Fourier Series

$f(t+T) = f(t) \quad \forall t$

T- period

$f(t+nT) = f(t) \quad \forall t$

$f(t) \longrightarrow$ periodic function.

$f(t) = \underline{\text{sum of sines \& cosines}}$

Trignomeric series

The following important results are usefull in derivation of constants of fourier series :-

① $\displaystyle\int_c^{c+2\pi} \cos mx \cos nx \, dx \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \end{cases}$

② $\displaystyle\int_c^{c+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \end{cases}$

③ $\displaystyle\int_c^{c+2\pi} \sin mx \cos nx \, dx = 0 \quad \forall \ m \,\&\, n$

④ $\displaystyle\int_c^{c+2\pi} \cos mx \, dx \begin{cases} 0 & m > 0 \\ 2\pi & m = 0 \end{cases}$

⑤ $\displaystyle\int_c^{c+2\pi} \sin mx \, dx = 0 \quad \forall \ m$

Euler's Formula to determine Fourier constants.

Let $f(x)$ be expanded in $(c, c+2\pi)$ as a trignometric series. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ —— ①

where the coefficient a_0, a_n & b_n are given by

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

is called fourier series or expansion of $f(x)$ & the coefficients a_0, a_n & b_n are called fourier constants

→ Integrating ① w.r.t x between c to $c+2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}\int_c^{c+2\pi} dx + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_c^{c+2\pi} \cos nx\,dx}_{0} + b_n \underbrace{\int_c^{c+2\pi} \sin nx\,dx}_{0}\right)$$

$$= \frac{a_0}{2}(2\pi) \cancel{+ \sum_{n=1}^{\infty}}$$

$$\int_c^{c+2\pi} f(x)\,dx = a_0\pi$$

$$\boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

Multiply ① with $\cos nx$ & integrate wrt x between c to $c+2\pi$

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\underbrace{\int_c^{c+2\pi}\cos nx\,dx}_{0} + \cancel{\sum_{n=1}^{\infty} a_n \int_c^{c+2\pi}} + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \underbrace{\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx}_{0}$$

$$\int_c^{c+2\pi} f(x)\cos nx = a_n\pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx.}$$

Similarly

Multiply ① with $\sin nx$ & integrate wrt x between c to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = b_n\pi$$

$$\boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

Dirichlet's Condition for Fourrier series expansion

Suppose $f(x)$ is defined in $c < x < c+2\pi$ is periodic with period 2π is

1. finite & single valued in the interval
2. has finite no. of maximas & minimas in the interval
3. finite number of discontinuities in the given interval
4. then ~~Fa~~ fourrier series of $f(x)$ converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ at every value of x.

Remark :- Tan x has infinite discontinuities hence does not posses fourier ~~expansion~~ series.

If $c = 0$

1. $f(x)$ is defined in $(0, 2\pi)$

then the fourrier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

2. If $c = -\pi$

then $f(x)$ is defined in $(-\pi, \pi)$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

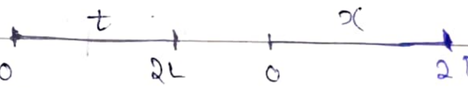
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

(3) ~~If $(c, c+2L)$~~

$f(x)$ be defined in $(0, 2L)$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$



$$\frac{x-0}{t-0} = \frac{x-2\pi}{t-2L}$$

$$x = \frac{\pi t}{L}$$

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

(4) $f(x)$ be defined in $(-L, L)$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Even & Odd Function

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ in the given interval.
And its said to be odd function if $f(-x) = -f(x)$ in the given interval

① $f(x)$ in $(-\pi, \pi) \longrightarrow$ even function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \cancel{\frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}}\,\underbrace{\sin nx}_{\text{odd}}\,dx$$

odd

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

odd x even ⟶ odd
odd x odd ⟶ even

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② f(x) defined in $(-\pi, \pi) \longrightarrow$ odd function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

∴

Fourier series will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

③ f(x) defined in $(-L, L) \longrightarrow$ even function

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx = \frac{2}{L}\int_{0}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

