Z Transform

Type I: Based on Definition

1. If $f(k) = \{2^0, 2^1, 2^2, 2^3, \dots \dots \dots .\}$find $Z\{f(k)\}$ Ans. $\frac{z}{z-2}$
2. Find the Z transform of
   (i) $f(k) = 1, k \geq 0, |z| > 1$ Ans. $\frac{z}{z-1}$
   (ii) $f(k) = a^k, k \geq 0, |z| > a$ Ans. $\frac{z}{z-a}$
   (iii) $f(k) = \frac{1}{2^k}, k \geq 0, |2z| > 1$ Ans. $\frac{2z}{2z-1}$
3. Find the Z transform of $f(k) = b^k, k < 0$ Ans. $\frac{z}{b-z}$
4. Find the Z transform of $f(k) = a^{|k|}$ Ans. $\frac{z(1-a^2)}{(1-az)(z-a)}$
5. Find the Z transform of Unit impulse function Ans. 1
6. Find the Z transform of Discrete Unit Step function Ans. $\frac{z}{z-1}$
7. Find the Z transform of $f(k) = \begin{cases} 5^k & k < 0 \\ 3^k & k \geq 0 \end{cases}$ Ans. $\frac{2z}{(5-z)(z-3)}$
8. Find the Z transform of f(k) = c^k cosαk, k ≥ 0 hence find cosαk
   Ans. $\frac{z(z-c\,cos\alpha)}{z^2-2czcos\alpha+c^2}$, $\frac{z(z-cos\alpha)}{z^2-2zcos\alpha+1}$
9. Find the Z transform of f(k) = c^k sinαk, k ≥ 0 hence find sinαk
   Ans. $\frac{czsin\alpha}{z^2-2czcos\alpha+c^2}$, $\frac{zsin\alpha}{z^2-2zcos\alpha+1}$
10. Find the Z transform of $f(k) = \sin\left(\frac{k\pi}{4} + a\right)$, $k \geq 0$
    Ans. $\frac{z\sin\left(\frac{\pi}{4}-a\right)+zsina}{z^2-\sqrt{2}z+1}$
11. Find the Z transform of f(k) = a^k , k ≥ 0 Ans. $\frac{z}{z-a}$
12. Find the Z transform of $f(k) = b^k, k \geq 0$
    [M16/CompIT/6M]
    Solution:
    We have,
    $Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$
    $Z\{b^k\} = \sum_0^{\infty} b^k z^{-k}$
    $= b^0z^0 + b^1z^{-1} + b^2z^{-2} + b^3z^{-3} + \cdots \dots \dots .$
    $= 1 + \frac{b}{z} + \frac{b^2}{z^2} + \frac{b^3}{z^3} + \cdots \dots ..$
    $= \left[1 - \frac{b}{z}\right]^{-1}$
    $= \left[\frac{z-b}{z}\right]^{-1}$

$= \frac{z}{z-b}$

13. Find the Z transform of f(k) = 3^k , k ≥ 0 Ans. $\frac{z}{z-3}$
14. Find the Z transform of $f(k) = 2^k, k < 0$ Ans. $\frac{z}{2-z}$
15. Find the Z transform of $f(k) = \left(\frac{1}{2}\right)^{|k|}$ Ans. $\frac{2z}{2-z} + \frac{1}{1-2z}$
16. Find the Z transform of $f(k) = \left(\frac{1}{3}\right)^{|k|}$

[N14/CompIT/5M]

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \sum_{-\infty}^{\infty} \left(\frac{1}{3}\right)^{|k|} z^{-k}$

$= \sum_{-\infty}^{-1} \left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^{k} z^{-k}$

$= \left[\dots\dots + \left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right)^1 z^1\right] + \left[\left(\frac{1}{3}\right)^0 z^0 + \left(\frac{1}{3}\right)^1 z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \cdots\right]$

$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots..\right] + \left[1 + \frac{1}{3z} + \frac{1}{3^2z^2} + \cdots\dots\right]$

$= \frac{z}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots.\right] + \left[1 + \frac{1}{3z} + \frac{1}{3^2z^2} + \cdots..\right]$

$= \frac{z}{3}\left[1 - \frac{z}{3}\right]^{-1} + \left[1 - \frac{1}{3z}\right]^{-1}$

$= \frac{z}{3}\left[\frac{3-z}{3}\right]^{-1} + \left[\frac{3z-1}{3z}\right]^{-1}$

$= \frac{z}{3}\left[\frac{3}{3-z}\right] + \left[\frac{3z}{3z-1}\right]$

$= \frac{z}{3-z} + \frac{3z}{3z-1}$

17. Find the Z transform of $f(k) = \left(\frac{1}{4}\right)^{|k|}$

[M18/Comp/6M]

Solution:

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\left(\frac{1}{4}\right)^{|k|}\right\} = \sum_{-\infty}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$

$= \sum_{-\infty}^{-1} \left(\frac{1}{4}\right)^{-k} z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{4}\right)^{k} z^{-k}$

$= \left[\dots\dots + \left(\frac{1}{4}\right)^3 z^3 + \left(\frac{1}{4}\right)^2 z^2 + \left(\frac{1}{4}\right)^1 z^1\right] + \left[\left(\frac{1}{4}\right)^0 z^0 + \left(\frac{1}{4}\right)^1 z^{-1} + \left(\frac{1}{4}\right)^2 z^{-2} + \cdots\right]$

$= \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots..\right] + \left[1 + \frac{1}{4z} + \frac{1}{4^2z^2} + \cdots\dots\right]$

$= \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots.\right] + \left[1 + \frac{1}{4z} + \frac{1}{4^2z^2} + \cdots..\right]$

$$= \frac{z}{4}\left[1-\frac{z}{4}\right]^{-1} + \left[1-\frac{1}{4z}\right]^{-1}$$
$$= \frac{z}{4}\left[\frac{4-z}{4}\right]^{-1} + \left[\frac{4z-1}{4z}\right]^{-1}$$
$$= \frac{z}{4}\left[\frac{4}{4-z}\right] + \left[\frac{4z}{4z-1}\right]$$
$$= \frac{z}{4-z} + \frac{4z}{4z-1}$$

18. Find the Z transform of $f(k) = a^{|k|}$ and hence find the Z transform of $f(k) = \left(\frac{1}{2}\right)^{|k|}$

[N13/CompIT/6M]

Solution:

We have,

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$$
$$Z\{a^{|k|}\} = \sum_{-\infty}^{\infty} a^{|k|}z^{-k}$$
$$= \sum_{-\infty}^{-1} a^{-k}z^{-k} + \sum_{0}^{\infty} a^{k}z^{-k}$$
$$= [\dots\dots + (a)^3z^3 + (a)^2z^2 + (a)^1z^1] + [(a)^0z^0 + (a)^1z^{-1} + (a)^2z^{-2} + \cdots]$$
$$= [az + a^2z^2 + a^3z^3 + \cdots..] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots\dots\right]$$
$$= az[1 + az + a^2z^2 + \cdots.] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots\dots\right]$$
$$= az[1-az]^{-1} + \left[1-\frac{a}{z}\right]^{-1}$$
$$= \frac{az}{1-az} + \left[\frac{z-a}{z}\right]^{-1}$$
$$\therefore Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a}$$

Put $a = \frac{1}{2}$,

$$Z\left\{\left(\frac{1}{2}\right)^{|k|}\right\} = \frac{\left(\frac{1}{2}\right)z}{1-\left(\frac{1}{2}\right)z} + \frac{z}{z-\frac{1}{2}}$$
$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{2z}{2z-1}$$
$$= \frac{z}{2-z} + \frac{2z}{2z-1}$$

19. Find the Z transform of $f(k) = \begin{cases} 4^k & for \quad k < 0 \\ 3^k & for \quad k \geq 0 \end{cases}$

[N17/Comp/6M][N19/Comp/6M]

Solution:

We have,

$$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$$
$$Z\{f(k)\} = \sum_{-\infty}^{-1} 4^k z^{-k} + \sum_{0}^{\infty} 3^k z^{-k}$$

$$= [\dots\dots..+4^{-3}z^{3}+4^{-2}z^{2}+4^{-1}z^{1}]+[3^{0}z^{0}+3^{1}z^{-1}+3^{2}z^{-2}+\cdots\dots\dots$$
$$= \left[\frac{z}{4}+\frac{z^2}{4^2}+\frac{z^3}{4^3}+\cdots\right]+\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots\dots..\right]$$
$$= \frac{z}{4}\left[1+\frac{z}{4}+\frac{z^2}{4^2}+\cdots\right]+\left[1-\frac{3}{z}\right]^{-1}$$
$$= \frac{z}{4}\left[1-\frac{z}{4}\right]^{-1}+\left[\frac{z-3}{z}\right]^{-1}$$
$$= \frac{z}{4}\left[\frac{4-z}{4}\right]^{-1}+\frac{z}{z-3}$$
$$= \frac{z}{4}\left[\frac{4}{4-z}\right]+\frac{z}{z-3}$$
$$= \frac{z}{4-z}+\frac{z}{z-3}$$

20. Find the Z transform of $f(k)=\begin{cases}3^k & for \quad k<0\\ 2^k & for \quad k\geq 0\end{cases}$

[M19/Comp/6M]

Solution:

