Module 2 : Relations & Functions

The concept of relations is of primary importance in computer science, especially in the study of data structures such as linked list, array, relational models etc.

If A is the set of all living human males and B is the set of all living human females, then the relation F (Father) can be defined between A and B.

Thus, if $x \in A$ and $y \in B$, then x is related to y by the relation F if x is the father of y, and we write xFy.

The order matters here, we refer to F as a relation from A to B.

We could also consider the relationship S and H from A to B by letting

$x \, S \, y$,

which means that x is a son of y, and $x \, H \, y$ means that x is the husband of y.

Formal Definition of Relations

Let A and B be two non-empty sets.
A relation R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a,b) \in R$, we say that
"a is related to b" by R, and we also write aRb.

If a is not related to b by R, we write $a \not R b$.

If $A = B$, we say that $R \subseteq A \times A$ is a <u>relation on A</u>, instead from A to A.

Examples:

1. Let $A = \{1, 2, 3\}$, $B = \{x, s\}$

Then $R = \{(1, x), (2, s), (3, x)\}$

is a relation from A to B.

2. Determine which of the following are relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$

a) $R_1 = \{(a, 1), (a, 2), (c, 2)\}$

b) $R_2 = \{(a, 2), (b, 1), (2, a)\}$

c) $R_3 = \{(c, 1), (c, 2), (a, 2)\}$

d) $R_4 = \{(b, 2)\}$

e) $R5 = \phi$

Sol:

Here $A = \{a, b, c\}$, $B = \{1, 2\}$

$\therefore A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

a) $\because R_1 \subseteq A \times B$

c) $\because R_3 \subseteq A \times B$

d) $\because R_4 \subseteq A \times B$

$\left.\right\}$ It is a relation

e) $\because R5 \subseteq A \times B \rightarrow$ Also a relation called as empty relation

b) $\because R2 \nsubseteq A \times B \rightarrow$ Not a Relation.

Domain of a Relation

Let A and B be two non-empty sets and R be a relation from A to B.

i.e $R \subseteq A \times B$, then the domain of R is a subset of A such that it is a collection of first elements of all ordered pairs of R.

i.e Domain of $R = \{a \mid (a, b) \in R\}$

Range of a Relation

If R is a relation from A to B, then the range of R is a subset of B such that it is a collection of second elements of all ordered pairs of R

i.e Range of $R = \{b \mid (a, b) \in R\}$

Example

Let $A = \{1, 2, 3\}$, $B = \{x, 5\}$

$R = \{(1, x), (2, 5), (3, x)\}$

$\therefore$ $Dom(R) = \{1, 2, 3\}$

$Ran(R) = \{x, 5\}$

Example

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$

aRb if and only if $a < b$

$\therefore$ $R = \{ \quad \}$

$Dom(R) = \{ \quad \}$

$Ran(R) = \{ \quad \}$

Example

Let $A = \{1, 2, 3\}$, $B = \{x, 5\}$

$R = \{(1, x), (2, 5), (3, x)\}$

$\therefore$ $Dom(R) = \{1, 2, 3\}$

$Ran(R) = \{x, 5\}$

Example

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$

aRb if and only if $a < b$

$\therefore$ $R = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$

$Dom(R) = \{1, 2, 3, 4\}$

$Ran(R) = \{2, 3, 4, 5\}$

Inverse Relation

Let R be a relation from A to B then the inverse relation on R is the collection of all (b,a) such that $(a,b) \in R$.

It is denoted by R^{-1}, defined as

$$R^{-1} = \{ (b,a) \mid (a,b) \in R \}$$

Example

Let R be the relation on $A = \{1,2,3,4\}$ defined by "x is less than y"

i.e 'R' is the relation $<$.

i) Write 'R' as a set of ordered pairs.

ii) Find R^{-1} of the relation R.

iii) Can R^{-1} be described in words?

Sol: i) Here $A = \{1, 2, 3, 4\}$

$R = \{(x, y) \mid xRy \text{ iff } x < y\}$

$= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

ii) $R^{-1} = \{(y, x) \mid (x, y) \in R\}$

$= \{(2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$

iii) R^{-1} can be described as a statement

" $xR^{-1}y$ iff $x > y$ "

i.e x is greater than y.

Example

Let S be the relation on the set N of the +ve integers defined by an equation $x + 3y = 13$

i.e $S = \{(x, y) \mid x + 3y = 13\}$

i) Write S as a set or ordered pairs

ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation

Sol: i) $S = \{ \quad \mid \quad \}$

$\therefore$

$\therefore \qquad \{ \qquad \}$

$\therefore$

Example

Let S be the relation on the set N of the +ve integers defined by an equation $x + 3y = 13$

i.e $S = \{(x, y) \mid x + 3y = 13\}$

i) Write S as a set or ordered pairs

ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation

Sol: i) $S = \{(x, y) \mid x + 3y = 13\}$

$\because x + 3y = 13$

$\therefore x = 13 - 3y$ $\{\because y$ cannot exceed $4\}$

$\therefore y = 1, 2, 3, 4$ corresponding to

$x = 10, 7, 4, 1$

$\therefore \quad S = \{(10,1), (7,2), (4,3), (1,4)\}$

ii) $S^{-1} = \{(y,x) \mid (x,y) \in S\}$

$= \{(1,10), (2,7), (3,4), (4,1)\}$

S^{-1} can be described as :-

" x is related to y iff $3x + y = 13$ "

i.e $x S^{-1} y$ iff $3x + y = 13$

Representation of Relations

1. Graphical & Tabular Form

A relation between two finite sets can be represented in a tabular form as well as in a graphical form

