Laplace Transformation

Def^n :- If $f(t)$ is a function of t defined for $t \geq 0$ and if the integral 0 to ∞

$$\int_0^\infty e^{-st} f(t)\, dt$$ - if it exist then it is a function of parameter s, this function of s is denoted as $F(s)$ is called Laplace Transform of $f(t)$ over the range of value of s for which the integral exist i.e

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$L \rightarrow$ Laplace Transform operator.

Sufficient Condition for existance of Laplace Transform

If the $f(t)$ defined for $t \geq 0$ is

- piecewise continous in the Interval $0 \leq t \leq N$ for every finite $N > 0$

- is of exponential ord α $(\alpha > 0)$ as $t \rightarrow \infty$ then the Laplace Transform of $f(t)$ exist for $s > \alpha$

NOTE: Exponential order: A func. $f(t)$ is said to be Exponential order $(\alpha > 0)$ as $t \rightarrow \infty$ if there exist a positive constants to M such that

$$|f(t)| \leq Me^{\alpha t} \qquad \forall t \geq t_0$$

(for all)

Eg $t^2 < e^{3t}$ $\forall$ $t > 0$

$f(t) = t^2$ $\alpha = 3$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

Laplace Transform of standard function

1) $f(t) = 1 \quad , t \geq 0$

$$L[f(t)] = \int_0^\infty e^{-st} \cancel{dt} f(t)\, dt$$

$$L[1] = \int_0^0 e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$L[1] = \frac{1}{s} \qquad s > 0$$

2) $f(t) = e^{at} \qquad a \rightarrow \text{constant}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(-a+s)t}\, dt$$

$$L[e^{at}] = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$L[e^{at}] = \frac{1}{s-a} \qquad s > a$$

$$L[e^{at}] = \frac{1}{s+a} \qquad s > -a$$

3) $f(t) = \sin at \qquad a \rightarrow \text{constant}$

$\longrightarrow$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} [-s\sin at - a\cos at] \right]_0^\infty$$

$$L[\sin at] = \frac{a}{s^2+a^2} \quad s>0$$

(when $s>0$ both $e^{-st}\sin at$ & $e^{-st}\cos at \to 0$ ~~as~~ $t \to \infty$)

2) $f(t) = \cos at$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{a^2+s^2} [-s\cos at + a\sin at] \right]_0^\infty$$

$$L[\cos at] = \frac{s}{a^2+s^2}, \quad s>0$$

Note: **Linearity Property**

1) $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$

2) $L[k f(t)] = k\, L[f(t)]$

★ $L[e^{iat}] = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$

$L[\cos at + i\sin at] = L[\cos at] + i\, L[\sin at]$

Equating Real & Imaginary Parts

$L[\cos at] = \frac{s}{s^2+a^2}$ $\quad$ $L[\sin at] = \frac{a}{s^2+a^2}$

5) $f(t) = \sinh(at)$

$\Rightarrow$

$$L[\sinh(at)] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L[e^{at}] - L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$L[\sinh(at)] = \frac{a}{s^2-a^2} \qquad s > |a|$$

6) $f(t) = \cosh(at)$

$\Rightarrow$

$$L[\cosh(at)] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a} + \frac{1}{s-a}\right]$$

$$L[\cosh(at)] = \frac{s}{s^2-a^2} \qquad s > |a|$$

7) $f(t) = t^n$, $\quad n \to +ve$ integer

$\Rightarrow$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

put $st = x$

$dx = s\,dt$

$$\int_0^\infty e^{-st}\, t^n\, dt = \int_0^\infty e^{-x} \frac{x^n}{s^n} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

$$= \frac{1}{s^{n+1}} \Gamma{n+1}$$

$$L[t^n] = \frac{1}{s^{n+1}} \Gamma{n+1} \qquad s>0$$

$$L[t^n] = \frac{n!}{s^{n+1}} \qquad (\because \; n \text{ is +ve' integer})$$

que ① $L[\sin^3(2t)]$

$$L[\sin^3(2t)] = L\left[\frac{3\sin(2t) - \sin(6t)}{4}\right]$$

$$= \frac{3}{4} L[\sin(2t)] - \frac{1}{4} L[\sin(6t)]$$

$$= \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$L[\sin^3(2t)] = \frac{3}{2}\left(\frac{1}{s^2+4} - \frac{1}{s^2+36}\right)$$

que ② $L[\cos t \cos 2t] = ?$

$$L[\cos t \cos 2t] = \frac{1}{2} L[\cos 3t] + \frac{1}{2} L[\cos t]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right]$$

$$= \frac{s(s^2+5)}{(s^2+9)(s^2+1)}$$

que ③ $L[e^{2t} \sinh^2(at)]$

$\Rightarrow$

$$\sinh^2(at) = \left(\frac{e^{at} - e^{-at}}{2}\right)^2 = \frac{1}{4}\left(e^{2at} + e^{-2at} - 2\right)$$

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$$e^{2t}\sinh^2(at) = \frac{1}{4}\left(e^{(2+2a)t} + e^{(2-2a)t} - 2e^{2t}\right)$$

$$L\left[e^{2t}\sinh^2(at)\right] = \frac{1}{4} L\left[e^{(2+2a)t} + e^{(2-2a)t} - 2e^{2t}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-2-2a} + \frac{1}{s-2+2a} - \frac{2}{s-2}\right]$$

Que (4) Find $L[f(t)]$ where $f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$

$\Rightarrow$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$$

$$= \int_0^{2\pi} e^{-st}\cos t\, dt + \int_{2\pi}^\infty e^{-st}(0)\, dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{2\pi}$$

$$= \frac{e^{-2\pi s}}{s^2+1}(-s) + \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1}\left[1 - e^{-2\pi s}\right]$$

$$L[f(t)] = \frac{s}{s^2+1}\left(1 - \frac{e^{-\pi s}}{e^{\pi s}}\right)$$

$$= \frac{s}{s^2+1}\left(\frac{e^{\pi s} - e^{-\pi s}}{2e^{\pi s}}\right)(2)$$

$$L[f(t)] = \frac{2s}{s^2+1}\,\frac{1}{e^{\pi s}}\,\sinh(\pi s)$$

Que (5) Evaluate $\int_0^{\infty} e^{-2t} \sin^3 t \, dt$

$$\int_0^{\infty} e^{-2t} \sin^3 t \, dt = L[\sin^3 t] \qquad s = 2$$

$$L[\sin^3 t] = L\left[\frac{3\sin t - \sin 3t}{4}\right]$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1}\right] - \frac{1}{4}\left[\frac{3}{s^2+9}\right]$$

$$\int_0^{\infty} e^{-2t} \sin^3 t \, dt = \frac{3}{4}\left(\frac{1}{5}\right) - \frac{3}{4}\left(\frac{1}{13}\right)$$

$$= \frac{3}{4}\left(\frac{13-5}{13\times 5}\right) = \frac{3}{\cancel{4}} \times \frac{\cancel{8}\,2}{13\times 5} = \frac{6}{65}$$

$$\int_0^{\infty} e^{-2t} \sin^3 t \, dt = \frac{6}{65}$$

Que (6) Evaluate $\int_0^{\infty} e^{-3t} t^5 \, dt$

$$\int_0^{\infty} e^{-3t} t^5 \, dt = L[t^5] = \frac{5!}{s^6} = \frac{5!}{3^6} = \frac{40}{243}$$

$s = 3$

Que (7) Evaluate $\int_0^{\infty} e^{-4t} \cosh^3 t \, dt$

$s = 4$ $\quad \int_0^{\infty} e^{-4t} \cosh^3 t \, dt = L[\cosh^3 t]$

$\frac{12}{35}$

$=$

Que ⑧ Find $L[\sin\sqrt{t}]$

$$L[\sin\sqrt{t}] = L\left[\sqrt{t} - \frac{\sqrt{t}^3}{3!} + \frac{\sqrt{t}^5}{5!} - \frac{\sqrt{t}^7}{7!} \cdots\right]$$

$$= L\left[t^{1/2} - \frac{t^{3/2}}{6} + \frac{t^{5/2}}{120} - \frac{t^{7/2}}{7!} \cdots\right]$$

$$= \left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{s^{5/2}.3!} + \frac{\Gamma 7/2}{s^{7/2}.5!} - \frac{\Gamma 9/2}{s^{9/2}\,7!} \cdots\right]$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}.3!} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}.5!} - \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{7!\,s^{9/2}}$$

$$= \frac{\sqrt{\pi}}{2.s^{3/2}}\left[1 - \frac{\left(\frac{1}{2^2 s}\right)}{1!} + \frac{\left(\frac{1}{2^2 s}\right)^2}{2!} - \frac{\left(\frac{1}{2^2 s}\right)^3}{3!} \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2.s^{3/2}}\, e^{-\frac{1}{2^2 s}} = \frac{\sqrt{\pi}}{2.s^{3/2}}\, e^{-1/4s}$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$$