We have,

$$Z\{f(k)\}=\sum_{-\infty}^{\infty} f(k)z^{-k}$$
$$Z\{f(k)\}=\sum_{-\infty}^{-1} 3^k z^{-k}+\sum_{0}^{\infty} 2^k z^{-k}$$
$$= [\dots\dots..+3^{-3}z^{3}+3^{-2}z^{2}+3^{-1}z^{1}]+[2^{0}z^{0}+2^{1}z^{-1}+2^{2}z^{-2}+\cdots\dots\dots$$
$$= \left[\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right]+\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\dots..\right]$$
$$= \frac{z}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\cdots\right]+\left[1-\frac{2}{z}\right]^{-1}$$
$$= \frac{z}{3}\left[1-\frac{z}{3}\right]^{-1}+\left[\frac{z-2}{z}\right]^{-1}$$
$$= \frac{z}{3}\left[\frac{3-z}{3}\right]^{-1}+\frac{z}{z-2}$$
$$= \frac{z}{3}\left[\frac{3}{3-z}\right]+\frac{z}{z-2}$$
$$= \frac{z}{3-z}+\frac{z}{z-2}$$

21. Find the Z transform of $f(k)=\begin{cases}-\left(-\frac{1}{4}\right)^k fork<0\\ \left(-\frac{1}{5}\right)^k fork\geq 0\end{cases}$

Ans. $\frac{4z}{1+4z}+\frac{5z}{5z+1}$

22. Find the Z transform of f(k) = c^k sinh αk, k ≥ 0 Ans. $\frac{cz\ sinh\alpha}{z^2-2czcosh\alpha+c^2}$

23. Show that $Z\{cos\alpha k\}=\frac{z^2-zcos\alpha}{z^2-2zcos\alpha+1}$

24. Show that $z\{cos2k\}=\frac{z^2-(cos2)z}{z^2-2(cos2)z+1}$

25. Find the Z transform of $cosk\frac{\pi}{2}$ Ans. $\frac{z^2}{z^2+1}$
26. Find $z\{\sin(\alpha k)\}, k \geq 0$ Ans. $\frac{zsin\alpha}{z^2-2zcos\alpha+1}$
27. Find the Z transform of $f(k) = \cos\left(\frac{k\pi}{3} + \alpha\right)$ $, k \geq 0$

Ans. $\frac{z\left[zcos\alpha-\cos\left(\frac{\pi}{3}-\alpha\right)\right]}{z^2-2\cos\frac{\pi}{3}.z+1}$

28. Find $Z\{f(k)\}$ where $f(k) = \cos\left(\frac{k\pi}{4} + a\right)$ where $k \geq 0$

Ans. $\frac{z\left[zcosa-\cos\left(a-\frac{\pi}{4}\right)\right]}{z^2-\sqrt{2}z+1}$

29. Find the Z transform of $f(k) = \cos(ak + b), k \geq 0$

Ans. $\frac{z[zcosb-\cos(a-b)]}{z^2-2zcosa+1}$

30. Find the Z transform of $f(k) = \sin(3k + 5)$ Ans. $\frac{z[-sin2+zsin5]}{z^2-2zcos3+1}$
31. Find the Z transform of $a^n, cosn\theta, sinn\theta$

Ans. $\frac{z}{z-a}, \frac{z(z-cos\theta)}{z^2-2zcos\theta+1}, \frac{zsin\theta}{z^2-2zcos\theta+1}$

Type II: Based on property

1. Find Z[2^k sin(3k + 2)], k ≥ 0 Ans. $\frac{z[2sin1+zsin2]}{z^2-4zcos3+4}$
2. Find $Z\left[3^k\cos\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)\right], k \geq 0$ Ans. $\frac{1}{\sqrt{2}}.\frac{z^2-3z}{z^2+9}$
3. Find Z transform of $2^k sinh3k, k \geq 0$
   [M17/CompIT/6M]
   Solution:
   We know that,
   $Z\{a^k\} = \frac{z}{z-a}$
   Now,
   $Z\{\sinh 3k\} = Z\left\{\frac{e^{3k}-e^{-3k}}{2}\right\}$
   $= \frac{1}{2}\, Z\{e^{3k} - e^{-3k}\}$
   $= \frac{1}{2}\left[\frac{z}{z-e^3} - \frac{z}{z-e^{-3}}\right]$
   $= \frac{1}{2}\left[\frac{z^2-ze^{-3}-z^2+ze^3}{z^2-e^3z-e^{-3}z+1}\right]$
   $= \frac{1}{2}\left[\frac{z(e^3-e^{-3})}{z^2-z(e^3+e^{-3})+1}\right]$
   $= \frac{1}{2}\left[\frac{z(2\sinh 3)}{z^2-z(2\cosh 3)+1}\right]$
   $Z\{\sinh 3k\} = \frac{z\sinh 3}{z^2-2z\cosh 3+1}$
   Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$
   $Z\{2^k \sinh 3k\} = \frac{\frac{z}{2}\sinh 3}{\left(\frac{z}{2}\right)^2-2\left(\frac{z}{2}\right)\cosh 3+1} = \frac{2z\; sinh3}{z^2-4zcosh3+4}$

4. Find Z[2^k cos(3k + 2)] , k ≥ 0
   [N15/CompIT/6M]
   Solution:
   $Z\{\cos(3k+2)\} = Z\{cos3k\; cos2 - sin3k\; sin2\}$
   $Z\{\cos(3k+2)\} = cos2\; Z\{cos3k\} - sin2\; Z\{sin3k\}$
   $Z\{\cos(3k+2)\} = cos2\;\left[\frac{z(z-cos3)}{z^2-2zcos3+1}\right] - sin2\;\left[\frac{zsin3}{z^2-2zcos3+1}\right]$
   By $Z\{cos\alpha k\} = \frac{z(z-cos\alpha)}{z^2-2zcos\alpha+1}, Z\{sin\alpha k\} = \frac{zsin\alpha}{z^2-2zcos\alpha+1}$
   $\therefore Z\{\cos(3k+2)\} = \frac{z^2cos2-zcos2\; cos3-z\; sin2\; sin3}{z^2-2zcos3+1}$
   $Z\{\cos(3k+2)\} = \frac{z^2cos2-z(cos2cos3+sin2sin3)}{z^2-2zcos3+1}$
   $Z\{\cos(3k+2)\} = \frac{z^2cos2-zcos1}{z^2-2zcos3+1}$
   Now, by Change of scale property $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$$Z\{2^k \cos(3k+2)\} = \frac{\left(\frac{z}{2}\right)^2 cos2 - \left(\frac{z}{2}\right)cos1}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)cos3 + 1}$$

$$Z\{2^k \cos(3k+2)\} = \frac{z^2 cos2 - 2z\, cos1}{z^2 - 4zcos3 + 4}$$

5. Find the Z transform of $e^{3k} sin2k$ Ans. $\frac{e^3 zsin2}{z^2 - 2e^3 zcos2 + e^6}$
6. Find Z[3^k sinhαk], k ≥ 0 Ans. $\frac{3zsinh\alpha}{z^2 - 6zcosh\alpha + 9}$
7. Find $Z\left[3^k \sin\left(k\frac{\pi}{2} + \frac{\pi}{4}\right)\right], k \geq 0$ Ans. $\frac{1}{\sqrt{2}} . \frac{z^2+3z}{z^2+9}$
8. Find $Z[k.e^{-ak}], k \geq 0$ Ans. $\frac{e^a z}{(e^a z-1)^2}$
9. Find Z [k^2 e^{-ak}] , k ≥ 0 Ans. $z.e^a.\frac{ze^a+1}{(ze^a-1)^3}$
10. Find the Z transform of $f(k) = k^2 - 2k + 3,\ k \geq 0$
Ans.$\frac{3z^3 - 7z^2 + 6z}{(z-1)^3}$
11. Find the Z transform of $f(k) = a^{k-1}, k \geq 0$ Ans. $\frac{1}{z-a}$
12. Find Z [k^2 a^{k-1}] , k ≥ 0 Ans. $\frac{z(z+a)}{(z-a)^3}$
13. Find Z [k^2 a^{k-1} U(k – 1)], k ≥ 0
[M16/CompIT/6M]
Solution:
By definition,

$$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$$

$$Z\{U(k)\} = \sum_0^\infty 1.z^{-k}$$
$$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots..]$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots$$
$$= \left[1 - \frac{1}{z}\right]^{-1}$$
$$= \frac{1}{1-\frac{1}{z}}$$
$$\therefore Z\{U(k)\} = \frac{z}{z-1}$$

By Change of Scale, $Z\{a^k U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}$

