Posets

Partially ordered relation: A relation R on a set A is called partial order if R is reflexive, antisymmetric & transitive.

The set A together with the partial order R is called a **partially ordered set** or simply a poset. It is denoted by (A, R).

Example

1. Let A be a set of positive integers and let R be a binary relation such that (a, b) is in R if a divides b.

Since any integer divides itself. R is reflexive.

Since a divides b means b does not divide a unless $a=b$, R is an antisymmetric relation.

Since a divides b, b divides c, then a divides c, so R is transitive.

Consequently, R is a partial ordered relation.

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2. Let Z^+ be the set of positive integers. The relation '$\leq$' is a partial order on Z^+ because for any element x.

i) $x \leq x$

ii) if $x \leq y$ & $y \leq x$, then $y = x$.

iii) If $x \leq y$ & $y \leq z$, then $x \leq z$

Dual of Poset

Let R be a partial order on a set A, and let R^{-1} be the inverse relation of R.

Then R^{-1} is also a partial order.

The poset (A, R^{-1}) is called the dual of the poset (A, R) and the partial order R^{-1} is called the dual of the partial order R.

Hasse Diagram

A graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing upward is known as the "Hasse Diagram" of the relation.

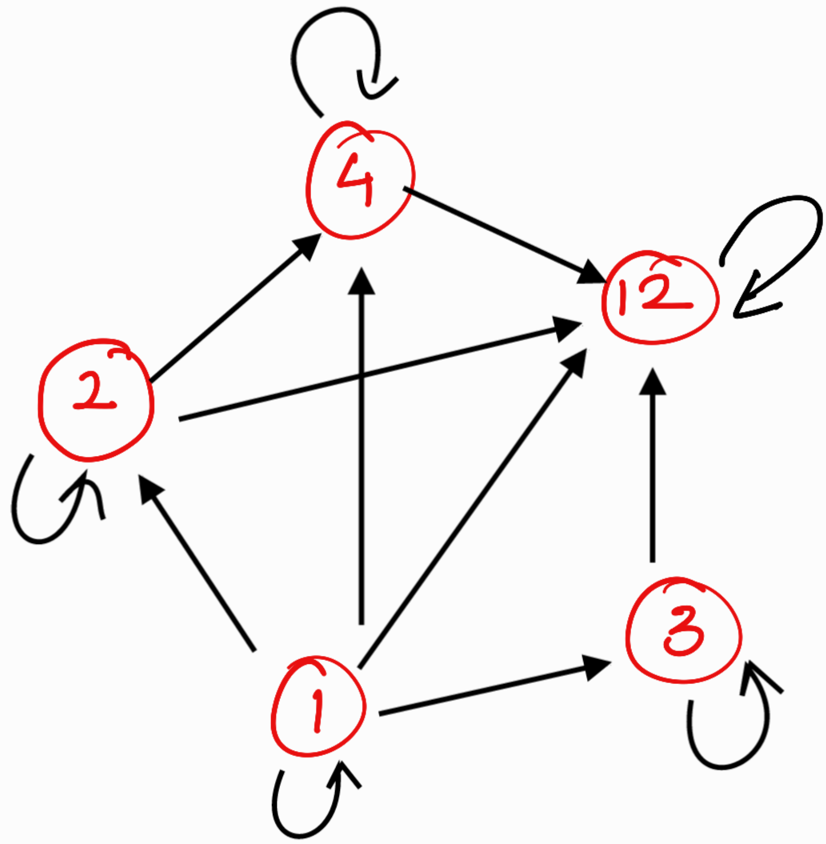
Solved Example

Draw all Hasse Diagrams of posets with three elements.

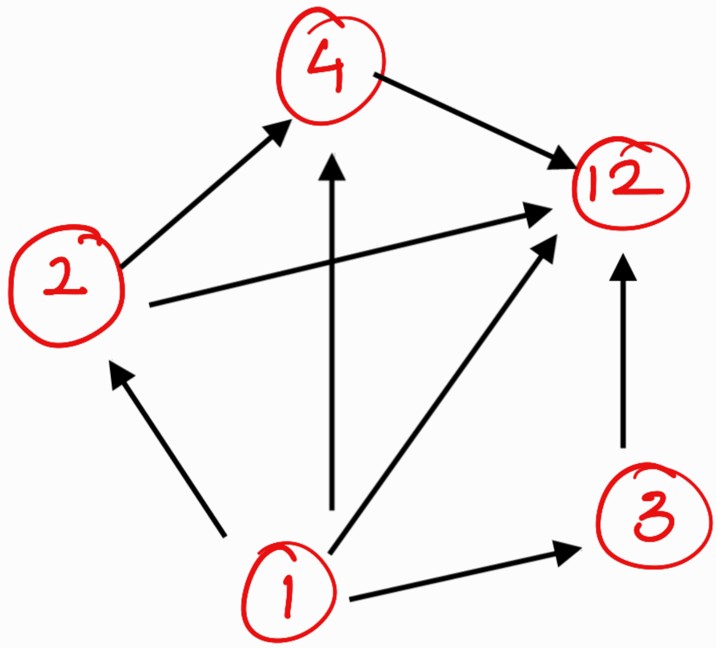
Draw Hasse diagram for the following relations on set $A = \{1, 2, 3, 4, 12\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (12,12), (1,2), (4,12), (1,3), (1,4), (1,12), (2,4), (2,12), (3,12)\}$

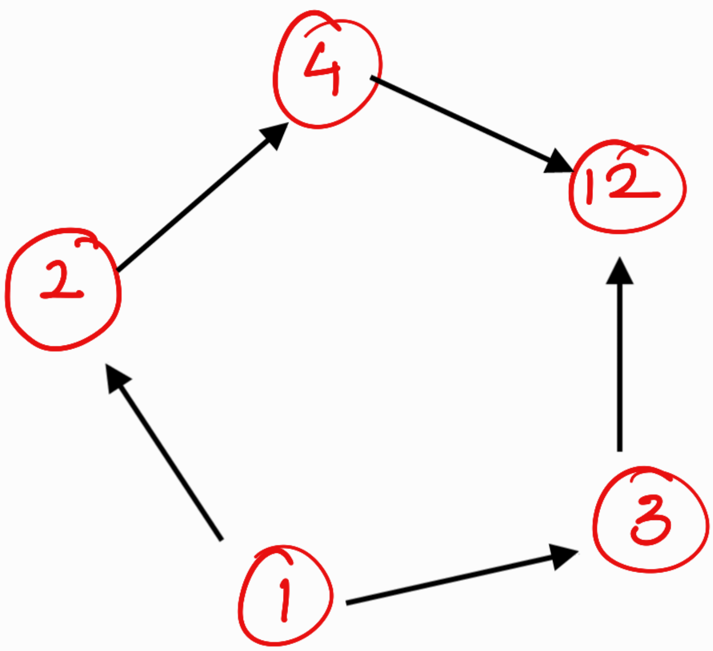
Sol: Digraph



Step 1: Remove Cycle



Step 2: Remove transitive edge

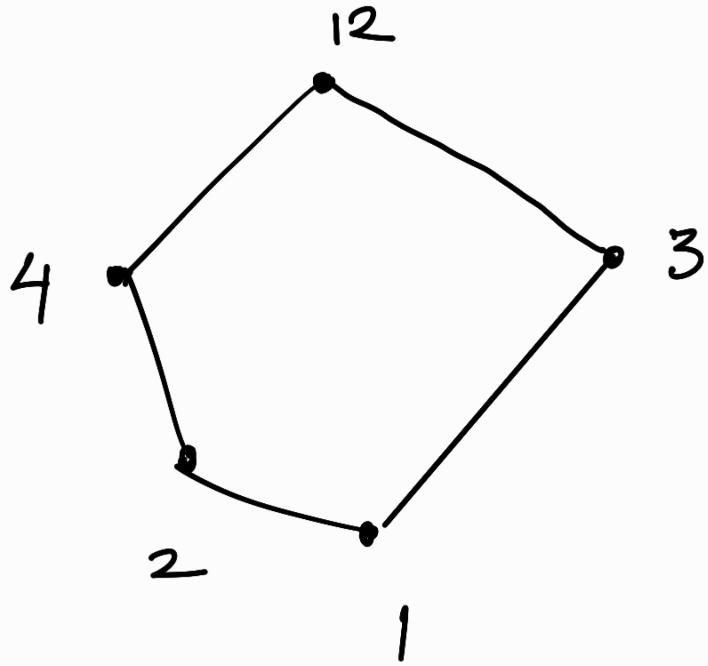


$1R2$, $2R4$ $\therefore$ $1R4$

$2R4$, $4R12$ $\therefore$ $2R12$

$1R4$, $4R12$ $\therefore$ $1R12$

Step 3 Circles are replaced by dots. Arrows are also removed.

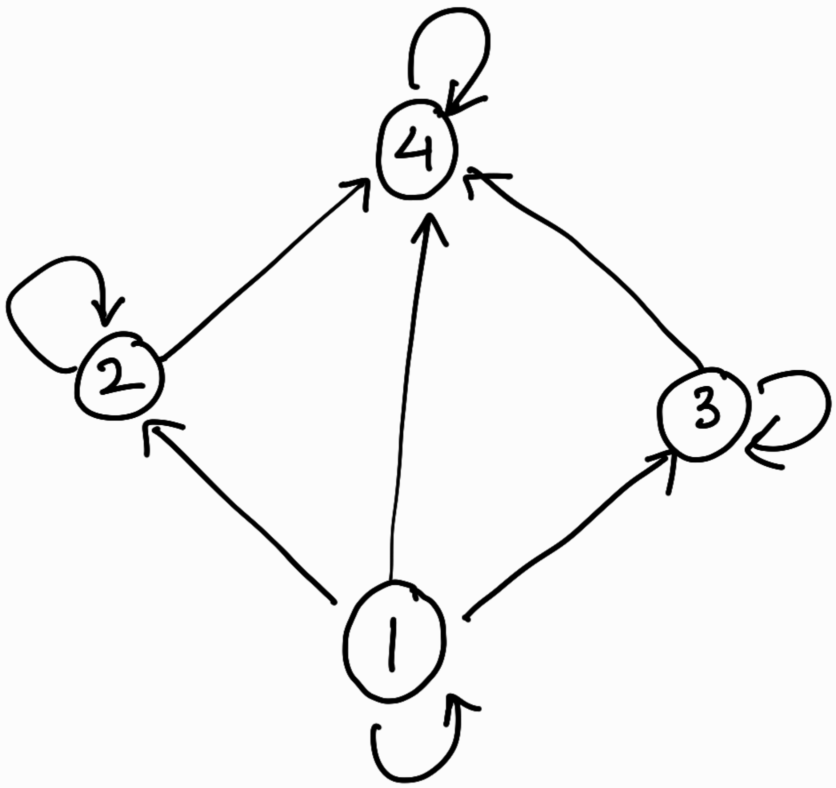


Q.2. Determine the Hasse Diagram of the relation R

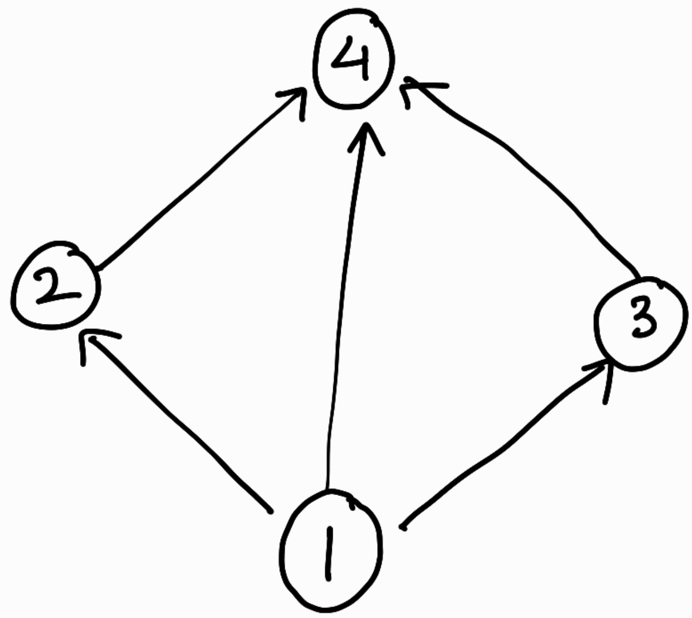
i) $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$

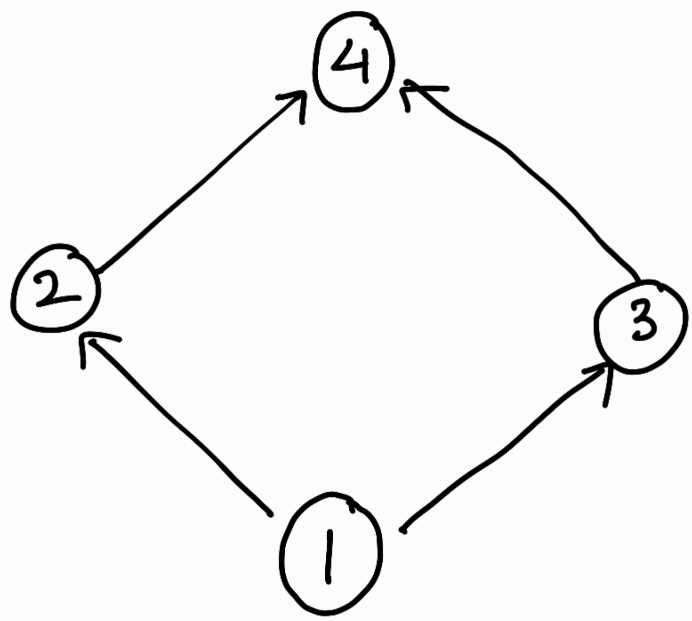
<u>Sol:</u> Digraph for the given relation set is



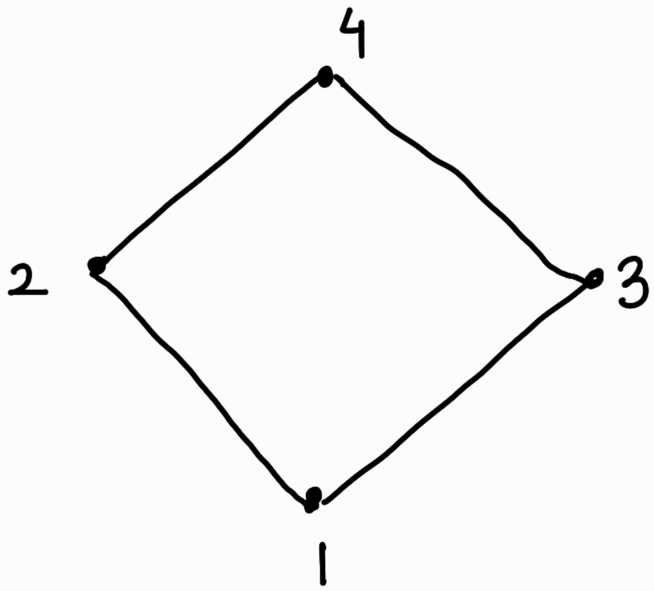
1) Remove Cycles



2) Remove Transitive Edge (1,4)



3.) Make sure that all edges are pointing upwards, then remove arrows from edges, replace circles by dots.

