Fourier series

1. Find a Fourier series to represent $f(x) = x^2$ in $(0\ ,\ 2\pi)$ and hence deduce that
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$
2. Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$. Deduce that
$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$$
3. Find the Fourier series for $f(x) = \sqrt{1 - \cos x}$ in $(0\ ,\ 2\pi)$ and hence deduce that
$$\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$$
4. Find the Fourier series for $f(x) = \frac{1}{2}(\pi - x)$ in $(0\ ,\ 2\pi)$, hence deduce that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$
5. Find the Fourier series for $f(x) = x$ in $(0\ ,\ 2\pi)$.
6. Find Fourier expansion of $f(x) = \begin{cases} 0 & , \quad -\pi < x < 0 \\ \sin x & , \quad 0 < x < \pi \end{cases}$ Hence deduce that,
$$\frac{1}{4}(\pi - 2) = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots$$
7. Find Fourier expansion of $f(x) = x^2$, $-\pi \le x \le \pi$ and hence prove that,
$$(i)\ \frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}\ , \qquad (ii)\ \frac{\pi^2}{12} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2}\ , \qquad (iii)\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$
By using Parseval's identity prove that $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$
8. Obtain Fourier series for
$$f(x) = \begin{cases} x + \dfrac{\pi}{2} & , \quad -\pi < x < 0 \\ \dfrac{\pi}{2} - x & , \quad 0 < x < \pi \end{cases}$$
Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Also deduce that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$
9. Find the Fourier series for $f(x) = |\sin x|$ in $(-\pi\ ,\pi)$
10. Find Fourier expansion of $f(x) = 2x - x^2$, $0 \le x \le 3$ whose period is 3.
11. Obtain Fourier series for $f(x) = \begin{cases} x & , \quad 0 < x < 1 \\ 1 - x & , \quad 1 < x < 2 \end{cases}$
12. Find the Fourier series for $f(x) = \begin{cases} \pi x & , \quad 0 < x < 1 \\ 0 & , \quad x = 1 \\ \pi(x - 2) & , \quad 1 < x < 2 \end{cases}$

Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
13. Find the Fourier expansion of $f(x) = \begin{cases} 2 & , \quad -2 < x < 0 \\ x & , \quad 0 < x < 2 \end{cases}$

14. Find the Fourier series for $f(x) = \begin{cases} -x & , \quad -1 < x < 0 \\ x & , \quad 0 < x < 1 \end{cases}$

15. Find the Fourier series for $f(x) = |x|$, $-k < x < k$ hence deduce that

$$\sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

16. Find the Fourier series for $f(x) = 1 - x^2$ in $(-1\ ,1)$

17. Find the Fourier expansion of $f(x) = \begin{cases} 0 & , \quad -2 < x < -1 \\ 1 + x & , \quad -1 < x < 0 \\ 1 - x & , \quad 0 < x < 1 \\ 0 & , \quad 1 < x < 2 \end{cases}$

18. Find the Fourier expansion of $f(x) = x^2 - 2$ $\quad in \ -2 \le x \le 2$

19. Obtain the expansion of $f(x) = x(\pi - x)$, $0 < x < \pi$ as a half range cosine series. Hence show that,

$$(i) \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad , \quad (ii) \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad , \quad (iii) \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

20. Expand $f(x) = lx - x^2$, $0 < x < l$ in half range sine series. Hence deduce that

$$(i)\ \frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \qquad (ii)\ \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots$$

21. Obtain half range sine series for $f(x) = x^2$ in $0 < x < 3$

22. Find half range sine series of period $2l$ for

$$f(x) = \begin{cases} \dfrac{2kx}{l} & , \quad 0 \le x \le \dfrac{l}{2} \\ \dfrac{2kx}{l}(l - x) & , \quad \dfrac{l}{2} \le x \le l \end{cases}$$

23. Obtain half range cosine series for $f(x) = sin\left(\frac{\pi x}{l}\right)$ in $0 < x < l$

24. Obtain half range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. Hence, find

$$\sum_{1}^{\infty} \frac{1}{n^2} \ and \ \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

25. Obtain half range sine series for $f(x) = \begin{cases} x & , \quad 0 < x < 1 \\ 2 - x & , \quad 1 < x < 2 \end{cases}$ hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Complex Form of Fourier series

1. Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(-l\ ,l)$
2. Obtain the complex form of Fourier series for $f(x) = \cosh ax + \sinh ax$ in $(-l\ ,l)$
3. Obtain the complex form of Fourier series for $f(x) = \cosh 3x + \sinh 3x$ in $(-3\ ,3)$
4. Find complex form of $f(x) = e^{x}$ in $(-\pi\ ,\pi)$
5. Obtain the complex form of Fourier series for $f(x) = \cosh x + \sinh x$ in $(-\pi\ ,\pi)$
6. Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(-1\ ,1)$
7. Show that the set of functions

$$1\ ,\sin\frac{\pi x}{l}\ ,\cos\frac{\pi x}{l}\ ,\sin\frac{2\pi x}{l}\ ,\cos\frac{2\pi x}{l}\ ,\dots$$

Form an orthogonal set in $(-L\ ,L)$ and construct an orthonormal set.

8. If $f(x) = C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)$ where $C_1\ ,C_2\ ,C_3$ constants and $\Phi_1\ ,\Phi_2\ ,\Phi_3$ are orthonormal sets on $(a\ ,b)$, show that

$$\int_a^b [f(x)]^2\, dx = {C_1}^2 + {C_2}^2 + {C_3}^2$$

9. Show that the set of functions $\cos x\ ,\cos 2x\ ,\cos 3x\ ,\dots$ is a set of orthogonal functions over $[-\pi\ ,\ \pi]$ Hence construct a set of orthonormal functions.
10. Prove that $\sin x\ ,\sin 2x\ ,\sin 3x\ ,\dots$ is orthogonal on $[0\ ,2\pi]$ and construct orthonormal set of functions.
11. Show that the set of functions $\Phi_n(x) = sin\left(\frac{n\pi x}{l}\right)$ $n = 1\ ,2\ ,3\ ,\dots$ is orthogonal in $(0\ ,l)$
12. Find the Fourier integral representation of $f(x) = \begin{cases} 0 & , \quad x < 0 \\ 1/2 & , \quad x = 0 \\ e^{-x} & , \quad x > 0 \end{cases}$
13. Find Fourier integral representation for $f(x) = \begin{cases} 1 - x^2 & for\ |x| \le 1 \\ 0 & for\ |x| > 1 \end{cases}$
14. Find Fourier integral representation for $f(x) = \begin{cases} e^{ax} & x \le 0\ ,a > 0 \\ e^{-ax} & x \ge 0\ ,a > 0 \end{cases}$

Hence show that

$$\int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2}\, d\omega = \frac{\pi}{2a} e^{-ax}\ ,\quad x > 0\ ,\quad a > 0$$

15. Find the Fourier cosine and sine integrals of the following function.

$$f(x) = \begin{cases} x & , \quad 0 \le x \le 1 \\ 2 - x & , \quad 1 \le x \le 2 \\ 0 & , \quad x > 2 \end{cases}$$

16. Express $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$ as Fourier sine integral and show that

$$\int_0^\infty \frac{\omega^3 \sin \omega x}{\omega^4 + 4}\, d\omega = \frac{\pi}{2} e^{-x} \cos x$$