right) + \frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \dots + \frac{z^k}{2^k} + \dots \right) \\ &= -\left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \dots + \frac{z^k}{3^{k+1}} + \dots \right) + \left(\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \dots + \frac{z^k}{2^{k+1}} + \dots \right) \\ &= -(3^{-1} + 3^{-2}z + 3^{-3}z^2 + \dots + 3^{-k-1}z^k + \dots) \\ &\quad + (2^{-1} + 2^{-2}z + 2^{-3}z^2 + \dots + 2^{-k-1}z^k + \dots) \end{aligned}$$

From the first series we find that the coefficient of $z^k = -3^{-k-1}$, $k \geq 0$.

∴ The coefficient of $z^{-k} = -3^{k-1}$, $k \leq 0$.

From the second series, we find that the coefficient of $z^k = 2^{-k-1}$, $k \geq 0$.

∴ The coefficient of $z^{-k} = 2^{k-1}$, $k \leq 0$

- (ii) If $2 < |z| < 3$ i.e. $2 < |z|$ ∴ $|2/z| < 1$ and $|z| < 3$ i.e. $|z/3| < 1$.

Hence, we take out 3 from the first fraction and z from the second fraction.

$$\therefore F(z) = \frac{1}{3[(z/3)-1]} - \frac{1}{z[1-(2/z)]} = -\frac{1}{3} \cdot \frac{1}{[1-(z/3)]} - \frac{1}{z} \cdot \frac{1}{[1-(2/z)]}$$

$$\begin{aligned} &= -\frac{1}{3} \left(1 - \frac{z}{3} \right)^{-1} - \frac{1}{z} \left(1 - \frac{2}{z} \right)^{-1} \\ &= -\frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \dots + \frac{z^k}{3^k} + \dots \right) - \frac{1}{z} \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \dots + \frac{2^{k-1}}{z^{k-1}} + \dots \right) \end{aligned}$$

$$\begin{aligned} &= -\left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \dots + \frac{z^k}{3^{k+1}} + \dots \right) - \left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \dots + \frac{2^{k-1}}{z^k} + \dots \right] \\ &= -[3^{-1} + 3^{-2}z + 3^{-3}z^2 + \dots + 3^{-k-1}z^k + \dots] - \left[\frac{1}{z} + \frac{2}{z^2} + \dots + \frac{2^{k-1}}{z^k} + \dots \right] \end{aligned}$$

- From the first series we find that the coefficient of $z^k = -3^{-k-1}$, $k \geq 0$.
(iii) If $|z| > 3$, clearly $|z| > 2$ i.e. $|z/3| > 1$ and $|z/2| > 1$ i.e. $|3/z| < 1$ and $|2/z| < 1$. Hence, we take out z from both fractions.

$$\begin{aligned} \therefore F(z) &= \frac{1}{z[1-(3/z)]} - \frac{1}{z[1-(2/z)]} = \frac{1}{z} \left(1 - \frac{3}{z} \right)^{-1} - \frac{1}{z} \left(1 - \frac{2}{z} \right)^{-1} \\ &= \frac{1}{z} \left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots + \frac{3^{k-1}}{z^{k-1}} + \dots \right) - \frac{1}{z} \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \dots + \frac{2^{k-1}}{z^{k-1}} + \dots \right) \\ &= \left(\frac{1}{z} + \frac{3}{z^2} + \frac{3^{k-1}}{z^k} + \dots \right) - \left(\frac{1}{z} + \frac{2}{z^2} + \dots + \frac{2^{k-1}}{z^k} + \dots \right) \end{aligned}$$

$$\begin{aligned} &\therefore \text{Coefficient of } z^{-k} = 3^{k-1} - 2^{k-1}, k \geq 1. \\ \therefore \quad \quad \quad &Z^{-1}[F(z)] = \{3^{k-1} - 2^{k-1}\}, k \geq 1 \end{aligned}$$

(iii) Linear repeated factors

When the linear factors are repeated, we use the above technique and expand $\frac{1}{(z-a)^2}$.

