10 Complex Form Of Fourier Series

1. Introduction

In this chapter we shall first derive complex from of Fourier Series. Then we shall define Fourier Integrals and solve some simple problems based on this.

2. Complex Form Of Fourier Series

Let $f(x)$ be defined in the interval $(C, C+2l)$. The complex form of Fourier Series for $f(x)$ in this interval is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ \text{where, } C_n &= \frac{1}{2l}\int_C^{C+2l} f(x)\, e^{-in\pi x/l}\, dx \end{aligned}} \quad n = 0, \pm 1, \pm 2, \ldots$$

Proof : Consider

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, a_0, a_n and b_n are as given in (3) on page 9-50.

$$= a_0 + \sum_1^{\infty} a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_1^{\infty} b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= a_0 + \sum_1^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/l} + \sum_1^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-in\pi x/l}$$

$$= C_0 + \sum_1^{\infty} C_n e^{in\pi x/l} + \sum_1^{\infty} C_{-n} e^{-in\pi x/l}$$

where, $C_n = \dfrac{a_n - ib_n}{2}$, $C_{-n} = \dfrac{a_n + ib_n}{2}$

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_0 = a_0 = \frac{1}{2l}\int_C^{C+2l} f(x)\,dx$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx - i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx \quad \left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}\,dx \qquad \text{.......... (1)}$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx + i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx \quad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{in\pi x/l}\,dx \qquad \text{.......... (2)}$$

Combining these results (1) and (2), we get

$$C_n = \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}\,dx \qquad n = 0, \pm 1, \pm 2, \ldots$$

Cor. 1 : If the interval is C to $C + 2\pi$ replacing l by π in the above result

$$\boxed{\begin{aligned} & f(x) = \sum_{-\infty}^{\infty} C_n\, e^{inx} \\ \text{where, } & C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\,e^{-inx}\,dx \end{aligned}}$$

Cor. 2 : If the interval is $(0, 2l)$, putting $C = 0$ in the above result

$$\boxed{\begin{aligned} & f(x) = \sum_{-\infty}^{\infty} C_n\, e^{in\pi x/l} \\ \text{where, } & C_n = \frac{1}{2l}\int_0^{2l} f(x)\,e^{-in\pi x/l}\,dx \end{aligned}}$$

Cor. 3 : If the interval is $(0, 2\pi)$, putting $l = \pi$ in the above corollary 2,

$$\boxed{\begin{aligned} & f(x) = \sum_{-\infty}^{\infty} C_n\, e^{inx} \\ \text{where, } & C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}\,dx \end{aligned}}$$

Cor. 4 : If the interval is $(-l, l)$, putting $C = -l$ in the above result

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}\, dx$$

Cor. 5 : If the interval is $(-\pi, \pi)$, putting $l = \pi$ in corollary 4,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

Ex. 1 : Obtain complex form of Fourier Series for $f(x) = e^{ax}$ in $(-\pi, \pi)$ where a is not an integer. **(M.U. 1998, 2001, 02, 04, 05, 08, 09)**

Hence deduce that when α is a constant other than an integer

(i) $\cos \alpha x = \dfrac{\sin \pi\alpha}{\pi} \sum \dfrac{(-1)^n \alpha}{(\alpha^2 - n^2)} \cdot e^{inx}$ **(M.U. 2009)**

(ii) $\sin \alpha x = \dfrac{\sin \pi\alpha}{i\pi} \sum \dfrac{(-1)^n \cdot n}{(\alpha^2 - n^2)} \cdot e^{inx}$ **(M.U. 2001, 02)**

Sol. : By corollary 5 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi}\right]$$

But $$e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi)$$
$$= (-1)^n + i(0) = (-1)^n$$

$$\therefore \quad C_n = \frac{1}{2\pi(a-in)}\left[(-1)^n e^{a\pi} - (-1)^n e^{-a\pi}\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right) = \frac{(-1)^n}{\pi(a-in)} \sin h\, a\pi$$

$$= \frac{(-1)^n \sin h\, a\pi}{\pi(a-in)} \cdot \frac{(a+in)}{(a+in)} = \frac{(-1)^n \sin h\, a\pi\,(a+in)}{\pi(a^2+n^2)}$$

Hence, $$e^{ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h\, a\pi \cdot (a+in)}{\pi(a^2+n^2)} e^{inx} \qquad \text{.......... (i)}$$

For deductions, replace a by $i\alpha$ in (i)

$$\therefore \quad e^{i\alpha x} = \sum \frac{(-1)^n \sin h\, i\alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n (i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha - n) \cdot e^{inx}$$

Now, replace a by $-i\alpha$ in (i)

$$\therefore \quad e^{-i\alpha x} = \sum \frac{(-1)^n \sin h(-i\alpha\pi)}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n (-i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha + in) \cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha + n) \cdot e^{inx}$$

$$\therefore \quad \cos \alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} = \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha) \cdot e^{inx}$$

$$= \frac{\sin \alpha\pi}{\pi} \sum \frac{(-1)^n \cdot \alpha}{(\alpha^2 - n^2)} \cdot e^{inx}$$

$$\therefore \quad \sin \alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2\pi i} = \sum \frac{(-1)^n \sin \alpha\pi}{\pi i(-\alpha^2+n^2)} \cdot (-n) \cdot e^{inx}$$

$$= \frac{\sin \alpha\pi}{\pi i} \sum \frac{(-1)^n \cdot n}{(\alpha^2 - n^2)} \cdot e^{inx}.$$

Ex. 2 : Obtain the complex form of Fourier Series for $f(x) = e^{-ax}$ in $(-\pi, \pi)$.

Sol. : Changing the sign of a, we get from the above result

$$f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h(-a\pi) \cdot (-a+in)}{\pi(a^2+n^2)} e^{inx}$$

But $\sin h(-a\pi) = \dfrac{e^{-a\pi} - e^{a\pi}}{2} = -\dfrac{e^{a\pi} - e^{-a\pi}}{2} = -\sin h\, a\pi$

$$\therefore \quad e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi \cdot (a - in)}{\pi(a^2 + n^2)} e^{inx}$$

Or proceeding as above obtain the result independently.

Ex. 3 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-\pi, \pi)$ where a is not an integer.