For example

$$\text{Let } A = \{a, b, c, d\}$$
$$R = \{\alpha, \beta, \gamma\}$$

& R is a relation from A to B

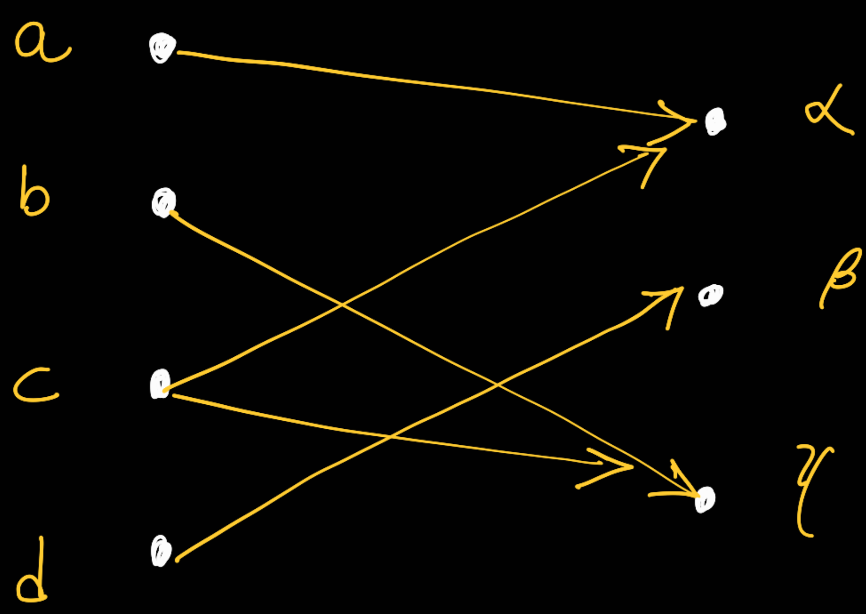
$$R = \{(a,\alpha), (b,\gamma), (c,\alpha), (c,\gamma), (d,\beta)\}$$

Above relation could be represented in a tabular form & graphical form

Tabular Form:

	α	β	γ
a	✓		
b			✓
c	✓		✓
d		✓	

Graphical Form



2. Matrix Form

We can represent a relation between two finite sets with matrices.

If $A = \{a_1, a_2 \ldots a_m\}$ & $B = \{b_1, b_2, \cdots b_n\}$ are finite sets containing m and n elements, respectively and R is a relation from A to B, we represent R by the $m \times n$ matrix $M_R = [m_{ij}]$, which is defined by,

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the <u>Matrix of R</u>.

It provides an easy way to check whether R has a given property or not.

<u>Examples</u>

Let $A = \{1, 2, 3\}$

$B = \{r, s\}$

Let R be a relation from set A to set B.

$$R = \{(1, r), (2, s), (3, r)\}$$

Then the matrix of R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Conversely, given sets A and B with $|A| = m$ and $|B| = n$, an $m \times n$ matrix whose entries are zeros and ones determines a relation as explained in the next example

Example

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Since M is 3×4, we let

$A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b_4\}$

Then $(a_i, b_j) \in R$ iff $m_{ij} = 1$

Thus $R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$

Example

Let $A = \{a, b, c, d\}$,

$B = \{1, 2, 3\}$

$R = \{(a,1), (a,2), (b,1), (c,2), (d,1)\}$

Find the relation matrix

Sol: M_R will have 4 rows & 3 columns

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array} \right] \end{array}$$

Example

Let $A = \{1, 2, 3, 4, 8\}$

$B = \{1, 4, 6, 9\}$

Let aRb iff a/b. Find the Relation Matrix.

Example

Let $A = \{1, 2, 3, 4, 8\} = B$

aRb iff $a+b \leq 9$. Find the relation matrix

Example

Let $A = \{a, b, c, d\}$ & let

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ find } R?$$

3. Digraphs

Definition:

If A is a finite set and R is a relation on A, we can also represent R pictorially as follows:-

1) Draw a small circle for each element of A and label the circle with the corresponding element of A. These circles are called <u>vertices</u>.

2) Draw an arrow, called an <u>edge</u> from vertex a_i to vertex a_j iff $a_i \, R \, a_j$.

The resulting pictorial representation of R is called a <u>directed graph</u> or <u>digraph</u> of R.

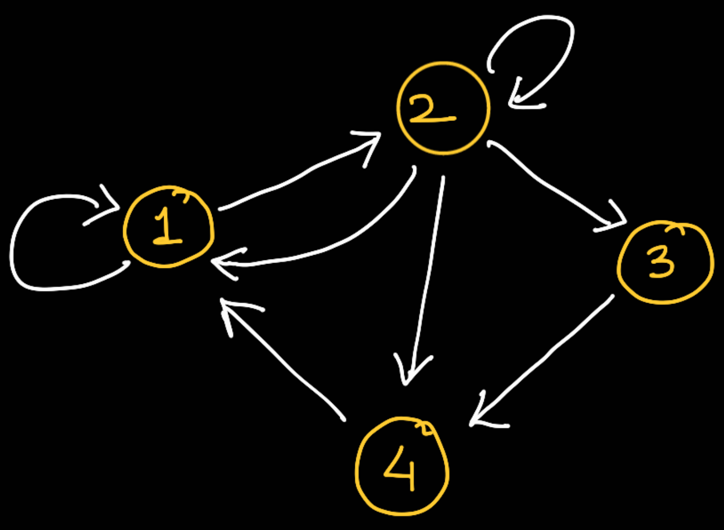
Thus, if R is a relation on A, the edges in the digraph of R correspond exactly to the pairs in R, and the vertices correspond exactly to the elements of set A.

Example

Q.1. Let $A = \{1, 2, 3, 4\}$. Let R is a relation from A to A.