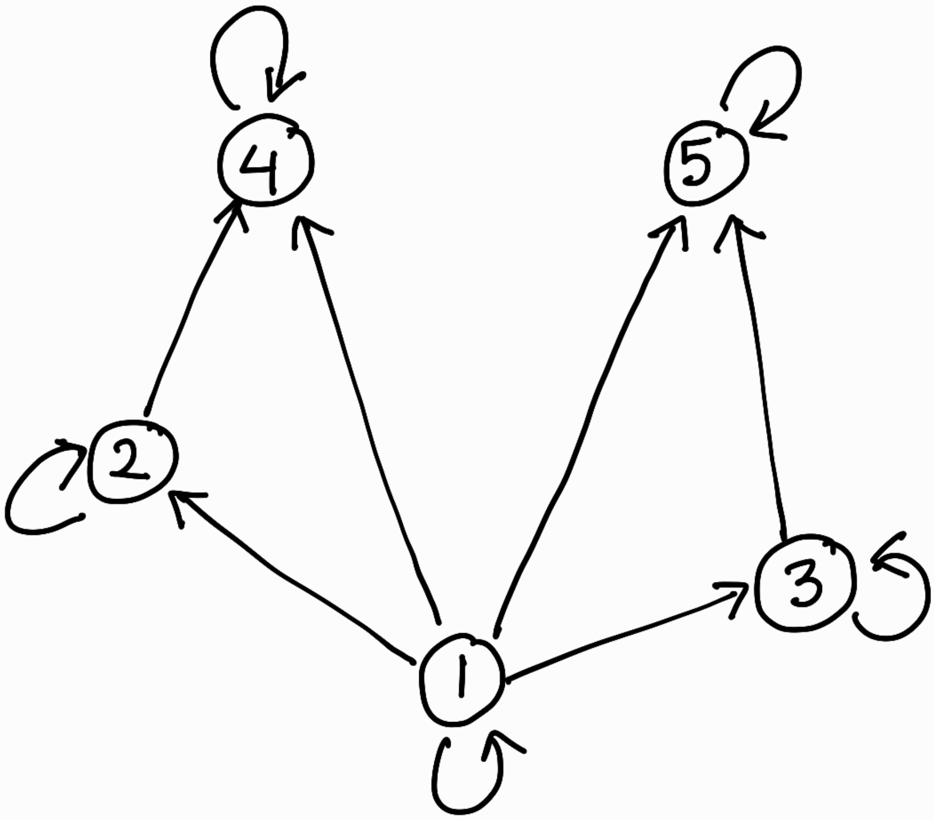


3. Determine the Hasse Diagram of the relation R

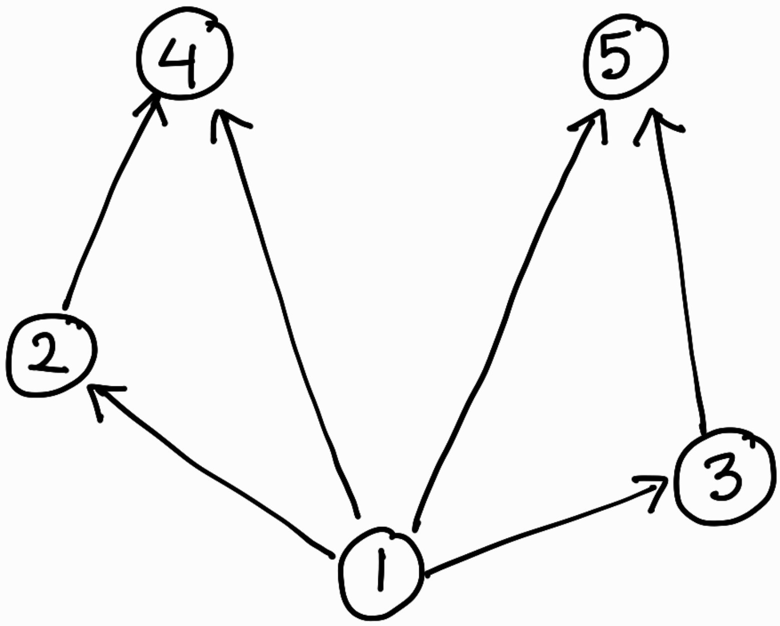
i) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,4), (3,5), (2,2), (3,3), (4,4), (5,5)\}$

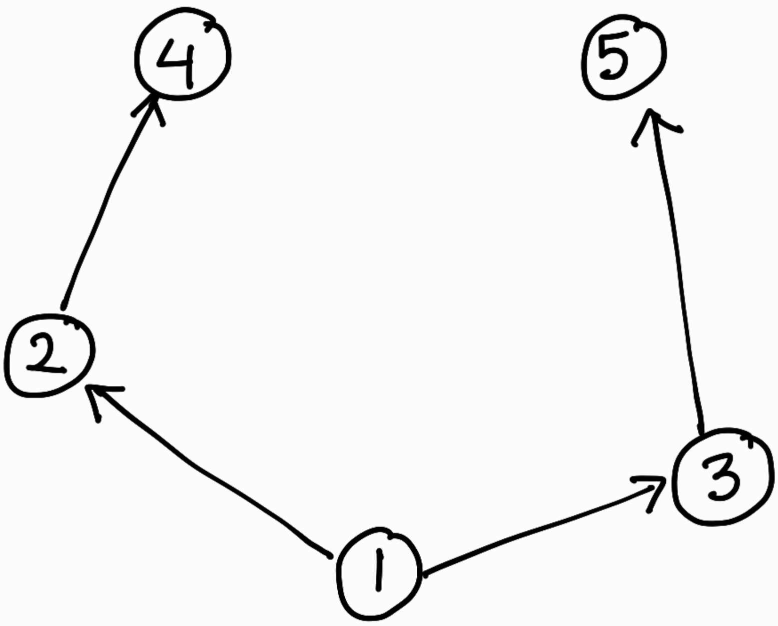
Sol. Digraph of the given relation set is



1. Remove Cycles

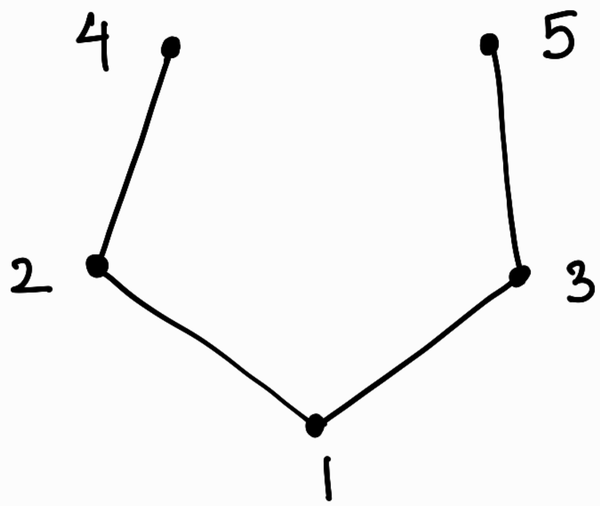


2. Remove Transitive edges (1,4), (1,5).



3. All edges are pointing upwards, remove arrows from edges, replace circles with dots.

Hasse Diagram



Q. 4. Let $A = \{a, b, c, d\}$ & $\times$ be a relation on A, whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i) Prove that R is partial order

ii) Draw Hasse Diagram of R.

<u>Sol:</u> $R = \{(a,a), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$

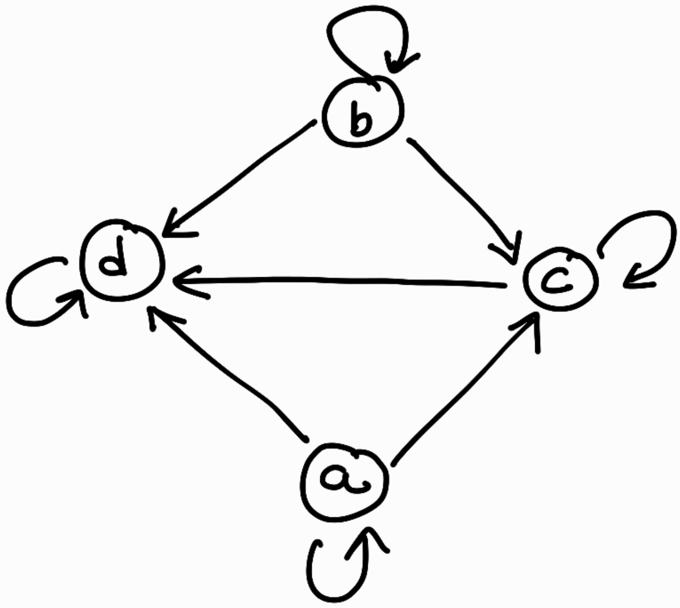
We can see that R is <u>reflexive</u> as $(a,a), (b,b), (c,c), (d,d) \in R$

R is also <u>antisymmetric</u> because it contains a and b such that if $a \neq b$, then $a \not R b$ or $b \not R a$.

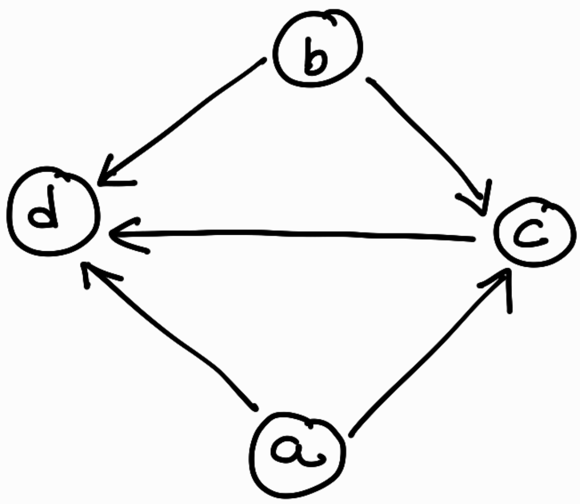
Also, R is <u>transitive</u> since it contains (a,d) & (b,d).

$\therefore$ <u>R is partial order</u>

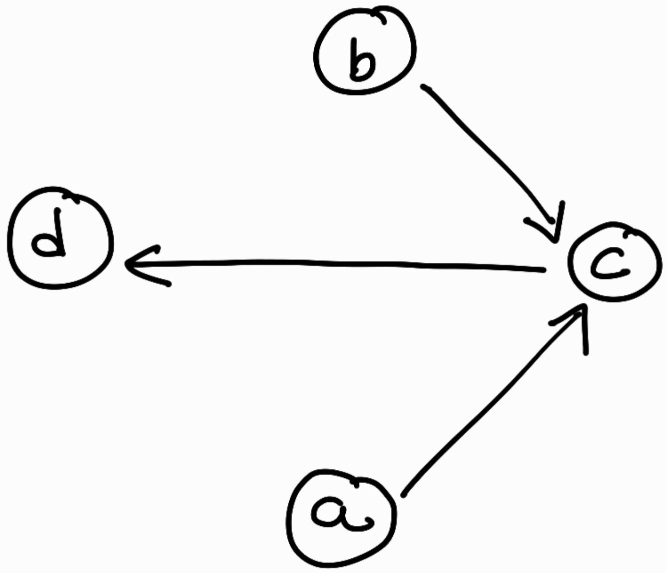
ii) Diagraph of the given relation is given below :-



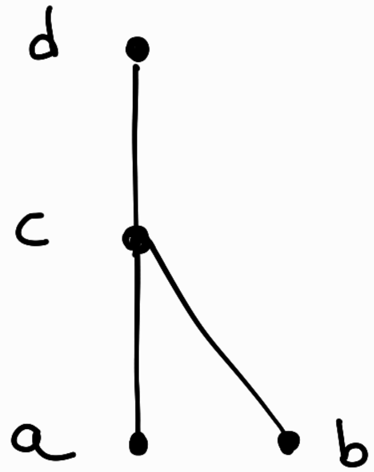
<u>Step 1</u>. Remove the cycles.



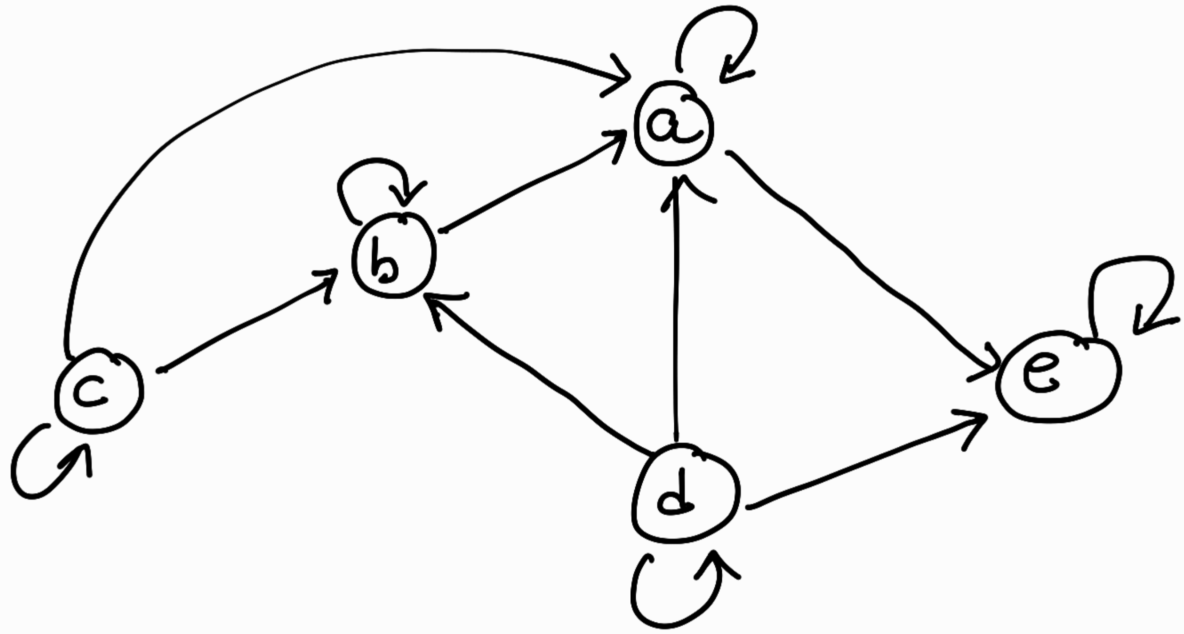
Step 2: Remove transitive edges (a,d) & (b,d)



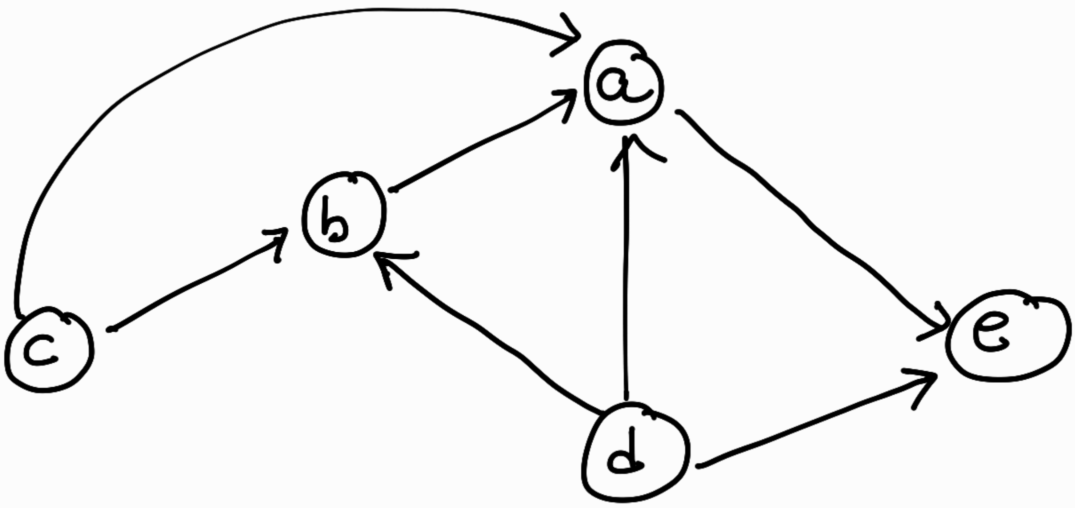
Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Q.5. Determine the Hasse Diagram of partial order having the given digraph.

