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Change of Scalar Property :

Theorem : For any non zero real number a

(i) $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$

(ii) $\mathcal{F}_s[f(ax)] = \frac{1}{a} F_s\left(\frac{\alpha}{a}\right) \quad a>0$

(iii) $\mathcal{F}_c[f(ax)] = \frac{1}{a} F_c\left(\frac{\alpha}{a}\right) \quad a>0$

proof :-

(i) If $a>0$

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{i\alpha x} dx$$

put $ax = t$

$dx = \frac{1}{a} dt$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{a} \int_{-\infty}^{\infty} f(t) e^{i\alpha \frac{t}{a}} dt$$

$$= \frac{1}{a} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\frac{\alpha}{a}(t)} dt \right)$$

$$= \frac{1}{a} F\left(\frac{\alpha}{a}\right)$$

If $a<0$

$$\mathcal{F}[f(ax)] = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{\alpha}{a}\right)t} \frac{dt}{a}$$

$$= \frac{-1}{a} F\left(\frac{\alpha}{a}\right)$$

$$= \frac{1}{|a|} F\left(\frac{\alpha}{a}\right)$$

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Modulation Property

Theorem:

(i) $\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a) + F(\alpha-a)]$

(ii) $\mathcal{F}_s[f(x)\cos ax] = 1/2[F_s(\alpha+a) + F_s(\alpha-a)]$

(iii) $\mathcal{F}_c[f(x)\cos ax] = \frac{1}{2}[F_c(\alpha+a) + F_c(\alpha-a)]$

(iv) $\mathcal{F}_s[f(x)\sin ax] = \frac{1}{2}[F_c(\alpha-a) - F_c(\alpha+a)]$

(v) $\mathcal{F}_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+\alpha) + F_s(a-\alpha)]$

Proof :- (i) $\mathcal{F}[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{i\alpha x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{i\alpha x}\,dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\,e^{i(\alpha+a)x}\,dx + \int_{-\infty}^{\infty} f(x)\,e^{i(\alpha-a)x}\,dx\right]$$

$$= \frac{1}{2}\cancel{\frac{1}{\sqrt{2\pi}}}[F(\alpha+a) + F(\alpha-a)]$$

(ii) $\mathcal{F}_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\,\sin\alpha x\,dx$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^{\infty} f(x)\left(\sin(\alpha+a)x + \sin(\alpha-a)x\right)dx$$

$$= \frac{1}{2}\left(\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\alpha+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\alpha-a)\,dx\right)$$

$$= \frac{1}{2}[F_s(\alpha+a) + F_s(\alpha-a)]$$

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Problem :-

Que① Find fourier transform of $\cos\left(\frac{x^2}{2}\right)$ & $\sin\left(\frac{x^2}{2}\right)$

→ $F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{\alpha^2}{4a^2}}$

put $-a^2 = \frac{i}{2}$

$\therefore a^2 = -\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

$\therefore a = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ using Demoiver's Theorem

$a = \frac{1}{2}(1-i)$

$\therefore F\left[e^{\frac{ix^2}{2}}\right] = \frac{1}{\frac{1}{2}(1-i)\sqrt{2}} \cdot e^{\frac{-\alpha^2}{4(-i/2)}}$

$= \frac{\sqrt{2}}{2}(1+i)\, e^{-i\alpha^2/2}$

$= \frac{(1+i)}{\sqrt{2}} e^{-\frac{i\alpha^2}{2}}$

$\therefore F\left[\cos\left(\frac{x^2}{2}\right) + i\sin\left(\frac{x^2}{2}\right)\right] = \left(\frac{1+i}{\sqrt{2}}\right)\left(\cos\frac{\alpha^2}{2} - i\sin\frac{\alpha^2}{2}\right)$

$\therefore F\left[\cos\left(\frac{x^2}{2}\right)\right] + iF\left[\sin\left(\frac{x^2}{2}\right)\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} + \sin\frac{\alpha^2}{2}\right] + \frac{i}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - \sin\frac{\alpha^2}{2}\right]$

$F\left[\cos\left(\frac{x^2}{2}\right)\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} + \sin\frac{\alpha^2}{2}\right]$

$F\left[\sin\left(\frac{x^2}{2}\right)\right] = \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha^2}{2} - \sin\frac{\alpha^2}{2}\right]$

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Que ② Evaluate Fourier Transform of $e^{-4x^2}\cos 4x$.

$\Rightarrow$

$$\mathcal{F}[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{\alpha^2}{4a^2}}$$

put $a=2$

$$\mathcal{F}[e^{-4x^2}] = \frac{1}{2\sqrt{2}} e^{-\frac{\alpha^2}{16}}$$

using modulation property

$$\mathcal{F}[e^{-4x^2}\cos 4x] = \frac{1}{2}\left[F(\alpha+4) + F(\alpha-4)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\sqrt{2}} e^{-\frac{(\alpha+4)^2}{16}} + \frac{1}{2\sqrt{2}} e^{-\frac{(\alpha-4)^2}{16}}\right]$$

$$= \frac{1}{4\sqrt{2}}\left[e^{-\left(\frac{\alpha+4}{4}\right)^2} + e^{-\left(\frac{\alpha-4}{4}\right)^2}\right]$$

Multiplication by x :-

Theorem:

(i) $\mathcal{F}[x^n f(x)] = (-i)^n \frac{d^n}{d\alpha^n} F(\alpha)$

(ii) $\mathcal{F}_s[x f(x)] = -\frac{d}{d\alpha} F_c(\alpha)$

(iii) $\mathcal{F}_c[x f(x)] = \frac{d}{d\alpha} F_s(\alpha)$

Proof :-

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$\frac{d^n}{d\alpha^n} F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{\partial^n}{\partial\alpha^n}\left(e^{i\alpha x}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ (ix)^n\ e^{i\alpha x}\ dx$$

$$\frac{d^n}{d\alpha^n} F(\alpha) = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, x^n\ e^{i\alpha x}\ dx$$

$$\frac{d^n}{d\alpha^n} F(\alpha) = i^n\ \mathcal{F}\left[x^n f(x)\right]$$

$$\therefore \mathcal{F}\left[x^n f(x)\right] = \frac{1}{i^n} \frac{d^n}{d\alpha^n} F(\alpha)$$

$$\therefore \mathcal{F}\left[x^n f(x)\right] = (-i)^n \frac{d^n}{d\alpha^n} F(\alpha)$$

(iii)

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\ \cos\alpha x\ dx$$

$$\frac{d}{d\alpha} F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\ (-\sin\alpha x)\ (x)\ dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x f(x)\ (\sin\alpha x)\ dx$$

$$= -\mathcal{F}_s\left[x f(x)\right]$$

$$\therefore \mathcal{F}_s\left[x f(x)\right] = -\frac{d}{d\alpha} F_c(\alpha)$$