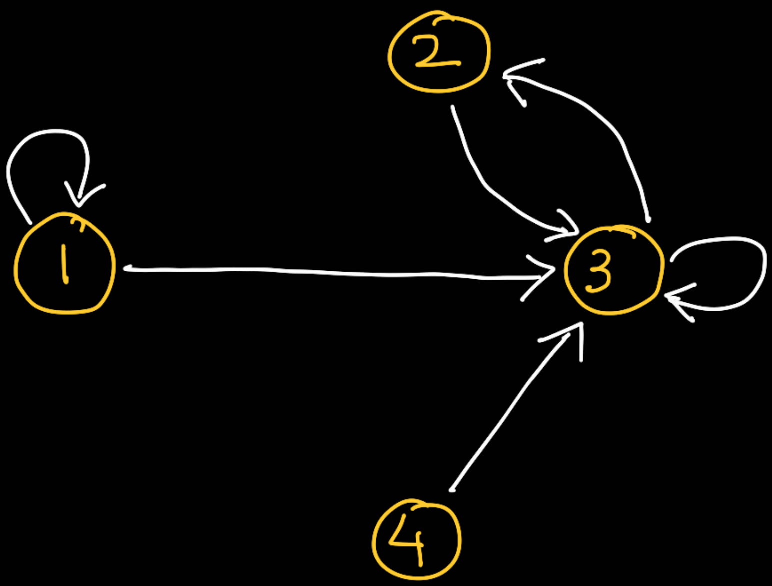
$$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$$

Then the digraph of R is as follows:-



Example 2

Find the relation determined by the following digraph.



Sol:

Since $a_i R a_j$ if and only if there is an edge from a_i to a_j, we have

$$R = \{(1,1), (1,3), (2,3), (3,2), (3,3), (4,3)\}$$

Degree of Vertex

An important concept for relations is inspired by visual form of digraphs.

If R is a relation on a set A and $a \in A$, then the in-degree of a is the number of $b \in A$ such that $(b, a) \in R$.

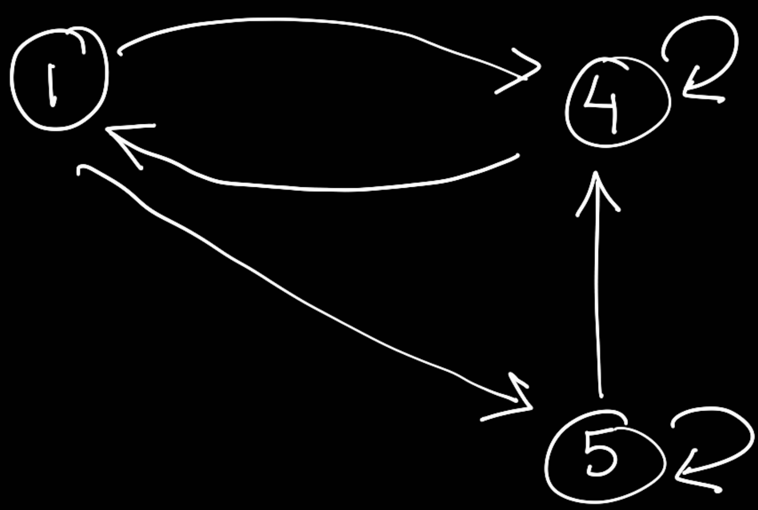
The out-degree of a is the number of $b \in A$ such that $(a, b) \in R$.

What this means, in terms of the digraph of R, is that the in-degree of a vertex is the number of edges terminating at the vertex.

The out-degree of a vertex is the no. of edges leaving the vertex.

Example: 1

Let $A = \{1,4,5\}$ & let R be given by the digraph as shown. Find M_R and R.



Sol: $$M_R = \begin{array}{c|ccc} & 1 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 1 \end{array}$$

$$R = \{(1,4),(1,5),(4,1),(4,4),(5,4),(5,5)\}$$

Example 2

Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2,3), (3,2), (3,4), (3,5), (4,3), (4,4), (4,5)\}$. Draw its digraph.

Example 3

Let $A = \{a, b, c, d\}$ and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw the digraph of R.

Example 4

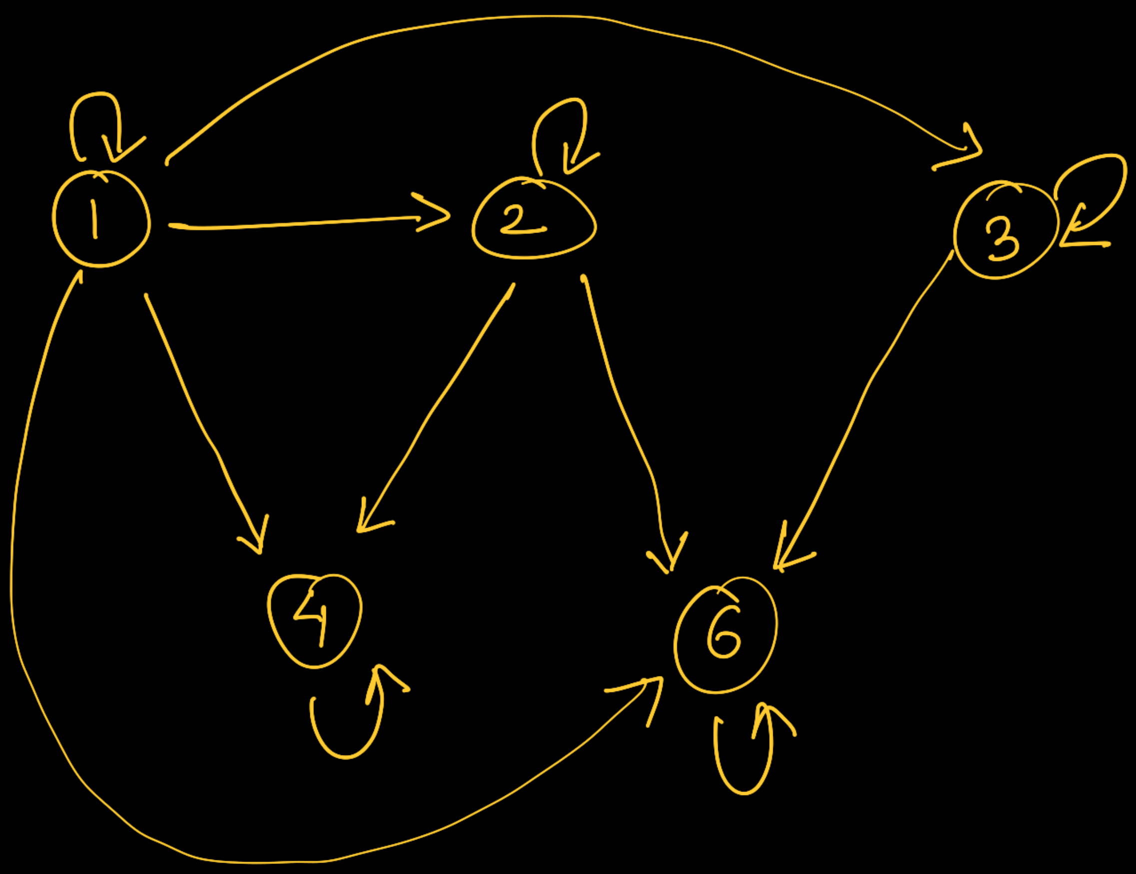
Draw the graphical representation of the relation "less than" on $\{1,2,3,4\}$