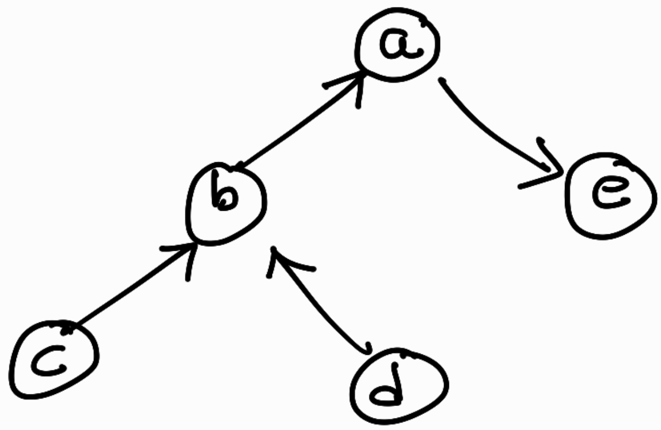


Sol: Step 1. Remove Cycles

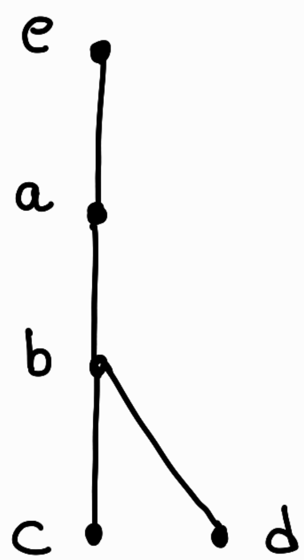


Step 2: Remove transitive edges

(c, a), (d, a), (d, e) .



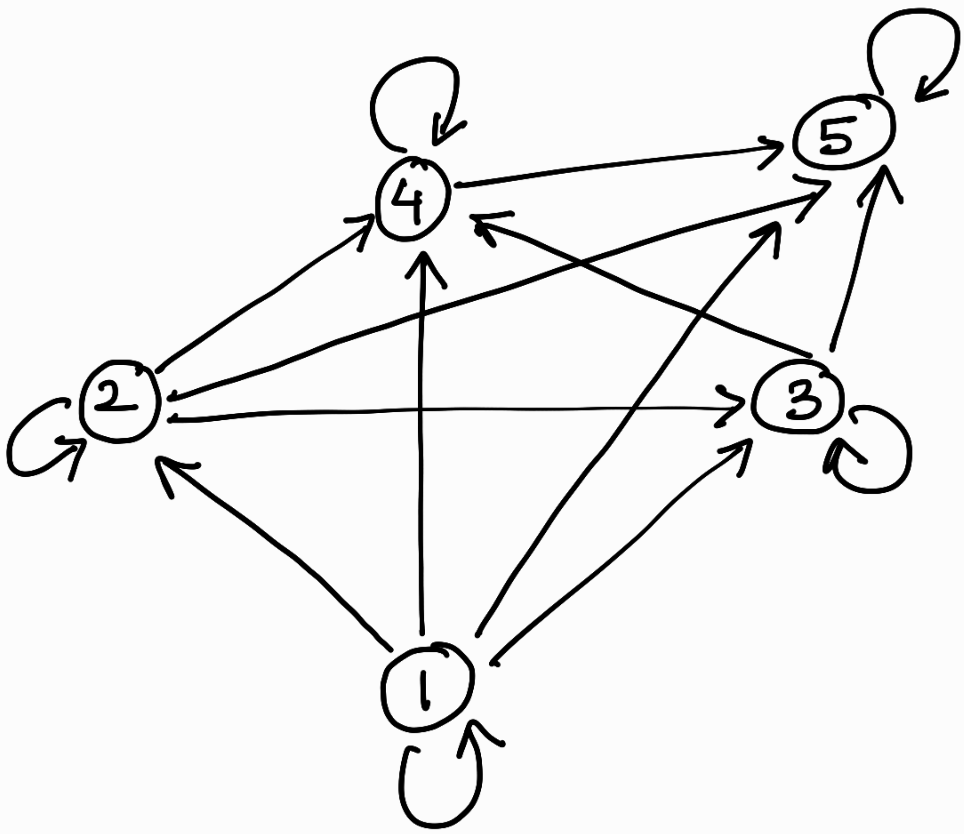
Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Q.6 Determine the Hasse Diagram of the relation on $A = \{1,2,3,4,5\}$, whose matrix is shown:-

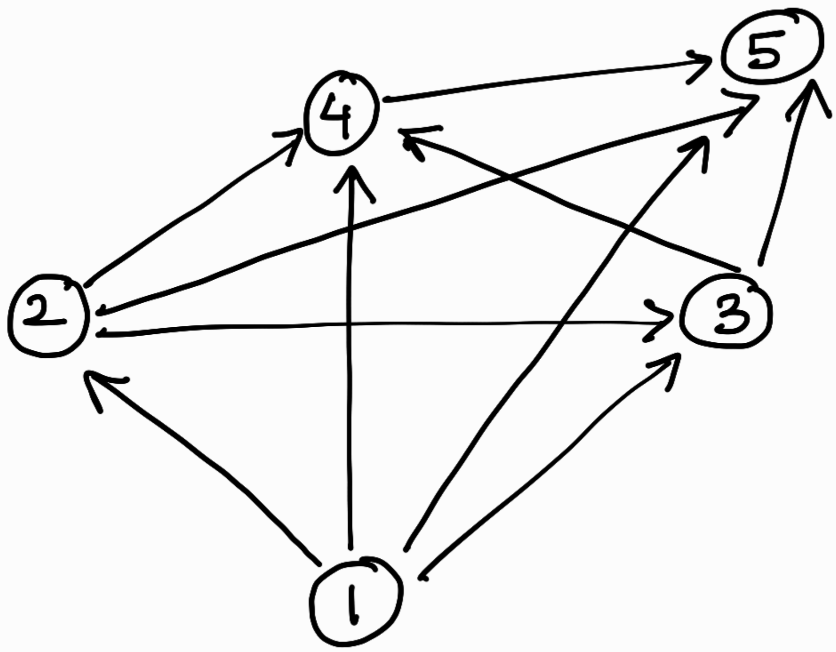
a) $$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: Digraph for the given matrix is



Step 1

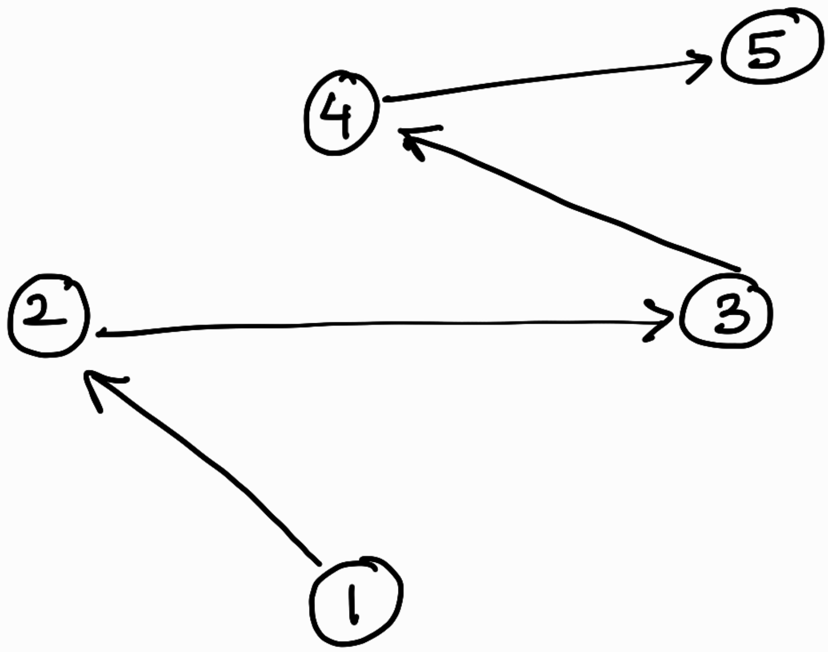
Remove Cycles



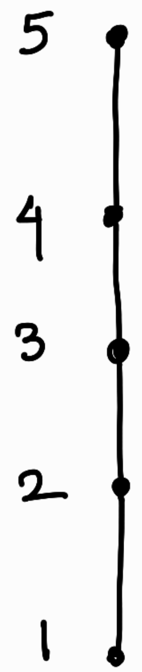
Step 2

Remove Transitive edges

(2,5), (1,3), (2,4), (1,5), (1,4), (3,5)



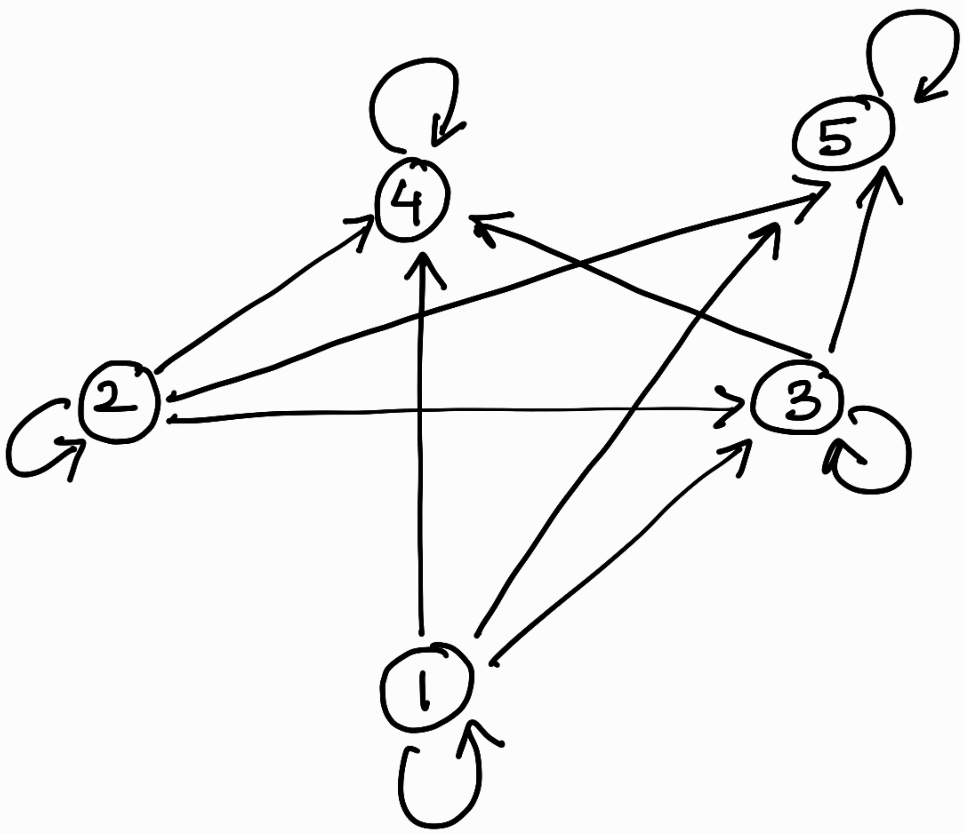
<u>Step 3:</u> Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



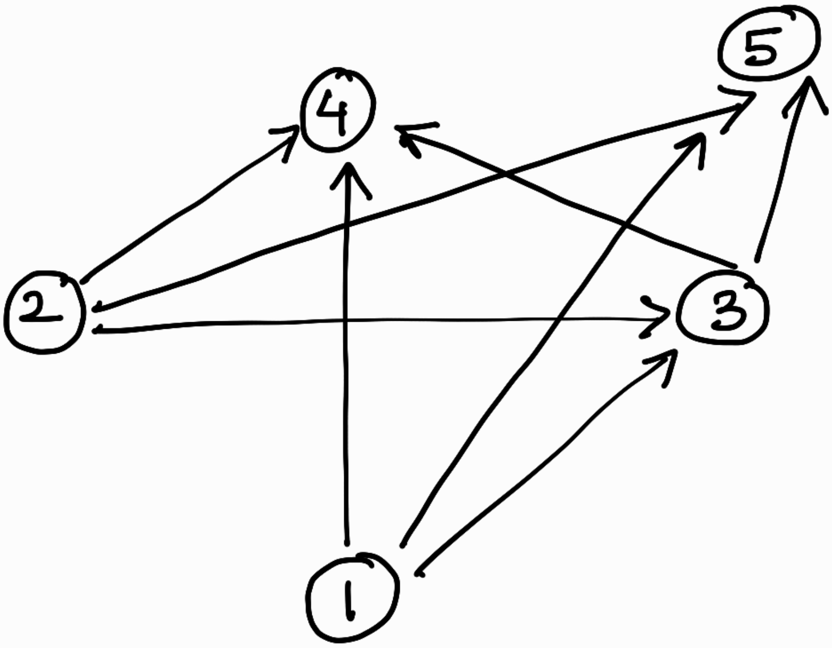
b)

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: Digraph for the given matrix is

