Ex. 1 : Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} \quad \text{and hence, evaluate}$$

(i) $\int_{-\infty}^{\infty} \frac{\sin sk \cos sx}{s} ds$, (ii) $\int_{-\infty}^{\infty} \frac{\sin ks}{s} ds$, **(M.U. 2009)**

(iii) $\int_{-\infty}^{\infty} \frac{\sin s}{s} ds$ (iv) $\int_{0}^{\infty} \frac{\sin ks}{s} ds$

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} 1 \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-k}^{k} = \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{s} \left[\frac{e^{isk} - e^{-isk}}{2i} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sk \text{ for } s \neq 0$$

For $s = 0$, $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} dx = \frac{1}{\sqrt{2\pi}} [k + k] = \frac{2k}{\sqrt{2\pi}}$

Now, we use inverse Fourier Transform. We know that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

then, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds$

(i) $\therefore f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sk \cdot e^{-isx} \cdot ds$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\cos sx - i \sin sx)}{s} \sin sk \cdot ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin sx \sin sk}{s} ds$$

The second integral being odd is zero.

$$\therefore f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} \cdot ds$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} ds = \begin{cases} \pi, & |x| < k \\ 0, & |x| > k \end{cases}$$

(ii) In the above result, if we put $x = 0$, we put

$$\int_{-\infty}^{\infty} \frac{\sin ks}{s} ds = \pi \quad \therefore 2 \int_{0}^{\infty} \frac{\sin ks}{s} ds = \pi$$

$$\therefore \int_{0}^{\infty} \frac{\sin ks}{s} ds = \frac{\pi}{2}.$$

Note ....

From the result (ii) above, we get

$$\boxed{\int_0^{\infty} \frac{\sin kx}{x} dx = \frac{\pi}{2}}$$

This is an important integral and can be used as a standard result when required. You are advised to memorise it and also the following result.

(iii) In the above result put $k = 1$,

$$\therefore \quad \boxed{\int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}}$$

Ex. 2 : Find the Fourier transform of

$$f(x) = \begin{cases} (1-x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and hence evaluate $\int_0^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cdot \cos\frac{x}{2} dx.$ **(M.U. 2008, 09)**

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underset{u}{(1-x^2)} \underset{v}{e^{isx}} dx$$

Integrating by parts, the integral I is given by

$$I = (1-x^2) \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is}(-2x) dx$$

$$= (1-x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is}\left[x \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is} \cdot 1 \cdot dx\right]$$

$$= (1-x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is} \cdot \left[x \cdot \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\right]$$

$$\therefore \quad F(s) = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{(1-x^2)e^{isx}}{is}\right)_{-1}^{+1} + \frac{2}{is}\left(\frac{x\,e^{isx}}{is}\right)_{-1}^{+1} + \frac{2}{is}\left(\frac{e^{isx}}{s^2}\right)_{-1}^{+1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \frac{4}{s^2}\left(\frac{e^{is}+e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is}-e^{-is}}{2i}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{4}{s^2}\cos s + \frac{4}{s^3}\sin s\right]$$

$$= -2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{s\cos s - \sin s}{s^3}\right)$$

Now, we use inverse Fourier Transform. We know that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

then, $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx}\, ds$$

$$\therefore \quad f(x) = \frac{1}{\sqrt{2\pi}} \left(-2 \cdot \sqrt{\frac{2}{\pi}}\right) \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3}\right) e^{-isx}\, ds$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \cos sx \left(\frac{s \cos s - \sin s}{s^3}\right) dx$$

$$+ i \frac{2}{\pi} \int_{-\infty}^{\infty} \sin sx \left(\frac{s \cos - \sin s}{s^3}\right) ds$$

Now, the second integral being odd is zero.

$$\therefore \quad f(x) = \begin{cases} (1-x^2), & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \cos sx \left(\frac{s \cos s - \sin s}{s^3}\right) ds$$

Now, we put $x = \frac{1}{2}$,

$$\therefore \quad \frac{3}{4} = -\frac{2}{\pi} \int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right)\left(\frac{s \cos s - \sin s}{s^3}\right) ds$$

$$\therefore \quad \int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right) \cdot \left(\frac{s \cos s - \sin s}{s^3}\right) ds = -\frac{3\pi}{8}$$

$$\therefore \quad \int_{0}^{\infty} \cos\left(\frac{x}{2}\right) \cdot \left(\frac{x \cos x - \sin x}{x^3}\right) dx = -\frac{3\pi}{16}.$$

EXERCISE - III

Find the inverse Fourier Transform of $F(s) = e^{-|s|a}$.

$$\left[\textbf{Ans. :} \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + x^2}\right]$$

(**Hint :** $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-ix)s}\, ds + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+ix)s}\, ds$

For $|s| = -s$ if $s \leq 0$ and $|s| = s$ if $s \geq 0$.)

6. Fourier Sine Transform

The **Infinite Fourier Sine Transform** of $f(x)$, $0 < x < \infty$, denoted by $F_s(s)$ is defined by