Shri Vile Parle Kelavani Mandal's
DWARKADAS J. SANGHVI COLLEGE OF ENGINEERING
Autonomous College Affiliated to the University of Mumbai
NAAC Accredited with 'A' Grade (CGPA 3.18)

RUBRICS INDEX

Academic Year 2022-23

Name: Shasvat Shah
Department /Branch: Computer Engineering
Course: Engineering Mathematics-III Tutorial

SAP ID: 60004220126
Division: B
Course Code: DJ19CET301

At the end of the course, the student will be able to...		PO'S	Blooms Level
DJ19CET301.1 CO1	Use Laplace and inverse Laplace Transform to the Ordinary Differential Equations.	PO1 PO2	Understand Apply
DJ19CET301.2 CO2	Expand the periodic function by using Fourier series and complex form of Fourier series.	PO1	Understand
DJ19CET301.3 CO3	Apply Fourier Transform in the future subjects like signal processing.	PO1 PO2	Understand Apply
DJ19CET301.4 CO4	Apply the concept of Z- transformation and its inverse of the given sequence.	PO1	Understand

Performance Indicators (Maximum 5 marks per indicator)	Tut 1	Tut 2	Tut 3	Tut 4	Tut 5	Tut 6	Tut 7	Tut 8	Tut 9	Tut 10	Total
Course Outcome	CO1	CO1	CO1	CO1	CO2	CO2	CO3	CO4			
1. Knowledge											
2. Describe											
3. Demonstration											
4. Interpret / Develop											
5. Attitude towards learning											
Total (out of 25 marks)	21	21	21	21	21	21	21	21			21

Outstanding (5), Excellent (4), Good (3), Fair (2), Needs improvement (1)

$\underbrace{\hspace{10em}}_{5}$

Signature of the Teacher   Head of the Department   Principal

Name of the Teacher: Dr. (Mrs.) Gayatri Pandey

Date: 12/12/2022

Tutorial 1 Date :

SAP : 60004220126

LAPLACE TRANSFORM

1 Find the laplace transform of $f(t) = \begin{cases} \cos t &, 0 < t < \pi \\ \sin t &, t > \pi \end{cases}$

By deflnation of laplace

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[f(t)] = \int_0^\pi e^{-st} \cancel{[f(t)]} \cos t \, dt + \int_\pi^\infty e^{-st} \cdot \sin t \, dt \quad \text{—} (1)$$

Now,

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a\cos bx + b\sin bx]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a\sin bx - b\cos bx]$$

$$\therefore \int_0^\pi e^{-st} \cdot \cos t \, dt = \left[\frac{e^{-st}}{s^2+1} [-s\cos t + \sin t] \right]_0^\pi$$

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$$= \left[\frac{e^{-\pi s}}{s^2+1} [-s\cos \pi + \sin \pi] \right] - \left[\frac{e^{0}}{s^2+1} (-s) \right]$$

$$= \left[\frac{e^{-\pi s}}{s^2+1} (s) - \frac{e^{0}}{s^2+1} (-s) \right]$$

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$$= \frac{s}{s^2+1}\left(e^{-\pi s}+1\right)$$

$$\int_{\pi}^{\infty} e^{-st} \cdot \sin t \, dt = \left[\frac{e^{-st}}{s^2+1}\left(-s\sin t - \cos t\right)\right]_{\pi}^{\infty}$$

$$= \left[0 - \frac{e^{-s\pi}}{s^2+1}\left[-s(0) - (-1)\right]\right]$$

$$= \frac{-e^{-\pi s}}{s^2+1}$$

from ①

$$L\left[f(t)\right] = \frac{s}{(s^2+1)}\left(e^{-\pi s}+1\right) + \left(\frac{-e^{-\pi s}}{s^2+1}\right)$$

$$= \frac{se^{-\pi s} + s - e^{-\pi s}}{s^2+1}$$

$$= \frac{1}{s^2+1}\left[s + e^{-\pi s}(s-1)\right]$$

2 $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \ldots\ldots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!}$$

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$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} \dots$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[\frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!}\right]$$

Now .. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

$$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} \dots$$

$$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1/2}{2}\frac{\Gamma(1/2)}{s^{3/2}} + \frac{1}{4!}\frac{3/2 \times 1/2\ \Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2 \times 3/2 \times 1/2\ \Gamma(1/2)}{s^{7/2}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}\left[1 - \frac{1/2}{2s} + \frac{3/2 \times 1/2}{4! \cdot s^2} - \frac{1}{6!}\frac{5/2 \times 3/2 \times 1/2}{s^3} \dots\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!\,(4s)^2} - \frac{1}{3!\,(4s)^3} + \dots\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(e^{-1/4s}\right) \qquad \left[\because e^{-x} = 1 - \frac{1}{x} + \frac{1}{2!\,x^2} - \frac{1}{3!\,x^3} \dots\right]$$

3 Evaluate by Laplace Transform $\int_0^\infty e^{-3t} \cdot \sin^3 t \, dt$.

Now, $\int_0^\infty e^{-3t} \sin^3 t \, dt = L[\sin^3 t]$, $s = 3$

$\because$ By defination.

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we know, $\sin^3 t = \dfrac{3\sin t - \sin 3t}{4}$

$$L[\sin^3 t] = L\left[\frac{3\sin t - \sin 3t}{4}\right]$$

$$= \frac{1}{4}\left[L[3\sin t]_{s=3} - L[\sin 3t]_{s=3}\right]$$

$$= \frac{1}{4}\left[\frac{3}{s^2+1}_{s=3} - \frac{3}{s^2+9}_{s=3}\right]$$

$$= \frac{1}{4}\left[\frac{3}{10} - \frac{\cancel{3}\,1}{\cancel{18}\,6}\right]$$

$$= \frac{1}{4}\left[\frac{18-10}{60}\right]$$

$$= \frac{1}{30}$$

4 If $L[\sin\sqrt{t}] = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s}$ find $L[\sin(2\sqrt{t})]$

Its given, $L[\sin\sqrt{t}] = \dfrac{1}{2s}\sqrt{\dfrac{\pi}{s}}\, e^{-1/4s}$

$$L[\sin^2\sqrt{t}] = L[\sin\sqrt{4t}]$$

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By change of order of scaler, here $F(s) = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$
so, by change of order of scalar,

$$L\left[\sin(2\sqrt{t})\right] = \frac{1}{4} F\left(\frac{s}{4}\right)$$

$$= \left[\frac{1}{2\frac{s}{4}}\sqrt{\frac{\pi}{s/4}}\; e^{-1/4 \times 4/s}\right] \times \frac{1}{4}$$

$$= \left[\frac{2}{s} \times 2\sqrt{\frac{\pi}{s}}\; e^{-1/s}\right] \times \frac{1}{4}$$

$$= \frac{1}{s}\sqrt{\frac{\pi}{s}}\; e^{-1/s}$$

5 $L\left[e^{-3t} . \cosh 4t \sin 3t\right]$

$L\left[\sin 3t\right] = \frac{3}{s^2+9}$ By formula.