By Shifting Property,

$$Z\{a^{k-1}U(k-1)\} = z^{-1}.\frac{z}{z-a} = \frac{1}{z-a}$$

By Multiplication by k,

$$Z\{k\ a^{k-1}U(k-1)\} = -z\frac{d}{dz}\left[\frac{1}{z-a}\right]$$

$$= -z\left[-\frac{1}{(z-a)^2}\right]$$

$$Z\{k\, a^{k-1}U(k-1)\} = \frac{z}{(z-a)^2}$$

By Multiplication by k,

$$Z\{k^2 a^{k-1}U(k-1)\} = -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right]$$

$$= -z\left[\frac{(z-a)^2[1]-z[2(z-a)]}{(z-a)^4}\right]$$

$$= -z\left[\frac{z-a-2z}{(z-a)^3}\right]$$

$$= -\frac{z(-z-a)}{(z-a)^3}$$

$$Z\{k^2 a^{k-1}U(k-1)\} = \frac{z(z+a)}{(z-a)^3}$$

14. Find the Z transform of $\delta(k-n)$ where $\delta(k) = \begin{cases} 1 & k=0 \\ 0 & otherwise \end{cases}$

    Ans. $\frac{1}{z^n}$

15. Find Z [(k + 1)a^k] , k ≥ 0 Ans. $\frac{z^2}{(z-a)^2}$

16. Find $Z\{2^k k^2\}$ Ans. $\frac{2z(z+2)}{(z-2)^3}$

17. Find Z [k 2^k + k 3^k], k ≥ 0 Ans. $\frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}$

18. Find the Z transform of (i) $4^k\delta(k-1)$ and (ii) $U(k-1)$

    where $\delta(k) = \begin{cases} 1 & k=0 \\ 0 & otherwise \end{cases}$ and $U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & otherwise \end{cases}$

    Ans. $\frac{4}{z}, \frac{1}{z-1}$

Type III: Convolution Theorem

1. State convolution Theorem for z transform hence if $f(k) = U(k)$ & $g(k) = 2^k U(k)$, find $Z\{f(k) * g(k)\}$ Ans. $\frac{z^2}{(z-1)(z-2)}$

2. State convolution Theorem for z transform hence if $f(k) = \frac{1}{3^k}, k \geq 0$ & $g(k) = \frac{1}{4^k}, k \geq 0$, find $Z\{f(k) * g(k)\}$

[M15/CompIT/5M]

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z). G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k) z^{-k}$

$Z\left\{\frac{1}{3^k}\right\} = \sum_0^{\infty} \frac{1}{3^k}. z^{-k}$

$= \frac{z^0}{3^0} + \frac{z^{-1}}{3^1} + \frac{z^{-2}}{3^2} + \frac{z^{-3}}{3^3} + \cdots ..$

$= 1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots ….$

$= \left[1 - \frac{1}{3z}\right]^{-1}$

$= \frac{1}{1 - \frac{1}{3z}}$

$Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}$

$\therefore Z\{f(k)\} = Z\left\{\frac{1}{3^k}\right\}$

$\therefore F(z) = \frac{3z}{3z-1}$

Similarly,

$Z\left\{\frac{1}{4^k}\right\} = \frac{4z}{4z-1}$

$\therefore Z\{g(k)\} = Z\left\{\frac{1}{4^k}\right\}$

$\therefore G(z) = \frac{4z}{4z-1}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z). G(z) = \frac{3z}{3z-1} . \frac{4z}{4z-1} = \frac{12z^2}{(3z-1)(4z-1)}$

3. State convolution Theorem for z transform hence if $f(k) = 4^k U(k) \& g(k) = 5^k U(k)$, find $Z\{f(k) * g(k)\}$

[M14/CompIT/6M]

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z)$

By definition,

$U(k) = \begin{cases} 1 & for\ k \geq 0 \\ 0 & for\ k < 0 \end{cases}$

$Z\{U(k)\} = \sum_0^\infty 1.z^{-k}$

$= [z^0 + z^{-1} + z^{-2} + z^{-3} + \cdots \ldots..]$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \ldots.$

$= \left[1 - \frac{1}{z}\right]^{-1}$

$= \frac{1}{1-\frac{1}{z}}$

$\therefore Z\{U(k)\} = \frac{z}{z-1}$

By Change of Scale,

$Z\{a^k U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a}-1} = \frac{z}{z-a}$

Now,

$Z\{f(k)\} = Z\{4^k U(k)\}$

$F(z) = \frac{z}{z-4}$

Also,

$Z\{g(k)\} = Z\{5^k U(k)\}$

$G(z) = \frac{z}{z-5}$

By Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{z}{z-4}.\frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}$

4. State convolution Theorem for z transform hence if $f(k) = \frac{1}{2^k}, k \geq 0$ $\&g(k) = cosk\pi, k \geq 0$, find $Z\{f(k) * g(k)$ **[N18/Comp/6M]**

Solution:

If $Z\{f(k)\} = F(z)$ and $Z\{g(k) = G(z)$

Then by Convolution Theorem,

$Z\{f(k) * g(k)\} = F(z).G(z)$

We have,

$Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)z^{-k}$

$Z\left\{\frac{1}{2^k}\right\} = \sum_0^\infty \frac{1}{2^k}.z^{-k}$

$= \frac{z^0}{2^0} + \frac{z^{-1}}{2^1} + \frac{z^{-2}}{2^2} + \frac{z^{-3}}{2^3} + \cdots ..$

$= 1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \frac{1}{(2z)^3} + \cdots \ldots.$

$$= \left[1 - \frac{1}{2z}\right]^{-1}$$
$$= \frac{1}{1-\frac{1}{2z}}$$
$$Z\left\{\frac{1}{2^k}\right\} = \frac{2z}{2z-1}$$
$$\therefore Z\{f(k)\} = Z\left\{\frac{1}{2^k}\right\}$$
$$\therefore F(z) = \frac{2z}{2z-1}$$

Also,

We know that,

$$Z\{a^k\} = \frac{z}{z-a}$$

Now,

$$Z\{cosk\pi\} = Z\left\{\frac{e^{\pi ik}+e^{-\pi ik}}{2}\right\}$$
$$= \frac{1}{2}\, Z\left\{e^{\pi ik} + e^{-\pi k}\right\}$$
$$= \frac{1}{2}\left[\frac{z}{z-e^{\pi i}} + \frac{z}{z-e^{-\pi i}}\right]$$
$$= \frac{1}{2}\left[\frac{z^2-ze^{-\pi i}+z^2-ze^{\pi i}}{z^2-e^{\pi i}z-e^{-\pi i}z+1}\right]$$
$$= \frac{1}{2}\left[\frac{2z^2-z\left(e^{\pi i}-e^{-\pi i}\right)}{z^2-z\left(e^{\pi i}+e^{-\pi i}\right)+1}\right]$$
$$= \frac{1}{2}\left[\frac{2z^2-z(2\,cos\pi)}{z^2-z(2cos\pi)+1}\right]$$
$$Z\{cosk\pi\} = \frac{z^2+z}{z^2+2z+1} = \frac{z(z+1)}{(z+1)^2} = \frac{z}{z+1}$$
$$G(z) = \frac{z}{z+1}$$

By Convolution Theorem,

$$Z\{f(k) * g(k)\} = F(z).G(z) = \frac{2z}{2z-1}.\frac{z}{z+1}$$

5. State convolution Theorem for z transform hence if $f(k) = 3^k, k \geq 0$ &$g(k) = 4^k, k \geq 0$, find $Z\{f(k) * g(k)\}$ Ans. $\frac{z^2}{(z-3)(z-4)}$

6. Find Z[f(k)] where$f(k) = \frac{1}{2^k} * \frac{1}{3^k}$ Ans. $\frac{6z^2}{(2z-1)(3z-1)}$

Type IV: Binomial Expansion (Inverse Z transform)

1. $Z^{-1}\left[\frac{1}{z-a}\right]$ for $|z| > a$ and for $|z| < a$

Ans. $a^{k-1}, k > 0$ and – $a^{k-1}, k \leq 0$

2. $Z^{-1}\left[\frac{z}{z-a}\right]$for $|z| < a$ and for $|z| > a$

Ans. $a^{k}, k \geq 0$ and – $a^{k}, k < 0$

3. $Z^{-1}\left[\frac{z}{z-5}\right]$for $|z| < 5$

[M19/Comp/4M]

Solution:

Let $F(z) = \frac{z}{z-5}$

$F(z) = \frac{z}{-5+z}$

$F(z) = \frac{z}{(-5)\left(1-\frac{z}{5}\right)}$

$F(z) = -\frac{z}{5}\left[1-\frac{z}{5}\right]^{-1}$

$F(z) = -\frac{z}{5}\left[1+\frac{z}{5}+\frac{z^2}{5^2}+\cdots\ldots.\right]$

$F(z) = -\frac{z}{5}-\frac{z^2}{5^2}-\frac{z^3}{5^3}-\cdots\ldots$

$F(z) = -5^{-1}z^1 - 5^{-2}z^2 - 5^{-3}z^3 - \cdots \ldots.$

Thus, coefficient of $z^k = -5^{-k}, k > 0$

$\therefore$ coefficient of $z^{-k} = -5^{k}, k < 0$

Thus, $Z^{-1}\left[\frac{z}{z-5}\right] = -5^{k}\quad,\quad k < 0$

4. $Z^{-1}\left[\frac{1}{(z-a)^2}\right]$

for (a)$|z| < a$ **[N17/Comp/4M]**

(b) $|z| > a$

Solution:

Let $F(z) = \frac{1}{(z-a)^2}$

$F(z) = \frac{1}{(-a+z)^2}$

$F(z) = \frac{1}{(-a)^2\left(1-\frac{z}{a}\right)^2}$

$F(z) = \frac{1}{a^2}\left[1-\frac{z}{a}\right]^{-2}$

$F(z) = \frac{1}{a^2}\left[1+2\frac{z}{a}+3\frac{z^2}{a^2}+4\frac{z^3}{a^3}+\cdots\ldots.\right]$