Que ⑨

$$\boxed{\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du}$$

$$L[\operatorname{erf}(\sqrt{t})]$$

$$= L\left[\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right]$$

$$= L\left[\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} \cdots\right)du\right]$$

$$= L\left[\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} \cdots\right.\right]$$

$$= \frac{2}{\sqrt{\pi}} L\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5 \cdot 2!} - \frac{t^{7/2}}{7 \cdot 3!} \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{s^{5/2} \cdot 3} + \frac{\Gamma 7/2}{s^{7/2} \cdot 5} - \frac{\Gamma 9/2}{s^{9/2}\, 7 \cdot 3!} \cdots \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\frac{1/2 \sqrt{\pi}}{s^{3/2}} - \frac{3/2 \cdot 1/2 \sqrt{\pi}}{3 \cdot s^{5/2}} + \frac{5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi}}{5 \cdot 2!\, s^{7/2}} - \frac{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi}}{7\, 3!\, s^{9/2}} \right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2s} + \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{s^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{1}{s^3} \cdots \right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 \cdots$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s} \right)^{-1/2}$$

$$= \frac{1}{s\sqrt{s+1}}$$

$$L[\text{erf}(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$$

Properties of Laplace Transform

(1) First Shifting Theorem

$\Rightarrow$

If $L[f(t)] = F(s)$ then,

$$L[e^{at} f(t)] = F(s-a)$$

$$L[e^{-at} f(t)] = F(s+a)$$

Proof

$$L[e^{at} f(t)] = \int_0^{\infty} e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt$$

put $s-a = u$

$$= \int_0^{\infty} e^{-ut} f(t)\, dt = F(u)$$
$$= F(s-a)$$

$$L[e^{at} f(t)] = F(s-a)$$

$$L[e^{-at} f(t)] = F(s+a)$$

que ① $L[e^{-at} \underbrace{\sin bt}_{f(t)}]$

$\Rightarrow$

$$L[\underbrace{\sin bt}_{f(t)}] = \frac{b}{s^2+b^2} = F(s)$$

by First Shifting Theorem

$$L[e^{-at} \sin bt] = F(s+a) = \frac{b}{(s+a)^2+b^2}$$

que ② $L[e^{t} \cos t \sin t]$

$$L[\cos t \sin t] = L\left[\frac{\sin 2t}{2}\right] = \frac{1}{2} \cdot \frac{2}{s^2+4} = \frac{1}{s^2+4}$$

by FST

$$L[e^{t} \cos t \sin t] = \frac{1}{(s-1)^2+4}$$

que ③ $L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

que ④ $L\left[e^{-5t}\,\text{erf}\,(\sqrt{t})\right] = \frac{1}{(s+5)\sqrt{s+6}}$

que ⑤ $L\left[e^{-3t}\sin 3t\sinh 4t\right] = \frac{1}{2}\left[\frac{3}{(s-1)^2+9} - \frac{3}{(s+7)^2+9}\right]$

que ⑥ Evaluate $\int_0^\infty e^{-3t}\,\text{erf}\,(\sqrt{t})\,dt = \frac{1}{6}$

que ⑦ Evaluate $\int_0^\infty \text{erf}\,(\sqrt{t})\,e^{-t}\,dt = \frac{1}{\sqrt{2}}$

que ⑧ $L\left[e^{-2t}\,\text{erf}\,(\sqrt{t})\right] = \frac{1}{(s+2)\sqrt{s+3}}$

que ⑨ $L\left[e^{-4t}\sinh t\sin t\right]$

ans ③ $L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$$L\left[\frac{e^{t/2}-e^{-t/2}}{2}\left(\sin\frac{\sqrt{3}}{2}t\right)\right]$$

$$\frac{1}{2}L\left[e^{t/2}\sin\frac{\sqrt{3}}{2}t - e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right]$$

$$\frac{1}{2}\left[L\left(e^{t/2}\sin\frac{\sqrt{3}}{2}t\right) - L\left(e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right)\right]$$

using FST

$$\frac{1}{2}\left[\frac{\sqrt{3}/2}{(s-1/2)^2+3/4} - \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right]$$

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ans ④ $L\left[e^{-5t}\,erf(\sqrt{t})\right]$

ans ③ $L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$L\left[\left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$\frac{1}{2}L\left[e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)-e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$L\left[\sin\left(\frac{\sqrt{3}}{2}t\right)\right]=\frac{\sqrt{3}/2}{s^2+3/4}=\frac{\sqrt{3}/2}{\frac{4s^2+3}{4}}$

FST
$L\left[e^{t/2}\sin(\sqrt{3}/2\,t)\right]=\frac{\sqrt{3}/2}{(s-1/2)^2+3/4}$

$L\left[e^{-t/2}\sin(\sqrt{3}/2\,t)\right]=\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}$

$s^2+\frac{1}{4}+s$

$=s^2+\frac{1}{4}+s$

$\frac{\sqrt{3}/2}{2}\left[\frac{1}{(s-1/2)^2+3/4}-\frac{1}{(s+1/2)^2+3/4}\right]$

$\frac{\sqrt{3}}{4}\left[\frac{(s+1/2)^2+\cancel{3/4}-(s-1/2)^2-\cancel{3/4}}{\left[(s-1/2)^2+3/4\right]\left[(s+1/2)^2+3/4\right]}\right]$

$\frac{\sqrt{3}}{4}\left[\frac{2s}{\left((s-1/2)^2+3/4\right)\left((s+1/2)^2+3/4\right)}\right]$

ans ④ $L\left[e^{-5t}\,erf(\sqrt{t})\right]$

$L\left[erf(\sqrt{t})\right]=\frac{1}{s\sqrt{s+1}}$

$L\left[e^{-5t}\,erf(\sqrt{t})\right]=\frac{1}{(s+5)\sqrt{s+6}}$ By FST que ⑤ $L\left[e^{-3t}\sin 3t \sinh 4t\right]$ $\qquad \sinh 4t = \dfrac{e^{4t} - e^{-4t}}{2}$

$$L\left[\frac{e^{t} - e^{-7t}}{2}(\sin 3t)\right]$$

$$\frac{1}{2} L\left[e^{t}\sin 3t - e^{-7t}\sin 3t\right]$$

$$L\left[\sin 3t\right] = \frac{3}{s^2+9} \qquad L\left[e^{t}\sin 3t\right] = \frac{3}{(s-1)^2+9}$$

$$L\left[e^{-7t}\sin 3t\right] = \frac{3}{(s+7)^2+9}$$

$$\frac{1}{2}\left[\frac{3}{(s-1)^2+9} - \frac{3}{(s+7)^2+9}\right].$$

que ⑥ $\displaystyle\int_0^\infty e^{-3t}\,\mathrm{erf}(\sqrt{t})\,dt = L\left[\mathrm{erf}(\sqrt{t})\right] = \frac{1}{s\sqrt{s+1}} = \frac{1}{3\sqrt{4}} = \frac{1}{6}$

$s = 3$

que ⑦ $\displaystyle\int_0^\infty \mathrm{erf}(\sqrt{t})\,e^{-t}\,dt = L\left[\mathrm{erf}(\sqrt{t})\right] = \frac{1}{s\sqrt{s+1}} = \frac{1}{\sqrt{2}}$

$s = 1$

que ⑧ $L\left[e^{-2t}\mathrm{erf}(\sqrt{t})\right] = \dfrac{1}{s\sqrt{s+1}} = \dfrac{1}{(s+2)\sqrt{s+3}}$

que ⑨ $L\left[e^{-4t}\sinh t \sin t\right]$ $\qquad \sinh t = \dfrac{e^{t} - e^{-t}}{2}$

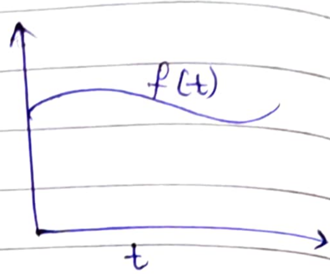
$$L\left[\frac{e^{-3t} - e^{-5t}}{2}(\sin t)\right] \qquad L\left[\sin t\right] = \frac{1}{s^2+1}$$

$$\frac{1}{2} L\left[e^{-3t}\sin t - e^{-5t}\sin t\right]$$

$$\frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right]$$

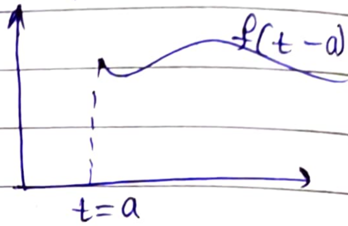
② Second Shifting Theorem

If $L[f(t)] = F(s)$



& $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$



Que ① Find $L[f(t)]$ where, $f(t) = \begin{cases} (t-3)^4 & t>3 \\ 0 & t<3 \end{cases}$

→

$L[t^4] = \frac{4!}{s^5} = F(s)$

By SST $\quad a = 3$

$L[f(t)] = e^{-3s} F(s) = e^{-3s} \frac{4!}{s^5}$

Que ② Find $L[f(t)]$ where, $f(t) \begin{cases} \cos(t-\pi/3) & t>\pi/3 \\ 0 & t<\pi/3 \end{cases}$

→

$L[\cos t] = \frac{s}{s^2+1}$

By SST $\quad a = \pi/3$

$L[f(t)] = e^{-\pi/3 s} \frac{s}{s^2+1}$

Que ③ Find $L[f(t)]$ where, $f(t) \begin{cases} (t-1)^2 & t>1 \\ 0 & 0<t<1 \end{cases}$

→

$$L[t^2] = \frac{2}{s^3}$$

By SST.