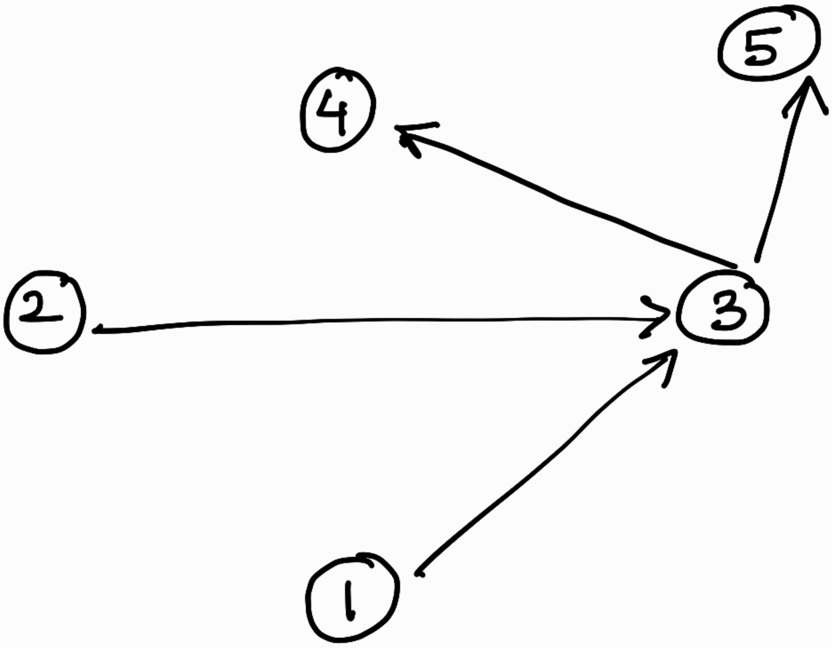


Step 1: Remove Cycles

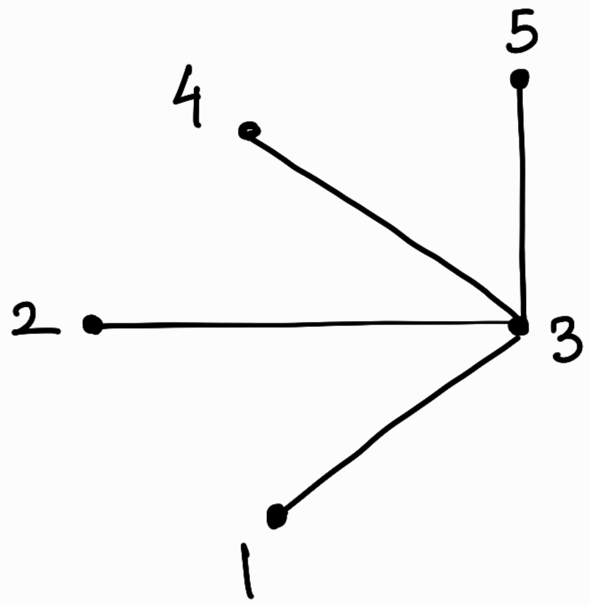


Step2: Remove Transitive edges

(2,4), (1,4), (1,5), (2,5)



Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Practice Questions

1. Draw the Hasse Diagram for divisibility on the set

   i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

   ii) $\{1, 2, 3, 4, 5, 7, 11, 13\}$

2. Draw Hasse Diagram for following relation, what the diagram is called as? Justify.

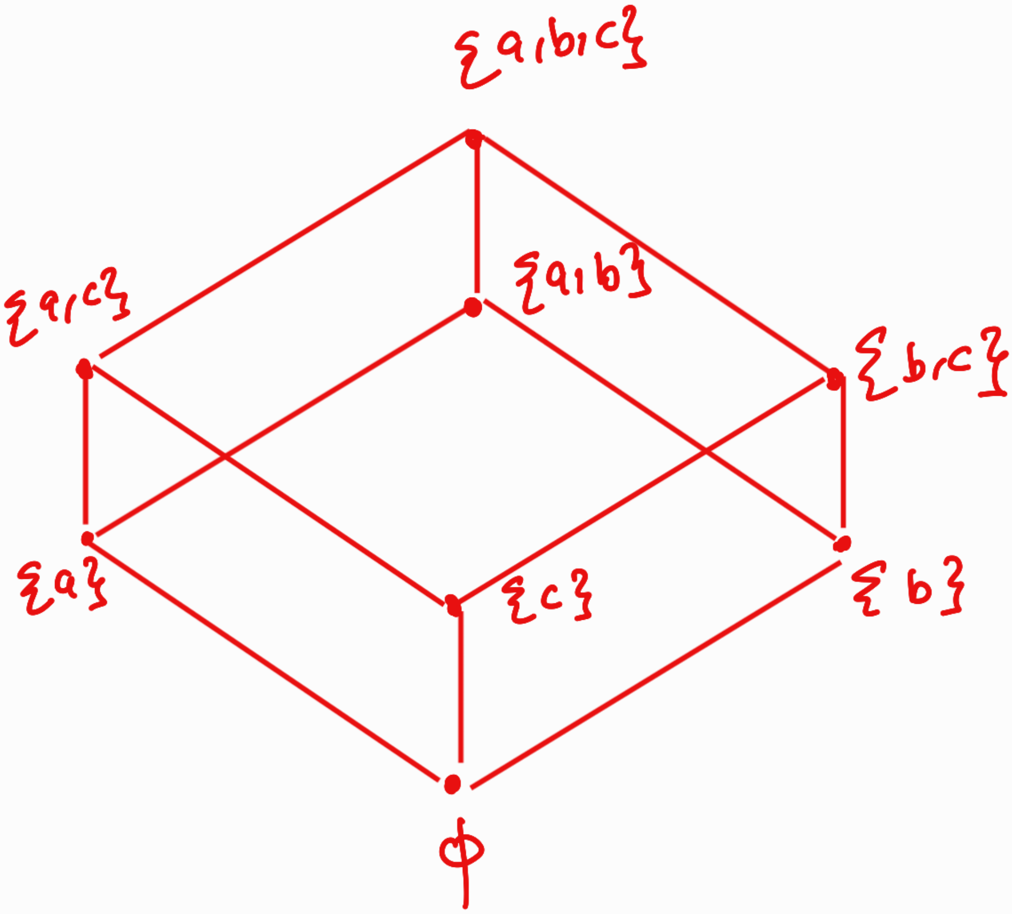
   Let $A = \{a, b, c, d, e\}$ &

   $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, c), (c, d), (d, e), (a, c), (a, d), (a, e), (b, d), (b, e), (c, e)\}$

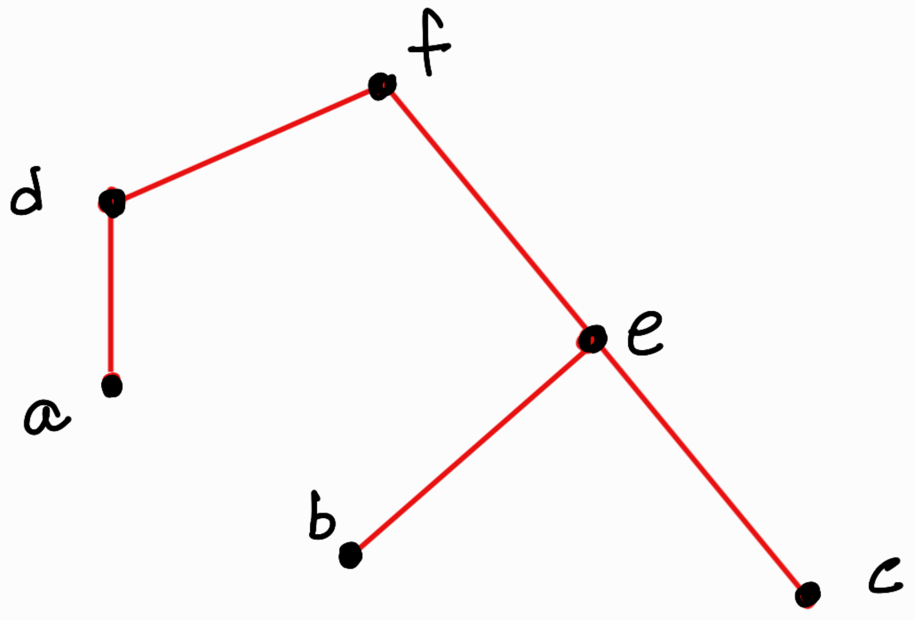
Q. $S = \{a, b, c\}$ and A is a poset under $\subseteq$.

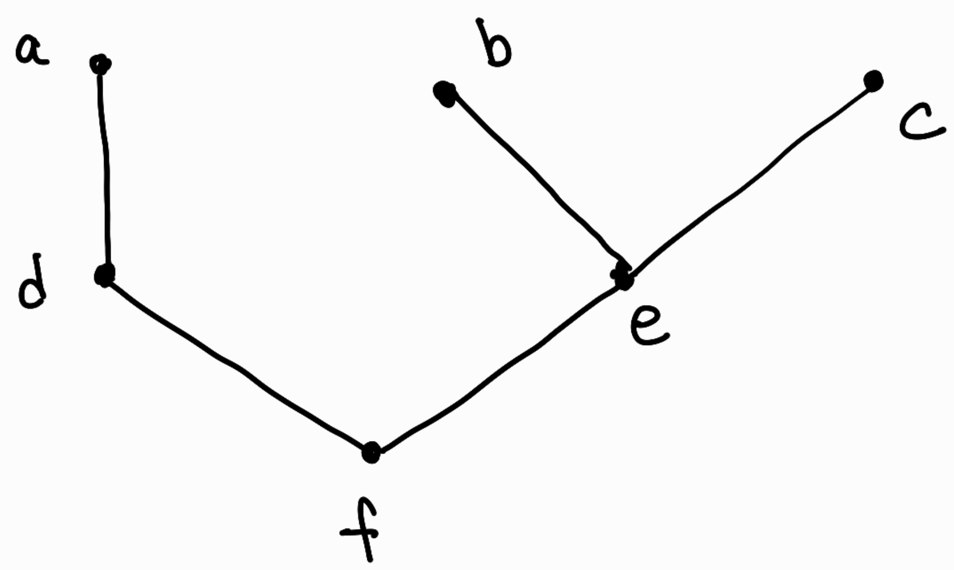
Draw Hasse Diagram of poset $(A, \subseteq)$

Sol: $A = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\})$



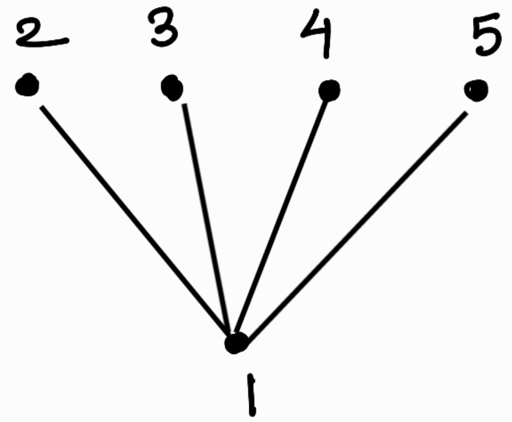
Q. Draw Hasse diagram of dual poset of the poset whose Hasse diagram is given :-





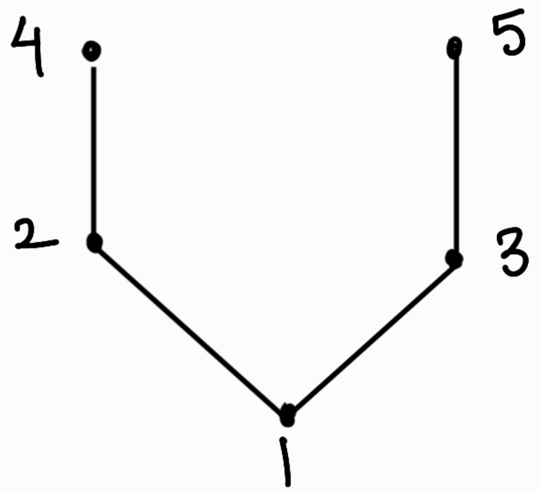
Q. Determine matrix of partial order whose Hasse Diagrams are:-

a)



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

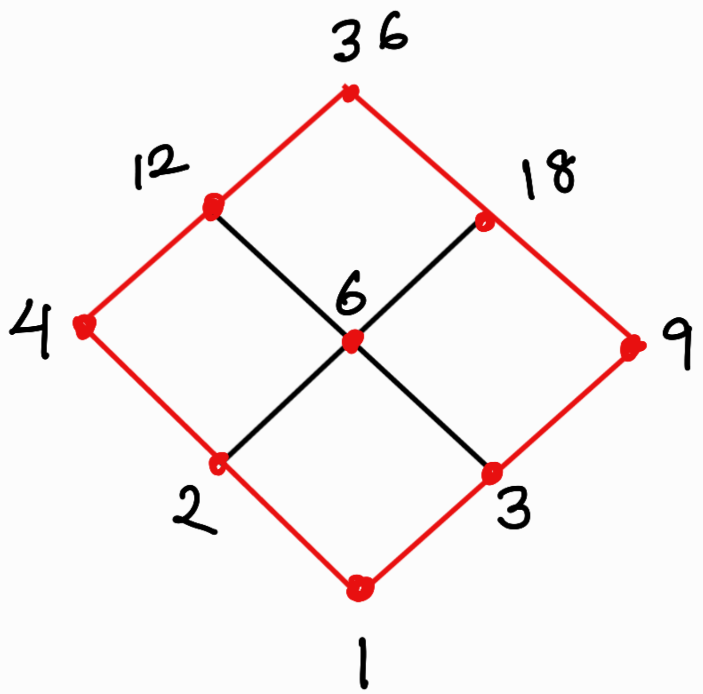
b)



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Q. Draw a Hasse Diagram for the set D_{36}.

Sol. $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



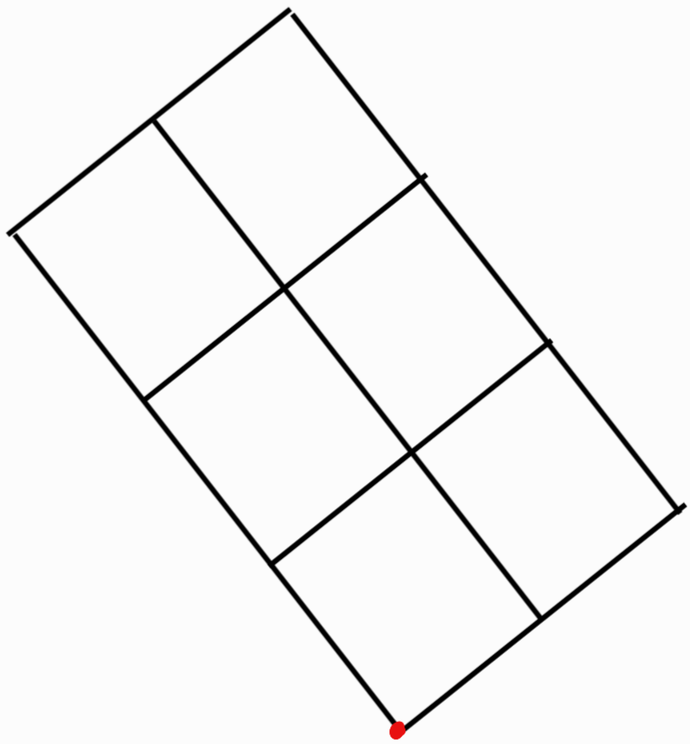
$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (1,18), (1,36), (2,2), (2,4), (2,6), (2,12), (2,18), (2,36), (3,3), (3,6), (3,9), (3,12), (3,18), (3,36), (4,4), (4,12), (4,36), (6,6), (6,12), (6,18), (6,36), (9,9), (9,18), (9,36), (12,12)$
$(12,36), (18,18), (18,36), (36,36)\}$

Q. Draw a Hasse Diagram for the set D_{72}

Sol.