$F(z) = \frac{1}{a^2}+\frac{2z}{a^3}+\frac{3z^2}{a^4}+\frac{4z^4}{a^5}+\cdots.$

$F(z) = 1.a^{-2}z^0 + 2.a^{-3}z^1 + 3a^{-4}z^2 + 4a^{-5}a^4 + \cdots \ldots.$

Thus, coefficient of $z^k = (k+1)a^{-(k+2)}, k \geq 0$

$\therefore$ coefficient of $z^{-k} = (-k+1)a^{-(-k+2)}, k \leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = -(k-1)a^{k-2} \quad , \quad k \leq 0$

5. $Z^{-1}\left[\frac{1}{(z-5)^2}\right]$ for $|z| < 5$

[M18/Comp/4M]

Solution:

Let $F(z) = \frac{1}{(z-5)^2}$

$$F(z) = \frac{1}{(-5+z)^2}$$

$$F(z) = \frac{1}{(-5)^2\left(1-\frac{z}{5}\right)^2}$$

$$F(z) = \frac{1}{5^2}\left[1 - \frac{z}{5}\right]^{-2}$$

$$F(z) = \frac{1}{5^2}\left[1 + 2\frac{z}{5} + 3\frac{z^2}{5^2} + 4\frac{z^3}{5^3} + \cdots \ldots.\right]$$

$$F(z) = \frac{1}{5^2} + \frac{2z}{5^3} + \frac{3z^2}{5^4} + \frac{4z^4}{5^5} + \cdots.$$

$$F(z) = 1.5^{-2}z^0 + 2.5^{-3}z^1 + 35^{-4}z^2 + 45^{-5}a^4 + \cdots \ldots$$

Thus, coefficient of $z^k = (k+1)5^{-(k+2)}, k \geq 0$

$\therefore$ coefficient of $z^{-k} = (-k+1)5^{-(-k+2)}, k \leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-5)^2}\right] = -(k-1)5^{k-2} \quad , \quad k \leq 0$

6. $Z^{-1}\left[\frac{1}{(z-5)^3}\right]$for $|z| > 5$ Ans. $\frac{(k-1)(k-2)}{2}.5^{k-3}, k \geq 3$

7. $Z^{-1}\left[\frac{1}{(z-5)^3}\right]$for $|z| < 5$

[N16/CompIT/6M]

Solution:

Let $F(z) = \frac{1}{(z-5)^3}$

$$F(z) = \frac{1}{(-5+z)^3}$$

$$F(z) = \frac{1}{(-5)^3\left(1-\frac{z}{5}\right)^3}$$

$$F(z) = -\frac{1}{5^3}\left[1 - \frac{z}{5}\right]^{-3}$$

$$F(z) = -\frac{1}{5^3}\left[1 + (-3)\left(-\frac{z}{5}\right) + \frac{(-3)(-3-1)}{2!}\left(-\frac{z}{5}\right)^2 + \cdots \ldots.\right]$$

$$F(z) = -\frac{1}{5^3}\left[1 + 3\frac{z}{5} + 6\frac{z^2}{5^2} + 10\frac{z^3}{5^3} + \cdots..\right]$$

$$F(z) = -\frac{1}{5^3} - 3\frac{z}{5^4} - 6\frac{z^2}{5^5} - 10\frac{z^3}{5^6} - \cdots \ldots$$

$F(z) = -1.z^0.5^{-3} - 3.z^1.5^{-4} - 6.z^2.5^{-5} - 10.z^3.5^{-6} - \cdots ....$

Thus, coefficient of $z^k = -\frac{(k+1)(k+2)}{2}\ 5^{-(k+3)}, k \geq 0$

$\therefore$ coefficient of $z^{-k} = -\frac{(-k+1)(-k+2)}{2}\ 5^{-(-k+3)}, k \leq 0$

Thus, $Z^{-1}\left[\frac{1}{(z-5)^3}\right] = -\frac{(k-1)(k-2)}{2}\ 5^{k-3}\ ,\ k \leq 0$

8. $Z^{-1}\left[\frac{1}{(z-1)^2}\right]$ for $|z| < 1$ and $|z| > 1$

Ans. $-k+1, k \leq 0$ and $k-1, k \geq 2$

9. $Z^{-1}\left[\frac{1}{(z-1)^2}\right]$ for $|z| > 1$

[M19/Comp/4M]

Solution:

Let $F(z) = \frac{1}{(z-1)^2}$

$F(z) = \frac{1}{(z)^2\left(1-\frac{1}{z}\right)^2}$

$F(z) = \frac{1}{z^2}\left[1 - \frac{1}{z}\right]^{-2}$

$F(z) = \frac{1}{z^2}\left[1 + 2.\frac{1}{z} + 3.\frac{1}{z^2} + 4.\frac{1}{z^3} + \cdots ....\right]$

$F(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \frac{4}{z^5} + \cdots.$

$F(z) = 1.z^{-2} + 2.z^{-3} + 3z^{-4} + 4z^{-5} + \cdots ....$

Thus, coefficient of $z^{-k} = (k-1),\ k \geq 2$

Thus, $Z^{-1}\left[\frac{1}{(z-1)^2}\right] = (k-1)\ ,\ k \geq 2$

Type V: Partial Fractions (Inverse Z transform)

1. $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ if ROC is $2 < |z| < 3$

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - 3B = 1$

On solving, we get $A = 1, B = -1$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

For the ROC $2 < |z| < 3$

$F(z) = \frac{1}{-3+z} - \frac{1}{z-2}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots..\right] - \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots..\right]$

$F(z) = \left[-\frac{1}{3}-\frac{z}{3^2}-\frac{z^2}{3^3}-\frac{z^3}{3^4}-\cdots...\right] + \left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}-\frac{2^3}{z^4}-\cdots...\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k > 0 \end{cases}$$

2. $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ if ROC is $|z| > 3$

[N17/Comp/4M]

Solution:

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$F(z) = \frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots..\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \cdots ...\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots ...\right]$

$F(z) = [3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} - \cdots.] + [-2^0 z^{-1} - 2^1 z^{-2} - 2^2 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0$

3. $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right], |z| > 2$

[M16/CompIT/6M][N16/CompIT/6M]

Solution:

We have,

$F(z) = \frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$z = A(z-2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - B = 0$

On solving, we get

$A = -1, B = 2$

$F(z) = -\frac{1}{z-1} + \frac{2}{z-2}$

$F(z) = -\frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{2}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{z}\left[1 - \frac{1}{z}\right]^{-1} + \frac{2}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots..\right] + \frac{2}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots.\right]$

$F(z) = [-z^{-1} - z^{-2} - z^{-3} + \cdots.] + [2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = -1, k > 0$

From second series,

Coefficient of $z^{-k} = 2^k, k > 0$

Thus,

$Z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\} = 2^k - 1, k > 0$

4. $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right], 1 < |z| < 2$

Solution:

We have,

$F(z) = \frac{z}{(z-1)(z-2)}$

Let $\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$z = A(z-2) + B(z-1)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - B = 0$

On solving, we get

$A = -1, B = 2$

$F(z) = -\frac{1}{z-1} + \frac{2}{2-z}$

$F(z) = -\frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{2}{2\left[1-\frac{z}{2}\right]}$

$F(z) = -\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{2}{2}\left[1-\frac{z}{2}\right]^{-1}$

$F(z) = -\frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots ..\right] + \left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots .\right]$

$F(z) = [-z^{-1} - z^{-2} - z^{-3} + \cdots .] + [2^0 z^0 + 2^{-1}z^1 + 2^{-2}z^2 + \cdots ..]$

From first series,

Coefficient of $z^{-k} = -1 , k > 0$

From second series,

Coefficient of $z^{k} = 2^{-k} , k \geq 0$

Coefficient of $z^{-k} = 2^{k}, k \leq 0$

Thus,

$Z^{-1}\left\{\frac{z}{(z-1)(z-2)}\right\} = \begin{cases} 2^k & k \leq 0 \\ -1 & k > 0 \end{cases}$

5. $Z^{-1}\left[\frac{z}{(z-3)(z-2)}\right]$ if ROC is $|z| > 3$

[M18/Comp/4M]

Solution:

We have,

$F(z) = \frac{z}{(z-3)(z-2)}$

Let $\frac{z}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$z = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 1$

$-2A - 3B = 0$

On solving, we get $A = 3, B = -2$

$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$

ROC is $|z| > 3$

$F(z) = \frac{3}{z\left[1-\frac{3}{z}\right]} - \frac{2}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{3}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{2}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{3}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots..\right] - \frac{2}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[\frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \frac{3^4}{z^4} + \cdots \ldots\right] + \left[-\frac{2}{z} - \frac{2^2}{z^2} - \frac{2^3}{z^3} - \frac{2^4}{z^4} - \cdots \ldots\right]$

$F(z) = [3^1z^{-1} + 3^2z^{-2} + 3^3z^{-3} - \cdots.] + [-2^1z^{-1} - 2^2z^{-2} - 2^3z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^k, k > 0$

From second series,

Coefficient of $z^{-k} = -2^k, k > 0$

Thus,

$Z^{-1}\left\{\frac{z}{(z-3)(z-2)}\right\} = 3^k - 2^k, k > 0$

6. $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ if ROC is (i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

[N13/CompIT/8M]

Solution:

We have,

$F(z) = \frac{1}{(z-3)(z-2)}$

Let $\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$1 = A(z-2) + B(z-3)$

Comparing the coefficients, we get

$A + B = 0$

$-2A - 3B = 1$

On solving, we get $A = 1, B = -1$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

(i) $|z| < 2$

$F(z) = \frac{1}{-3+z} - \frac{1}{-2+z}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{-2\left[1-\frac{z}{2}\right]}$

$F(z) = -\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} + \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots..\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots..\right]$

$F(z) = \left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} - \cdots \dots\right] + \left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \cdots \dots\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [2^{-1}z^0 + 2^{-2}z^1 + 2^{-3}z^2 + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^k = 2^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = 2^{k-1}, k \leq 0$

Thus, $Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 2^{k-1} - 3^{k-1}, k \leq 0$

(ii) $2 < |z| < 3$

$F(z) = \frac{1}{-3+z} - \frac{1}{z-2}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = -\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots..\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} - \cdots \dots\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots \dots\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}z^1 - 3^{-3}z^2 - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^k = -3^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -3^{-(-k+1)}, k \leq 0$

i.e. Coefficient of $z^{-k} = -3^{k-1}, k \leq 0$

From second series,

Coefficient of $z^{-k} = -2^{k-1}, k > 0$

Thus,

$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k > 0 \end{cases}$

(iii) $|z| > 3$

$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$F(z) = \frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{2}{z}\right]}$

$F(z) = \frac{1}{z}\left[1 - \frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots..\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots..\right]$

$F(z) = \left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \cdots \dots\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots \dots\right]$

$F(z) = [3^0z^{-1} + 3^1z^{-2} + 3^2z^{-3} - \cdots.] + [-2^0z^{-1} - 2^1z^{-2} - 2^2z^{-3} + \cdots]$

From first series,

Coefficient of $z^{-k} = 3^{(k-1)}, k > 0$
From second series,
Coefficient of $z^{-k} = -2^{k-1}, k > 0$
Thus,
$Z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = 3^{k-1} - 2^{k-1}, k > 0$

7. $Z^{-1}\left[\frac{z+2}{z^2-2z+1}\right], |z| > 1$
[M14/CompIT/8M][N15/CompIT/6M]
Solution:
We have,
$F(z) = \frac{z+2}{z^2-2z+1}$
$F(z) = \frac{z+2}{(z-1)^2}$
$F(z) = \frac{z-1+1+2}{(z-1)^2}$
$F(z) = \frac{z-1}{(z-1)^2} + \frac{3}{(z-1)^2}$
$F(z) = \frac{1}{z-1} + \frac{3}{(z-1)^2}$
$F(z) = \frac{1}{z\left[1-\frac{1}{z}\right]} + \frac{3}{\left(z\left[1-\frac{1}{z}\right]\right)^2}$
$F(z) = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} + \frac{3}{z^2}\left[1-\frac{1}{z}\right]^{-2}$
$F(z) = \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots.\right] + \frac{3}{z^2}\left[1+\frac{2}{z}+\frac{3}{z^2}+\frac{4}{z^3}+\cdots..\right]$
$F(z) = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots..\right] + \left[\frac{3}{z^2}+\frac{3.2}{z^3}+\frac{3.3}{z^4}+\frac{3.4}{z^5}+\cdots.\right]$
$F(z) = [z^{-1}+z^{-2}+z^{-3}+\cdots\ldots] + [3.1z^{-2}+3.2z^{-3}+3.3z^{-4}+\cdots.]$
$F(z) = (3.0+1)z^{-1} + (3.1+1)z^{-2} + (3.2+1)z^{-3} + \cdots\ldots\ldots$
Coefficient of $z^{-k} = (3(k-1)+1), k > 0$
Coefficient of $z^{-k} = 3k-2, k > 0$
Thus,
$Z^{-1}\left\{\frac{z+2}{z^2-2z+1}\right\} = 3k-2, k > 0$

8. $Z^{-1}\left[\frac{2z^2-10z+13}{(z-3)^2(z-2)}\right], 2 < |z| < 3$
Solution:
We have,
$F(z) = \frac{2z^2-10z+13}{(z-3)^2(z-2)}$
Let $\frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{A}{z-3} + \frac{B}{(z-3)^2} + \frac{C}{z-2}$
$2z^2 - 10z + 13 = A(z-3)(z-2) + B(z-2) + C(z-3)^2$

$2z^2 - 10z + 13 = A(z^2 - 5z + 6) + B(z - 2) + C(z^2 - 6z + 9)$

Comparing the coefficients, we get

$A + 0B + C = 2$

$-5A + B - 6C = -10$

$6A - 2B + 9C = 13$

On solving, we get

$A = 1, B = 1, C = 1$

$F(z) = \frac{1}{z-3} + \frac{1}{(z-3)^2} + \frac{1}{z-2}$

For ROC, $2 < |z| < 3$

$F(z) = \frac{1}{-3+z} + \frac{1}{(-3+z)^2} + \frac{1}{z-2}$

$F(z) = \frac{1}{-3\left[1-\frac{z}{3}\right]} + \frac{1}{(-3)^2\left[1-\frac{z}{3}\right]^2} + \frac{1}{z\left(1-\frac{2}{z}\right)}$

$F(z) = -\frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} + \frac{1}{9}\left[1 - \frac{z}{3}\right]^{-2} + \frac{1}{z}\left[1 - \frac{2}{z}\right]^{-1}$

$F(z) = -\frac{1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots\right] + \frac{1}{9}\left[1 + 2\frac{z}{3} + 3\frac{z^2}{3^2} + \cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots ..\right]$

$F(z) = [-3^{-1}z^0 - 3^{-2}.z^1 - 3^{-3}.z^2 - 3^{-4}z^3 \ldots.]$
$\quad +[1.3^{-2}z^0 + 2.3^{-3}z^1 + 3.3^{-4}.z^2 + \cdots \ldots.]$
$\quad +[2^0z^{-1} + 2^1z^{-2} + 2^2z^{-3} + \cdots \ldots.]$

$F(z) = [(1.3^{-2} - 3^{-1})z^0 + (2.3^{-3} - 3^{-2})z^1 + (3.3^{-4} - 3^{-3})z^2 + \cdots ..]$
$\quad +[2^0z^{-1} + 2^1z^{-2} + 2^2z^{-3} + \cdots \ldots.]$

From first series,

Coefficient of $z^k = (k+1).3^{-(k+2)} - 3^{-(k+1)}, k \geq 0$

$= [k + 1 - 3]3^{-k-2}$

$= [k - 2]3^{-k-2}, k \geq 0$

Coefficient of $z^{-k} = [-k - 2]3^{k-2}\ , k \leq 0$

From second series,

Coefficient of $z^{-k} = 2^{k-1}\ , k > 0$

Thus, $Z^{-1}\left\{\frac{2z^2-10z+13}{(z-3)^2(z-2)}\right\} = \begin{cases}[-k-2]3^{k-2} & , k \leq 0 \\ 2^{k-1} & , k > 0\end{cases}$

9. $Z^{-1}\left[\frac{3z^2+2z}{z^2-3z+2}\right]$ for $1 < |z| < 2$

Solution:

We have,

$F(z) = \frac{3z^2+2z}{(z-1)(z-2)}$

$\frac{F(z)}{z} = \frac{3z+2}{(z-1)(z-2)}$

Let $\frac{3z+2}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$3z + 2 = A(z - 2) + B(z - 1)$

Comparing the coefficients, we get

$A + B = 3$
$-2A - B = 2$
On solving, we get
$A = -5, B = 8$
$\frac{F(z)}{z} = -\frac{5}{z-1} + \frac{8}{z-2}$
$F(z) = -\frac{5z}{z-1} + \frac{8z}{z-2}$
For ROC $1 < |z| < 2$
$F(z) = -\frac{5z}{z-1} + \frac{8z}{-2+z}$
$F(z) = -\frac{5z}{z\left(1-\frac{1}{z}\right)} + \frac{8z}{-2\left(1-\frac{z}{2}\right)}$
$F(z) = -5\left[1 - \frac{1}{z}\right]^{-1} - 4z\left[1 - \frac{z}{2}\right]^{-1}$
$F(z) = -5\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots..\right] - 4z\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots.\right]$
$F(z) = [-5z^0 - 5z^{-1} - 5z^{-2} + \cdots.] + [-4.\,2^0 z^1 - 4.\,2^{-1}z^2 - 4.\,2^{-2}z^3 +]$
From first series,
Coefficient of $z^{-k} = -5$, $k \geq 0$
From second series,
Coefficient of $z^k = -4.2^{-(k-1)}$, $k > 0$
Coefficient of $z^{-k} = -4.2^{k+1}, k < 0$
Coefficient of $z^{-k} = -8.2^k, k < 0$
Thus,
$Z^{-1}\left\{\frac{3z^2+2z}{z^2-3z+2}\right\} = \begin{cases} -8.2^k & k < 0 \\ -5 & k \geq 0 \end{cases}$

10. $Z^{-1}\left[\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right]$ for (i) $|z| > 1$ (ii) $|z| < \frac{1}{2}$ (iii) $\frac{1}{2} < |z| < 1$

Solution:
We have,
$F(z) = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$
$\frac{F(z)}{z} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)}$
Let $\frac{z}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$
$z = A\left(z - \frac{1}{2}\right) + B(z-1)$
Comparing the coefficients, we get
$A + B = 1$
$-\frac{1}{2}A - B = 0$