$L\left[e^{-3t} \cosh 4t . \sin 3t\right]$

$$= L\left[e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t\right]$$

$$= \frac{1}{2} L\left[\left(e^{t} + e^{-7t}\right)\sin 3t\right]$$

$$= \frac{1}{2} L\left[e^{t} . \sin 3t + e^{-7t}\sin 3t\right]$$

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By F.S.T (First Shifting Property)

$$= \frac{1}{2}\left[\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+1-2s+9} + \frac{3}{s^2+49+14s+9}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-2s+10} + \frac{3}{s^2+14s+58}\right]$$

$$= \frac{1}{2}\left[\frac{3s^2+42s+174+3s^2-6s+30}{(s^2-2s+10)(s^2+14s+58)}\right]$$

$$= \frac{3}{2}\left[\frac{2s^2+12s+68}{(s^2-2s+10)(s^2+14s+58)}\right]$$

$$= \frac{3s^2+18s+102}{(s^2-2s+10)(s^2+14s+58)}$$

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Tutorial 2

Date !

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LAPLACE TRANSFORM

1 Evaluate $\int_0^\infty \frac{t^2 \sin t}{e^{2t}} \, dt$

$$= \int_0^\infty e^{-2t} \cdot t^2 \cdot \sin t \; dt .$$

$$= L\left[t^2 \cdot \sin t\right]_{s=2}$$

Now, $L[\sin t] = \frac{1}{s^2+1}$ ... By defination.

By multiplication of order of power of t

$$L\left[t^2 \cdot \sin t\right]_{s=2} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{1+s^2}\right)$$

$$= 1 \times \frac{d}{ds}\left(\frac{-2s}{(1+s^2)^2}\right)$$

$$= -1 \left[\frac{(1+s^2)^2(20) - 2(1+s^2)(2s)(2s)}{(1+s^2)^4}\right]$$

$$= -\left[\frac{2+2s^2-8s^2}{(1+s^2)^3}\right].$$

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$$= -\left[\frac{2 - 6s^2}{[1+s^2]^3}\right]$$

$\therefore$ puting $s = 2$

$$= \frac{24 - 2}{1+4}$$

$$= \frac{22}{125}$$

2 Find $L\left[e^{-2t}\, t^4 \cdot \sinh(4t)\right]$

we know $\sinh 4t = \dfrac{e^{4t} - e^{-4t}}{2}$

$$L\left[e^{-2t} \cdot t^4 \cdot \sinh(4t)\right] = L\left[t^4 e^{-2t}\left(\frac{e^{4t} - e^{-4t}}{2}\right)\right]$$

$$= \frac{1}{2} L\left[t^4\left(e^{2t} - e^{-6t}\right)\right]$$

$L\left[t^4\right] = \dfrac{4!}{s^5}$ $\boxed{\text{By defination}}$

By F.S.T (First Shifting Theorum)

$$= \frac{1}{2}\left[\frac{4!}{(s-2)^5} - \frac{4!}{(s+6)^5}\right]$$

$$= \frac{4!}{2}\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right] \rightarrow 12\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right]$$

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3

Now, $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

By Scalar Property...

$$L[J_0(at)] = \frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}} \times \frac{1}{a}$$

$$= \frac{a}{\sqrt{a^2+s^2}} \times \frac{1}{a} = \frac{1}{\sqrt{s^2+a^2}}$$

By Multiplication by power of t.

$$L[t\,J_0(at)] = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)$$

$$= (-1)\left(\frac{-1}{2} \times \frac{2s}{(s^2+a^2)^{3/2}}\right)$$

$$= \frac{s}{(s^2+a^2)^{3/2}}$$

4 Prove! $\int_0^\infty e^{-\sqrt{2}t}\,\frac{\sin t \cdot \sinh t}{t}\,dt = \frac{\pi}{8}$

$$\int_0^\infty \frac{e^{-\sqrt{2}t}}{t}\left(\frac{e^t - e^{-t}}{2}\right)\sin t \quad dt.$$

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$$= \int_0^\infty \frac{e^{(1-\sqrt{2})t}\sin t}{2t} - \frac{e^{-(1+\sqrt{2})}\sin t}{2t}\, dt.$$

$$= L\left[\frac{\sin t}{2t}\right]_{s=\sqrt{2}-1} - L\left[\frac{\sin t}{2t}\right]_{s=\sqrt{2}+1} \quad \text{——} (1)$$

for $L\left[\frac{\sin t}{2t}\right]$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin t}{2t}\right] = \frac{1}{2}\int_s^\infty \frac{1}{1+s^2}\, ds.$$

$$= \frac{1}{2}\left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1} s\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

put values of s in the following

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(\sqrt{2}-1) - \frac{\pi}{2} + \tan^{-1}(\sqrt{2}+1)\right] \rightarrow \frac{1}{2}\times\frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

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5 $L\left[\int_0^t u e^{-3u} \sin 4u \, du\right]$

for $L\left[u e^{-3u} . \sin 4u\right]$

now, $L\left[\sin 4u\right] = \frac{4}{s^2+16}$ [By defination]

So, $L\left[e^{-3u} . \sin 4u\right] = \frac{4}{(s+3)^2+16}$ [By F.S.T] First Shifting Theorum.

So, $L\left[u . e^{-3u} . \sin 4u\right] = (-1) \frac{d}{ds}\left(\frac{4}{s^2+6s+9+16}\right)$

$= (-1)\left[\frac{-4(2s+6)}{(s^2+6s+25)^2}\right]$

By Laplace of integration.

$L\left[\int_0^t u e^{-3u} \sin 4u \, du\right] = \frac{1}{s}\left[\frac{8s+24}{(s^2+6s+25)^2}\right]$

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Tutorial 3

Date :

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Inverse Laplace Transform & Convolution Theorum.

1 $L^{-1}\left[\frac{3s+5}{9s^2-25}\right]$, find this

so $L^{-1}\left[\frac{3s+5}{9s^2-25}\right] = L^{-1}\left[\frac{3s}{9s^2-25}\right] + L^{-1}\left[\frac{5}{9s^2-25}\right]$

$= 3L^{-1}\left[\frac{s}{9(s^2-25/9)}\right] + \frac{5}{9}L^{-1}\left[\frac{1}{s^2-25/9}\right]$

$= \frac{1}{3}L^{-1}\left[\frac{s}{s^2-(5/3)^2}\right] + \frac{5}{9}L^{-1}\left[\frac{1}{s^2-(5/3)^2}\right]$

we know,

$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh(at)}{a}$, $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$\therefore L^{-1}\left[\frac{3s+5}{9s^2-25}\right] = \frac{1}{3}\left[\cosh 5/3t\right] + \frac{5}{9}\left[\frac{\sinh 5/3t}{5/3}\right]$

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$= \frac{1}{3}\left[\cosh 5/3t\right] + \frac{5}{9} \times \frac{3}{5}\left[\sinh 5/3t\right]$

$= \frac{1}{3}\left[\cosh 5/3t + \sinh 5/3t\right] = \frac{1}{3}e^{5/3t}$

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2 Find $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$

so,

$$L^{-1}\left[\frac{s+2}{s^2+4s+4-4+7}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+3}\right]$$

$$L^{-1}\left[\frac{s+2}{(s+2)^2+(\sqrt{3})^2}\right]$$ $\therefore$ By First Shifting Property

$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = e^{-2t}\, L^{-1}\left[\frac{s}{s^2+(\sqrt{3})^2}\right]$$

Now, $$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = e^{-2t}\cdot\cos\sqrt{3}\,t$$ $\because a = \sqrt{3}$