Example 5

Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find matrix of R. Find inverse relation of R.

Example

Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find matrix of R. Find inverse relation of R.



Example 6

Given $A = \{1, 2, 3, 4\}$ & $B = \{x, y, z\}$.

Let R be the following relation from A to B:

$$R = \{(1,y), (1,z), (3,y), (4,x), (4,z)\}$$

i) Determine the matrix of the relation

ii) Draw the digraph of R

iii) Find R^{-1}

iv) Determine domain & range of R.

Paths

Suppose that R is a relation on a set A. A <u>path of length n</u> in R from a to b is a finite sequence $\pi : a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that

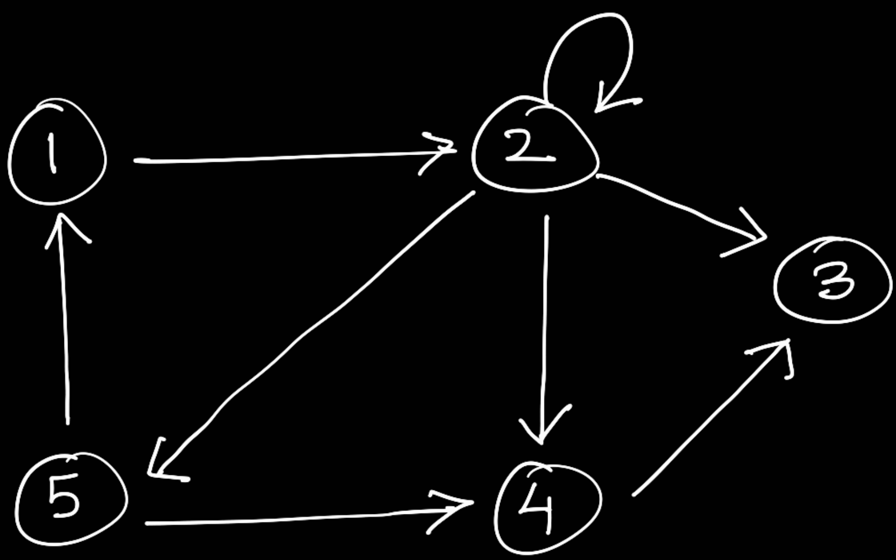
$$a R x_1, \; x_1 R x_2, \; \ldots\ldots, \; x_{n-1} R b$$

Note that a path of length n involves $n+1$ elements of A, although they are not necessarily distinct.

A path is most easily visualized with the aid of the digraph of the relation.

It appears as a geometric 'path' or succession of edges in such a digraph, where the indicated directions of the edges are followed, and in fact a path derives its name from this representation.

For example



Then π: 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3.

π_2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself.

π_3 : 2, 2 is a path of length 1 from vertex to itself.

A path that begins & ends at the same vertex is called a cycle.

π_2 & π_3 are cycles of length 3 & 1 respectively.

If $\underline{n}$ is a fixed positive integer, we define a relation R^n on A as follows:-

$\underline{xR^n}y$ means there is a path of length n from x to y in R.

We may also define a relation R^∞ on A, by letting $x R^\infty y$ which means that there is some path in R from x to y.

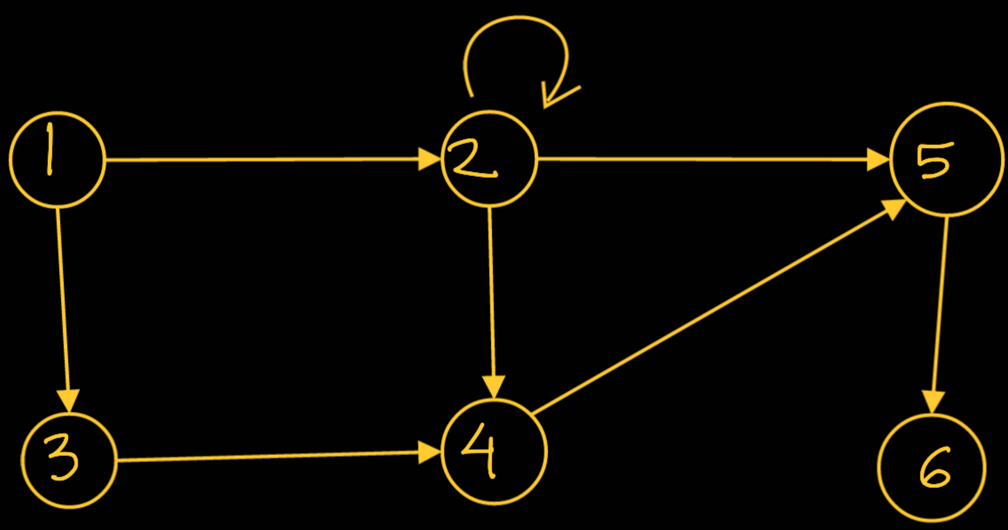
The length of such a path, in general, will depend on x & y.