$$L[f(t)] = e^{-s} F(s) = e^{-s} \cdot \frac{2}{s^3}$$

(3) Multiplication By Power of t

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

que (1) $L[t \sin at]$

$$L[\sin at] = \frac{a}{s^2+a^2} = F(s)$$

$\therefore$ By Multipl^n by power of t

$$L[t \sin at] = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= (-1)\ a -(s^2+a^2)^{-2}\ (2s)$$

$$= 2as\ (s^2+a^2)^{2}$$

$$L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$$

que (2) $L[\sin at - at \cos at]$

$$L[\sin at] - a\,L[t \cos at]$$

$$\frac{a}{s^2+a^2} - a\ L[t \cos at]$$

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$L[t\cos at] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$

$= (-1)\left(\frac{s^2+a^2 - s(2s)}{(s^2+a^2)^2}\right)$

$= (-1)\frac{(s^2+a^2-2s^2)}{(s^2+a^2)^2}$

$= \frac{s^2-a^2}{(s^2+a^2)^2}$

$\frac{a}{s^2+a^2} - a\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right) = \frac{2a^3}{(s^2+a^2)^2}$

que (3) $L[t e^{3t} \sin 2t]$

$L[\sin 2t] = \frac{2}{s^2+4}$

$L[t\sin 2t] = (-1)\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$

$= (-1)2 \cdot -(s^2+4)^{-2}(2s)$

$= \frac{4s}{(s^2+4)^2}$

By FST

$L[t e^{3t}\sin 2t] = \cancel{\frac{4s}{(s^2+4)^2}} \quad \frac{4(s-3)}{[(s-3)^2+4]^2}$

$= \cancel{4(s-3)}$

que (4) Evaluate $\int_0^\infty t e^{-t} \sin t \, dt = L[t\sin t]$

By defination $\quad s=1$

By Multiplication by power of t

$$L[t \sin t] = (-1) \frac{d}{ds}\left(\frac{1}{s^2+1}\right) \quad s=1$$

$$= \frac{2s}{(s^2+1)^2} \quad s=1$$

$$= \frac{1}{2}$$

Que (5)

Evaluate $\int_0^\infty t^2 e^{-t} \sin t \, dt$

$$= L[t^2 \sin t] \quad s=1$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{1+s^2}\right)$$

$$= (-1)^2 \frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right)$$

$$= \frac{2(s^2+1)^2 - 2s \cdot 2(s^2+1)}{(s^2+1)^4}$$

$$= \frac{2(s^4+1+2s^2) - 4s^3 - 4s}{(s^2+1)^4} \quad s=1$$

$$= \frac{1}{2}$$

Que (6) $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Que (7) $\int_0^\infty t\, e^{-3t} \sin t \, dt = L[t \sin t] = \frac{3}{50}$ $\quad s=3$

Que (8) $L[\sin 3t \; e^{4t} . t]$

Que (9) $L[e^{-2t} t^4 \sinh 4t]$ → Put $\sinh 4t = \frac{e^{+4t} - e^{-4t}}{2}$

Que (10) $L[t\, e^{3t} \operatorname{erf} \sqrt{t}]$ Ans:- $\frac{1}{2}\left\{\frac{4!}{(s-2)^5} - \frac{4!}{(s+6)^5}\right\}$

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que (11) $L[t^4 \cosh 4t] = \frac{1}{2}\left\{\frac{4!}{(s-4)^5} + \frac{4!}{(s+4)^5}\right\}$

que (12) $L[t^2 \sin 2t]$

que (13) $L[t^5 \cosh 4t] \longrightarrow FST = \frac{1}{2} L[e^{4t} t^5 + e^{-4t} t^5$

ans (6) $\Rightarrow$ $\int_0^\infty t^3 e^{-t} \sin t \, dt = L[t^3 \sin t]$

$= (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$

$= (-1) \frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$

$= (-1) \frac{d}{ds}\left[\frac{-2(s^2+1)^2 + 2s\, 2(s^2+1)(2s)}{(s^2+1)^4}\right]$

$\cancel{\frac{-2(s^4+1+2s^2) + 8s^2(s^2+1)}{(s^2+1)^4}}$

$= (-1) \frac{d}{ds}\left[\frac{-2(s^2+1) + 8s^2}{(s^2+1)^3}\right]$

$= (-1) \frac{d}{ds}\left[\frac{6s^2-2}{(s^2+1)^3}\right] = (-1)\left[\frac{12s(s^2+1)^3 - (6s^2-2)\,3\,(s^2+1)^2\,2s}{(s^2+1)^6}\right]$

$= (-1)\left[\frac{12s(s^2+1) - 6s(6s^2-2)}{(s^2+1)^4}\right]$

que (8) $L[\sin 3t\, e^{4t}\, t]$

$L[\sin 3t] = \frac{3}{s^2+9}$

$L[t \sin 3t] = (-1) \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-1)\left[\frac{-3(2s)}{(s^2+9)^2}\right]$

$= \frac{6s}{(s^2+9)^2}$

$L[e^{4t} \sin 3t\, t] = \frac{6(s-4)}{[(s-4)^2+9]^2}$

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que 9) $L\left[e^{-2t}\,t^4\sinh 4t\right]$ $\qquad \sinh 4t = \dfrac{e^{4t}-e^{-4t}}{2}$

$$L\left[t^4\left(\frac{e^{2t}-e^{-6t}}{2}\right)\right]$$

$$\frac{1}{2}L\left[t^4e^{2t}-t^4e^{-6t}\right]$$

$$\frac{1}{2}\left[\frac{4!}{s^5}-\frac{4!}{s^5}\right] = \cancel{\frac{1}{2}\left[\frac{4!}{(-2)^5}-\frac{4!}{(6)^5}\right]}$$

$$\frac{1}{2}\left[\frac{4!}{(s-2)^5}-\frac{4!}{(s+6)^5}\right]$$

que 10) $L\left[t\,e^{3t}\operatorname{erf}\sqrt{t}\right]$

$$L\left[\operatorname{erf}\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$$

$$L\left[t\operatorname{erf}\sqrt{t}\right] = (-1)^1\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right) = (-1)\left[\frac{-\left(\sqrt{s+1}+s\frac{1}{2\sqrt{s+1}}\right)}{s^2(s+1)}\right]$$

$$= (+1)\left[\frac{2(s+1)+s}{2\sqrt{s+1}\;s^2(s+1)}\right]$$

$$= (+1)\left[\frac{3s+2}{2s^2(s+1)^{3/2}}\right]$$

$$L\left[e^{3t}\operatorname{erf}(\sqrt{t})t\right] = (+1)\;\frac{3(s-3)+2}{2(s-3)^2(s-2)^{3/2}}$$

que 11) $L\left[t^4\cosh 4t\right] = L\left[t^4\left(\dfrac{e^{4t}+e^{-4t}}{2}\right)\right]$

$$= \frac{1}{2}L\left[t^4e^{4t}+t^4e^{-4t}\right]$$

$$= \frac{1}{2}\left[\frac{4!}{s^5}+\frac{4!}{s^5}\right]$$

$$= \frac{1}{2}\left[\frac{4!}{(s-4)^5}+\frac{4!}{(s+4)^5}\right]$$

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Que (12) $L[t^2 \sin 2t]$ = $\Rightarrow$ $L[\sin 2t] = \dfrac{2}{s^2+4}$

$$L[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{d}{ds}\left[\frac{-2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \frac{-4(s^2+4)^2 + 4s \cdot 2(s^2+4)2s}{(s^2+4)^4}$$

$$= \cancel{-4+16s^2} \quad \frac{-4(s^2+4)+16s}{(s^2+4)3}$$

$$= \frac{-4s^2-16+16s}{(s^2+4)3}$$

Que (13) $L[t^5 \cosh 4t] = L\left[t^5\left(\dfrac{e^{4t}+e^{-4t}}{2}\right)\right]$

$$= \frac{1}{2} L\left[t^5 e^{4t} + t^5 e^{-4t}\right]$$

$$= \frac{1}{2}\left[\frac{5!}{s^6} + \frac{5!}{s^6}\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-4)^6} + \frac{5!}{(s+4)^6}\right]$$

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④ Division by t :-

If $L[f(t)] = F(s)$ & $\lim_{t\to 0} \frac{f(t)}{t}$ exist

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Que ① $L\left[\frac{\cos at - \cos bt}{t}\right]$ where $a \neq b$

$\Rightarrow$

$$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

By Division by t property

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty F(s)\,ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

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ans. ② $L\left[\frac{1-\cos at}{t}\right]$

$$L[1-\cos at] = \frac{1}{s} - \frac{s}{a^2+s^2} = \frac{s^2+a^2-s^2}{s(a^2+s^2)}$$

$$= \frac{a^2}{s(a^2+s^2)}$$

$$L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty \frac{a^2}{s(a^2+s^2)}\, ds$$

$$= \int_s^\infty \frac{1}{s} - \frac{s}{a^2+s^2}\, ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{\sqrt{a^2+s^2}}\right) - \cancel{\log\left(\frac{s}{\sqrt{2s^2}}\right)}\ \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{1}{\sqrt{a^2/s^2+1}}\right) - \cancel{\log\left(\frac{s}{\sqrt{2}\,s}\right)}$$

$$= 0 - \cancel{\log\left(\frac{1}{\sqrt{2}}\right)} \quad \cdot\log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

$$= \log(\sqrt{2})$$

Que ③ $L\left[\frac{\sin t}{t}\right]$

$$L[\sin t] = \frac{1}{1+s^2}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} \cdot ds$$

$$= \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$= \underline{\underline{\cot^{-1} s}}$$