3 Find $L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$

so, using partial fractions,

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{A}{(s+3)} + \frac{Bs+C}{s^2+1}$$

$$\frac{5s^2+8s-1}{\cancel{(s+3)}\cancel{(s^2+1)}} = \frac{A(s^2+1)}{\cancel{(s+3)}} + \frac{(Bs+C)(s+3)}{\cancel{(s^2+1)}}$$

$$5s^2+8s-1 = A(s^2+1) + (Bs+C)(s+3)$$

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put $s = -3$

$$20 = 10(A)$$
$$A = 2$$

put $s = 0$

$$-1 = A + 3C$$
$$C = -1$$

put $s = 1$

$$12 = 2A + (B+C)(4)$$
$$12 = 4 + 4B - 4$$
$$12 = 4B$$
$$3 = B$$

$$\frac{5s^2 + 8s - 1}{(s+3)(s^2+1)} = \frac{2}{s+3} + \frac{3s - 1}{s^2+1}$$

so, $L^{-1}\left[\frac{2}{s+3} + \frac{3s-1}{s^2+1}\right]$ ∴ By Linearity Property.

$$= L^{-1}\left[\frac{2}{s+3}\right] + L^{-1}\left[\frac{3s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

Now, $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$ $L^{-1}\left[\frac{s}{s^2+1^2}\right] = \cos at$ $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$$= 2e^{-3t} + 3\cos t - \sin t$$

$$L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = 2e^{-3t} + 3\cos t - \sin t$$

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4 Find $L^{-1}\left[\cot^{-1}(s+1)\right]$

we know,

$$L^{-1}\left[F(s)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$$

$$L^{-1}\left[\cot^{-1}(s+1)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}(s+1)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\frac{\pi}{2} - \tan^{-1}(s+1)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= \frac{1}{t} e^{-t}\left[\frac{1}{s^2+1}\right]$$ $\therefore$ By F.S.T

we know, $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$$= \frac{1}{t} e^{-t} \sin t$$

$$L^{-1}\left[\cot^{-1}(s+1)\right] = \frac{1}{t} e^{-t} \sin t$$

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5 Find using convolution thorum $L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right]$

$$L^{-1}\left[\frac{s}{(s^2+4)} \times \frac{1}{s^2+1^2}\right]$$

Here,

$$F(s) = \frac{s}{s^2+4} \qquad G(s) = \frac{1}{s^2+1}$$

$$f(t) = \cos 2t \qquad g(t) = \sin t$$

$\therefore$ By convolution theorum

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = f(t) * g(t) \quad \text{—— (1)}$$

where, $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

$$\cos(2t) * \sin(t) = \int_0^t \cos 2u \cdot \sin(t-u)\, du.$$

$$= \int_0^t \frac{1}{2}\left[\sin(t-u-2u) + \sin(2u+t-u)\right]$$

$$= \frac{1}{2}\int_0^t \sin(t-3u) + \sin(u+t)\, du.$$

$$= \frac{1}{2}\left[\int_0^t \sin(t-3u)\, du + \int_0^t \sin(u+t)\, du.\right]$$

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$$= \frac{1}{2}\left[\frac{-\cos(t-3u)}{-3} + \frac{-\cos(t+u)}{1}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\cos(-2t)}{3} - \cos 2t\right] - \left[\frac{\cos t}{3} - \cos t\right]$$

$$= \frac{1}{2}\left[\frac{\cos 2t}{3} - \cos 2t - \frac{\cos t}{3} + \cos t\right]$$

$$= \frac{1}{6}\left[\cos 2t - \cos t - 3\cos 2t + 3\cos t\right]$$

$$= \frac{1}{6}\left[2\cos t - 2\cos 2t\right]$$

$$= \frac{1}{3}\left[\cos t - \cos 2t\right]$$

$$L\left[\frac{s}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}(\cos t - \cos 2t)$$

5 Find using Convolution thorum $L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$

$$L^{-1}\left[\frac{1}{(s-2)^4 (s+3)}\right]$$

Here,

$$F(s) = \frac{1}{(s-2)^4}, \quad G(s) = \frac{1}{s+3}$$

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So, $f(t) = e^{2t} L^{-1}\left[\frac{1}{s^4}\right]$ $\qquad g(t) = e^{-3t}$

∴ By F.S.T

$$f(t) = e^{2t}\,\frac{t^3}{6}$$

so by convolution theorum.

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = f(t) * g(t)$$

where, $f(t) * g(t) = \int_0^t e^{2u} \cdot u^3/6 \cdot e^{-3(t-u)}\, du.$

$$= \int_0^t e^{2u} \cdot \frac{u^3}{6} \cdot e^{-3t} \cdot e^{3u}\, du.$$

$$= \int_0^t e^{5u} \cdot e^{-3t} \cdot \frac{u^3}{6}\, du$$

$$= \frac{e^{-3t}}{6} \int_0^t \frac{u^3}{6} \cdot e^{5u}\, du.$$

$$= \frac{e^{-3t}}{6}\left[\frac{3u^3 \cdot e^{5u}}{5} - \frac{36u^2 \cdot e^{5u}}{25} + \frac{6u\, e^{5u}}{125} - \frac{6e^{5u}}{625}\right]_0^t$$

$$= \frac{e^{-3t}}{6}\left[\frac{3t^3 \cdot e^{5t}}{5} - \frac{36t \cdot e^{5t}}{25} + \frac{6 \cdot e^{5t}}{125} - \frac{6e^{5t}}{625}\right] - \left[\frac{-6}{625}\right]$$

$$= \frac{e^{-3t}}{6} \cdot \frac{e^{5t}}{5}\left[3t^3 - \frac{6t}{5} + \frac{6}{25} - \frac{1}{125}\right] - \left[\frac{-6}{625}\right]$$

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$$\frac{e^{2t}}{30}\left[t^3 - \frac{3t^2}{5} + \frac{6t}{25} - \frac{6}{125} + \frac{6e^{-5t}}{625}\right]$$

$$= \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right]$$

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Tutorial 4 Date :

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Application, Heavy Side and Direct Delta

1

$(D^2 - D - 2)\, y = 20 \sin 2t$, $y(0) = 1$ $y'(0) = 2$

Taking Laplace on both sides.

$L\left[(D^2 - D - 2)y\right] = L\left[20 \sin 2t\right]$

$L\left[D^2y - Dy - 2y\right] = L\left[20 \sin 2t\right]$

$L\left[D^2y\right] - L\left[Dy\right] - 2L\left[y\right] = \frac{20 \times 2}{s^2 + 4}$

$s^2 L\left[y(t)\right] - s\,y(0) - y'(0) - sL\left[y(t)\right] + y(0) - 2L\left[y(t)\right] = 40/s^2+4$

$L\left[y(t)\right]\left[s^2 - s - 2\right] - s - 2 + 1 = \frac{40}{s^2 + 4}$

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$L\left[y(t)\right](s^2 - s - 2) = \frac{40}{s^2+4} + (s+1)$

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$L\left[y(t)\right] = \frac{40}{(s^2+4)(s^2 - s - 2)} + \frac{s+1}{s^2 - s - 2}$