The relation R^∞ is sometimes called as the "Connectivity" relation for R.

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the relation whose digraph is shown. Find R^2 and draw diagraph of the relation R^2.

Sol: To compute R^2 on A, $x R^2 y$ means that there is a pair of length 2 from x to y in R.



$1 R^2 2$ Since $1R2$ and $2R2$

$1 R^2 4$ Since $1R2$ and $2R4$

$1 R^2 5$ Since $1R2$ and $2R5$

$2 R^2 2$ Since $2R2$ & $2R2$

$2 R^2 4$ Since $2R2$ & $2R4$

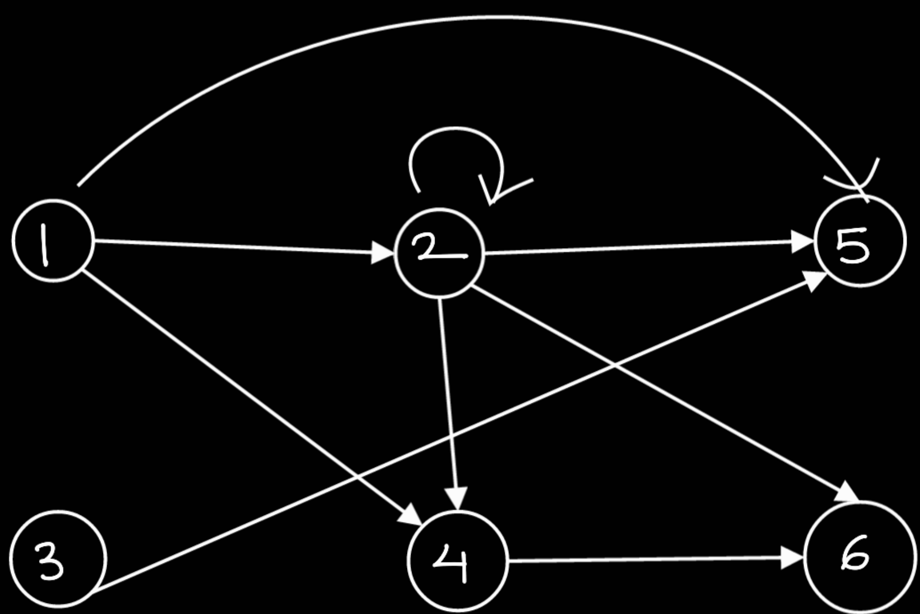
$2 R^2 5$ Since $2R2$ & $2R5$

$2 R^2 6$ Since $2R5$ & $5R6$

$3 R^2 5$ Since $3R4$ & $4R5$

$4 R^2 6$ Since $4R5$ & $5R6$

Hence, we can obtain the digraph for R^2 as follows



$\therefore R^2 = \{(1,2), (1,4), (1,5), (2,2), (2,4), (2,5), (2,6), (3,5), (4,6)\}$

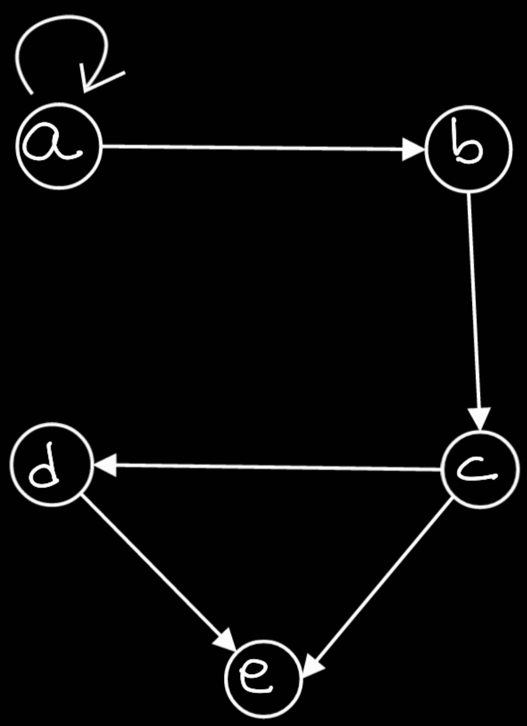
Example

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$

Compute i) R^2

ii) R^∞



i) The digraph is as shown above.

$x R^2 y$ means that there is a path of length n from x to y in R.

$a R^2 a$ Since aRa & aRa

$a R^2 b$ Since aRa & aRb

$a R^2 c$ Since aRb & bRc

$b R^2 d$ Since bRc & cRd

$b R^2 e$ Since bRc & cRe

$c R^2 e$ Since cRd & dRe

Hence $R^2 = \{(a,a), (a,b), (a,c), (b,d), (b,e), (c,e)\}$

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ii) To compute R^∞, we need all ordered pairs of vertices for which there is a path of any length from the first vertex to the second.

$$R^\infty = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)\}$$

If $|R|$ is large, it can be tedious and perhaps difficult to compute R^∞, or even R^2, from the set representation of R.

However, M_R can be used to accomplish these tasks more efficiently.

Let R be a relation or a finite set $A = \{a_1, a_2, \ldots a_n\}$, and let M_R be the $n \times n$ matrix representing R.

Let's see how the matrix M_R of R^2 can be computed from M_R.