Que ④ $L\left[\frac{e^{-4t}\sin 3t}{t}\right]$

$\Rightarrow$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2+9} \, ds$$

$$= \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$L\left[\frac{\sin 3t}{t}\right] = \cot^{-1}\left(\frac{s}{3}\right)$$

By FST

$$L\left[\frac{e^{-4t}\sin 3t}{t}\right] = \cot^{-1}\left(\frac{s+4}{3}\right)$$

Que ⑤ Evaluate $\int_0^\infty e^{-2t} \frac{\sinh t}{t} \, dt$

$$\Rightarrow \int_s^\infty e^{-2t} \frac{\sinh t}{t}\, dt = L\left[\frac{\sinh t}{t}\right] \quad s=2$$

$$= \int_s^\infty \frac{1}{s^2-1}\, ds \quad s=2$$

$$= \int_s^\infty \frac{1}{(s-1)(s+1)}\, ds \quad s=2$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s+1}\right) ds \quad s=2$$

$$= \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty \quad s=2$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left(\frac{1-1/s}{1+1/s}\right) - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= -\frac{1}{2}\log\left(\frac{s-1}{s+1}\right) \quad s=2$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$\int_s^\infty e^{-2t} \frac{\sinh t}{t}\, dt = \frac{1}{2}\log(3)$$

que ⑥ $L\left[\frac{1-\cos 2t}{t}\right]$

$$L[1-\cos 2t] = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left[\frac{1-\cos 2t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left(\frac{1}{\sqrt{1+4/s^2}}\right) - \log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

que (7) $L\left[\frac{\sin^2 t}{t}\right]$

$$\frac{1}{2} L\left[\frac{1-\cos 2t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

que (8) Evaluate $\int_0^\infty \frac{e^{-2t}\sin t}{t}\,dt$

que (9) Evaluate $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

que (10) Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$ ★

que (11) Prove that $\int_0^\infty e^{-\sqrt{2}t}\,\frac{\sin t \sinh t}{t}\,dt = \frac{\pi}{8}$ ★

que (12) Prove that $\int_0^\infty \frac{\sin^2 t}{t}\,e^{-t}\,dt = \log\sqrt[4]{5}$

que (13) $L\left[\frac{e^{-2t}\sin 2t \cosh t}{t}\right]$ ★

que (14) $L\left[t^{-1} e^{-t} \sin t\right]$

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Que (15) $L\left[\frac{1-\cos t}{t}\right]$ ans:- $\log\left(\sqrt{1+\frac{1}{s^2}}\right)$

Que (16) $L\left[\frac{1-\cos at}{t^2}\right]$ ~~ans~~ = ★

Propery 5 Laplace Transform of Derivation

$$L\left[\frac{d^n(f(t))}{dt^n}\right] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \dots s f^{n-2}(0) - f^{n-1}(0)$$

Que (1) $L[t \sin at] = \frac{2as}{(s^2+a^2)^2}$

Find $L[at\cos at + \sin at]$

$\Rightarrow$

$$L[at\cos at + \sin at] = L\left[\frac{d}{dt} t \sin at\right]$$
$$= s\, L[t \sin at] - f(0)$$
$$= s\left(\frac{2as}{(s^2+a^2)^2}\right) - 0$$
$$L[at\cos at + \sin at] = \frac{2as^2}{(s^2+a^2)^2}$$

Que (2) Find $L\left[\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y\right]$ Given that
$y(0) = 2$
$y'(0) = -4$

$\Rightarrow$

$$L\left[\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 5y\right] = \left\{s^2 L[y(t)] - s\,y(0) - y'(0)\right\}$$
$$- 3\left\{sL[y(t)] - y(0)\right\}$$
$$+ 5L[y(t)]$$

$= L[y(t)](s^2-3s+5) - (y(0))(s-3) - y'(0)$

$= L[y(t)](s^2-3s+5) - 2s + 6 + 4$

$= (s^2-3s+5)\, L[y(t)] - 2s + 10$

que ③ $L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right]$

$= s\, L\left[\frac{\sin t}{t}\right] - \lim_{t \to 0} \frac{\sin t}{t}$

$= s \cot^{-1}(s) - 1$

que ④ find $L[f(t)]$, $4f^2 + f = 0$, $f(0) = 0$, $f'(0) = 2$

$\Rightarrow$

$L[4f'' + f] = L[0] = 0$

$4L[f''] + L[f(t)] = 0$

$\{4s^2 L[f(t)] - 4sf(0) - 4f'(0)\} + L[f(t)] = 0$

$4s^2 L[f(t)] - 8 + L[f(t)] = 0$

$L[f(t)](4s^2+1) = 8$

$L[f(t)] = \frac{8}{4s^2+1}$

Property 6 Lapalace Transformation of Integration

$L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s} L[f(t)]$

$= \frac{1}{s} F(s)$

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que ① $L\left[\int_0^t \frac{\sin x}{x}\, dx\right]$

$\Rightarrow$ $L\left[\frac{\sin t}{t}\right] = \cot^{-1} s$

$L\left[\int_0^t \frac{\sin x}{x}\, dx\right] = \frac{\cot^{-1} s}{s}$

que ② $L\left[\int_0^t u^2 e^{-u}\, du\right]$

$L\left[u^2 e^{-u}\right]$

$L\left[t^2\right] = \frac{2}{s^3}$

By FST

$L\left[t^2 e^{-t}\right] = \frac{2}{(s+1)^3}$

$L\left[\int_0^t u^2 e^{-u}\, du\right] = \frac{2}{s(s+1)^3}$

que ③ $L\left[\int_0^t u \cosh u\, du\right]$

$\Rightarrow$ $L[\cosh u] = \frac{s}{s^2-1}$

$L[u \cosh u] = (-1)\frac{d}{ds}\left(\frac{s}{s^2-1}\right)$

$= (-1)\left[\frac{(s^2-1) - s(2s)}{(s^2-1)^2}\right]$

$L[u \cosh u] = \frac{s^2+1}{(s^2-1)^2} \cdot P(s)$

$L\left[\int_0^t u \cosh u\, du\right] = \frac{s^2+1}{s(s^2-1)^2}$

que ④ $L\left[\int_0^t \frac{e^t \sin t}{t}\, dt\right]$

$\Rightarrow$

$L\left[\frac{\sin t}{t}\right] = \cot^{-1} s$

By FST

$L\left[\frac{e^t \sin t}{t}\right] = \cot^{-1}(s-1)$

$L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right] = \frac{\cot^{-1}(s-1)}{s}$

que ⑤ $L\left[\int_0^t x^2 e^x\, dx\right]$

$L[t^2] = \frac{2}{s^3}$

$L[e^x x^2] = \frac{2}{(s-1)^3}$

$L\left[\int_0^t x^2 e^x\, dx\right] = \frac{2}{s(s-1)^3}$

que ⑥ $L\left[\int_0^t u e^{-3u} \sin 4u\, du\right]$

$L[\sin 4u] = \frac{4}{s^2+16}$

$L[u \sin 4u] = (-1) \frac{d}{ds}\left(\frac{4}{s^2+16}\right)$

$= \frac{8s}{s^2+16}$

By FST

$L[u e^{-3u} \sin 4u] = \frac{8(s+3)}{(s+3)^2+16}$

$$L\left[\int_0^t u e^{-3u} \sin 4u \, du\right] = \frac{8(s+3)}{s[(s+3)^2+16]}$$

que ⑦ $$L\left[\int_0^t e^{-t} \frac{\sin t}{t} dt\right] = \frac{\cot^{-1}(s+1)}{s}$$

que ⑧ $$L\left[\int_0^t e^{2t} \frac{\sin t}{t} dt\right] = \frac{\cot^{-1}(s-2)}{s}$$

que ⑨ $$L\left[\int_0^t \frac{\sin 4t}{t} e^{t} dt\right]$$

$$L[\sin 4t] = \frac{4}{s^2+16}$$

$$L\left[\frac{\sin 4t}{t}\right] = \int_s^\infty \frac{4}{s^2+16} ds$$

$$= 4\left[\frac{1}{4}\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{4}\right)$$

By FST

$$L\left[\frac{\sin 4t}{t} e^{t}\right] = \frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{4}\right)$$

$$L\left[\int_0^t \frac{\sin 4t}{t} e^{t} dt\right] = \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s-1}{4}\right)\right]$$

que ⑩ $$L\left[\int_0^t \frac{1-e^{-u}}{u} du\right]$$

que (ii) $L\left[\cosh t \int_0^t e^t \cosh t \, dt\right] = L\left[\frac{e^t f(t) + e^{-t} f(t)}{2}\right]$

$\rightarrow$

$L[\cosh t] = \frac{s}{s^2-1}$

By FST

$L[e^t \cosh t] = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s(s-2)}$

$L\left[\int_0^t e^t \cosh t \, dt\right] = \frac{s-1}{s^2(s-2)} = F(s)$

$L\left[\cosh t \int_0^t e^t \cosh t \, dt\right] = L\left[\frac{e^t f(t) + e^{-t} f(t)}{2}\right]$

