is org over $(0, L)$. & Find O.N. set. — norm will be $\sqrt{\frac{L}{2}}$.

* Fourier Integral:-

Let $f(x)$ satisfy:- i) Direchlet's conditions in every finite interval $(-L, L)$.

ii) $\int_{-\infty}^{\infty} |f(x)| \cdot dx$ converges, i.e, $f(x)$ is absolutely integrable in $-\infty$ to ∞, then Fourier Integral Theorem states that

$$f(x) = \int_{\lambda=0}^{\infty} (A_\lambda \cdot \cos \lambda x + B_\lambda \sin \lambda x) \cdot d\lambda. \quad \text{—— (1)}$$

where,

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cdot \cos \lambda u \cdot du.$$

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cdot \sin \lambda u \cdot du$$

$$\therefore f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{u=-\infty}^{\infty} f(u) \cos (u-x)\lambda \cdot du \cdot d\lambda.$$

OR

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \cdot \cos(u-x)\lambda \cdot du \cdot d\lambda.$$

Result ① holds if x is a point of continuity. if x is discontinuity, we must replace $f(x)$ by $f(x) = \frac{f(x^+) + f(x^-)}{2}$ as in case of F.S.

RHS of ① is called F.I. representation of $f(x)$.

→ If $f(x)$ is odd func. then, $A(\lambda) = 0$.

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty f(u) \cdot \sin \lambda u \cdot du$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(u) \sin \lambda u \cdot du \right) \sin \lambda x \cdot d\lambda$$

This is called Fourier sine integral representation.

→ If $f(x)$ is even func., then, $B(\lambda) = 0$.

$$A(\lambda) = \frac{2}{\pi} \int_0^\infty f(u) \cos \lambda u \cdot du$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(u) \cos \lambda u \cdot du \right) \cos \lambda x \cdot d\lambda$$

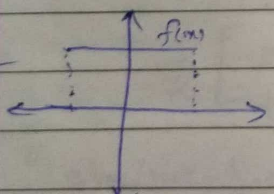
This is called Fourier cosine integral representation.

Q) Define F.I (sin & cos) & express $f(x)$ as FI where

$$f(x) = \begin{cases} 1 & , -1 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \cdot d\lambda$.

Solution



Function is even ∵ curve symmetric about y axis

∴ we express func. as F.C.I.

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \left[\int_0^1 1 \cdot \cos \lambda u \cdot du \right] \cdot \cos \lambda x \cdot d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda}{\lambda} \cdot \cos \lambda x \cdot d\lambda$$

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Now, $\int_0^\infty \frac{\sin\lambda}{\lambda}\cos\lambda x \cdot d\lambda = \frac{\pi}{2} f(x)$.

$$= \begin{cases} \frac{\pi}{2} , & |x| < 1 \\ 0 , & |x| > 1 \end{cases}$$

and at $|x| = 1$ (point of discontinuity), value of integral is $\frac{\pi}{2}\left[\frac{1}{2}(f(1^+) + f(1^-))\right] = \frac{\pi}{4}$.

$\therefore \int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda} \cdot d\lambda = \begin{cases} \frac{\pi}{2} , & |x| < 1 \\ 0 , & |x| > 1 \\ \frac{\pi}{4} , & |x| = 1 \end{cases}$

Q2.] Define Fourier sin & cos integral & find, F.C.I of fun $f(x) = e^{-ax}$, Hence, ST $\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1^2} \cdot d\lambda = \frac{\pi}{2} e^{-x}, x \geq 0$.

$\therefore$ F.C is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(u) \cdot \cos\lambda u \cdot du\right) \cdot \cos\lambda x \cdot d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty e^{-au} \cdot \cos\lambda u \cdot du\right) \cdot \cos\lambda x \cdot d\lambda$$

$$\cancel{= \frac{2}{\pi} \int_0^\infty \left[\frac{\sin\lambda u}{\lambda}\right]_0^\infty e^{-ax} \cos\lambda x \cdot d\lambda}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+\lambda^2} \cos\lambda x \cdot d\lambda$$

Let $a = 1$, $\therefore e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos\lambda x}{1^2+\lambda^2} \cdot d\lambda$