$= \frac{40 + (s+1)(s^2+4)}{(s^2+4)(s^2 - s - 2)}$

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$$= \frac{40 + s^3 + 4s + s^2 + 4}{(s^2+4)(s^2-s-2)}$$

$$= \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s - 2)(s+1)}$$

Applying Partial function,

$$\frac{s^3+s^2+4s+44}{(s+1)(s-2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$$

$$s^3+s^2+4s+44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2)$$

put $s = 2$,

$$8+4+8+44 = B(3)(8) = \boxed{\frac{8}{3} = B}$$

put $s = -1$

$$-1+1-4+44 = A(-3)(5) = \boxed{\frac{-8}{3} = A}$$

put $s = 0$

$$44 = -8A + 4B + D(-2) = \boxed{-6 = D}$$

put $s = 1$

$$1+1+4+44 = -5A + 10B - 2C + 12 = \boxed{1 = C}$$

$$L\,\boxed{y(t)} = \frac{-8}{3(s+1)} + \frac{8}{3(s-2)} + \frac{s-6}{s^2+4}$$

Taking Inverse laplace transform,

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$$y(t) = -\frac{8}{3}e^{-t} + \frac{8}{3}e^{2t} + \cos 2t - 3\sin 2t$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}, \quad L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at, \quad L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$$

2 $2x' + y' = 5e^t \qquad x(0) = 0, \quad y(0) = 0$

$y' - 3x = 5$

Solving both equations simultaneously

Taking laplace transform on both sides and both equations,

$$L[2x' + y'] = L[5e^t], \quad L[-3x' + y'] = L[5] \quad -① $$

$$2L[x'] + L[y'] = 5L[e^t]$$

$$2[sL[x(t)] - x[0]] + sL[y(t)] - y[0] = \frac{5}{s-1} \quad \left[\because L[e^{at}] = \frac{1}{s-a}\right]$$

$$2sL[x(t)] + sL[y(t)] = \frac{5}{s-1}$$

Let $L(x(t)]$ be X & $L[y(t)]$ be Y

$$2sX + 5y = \frac{5}{s-1}$$

from 1

$$L[-3x' + y'] = L[5]$$

$$-3L[X'] + L[Y'] = 5L[1]$$

$$-3[sL(x(t)) - x(0)] + [sL(y(t)) - y(0)] = \frac{5}{s}$$

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$$-3sL[x(t)] + sL(y(t)) = \frac{5}{3s}$$

Let $L[x(t)]$, $L[y(t)]$ be X, Y respectively.

$$-3s(X) + s(Y) = \frac{5}{s} \quad \text{—③}$$

Subtracting ② and ③

$$X(2s) + Y(s) = 5/s-1$$
$$X(-3s) + Y(s) = 5/s$$
$$- \qquad - \qquad -$$

$$X(5s) = \frac{5}{s-1} - \frac{5}{s}$$

$$L[x(t)](5s) = \frac{5}{s-1} - \frac{5}{s}$$

$$L[x(t)] = \frac{1}{s(s-1)} - \frac{1}{s^2}$$

$$= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \quad \text{—④}$$

Taking inverse laplace transform.

$$x(t) = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right]$$

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$x(t) = e^{-t} - 1 - t$

i.e $x(t) = e^{-t} - t - 1$

from (2)

$$Y(s) = \frac{5}{s-1} - X(2s)$$

Putting Y, X as $L[y(t)]$ & $L[x(t)]$ resp.

$$s L(y(t)) = \frac{5}{s-1} - L[x(t)](2s)$$

As $L[x(t)] = \dfrac{1}{s^2(s-1)}$

$$(s) L[y(t)] = \frac{5}{s-1} - \frac{1}{s^2(s-1)} (2s)$$

$$(s) L[y(t)] = \frac{5}{s-1} - \frac{2}{s(s-1)}$$

$$(s) L[y(t)] = \frac{5}{s-1} - 2\left[\frac{1}{s-1} - \frac{1}{s}\right]$$

$$L[y(t)] = \frac{3}{s(s-1)} + \frac{2}{s^2}$$

$$L[y(t)] = 3\left[\frac{1}{s-1} - \frac{1}{s}\right] + \frac{2}{s^2}$$

Taking Inverse Laplace transform.

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$$y(t) = 3L^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = 3e^{-t} - 3 + 2t$$

$$y(t) = 3e^{t} + 2t - 3$$

$$\therefore \; x(t) = e^{t} - t - 1$$
$$y(t) = 3e^{t} + 2t - 3$$

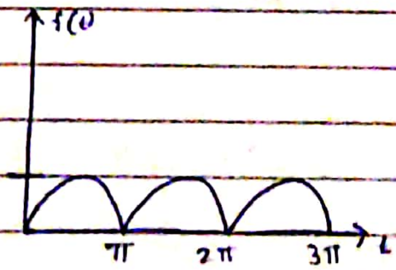
3

$f(t) = |\sin pt|, \quad t \geq 0$

here $|\sin pt|$ is periodic function $f(t + \pi/e) = f(t), \forall t$

period $(t) = \pi/p$

for periodic function.



$$L[f(t)] = \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s\pi/p}} \int_0^{\frac{\pi}{p}} e^{-st} \sin pt \, dt$$

$$= \frac{1}{1-e^{-s\pi/p}} \left[\frac{e^{-st}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$= \frac{1}{1-e^{-s\pi/p}} \left[\frac{e^{-\pi s/p}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]$$

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$$= \frac{1}{1-e^{-s\pi/p}} \left[\frac{e^{-\pi s/p}}{s^2+p^2} [p] - \frac{(-p)}{s^2+p^2} \right]$$

$$= \frac{1}{1-e^{-s\pi/p}} \left[\frac{e^{-\pi s/p}}{s^2+p^2} [p] - \frac{(-p)}{s^2+p^2} \right]$$

$$= \frac{1}{1-e^{-s\pi/p}} \times \frac{p}{s^2+p^2} \left[e^{-\pi s/p} + 1 \right]$$

$$= \frac{p}{s^2+p^2} \frac{\left(1+e^{-\pi s/p}\right)}{\left(1-e^{-\pi s/p}\right)}$$

$$= \frac{p}{s^2+p^2} \left(\frac{e^{\pi s/2p} + e^{-\pi s/2p}}{e^{\pi s/2p} + e^{-\pi s/2p}} \right)$$

$$= \frac{p}{s^2+p^2} \cosh\left(\frac{\pi s}{2p}\right)$$

4

$$\int_0^\infty e^{-t} \left(1+2t-t^2+t^3\right) H(t-1)\, dt$$

By defination

$$L\left[\left(1+2t-t^2+t^3\right) H(t-1)\right]_{s=1}$$

Now, $L[f(t)\, H(t-a)] = e^{-as} [f(t+a)]$

$\therefore \quad L\left[\left(1+2t-t^2+t^3\right) H(t-1)\right]$

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$$= e^{-s} L\left[1+2(t+1)-(t+1)^2+(t+1)^3\right]$$

$$= e^{-s} L\left[1+2t+2-t^2-2t-1+t^3+1+3t^2+3t\right]$$

$$= e^{-s} L\left[t^3+2t^2+3t+3\right]$$

$$= e^{-s}\left[\frac{3!}{s^4}+\frac{2\cdot 2!}{s^3}+\frac{3!!}{s^2}+\frac{3}{s}\right]$$

Now, putting $s = 1$

$$= e^{-1}\left[3!+4+3+3\right] = e^{-1}(16) = 16/e$$

$$\therefore \int_0^\infty e^{-t}\left(t^3-t^2+2t+1\right)H(t-1)\,dt = \frac{16}{e}$$