$C = [C_{ij}]$ defined by

$$C_{ij} = \begin{cases} 1 & \text{If } a_{ik} = 1 \text{ \& } b_{kj} = 1 \text{ for some } K,\ 1 \le K \le p \\ 0 & \text{otherwise} \end{cases}$$

This multiplication is similar to ordinary matrix multiplication.

The preceding formula states that for any i and j, the element C_{ij} of $C = A \odot B$ can be computed in the following way.

1. Select row i of A and column j of B, and arrange them side by side.

2. Compare corresponding entries. If even a single pair of corresponding entries consists of two 1's, then $C_{ij} = 1$.

   If this is not the case, then $C_{ij} = 0$

Example

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$

Then

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\because \quad M_R^2 = M_R \cdot M_R$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we compute M_{R^2} from R^2, we obtain the same result.

It is often easier to compute R^2 by computing $M_R \odot M_R$ instead of searching the digraph of R for all vertices that can be joined by a path of length 2.

Similarly, we can show that

$$M_{R^3} = M_R \odot (M_R \odot M_R)$$

Composition of Paths

Let $\pi_1 : a, x_1, x_2, \ldots\ldots x_{n-1}, b$ be a path in a relation R of length n from a to b, and

Let $\pi_2 : b, y_1, y_2, \ldots\ldots, y_{m-1}, c$ be a path in R of length m from b to c.

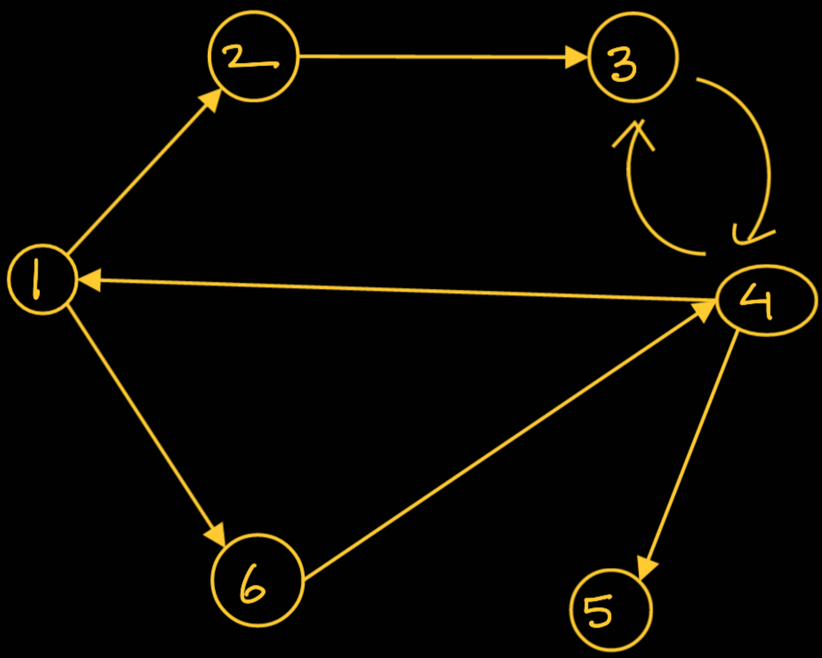
Then, the composition of π_1 and π_2 is the path $a, x_1, x_2 \ldots b, y_1, y_2 \ldots y_{m-1}, c$ of length $n+m$, which is denoted by $\pi_2 \pi_1$.

This is a path from a to c.

Example:

Consider the relation whose digraph is given and the paths

$\pi_1 : 1, 2, 3$ & $\pi_2 : 3, 5, 6, 2, 4$.



i) List all path of length 3 starting from vertex 3.

ii) List all paths of length 3

Sol: i) All paths of length 3 starting from vertex 3.

π_1 : 3, 3, 3, 3

π_2 : 3, 3, 4, 3

π_3 : 3, 3, 4, 5

π_4 : 3, 3, 3, 4

π_5 : 3, 4, 3, 4

π_6 : 3, 4, 1, 2

π_7 : 3, 4, 1, 6

π_8 : 3, 4, 3, 3

π_9 : 3, 3, 4, 1

ii) List all paths of length 3

Total 30 paths.

e.g. π_1 : 1, 2, 3, 3
π_2 : 1, 2, 3, 4
π_3 : 1, 6, 4, 5
π_4 : 1, 6, 4, 3
π_5 : 1, 6, 4, 1

Starting from Vertex 1.

Example:
Compute R^2 and draw the digraph of R^2.

Sol: R^2

$1 R^2 3$ Since $1 R 2$ & $2 R 3$
$1 R^2 4$ Since $1 R 6$ & $6 R 4$
$2 R^2 3$ Since $2 R 3$ & $3 R 3$
$2 R^2 4$ Since $2 R 3$ & $3 R 4$

likewise we will also get

$3R^2 1$, $3R^2 3$, $3R^2 3$, $3R^2 4$, $3R^2 5$,
$4R^2 2$, $4R^2 3$, $4R^2 4$, $4R^2 6$, $6R^2 1$,
$6R^2 3$, $6R^2 5$.

Diagram:

Find $M_R^{\,2}$

$$M_R^{\,2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Find i) R^{∞}

ii) $M_R^{\,\infty}$

i) $R^{\infty} = \{ (a,c), (a,b), (a,d), (a,e), (a,f),$
$(b,b), (b,f), (b,d), (b,c), (b,e), (c,b),$
$(c,d), (c,e), (c,f), (c,c), (d,b),$
$(d,c), (d,e), (d,f), (d,d), (e,f),$
$(e,d), (e,b), (e,c), (e,e), (f,b),$
$(f,d), (f,c), (f,e), (f,f) \}$

ii)

$$M_R^{\infty} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example: Let $A = \{1, 2, 3, 4, 5\}$ & R be the relation defined by $a R b$ iff $a < b$, compute R, R^2, R^3 & draw digraphs for the same.