HW Q) Find F.I. for $f(x) = \begin{cases} 1-x^2 , & |x| \leq 1 \\ 0 , & |x| > 1 \end{cases}$

$f(-x) = f(x)$ $\therefore$ F.C.I $\because$ even func. symmetric to Y axis

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \left(\underbrace{\int_0^\infty f(u) \cdot \cos\lambda u \cdot du}\right) \cos\lambda x \cdot d\lambda$$

$$\int_0^1 (1-u^2)\cos\lambda u \, du = \cancel{\left[(1-u^2)\frac{\sin\lambda u}{\lambda}\right]_0^1} - \int_0^1 2u \cdot \frac{\sin\lambda u}{\lambda} \, du$$

$$= -\frac{2}{\lambda}\left[-u \cdot \frac{\cos\lambda u}{\lambda} + \frac{\sin\lambda u}{\lambda^2}\right]_0^1$$

$$= \frac{-2}{\lambda}\left[-\frac{\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2}\right]$$

$$= \frac{+2}{\lambda^2}\left(\frac{\sin\lambda}{\lambda} - \cos\lambda\right).$$

* Fourier Transform :- *

F.I.R of $f(x)$ is given by $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot\cos(t-x)\alpha \, dt \, d\alpha$ — ②

As $\cos[\alpha(t-x)]$ is even func. of α & $\sin[\alpha(t-x)]$ is an odd func. of α, we have, $0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot\sin\alpha(t-x) \, dt \, d\alpha$ — ③

$\therefore$ ② $- i$③ will give

$\rightarrow$ $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\alpha(t-x)} \, dt \, d\alpha$ — ④.

This is complex form of F.I.

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}}\int_{\alpha=-\infty}^{\infty} e^{ix\alpha}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-i\alpha t} \, dt\right)}_{F(\alpha)} d\alpha \quad — ⑤$$

$F(\alpha)$ is the Fourier Transform of $f(x)$ through the Kernal function $e^{-i\alpha t}$ & it is denoted by $\mathcal{F}[f(t)] = F(\alpha)$

if $F(\alpha)$ exists, then eq. ⑦ gives inverse Fourier Transf. of $F(\alpha)$.

$$\therefore \mathcal{F}^{-1}[F(\alpha)] = f(x).$$

*) If $f(x)$ is odd func, then $f(x)$ is :-

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(t)\cdot\sin\alpha t \, dt\right)\cdot\sin\alpha x \, d\alpha.$$

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$\therefore \mathcal{F}_s[f(t)] = F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \sin\alpha t \cdot dt.$

*) If $f(x)$ is an even func., then F.C.I is:

$f(x) = \frac{2}{\pi} \int_0^\infty \left(\int_0^\infty f(t) \cdot \cos\alpha t \cdot dt \right) \cdot \cos\alpha x \cdot d\alpha.$

$\therefore \mathcal{F}_c[f(t)] = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \cos\alpha t \cdot dt.$

Problems: 1) Find F.C.T of $f(x) = e^{-ax}$, and hence evaluate $\int_0^\infty \frac{\cos\alpha x}{\alpha^2+a^2} \cdot d\alpha.$

$\therefore \mathcal{F}_c[e^{-ax}] = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cdot \cos\alpha x \cdot dx.$

$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\alpha^2} (-a\cos\alpha x + \alpha \sin\alpha x) \right]_0^\infty$

$= \frac{a}{a^2+\alpha^2} \sqrt{\frac{2}{\pi}} = F_c(\alpha).$

$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cdot \cos\alpha x \cdot d\alpha.$

$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+\alpha^2} \cdot \cos\alpha x \cdot d\alpha.$

$\therefore \int_0^\infty \frac{\cos\alpha x}{a^2+\alpha^2} \cdot d\alpha = \frac{\pi}{2a} e^{-ax}$

2) Find F.S.T of $f(x) = \begin{cases} 0, & 0 < x < a \\ 1, & a < x < b \\ 0, & x > b \end{cases}$

and hence evaluate $\int_0^\infty \frac{\cos a\alpha - \cos b\alpha}{\alpha} \cdot \sin\alpha x \cdot d\alpha.$

$\therefore F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_a^b \sin\alpha t \cdot dt = \left[\frac{-\cos\alpha t}{\alpha} \right]_a^b = \frac{-(\cos b\alpha - \cos a\alpha)}{\alpha}.$

$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos a\alpha - \cos b\alpha}{\alpha} \cdot \sin\alpha x \cdot d\alpha = \begin{cases} 0, & 0 < x < a \\ \frac{\pi}{2}, & a < x < b. \\ 0, & x > b. \end{cases}$

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3.) Find F.T. of $f(x) = e^{-|x|}$ & hence evaluate $\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} \cdot d\alpha$ HW

Let $F(\alpha)$ be F.T.