Sol. : As proved in example 1,

$$e^{ax} = \sum \frac{(-1)^n \sin h\, a\pi \cdot (a + in)}{\pi(a^2 + n^2)} e^{inx}$$

Changing the sign of a, we get,

$$e^{-ax} = \sum \frac{(-1)^n \sin h(-a\pi) \cdot (-a + in)}{\pi(a^2 + n^2)} e^{inx}$$

But $\sin h(-a\pi) = -\sin h\, a\pi$.

$$e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi \cdot (a - in)}{\pi(a^2 + n^2)} e^{inx}$$

Now, $\cos h\, ax = \dfrac{e^{ax} + e^{-ax}}{2}$

$$= \frac{\sin h\, a\pi}{2} \left[\sum \frac{(-1)^n (a + in)}{(a^2 + n^2)} e^{inx} + \sum \frac{(-1)^n (a - in)}{(a^2 + n^2)} e^{inx} \right]$$

$$\cos h\, ax = a \cdot \sin h\, a\pi \sum \frac{(-1)^n}{(a^2 + n^2)} e^{in\pi x}$$

Ex. 4 : Obtain the complex form of Fourier Series for $f(x) = e^{ax}$ in $(-l, l)$.

(M.U. 1993, 2003)

Sol. : By corollary 4 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{.......... (1)}$$

where, $C_n = \dfrac{1}{2l} \displaystyle\int_{-l}^{l} f(x)\, e^{-in\pi x/l}$

$$= \frac{1}{2l} \int_{-l}^{l} e^{ax} \cdot e^{-in\pi x/l} \cdot dx$$

$$= \frac{1}{2l} \int_{-l}^{l} e^{(a - in\pi/l)x}\, dx = \frac{1}{2l} \left[\frac{e^{(a - in\pi/l)x}}{(a - in\pi/l)} \right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)l} - e^{-(a-in\pi/l)l}}{(a-in\pi/l)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{al}\cdot e^{-in\pi} - e^{-al}\cdot e^{in\pi}}{(al-in\pi)}\right]$$

Now, $e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$= (-1)^n + i - 0 = (-1)^n$$

$$\therefore \quad C_n = \frac{e^{al}(-1)^n - e^{-al}(-1)^n}{2(al-in\pi)} = \frac{(-1)^n(e^{al} - e^{-al})}{2(al-in\pi)}$$

$$= \frac{(-1)^n\cdot \sin h\, al}{al-in\pi}\cdot\frac{al+in\pi}{al+in\pi}$$

$$= \frac{(-1)^n \sin h\, al\,(al+in\pi)}{a^2l^2+n^2\pi^2}$$

$$\therefore \quad e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n \sin h\, al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 5 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-l, l)$. **(M.U. 2002, 05, 09)**

Sol. : By example (4) above

$$e^{ax} = \sum\frac{(-1)^n \sin h\, al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Changing the sign of a, we get

$$e^{-ax} = \sum\frac{(-1)^n \sin h(-al)(-al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

But $\sin h(-x) = \dfrac{e^{-x}-e^{x}}{2} = -\left(\dfrac{e^{x}-e^{-x}}{2}\right) = -\sin h\, x$

$$\therefore \quad e^{-ax} = \sum\frac{(-1)^n \sin h\, al\,(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

$$\therefore \quad \cos h\, ax = \frac{e^{ax}+e^{-ax}}{2}$$

$$= \frac{\sin h\, al}{2}\left[\sum\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l} + \sum\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}\right]$$

$$= al\sin h\, al\sum\frac{(-1)^n}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 6 : Obtain complex form of Fourier Series for $f(x) = \sin h\, ax$ in $(-l, l)$.

Sol. : Using the results obtained in example 5, we get

$$\sin h\, ax = \frac{e^{ax} - e^{-ax}}{2}$$

$$= \frac{\sin h\, al}{2}\left[\sum \frac{(-1)^n (al + i n\pi)}{a^2l^2 + n^2\pi^2} e^{i n\pi x/l} - \sum \frac{(-1)^n (al - i n\pi)}{a^2l^2 + n^2\pi^2} e^{i n\pi x/l}\right]$$

$$= \sin h\, al \sum \frac{(-1)^n\, i n\pi}{a^2l^2 + n^2\pi^2} e^{i n\pi x/l}$$

Ex. 7 : Obtain complex form of Fourier Series for

$f(x) = \cos h\, ax + \sin h\, ax$ in $(-l, l)$. **(M.U. 2002, 07, 08)**

Sol. : By using the results obtained in examples 5 and 6,

$$f(x) = \cos h\, ax + \sin h\, ax$$

$$= \sin h\, al \sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{i n\pi x/l}$$

Ex. 8 : Obtain complex form of Fourier Series for

$f(x) = \cos h\, 3x + \sin h\, 3x$ in $(-3, 3)$. **(M.U. 2008)**

Sol. : We have by corollary 4

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{i n\pi x/l} \quad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-i n\pi x/l}\, dx.$$

Here, $l = 3$

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{i n\pi x/3} \quad \text{where } C_n = \frac{1}{6}\int_{-3}^{3} f(x) \cdot e^{-i n\pi x/3}\, dx$$

But $\quad f(x) = \cos h\, 3x + \sin h\, 3x$

$$= \frac{e^{3x} + e^{-3x}}{2} + \frac{e^{3x} - e^{-3x}}{2} = e^{3x}$$

$$\therefore \quad C_n = \frac{1}{6}\int_{-3}^{3} e^{3x} \cdot e^{-i n\pi x/3}\, dx$$

$$= \frac{1}{6}\int_{-3}^{3} e^{(3 - i n\pi/3)x}\, dx = \frac{1}{6}\left[\frac{e^{(3 - i n\pi/3)x}}{3 - in\pi/3}\right]_{-3}^{3}$$

$$= \frac{1}{6} \cdot \frac{3}{(9 - i n\pi)}\left[e^{9 - in\pi} - e^{-9 + in\pi}\right]$$

But $\quad e^{\pm i n\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm 0 = (-1)^n$

$$\therefore \; C_n = \frac{1}{2(9-in\pi)}[(-1)^n \cdot e^9 - (-1)^n e^{-9}]$$

$$= \frac{1}{2} \cdot \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} \cdot [e^9 - e^{-9}]$$

$$= \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2} \sinh 9$$

$\therefore$ Complex form of Fourier Series for

$$f(x) = \sinh 9 \cdot \sum_{-\infty}^{\infty} \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2} \cdot e^{in\pi x/3}.$$

Ex. 9 : Find complex form of Fourier Series for cos ax, where a is not an integer in $(-\pi, \pi)$.