Q. Draw a Hasse Diagram for the set D_{72}

Sol. $D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$



$\Rightarrow \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,9), (1,12), (1,18), (1,24), (1,36), (1,72), (2,2), (2,4), (2,6), (2,8), (2,12), (2,18), (2,24), (2,36), (2,72), (3,3), (3,6), (3,9), (3,12), (3,18), (3,24), (3,36), (3,72), (4,4), (4,8), (4,12), (4,24), (4,36), (4,72), (6,6), (6,12), (6,18),$

$(6,24), (6,36), (6,72), (8,8), (8,24),$

$(8,72), (9,9), (9,18), (9,36), (9,72),$

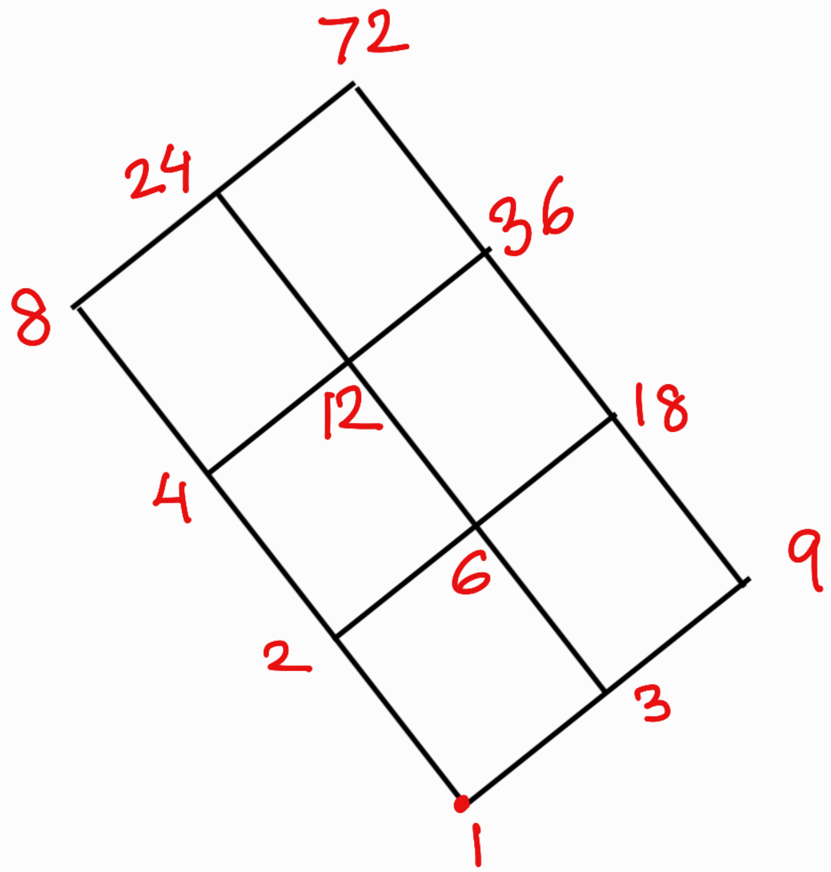
$(12,12), (12,24), (12,36), (12,72),$

$(18,18), (18,36), (18,72), (24,24),$

$(24,72), (36,36), (36,72), (72,72) \}$

Q. Draw a Hasse Diagram for the set D_{72}

Sol. $D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

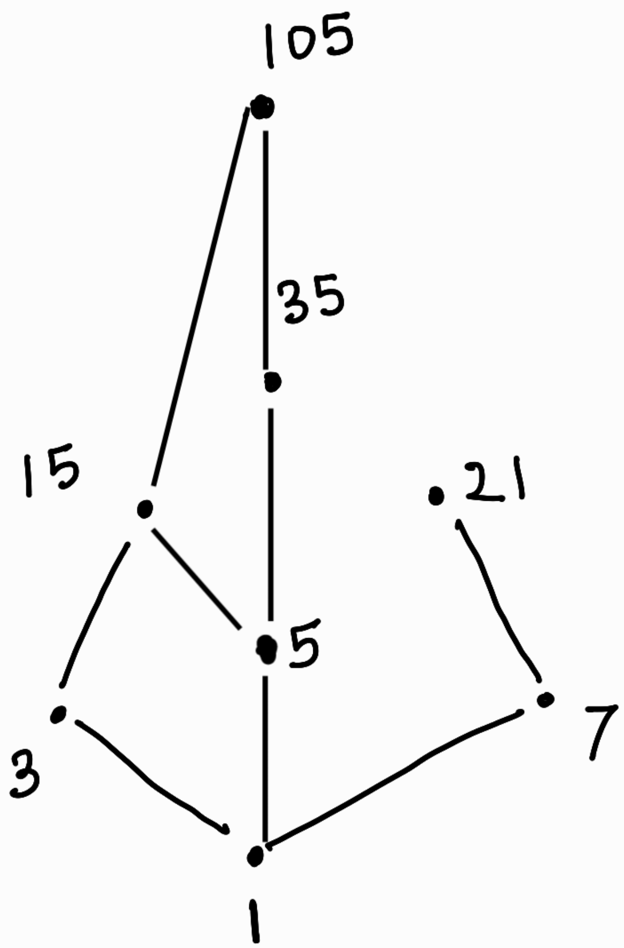


$R \Rightarrow \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,9),$
$(1,12), (1,18), (1,24), (1,36), (1,72),$
$(2,2), (2,4), (2,6), (2,8), (2,12), (2,18),$
$(2,24), (2,36), (2,72), (3,3), (3,6),$
$(3,9), (3,12), (3,18), (3,24), (3,36),$
$(3,72), (4,4), (4,8), (4,12), (4,24),$
$(4,36), (4,72), (6,6), (6,12), (6,18),$

$(6,24)$, $(6,36)$, $(6,72)$, $(8,8)$, $(8,24)$,
$(8,72)$, $(9,9)$, $(9,18)$, $(9,36)$, $(9,72)$,
$(12,12)$, $(12,24)$, $(12,36)$, $(12,72)$,
$(18,18)$, $(18,36)$, $(18,72)$, $(24,24)$,
$(24,72)$, $(36,36)$, $(36,72)$, $(72,72)\}$

Q. Draw Hasse Diagram for D_{105}

Sol: $D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$

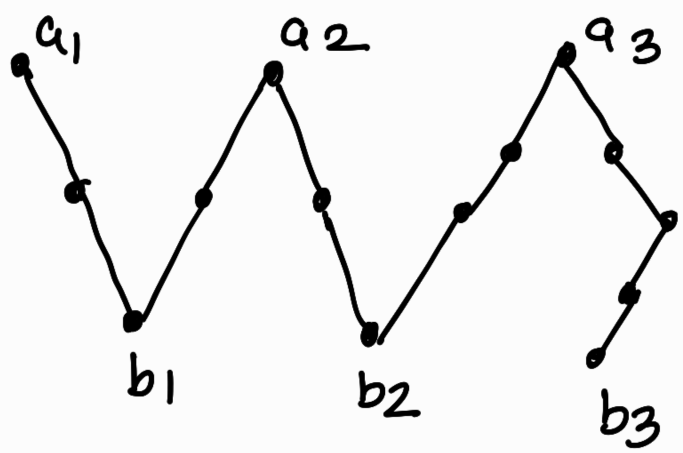


Maximal & Minimal Element.

An element $a \in A$ is called a maximal element of A if there is no element c in A such that $a < c$.

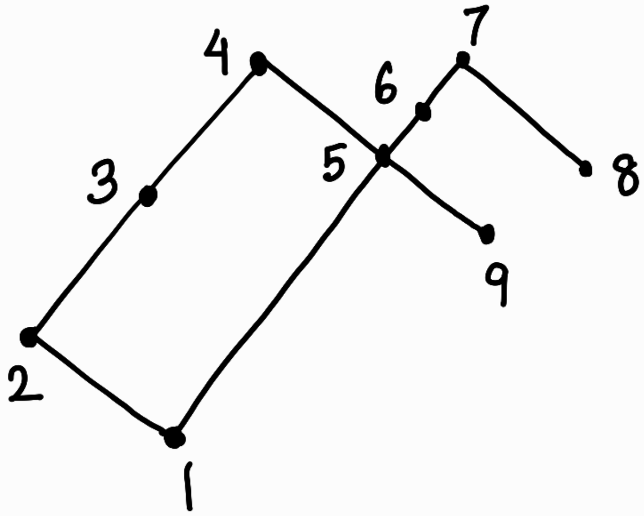
An element $b \in A$ is called a minimal element of A if there is no element c in A such that $c < b$.

Example



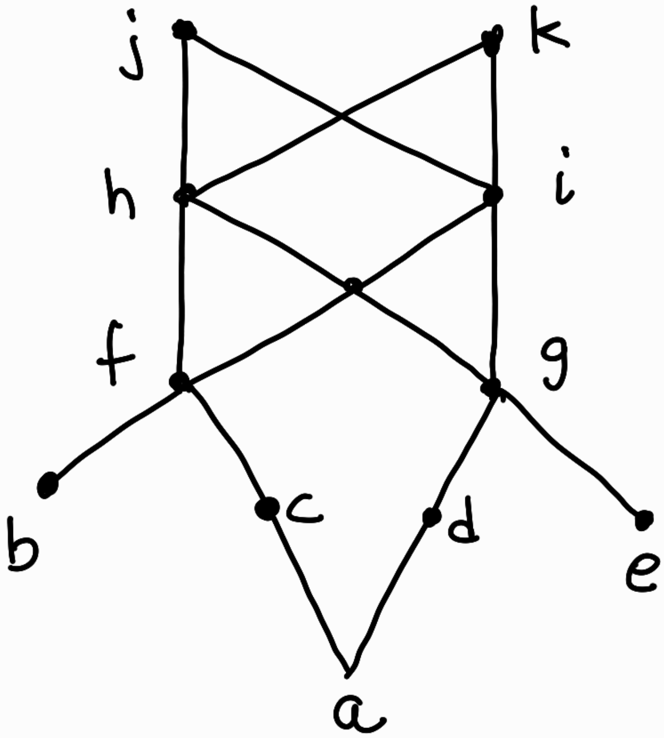
Maximal Elements : a_1, a_2, a_3

Minimal Elements : b_1, b_2, b_3



Maximal : 4, 7

Minimal : 1, 9, 8



Maximal : j, k

Minimal : b, a, e

Greatest & Least Element

An element $a \in A$ is called a greatest element of A if $\underline{x \leq a}$ for all $\underline{x \in A}$.

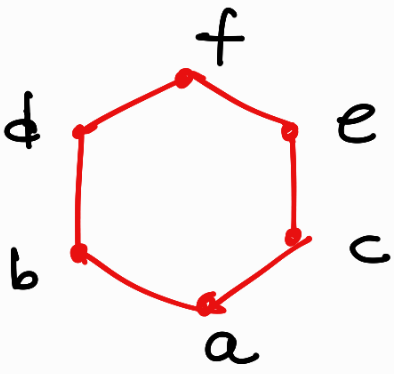
An element $a \in A$ is called a least element of A if $\underline{a \leq x}$ for all $\underline{x \in A}$.

A poset has atmost one greatest element and atmost one least element.

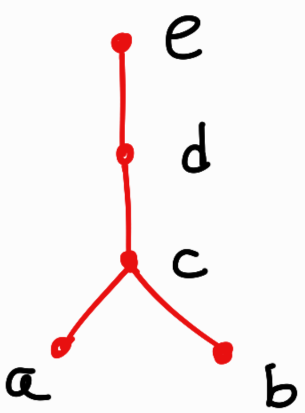
The greatest elements of a poset, if it exists, is denoted by I and is often called as the <u>Unit</u> element.

Similarly, the least element of a poset, if it exists, is denoted by '0' and is called as the <u>Zero</u> Element.

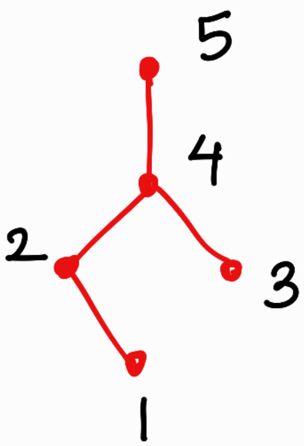
Examples



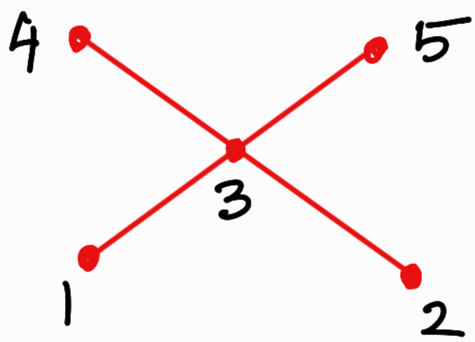
I = f
O = a



I = e
O = None



I = 5
O = 1



I = None
O = None

Greatest Element is also called as universal upper bound.

Least Element is also called as universal lower bound.

Upper Bound: Consider a poset A and a subset B of A. An element $a \in A$ is called an upper bound of B if $b \leq a$ for all $b \in B$.

Lower Bound: An element $a \in A$ is called a lower bound of B if $a \leq b$ for all $b \in B$.

Least Upper Bound (LUB)

Let A be a poset & B be a subset of A.

An element $a \in A$ is called a Least Upper Bound (LUB) of B if a is an upperbound of B and $a \leq a'$, whenever a' is an upper bound of B.

Thus $a = (LUB)(B)$ if $b \leq a$

for all $b \in B$ and if

whenever $a' \in A$ is also an upper bound of B.

Then $a \leq a'$.

Greatest Lower Bound (GLB)

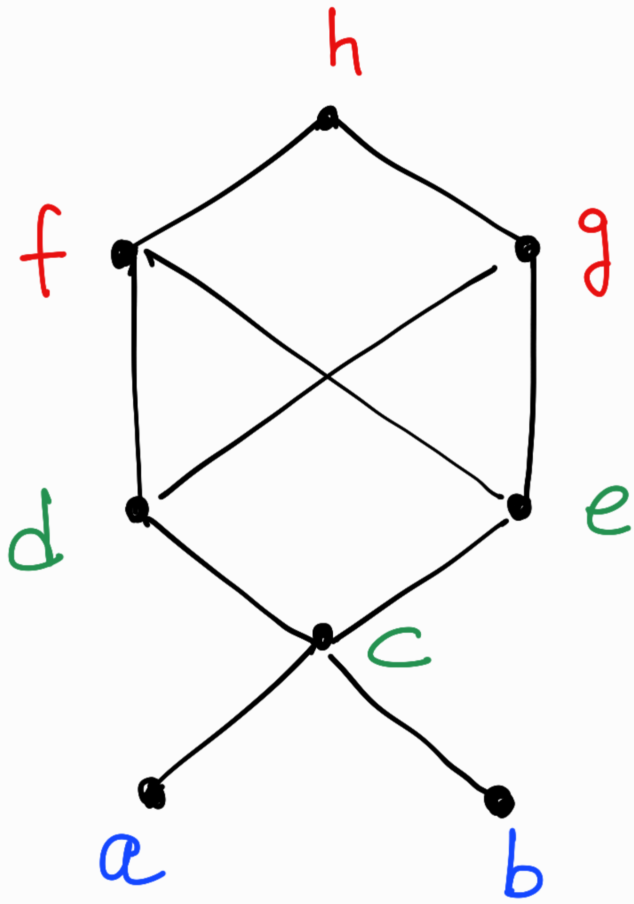
Let A be a poset & B be a subset of A.