Types of Relations

1) Reflexive Relations

A relation R on a set A is reflexive if for 'every' element $a \in A$, $a R a$

i.e $(a, a) \in R$.

R is not reflective if for 'some' element $a \in A$, $a \not R a$,

i.e $(a, a) \notin R$.

Example:

1. Let $A = \{a, b\}$ and let

$R = \{(a,a), (a,b), (b,b)\}$

Then R is reflexive

2. Let $A = \{1, 2\}$ & let

$R = \{(1,1), (1,2)\}$. R is not reflexive

since $(2,2) \notin R$.

We can identify a reflexive relation by its matrix as follows. The matrix of a reflexive relation must have all 1's on its main diagonal.

Similarly, we can characterize the digraph of reflexive relation as follows.

A reflexive relation has a cycle of length 1 at every vertex.

Finally, we may note that if R is reflexive on a set A,

then $\text{Dom}(R) = \text{Ran}(A) = A$

2) Irreflexive Relations

A relation R on a set A is <u>irreflexive</u> if $a \not R a$.

i.e $(a,a) \notin R$ for every $a \in A$.

Thus R is irreflexive if no element is related to itself.

<u>Examples</u>

1) Let $A = \{1,2\}$ and let

$R = \{(1,2),(2,1)\}$

Then R is irreflexive since

$(1,1), (2,2) \notin R$.

2) Let $A = \{1,2\}$ and let

$R = \{(1,2),(2,2)\}$

Then R is not <u>irreflexive</u> since

$(2,2) \in R$

Note that R is not <u>reflexive</u> either

since $(1,1) \notin R$

3. Symmetric Relations

A relation R on a set A is <u>symmetric</u> if whever aRb, then bRa.

It then follows that R is <u>not symmetric</u> if we have some a and $b \in A$ with aRb, but $b \not R a$

<u>Examples</u>

1. Let A be set of people. Let <u>aRb</u> if a is a friend of b.

   Then obviously b is related to a.

   Hence the relation of being "<u>friend</u>" is a symmetric relation.

2. Let A be a set of lines in a plane.

For lines $l_1, l_2 \in A$, let $l_1 R l_2$ if l_1 is parallel to l_2.

Then $l_2 R l_1$, since the relation of being "parallel to" is a symmetric relation.

---

3. Let A be a set of people and let aRb if a is brother of b.

Then this is not a symmetric relation since b can be the sister of a.

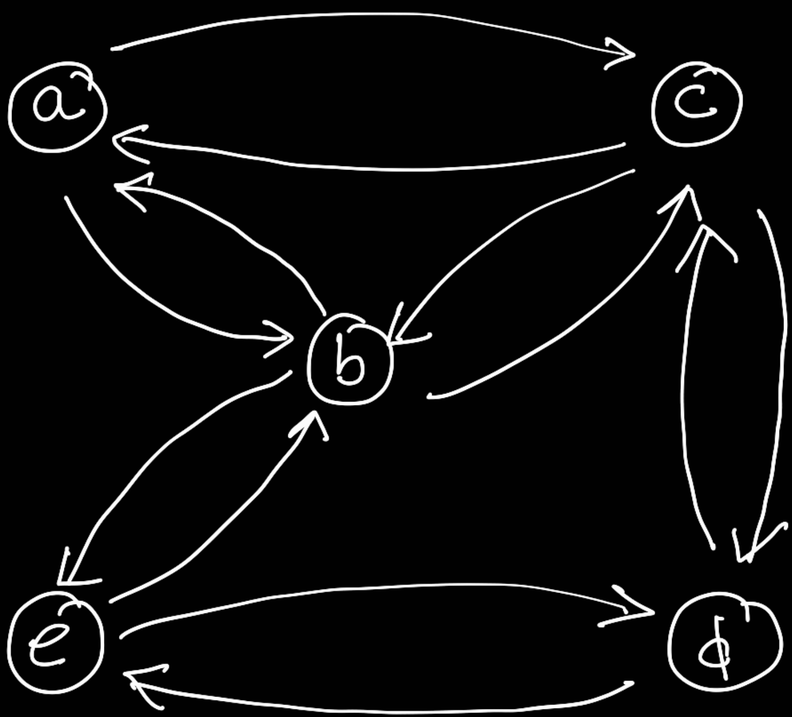
This relation will be symmetric only if A is the set of males

Digraph of Symmetric Relation

Let $A = \{a, b, c, d, e\}$ and let R be the symmetric relation given by,

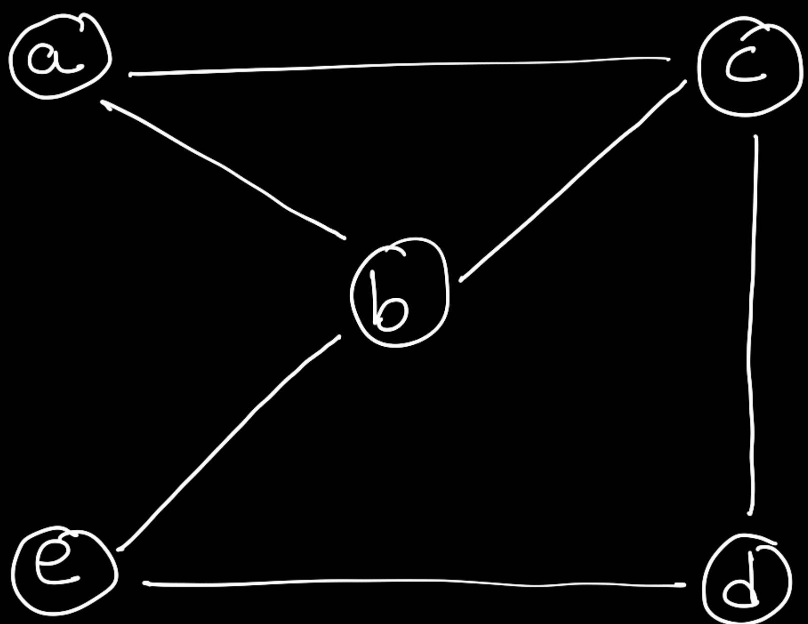
$$R = \{(a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (b,e), (e,b), (e,d), (d,e), (c,d), (d,c)\}$$

The usual digraph of R is shown below. Each undirected edge corresponds to two ordered pairs in the relation R.