Sol. : We have $\cos ax = \dfrac{e^{aix} + e^{-aix}}{2}$

Replacing a by ai in example 1, we get

$$e^{aix} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh ai\pi \cdot (ai + in)}{\pi(-a^2+n^2)} e^{inx} \quad \left[\because -a^2+n^2 \neq 0\right]$$

$$= \frac{-i \sinh ai\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx}$$

Since, $\sinh ix = i \sin x$, we get

$$e^{aix} = \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx}$$

Changing the sign of a,

$$e^{-aix} = \frac{\sin(-a\pi)}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx}$$

$$= -\frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx}$$

$$= \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}$$

$$\therefore \; \cos ax = \frac{e^{aix} + e^{-aix}}{2}$$

$$= \frac{\sin a\pi}{2\pi} \left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} + \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}\right]$$

$$= \frac{\sin a\pi}{2\pi} \sum \frac{(-1)^n \cdot 2a}{(a^2-n^2)} e^{inx}$$

$$\therefore \quad \cos ax = \frac{a \sin a\pi}{\pi} \sum \frac{(-1)^n}{(a^2 - n^2)} e^{inx}.$$

Ex. 10 : Find complex from of Fourier Series for sin ax, where a is not an integer in $(-\pi, \pi)$. **(M.U. 2004)**

Sol. : Using the results obtained in the above example, we get,

$$\sin ax = \frac{e^{aix} - e^{-aix}}{2i}$$

$$= \frac{\sin a\pi}{2\pi i} \left[\sum \frac{(-1)^n (a+n)}{(a^2 - n^2)} e^{inx} - \sum \frac{(-1)^n (a-n)}{(a^2 - n^2)} e^{inx} \right]$$

$$= \frac{\sin a\pi}{\pi i} \sum (-1)^n \cdot \frac{n}{(a^2 - n^2)} \cdot e^{inx}$$

Ex. 11 : Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(0, a)$. **(M.U. 2003)**

Sol. : By corollary (2), page 10-2, putting $2l = a$ *i.e.* $l = a/2$, we get

$$f(x) = \sum_{-\infty}^{\infty} C_n \cdot e^{2in\pi x/a} \qquad \text{where,} \quad C_n = \frac{1}{a} \int_0^a e^{ax} \cdot e^{-2in\pi x/a} \, dx$$

$$\therefore \quad C_n = \frac{1}{a} \int_0^a e^{(a - 2in\pi/a)x} dx$$

$$= \frac{1}{a} \left[\frac{e^{(a-2in\pi/a)x}}{(a - 2in\pi/a)} \right]_0^a$$

$$= \frac{1}{a} \cdot \frac{a}{(a^2 - 2in\pi)} \cdot \left[e^{(a^2 - 2in\pi)} - 1 \right]$$

$$= \frac{1}{(a^2 - 2in\pi)} \left[e^{a^2} \cdot e^{-2in\pi} - 1 \right]$$

$$= \frac{1}{(a^2 - 2in\pi)} (e^{a^2} - 1) \qquad \left[\because e^{-2in\pi} = \cos 2n\pi - i \sin 2n\pi = 1 \right]$$

$$\therefore \; e^{ax} = (e^{a^2} - 1) \sum_{-\infty}^{\infty} \frac{e^{2in\pi x/a}}{(a^2 - 2in\pi)}.$$

Ex. 12 : Find the complex form of Fourier Series for

$$f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$$ **(M.U. 2008)**

Sol. : By corollary 2, the complex form of Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{i n\pi x/l} \quad \text{where, } C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-i n\pi x/l}\, dx$$

$$\therefore \; C_n = \frac{1}{2l}\left[\int_0^l 0 \cdot dx + \int_l^{2l} a\, e^{-i n\pi x/l}\, dx\right] \quad \text{.......... (1)}$$

$$= \frac{a}{2l}\int_l^{2l} e^{-i n\pi x/l}\, dx = \frac{a}{2l}\left[\frac{e^{-i n\pi x/l}}{-i n\pi/l}\right]_l^{2l}$$

$$= \frac{-a}{2 i n\pi}\left[e^{-2 i n\pi} - e^{-i n\pi}\right] \text{ except when } n = 0. \quad \text{.......... (2)}$$

Case 1 : Where $n = 0$ from (1)

$$C_0 = \frac{1}{2l}\left[\int_0^l 0\, dx + \int_l^{2l} a\, e^0\, dx\right]$$

$$\therefore \; C_0 = \frac{1}{2l}\int_l^{2l} a\, dx = \frac{a}{2l}[2l - l]$$

$$= \frac{al}{2l} = \frac{a}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \ldots\ldots$ from (2)

$$C_1 = \frac{-a}{2 i\pi}\left[e^{-2i\pi} - e^{-i\pi}\right]$$

$$= \frac{-a}{2i\pi}[\cos(-2\pi) + i\sin(-2\pi) - \cos(-\pi) - i\sin(-\pi)]$$

$$= \frac{-a}{2i\pi}[1 + i(0) - (-1) + i(0)] = \frac{-a}{2i\pi} \cdot 2 = \frac{ai}{\pi}$$

$$C_{-1} = \frac{a}{2i\pi}[\cos 2\pi + i\sin 2\pi - \cos\pi - i\sin\pi]$$

$$= \frac{a}{2i\pi}[1 + i(0) - (1) + i\,0] = \frac{a}{2i\pi} \cdot 2 = -\frac{ai}{\pi}$$

Similarly, $\quad C_3 = \dfrac{ia}{3\pi}, \quad C_{-3} = -\dfrac{ai}{3\pi}$

$\quad C_5 = \dfrac{ia}{5\pi}, \quad C_{-5} = -\dfrac{ai}{5\pi}$

Case 3 : When $n = \pm 2, \pm 4, \ldots\ldots$

$$C_2 = \frac{-a}{4i\pi}\left[e^{-4i\pi} - e^{-2i\pi}\right]$$

$$= \frac{-a}{4i\pi}[\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) + i\sin(-2\pi)]$$

$$\therefore \quad C_2 = \frac{-a}{4i\pi}[1 + i(0) - (1) - i(0)] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots\ldots = 0$

$$\therefore \quad f(x) = \frac{a}{2} + \frac{ai}{\pi}\left[(e^{u} - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \ldots\right] \text{ where } u = \frac{i\pi x}{l}$$