∴

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

④ f(x) defined in $(-L, L) \longrightarrow$ odd function

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

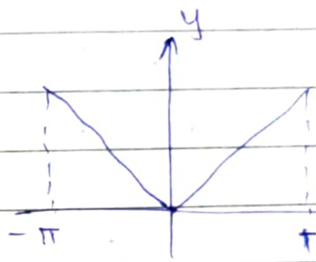
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

∴ fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Examples

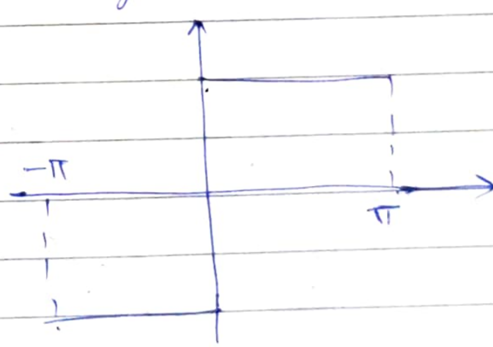
① $f(x) \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$



even ~~odd~~ function

★ Even function is symmetric wrt y-axis
★ Odd function is symmetric wrt origin

② $f(x) = \begin{cases} -a & -\pi < x < 0 \\ a & 0 < x < \pi \end{cases}$



③ $f(x) = x^2$ in $(-3, 3)$
$\longrightarrow$ even

④ $f(x) = x^2$ in $(0, 3)$ $\longrightarrow$ neither even nor odd

Que ① Obtain the fourier series to represent $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$ & hence evaluate the sum

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - - - - - = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - - = \frac{\pi^2}{8}$

$\Rightarrow$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{-3} \right]_0^{2\pi}$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx\,dx$$

$$a_n = \frac{1}{4\pi} \left\{ \underbrace{\left[(\pi - x)^2 \frac{\sin nx}{n} \right]_0^{2\pi}}_{0} - \int_0^{2\pi} 2(\pi - x)(-1) \frac{\sin nx}{n}\,dx \right\}$$

$$a_n = \frac{1}{4\pi} \int_0^{2\pi} 2(\pi - x) \frac{\sin nx}{n}\,dx$$

$$a_n = \frac{1}{2\pi n} \int_0^{2\pi} (\pi - x) \sin nx\,dx$$

$$a_n = \frac{1}{2\pi n} \left\{ \left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2} \right]_0^{2\pi} \right\} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4} \sin nx \, dx$$

$$b_n = \frac{1}{4\pi}\left\{\left[(\pi-x)^2\frac{\cos nx}{-n}\right]_0^{2\pi} - \int_0^{2\pi} 2(\pi-x)(-1)\frac{\cos nx}{n}\,dx\right\}$$

$$b_n = \frac{-1}{2\pi n}\int_0^{2\pi}(\pi-x)\cos nx \, dx$$

$$= -\frac{1}{2\pi n}\left[(\pi-x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$b_n = 0$$

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \quad \text{—— (1)}$$

$\therefore$

$$\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \left[\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

(i) put $x=0$ in (1)

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

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$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots=\frac{\pi^2}{6} \quad \text{——— (2)}$$

(ii) put $x=\pi$

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$\therefore 0=\frac{\pi^2}{12}+\left(-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots\cdots\right)$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\cdots\cdots=\frac{\pi^2}{12} \quad \text{——— (3)}$$

(iii) (2) + (3)

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$$

que(2) Expand $f(x)=x\sin x,\ 0<x<2\pi$ as a fourier series

$\Rightarrow$

we have in $(0,2\pi)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x\sin x\,dx$$

$$=\frac{1}{\pi}\Big[-x\cos x+\sin x\Big]_0^{2\pi}$$

$$\underline{a_0=-2}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx \quad \text{—— (1)}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x \left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$a_n = \frac{1}{2\pi}\left[\int_0^{2\pi} x \sin(n+1)x - \int_0^{2\pi} x \sin(n-1)x \, dx\right]$$

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} - \left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}\right\}$$

$n > 1$

$$a_n = \frac{-\cos 2\pi(n+1)}{(n+1)} + \frac{\cos(2\pi(n-1))}{n-1}$$

$$= \frac{-1}{n+1} + \frac{1}{n-1}$$

$$\boxed{a_n = \frac{2}{n^2-1}} \qquad n > 1$$

put $n = 1$ in (1)

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$a_1 = \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left(\frac{-2\pi}{2}\right) = \frac{-1}{2}$$

$$\boxed{a_1 = \frac{-1}{2}}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx \quad \text{——— (2)}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$b_n = \frac{1}{2\pi}\left\{\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right\}$$

$$b_n = \frac{1}{2\pi}\left\{\left[\frac{x\sin(n-1)x}{(n-1)} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi} - \left[\frac{x\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}\right\}$$

$$b_n = 0, \quad n > 1$$

put $n = 1$ in (2).