By FST

$L\left[\cosh t \int_0^t e^t \cosh t \, dt\right] = \frac{1}{2}\left\{\frac{s-1-1}{(s-1)^2(s-3)} + \frac{s+1-1}{(s+1)^2(s+1-2)}\right\}$

$= \frac{1}{2}\left\{\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right\}$

Property 7 Change of Scalar

If $L[f(t)] = F(s)$

$F[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

que (1) Given that $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

Prove that $\int_0^\infty t e^{-3t} J_0(4t) \, dt = \frac{3}{125}$

$\Rightarrow$

$\Rightarrow$ $\int_0^\infty t e^{-3t} J_0(4t)\, dt = \int_0^\infty e^{-3t} (t J_0(4t))\, dt$

$= L[t J_0(4t)]$ $\quad s = 3$

$L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

By change of scalar

$L[J_0(4t)] = \frac{1}{4} \frac{1}{\sqrt{1+(s/4)^2}} = \frac{1}{\sqrt{s^2+16}}$

$L[t J_0(4t)] = (-1) \frac{d}{ds} \frac{1}{\sqrt{s^2+16}}$

$= \frac{1}{2} \frac{2s}{(s^2+16)^{3/2}} = \frac{s}{(s^2+16)^{3/2}}$

$\int_0^\infty t e^{-3t} J_0(4t) = \frac{s.}{(s^2+16)^{3/2}}$ $\quad s = 3$

$= \frac{3}{125}$

Que (2) find $L[t\, erf(2\sqrt{t})]$, where $L[erf(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$

$\Rightarrow$

$L[erf(2\sqrt{t})] = L[erf(\sqrt{4t})] = \frac{1}{s/4 \sqrt{s/4+1}}$

$= \frac{2}{s\sqrt{s+4}}$

$L[t\, erf(2\sqrt{t})] = (-1) \frac{d}{ds} \left(\frac{2}{s\sqrt{s+4}} \right)$

$$L[t\,erf(2\sqrt{t})] = (-2)\left[\frac{-1}{s^2}\frac{1}{\sqrt{s+4}} + \frac{-1}{2(s+4)^{3/2}}\frac{1}{s}\right]$$

$$= \frac{3s+8}{s^2(s+4)^{3/2}}$$

Que ③ Find $L[t\,J_0(at)]$ given $L[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$

$$\Rightarrow \quad L[J_0(at)] = \frac{1}{a}\frac{1}{\sqrt{1+(s/a)^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

$$L[t\,J_0(at)] = (-1)\frac{d}{ds}\frac{1}{\sqrt{s^2+a^2}}$$

$$= \frac{s}{(s^2+a^2)^{3/2}}$$

Inverse Laplace Transform

If $L[f(t)] = F(s)$ then $f(t)$ is called the inverse Laplace transform of $F(s)$ is denoted by $L^{-1}[F(s)]$

$\therefore \quad f(t) = L^{-1}[F(s)]$.

$L^{-1} : s \to t$

ILT → operator

1) $L^{-1}\left[\frac{1}{s}\right] = 1$ 2) $L^{-1}\left[\frac{1}{s^2}\right] = t$

3) $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!}$

1) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ n is +ve integer

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

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2) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

4) $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

5) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6) $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh(at)}{a}$

7) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh(at)$

8) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{\sin at - at\cos at}{2a^3}$

9) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$

10) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{\sin at + at\cos at}{2a}$

11) $L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = \cos at - \frac{1}{2}at\sin at$

12) $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

13) $L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right] = \frac{at\cosh at - \sinh at}{2a^3}$

14) $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{t\sinh(at)}{2a}$

que (15) $L^{-1}\left[\frac{s^2}{(s^2-a^2)^2}\right] = \frac{at\cosh(at)+\sinh at}{2a}$

que (16) $L^{-1}\left[\frac{s^3}{(s^2-a^2)^2}\right] = \cosh at + \frac{1}{2}at\sinh at$

que (17) $L^{-1}\left[\frac{s^2+a^2}{(s^2-a^2)^2}\right] = t\cosh(at)$

Proof

1) $$L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s(s^2+a^2-a^2)}{(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2} - a^2\frac{s}{(s^2+a^2)^2}\right]$$

$$= \cos at - a^2\left(\frac{t\sin at}{2a}\right)$$

$$L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = \cos at - \frac{1}{2}at\sin at$$

2) $$L^{-1}\left[\frac{s^3}{(s^2-a^2)^2}\right] = L^{-1}\left[\frac{s(s^2-a^2+a^2)}{(s^2-a^2)^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2-a^2} + a^2\frac{s}{(s^2-a^2)^2}\right]$$

$$= \cosh(at) + a^2\,\frac{t\sinh(at)}{2a}$$

$$L^{-1}\left[\frac{s^3}{(s^2-a^2)^2}\right] = \cosh(at) + \frac{1}{2}at\sinh(at)$$

★ Problems

Ⓐ Using Table

1) $L^{-1}\left[\frac{2s^2+3s+4}{5s^3}\right]$

$\frac{2}{5}L^{-1}\left[\frac{1}{s}\right]+\frac{3}{5}L^{-1}\left[\frac{1}{s^2}\right]+\frac{4}{5}L^{-1}\left[\frac{1}{s^3}\right]$

$\frac{2}{5}+\frac{3}{5}(t)+\frac{2}{5}(2t^2)$

$\frac{4t^2+3t+2}{5}$

2) $L^{-1}\left[\frac{1}{2s-3}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s-3/2}\right]$

$=\frac{1}{2}e^{3/2\,t}$

3) $L^{-1}\left[\frac{2s+4}{s^2+9}\right]=2L^{-1}\left[\frac{s}{s^2+9}\right]+4L^{-1}\left[\frac{1}{s^2+9}\right]$

$=2\cos 3t+\frac{4}{3}\sin 3t$

4) $L^{-1}\left[\frac{2s+6}{s^2+4}\right]=2L^{-1}\left[\frac{s}{s^2+4}\right]+6L^{-1}\left[\frac{1}{s^2+4}\right]$

$=2\cos 2t+3\sin 2t$

5) $L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right]=\frac{3}{2}L^{-1}\left[\frac{s^4-2s^2+1}{s^5}\right]$

$=\frac{3}{2}\left[L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left(\frac{1}{s^3}\right)+L^{-1}\left(\frac{1}{s^5}\right)\right]$

$=\frac{3}{2}\left[1-2\left(\frac{t^2}{2}\right)+\frac{t^4}{4!}\right]$

$=\frac{t^4}{16}-\frac{3}{2}t^2+\frac{3}{2}$

(B) Use of First Shifting theorem

If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[F(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[F(s)]$$

Similarly $L^{-1}[F(s-a)] = e^{at} f(t) = e^{at} L^{-1}[F(s)]$

1) $L^{-1}\left[\frac{2s+3}{(s+4)^2}\right] = L^{-1}\left[\frac{2(s+4-4)+3}{(s+4)^2}\right]$

$$= e^{-4t} L^{-1}\left[\frac{2(s-4)+3}{s^2}\right]$$

using FST

$$= e^{-4t} L^{-1}\left[\frac{2s-5}{s^2}\right]$$

$$= e^{-4t}\left[L^{-1}\left[\frac{2}{s}\right] - 5L^{-1}\left[\frac{1}{s^2}\right]\right]$$

$$= e^{-4t}[2-5t]$$

2) $L^{-1}\left[\frac{1}{s^2+2s+2}\right] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

using FST replace s+1 by s

$$L^{-1}\left[\frac{1}{s^2+2s+2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t}\sin t$$

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~~Ans~~ (3) $L^{-1}\left[\frac{1}{(2s-5)^3}\right] = \frac{1}{8} L^{-1}\left[\frac{1}{(s-5/2)^3}\right]$

$= \frac{1}{8} e^{5/2 t} L^{-1}\left(\frac{1}{s^3}\right)$ using FST

$= \frac{1}{8} e^{5/2 t} \frac{t^2}{2}$

$= \frac{1}{16} e^{5/2 t} t^2$

(4) $L^{-1}\left[\frac{s^2}{(2s+5)^4}\right] = \frac{1}{16} L^{-1}\left[\frac{s^2}{(s+5/2)^4}\right]$