5

$$L\left[\cos t\left[H(t-\pi/2)-H(t-3\pi/2)\right]\right]$$

$$L\left[\cos t\left[H(t-\pi/2)\right]\right]-L\left[\cos t\left[H(t-3\pi/2)\right]\right]$$

Now $L\left[f(t)H(t-a)\right] = e^{-as}F(s+a)$

$$= e^{-s\pi/2} L\left[\cos(t+\pi/2)\right]-e^{-3\pi s/2} L\left[\cos(t+3\pi/2)\right]$$

$$= e^{-st/2} L\left[-\sin t\right]-e^{-3\pi s/2} L\left[\sin t\right]$$

$$= e^{-s\pi/2}\left(\frac{1}{s^2+1}\right)-e^{-3\pi s/2}\left(\frac{1}{s^2+1}\right)$$

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$$= \frac{-1}{s^2+1}\left[e^{\pi s/2} + e^{3\pi s/2}\right]$$

6

$$L\left[t^2 H(t-2) - \cos nt\ \delta(t-4)\right]$$

Now, $L\left[f(t)\,H(t-a)\right] = e^{-as}\,L\left[f(t+a)\right]$

$$L\left[f(t)\ \delta(t-a)\right] = e^{-as}\,f(a)$$

$$\therefore\ L\left[t^2 H(t-2)\right] = L\left[\cos nt\ \delta(t-4)\right]$$

$$= e^{-2s}\,L\left[t^2+2t+4\right] - e^{-4s}\cos n4$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cos h4$$

$\therefore$

$$L\left[t^2 H(t-2) - \cos nt\,\delta(t-4)\right] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cos n4$$

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Tutorial 5

Date

Roll no : 60004220126

FOURIER SERIES

1. Find fourier series for $f(x)$ in $(0, 2\pi)$

$$f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi \leq x < 2\pi \end{cases}$$

Here fourier series in $(0, 2\pi)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + b_n \sin nx$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} 2\pi - x\, dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{2} \right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2} \right)_{\pi}^{2\pi} \right]$$

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$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \left[\frac{2\pi^2}{2} - \frac{4\pi^2}{2} + 2\pi^2 \right]$$

$$= \pi^2 \times \frac{1}{\pi} = \pi$$

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$$a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cdot \cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x)(\cos nx)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{ \cancel{(x)\left(\frac{\sin nx}{n}\right)} - (1)\left(\frac{-\cos nx}{n^2}\right)\right\} + \left\{ \cancel{(2\pi - x)\left(\frac{\sin nx}{n}\right)} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)_0^{\pi} + \left(-\frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 - (+1) - (-1)^n\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{-4}{\pi n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cdot \sin nx \, dx + \int_{\pi}^{2\pi} 2\pi - x \cdot \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-x \cos nx}{n}\right)_0^{\pi} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi} + \left(\frac{(2\pi - x)\cos nx}{n}\right)_{\pi}^{2\pi} - \left[\frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}[0] = 0$$

fourier series of $f(x) =$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \cos nx$$

$$f(x) = \frac{\pi}{2}\left[\frac{-4}{\pi}\cos x - \frac{4}{\pi 9}\cos 3x - \frac{4}{\pi 25}\cos 5x \dots\right]$$

If $n =$ odd

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \dots\right]$$

If $n =$ even (Here $n =$ even)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

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2 Obtain fourier series.

$$f(x) = \begin{cases} \pi/2 + x & -\pi < x < 0 \\ \pi/2 - x & 0 < x < \pi \end{cases}$$

deduce in,

$\rightarrow \dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots$

$\rightarrow \dfrac{\pi^4}{96} = \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4}$

$\rightarrow$ put $x = -x$

$$f(-x) = \begin{cases} \pi/2 - x & -\pi < -x < 0 \\ \pi/2 + x & 0 < -x < \pi \end{cases}$$

$$\therefore \begin{cases} \pi/2 - x & +\pi > x > 0 \\ \pi/2 + x & 0 > x > -\pi \end{cases}$$

$\therefore f(x)$ is even $\therefore b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{2} - x \quad dx$$

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$$= \frac{2}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[+\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[-\left((-1)^n - 1\right)\right]$$

$$= -\frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{4}{\pi n^2}, & n \text{ is odd} \\ 0, & n \text{ is even,} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\cos nx \qquad \{\text{for even } f(x) = 0\}$$

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$$f(x) = \frac{4}{\pi}\cos x + \frac{4}{\pi 3^2}\cos 3x + \frac{4}{\pi 5^2}\cos 5x + \dots \quad \text{—(1)}$$

put $x = 0$

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} + \frac{4}{\pi 3^2} + \frac{4}{\pi 5^2} + \dots \quad [\text{from } (2)]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

By Parsvels Identity. for even function

$$\frac{2}{\pi}\int_0^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)^2 dx = \frac{16}{\pi^2} + \frac{16}{\pi^2 3^4} + \frac{16}{\pi^2 5^4} + \dots$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{4} + x^2 - \pi x\right)dx = \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right]$$

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$$\frac{\pi}{8}\left[\frac{\pi^3}{4} + \frac{\pi^3}{3} - \frac{\pi^3}{2}\right] = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^2} + \dots$$

Hence deduced.

3 Find half range cosine series for $f(x) = e^x$, $0 < x < 1$

$f(x) = e^x$ Half range cosine series in $(0,1)$ is given

where,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right)$$ here $l = 1$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 e^x\,dx = \left[2e^x\right]_0^1$$

$$= 2e^x - 2$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos\frac{n\pi x}{l}\,dx$$

$$= 2\int_0^1 e^x \cdot \cos(n\pi x)\,dx$$

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$$= 2\left[\frac{e^{x}}{1+n^2\pi^2}\left(\cos(n\pi x) + n\pi \sin(n\pi x)\right)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]$$

$$F(x) = (e-1) + \sum_{n=1}^{\infty} \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]\cos(n\pi x)$$

$$= (e-1) + 2\sum_{n=1}^{\infty} \frac{\left[(-1)^n e - 1\right]}{1+n^2\pi^2}\cos(n\pi x)$$

4 Obtain half rang sine series to represent.

$$f(x) = \begin{cases} 2x/3 & 0 \leq x \leq \pi/3 \\ \pi - x/3 & \pi/3 \leq x \leq \pi \end{cases}$$

$$F(x) = \sum_{n=1}^{\infty} bn \cdot \sin(nx)$$

where $bn = \frac{2}{\pi}\int_0^{\pi} f(x)\, \sin(nx)\, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/3} \frac{2x}{3}\sin(nx)\, dx + \int_{\pi/3}^{\pi}\left(\pi - \frac{x}{3}\right)\sin(nx)\, dx\right.$$