$\frac{1}{(z-a)^3}$ by Binomial Theorem as illustrated in the following examples.

Example 1 : Find the inverse Z-transform of $F(z) = \frac{z+2}{z^2-2z+1}$, $|z| > 1$. (M.U. 2008, 14)

$$\text{Sol. : We have } F(z) = \frac{z+2}{z^2-2z+1} = \frac{z+2}{(z-1)^2} = \frac{3}{(z-1)^2} + \frac{1}{z-1}$$

Since, $|z| > 1$, $\frac{1}{|z|} < 1$. ∴ We take out z.

$$\begin{aligned} \therefore F(z) &= \frac{3}{z^2[1-(1/z)]^2} + \frac{1}{z[1-(1/z)]} = \frac{3}{z^2} \left(1 - \frac{1}{z} \right)^{-2} + \frac{1}{z} \left(1 - \frac{1}{z} \right)^{-1} \\ &= \frac{3}{z^2} \left(1 - (-2) \cdot \frac{1}{z} + \frac{(-2)(-3)}{2!} \cdot \frac{1}{z^2} - \frac{(-2)(-3)(-4)}{3!} \cdot \frac{1}{z^3} + \dots \right) \\ &\quad + \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\ &= \frac{3}{z^2} \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots + \frac{k-1}{z^{k-2}} + \dots \right) + \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^k} + \dots \right) \\ &= 3 \left(\frac{1}{z^2} + \frac{2}{z^3} + \dots + \frac{k-1}{z^{k-1}} + \dots \right) + \left(\frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^k} \right) \end{aligned}$$

$$= \frac{1}{z} + \frac{3+1}{z^2} + \dots + \frac{3k-3+1}{z^k} + \dots$$

$$\therefore F(z) = \frac{1}{z} + \frac{4}{z^2} + \frac{7}{z^3} + \dots + \frac{3k-2}{z^k} + \dots$$

\therefore Coefficient of $z^{-k} = 3k-2$, $k \geq 1$.

$$\therefore Z^{-1}[F(z)] = \{3k-2\}, k \geq 1.$$

Example 2: Find the inverse Z-transform of $\frac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$, $2 < |z| < 3$.

$$\text{Sol. : We have } F(z) = \frac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

Since, $2 < |z|$, $|2/z| < 1$ and since $|z| < 3$, $|z/3| < 1$.

\therefore We take out z from the first bracket and 3 out from the last two bracket.

$$\begin{aligned} \therefore F(z) &= \frac{1}{z} \cdot \frac{1}{[1-(2/z)]} + \frac{1}{3} \cdot \frac{1}{[(z/3)-1]} + \frac{1}{9} \cdot \frac{1}{[(z/3)-1]^2} \\ &= \frac{1}{z} \cdot \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9} \left(1 - \frac{z}{3}\right)^{-2} \\ &= -5(4 + 4^2 z + \dots + 4^k z^{k-1} + \dots) - \frac{4}{z} \left(1 + \frac{1}{5z} + \frac{1}{(5z)^2} + \dots\right) \\ &= -5(4 + 4^2 z + \dots + 4^k z^{k-1} + \dots) - \frac{4}{z} \left(1 + \frac{1}{5z} + \dots + \frac{1}{5^k} \cdot \frac{1}{z^k} + \dots\right) \\ &= \frac{1}{z} \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots + \frac{2^{k-1}}{z^{k-1}} + \dots\right) - \frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \dots + \frac{z^k}{3^k} + \dots\right) \\ &\quad + \frac{1}{9} \left(1 + 2 \cdot \frac{z}{3} + 3 \cdot \frac{z^2}{3^2} + \dots + (k+1) \frac{z^k}{3^k} + \dots\right) \end{aligned}$$

\therefore From the first series we find that the coefficient of z^{-k} is 2^{k-1} , from the second series, we find that the coefficient of z^k is $-5 \cdot 4^k$, from the third series, we find that the coefficient of z^k is $\frac{1}{3^{k+1}}$.