$\therefore F(\alpha) = \frac{1}{\sqrt{2\pi}} \int f(x) \cdot e^{i\alpha x} \cdot dx = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x} \cdot e^{i\alpha x} \cdot dx + \int_{0} e^{-x} \cdot e^{i\alpha x} \cdot dx \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x(1+i\alpha)} \cdot dx + \int_{0}^{\infty} e^{x(i\alpha-1)} \cdot dx \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{x(1+i\alpha)}}{1+i\alpha} \right]_{-\infty}^{0} + \left[\frac{e^{x(i\alpha-1)}}{i\alpha-1} \right]_{0}^{\infty} \right\}$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{1+i\alpha} + \frac{1}{i\alpha-1} \right]$

$= \frac{+1}{\sqrt{2\pi}} \left[\frac{i\alpha+1+1-i\alpha}{+\alpha^2+1} \right] = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\alpha^2}$

$\therefore f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} \cdot e^{-i\alpha x} \cdot d\alpha$

$\therefore$ Hence evaluate $= \pi \cdot e^{-|x|}$.

4.) $\mathcal{F}[e^{-a^2x^2}] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i\alpha x} \cdot dx.$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{i\alpha x} \cdot dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left\{ (ax - \frac{i\alpha}{2a})^2 + \frac{\alpha^2}{4a^2} \right\}} \cdot dx$

Let $ax - \frac{i\alpha}{2a} = v$ $\therefore a \cdot dx = dv.$

$= \frac{e^{-\frac{\alpha^2}{4a^2}}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{2 \cdot e^{-\frac{\alpha^2}{4a^2}}}{a \cdot \sqrt{2\pi}} \int_{0}^{\infty} e^{-v^2} \cdot dv.$

$= \frac{e^{-\frac{\alpha^2}{4a^2}}}{a\sqrt{2}}.$

$\therefore \mathcal{F}[e^{-x^2/2}] = e^{-\frac{\alpha^2}{2}}$ Hence it is the function itself.

Remark: If a transformation of $f(x)$ is equal to $F(\alpha)$, then $f(x)$ is called self reciprocal wrt the transformation.

The above function is self reciprocal wrt F.C.T also

HW

Q2) P.T. $x \cdot e^{-x^2/2}$ is self reciprocal under F.S.T.

Q6) Find F.T. of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$

Hence deduce that $\int_0^\infty \frac{\sin t}{t} \cdot dt = \frac{\pi}{2}$.

$\rightarrow$ $F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{i\alpha x} \cdot dx$.

$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi} \cdot i\alpha} \left[e^{i\alpha a} - e^{-i\alpha a} \right]$.

$= \sqrt{\frac{2}{\pi}} \cdot \sin \alpha a$

$\therefore f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) \cdot e^{-i\alpha x} \cdot d\alpha$.

* Properties of F.T:-

1) Linearity:-

i) $\mathcal{F}[a f(x) + b \cdot g(x)] = a \cdot \mathcal{F}[f(x)] + b \cdot \mathcal{F}[g(x)]$.

$\sim$ly for $\mathcal{F}_c$ & $\mathcal{F}_s$.

ii) Shifting Theorems:- (shift in x).

If $\mathcal{F}[f(x)] = F(\alpha)$, then:-

$\mathcal{F}[f(x-a)] = e^{i\alpha a} \cdot F(\alpha)$. - this is for $F(\alpha)$ + & $f(x)$ -

If $F(\alpha)$ me $e^{-i\alpha x}$, then, $\mathcal{F}[f(x-a)] = e^{-i\alpha a} \cdot F(\alpha)$.