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$$= \frac{2}{\pi}\left[\frac{2}{3}\left(\frac{x\cos nx}{n}\right)_0^{\pi/3} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi/3}\right] + \frac{1}{3}\left[\frac{2\pi}{3}\frac{\cos n\pi}{3} - \left(\frac{\sin nx}{n^2}\right.\right.$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{-\pi}{3n}\cos\left(\frac{n\pi}{3}\right) + \frac{\sin n\pi/3}{n^2}\right] + \frac{1}{3}\left[\frac{2\pi}{3n}\frac{\cos n\pi}{3} + \frac{\sin n\pi/3}{n^2}\right.\right.$$

$$= \frac{2}{\pi}\left[\left(\frac{-2\pi}{9}\cos\left(\frac{n\pi}{3}\right) + \frac{2}{3n^2}\sin\left(\frac{n\pi}{3}\right)\right] + \frac{2\pi}{9}\cos\frac{n\pi}{3} + \frac{\sin n\pi/3}{n^2 3}\right.$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{\pi n}{3}\right)\right]$$

$\therefore$ Half range sin series is :-

$$F(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{3}\right)\sin(nx)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)\sin(nx)$$

5 Find fourier expansion of $f(x) = 4 - x^2$ in $(0,2)$. Also state the values for $x = 0, 1, 2, \frac{10}{8}, 1$, Hence deduce

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

here $2l = 2 \quad \therefore \quad l = 1$

Now, fourier series in $(0,2)$ is given by

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$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right) + b_n \cdot \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

where,

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\int_0^{2} 4 - x^2 \cdot dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left[8 - \frac{8}{3}\right]^2 = \frac{16}{6} = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^2 f(x) \cdot \cos(n\pi x/l)\,dx$$

$$= \frac{1}{1}\int_0^2 (4-x^2) \cdot (\cos n\pi x/l)\,dx$$

$$= 1\left[(4-x^2)\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - (-2x)\left(\frac{-\cos n\pi x/l}{n^2\pi^2/l^2}\right)\right.$$

$$= \frac{-2l^2}{n^2\pi^2}\left[x \cdot \cos\frac{n\pi x}{l}\right]_0^2$$

$$= \frac{-2}{n^2\pi^2}(2)(1) = \frac{-4}{n^2\pi^2}$$

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$$b_n = \frac{1}{l} \int_0^2 f(x) \sin n\pi x \, dx$$

$$= 1 \int_0^2 4 - x^2 \cdot \sin n\pi x \, dx$$

$$= \left[\frac{(4-x^2)(-\cos n\pi x)}{n\pi} - \cancel{\frac{(-2x)(-\sin n\pi x)}{n^2\pi^2}} \right]_0^2$$

$$= \frac{1}{n\pi} \left[4 \right] = \frac{4}{n\pi}$$

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2\pi^2} \right) \cos(n\pi x) + \frac{4}{n\pi} \sin(n\pi x)$$

$$f(x) = \frac{8}{3} + \frac{-4}{\pi^2} \left[1 \cos \pi x + \frac{1}{2^2} \cos 2\pi x + \frac{1}{3^2} \cos 3\pi x \ldots \right.$$

$$+ \frac{4}{\pi} \left[1 \sin \pi x + \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x \ldots \right.$$

put $x = 0$

$$4 = \frac{8}{3} - \frac{4}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right]$$

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$$-\frac{1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots\right.$$

$$-\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Put $x = 2$

$$0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

put $x = 10$

$$4 - 100 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$-\frac{246}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

put $x = 11$

$$4 - 121 - \frac{8}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{359}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

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$$\frac{2}{3} - \frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

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Tutorial 6 Date

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Complex Form of Fourier Series.

1 Find the complex form of fourier series of

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 1, & 1 < x < 2 \end{cases}$$

In the range $x \in (c, c+2l)$ complex form of fourier integral is given as.

$$f(x) = c_0 + \sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/l}, \quad \text{where } c_n = \frac{1}{2l}\int_c^{c+2l} f(x) e^{-in\pi x/l} dx$$

$$c_n = \frac{1}{2L}\int_c^{c+2L} f(x) e^{-in\pi x/l} dx \qquad \text{here } l = 1$$

$$= \frac{1}{2}\left[\int_0^1 x^2 \cdot e^{-in\pi x/l} dx + \int_1^2 e^{-in\pi x/l} dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{x^2 e^{-in\pi x}}{-in\pi} - \frac{2x e^{-in\pi x}}{(-in\pi)^2} + \frac{2e^{-in\pi x}}{(-in\pi)^3}\right]_0^1 + \left[\frac{e^{-in\pi x}}{-in\pi}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi}}{-in\pi} - \frac{2e^{-in\pi}}{(in\pi)^2} + \frac{2e^{-in\pi}}{(-in\pi)^3} - \frac{2}{(-in\pi)^3}\right] + \left[\frac{e^{-in\pi 2}}{-in\pi} - \frac{e^{-in\pi}}{-in\pi}\right]$$

$$c_n = \frac{1}{2}\left[\frac{2e^{-in\pi}}{n^2\pi^2} + \frac{2ie^{-in\pi} - 2i}{n^3\pi^3} + \frac{ie^{-i2n\pi}}{n\pi}\right]$$

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Now, $e^{-in\pi} = (-1)^n$ & $e^{-2in\pi} = +1$

$$C_n = \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i\left((-1)^n - 1\right)}{n^3\pi^3} + \frac{i}{n\pi}\right]$$

$$C_0 = \frac{1}{2L}\int_c^{c+2L} f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x^2\,dx + \int_1^2 1\,dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{x^3}{3}\right]_0^1 + \left[x\right]_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + 2 - 1\right]$$

$$= \frac{2}{3}$$

Substitute C_0 and C_n in equation

$$f(x) = \frac{2}{3} + \sum_{n=-\infty}^{n=\infty} \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i\left((-1)^n - 1\right)}{n^3\pi^3} + \frac{i}{n\pi}\right]$$

here $n \neq 0$ as numerator $\to \infty$ at $n = 0$

$$f(x) = 2/3 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} 1/2\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i\left((-1)^n - 1\right)}{n^3\pi^3} + \frac{i}{n\pi}\right]$$

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2 Show that set of functions $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ is orthogonal set on interval $0 \leq x \leq L$ and find corresponding orthonormal set.

Let $f_1(x) = \sin(n\pi x/L)$ $n = 1, 2, 3 \ldots$

& $f_2(x) = \sin(m\pi x/L)$ $m = 1, 2, 3 \ldots$

for the function $\phi_n(x)$ to be orthogonal to each other in interval $0 \leq x \leq L$ the inner product i.e $(f_1(x), f_2(x))$ should be 0.

i.e $[f_1(x)\, f_2(x)] = \int_0^L f_1(x)\, f_2(x)\, dx = 0$

so,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \quad 0 \qquad \text{Here } m \neq n$$

$$= \int_0^L \frac{1}{2} \times \left[\cos\left(\frac{(m-n)\pi x}{L}\right)\right] - \left[\cos\left(\frac{(m+n)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)\pi x/L)}{\pi(m-n)/L} - \frac{\sin((m+n)\pi x/L)}{\pi(m+n)/L}\right]_0^L$$

$$= \frac{1}{2}[0 - 0] = 0$$

Hence the given set of functions is orthogonal in $0 \leq x \leq L$ as inner product is zero.

Now, their corresponding orthonormal sets.