Ex. 13 : If $f(x) = 1 \quad 0 < x < 1$

$\quad = 0 \quad 1 < x < 2$

$f(x+2) = f(x)$, find complex form of Fourier Series. **(M.U. 1996)**

Sol. : By corollary 2, the complex form of Fourier Series in $(0, 2l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_n = \dfrac{1}{2l}\displaystyle\int_0^{2l} f(x)\, e^{-in\pi x/l} dx$

Since, in this case $l = 1$, $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$

and $\quad C_n = \dfrac{1}{2}\displaystyle\int_0^2 f(x)\, e^{-in\pi x} dx \quad$ .... (1)

$$= \frac{1}{2}\left[\int_0^1 1 \cdot e^{-in\pi x} dx + \int_1^2 0 \cdot e^{-in\pi x} dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_0^1 = \frac{1}{-2in\pi}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[1 - e^{-in\pi}\right] \text{ except when } n = 0. \quad \text{.......... (2)}$$

Case 1 : When $n = 0$, from (1)

$$C_0 = \frac{1}{2}\left[\int_0^1 1 \cdot dx + \int_1^2 0 \cdot dx\right]$$

$$= \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$ from (2)

$$C_1 = \frac{1}{2i\pi}\left[1 - e^{-i\pi}\right] = \frac{1}{2i\pi}[1 - \{\cos\pi - i\sin\pi\}]$$

$$= \frac{1}{2i\pi}[1 - (-1)] = \frac{2}{2i\pi} = \frac{1}{i\pi}$$

$$C_{-1} = \frac{1}{-2i\pi}\left[1 - e^{i\pi}\right] = -\frac{1}{2i\pi}[1 - \{\cos\pi + i\sin\pi\}]$$

$$\therefore \quad C_{-1} = -\frac{1}{2i\pi}[1-(-1)] = -\frac{1}{i\pi}$$

Similarly, $C_3 = \frac{1}{3i\pi}, \quad C_{-3} = -\frac{1}{3i\pi},$

$C_5 = \frac{1}{5i\pi}, \quad C_{-5} = -\frac{1}{5i\pi}, \ .....$

Case 3 : When $n = \pm 2, \pm 4, ...$ from (2)

$$C_2 = \frac{1}{4i\pi}\left[1-e^{-2i\pi}\right] = \frac{1}{4i\pi}[1-\{\cos 2\pi + i\sin 2\pi\}]$$

$$= \frac{1}{4i\pi}[1-1] = 0$$

Similarly, $C_{-2} = 0$

and $C_4 = C_{-4} = C_6 = C_{-6} = ... = 0$

Hence, $f(x) = \frac{1}{2} + \frac{1}{i\pi}(e^{i\pi x} - e^{-i\pi x}) + \frac{1}{3i\pi}(e^{3i\pi x} - e^{-3i\pi x}) + ...$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right) + \frac{1}{3}\left(\frac{e^{3i\pi x} - e^{-3i\pi x}}{2i}\right) + ...\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + ...\right]$$

Ex. 14 : Find the complex form of the Fourier series for $f(x) = 2x$ in $(0, 2\pi)$. **(M.U. 2009)**

Sol. : By corollary (3), page 10-2,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx$

Hence, $C_n = \frac{1}{2\pi}\int_0^{2\pi} 2x \cdot e^{-inx}\, dx$ .... (1)

$$= \frac{1}{\pi}\int_0^{2\pi} x\, e^{-inx}\, dx$$

$$= \frac{1}{\pi}\left[x \cdot \frac{e^{-inx}}{-in} - \int \frac{e^{-inx}}{-in} \cdot 1 \cdot dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x}{in}e^{-inx} - \frac{e^{-inx}}{(in)^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{e^{-in2\pi}}{n^2} - 0 - \frac{1}{n^2}\right] \quad \text{for } n \neq 0,\ (\because i^2 = -1)$$

$$= -\frac{2}{in} + \frac{\cos 2n\pi - i\sin 2n\pi}{n^2} - \frac{1}{n^2} \quad \text{for } n \neq 0$$

$$= \frac{2i}{n} + \frac{1}{n^2} - \frac{1}{n^2} \quad \text{for } n \neq 0$$

$$= \frac{2i}{n} \quad \text{for } n \neq 0$$

For $n = 0$, we get from (1),

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} 2x \cdot e^0\, dx = \frac{1}{2\pi}\left[x^2\right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \cdot 4\pi^2 = 2\pi.$$

Hence, $f(x) = 2\pi + 2i \sum_{n=-\infty}^{\infty} \frac{1}{n} \cdot e^{inx}, \quad \text{for } n \neq 0$

Putting $n = \pm 1, \pm 2, \pm 3, \ldots$, we get

$$f(x) = 2\pi + 2i\left[\frac{e^{ix} - e^{-ix}}{1} + \frac{1}{2}\left(\frac{e^{2ix} - e^{-2ix}}{1}\right) + \frac{1}{3}\left(\frac{e^{3ix} - e^{-3ix}}{1}\right) + \ldots\right]$$

$$= 2\pi + 4i^2\left[\frac{(e^{ix} - e^{-ix})}{2i} + \frac{1}{2} \cdot \frac{(e^{2ix} - e^{-2ix})}{2i} + \frac{1}{3} \cdot \frac{(e^{3ix} - e^{-3ix})}{2i} + \ldots\right]$$

$$= 2\pi - 4\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

EXERCISE - I

1. Find complex form of Fourier Series for $f(x) = e^{-x}$ in $(-1, 1)$.

(M.U. 2009) $\left[\textbf{Ans. :} \ f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi) \sinh 1}{1 + n^2\pi^2} e^{in\pi x}\right]$

2. Find the complex form of $f(x) = e^x$ in $(-\pi, \pi)$. **(M.U. 1999, 2000, 04)**

$\left[\textbf{Ans. :} \ f(x) = \sum \frac{(-1)^n \sinh \pi (1 + in)}{\pi(1^2 + n^2)} e^{inx}\right]$

3. Obtain complex form of Fourier series for $f(x) = e^{ax}, \quad -1 < x < 1$. **(M.U. 1996, 2003)**

$\left[\textbf{Ans. :} \ f(x) = \sum \frac{(-1)^n \sinh a (a + in\pi)}{a^2 + n^2\pi^2} e^{in\pi}\right]$

4. Find the complex form of Fourier Series for $f(x) = \sin hx$ in $(-l, l)$. **(M.U. 1997)**

$$\left[\textbf{Ans. :}\ f(x) = \sin h\,l \cdot (i\,\pi) \sum \frac{(-1)^n n}{l^2 + n^2\pi^2}\, e^{in\pi x/l}\right]$$

5. Find the complex form of Fourier Series for $f(x) = \cos h\,2x + \sin h\,2x$ in $(-5, 5)$. **(M.U. 1999, 2003)**

(**Hint :** In solved example 6, put $a = 2$ and $l = 5$.)