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$b_1 = \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$b_1 = \pi$$

Fourier series is

$$f(x) = -\frac{2}{2} + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}\cos nx - \frac{1}{2}\cos x + \pi\sin x\right)$$

$$\boxed{x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + 2\sum_{n=2}^{\infty}\left(\frac{1}{n^2-1}\cos nx\right)}.$$

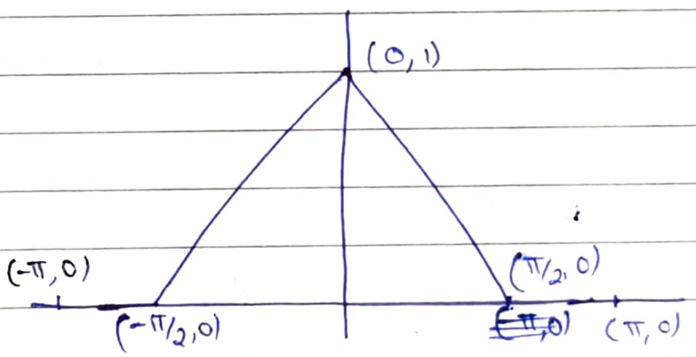
(3) $f(x) \begin{cases} 1+\frac{2x}{\pi} & -\pi < x \\ 1-\frac{2x}{\pi} & 0 \le x < \pi \end{cases}$

Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$\Rightarrow$

The graph of $f(x)$



From the diagram it is observed that $f(x)$ is an even function

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi}\right] = 0$$

$$a_0 = 0$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\cos nx - \frac{2}{\pi}x\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n} - \frac{2}{\pi}\frac{x\sin nx}{2} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{-4}{\pi^2 n^2}\left[\cos n\pi - 1\right]$$

$$= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{8}{\pi^2 n^2} & n \text{ is odd} \end{cases}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

put $x=0$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}}$$

que (4) Find fourier Expansion for the func $f(x) = x - x^2$, $-1 < x < 1$

$\Rightarrow$

Fourier expansion in $(-1, 1)$ $L = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$\because L = 1$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$$a_0 = \int_{-1}^{1} (x - x^2)\,dx$$

$$a_0 = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{3} - \left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}} - \frac{1}{3} = \frac{-2}{3}$$

$$\boxed{a_0 = \frac{-2}{3}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x \,dx$$

$$a_n = \int_{-1}^{1} x \cos n\pi x \,dx - \int_{-1}^{1} x^2 \cos n\pi x \,dx$$

product of x & $\cos nx$ is odd function.

$$a_n = 0 - 2\int_{0}^{1} x^2 \cos n\pi x \,dx$$

$$a_n = -2\left\{\left[\frac{x^2 \sin n\pi x}{n\pi}\right]_0^1 - \int_0^1 \frac{2x \sin n\pi x}{n\pi}\right\}$$

$$a_n = -2\left\{\frac{\sin n\pi}{n\pi} - \left[\frac{-2x\cos n\pi x}{n^2\pi^2} + \frac{2\sin n\pi x}{n^2\pi^2}\right]_0^1\right\}$$

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$$a_n = \frac{4}{n^2\pi^2}(-\cos n\pi)$$

$$a_n = \frac{-4}{n^2\pi^2}(-1)^n$$

$$a_n = \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \int_{-1}^{1}(x-x^2)\sin n\pi x\, dx$$

$$= \underbrace{\int_{-1}^{1} x\sin n\pi x\, dx}_{\text{even}} - \underbrace{\int_{-1}^{1} x^2 \sin n\pi x\, dx}_{\text{odd}}$$

$$= 2\int_{0}^{1} x\sin n\pi x\, dx$$

$$= 2\left[\frac{-x\cos \pi x}{\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= -\frac{2\cos n\pi}{n\pi}$$

$$= \frac{+2}{-n\pi}(-1)^{n+1}$$

Fourier expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \left(-\frac{2}{3}\right)\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x) + \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

$$f(x) = \frac{-1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\cos n\pi x + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x$$