On solving, we get $A = 2, B = -1$

$\frac{F(z)}{z} = \frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}$

$F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$

(i) $|z| > 1$

$F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$

$F(z) = \frac{2z}{z\left[1-\frac{1}{z}\right]} - \frac{z}{z\left[1-\frac{1}{2z}\right]}$

$F(z) = 2\left[1 - \frac{1}{z}\right]^{-1} - \left[1 - \frac{1}{2z}\right]^{-1}$

$F(z) = 2\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots ..\right] - \left[1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \frac{1}{2^3z^3} + \cdots ..\right]$

$F(z) = [2z^0 + 2z^{-1} + 2z^{-2} + \cdots .] + [-2^0z^0 - 2^{-1}z^{-1} - 2^{-2}z^{-2} + \cdots]$

From first series,

Coefficient of $z^{-k} = 2, \quad k \geq 0$

From second series,

Coefficient of $z^{-k} = -2^{-k}, k \geq 0$

Thus, $Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = 2 - 2^{-k}, k \geq 0$

(ii) $|z| < \frac{1}{2}$

$F(z) = \frac{2z}{-1+z} - \frac{z}{-\frac{1}{2}+z}$

$F(z) = \frac{2z}{-[1-z]} - \frac{z}{-\frac{1}{2}[1-2z]}$

$F(z) = -2z[1-z]^{-1} + 2z[1-2z]^{-1}$

$F(z) = -2z[1 + z + z^2 + z^3 + \cdots ..] + 2z[1 + 2z + 2^2z^2 + 2^3z^3 + \cdots ..]$

$F(z) = [-2z - 2z^2 - 2z^3 - \cdots ...] + [2z + 2^2z^2 + 2^3z^3 + \cdots ...]$

From first series,

Coefficient of $z^k = -2, k > 0$

Coefficient of $z^{-k} = -2, k < 0$

From second series,

Coefficient of $z^k = 2^k, k > 0$

Coefficient of $z^{-k} = 2^{-k}, k < 0$

Thus,

$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = 2^{-k} - 2 \quad , \quad k < 0$

(iii) $\frac{1}{2} < |z| < 1$

$F(z) = \frac{2z}{-1+z} - \frac{z}{z-\frac{1}{2}}$

$F(z) = \frac{2z}{-[1-z]} - \frac{z}{z\left[1-\frac{1}{2z}\right]}$

$F(z) = -2z[1-z]^{-1} - \left[1-\frac{1}{2z}\right]^{-1}$

$F(z) = -2z[1+z+z^2+z^3+\cdots..] - \left[1+\frac{1}{2z}+\frac{1}{2^2z^2}+\frac{1}{2^3z^3}+\cdots..\right]$

$F(z) = [-2z-2z^2-2z^3-\cdots...] + [-2^0z^0-2^{-1}z^{-1}-2^{-2}z^{-2}+\cdots]$

From first series,

Coefficient of $z^k = -2, k > 0$

Coefficient of $z^{-k} = -2, k < 0$

From second series,

Coefficient of $z^{-k} = -2^{-k}, k \geq 0$

Thus,

$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = \begin{cases} -2 & k < 0 \\ -2^{-k} & k \geq 0 \end{cases}$

11. $Z^{-1}\left[\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right]$ for $\frac{1}{2} < |z| < 1$

Solution:

We have,

$F(z) = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$

$\frac{F(z)}{z} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)}$

Let $\frac{z}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$

$z = A\left(z-\frac{1}{2}\right) + B(z-1)$

Comparing the coefficients, we get

$A + B = 1$

$-\frac{1}{2}A - B = 0$

On solving, we get $A = 2, B = -1$

$\frac{F(z)}{z} = \frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}$

$F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$

For ROC $\frac{1}{2} < |z| < 1$

$F(z) = \frac{2z}{-1+z} - \frac{z}{z-\frac{1}{2}}$

$F(z) = \frac{2z}{-[1-z]} - \frac{z}{z\left[1-\frac{1}{2z}\right]}$

$F(z) = -2z[1-z]^{-1} - \left[1 - \frac{1}{2z}\right]^{-1}$

$F(z) = -2z[1 + z + z^2 + z^3 + \cdots..] - \left[1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \frac{1}{2^3z^3} + \cdots..\right]$

$F(z) = [-2z - 2z^2 - 2z^3 - \cdots\ldots] + [-2^0z^0 - 2^{-1}z^{-1} - 2^{-2}z^{-2} + \cdots]$

From first series,

Coefficient of $z^k = -2, k > 0$

Coefficient of $z^{-k} = -2, k < 0$

From second series,

Coefficient of $z^{-k} = -2^{-k}, k \geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right\} = \begin{cases} -2 & k < 0 \\ -2^{-k} & k \geq 0 \end{cases}$$

12. Find the inverse z transform of $F(z) = \frac{z^3}{(z-3)(z-2)^2}$ (i) $2 < |z| < 3$ (ii) $|z| > 3$

[N14/CompIT/6M]

Solution:

We have,

$F(z) = \frac{z^3}{(z-3)(z-2)^2}$

$\frac{F(z)}{z} = \frac{z^2}{(z-3)(z-2)^2}$

Let $\frac{z^2}{(z-3)(z-2)^2} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$

$z^2 = A(z-2)^2 + B(z-3)(z-2) + C(z-3)$

$z^2 = A(z^2 - 4z + 4) + B(z^2 - 5z + 6) + C(z-3)$

Comparing the coefficients, we get

$A + B = 1$

$-4A - 5B + C = 0$

$4A + 6B - 3C = 0$

On solving, we get

$A = 9, B = -8, C = -4$

$\frac{F(z)}{z} = \frac{9}{z-3} - \frac{8}{z-2} - \frac{4}{(z-2)^2}$

$F(z) = \frac{9z}{z-3} - \frac{8z}{z-2} - \frac{4z}{(z-2)^2}$

(i) $2 < |z| < 3$

$F(z) = \frac{9z}{-3+z} - \frac{8z}{z-2} - \frac{4z}{(z-2)^2}$

$F(z) = \frac{9z}{-3\left[1-\frac{z}{3}\right]} - \frac{8z}{z\left[1-\frac{2}{z}\right]} - \frac{4z}{\left(z\left[1-\frac{2}{z}\right]\right)^2}$

$F(z) = -3z\left[1-\frac{z}{3}\right]^{-1} - 8\left[1-\frac{2}{z}\right]^{-1} - \frac{4}{z}\left[1-\frac{2}{z}\right]^{-2}$

$F(z) = -3z\ \left[1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\cdots\right] - 8\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right] - \frac{4}{z}\left[1+\frac{2.2}{z}+\frac{3.2^2}{z^2}+\cdots..\right]$

$F(z) = [-3.3^0.z - 3.3^{-1}.z^2 - 3.3^{-2}.z^3 - \cdots.]$
$\quad +[-8.2^0.z^0 - 8.2^1.z^{-1} - 8.2^2.z^{-2} - \cdots..]$
$\quad +[-4.1.2^0.z^{-1} - 4.2.2^1.z^{-2} - 4.3.2^2.z^{-3} - \cdots ....]$

$F(z) = [-3.3^0.z - 3.3^{-1}.z^2 - 3.3^{-2}.z^3 - \cdots.]$
$\quad +[(-8.2^0 - 4.0.2^{-1})z^0 + (-8.2^1 - 4.1.2^0)z^{-1} + (-8.2^2 - 4.2.2^1)z^{-2} + \cdots..]$

From first series,

Coefficient of $z^k = -3.3^{-(k-1)}, k > 0$

Coefficient of $z^{-k} = -3.3^{k+1}, k < 0$

Coefficient of $z^{-k} = -3^{k+2}, k < 0$

From second series,

Coefficient of $z^{-k} = (-8.2^k - 4.k.2^{k-1})\ , k \geq 0$

Coefficient of $z^{-k} = -8.2^k - 2k.2^k$

Coefficient of $z^{-k} = -(2k+8)2^k, k \geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^3}{(z-3)(z-2)^2}\right\} = \begin{cases} -3^{k+2} & ,k < 0 \\ -(2k+8)2^k & ,k \geq 0 \end{cases}$$

(ii) $|z| > 3$

$F(z) = \frac{9z}{z-3} - \frac{8z}{z-2} - \frac{4z}{(z-2)^2}$

$F(z) = \frac{9z}{z\left[1-\frac{3}{z}\right]} - \frac{8z}{z\left[1-\frac{2}{z}\right]} - \frac{4z}{\left(z\left[1-\frac{2}{z}\right]\right)^2}$

$F(z) = 9\left[1-\frac{3}{z}\right]^{-1} - 8\left[1-\frac{2}{z}\right]^{-1} - \frac{4}{z}\left[1-\frac{2}{z}\right]^{-2}$

$F(z) = 9\ \left[1+\frac{3}{z}+\frac{3^2}{z^2}+\frac{3^3}{z^3}+\cdots\right] - 8\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right] - \frac{4}{z}\left[1+\frac{2.2}{z}+\frac{3.2^2}{z^2}+\cdots..\right]$