6. Find the complex form of Fourier Series of $\cos h\ 3x + \sin h\ 3x$ in $(-\pi, \pi)$. **(M.U. 1999, 2003)**

7. Obtain the complex form of Fourier Series for $f(x) = e^{2x}$ in $(0, 2)$. **(M.U. 2003)**

8. Find complex form of Fourier Seires of $f(x) = \cos hx + \sin hx$ in $(-\pi, \pi)$. **(M.U. 2003, 09)**

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3. Orthogonality, Orthonormality

We first define these terms and then solve some problems based on this definition.

Definition 1 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots\ f_n(x) \ldots$ is said to be **orthogonal** on (a, b) if

$$\int_a^b f_m(x)\, f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), \ldots\ f_n(x) \ldots$ is orthogonal on (a, b) if

$$\boxed{\begin{aligned} &\int_a^b f_m(x)\, f_n(x)\, dx = 0 \quad \text{if } m \neq n \\ \text{and}\quad &\int_a^b [f_m(x)]^2\, dx \neq 0 \end{aligned}}$$

Definition 2 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots\ f_n(x) \ldots$ is said to be **orthonormal** on (a, b) if

$$\int_a^b f_m(x)\, f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ = 1, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), f_3(x), \ldots\ f_n(x) \ldots$ is said to be **orthonormal** on (a, b) if

$$\boxed{\begin{aligned} &\int_a^b f_m(x)\, f_n(x)\, dx = 0 \quad \text{if } m \neq n \\ \text{and}\quad &\int_a^b [f_m(x)]^2\, dx = 1 \end{aligned}}$$

Note ....

Every orthonormal set of functions is orthogonal but the converse may not be true.

Ex. 1 : Show that the set of functions $\cos nx$, $n = 1, 2, 3, ...$ is orthogonal on $(0, 2\pi)$. **(M.U. 1994)**

Sol. : We have $f_n(x) = \cos nx$

$$\therefore \int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos mx \cdot \cos nx\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\cos (m+n)\, x + \cos (m-n)\, x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin (m+n)\, x}{m+n} + \frac{\sin (m-n)\, x}{m-n}\right]_0^{2\pi}$$

Now, two cases arise

Case 1 : When $m \neq n$, then

$$\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : When $m = n$, then

$$\int_0^{2\pi} f_n(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos^2 nx\, dx$$

$$\therefore \int_0^{2\pi} \left[f_n(x)\right]^2 dx = \int_0^{2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx$$

$$\therefore \int_0^{2\pi} \left[f_n(x)\right]^2 dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 0 \quad .... (1)$$

Since, $$\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

the given set of functions are orthogonal over $[\,0, 2\pi\,]$.

Ex. 2 : How can you construct orthonormal set of functions from the set given in the above example number 1?

Sol. : If the set of functions is to be orthonormal, we should have

$$\int_0^{2\pi} \left[f_n(x)\right]^2 dx = 1$$

For this we divide (1) by π and write it as

$$\int_0^{2\pi} \frac{1}{\pi}\left[f_n(x)\right]^2 dx = \pi \cdot \frac{1}{\pi} = 1$$

i.e. $$\int_0^{2\pi} \frac{1}{\sqrt{\pi}} f_n(x) \cdot \frac{1}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where

$$\phi_n(x) = \frac{1}{\sqrt{\pi}} \cos n x$$

Hence, the required orthonormal set is

$$\frac{1}{\sqrt{\pi}} \cos x, \ \frac{1}{\sqrt{\pi}} \cos 2x, \ \frac{1}{\sqrt{\pi}} \cos 3x, \ .....$$

Ex. 3 : Show that the set of functions $\sin(2n+1)x$, $n = 0, 1, 2, ...$ is orthogonal over $[0, \pi/2]$. Hence, construct orthonormal set of functions.

(M.U. 1993, 2002, 04, 06, 08, 09)

Sol. : We have $f(x) = \sin(2n+1)x$.

$$\therefore \quad \int_0^{\pi/2} f_m(n) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin(2m+1)x \cdot \sin(2n+1)x\, dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left[\cos(2m+2n+2)x - \cos(2m-2n)x\right] dx$$

$$= -\frac{1}{2}\left[\frac{\sin(2m+2n+2)x}{(2m+2n+2)} - \frac{\sin(2m-2n)x}{(2m-2n)}\right]_0^{\pi/2}$$

Now, two cases arise

Case 1 : If $m \neq n$, then

$$\int_0^{\pi/2} f_m(x) \cdot f_n(x) = 0$$

Case 2 : If $m = n$, then

$$\int_0^{\pi/2} f_n(x) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin^2(2n+1)x\, dx$$

$$\therefore \quad \int_0^{\pi/2} [f_n(x)]^2\, dx = \int_0^{\pi/2} \left(\frac{1-\cos 2(2n+1)x}{2}\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2(2n+1)x}{2(2n+1)}\right]_0^{\pi/2}$$

$$\therefore \quad \int_0^{\pi/2} [f_n(x)]^2\, dx = \frac{\pi}{4} \neq 0 \qquad \text{.........} (1)$$

Since,

$$\int_0^{\pi/2} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

the given set of functions is orthogonal over $[0, \pi/2]$.