An element $a \in A$ is called a Greatest Lower Bound (GLB) of B if a is a lower bound of B

Thus $a = (GLB)(B)$ if $a \leq b$ for all $b \in B$ and if whenever $a' \in A$ is also a lower bound of B.

Then $a' \leq a$.

Example



$A = \{a, b, c, d, e, f, g, h\}$

$B_1 \subseteq A$

$B_2 \subseteq A$

i) $B_1 = \{a, b\}$

ii) $B_2 = \{c, d, e\}$

Find Upper bound, Lower bound,
LUB & GLB of B_1 & B_2.

Sol:

i) Upper bounds of set B_1 are
c, d, e, f, g, h

Least Upper Bound is 'c'.

There is no Lower bound of set B_1, so no GLB as well.

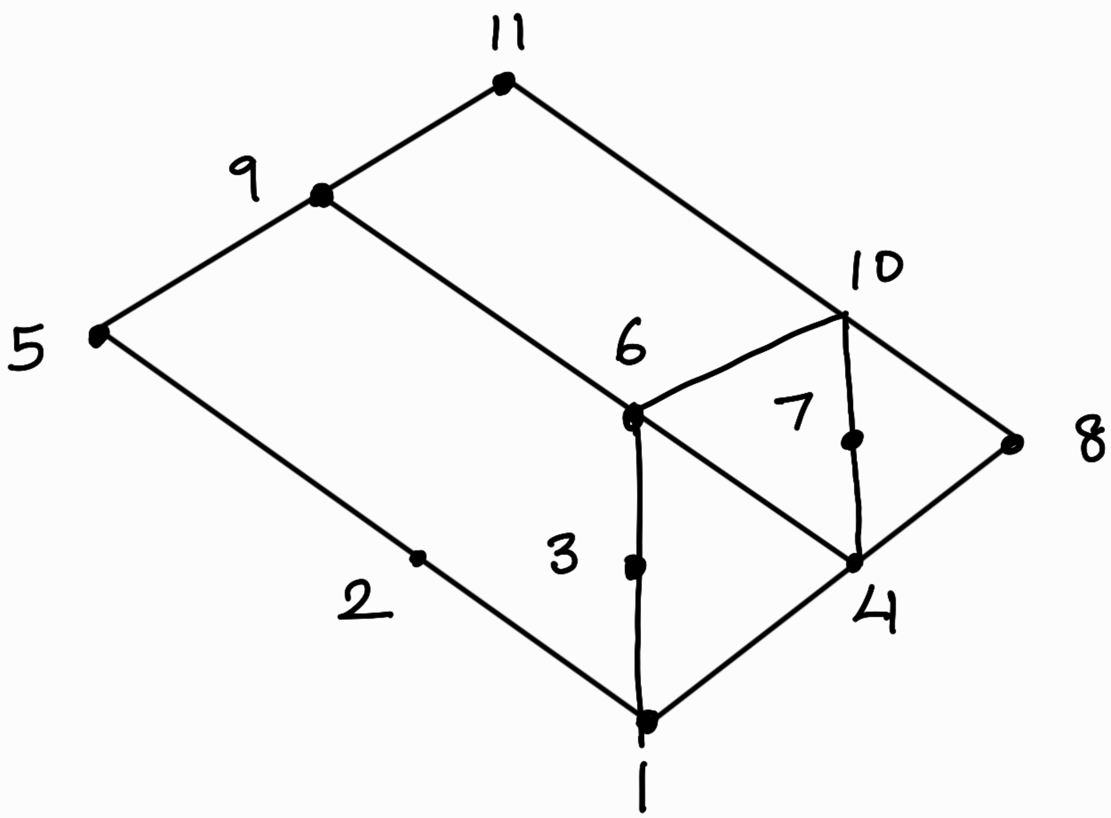
ii) Upper bounds of set B_2 are
f, g & h.

There is no least upper bound in B_2.

Lower bounds are c, a & b.

Greatest lower bound is c.

Q. Let $A = \{1, 2, 3, 4, 5, \ldots, 11\}$ be the poset whose Hasse diagram is shown below. Find LUB & GLB of $B = \{6, 7, 10\}$ if they exist.



Upper Bounds of B ⇒ 10, 11

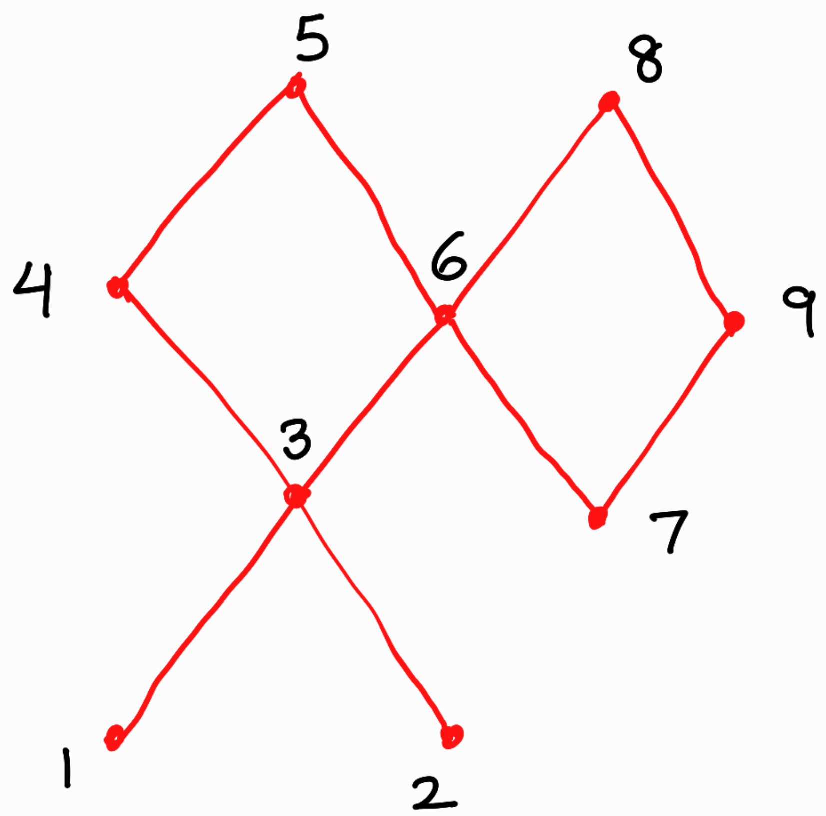
Least Upper Bound ⇒ 10

Lower Bounds of B ⇒ 1, 4

Greatest Lower Bound = 4

Q. Let A be the poset whose Hasse diagram is as shown.

$A = \{1, 2, \dots 9\}$. Find GLB, LUB of set $B = \{3, 4, 6\}$



Lower Bounds of B are $\{3, 1, 2\}$

GLB is 3

Upper Bounds of B are 5

LUB is 5

Q. Find the GLB & LUB of the set (3,9,12) and (1,2,4,5,10) if they exist in the poset (Z, +, /) where / is a relation of divisibility

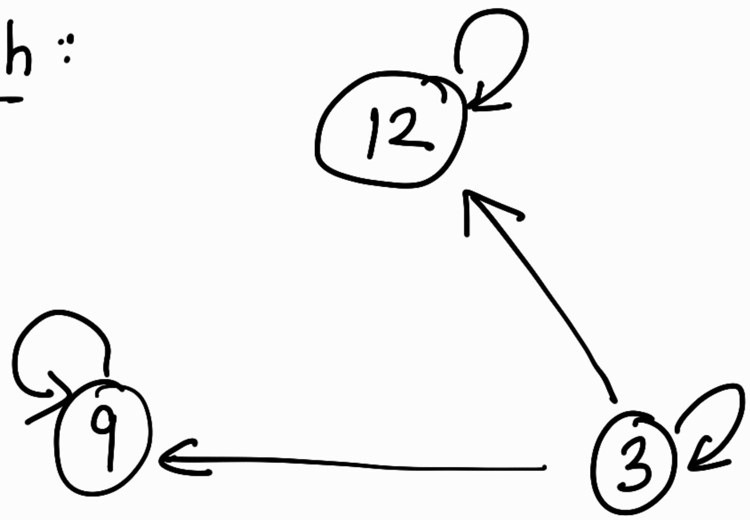
<u>Sol:</u> Set $A = \{3, 9, 12\}$

$$R = \{(3,3), (3,9), (3,12), (9,9), (12,12)\}$$

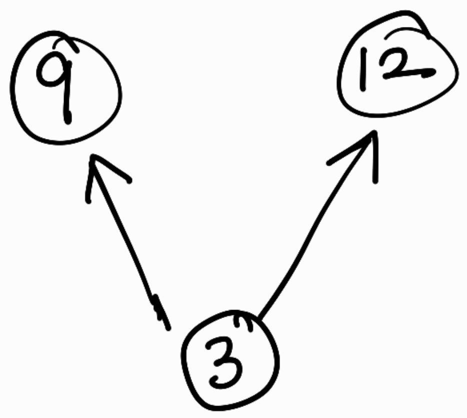
<u>Matrix</u> $M_R =$

$$\begin{array}{c} \\ 3 \\ 9 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 9 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

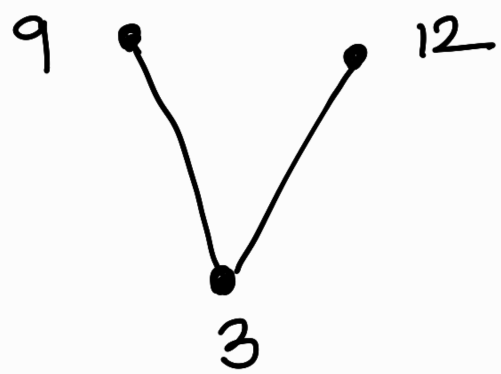
<u>Digraph</u>:



Remove transitive edges & cycles



Hasse Diagram



LUB:

	3	9	12
3	3	9	12
9	9	9	—
12	12	—	12

GLB:

	3	9	12
3	3	3	3
9	3	9	3
12	3	3	12

GLB of $(3, 9, 12) = 3$

LUB of $(3, 9, 12) = 36$ Why??

Because 36 is a number which is divisible by all the 3 numbers, i.e 3, 9 & 12.

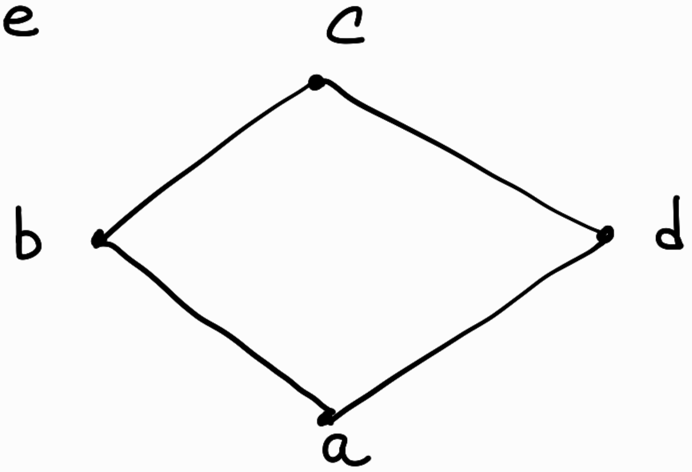
Lattices

A lattice is a poset in which every subset $\{a, b\}$ consisting of two elements, has a least upper bound and a greatest lower bound.

We denote LUB by $a \vee b$ and call it the join of a & b.

Similarly, we denote GLB by $a \wedge b$ and we call it as the meet of a and b.

Example



Determine whether the above Hasse diagram represents a lattice or not.

Sol. LUB:

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

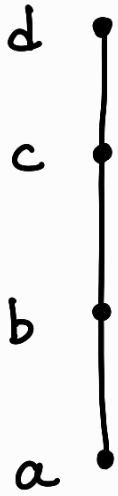
GLB:

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

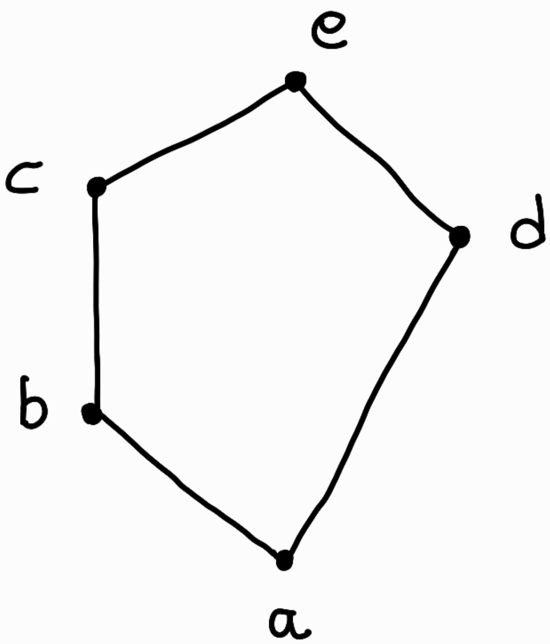
Here, every subset has a least upper bound and greatest lower bound, hence it is a lattice.

Practice

1.



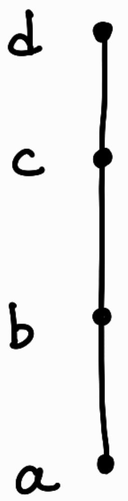
2.



Here, every subset has a least upper bound and greatest lower bound, hence it is a lattice.

Practice

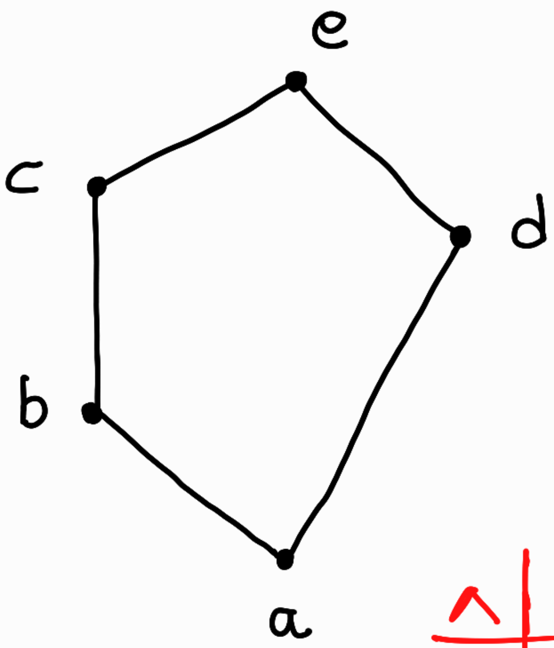
1.