An undirected edge between a and b, in the graph of a symmetric relation R, corresponds to a set $\{a, b\}$ such that $(a, b) \in R$ and $(b, a) \in R$.

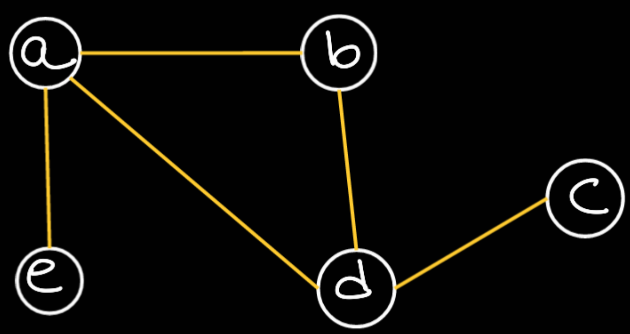
Sometimes we will also refer to such a set $\{a, b\}$ as an 'undirected edge' of the relation R and call a and b 'adjacent vertices'.



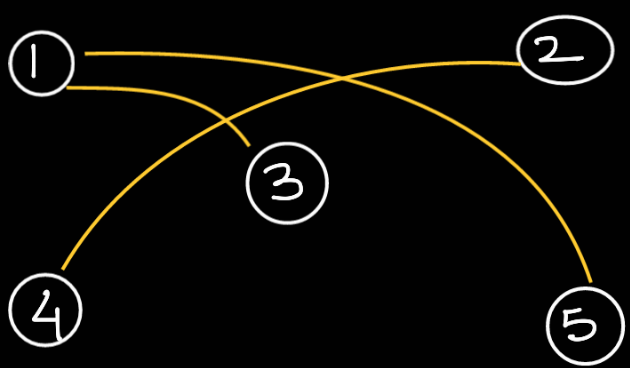
4. Connected Relation

A symmetric relation R on a set A is called connected if there is a path from any element of A to any other element of A.

This simply means that the graph of R is all in one piece.



Connected Symmetric Relation



Unconnected Symmetric Relation

5. Asymmetric Relation

A relation R on a set A is asymmetric if whenever aRb, then $b \not R a$.

It then follows that R is not asymmetric if we have some a and $b \in A$ with both aRb & bRa.

Examples :

1. Let A be the set of real numbers and let R be the relation '$<$'.

   If $a < b$, then $b \not< a$,

   so '$<$' is asymmetric.

2. Let $A = \{1, 2, 3, 4\}$ and let

   $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$

   then R is not asymmetric, since $(2, 2) \in R$

6. Antisymmetric Relations

A relation R on a set A is <u>antisymmetric</u> if whenever $a \neq b$, then either $a \not R b$ or $b \not R a$.

OR

if whenever $\underline{aRb}$ and $\underline{bRa}$, then $a = b$

Contrapositive

<u>Example:</u>

Let $A = \{1,2,3,4\}$ and let $R = \{(1,2),(2,2),(3,4),(4,1)\}$ Is it symmetric, asymmetric or antisymmetric?

<u>Sol:</u> <u>Symmetry</u>: R is not symmetric, since $(1,2) \in R$, but $(2,1) \notin R$.

<u>Asymmetry</u>: R is not asymmetric, since $(2,2) \in R$

<u>Antisymmetry</u>: R is antisymmetric, since if $a \neq b$, either $(a,b) \notin R$ or $(b,a) \notin R$.

7. Transitive Relations

A relation R on a set A is transitive if whenever aRb and bRc, then aRc. It follows that a relation R is not transitive if there exists a, b and c in A so that aRb and bRc but $a \not R c$.

Example

Let $A = \mathbb{Z}$, the set of integers and let R be the relation less than.

Sol: To see whether R is transitive, we assume that aRb & bRc. Thus $a < b$ & $b < c$. It then follows that $a < c$, so aRc.

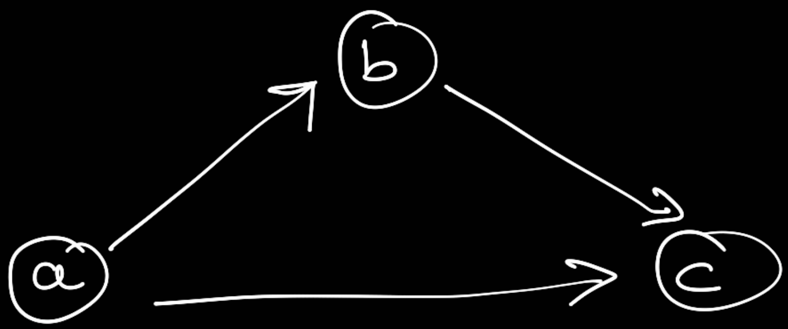
Hence R is transitive.

Digraph of a transitive relation :-

Let $A = \{a, b, c\}$ and let

$R = \{(a,b), (b,c), (a,c)\}$

$\therefore$



8. Identity Relation

Identity relation I on set A is reflexive, transitive and symmetric.

Example

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

9. Void Relation

It is given by $R : A \rightarrow B$ such that $R = \phi \ (\subseteq A \times B)$ is a null/void relation.

10. Universal Relation

A relation $R : A \rightarrow B$ such that $R = A \times B \ (\subseteq A \times B)$ is a universal relation.