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$$\left[\sin\left(\frac{n\pi x}{l}\right)\right]^2 = \int_0^L \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2 dx$$

$$= \int_0^L \left[\frac{1 + \cos\left(2n\pi x/L\right)}{2}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2n\pi x/L}{2n\pi/L}\right]_0^L$$

$$= \frac{1}{2}\left[L - 0\right]$$

$$= L/2$$

Corrusponding orthonormal set

$$\left[\sqrt{\frac{2}{l}} \quad \phi_n x\right]$$

3 Find fourier integral representation of function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

fourier integral representation of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(u) \cos \lambda (u - x)\, du\, d\lambda$$

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$$f(x) = \frac{1}{\pi}\int_0^\infty \Big\{ \int_{-\infty}^{\infty} \cancel{f(u)}\; 0\, du + \int_{-\infty}^{\infty} e^{-u} \cos\lambda(u-x)\, du + 0 \Big\}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \Big\{ 0 + \int_0^\infty e^{-u} (\cos\lambda u \cos\lambda x + \sin\lambda u \sin\lambda x)\, du + 0 \Big\}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \Big\{ \int_0^\infty e^{-u} \cos\lambda u \cos\lambda x\, du + \int_0^\infty e^{-u} \sin\lambda u \sin\lambda x\, du \Big\}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \Big\{ \Big[\frac{\cos\lambda x\, e^{-u}}{1+\lambda^2} \big[-\cos\lambda u + \lambda\sin\lambda u\big] \Big]_0^\infty - \Big[\frac{\sin\lambda x}{1+\lambda^2}\, e^{-u} \big[-\sin\lambda u - \lambda\cos\lambda u\big] \Big]_0^\infty \Big\}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2} + \frac{\sin\lambda x(\lambda)}{1+\lambda^2}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\, d\lambda$$

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Tutorial 7

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FOURIER TRANSFORM, FOURIER SINE & COSINE TRANSFORM

1 Find Fourier transform of

$$f(x) = \begin{cases} 1 + \frac{x}{a}, & -a < x < 0 \\ 1 - \frac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

put $x = -x$ in defination of $f(x)$

$$f(-x) = \begin{cases} 1 + \frac{x}{a} & -a < x < 0 \\ 1 - \frac{x}{a} & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

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$\because f(x) = f(-x) \Rightarrow$ The function $f(x)$ is even

$\therefore$ Function is even, the fourier transform of $f(x)$ = fourier cosine transform of $f(x)$ denoted by $F_c(\alpha)$

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$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \alpha t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{a} \left(1 - \frac{t}{a}\right) \cos \alpha t + \int_0^{\infty} \cancel{0}^{0} \, dt \right.$$

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$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \cos \alpha b \, dt - \frac{1}{a} \int_0^a t \cdot \cos \alpha t \, dt \right]$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{\sin \alpha t}{\alpha} \right\}_0^a - \frac{1}{a} \left\{ \frac{t \cdot \sin \alpha t}{\alpha} - \left(- \frac{\cos \alpha t}{\alpha} \right) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin a \alpha}{\alpha} - \frac{\sin a \alpha}{\alpha} - \frac{\cos a \alpha}{a \alpha^2} + \frac{1}{a \alpha^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a \alpha^2} \right] (1 - \cos a \alpha)$$

but,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

Taking $\sqrt{\frac{2}{\pi}}$ with $F_c(\alpha)$

$$F_c(\alpha) = \frac{2}{\pi} \alpha \frac{1}{a r^2} \times 2 \sin^2 \left(\frac{\alpha}{2} \right)$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{4}{a r^2} \sin^2 \left(\frac{\alpha}{2} \right) \cos \alpha x \, d\alpha$$

$$F_c(\alpha) = \frac{4}{a r^2} \sin^2 \left(\frac{\alpha}{2} \right)$$

2 Find the fourier sine and cosine transform of

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - \pi & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

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→ let the fourier cosine and sine transforms be denoted by $F_c(\alpha)$ and $F_s(\alpha)$ respectively.

Now, using formula for $F_c(\alpha)$ we get,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos\alpha t\,dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \cos\alpha t\,dt + \int_1^2 (2-t)\cos\alpha t\,dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{t\sin\alpha t}{\alpha} - \int \frac{\sin\alpha t}{\alpha} \right]_0^1 \right.$$

$$+ \left[\frac{2\sin\alpha t}{\alpha} - \left\{ \frac{t\sin\alpha t}{\alpha} - \int \frac{\sin\alpha t}{\alpha} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha} - \frac{1}{\alpha^2} + \left[\frac{2\sin 2\alpha}{\alpha} - \frac{2\sin 2\alpha}{\alpha} - \frac{\cos 2\alpha}{\alpha^2} \right. \right.$$

$$\left. \left. \left(\frac{2\sin\alpha}{\alpha} - \frac{\sin\alpha}{\alpha} - \frac{\cos\alpha}{\alpha} \right) \right] \right]$$

$$= F_c(\alpha) = \sqrt{\frac{2}{\pi}} \left\{ \frac{2\cos\alpha - 1 - \cos 2\alpha}{\alpha^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{2\cos\alpha - 2\cos^2\alpha}{\alpha^2} \right\}$$

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$$= 2\sqrt{\frac{2}{\pi}} \frac{\cos\alpha(1-\cos\alpha)}{\alpha^2}$$

However, $f(x) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha x\, d\alpha$. $\frac{2}{\pi}$ were distributed below inner and outer integral combining $\sqrt{2/\pi}$ with outer we get.

$$F_c(\alpha) = \frac{2}{\alpha^2}\cos\alpha(1-\cos\alpha)$$

also, $F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\alpha t\, dt$.

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cdot\sin\alpha t\,dt + \int_1^2 (2-x)\sin\alpha t\,dt\right]$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left\{\left[\frac{-x\cdot\cos\alpha t}{\alpha} - \int_0 -\frac{\cos\alpha t}{\alpha}\right]_0^1 + \left[\frac{(2-x)(\cos\alpha t)}{\alpha} - \int \frac{(-1)(-\cos\alpha t)}{\alpha}\right]\right.$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left\{\frac{-\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2} - 0 + \left\{0 - \frac{\sin 2\alpha}{\alpha^2} - \left[\frac{\cos\alpha}{\alpha} - \frac{\sin t}{\alpha^2}\right]\right\}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin\alpha}{\alpha^2} + \frac{\sin 2\alpha}{\alpha^2}\right]$$

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$$= \sqrt{\frac{2}{\pi}} \frac{\sin\alpha}{\alpha^2} (1 + \cos\alpha)$$

But $\sqrt{\frac{2}{\pi}}$ is part of outer integral

$$F_s(\alpha) = \frac{2}{\alpha^2} \sin\alpha (1 + \cos\alpha)$$

3. Find the fourier sine and cosine transform of i) x^{m-1} ii) $\frac{1}{\sqrt{x}}$

let $F_s(\alpha)$ be the fourier transform.