∨	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

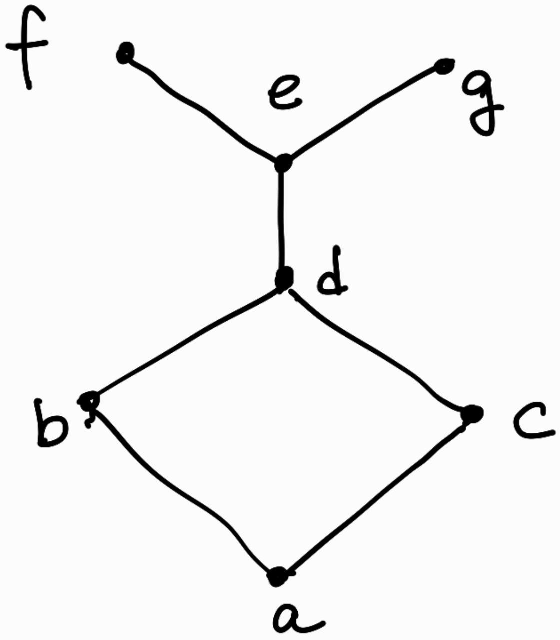
2.



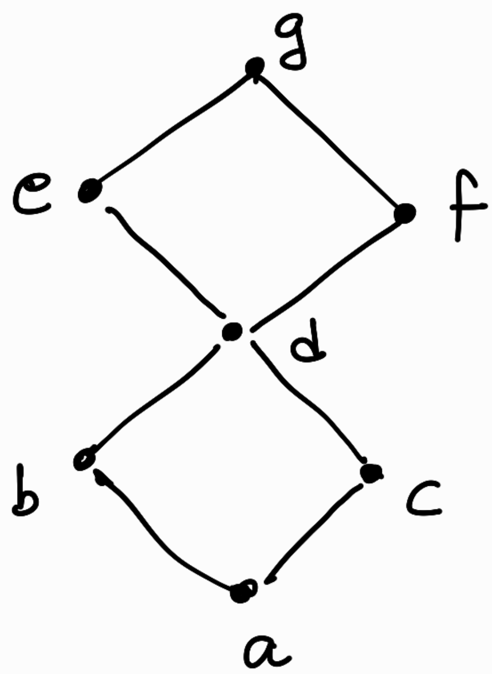
∨	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	c	c	c	e	e
d					
e					

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	a	b
c	a	b	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

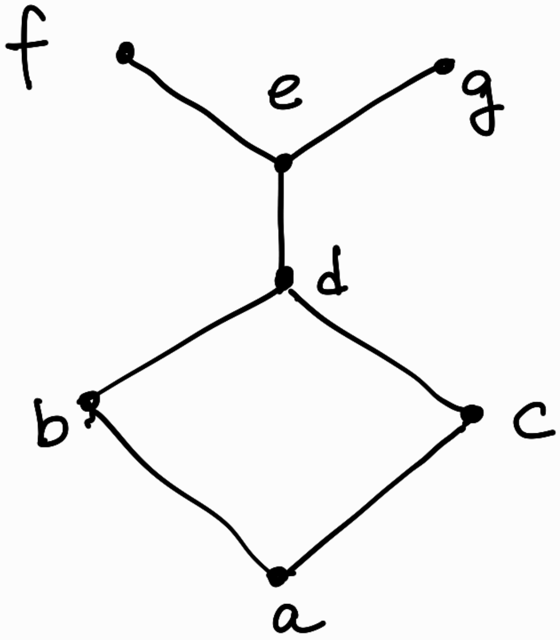
3.



4.

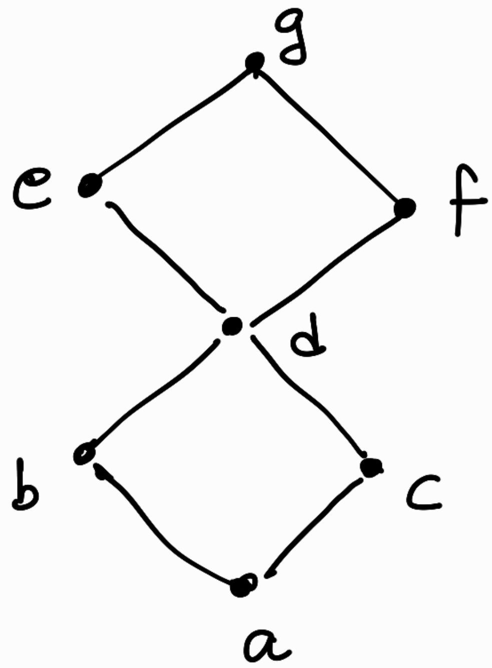


3.



$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	–
g	g	g	g	g	g	–	g

4.



$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

Properties of Lattices

Let L be a lattice then,

1. <u>Idempotent</u> Property

   a) $a \vee a = a$

   b) $a \wedge a = a$

2. <u>Commutative</u> Property

   a) $a \vee b = b \vee a$

   b) $a \wedge b = b \wedge a$

3. <u>Associative</u> Property

   a) $a \vee (b \vee c) = (a \vee b) \vee c$

   b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. <u>Absorption Property</u>

   a) $a \vee (a \vee b) = a$

   b) $a \wedge (a \vee b) = a$

Types of Lattices

1. Bounded Lattice

A lattice L is said to be bounded if it has a greatest element I and a least element O. If L is a bounded lattice, then for all $a \in A$.

$$O \leq a \leq I$$

$$a \vee O = a \quad , \quad a \wedge O = O$$

$$a \vee I = I \quad , \quad a \wedge I = a$$

$\vee \rightarrow$ LUB

$\wedge \rightarrow$ GLB

2. Distributive Lattice

A lattice L is called distributive if for any elements a, b and c in L, we have the following distributive properties.

a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

b) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

3. Complemented Lattice

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$.

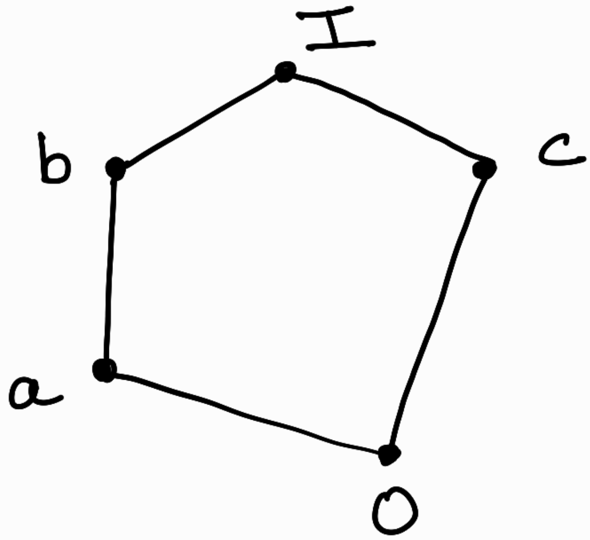
An element $a' \in L$ is called a complement of a if

$$a \vee a' = I \quad \text{and} \quad a \wedge a' = 0$$

A lattice L is called complemented if it is bounded and if every element in L has a complement.

Example:

Show that this is a complemented lattice



Sol.

$$a \vee c = I$$

$$a \wedge c = 0$$

$$b \vee c = I$$

$$b \wedge c = 0$$

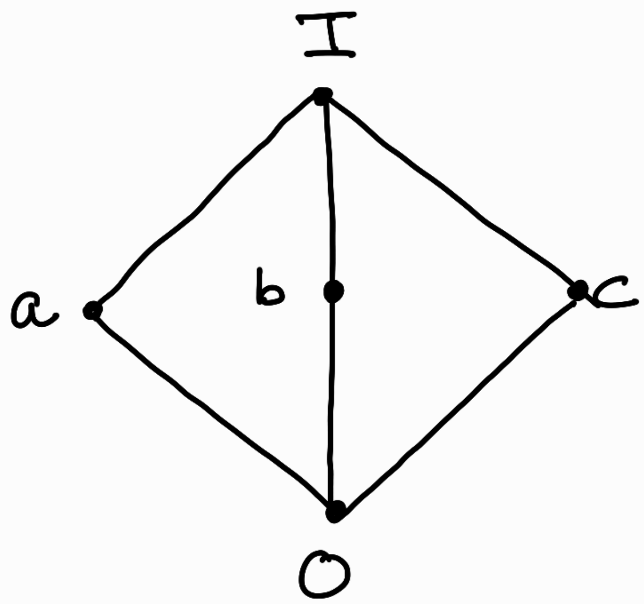
$\therefore$ Complement of a & b are c.

$\therefore$ Complement of c are b & a.

$\therefore$ Every element has a complement

$\therefore$ Above lattice is a complemented lattice.

Q. Check if the following is a complemented lattice or not.



<u>Sol.</u> $a \vee b = I$, $a \wedge b = O$

$\therefore$ Complement of a is b

$\therefore$ Complement of b is a

Also $a \vee c = I$, $a \wedge c = O$

$\therefore$ Complement of a is C

$\therefore$ Complement of c is a.

$\therefore$ Every element has a complement.

So above lattice is complemented

Functions

Definition:

Let A and B be non-empty sets. A function f from A to B, denoted as $f : A \to B$, is a relation from A to B such that for every $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in f$.

Normally if $(a,b) \in f$, we write $f(a) = b$

If $f(a) = b$ & $f(a) = c$

then $b = c$

Examples:

1. Let $A = \{1, 2, 3\}$ & $B = \{p, q, r\}$ & $f = \{(1, p), (2, q), (3, r)\}$.

Determine whether the given relation is a function.

Sol:

$$f(1) = \{p\}$$

$$f(2) = \{q\}$$

$$f(3) = \{r\}$$

'f' is a function, since each set $f(n)$ consists of a single element.

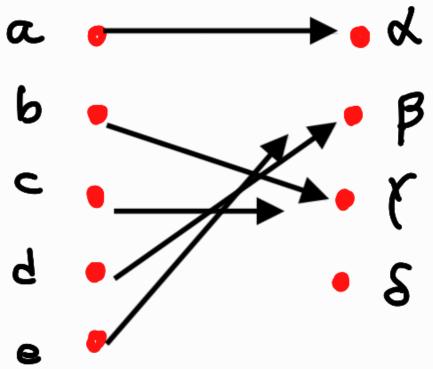
A function can be represented in graphical & tabular form.

For example :-

Let A be a set of houses and B be a set of colors. Then a function from A to B is an assignment of colours for painting the houses. i.e

$f(a) = \alpha \qquad f(b) = \gamma \qquad f(c) = \gamma$

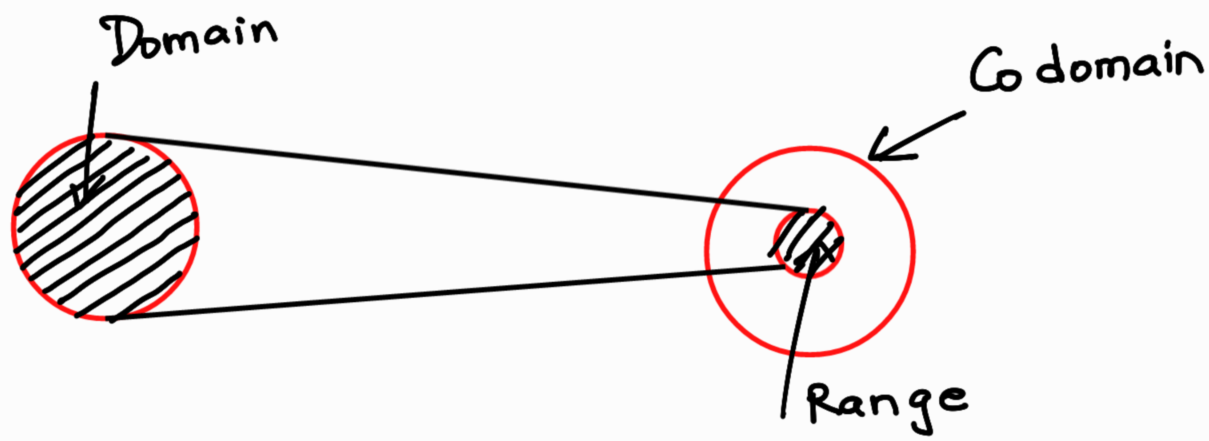
$f(d) = \beta \qquad f(e) = \beta$



Graphical

	α	β	γ	δ
a	✓			
b			✓	
c			✓	
d		✓		
e		✓		

Tabular



The set A is called as the domain of f, denoted as $Dom(f)$.

The set B is called as the codomain and the set $\{f(a) \mid a \in A\}$, which is the subset of B, is called as the range of f, denoted by $Ran(f)$

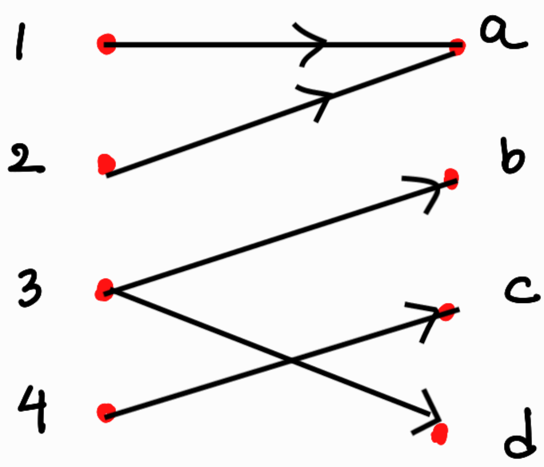
The element a is called an argument of the function f and $f(a)$ is called the value of the function for the argument a.

Types of Functions

1. Onto or Surjective Functions

A function from A to B is said to be an onto function if every element of B is the image of one or more elements of A.

Onto function is also called surjective or 'f' is ONTO if Ran(f) = B.



Example : Let $A = \{1,2,3,4\}$ & $B = \{a,b,c,d\}$

and $f = \{(1,a), (2,a), (3,d), (4,c), (3,b)\}$

$f(1) = a$, $f(2) = a$, $f(3) = d$,

$f(4) = c$, $f(3) = b$

$\therefore$ $\text{Ran}(f) = \{a, b, c, d\} = B$

So this function is onto or surjective function.