$= \frac{1}{16} e^{-5/2 t} L^{-1}\left[\frac{(s-5/2)^2}{s^2}\right]$

$= \frac{1}{16} e^{-5/2 t} L^{-1}\left[\frac{s^2-5s+25/4}{s^4}\right]$

(5) $L^{-1}\left[\frac{2s+5}{4s^2-4s+17}\right]$

$L^{-1}\left[\frac{2s+5}{4(s^2-s+17/4)}\right]$

$\frac{1}{4} L^{-1}\left[\frac{2s+5}{s^2-s+17/4}\right]$

$\frac{1}{4} L^{-1}\left[\frac{2s+5}{(s-1/2)^2+4}\right]$

Using FST

$\frac{1}{4} e^{t/2} L^{-1}\left[\frac{2(s+1/2)+5}{s^2+4}\right]$

$$= \frac{1}{4} e^{t/2} L^{-1}\left[\frac{2s+6}{s^2+4}\right]$$

$$= \frac{1}{4} e^{t/2} \left[2\cos 2t + 3\sin 2t\right]$$

(6) $$L^{-1}\left[\frac{3s-2}{7-12s-4s^2}\right]$$

$$= \frac{-1}{4} L^{-1}\left[\frac{3s-2}{s^2+3s-7/4}\right]$$

$$= \frac{-1}{4} L^{-1}\left[\frac{3s-2}{(s+3/2)^2-4}\right]$$

using FST

$$= \frac{-1}{4} e^{-3/2t} L^{-1}\left[\frac{3(s-3/2)-2}{s^2-4}\right]$$

$$= \frac{-1}{4} e^{-3/2t} L^{-1}\left[\frac{3s-13/2}{s^2-4}\right]$$

$$= \frac{-1}{4} e^{-3/2t}\left(3\cosh(2t) - \frac{13}{2}\frac{\sinh(2t)}{2}\right)$$

$$= \frac{1}{16} e^{-3/2t}\left(13\sinh(2t) - 12\cosh(2t)\right)$$

(7) $$L^{-1}\left[\frac{s^3-3s^2+8s-6}{(s^2-2s+2)^2}\right]$$

$$L^{-1}\left[\frac{s^3-3s^2+8s-6}{((s-1)^2+1)^2}\right]$$

By using FST

$$= e^{t} L^{-1}\left[\frac{(s+1)^3 - 3(s+1)^2 + 8(s+1) - 6}{(s^2+1)^2}\right]$$

$$= e^{t} L^{-1}\left[\frac{s^3+1+3s^2+3s-3(s^2+1+2s)+8s+8-6}{(s^2+1)^2}\right]$$

$$= e^{t} L^{-1}\left[\frac{s^3+1+\cancel{3s^2}+\cancel{3s}-\cancel{3s^2}-3-\cancel{6s}+\cancel{8}s+2}{(s^2+1)^2}\right]$$

$$= e^{t} L^{-1}\left[\frac{s^3+5s}{(s^2+1)^2}\right]$$

$$= e^{t} L^{-1}\left[\frac{s(s^2+1)+4s}{(s^2+1)^2}\right]$$

$$= e^{t} L^{-1}\left[\frac{s}{(s^2+1)} + 4\frac{s}{(s^2+1)^2}\right]$$

$$= e^{t}\left(\cos t + 4\frac{t\sin t}{2}\right)$$

$$= e^{t}\left(\cos t + 2t\sin t\right)$$

(8) $$L^{-1}\left[\frac{s+1}{s^2+s+1}\right] = L^{-1}\left[\frac{s+1/2+1/2}{(s+1/2)^2+3/4}\right]$$

$$= e^{-t/2} L^{-1}\left[\frac{s+1/2}{s^2+3/4}\right]$$

$$= e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\cdot\frac{\sin(\sqrt{3}/2\, t)}{\sqrt{3}/2}\right)$$

$$= e^{-t/2}\left(\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right)$$

Using Second Shifting Theorem.

If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$$

Problems

① $L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right] = L^{-1}[e^{-as}F(s)]$

$\Rightarrow$

$$L^{-1}\left[\frac{1}{(s-2)^4}\right]$$

using FST

$$e^{2t}L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}\frac{t^3}{3!}$$

using SST

$$L^{-1}\left[e^{-3s}\frac{1}{(s-2)^4}\right] = \begin{cases} e^{2(t-3)}\frac{(t-3)^3}{3!} & t>3 \\ 0 & t<3 \end{cases}$$

② $L^{-1}\left[\frac{s\,e^{-\frac{4}{5}\pi s}}{s^2+25}\right]$

$$L^{-1}\left[\frac{s}{s^2+25}\right] = \cos(5t)$$

using SST

$$L^{-1}\left[e^{-4/5\pi s}\frac{s}{s^2+25}\right] = \begin{cases} \cos 5(t-4/5\pi) & t > 4\pi/5 \\ 0 & t < 4\pi/5 \end{cases}$$

③ $L^{-1}\left[\frac{8}{s^2+4}e^{-3s}\right]$

$$L^{-1}\left[\frac{8}{s^2+4}\right] = \frac{8\sin 2t}{2} = 4\sin 2t$$

$$L^{-1}\left[e^{-3s}\frac{8}{s^2+4}\right] = \begin{cases} 4\sin 2(t-3) & t > 3 \\ 0 & t < 3 \end{cases}$$

Using Multiplication by power of t

$$L[t f(t)] = -\frac{d}{ds}F(s)$$

$$\therefore -t f(t) = L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$\therefore f(t) = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$\therefore \boxed{L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

→ useful when F(S) involves logarithm or inverse circular function

$$L[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}F(s) = (-1)^n F^n(s)$$

$L^{-1}[F^n(s)] = (-1)^n t^n L^{-1}[F(s)]$

Using L.T of derivatives

If $L^{-1}[F(s)] = f(t)$ & $f(0) = 0$

$$L[f'(t)] = sL[f(t)] - f(0)$$
$$= sF(s) - 0$$

$$L^{-1}[sF(s)] = f'(t)$$

$$L^{-1}[s^n F(s)] = \frac{d^n}{dt^n} f(t) \quad (f(0) = f'(0) = f''(0) \text{ --- } f^{n-1}(0) = 0)$$

$\downarrow$

Multiplication by s

Using division by t

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

$$\therefore L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t} = \frac{L^{-1}[F(s)]}{t}$$

$$\boxed{L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]}$$

$\rightarrow$ when you know the integration of the function.

Using LT of Integration

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

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$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt = \int_0^t L^{-1}[F(s)]\,dt$$

$\downarrow$ Division by

Division H.W

que (8) $\Rightarrow$ $\int_0^\infty e^{-2t}\frac{\sin t}{t}\,dt = L\left[\frac{\sin t}{t}\right]$

$$L[\sin t] = \frac{1}{1+s^2}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{1+s^2}\,ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

que (9) $\Rightarrow$ $\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = L\left[\frac{\sin t}{t}\right]$

$$= \cot^{-1}s$$

que (10) $\Rightarrow$ $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt =$

que (11) $\Rightarrow$ $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t\,\sinh t}{t}\,dt = L\left[\frac{\sin t\,\sinh t}{t}\right]$

$$L[\sin t\,\sinh t] = L\left[\sin t\left(\frac{e^t - e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2}L\left[\sin t\,(e^t - e^{-t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$L\left[\sin t \sinh t\right] = \frac{1}{2}\left[\frac{(s+1)^2+1-(s-1)^2-1}{\left[(s-1)^2+1\right]\left[(s+1)^2+1\right]}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+1+2s-s^2-1+2s}{\left[(s-1)^2+1\right]\left[(s+1)^2+1\right]}\right]$$

$$= \frac{2s}{\left[(s-1)^2+1\right]\left[(s+1)^2+1\right]}$$

$$L\left[\frac{\sin t \sinh t}{t}\right] = \int_s^\infty \frac{2s}{\left[(s-1)^2+1\right]\left[(s+1)^2+1\right]}$$

$$= \int_s^\infty \frac{2s}{(s^2-2s+2)(s^2+2s+2)}$$

$s^2-2s+2 = t$

$dt = (2s-2)\, ds$

Que (1) $\displaystyle\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t}\, dt$

$$\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t}{t}\left(\frac{e^t - e^{-t}}{2}\right) dt$$

$$\int_0^\infty \left[\frac{e^{(1-\sqrt{2})t} - e^{(-1-\sqrt{2})t}}{2}\right]\frac{\sin t}{t}\, dt$$

$$\frac{1}{2}\int_0^\infty e^{(1-\sqrt{2})t}\frac{\sin t}{t} - e^{(-1-\sqrt{2})t}\frac{\sin t}{t}$$

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que (12) $\int_0^\infty \frac{\sin^2 t}{t} e^{-t}\,dt = L\left[\frac{\sin^2 t}{t}\right]$

$$L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s^2}+4-\cancel{s^2}}{s(s^2+4)}\right)$$

$$= \frac{2}{s(s^2+4)}$$

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty \frac{2}{s(s^2+4)}\,ds$$

$$\frac{2}{s(s^2+4)} = \frac{1/2}{s} + \frac{x}{s^2+4} \qquad = \frac{1/2}{s} - \frac{1/2\, s}{s^2+4}$$

$$= \frac{1}{2s} + \frac{x}{s^2+4}$$

$$\frac{s^2+4+2s(x)}{2s(s^2+4)} \quad (-1/2\, s)$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{2s}{2(s^2+4)}\,ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty \qquad s=1$$

$$= \frac{1}{2}\left(-\log\frac{1}{\sqrt{5}}\right)$$

$$= \frac{1}{2}\log\sqrt{5}$$

Que (13) $L\left[e^{-2t}\,\frac{\sin 2t\,\cosh t}{t}\right]$

$L\left[e^{-2t}\,\frac{\sin 2t}{t}\left(\frac{e^{t}+e^{-t}}{2}\right)\right]$

$L\left[\left(\frac{e^{-t}+e^{-3t}}{2}\right)\frac{\sin 2t}{t}\right]$

$L[e^{-t}\sin t] = \frac{1}{1+(s+1)^2}$

$L[e^{-3t}\sin 2t] = \frac{2}{4+(s+3)^2}$

$\frac{1}{2}\,L\left[\frac{e^{-t}\sin t}{t}+\frac{e^{-3t}\sin 2t}{t}\right]$

$\frac{1}{2}\int_s^\infty \frac{1}{1+(s+1)^2}+\frac{2}{4+(s+3)^2}\,ds$

① $L^{-1}\left[\log\left(1+\frac{a}{s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{a}{s}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left[\log(s+a)-\log s\right]\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s}\right]$