Now, if the set is to be orthonormal, then we should have,

$$\int_0^{\pi/2} [f_n(n)]^2\, dx = 1$$

For this, we divide (1) by $\pi / 4$ and write it as

$$\int_0^{\pi/2} \frac{4}{\pi}[f_n(x)]^2\, dx = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1$$

i.e. $$\int_0^{\pi/2} \frac{2}{\sqrt{\pi}} f_n(x) \cdot \frac{2}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where

$$\phi_n(x) = \frac{2}{\sqrt{\pi}} \sin(2n+1)\, x$$

Hence, the required orthonormal set of functions is

$$\frac{2}{\sqrt{\pi}} \sin x,\ \frac{2}{\sqrt{\pi}} \sin 3x,\ \frac{2}{\sqrt{\pi}} \sin 5x,\ \ldots$$

Ex. 4 : Is $S = \left\{\sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \ldots\right\}$ orthogonal in (0, 1) ? **(M.U. 2004)**

Sol. : We have

$$\int_0^1 f_m(x) \cdot f_n(x)\, dx = \int_0^1 \sin\frac{(2m+1)\pi x}{4} \sin\frac{(2n+1)\pi x}{4} \cdot dx$$

$$= -\frac{1}{2}\left[\int_0^1 \left[\cos\frac{(m+n+1)\pi x}{2} - \cos\frac{(m-n)\pi x}{2}\right] dx\right]$$

$$= -\frac{1}{2}\left[\sin\frac{(m+n+1)\pi x}{2} \cdot \frac{2}{(m+n+1)\pi}\right.$$

$$\left. - \sin\frac{(m-n)\pi x}{2} \cdot \frac{2}{(m-n)\pi}\right]_0^1 \neq 0$$

$\therefore$ S is not orthogonal.

Ex. 5 : Show that the set of functions

$$\sin\left(\frac{\pi x}{2L}\right),\ \sin\left(\frac{3\pi x}{2L}\right),\ \sin\left(\frac{5\pi x}{2L}\right),\ \ldots \text{ is orthogonal over } (0, L).$$

(M.U. 1996, 2002, 05, 06, 09)

Sol. : We have $f_n(x) = \sin\frac{(2n+1)\pi x}{2L}$, $n = 0, 1, 2, \ldots$

$$\therefore \int_0^L f_m(x) \cdot f_n(x)\, dx = \int_0^L \sin\frac{(2m+1)\pi x}{2L} \cdot \sin\frac{(2n+1)\pi x}{2L}\, dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{2m+2n+2}{2L}\right)\pi x - \cos\left(\frac{2m-2n}{2L}\right)\pi x\right] dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{m+n+1}{L}\right)\pi x - \cos\left(\frac{m-n}{L}\right)x\right]dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n+1)/L\}\pi x}{(m+n+1)\pi/L} - \frac{\sin\{(m-n)/L\}\pi x}{(m-n)\pi/L}\right]_0^L$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n+1)\pi}{(m+n+1)\pi/L} - \frac{\sin(m-n)\pi}{(m-n)\pi/L}\right] \qquad \text{.......... (1)}$$

Now, two cases arise.

Case 1 : If $m \neq n$, then since m, n are integers from (1)

$$\int_0^L f_m(x) \cdot f_n(x)\,dx = 0$$

Case 2 : If $m = n$, then

$$\int_0^L f_n(x) \cdot f_n(x)\,dx = \int_0^L \sin^2 \frac{(2n+1)\pi x}{2L}\,dx$$

$$\therefore \quad \int_0^L [f_n(x)]^2\,dx = \int_0^L \left[\frac{1-\cos 2\{(2n+1)/2L\}\pi x}{2}\right]dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2\{(2n+1)/2L\}\pi x}{(2n+1)\pi/L}\right]_0^L$$

$$\therefore \quad \int_0^L [f_n(x)]^2\,dx = \frac{1}{2}\left[x - \frac{\sin\{(2n+1)/L\}\pi x}{(2n+1)\pi/L}\right]_0^L$$

$$= \frac{L}{2} \neq 0$$

Since, $\int_0^L f_m(x) \cdot f_n(x)\,dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal over $[0, L]$.

Ex. 6 : Show that the set of functions

$$1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \ldots..$$

form an orthogonal set in $(-L, L)$ and construct an orthonormal set.

(M.U. 1998, 2002)

Sol. : Let $f_n(x) = \sin\frac{n\pi x}{L}$, $n = 0, 1, 2, \ldots..$

$$g_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots..$$

(a) $\int_{-L}^{L} f_m(x) \cdot f_n(x)\, dx = \int_{-L}^{L} \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L}\, dx$ .... (1)

$$= -\frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m+n)\,\pi x}{L} - \frac{\cos(m-n)\,\pi x}{L}\right]dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\,\pi x}{(m+n)\,\pi/L} - \frac{\sin\{(m-n)/L\}\,\pi x}{(m-n)\,\pi/L}\right]_{-L}^{L}$$

Now, two cases arise.

Case 1 : If $m \neq n$, then

$$\int_{-L}^{L} f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : If $m = n$, then from (1)

$$\int_{-L}^{L} f_n(x)\, dx \cdot f_n(x)\, dx = \int_{-L}^{L} \sin^2\frac{n\pi x}{L}\, dx$$

$$\therefore \quad \int_{-L}^{L} [f_n(x)]^2\, dx = \int_{-L}^{L}\left(\frac{1-\cos(2n\pi x/L)}{2}\right)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2n\pi x)/L}{2n\pi/L}\right]_{-L}^{L} = L \neq 0$$

(b) $\int_{-L}^{L} g_m(x) \cdot g_n(x)\, dx = \int_{-L}^{L} \cos\frac{m\pi x}{L} \cdot \cos\frac{n\pi x}{L}\, dx$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m+n)\,\pi x}{L} + \cos\frac{(m-n)\,\pi x}{L}\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\,\pi x}{(m+n)\pi/L} + \frac{\sin\{(m-n)/L\}\,\pi x\}}{(m-n)\,\pi/L}\right]_{-L}^{L}$$

Again two cases arise

Case 1 : When $m \neq n$, then

$$\int_{-L}^{L} g_m(x) \cdot g_n(x)\, dx = 0$$

Case 2 : If $m = n$, then

$$\int_{-L}^{L} g_n(x) \cdot g_n(x)\, dx = \int_{-L}^{L} \cos^2\frac{n\pi x}{L}\, dx$$

$$\therefore \quad \int_{-L}^{L} [g_n(x)]^2\, dx = \int_{-L}^{L}\left(\frac{1+\cos(2n\pi x/L)}{2}\right)dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_{-L}^{L} = L \neq 0$$