2. One to One or Injective Function

A function from A to B is said to be a one to one function if no two elements of A have the same image

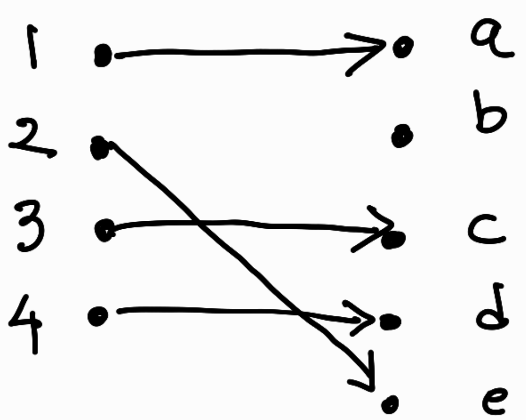
Example

Let $A = \{1, 2, 3, 4\}$ & $B = \{a, b, c, d, e\}$

& $f = \{(1, a), (2, e), (3, c), (4, d)\}$

$\therefore$ $f(1) = a$, $f(2) = e$

$f(3) = c$, $f(4) = d$



3. One to One Onto Function or Bijective Function.

A function from A to B is said to be a one to one onto function if it is both an onto and one to one function.

One to One Onto function is also called as bijective function.

Example.

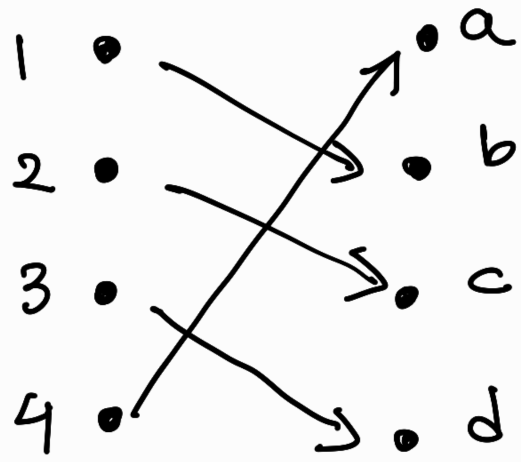
Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$f = \{(1, b), (2, c), (3, d), (4, a)\}$

$f(1) = b$, $f(2) = c$,

$f(3) = d$, $f(4) = a$

Given function f is one to one onto or bijective function



4. Everywhere Defined Function

A function from A to B is said to be everywhere defined if

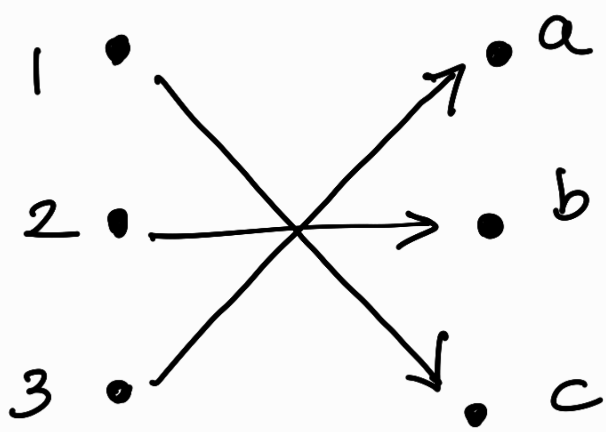
$Dom(f) = A$.

Example: Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $f = \{(1, c), (2, b), (3, a)\}$

$f(1) = c$, $f(2) = b$, $f(3) = a$

$\therefore$ $\mathrm{Dom}(f) = \{1, 2, 3\}$.

Thus given function f is everywhere defined function



Q.1. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$, $D = \{d_1, d_2, d_3, d_4\}$.

Consider the following three functions from A to B, A to D & B to C

a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$

b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$

c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$

Determine whether each function is one to one, onto or everywhere defined.

Sol. a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$

$$f_1(a_1) = b_2$$

$$f_1(a_2) = b_3$$

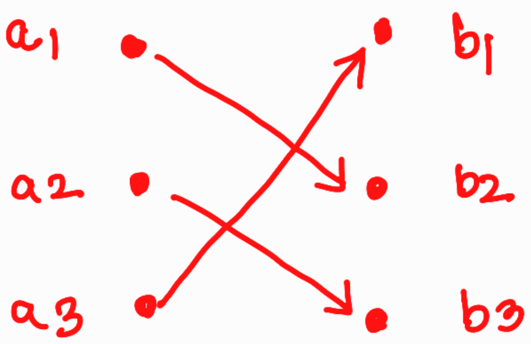
$$f_1(a_3) = b_1$$

f is everywhere defined because

$Dom(f_1) = A$

f is onto becayse $Ran(f_1) = B$.

f is one-to-one because no two elements of set B have same image.



So, f_1 is surjective, injective, bijective and everywhere defined function.

b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$

$f_2(a_1) = d_2$

$f_2(a_2) = d_1$

$f_2(a_3) = d_4$

f_2 is everywhere defined function because $Dom(f_2) = A$

f_2 is not onto function because $Ran(f_2) \neq D$

f_2 is one to one or <u>injective</u> function because no two elements of set D have same image.

c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$

$f_3(b_1) = c_2,$

$f_3(b_2) = c_2$

$f_3(b_3) = c_1$

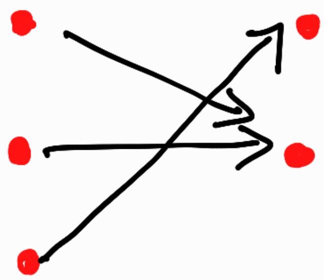
f_3 is everywhere defined function because

$$\text{Dom}(f_3) = B$$

f_3 is onto surjective function because

$$\text{Ran}(f_3) = C.$$

f_3 is not one to one injective because two elements of set B have same image.



Q.2. Is the following function one to one?

$f: Z \to Z$, where $f(x) = 2x - 1$

<u>Sol</u>: $a, b \in Z$ and

$f(a) = f(b)$, then

$2a - 1 = 2b - 1$

$a = b$

$\therefore$ f is one to one function.

Q.3. Comment whether the function f is one to one or onto.

Consider the function $f: N \to N$ where N is the set of natural numbers including zero.

$$f(j) = j^2 + 2$$

<u>Sol.</u> $N = \{0, 1, 2, 3, 4, \ldots\ldots\}$

$\therefore f(0) = 2$

$f(1) = 3$

$f(2) = 6$

$f(3) = 11$

$f(4) = 18$ ..... and so on.

For every number 'n' from 'N' we can find another 'n' so the given function is one to one.

But not every element 'n' of set 'N' is image of some element 'n'.

So the given function is <u>not onto</u>.

Q.4 Test whether the following function is one-to-one onto or both.

$f : Z \to Z$

$f(x) = x^2 + x + 1$

A function from 'A' to 'B' is one to one if no two elements of A have the same image.

Let, $X = -2$,

$$f(-2) = (-2)^2 + (-2) + 1$$

$$= 4 - 2 + 1 = 3$$

Let $X = 1$,

$$f(1) = 1^2 + 1 + 1 = 3$$

Elements 1 & -2 have same image so the function is not one to one.

A function from 'A' to 'B' is said to be an onto function of every element of B is image of one or more elements of A.

In the given function, not every element is image of an element from A. For example, "2" is not an image of any elements of A. So it is not 'onto' function.

Composition of Functions

Let f be a function from A to B (i.e $f: A \to B$) and g be a function from B to C (i.e $g: B \to C$).

Then the composition of f & g denoted as $\underline{g \circ f}$ is a relation from A to C, where

$$g \circ f(a) = g(f(a)).$$

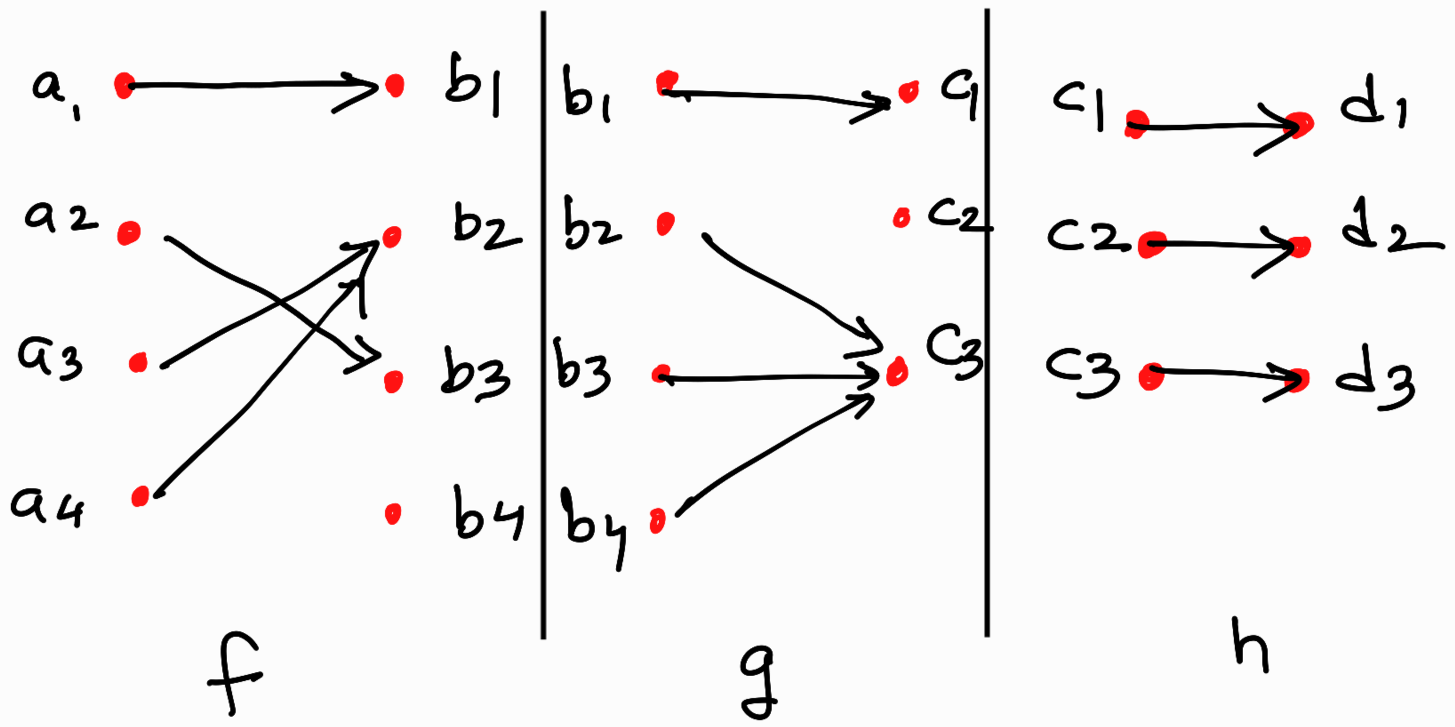
Q.1 $A = \{a_1, a_2, a_3, a_4\}$

$B = \{b_1, b_2, b_3, b_4\}$

$C = \{c_1, c_2, c_3\}$

$D = \{d_1, d_2, d_3\}$

i) For the function f & g, determine gof.



<u>Sol.</u> i) $f : A \to B$ i.e &

$f(a_1) = b_1$

$g : B \to C$ i.e

$g(b_1) = c_1$

$f(a_2) = b_3$ $\quad\quad g(b_2) = c_3,$

$f(a_3) = b_2$ $\quad\quad g(b_3) = c_3$

$f(a_4) = b_2$ $\quad\quad g(b_4) = c_3$

$$g \circ f = g(f(a_1)) = g(b_1) = c_1$$

$$= g(f(a_2)) = g(b_3) = c_3$$

$$= g(f(a_3)) = g(b_2) = c_3$$

$$= g(f(a_4)) = g(b_2) = c_3$$

$\therefore$ $g \circ f\,(a_1) = c_1$

$g \circ f\,(a_2) = c_3$

$g \circ f\,(a_3) = c_3$

$g \circ f\,(a_4) = c_3$

ii) For the function f, g & h, determine $h \circ (g \circ f)$ and $(h \circ g) \circ f$

Sol:

$$h \circ (g \circ f) = h(g \circ f(a_1))$$

$$= h(c_1)$$

$$= d_1$$

Sly

$$= h(g \circ f(a_2))$$

$$= h(c3)$$

$$= d3$$

Sly,

$$= h(g \circ f(a3))$$

$$= h(c3)$$

$$= d3$$

Also $= h(g\circ f(a_4))$

$= h(c_3)$

$= d_3$

Now $(h\circ g)\circ f$

$\Rightarrow h\circ g = h(g(b_1))$

$= h(c_1)$

$= \underline{d_1}$

$= h(g(b_2))$

$= h(c_3)$

$= \underline{d_3}$

$= h(g(b_3))$

$= h(c_3)$

$= \underline{d_3}$

$$= h(g(b4))$$

$$= h(c3)$$

$$= \underline{d3}$$

Now $(hog)of = (hog)(f(a_1))$

$$= hog(b_1)$$

$$= \underline{d_1}$$

$$= (hog)(f(a2))$$

$$= hog(b3)$$

$$= \underline{d3}$$

$$= (hog)(f(a3))$$

$$= hog(b2)$$

$$= \underline{d3}$$

$$=(hog)(f(a4))$$

$$= (hog)(b2)$$

$$= d3$$

Q-2 Consider the above function $f(x) = 2x-3$. Find a formula for the composition functions

i) $f^2 = f \circ f$ and

ii) $f^3 = f \circ f \circ f$

Sol i) $(f \circ f)(x) = f(f(x))$

$= f(2x-3)$

$= 2(2x-3) - 3$

$= 4x - 6 - 3$

$= 4x - 9$

ii) $(f \circ f \circ f)\,x = f(f(f(x)))$

$= f(f(2x-3))$

$= f(4x-9)$

$= 4(2x-3) - 9$

$= 8x - 12 - 9 = 8x - 21$

Q.3. Let $f: R \to R$

$f(x) = x^2 - 1$

$g(x) = 4x^2 + 2$

Find

i) $f \circ (g \circ f)$

ii) $g \circ (f \circ g)$

<u>Sol:</u> $f \circ (g \circ f)$

$= f(g(f(x)))$

$= f(g(x^2 - 1))$

$= f(4(x^2-1)^2 + 2)$

$= f(4(x^4 - 2x^2 + 1) + 2)$

$= f(4x^4 - 8x^2 + 4 + 2)$

$= f(4x^4 - 8x^2 + 6)$

$= (4x^4 - 8x^2 + 6)^2 - 1$

ii) $g \circ (f \circ g)$

$= g(f(g(x)))$

$= g(f(4x^2+2))$

$= g((4x^2+2)^2-1)$

$= g(16x^4+16x^2+4)$

$= 4(16x^4+16x^2+4)^2+2$

Q.4. Let the functions f, g & h are defined as follows:-

$f: R \to R \qquad f(x) = 4x-3$

$g: R \to R \qquad g(x) = x^2+1$

$h: R \to R \qquad h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Find

i) $f \circ f$ ii) $f \circ g$ iii) $g \circ f$ iv) $f \circ h$

v) $h \circ f$ vi) $g \circ h$ vii) $h \circ g$

Q.5. Functions f, g, h are defined on a set, $X = \{1, 2, 3\}$ as

$f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,3)\}$

$h = \{(1,1), (2,2), (3,1)\}$

i) Find fog, gof. Are they equal?

ii) Find $fogoh$ & $fohog$.