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\alpha x \, dx$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} \sin\alpha \, dx$$

now, $\sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} [\cos\alpha t - i\sin\alpha t] \, dt$

using $e^{-i\alpha t} = \cos\alpha t - i\sin\alpha t$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) - F_s(\alpha) \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) [\cos\alpha t - i\sin\alpha t] \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty t^{m-1} e^{i\alpha t}$$

let $i\alpha t = s$ $\quad dt = \frac{ds}{i\gamma}$

t	0	∞
s	S	∞

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$$= \sqrt{\frac{2}{\pi}} = \int_0^\infty \left(\frac{s}{i\alpha}\right)^{m-1} e^{-s} \frac{ds}{i\alpha}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{(ir)^m} \int_0^\infty e^{-s} s^{m-1} ds.$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{i r^m}\right) \Gamma m$$

$$= \sqrt{\frac{2}{\pi}} \frac{\Gamma m}{r^m} \left(\cos \frac{m\pi}{2} - i \sin \frac{\pi}{2}\right)$$

real port $= F_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sqrt{m}}{r^m} \cos \frac{m\pi}{2}$

imaginery part $= F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\Gamma m}{2^m} \sin \frac{m\pi}{2}$

put $m = \frac{1}{2}$ for $f(x) = \frac{1}{\sqrt{2}}$

$$F_c(r) = \sqrt{\frac{2}{\pi}} \frac{\Gamma m}{\alpha^{1/2}} \cos \frac{\pi}{9} = \frac{1}{\sqrt{\alpha}}$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\Gamma 1/2}{r^{1/2}} \sin \frac{\pi}{9} = \frac{1}{\sqrt{\alpha}}$$

Tutorial 8

Date :

SAP : 60004220126

Z - TRANSFORM

1 Find $z\{c^k \sin\alpha k\}$ from $z\{\sin\alpha k\}$, $k>0$

For z-transform of $\sin\alpha k$

$$z\{\sin\alpha k\} = z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, e^{-k}$$

Assuming $\sin\alpha k = 0$ from $(-\infty, -1)$, $k>0$ defined from $(0,\infty)$

$$z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \sin\alpha k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} - e^{-i\alpha k}}{2} \right) z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{e^{i\alpha k}}{z^k} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{e^{-i\alpha k}}{z^k}$$

$$= \frac{1}{2} \left[\sum_{k=0}^{\infty} \left(\frac{e^{i\alpha}}{z} \right)^k - \sum_{k=0}^{\infty} \left(\frac{e^{-i\alpha}}{z} \right)^k \right]$$

2

$$= \frac{1}{2} \left[\frac{1}{1 - e^{i\alpha}/z} - \frac{1}{1 - e^{-i\alpha}/z} \right]$$

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$$= \frac{1}{2} \left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}} \right]$$

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$$= \frac{1}{2}\left[\frac{z^2 - ze^{-i\alpha} - z^2 + ze^{i\alpha}}{z^2 - ze^{-i\alpha} - ze^{i\alpha} + 1}\right]$$

$$= \frac{1}{2}\left[\frac{z\left(e^{i\alpha} - e^{-i\alpha}\right)}{z^2 - ze^{i\alpha} - ze^{-i\alpha} + 1}\right]$$

$$= \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$F(z) = z\{\sin\alpha k\} = \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1} \quad \text{——} \; (1)$$

For $z\{c^k \sin\alpha k\}$,

By change of scalar property using (1)

2 $f(k) = \{ke^{-ak}\}, \quad k \geq 0$

By theorum,

$$z\{f(k)\} = f(z) = \sum_{k=-\infty}^{\infty} F(k) z^{-k}$$

$$z\{e^{-ak} f(k)\} = \sum e^{-ak} f(k) z^{-k}$$

$$= \sum f(k) (e^a z)^{-k}$$

$$= f(e^a z)$$

We know that $U(k) = 1 \quad k > 0$ then $z\{U(k)\} = z/z-1$

By above theorum $z\{e^{-ak} U(k)\} = z\{e^{-ak}\} = \dfrac{e^a z}{e^a z - 1}$

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Using multiplication by t property

$$z\{ke^{-ak}\} = -z \frac{d}{dz}\left(\frac{e^a z}{e^a z - 1}\right)$$

$$= -z\left[\frac{(e^a z - 1)e^a - z(e^a)(e^a)}{(e^a z - 1)^2}\right]$$

$$= -ze^a\left[\frac{(e^a z - 1) - ze^a}{(e^a z - 1)^2}\right]$$

$$= \frac{e^a z}{(e^a z - 1)^2}$$

$$z\{ke^{-ak}\} = \frac{e^a z}{(e^a z - 1)^2}$$

3 $z\{{}^nC_k\}$ $0 \leq k \leq n$

$f(k) = {}^nC_k$, finite fourier transform of $f(x)$ is given by

$$z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} {}^nC_k z^{-k}, \quad k[0, n]$$

$$= {}^nC_0 + {}^nC_1 \frac{1}{z} + {}^nC_2 \frac{1}{z^2} + {}^nC_3 \frac{1}{z^3} + \ldots$$

$$= \left(1 + \frac{1}{z}\right)^n$$

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The series being finite is convergent if $x \neq 0$

$$z\left\{{}^{n}C_{k}\right\} = \left(1 + \frac{1}{z}\right)^{n}$$

ROC is all z except the origin.

4 $f(t) = \dfrac{2t^2 - 10t + 13}{(t-3)^2\,(t-2)}$, $2|t| < 3$

Given $f(t) = \dfrac{2t^2 - 10t + 13}{(t-3)^2\,(t-2)}$

using partial fractions.

$$\frac{2t^2 - 10t + 13}{(t-3)^2\,(t-2)} = \frac{A}{(t-3)} + \frac{B}{(t-3)^2} + \frac{C}{(t-2)}$$

$$2t^2 - 10t + 13 = A(t-3)^2(t-2) + B\cancel{(t-3)}(t-2) + C(t-3)^2$$

we get $A = B = C = 1$

$$f(t) = \frac{1}{t-2} + \frac{1}{t-3} + \frac{1}{(t-3)^2} = \frac{2t^2 - 10t + 13}{(t-3)^2\,(t-2)}$$

$\therefore$ $2 < |t| \rightarrow \left|\frac{2}{t}\right|$

$|t| < 3 \Rightarrow \left|\frac{t}{3}\right| <$

$$f(t) = \frac{1}{t}\left[\frac{1}{1 - 2/t}\right] + \frac{1}{3}\left[\frac{1}{t/3 - 1}\right] + \frac{1}{9}\left[\frac{1}{(t/3 - 1)^2}\right]$$

$$= \frac{1}{t}\left[1 - \frac{2}{t}\right]^{-1} - \frac{1}{3}\left(1 - \frac{t}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{t}{3}\right)^{-2}$$

$$= \frac{1}{t}\left[1 + \frac{2}{t} + \frac{2^2}{t^2} \ldots\right] - \frac{1}{3}\left[1 + \frac{t}{3} + \frac{t^2}{3^2} + \ldots\right] + \frac{1}{9}\left[1 + 2\left(\frac{t}{3}\right) + 3\left(\frac{t^2}{3^2}\right) \ldots\right]$$

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1st series $z^{-k} = 2^{k-1}$

2nd & 3rd series $\rightarrow \dfrac{k+1}{3^{k+2}} - \dfrac{1}{3^{k+1}} = \dfrac{k-2}{3^{k+2}}, \quad k \geq 0$

Hence coefficient of $z^{-k} = \dfrac{-k-2}{3^{-k+2}}, \quad k \leq 0$

Hence, $z^{-1}\left[F(z)\right] = \left\{2^{k-1}\right\}, \quad k \geq 1$

$$= \left\{\frac{-k-2}{3^{-k+2}}\right\}, \quad k \leq 0$$

$z\left\{c^k \sin \alpha k\right\} = F(3/c)$

$$= \frac{(z/c)\sin\alpha}{(z/c)^2 - 2(z/c)\cos\alpha + 1}$$

$$= \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + z^2}, \quad \text{where } |z| > |c|$$

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