(c) $\int_{-L}^{L} f_m(x) \cdot g_n(x)\, dx = \int_{-L}^{L} \sin\frac{m\pi x}{L} \cdot \cos\frac{n\pi x}{L}\, dx$

$$= \frac{1}{2}\int_{-L}^{L}\left[\frac{\sin(m+n)\pi x}{L} + \frac{\sin(m-n)\pi x}{L}\right] dx$$

$$= \frac{1}{2}\left[-\frac{\cos\{(m+n)/L\}\pi x}{(m+n)/L} - \frac{\cos\{(m-n)/L\}\pi x}{(m-n)/L}\right]_{-L}^{L}$$

Again two cases arise

Case 1 : If $m \neq n$, then

$$\int_{-L}^{L} f_m(x) \cdot g_n(x)\, dx = 0$$

Case 2 : If $m = n$, then

$$f_n(x) \cdot g_n(x) = [f_n(x)]^2 \text{ or } [g_n(x)]^2$$

and we have already proved above that

$$\int_{-L}^{L} [f_n(x)]^2\, dx = L \text{ and } \int_{-L}^{L} [g_n(x)]^2\, dx = L \text{ and } \int_{-L}^{L} 1\, dx = 2L$$

Hence, the given sequence is orthogonal.

For orthonormality the value of the integral L must be 1. Hence, if each term is devided by $\sqrt{L}$ then the set will be orthonormal. Hence, the set $\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin x, \frac{1}{\sqrt{L}}\cos x, \frac{1}{\sqrt{L}}\sin 2x, \ldots$ is an orthonormal set.

Ex. 7 : Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = (3x^2 - 1)/2$ are orthogonal over $(-1, 1)$. **(M.U. 1997, 2002, 03, 04, 08)**

Sol. : We have $\int_{-1}^{1} f_1(x) \cdot f_2(x)\, dx = \int_{-1}^{1} x\, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$

$$\int_{-1}^{1} f_1(x) \cdot f_3(x)\, dx = \int_{-1}^{1} \frac{1}{2}(3x^2 - 1)\, dx$$

$$= \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x) \cdot f_3(x)\, dx = \int_{-1}^{1} \frac{x}{2}(3x^2 - 1)\, dx = \frac{1}{2}\int_{-1}^{1}(3x^3 - x)\, dx$$

$$= \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} = 0$$

Further $\int_{-1}^{1} f_1(x) \cdot f_1(x)\, dx = \int_{-1}^{1} 1 \cdot 1 \cdot dx = [x]_{-1}^{1} = 2 \neq 0$

$$\int_{-1}^{1} f_2(x) \cdot f_2(x)\, dx = \int_{-1}^{1} x \cdot x\, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 0$$

$$\int_{-1}^{1} f_3(x) \cdot f_3(x)\, dx = \int_{-1}^{1} \frac{1}{4}(9x^2 - 6x + 1)\, dx$$

$$= \frac{1}{4}\left[3x^3 - 3x^2 + x\right]_{-1}^{1} = 2 \neq 0$$

Hence, the given set is orthogonal over [−1, 1].

Ex. 8 : Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on (−1, 1). Determine the constants a and b such that the function $f_3(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on that interval. **(M.U. 2003, 05, 06, 07, 09)**

Sol. : We have already proved the first part above.

Now, if $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ we should have,

(i) $\int_{-1}^{1} f_1(x) \cdot f_3(x)\, dx = 0 \quad \therefore \quad \int_{-1}^{1} 1 \cdot (-1 + ax + bx^2)\, dx = 0$

$$\therefore \quad \left[-x + \frac{ax^2}{2} + \frac{bx^3}{3}\right]_{-1}^{1} = 0 \quad \therefore \quad \left(-1 + \frac{a}{2} + \frac{b}{3}\right) - \left(+1 + \frac{a}{2} - \frac{b}{3}\right) = 0$$

$$\therefore \quad -2 + \frac{2b}{3} = 0 \quad \therefore \quad b = 3.$$

And **(ii)** $\int_{-1}^{1} f_2(x) \cdot f_3(x)\, dx = 0 \quad \therefore \quad \int_{-1}^{1} x \cdot (-1 + ax + bx^2)\, dx = 0$

$$\therefore \quad \left[\frac{-x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}\right]_{-1}^{1} = 0 \quad \therefore \left(-\frac{1}{2} + \frac{a}{3} + \frac{b}{4}\right) - \left(\frac{1}{2} - \frac{a}{3} + \frac{b}{4}\right) = 0$$

$$\therefore \quad \frac{2a}{3} = 0 \quad \therefore \quad a = 0$$

$$\therefore \quad f_3(x) = 3x^2 - 1$$

Now, $\int_{-1}^{1} [f_3(x)]^2\, dx = \int_{-1}^{1} (3x^2 - 1)^2\, dx$

$$= \int_{-1}^{1} (9x^2 - 6x + 1)\, dx = \left[3x^3 - 3x^2 + x\right]_{-1}^{1}$$

$$= (3 - 3 + 1) - (-3 - 3 - 1) = 8 \neq 0$$

The required function $f_3(x) = 3x^2 - 1$.

Ex. 9 : If $f_i(x)$, $i = 1, 2, 3, \ldots$ is a set of orthogonal functions in $[a, b]$ and $g(x) = \sum_{i=1}^{\infty} a_i\, f_i(x)$, then find a_i. **(M.U. 2003)**

Sol. : We have

$$g(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \ldots . \infty$$

Multiply both sides by $f_1(x)$,

$$\therefore \quad f_1(x) g(x) = a_1 [f_1(x)]^2 + a_2 f_1(x) \cdot f_2(x) + a_3 f_1(x) f_3(x) + \ldots . \infty \quad \ldots\ldots\ldots (1)$$

Now, integrate both sides w.r.t. x from a to b.

$$\therefore \quad \int_a^b f_1(x) g(x)\, dx = a_1 \int_a^b [f_1(x)]^2 dx + a_2 \int_a^b f_1(x) \cdot f_2(x)\, dx + a_3 \int_a^b f_1(x) \cdot f_3(x)\, dx + \ldots . \infty \quad \ldots\ldots\ldots (2)$$

But by definition of orthogonal functions

$$\int_a^b f_m(x) f_n(x) = 0 \text{ if } m \neq n$$

and $\int_a^b [f_m(x)]^2 dx \neq 0$

Hence, on the r.h.s. of (2) all the integrals except the first are zero and the first not zero.

$$\therefore \quad \int_a^b f_1(x) g(x)\, dx = a_1 \int_a^b [f_1(x)]^2 dx$$

$$\therefore \quad a_1 = \frac{\int_a^b f_1(x) \cdot g(x)\, dx}{\int_a^b [f_1(x)]^2 dx}$$

Similarly, by multiplying (1) successivaly by $f_2(x)$, $f_3(x)$ ..... and integrating both sides w.r.t. x from a to b, we can obtain the values of a_2, a_3, ...