\therefore From the first series, coefficient of $z^{-k} = 2^{k-1}$, $k \geq 1$.
From second and third series,

$$\begin{aligned} \text{Coefficient of } z^k &= \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}} = \frac{k+1}{3^{k+2}} - \frac{3}{3^{k+2}} \\ &= \frac{k-2}{3^{k+2}}, k \geq 0 \end{aligned}$$

\therefore Coefficient of $z^{-k} = \frac{-k-2}{3^{-k+2}}$, $k \leq 0$.

Hence, $Z^{-1}[F(z)] = \{2^{k-1}\}$, $k \geq 1$

$$= \begin{cases} \frac{-k-2}{3^{-k+2}}, & k \leq 0. \end{cases}$$

$$\begin{aligned} \therefore F(z) &= 4(5z + 5^2 z^2 + \dots + 5^k z^k + \dots) - 5(4z + 4^2 z^2 + \dots + 4^k z^k + \dots) \\ &= 4(5z + 5^2 z^2 + \dots + 5^k z^k + \dots) - 5(4z + 4^2 z^2 + \dots + 4^k z^k + \dots) \end{aligned}$$

$$\begin{aligned} \text{Example 3 : Find the inverse Z-transform of } F(z) &= \frac{z^2}{[z - (1/4)][z - (1/5)]} \\ \text{(i) } \frac{1}{5} &< |z| < \frac{1}{4}, \quad \text{(ii) } |z| < \frac{1}{5}. \\ \text{Sol. : (i) Since, the degree of the numerator is equal to the degree of the denominator, we write,} \\ \frac{F(z)}{z} &= \frac{z}{[z - (1/4)][z - (1/5)]} = \frac{5}{z - (1/4)} - \frac{4}{z - (1/5)} \\ \text{Now, } \frac{1}{5} &< |z| \quad \therefore \frac{1}{|5z|} < 1; \quad |z| < \frac{1}{4} \quad \therefore |4z| < 1. \\ \therefore \frac{F(z)}{z} &= \frac{5z}{4z-1} - \frac{4z}{z-1} = -\frac{20}{1-4z} - \frac{4}{z-1-(1/5z)} \\ &= -20(1-4z)^{-1} - \frac{4}{z} \left(1 - \frac{1}{5z}\right)^{-1} \\ &= -20(1+4z+4^2 z^2 + \dots) - \frac{4}{z} \left(1 + \frac{1}{5z} + \frac{1}{(5z)^2} + \dots\right) \\ &= -5(4 + 4^2 z + \dots + 4^k z^{k-1} + \dots) - \frac{4}{z} \left(1 + \frac{1}{5z} + \dots + \frac{1}{5^k} \cdot \frac{1}{z^k} + \dots\right) \end{aligned}$$

In the first series the coefficient of $z^k = -5 \cdot 4^k$, $k \geq 1$.
 \therefore The coefficient of $z^{-k} = -5 \cdot 4^k$, $k \leq -1$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0.$$

In the second series the coefficient of $z^{-k} = -4 \cdot \left(\frac{1}{5}\right)^k$, $k \geq 0$.

$$\therefore Z^{-1}[F(z)] = \begin{cases} -5 \cdot \left(\frac{1}{4}\right)^k & k < 0 \\ -4 \cdot \left(\frac{1}{5}\right)^k & k \geq 0 \end{cases}$$

(iii) Since, $|z| < 1/5$, clearly $|z| < 1/4$

$\therefore |5z| < 1$ and $|4z| < 1$.