⑤ Find fourier series expansion $f(x) = 2x - x^2$ in $(0, 3)$

→ Here $2L = 3$ $\quad L = 3/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$\because L = 3/2$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x)\,dx = \frac{2}{3} \int_0^{3} 2x - x^2 \, dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3$$

$a_0 = 0$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left\{ \int_0^3 2x \cos\left(\frac{2n\pi x}{3}\right) dx - \int_0^3 x^2 \cos \frac{2n\pi x}{3} \, dx \right]$$

$$a_n = \frac{-9}{n^2 \pi^2}$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{3} \int_0^3 (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \int_0^3 2x \sin\left(\frac{2n\pi x}{3}\right) dx - \frac{2}{3} \int_0^3 x^2 \sin \frac{2 n\pi x}{3} \, dx$$

$$= \frac{4}{3} \left[\frac{x \cos 2/3\, n\pi x}{2/3\, n\pi} + \frac{\sin 2/3\, n\pi x}{(2/3\, n\pi)^2} \right]_0^3 -$$

$$= \frac{2}{3} \left\{ \left[\frac{-x^2 \cos 2/3\, n\pi x}{2/3\, n\pi} \right]_0^3 - \int_0^3 \frac{2x \cos 2/3\, n\pi x \, dx}{2/3\, n\pi x} \right\}$$

$$= \frac{3\cos 2n\pi}{n\pi} + \frac{2}{n\pi}\left\{\left[\frac{x\sin 2/3 n\pi x}{2/3 n\pi} + \frac{\cos 2/3 n\pi x}{(2/3 n\pi)^2}\right]_0^3\right\}$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cdot\cos\left(\frac{2n\pi}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2}{3}n\pi x\right)\right)$$

$$= \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)$$

Half Range Series

$\left.\begin{matrix}\sin nx \\ \cos nx\end{matrix}\right\}$ period 2π

$f(x)$ be defined in $(0, \pi)$

① $f(x) = \begin{cases} f(x) & 0 < x < \pi \\ f(-x) & -\pi < x < 0 \end{cases}$

$f(x) =$ even function

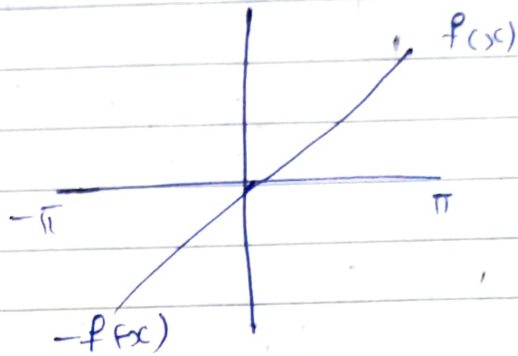
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} F(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

(2) $f(x) \begin{cases} f(x) & 0 < x < \pi \\ -f(-x) & -\pi < x < 0 \end{cases}$

$f(x) =$ odd function



$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

IMP

For (0, L)

→ Half range sine wave is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

→ Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L} \int_0^{L} f(x) \, dx$$

$$a_n = \frac{2}{L} \int_0^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Parseval's Identity (or Theorem)

If fourier series of $f(x)$ in $(-L, L)$ converges uniformly to $f(x)$ then

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof :-

The fourier series of $f(x)$ in $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right) \quad \text{— (1)}$$

where $$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \quad \text{— (2)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{— (3)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{— (4)}$$

Multiply (1) with $f(x)$ & integrate from $-L$ to L

$$\therefore \int_{-L}^{L}(f(x))^2\,dx = \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty} a_n\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$+ \sum_{n=1}^{\infty} b_n\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

From (2), (3), (4)

$$\int_{-L}^{L} f(x)^2\,dx = \frac{a_0}{2}(La_0) + \sum_{n=1}^{\infty} a_n(La_n) + b_n(Lb_n)$$