$F(z) = [9.3^0.z^0 + 9.3^1.z^{-1} + 9.3^2.z^{-2} + \cdots.]$
$\quad +[-8.2^0.z^0 - 8.2^1.z^{-1} - 8.2^2.z^{-2} - \cdots..]$
$\quad +[-4.1.2^0.z^{-1} - 4.2.2^1.z^{-2} - 4.3.2^2.z^{-3} - \cdots ....]$

$F(z) = [9.3^0.z^0 + 9.3^1.z^{-1} + 9.3^2.z^{-2} + \cdots.]$
$\quad +[(-8.2^0 - 4.0.2^{-1})z^0 + (-8.2^1 - 4.1.2^0)z^{-1} + (-8.2^2 - 4.2.2^1)z^{-2} + \cdots..]$

From first series,

Coefficient of $z^{-k} = 9.3^k, k \geq 0$

From second series,

Coefficient of $z^{-k} = (-8.2^k - 4.k.2^{k-1})\ , k \geq 0$

Coefficient of $z^{-k} = -8.2^k - 2k.2^k$

Coefficient of $z^{-k} = -(2k+8)2^k, k \geq 0$

Thus,

$$Z^{-1}\left\{\frac{z^3}{(z-3)(z-2)^2}\right\} = 9.3^k - (2k+8)2^k\ , k \geq 0$$

13. $Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right], \frac{1}{5} < |z| < \frac{1}{4}$

Solution:

We have,

$F(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$

Let $\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{5}}$

$z = A\left(z-\frac{1}{5}\right) + B\left(z-\frac{1}{4}\right)$

Comparing the coefficients, we get

$A + B = 1$

$-\frac{A}{5} - \frac{B}{4} = 0$

On solving, we get

$A = 5, B = -4$

$F(z) = \frac{5}{z-\frac{1}{4}} - \frac{4}{z-\frac{1}{5}}$

For ROC $\frac{1}{5} < |z| < \frac{1}{4}$

$F(z) = \frac{5}{z-\frac{1}{4}} - \frac{4}{-\frac{1}{5}+z}$

$F(z) = \frac{5}{z\left[1-\frac{1}{4z}\right]} - \frac{4}{-\frac{1}{5}[1-5z]}$

$F(z) = \frac{5}{z}\left[1-\frac{1}{4z}\right]^{-1} + 20[1-5z]^{-1}$

$F(z) = \frac{5}{z}\left[1+\frac{1}{4z}+\frac{1}{4^2z^2}+\frac{1}{4^3z^3}+\cdots..\right] + 20[1+5z+5^2z^2+5^3z^3+..]$

$F(z) = [5.4^0z^{-1} + 5.4^{-1}z^{-2} + 5.4^{-2}z^{-3} + \cdots\ldots]$
$+[20.5^0z^0 + 20.5^1z^1 + 20.5^2z^2 + \cdots\ldots]$

From first series,

Coefficient of $z^{-k} = 5.4^{-(k-1)}, k > 0$

Coefficient of $z^{-k} = 20.4^{-k}, k > 0$

Coefficient of $z^{-k} = 5.4^{-k+1}, k > 0$

From second series,

Coefficient of $z^{k} = 20.5^{k}, k \geq 0$

Coefficient of $z^{-k} = 20.5^{-k}, k \leq 0$

Coefficient of $z^{-k} = 4.5^{-k+1}, k \leq 0$

Thus,

$$Z^{-1}\left\{\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right\} = \begin{cases} 4.5^{-k+1} & k \leq 0 \\ 5.4^{-k+1} & k > 0 \end{cases}$$

14. $Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right], \frac{1}{3}<|z|<\frac{1}{2}$

[M15/CompIT/6M]

Solution:

We have,

$$F(z)=\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

Let $\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z-\frac{1}{3}}$

$$1=A\left(z-\frac{1}{3}\right)+B\left(z-\frac{1}{2}\right)$$

Comparing the coefficients, we get

$$A+B=0$$

$$-\frac{A}{3}-\frac{B}{2}=1$$

On solving, we get

$$A=6, B=-6$$

$$F(z)=\frac{6}{z-\frac{1}{2}}-\frac{6}{z-\frac{1}{3}}$$

$$F(z)=\frac{6}{-\frac{1}{2}+z}-\frac{6}{z-\frac{1}{3}}$$

$$F(z)=\frac{6}{-\frac{1}{2}[1-2z]}-\frac{6}{z\left[1-\frac{1}{3z}\right]}$$

$$F(z)=-12[1-2z]^{-1}-\frac{6}{z}\left[1-\frac{1}{3z}\right]^{-1}$$

$$F(z)=-12[1+2z+2^2z^2+2^3z^3+\cdots..]-\frac{6}{z}\left[1+\frac{1}{3z}+\frac{1}{3^2z^2}+\frac{1}{3^3z^3}+..\right]$$

$$F(z)=[-12.2^0.z^0-12.2^1.z^1-12.2^2.z^2-\cdots\ldots]$$
$$+[-6.3^0.z^{-1}-6.3^{-1}.z^{-2}-6.3^{-2}.z^{-3}-\cdots\ldots]$$

From first series,

Coefficient of $z^k=-12.2^k$, $k\geq 0$

Coefficient of $z^{-k}=-12.2^{-k}$, $k\leq 0$

From second series,

Coefficient of $z^{-k}=-6.3^{k-1}$, $k>0$

Thus,

$$Z^{-1}\left\{\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right\}=\begin{cases}-12.2^{-k} & k\leq 0\\ -6.3^{k-1} & k>0\end{cases}$$

15. $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

Solution:

We have,

$F(z) = \frac{8z^2}{(2z-1)(4z-1)} = \frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z-\frac{1}{4}\right)}$

$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$

Let $\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$

$z = A\left(z - \frac{1}{4}\right) + B\left(z - \frac{1}{2}\right)$

Comparing the coefficients, we get

$A + B = 1$

$-\frac{A}{4} - \frac{B}{2} = 0$

On solving, we get $A = 2, B = -1$

$\frac{F(z)}{z} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$

(i) For $|z| < \frac{1}{4}$

$F(z) = \frac{2z}{-\frac{1}{2}+z} - \frac{z}{-\frac{1}{4}+z}$

$F(z) = \frac{2z}{-\frac{1}{2}(1-2z)} - \frac{z}{-\frac{1}{4}(1-4z)}$

$F(z) = -4z[1-2z]^{-1} + 4z[1-4z]^{-1}$

$F(z) = -4z[1 + 2z + 2^2z^2 + \cdots..] + 4z[1 + 4z + 4^2z^2 + \cdots....]$

$F(z) = [-4.2^0z^1 - 4.2^1z^2 - 4.2^2z^3 - \cdots.] + [4^1z^1 + 4^2z^2 + 4^3z^3 + \cdots.]$

From I series,

Coefficient of $z^k = -4.2^{k-1}, k > 0$

Coefficient of $z^{-k} = -4.2^{-k-1}, k < 0$

Coefficient of $z^{-k} = -2.2^{-k}, k < 0$

From II series,

Coefficient of $z^k = 4^k, k > 0$

Coefficient of $z^{-k} = 4^{-k}, k < 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = 4^{-k} - 2.2^{-k}, k < 0$

(ii) For $\frac{1}{4} < |z| < \frac{1}{2}$,

$F(z) = \frac{2z}{-\frac{1}{2}+z} - \frac{z}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{-\frac{1}{2}(1-2z)} - \frac{z}{z\left(1-\frac{1}{4z}\right)}$

$F(z) = -4z[1-2z]^{-1} - \left[1 - \frac{1}{4z}\right]^{-1}$

$F(z) = -4z[1 + 2z + 2^2z^2 + \cdots..] - \left[1 + \frac{1}{4z} + \frac{1}{4^2z^2} + \cdots \ldots.\right]$

$F(z) = [-4.2^0z^1 - 4.2^1z^2 - 4.2^2z^3 - \cdots.] + [-4^0z^0 - 4^{-1}z^{-1} - 4^{-2}z^{-2} + \cdots.]$

From I series,

Coefficient of $z^k = -4.2^{k-1}, k > 0$

Coefficient of $z^{-k} = -4.2^{-k-1}, k < 0$

Coefficient of $z^{-k} = -2.2^{-k}, k < 0$

From II series,

Coefficient of $z^{-k} = -4^{-k}, k \geq 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = \begin{cases} -2.2^{-k} & k < 0 \\ -4^{-k} & k \geq 0 \end{cases}$

(iii) For $|z| > \frac{1}{2}$

$F(z) = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$

$F(z) = \frac{2z}{z\left(1-\frac{1}{2z}\right)} - \frac{z}{z\left(1-\frac{1}{4z}\right)}$

$F(z) = 2\left[1 - \frac{1}{2z}\right]^{-1} - \left[1 - \frac{1}{4z}\right]^{-1}$

$F(z) = 2\left[1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \cdots..\right] - \left[1 + \frac{1}{4z} + \frac{1}{4^2z^2} + \cdots \ldots.\right]$

$F(z) = [2.2^0z^0 + 2.2^{-1}z^{-1} + 2.2^{-2}z^{-2} + \cdots.] + [-4^0z^0 - 4^{-1}z^{-1} - 4^{-2}z^{-2} + \cdots.]$

From I series,

Coefficient of $z^{-k} = 2.2^{-k}, k \geq 0$

From II series,

Coefficient of $z^{-k} = -4^{-k}, k \geq 0$

$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = 2.2^{-k} - 4^{-k}, k \geq 0$