$$\begin{aligned} \therefore \frac{F(z)}{z} &= \frac{5}{z - (1/4)} - \frac{4}{z - (1/5)} = \frac{20}{4z-1} - \frac{20}{5z-1} \\ &= -\frac{20}{1-4z} + \frac{20}{1-5z} = 20 \left[\frac{1}{1-5z} - \frac{1}{1-4z} \right] \\ &= 20[(1-5z)^{-1} - (1-4z)^{-1}] \\ &= 20[(1+5z+\dots+5^{k-1}z^{k-1}+\dots) - (1+4z+\dots+4^{k-1}z^{k-1}+\dots)] \\ &= 4 \cdot (5+5^2 \cdot z + \dots + 5^k z^{k-1} + \dots) - 5(4+4^2 \cdot z + \dots + 4^k \cdot z^{k-1} + \dots) \end{aligned}$$

$$\therefore F(z) = 4(5z + 5^2 z^2 + \dots + 5^k z^k + \dots) - 5(4z + 4^2 z^2 + \dots + 4^k z^k + \dots)$$

In the first series the coefficient of $z^k = 4 \cdot 5^k$, $k \geq 1$.
 In the second series the coefficient of $z^{-k} = 4 \cdot 5^{-k}$, $k \leq -1$

$$\therefore \text{The coefficient of } z^{-k} = 4 \left(\frac{1}{5}\right)^k, \quad k < 0$$

$$= 4 \left(\frac{1}{5}\right)^k, \quad k < 0$$

$$\text{In the second series the coefficient of } z^k = -5 \cdot 4^k, \quad k \geq 1$$

$$\therefore \text{The coefficient of } z^{-k} = -5 \cdot 4^{-k}, \quad k \leq 1$$

$$= -5 \cdot \left(\frac{1}{4}\right)^k, \quad k < 0$$

$$\therefore Z^{-1}[F(z)] = \begin{cases} 4 \left(\frac{1}{5}\right)^k & k < 0 \\ -5 \left(\frac{1}{4}\right)^k & k < 0 \end{cases}$$

EXERCISE - VI

Find the inverse Z-transforms of the following:

$$1. \frac{1}{z-1}, |z| < 1, |z| > 1 \quad 2. \frac{1}{z-3}, |z| < 3, |z| > 3$$

$$3. \frac{z}{z-1}, |z| < 1, |z| > 1 \quad 4. \frac{z}{z-a}, |z| < a, |z| > a, a > 0$$

$$5. \frac{1}{(z-1)^2}, |z| < 1, |z| > 1 \quad 6. \frac{1}{(z-5)^2}, |z| < 5, |z| > 5$$

$$7. \frac{1}{(z-3)^3}, |z| < 3, |z| > 3 \quad 8. \frac{1}{(z-1)^3}, |z| < 1, |z| > 1$$

$$9. \frac{z}{z^2-3z+2}, |z| > 2 \quad 10. \frac{z}{[z-(1/4)][z-(1/5)]}, \frac{1}{5} < |z| < \frac{1}{4}$$

$$11. \frac{z}{(z-2)(z-3)}, |z| < 2, 2 < |z| < 3, |z| > 3$$

$$12. \frac{1}{[z-(1/2)][z-(1/3)]} \quad (i) \frac{1}{3} < |z| < \frac{1}{2}, \quad (ii) \frac{1}{2} < |z| \quad (\text{M.U. 2015})$$

$$13. \frac{3z^2+2z}{z^2-3z+2}, 1 < |z| < 2 \quad 14. \frac{z^3}{(z-1)(z-2)^2}, |z| > 2$$

$$15. \frac{z^3}{(z-3)(z-2)^2}, |z| > 3 \quad (\text{M.U. 2014})$$

$$16. \frac{z^2}{(z-1)[z-(1/2)]}; \quad |z| < \frac{1}{2}, \quad \frac{1}{2} < |z| < 1, |z| > 1$$

[Ans.: (1) (a) $-1, k \leq 0$; (b) $1, k \geq 1$;
 (2) (a) $-3^{k-1}, k \leq 0$; (b) $3^{k-1}, k \geq 1$;
 (3) (a) $-1, k < 0$; (b) $1, k \geq 0$;
 (4) (a) $-a^k, k < 0$; (b) $a^k, k \geq 0$;
 (5) (a) $-k+1, k \leq 0$; (b) $k-1, k \geq 2$,