Thus, in general, we have

$$a_i = \frac{\int_a^b f_i(x) g(x)\, dx}{\int_a^b [f_i(x)]^2 dx}.$$

Ex. 10 : If $f(x) = C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)$, where C_1, C_2, C_3 constants and Φ_1, Φ_2, Φ_3 are orthonormal sets on (a, b), show that

$$\int_a^b [f(x)]^2 dx = C_1^2 + C_2^2 + C_3^2$$ **(M.U. 2002, 07, 08)**

Sol. : Since, Φ_1, Φ_2, Φ_3 are orthonormal

$$\int_a^b [\Phi_1(x)]^2 dx = \int_a^b [\Phi_2(x)]^2 dx = \int_a^b [\Phi_3(x)]^2 dx = 1 \quad \ldots\ldots\ldots (1)$$

and $$\int_a^b \Phi_m(x) \Phi_n(x)\, dx = 0 \text{ when } m \neq n \quad \ldots\ldots\ldots (2)$$

Now $\int_a^b [f(x)]^2 dx = \int_a^b [C_1 \Phi_1(x) + C_2 \Phi_2(x) + C_3 \Phi_3(x)]^2 dx$

$$= \int_a^b [C_1^2 \{\Phi_1(x)\}^2 + C_2^2 (\Phi_2(x))^2 + C_3^2 \{\Phi_3(x)\}^2 + 2 C_1 C_2 \Phi_1(x) \Phi_2(x)$$

$$+ 2 C_1 C_3 \Phi_1(x) \Phi_3(x) + 2 C_2 C_3 \Phi_2(x) \Phi_3(x)] dx$$

$$= C_1^2 \int_a^b [\Phi_1(x)]^2 dx + C_2^2 \int_a^b [\Phi_2(x)]^2 dx$$

$$+ C_3^2 \int_a^b [\Phi_3(x)]^2 dx + 2 C_1 C_2 \int_a^b \Phi_1(x) \Phi_2(x) dx$$

$$+ 2 C_1 C_3 \int_a^b \Phi_1(x) \Phi_3(x) dx + 2 C_2 C_2 \int_a^b \Phi_1(x) \Phi_2(x) dx$$

$= C_1^2 + C_2^2 + C_3^2$ by (1) and (2).

EXERCISE - II

1. Show that the set of functions $\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$. **(M.U. 2003)**

2. Show that the set of functions $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$.

3. Show that the set of functions $\cos x$, $\cos 2x$, $\cos 3x$, ... is a set of orthogonal functions over $[-\pi, \pi]$. Hence, construct a set of orthonormal functions. **(M.U. 1995, 98, 2005, 08)**

4. Prove that $\sin x$, $\sin 2x$, $\sin 3x$, ... is orthogonal on $[0, 2\pi]$ and construct orthonormal set of functions. **(M.U. 1997, 99, 2003, 05, 09)**

5. Is the set of functions $\sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right), \sin\left(\frac{5\pi x}{l}\right), \ldots$ orthogonal over $(0, l)$. **(M.U. 2003)** **[Ans. : Yes]**

6. Is the set of functions $\cos x$, $\cos 3x$, $\cos 5x$, ... orthogonal over $(0, \pi/2)$. **(M.U. 2003) [Ans. : No]**

7. Show that the set of functions $\sin x$, sine $2x$, $\sin 3x$, ... is orthogonal on the interval $[0, \pi]$. **(M.U. 1999, 2003)**

8. Show that the set of functions $\cos x$, $\cos 2x$, $\cos 3x$, ... is orthogonal on $[-\pi, \pi]$. **(M.U. 2003, 06)**

9. Show that the set of functions 1, $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ... is orthogonal on $(0, 2\pi)$ but not on $(0, \pi)$.

How can you convert the set orthonormal on $(0, 2\pi)$? Write down the orthonormal set. **(M.U. 2003)**

[**Ans.** : $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \ldots$]

10. Show that the following set of functions is orthonormal on $(0, \infty)$

$$\left\{e^{-x/2}, e^{-x/2}(1-x), \frac{1}{2}e^{-x/2}(x^2-4x+2)\right\}$$ **(M.U. 2003, 04, 07)**

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4. Fourier Integral Theorem

If $f(x)$ satisfies Dirichlet's conditions (stated in chapter 9) in each finite interval $-l \le x \le l$ and if $f(x)$ is integrable in $-\infty$ to ∞ then Fourier Integral Theorem states that

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds} \quad \ldots\ldots\ldots (1)$$

We assume this result without proof.

Note ....

Unfortunately there is no uniformity in notation and in the use of constants before the integral.

5. Fourier Sine and Cosine Integrals

The above integral can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\{\cos\omega s\cos\omega x\}\,d\omega\,ds$$

$$+ \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\{\sin\omega s\sin\omega x\}\,d\omega\,ds$$

i.e. $$f(x) = \frac{1}{\pi}\int_0^\infty \cos\omega x\int_{-\infty}^\infty f(s)\cos\omega s\,d\omega\,ds$$

$$+\frac{1}{\pi}\int_0^\infty \sin\omega x\int_{-\infty}^\infty f(s)\sin\omega s\,d\omega\,ds$$

(a) Fourier Cosine Integral

When $f(x)$ is an even function, $f(s)$ **will be even but** $f(s)\sin\omega s$ **will be odd function and** $f(s)\cos\omega s$ **will be even function. Hence, the second integral will be zero and we will get**

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\,d\omega\,ds} \quad \ldots\ldots\ldots (2)$$

This is called **Fourier Cosine Integral.**