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Q(1) For $f(x) = x(\pi - x)$, find the Half range cosine series in $(0, \pi)$ & hence deduce that

i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

ii) Find value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$

⇒

Half range cosine series in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2)\,dx$

$$= \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\,dx$$

$$a_n = \cancel{\frac{2}{\pi}} \frac{2}{\pi} \left\{ \underbrace{\left[(\pi x - x^2) \frac{\sin nx}{n} \right]_0^{\pi}}_{0} - \int_0^{\pi} (\pi - 2x) \frac{\sin nx}{n}\,dx \right\}$$

$$a_n = \frac{-2}{\pi n} \int_0^{\pi} (\pi - 2x) \sin nx\,dx$$

$$a_n = \frac{-2}{n\pi} \left\{ \left[\frac{-(\pi - 2x) \cos nx}{x} + \frac{\sin nx}{n^2} (-2) \right]_0^{\pi} \right\}$$

$$a_n = \frac{-2}{\pi n^2} \left[(-\pi)(-\cos n\pi) - \pi(-1) \right]$$

$$a_n = \frac{2}{\pi n^2} (-\pi \cos n\pi - \pi)$$

$$a_n = \frac{2}{n^2} \left[-(-1)^n - 1 \right]$$

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$$a_n = \begin{cases} 0 & , n \text{ is odd} \\ -\frac{4}{n^2} & n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi^2}{6} - 4\left[\frac{1}{2^2}\cos 2x + \frac{1}{4^2}\cos 4x + \frac{1}{6^2}\cos 6x + \dots\right]$$

put $x = 0$

$$0 = \frac{\pi^2}{6} - 4\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} - \dots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

By Parseval's Identity for even function

$$\frac{2}{\pi}\int_0^{\pi} (f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\therefore LHS = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)^2\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\pi^2 x^2 + x^4 - 2x\pi^3\right)dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 x^3}{3} + \frac{x^5}{5} - x^2\pi^3\right]_0^{\pi}$$

$$LHS = \frac{\pi^4}{15}$$

$$\therefore \frac{\pi^4}{15} = \frac{\pi^4}{18} + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \dots\right]$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

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Prove that in $(0,\pi)$

$$\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \cdots\right]$$

→ Here $f(x)$ is $f(x) = e^{ax} - e^{-ax}$ in $(0,\pi)$

∴ we are required to express $f(x)$ as a half range sine series in $(0,\pi)$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (e^{ax}-e^{-ax})\sin nx\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} e^{ax}\sin nx\, dx - \int_0^{\pi} e^{-ax}\sin nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{e^{ax}}{a^2+n^2}(a\cancel{\sin nx} - n\cos nx)\right]_0^{\pi} - \left[\frac{e^{-ax}}{a^2+n^2}(-a\cancel{\sin nx} - n\cos nx)\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{1}{a^2+n^2}n - \frac{e^{-a\pi}}{a^2+n^2}(-n\cos(-n\pi)) - \frac{n}{a^2+n^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-n e^{a\pi}(-1)^n}{a^2+n^2} + \frac{n e^{-a\pi}(-1)^n}{a^2+n^2}\right\}$$

$$= \frac{2n(-1)^n}{(a^2+n^2)\pi}\left(-e^{a\pi} + e^{-a\pi}\right)$$

$$= \frac{-2n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$b_n = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$\therefore$ Half Range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)} \left(e^{a\pi} - e^{-a\pi}\right) \sin nx$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{(a^2+n^2)} \sin nx$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{(1^2+a^2)} - \frac{2\sin 2x}{(2^2+a^2)} + \frac{3\sin 3x}{(3^2+a^2)} - \cdots\right]$$

Que 3 $\quad f(x) = x^2$ in $(-\pi, \pi)$

Hence find the value of $\sum_{n=1}^{\infty} \frac{1}{n^4}$

even functn $\downarrow$ cosine series

Complex Form of Fourier Series

Let $f(x)$ be defined in $(c, c+2l)$ then the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{(1)}$$

but we know that,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}}{2} \qquad \sin\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}}{2i}$$