16. Find inverse Z transform of $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}, 3 < |z| < 4$

[M17/CompIT/6M][N18/Comp/6M]

Solution:

We have,

$F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$

Let $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$

$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-4)(z-2) + C(z-3)(z-2)$

$3z^2 - 18z + 26 = A(z^2 - 7z + 12) + B(z^2 - 6z + 8) + C(z - 5z + 6)$

Comparing the coefficients, we get

$A + B + C = 3$

$-7A - 6B - 5C = -18$

$12A + 8B + 6C = 26$

On solving, we get

$A = 1, B = 1, C = 1$

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$

For $3 < |z| < 4$,

$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{-4+z}$

$F(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{z\left(1-\frac{3}{z}\right)} + \frac{1}{-4\left(1-\frac{z}{4}\right)}$

$F(z) = \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1} + \frac{1}{z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{4}\left[1-\frac{z}{4}\right]^{-1}$

$F(z) = \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right] + \frac{1}{z}\left[1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots\right] - \frac{1}{4}\left[1+\frac{z}{4}+\frac{z^2}{4^2}+\cdots..\right]$

$F(z) = [2^0 z^{-1} + 2^1 z^{-2} + 2^2 z^{-3} + \cdots.]$
$+[3^0 z^{-1} + 3^1 z^{-2} + 3^2 z^{-3} + \cdots.]$
$+[-4^{-1} z^0 - 4^{-2} z^1 - 4^{-3} z^2 - \cdots..]$

From first series,

Coefficient of $z^{-k} = 2^{k-1}, k > 0$

From second series,

Coefficient of $z^{-k} = 3^{k-1}, k > 0$

From third series,

Coefficient of $z^{k} = -4^{-(k+1)}, k \geq 0$

Coefficient of $z^{-k} = -4^{k-1}, k \leq 0$

Thus,

$$Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\} = \begin{cases} -4^{k-1} & k \leq 0 \\ \{2^{k-1} + 3^{k-1}\} & k > 0 \end{cases}$$

17. Find inverse Z transform of $\frac{5z}{(2z-1)(z-3)}, \frac{1}{2} < |z| < 3$

[N19/Comp/6M]

Solution:

We have,

$F(z) = \frac{5z}{(2z-1)(z-3)}$

Let $\frac{5z}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$

$5z = A(z-3) + B(2z-1)$

Comparing the coefficients, we get

$A + 2B = 5$

$-3A - B = 0$

On solving, we get $A = -1, B = 3$

$F(z) = \frac{-1}{2z-1} + \frac{3}{z-3}$

$F(z) = -\frac{1}{2\left(z-\frac{1}{2}\right)} + \frac{3}{z-3}$

For $\frac{1}{2} < |z| < 3$

$$F(z) = -\frac{1}{2\left(z-\frac{1}{2}\right)} + \frac{3}{-3+z}$$

$$F(z) = -\frac{1}{2z\left(1-\frac{1}{2z}\right)} + \frac{3}{-3\left(1-\frac{z}{3}\right)}$$

$$F(z) = -\frac{1}{2z}\left[1-\frac{1}{2z}\right]^{-1} - \left[1-\frac{z}{3}\right]^{-1}$$

$$F(z) = -\frac{1}{2z}\left[1+\frac{1}{2z}+\frac{1}{2^2z^2}+\cdots..\right] - \left[1+\frac{z}{3}+\frac{z^2}{3^2}+\cdots....\right]$$

$$F(z) = \left[-\frac{1}{2z}-\frac{1}{2^2z^2}-\frac{1}{2^3z^3}-\cdots..\right] + \left[-1-\frac{z}{3}-\frac{z^2}{3^2}-\cdots.\right]$$

$$F(z) = [-2^{-1}z^{-1}-2^{-2}z^{-2}-2^{-3}z^{-3}-\cdots] + [-3^0z^0-3^{-1}z^1-3^{-2}z^2-..]$$

From I series,

Coefficient of $z^{-k} = -2^{-k}$, $k > 0$

From II series,

Coefficient of $z^{k} = -3^{-k}$, $k \geq 0$

Coefficient of $z^{-k} = -3^{k}$, $k \leq 0$

$$Z^{-1}\left\{\frac{5z}{(2z-1)(z-3)}\right\} = \begin{cases} -3^k & k \leq 0 \\ -2^{-k} & k > 0 \end{cases}$$

Type VI: Convolution Theorem

If $Z^{-1}\{F(z)\} = f(k)$ and $Z^{-1}\{G(z)\} = g(k)$

Then by convolution theorem,

$$Z^{-1}\{F(z) \cdot G(z)\} = f(k) * g(k)$$
$$= \sum_{k=0}^{n} f(k)g(n-k)$$

1. Find the inverse z transform of $\frac{z^2}{(z-a)(z-b)}$ by convolution method

Ans. $\frac{(a^{n+1}-b^{n+1})}{a-b}$

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-a}\right] = a^k \text{ and } Z^{-1}\left[\frac{z}{z-b}\right] = b^k$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-b}\right\} = \sum_{k=0}^{n} a^k b^{n-k}$$
$$= \sum_{k=0}^{n} a^k.b^n.b^{-k}$$
$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$
$$= b^n \left[\left(\frac{a}{b}\right)^0 + \left(\frac{a}{b}\right)^1 + \left(\frac{a}{b}\right)^2 + \cdots \ldots \left(\frac{a}{b}\right)^n\right]$$
$$= b^n \left[1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots \ldots . . \frac{a^n}{b^n}\right]$$
$$= b^n.\frac{1\left(\left(\frac{a}{b}\right)^{n+1} - 1\right]}{\left(\frac{a}{b}\right)-1} \qquad S_{n+1} = \frac{a(r^{n+1}-1)}{r-1}$$
$$= b^n.\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}$$

$$Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-b}\right\} = \frac{a^{n+1}-b^{n+1}}{a-b}$$

2. Find the inverse z transform of $\frac{z^2}{(z-1)(2z-1)}$ by convolution method

Ans. $\frac{2^{n+1}-1}{2^{n+1}}$

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1^k \text{ and } Z^{-1}\left[\frac{z}{2z-1}\right] = \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] = \frac{1}{2}.\left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1}$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$

$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{2z-1}\right\} = \sum_{k=0}^{n} 1^k \left(\frac{1}{2}\right)^{n-k+1}$$
$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n+1}.\left(\frac{1}{2}\right)^{-k}$$
$$= \left(\frac{1}{2}\right)^{n+1} \sum_{k=0}^{n} (2)^k$$
$$= \frac{1}{2^{n+1}}.\left[(2)^0 + (2)^1 + (2)^2 + \cdots \ldots (2)^n\right]$$
$$= \frac{1}{2^{n+1}}\left[1 + 2 + 2^2 + \cdots \ldots . + 2^n\right]$$
$$= \frac{1}{2^{n+1}}.\frac{1((2)^{n+1}-1]}{(2)-1} \qquad S_{n+1} = \frac{a(r^{n+1}-1)}{r-1}$$
$$= \frac{1}{2^{n+1}}.(2^{n+1}-1)$$
$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{2z-1}\right\} = \frac{2^{n+1}-1}{2^{n+1}}$$

3. Find the inverse z transform of $\frac{z^2}{(z-a)^2}$ by convolution method

Ans. $a^n(n+1)$

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-a}\right] = a^k$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$
$$Z^{-1}\left\{\frac{z}{z-a}.\frac{z}{z-a}\right\} = \sum_{k=0}^{n} a^k(a)^{n-k}$$
$$= \sum_{k=0}^{n}(a)^n$$
$$= a^n \sum_{k=0}^{n} 1$$
$$= a^n.[1 + 1 + 1 + \cdots \ldots (n+1)times]$$
$$= a^n[n+1]$$
$$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\} = a^n(n+1)$$

4. Find the inverse z transform of $\frac{z^3}{(z-1)^3}$ by convolution method

Ans. $\frac{(n+1)(n+2)}{2}$

Solution:

We know that,

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1^k$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k)g(n-k)$$
$$Z^{-1}\left\{\frac{z}{z-1}.\frac{z}{z-1}\right\} = \sum_{k=0}^{n} 1^k(1)^{n-k}$$
$$= \sum_{k=0}^{n}(1)^n$$

$$= 1^n \sum_{k=0}^{n} 1$$
$$= [1 + 1 + 1 + \cdots \ldots (n+1) times]$$
$$= [n + 1]$$

$$Z^{-1}\left\{\frac{z^2}{(z-1)^2}\right\} = (n+1) = (k+1)$$

By convolution theorem,

$$Z^{-1}\{F(z).G(z)\} = \sum_{k=0}^{n} f(k) g(n-k)$$

$$Z^{-1}\left\{\frac{z^2}{(z-1)^2}.\frac{z}{z-1}\right\} = \sum_{k=0}^{n}(k+1) \,.\, 1^{n-k}$$
$$= 1^n \sum_{k=0}^{n}(k+1)$$
$$= \sum_{k=0}^{n}(k+1)$$
$$= [1 + 2 + 3 + \cdots \ldots (n+1)]$$
$$= \frac{(n+1)(n+2)}{2}$$

$$Z^{-1}\left\{\frac{z^3}{(z-1)^3}\right\} = \frac{(n+1)(n+2)}{2}$$