$= -\frac{1}{t}\left(e^{-a\cancel{s}t} - 1\right)$

$= \left(\frac{1-e^{-a\cancel{s}t}}{t}\right)$

② $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left(\log(s^2+a^2) - \log s^2\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2}{s}\right]$

$= -\frac{1}{t}\left(2\cos at - 2\right)$

$= \frac{2(1-\cos at)}{t}$

③ $L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{a}{s}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+a^2/s^2}\left(\frac{-a}{s^2}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{+a}{s^2+a^2}\right]$

$= \frac{+1}{t}\sin at = \frac{\sin at}{t}$

(4) $L^{-1}\left[s\log\left(\frac{s-a}{s+a}\right)+2a\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(s\log\left(\frac{s-a}{s+a}\right)+2a\right)\right]$

$$= \frac{-1}{t} L^{-1}\left[\log\left(\frac{s-a}{s+a}\right) + s\left(\frac{1}{s-a} - \frac{1}{s+a}\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\log\left(\frac{s-a}{s+a}\right) + s\,\frac{2a}{(s-a)(s+a)}\right]$$

$$= \frac{-1}{t} L^{-1}\left[\log\left(\frac{s-a}{s+a}\right) + \frac{2as}{s^2-a^2}\right]$$

$$= \frac{-1}{t}\left[L^{-1}\left[\log\left(\frac{s-a}{s+a}\right)\right] + 2a\cosh at\right]$$

$$= \frac{-1}{t}\left[\frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\cancel{s}\log\left(\frac{s-a}{s+a}\right)\right)\right] + 2a\cosh at\right]$$

$$= \frac{-1}{t}\left[\frac{-1}{t} L^{-1}\left[\frac{2a}{s^2-a^2}\right] + 2a\cosh at\right]$$

$$= \frac{-1}{t}\left[\frac{-1}{t}\, 2\cancel{a}\sinh at + 2a\cosh at\right]$$

$$= \frac{2}{t^2}\sinh at - \frac{2a}{t}\cosh at$$

(5) $L^{-1}\left[\frac{1}{s^2(s+4)}\right]$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t F(t)\,dt$$

Division by s

$$= \int_0^t L^{-1}\left[\frac{1}{s(s+4)}\right] dt$$

$$= \int_0^t \left(\int_0^t L^{-1}\left[\frac{1}{s+4}\right] dt\right) dt$$

$$= \int_0^t \int_0^t e^{-4t}\,dt\,dt$$

$$= \int_0^t \left[\frac{e^{-4t}}{-4}\right]_0^t dt$$

$$= \int_0^t \left(\frac{e^{-4t}}{-4} - \frac{1}{-4}\right) dt$$

$$= \left(\frac{e^{-4t}}{16} + \frac{1}{4}t\right)_0^t = \boxed{\frac{e^{-4t}}{16} + \frac{t}{4} - \frac{1}{16}}$$

(6) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$$= t\,L^{-1}\left[\frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}\,ds\right]$$

$$= t\,L^{-1}\left[\frac{-1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty\right] = \frac{t}{2}\,L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cancel{t\,L^{-1}\left[\frac{-1}{2}\right.} = \frac{t\sin at}{2a}$$

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(7) $$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\,\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] dt$$

ILATE

$$= \int_0^t \frac{t\sin at}{2a}\, dt$$

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - \int_0^t \left(\frac{-\cos at}{a}\right)\right]$$

$$= \frac{1}{2a}\left[\frac{-\cos at\,(t)}{a} + \frac{1}{a}\,\frac{\sin at}{a}\right]$$

$$= \frac{1}{2a^2}\left(\frac{\sin at}{a} - t\cos at\right)$$

$$= \frac{1}{2a^3}\left(\sin at - at\cos at\right)$$

(8) $$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}\right]$$

$$= \frac{\sin at}{a} - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$= \frac{\sin at}{a} - a^2\,\frac{1}{2a^3}\left(\sin at - at\cos at\right)$$

$$= \frac{\sin at}{a} - \frac{1}{2a}\left(\sin at - at\cos at\right)$$

$$= \frac{1}{2}\,\frac{\sin at}{a} + \frac{t}{2}\cos at$$

Khushi Tobanpa

Partial Fraction Method

① $$L^{-1}\left[\frac{1}{s(s+a)}\right] = L^{-1}\left[\frac{1/a}{s} + \frac{-1/a}{s+a}\right]$$

$$= L^{-1}\left[\frac{1}{as} - \frac{1}{a(s+a)}\right]$$

$$= L^{-1}\left[\frac{1}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)\right]$$

$$= \frac{1}{a}\left(1 - e^{-at}\right)$$

$$= \frac{1 - e^{-at}}{a}$$

② $$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \cancel{L^{-1}\left[\quad\right]}$$

$$L^{-1}\left[\frac{1/b-a}{s+a} + \frac{1/a-b}{s+b}\right]$$

$$\frac{1}{b-a}\, L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$\frac{1}{b-a}\left(e^{-at} - e^{-bt}\right) = \frac{e^{-at} - e^{-bt}}{b-a}$$

③ $$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \cancel{L^{-1}\left[\frac{1/a^2}{s} \quad \frac{\quad}{s^2+a^2}\right]}$$

$$\frac{1}{s(s^2+a^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+a^2}$$

$$1 = A(s^2+a^2) + Bs^2 + Cs$$

$$= As^2 + Aa^2 + Bs^2 + Cs$$

$$1 = s^2(A+B) + \cancel{(B+C)}s + Aa^2$$

$C = 0$

$A = 1/a^2$ $\quad$ ~~B+C=0~~

$A + B = 0$ $\quad$ $B = -1/a^2$

$$L^{-1}\left[\frac{1/a^2}{s} + \frac{-1/a^2\, s}{s^2+a^2}\right]$$

$$\frac{1}{a^2}\, L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+a^2}\right]$$

$$\frac{1}{a^2}\left(1-\cos at\right)$$

~~other~~ (4) $L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)(s^2+9)}\right]$

Put $s^2 = u$

$$\frac{1}{(u+1)(u+4)(u+9)} = \frac{1/24}{u+1} + \frac{-1/15}{u+4} + \frac{+1/40}{u+9}$$

$=$

$\frac{1}{3\times 8}$

$u=-4$

$\frac{1}{-3\times 5}$

$u=-9$

$\frac{+1}{8\times 5}$

$$L^{-1}\left[\frac{1}{24(s^2+1)} - \frac{1}{15(s^2+4)} + \frac{1}{40(s^2+9)}\right]$$

$$\boxed{\frac{1}{24}\sin t - \frac{1}{15}\frac{\sin 2t}{2} + \frac{1}{40}\frac{\sin 3t}{3}}$$

(5) $L^{-1}\left[\frac{(s^2+1)}{(2s+3)^2(s-2)}\right]$

$$\frac{s^2+1}{(2s+3)^2(s-2)} = \frac{A}{s-2} + \frac{B}{2s+3} + \frac{C}{(2s+3)^2}$$

$$s^2+1 = A(2s+3)^2 + B(s-2)(2s+3) + C(s-2)$$

$A = \frac{5}{49}$ $B = \frac{29}{98}$ $C = \frac{-13}{14}$

$$L^{-1}\left[\frac{(s^2+1)}{(2s+3)^2(s-2)}\right] = \frac{5}{49} L^{-1}\left[\frac{1}{s-2}\right] + \frac{29}{98} L^{-1}\left[\frac{1}{2s\ldots}\right.$$

$$- \frac{13}{14} L^{-1}\left[\frac{1}{(2s+3)^2}\right]$$

$$= \frac{5}{49} e^{2t} + \frac{29}{98}\,\frac{1}{2} L^{-1}\left[\frac{1}{s+3/2}\right]$$

$$- \frac{13}{14}\,\frac{1}{4} L^{-1}\left[\frac{1}{(s+3/2)^2}\right]$$

$$= \frac{5}{49} e^{2t} + \frac{29}{196} e^{-3/2\,t} - \frac{13}{56} e^{-3/2\,t} L^{-1}\left[\frac{1}{s^2}\right]$$

By using FST

$$= \frac{5}{49} e^{2t} + \frac{29}{196} e^{-3/2\,t} - \frac{13}{56} e^{-3/2\,t}\, t$$

⑥ $$L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right]$$

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-3s+2}$$

$$4s^2-3s+5 = A(s^2-3s+2) + (Bs+C)(s+1)$$

$$= As^2 - 3As + 2A + Bs^2 + Cs + Bs + C$$

$$= (A+B)s^2 + s(-3A+C+B) + 2A+C$$

$A+B = 4$ $\qquad$ $A = 4-B$

$-3A+B+C = -3$

$2A+C = 5$

$\cancel{2(4-B)+C = 5}$

$\cancel{8-2B+C}$

$A = 2$

$B = 2$

$C = 1$

$$L^{-1}\left[\frac{2}{s+1} + \frac{2s+1}{s^2-3s+2}\right]$$

$$2L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{2s+1}{(s-3/2)^2-1/4}\right]$$

$$2e^{-t} + \underbrace{e^{3/2t}\, L^{-1}\left[\frac{2(s+3/2)+1}{s^2-1/4}\right]}_{\text{using FST}}$$

$$2e^{-t} + e^{3/2t}\, L^{-1}\left[\frac{2s+4}{s^2-1/4}\right]$$

$$2e^{-t} + e^{3/2t}\left(2\cosh\left(\frac{t}{2}\right) + 4\,\frac{\sinh(t/2)}{1/2}\right)$$

$$= 2e^{-t} + e^{3/2t}\left(2\cosh(t/2) + 8\sinh(t/2)\right)$$

Que 7 $L^{-1}\left[\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1}$$

$$5s^2+8s-1 = A(s^2+1) + (s+3)(Bs+C)$$

~~A = 3~~ ~~B = 8~~ ~~A=~~ $A = 2$

$B = 3$

$C = -1$

$$L^{-1}\left[\frac{2}{s+3} + \frac{3s-1}{s^2+1}\right] = 2L^{-1}\left[\frac{1}{s+3}\right] + L^{-1}\left[\frac{3s-1}{s^2+1}\right]$$

$$= 2e^{-3t} + L^{-1}\left[\frac{3s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^{-3t} + (3\cos t - \sin t)$$

que (8) $L^{-1}\left[\frac{\cancel{2s}\ (s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right]$

$s^2+2s = u$

$$\frac{u+3}{(u+2)(u+5)} = \frac{1/3}{u+2} + \frac{2/3}{u+5}$$

$\cancel{u+3}$

$$L^{-1}\left[\frac{1}{3(s^2+2s+2)} + \frac{2}{3}\frac{1}{(s^2+2s+5)}\right]$$

$$\frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{\cancel{3}1}{s^2+2s+5}\right]$$

$$\frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$\frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$\frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\sin 2t$$

$$\frac{e^{-t}}{3}\left[\sin t + 2\sin 2t\right]$$

Que. ⑨ $L^{-1}\left[\dfrac{s}{s^4+s^2+1}\right] = L^{-1}\left[\dfrac{s}{(s^2+1)^2-s^2}\right]$

$$= L^{-1}\left[\dfrac{s}{(s^2+1-s)(s^2+1+s)}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{(s^2+1-s)}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{(s^2+s+1)}\right]$$

$$\cancel{= \frac{1}{2} L^{-1}\left[\frac{1}{(s-1)^2+s}\right]}$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{(s-1/2)^2+3/4}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{(s+1/2)^2+3/4}\right]$$

$$= \frac{1}{2} e^{1/2 t} L^{-1}\left[\frac{1}{s^2+3/4}\right] - \frac{1}{2} e^{-1/2 t} L^{-1}\left[\frac{1}{s^2+3/4}\right]$$

$$= \frac{1}{2} e^{t/2} \sin\left(\frac{\sqrt{3}}{2}\right)t - \frac{1}{2} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}\right)t$$

$$= \frac{1}{\sqrt{3}/2} \sin\left(\frac{\sqrt{3}\,t}{2}\right)\left[\frac{e^{t/2}-e^{-t/2}}{2}\right]$$

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}\,t}{2}\right) \sinh\left(\frac{t}{2}\right)$$

~~Que~~ Problems

① $L^{-1}\left[\dfrac{4s+12}{s^2+8s+16}\right]$

② $L^{-1}\left[\dfrac{1}{2}\log\left(\dfrac{s-1}{s+1}\right)\right]$

③ $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right]$

④ $L^{-1}\left[2\tanh^{-1}s\right]$

⑤ $L^{-1}\left[\log\left(1+\dfrac{1}{s^2}\right)\right]$

⑩ $L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}\right]$

⑥ $L^{-1}\left[\dfrac{3s+1}{(s+1)(s^2+1)}\right]$

⑦ $L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right]$

⑧ $L^{-1}\left[\dfrac{(s+1)e^{-s}}{s^2+s+1}\right]$

⑨ $L^{-1}\left[\dfrac{s+29}{(s+4)(s^2+9)}\right]$

Defination of Convolution of two Function:-

Let $f(t)$ & $g(t)$ be two functions of t defined for $t \geq 0$ then the convolution of $f(t)$ & $g(t)$ is defined as the integral $\int_0^{t} f(u)g(t-u)\,du$ & is denoted by $f(t) * g(t)$

★ $$\boxed{f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du}$$

$$\boxed{L^{-1}[F(s)\,G(s)] = f(t) * g(t)}$$

~~It holds~~ Convolution of $f(t)$ & $g(t)$ is same as convolution of $g(t)$ & $f(t)$ i.e it is symmetric

~~Statem~~ Statement of Convolution Theorem:-

If $f(t)$ & $g(t)$ are the inverse Laplace transform of $F(s)$ & $G(s)$ respectively then the inverse Laplace transform of the product $F(s)\,G(s)$ is the convolution of $f(t)$ & $g(t)$ i.e

$$\boxed{L^{-1}[F(s)\,G(s)] = f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du}$$

__To prove__ $\quad L[f(t) * g(t)] = F(s) \cdot G(s)$

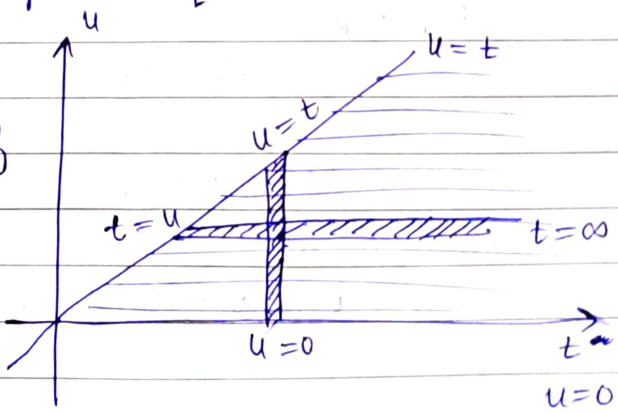
$$L[f(t) * g(t)] = \int_0^\infty e^{-st}[f(t) * g(t)]\,dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(u)\,g(t-u)\,du\right)dt$$

$$= \int_{t=0}^\infty \int_{u=0}^t e^{-st} f(u)\,g(t-u)\,du\,dt$$

$$= \iint_A e^{-st} f(u)\,g(t-u)\,du\,dt$$

where A is the region of (t, u) plane lying between the lines $u=0$, $u=t$ & positive quadrant as shown in fig.



After changing the order of integration

$L[f(t) * g(t)]$

$$L[f(t) * g(t)] = \int_{u=0}^{\infty}\left(\int_{t=u}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\right) du$$

In the inner integral put

$t - u = x$

$dt = dx$

t	u	∞
x	0	∞

$$= \int_{u=0}^{\infty} f(u)\left(\int_{x=0}^{\infty} e^{-s(u+x)} g(x)\, dx\right) du$$

$$= \int_{u=0}^{\infty} f(u)\, e^{-su}\left(\underbrace{\int_{x=0}^{\infty} e^{-sx} g(x)\, dx}_{G(s)}\right) du$$

$$= \int_{u=0}^{\infty} f(u)\, e^{-su}\, G(s)\, du$$

$$= G(s) \underbrace{\int_{0}^{\infty} e^{-su} f(u)\, du}_{F(s)}$$

$$= G(s)\, F(s)$$

$$L[f(t) * g(t)] = F(s)\, G(s)$$

Remark

$$L[f(t) * g(t)] = F(s)\, G(s)$$

take $g(t) = 1$ $\qquad G(s) = L[1] = \dfrac{1}{s}$

$$L[f(t) * 1] = \frac{F(s)}{s}$$

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

Problems :-

① Verify convolution theorem for the pair of functions $f(t) = t$, $g(t) = e^{at}$

$\Rightarrow$

To verify that

$$L[f(t) * g(t)] = F(s)\,G(s)$$

$f(t) = t$ $\qquad$ $g(t) = e^{at}$

$F(s) = \frac{1}{s^2}$ $\qquad$ $G(s) = \frac{1}{s-a}$

$$RHS = F(s)\,G(s) = \frac{1}{s^2(s-a)} \quad \text{—— ①}$$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \underbrace{u}_{I}\ \underbrace{e^{a(t-u)}}_{II}\,du$$

$$= \cancel{e^{a(t-u)}}\ e^{at} \int_0^t u\, e^{-au}\,du$$

$$= e^{at}\left[u\,\frac{e^{-au}}{-a} - \int \frac{e^{-au}}{-a}\,du\right]$$

$$= e^{at}\left[\frac{u\,e^{-au}}{-a} - \frac{e^{-au}}{a^2}\right]_0^t$$

$$= e^{at}\left[-\frac{t\,e^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}\left(e^{at} - at - 1\right)$$

$$LHS = L\left[f(t) * g(t)\right]$$

$$= L\left[\frac{1}{a^2}\left(e^{at} - at - 1\right)\right]$$

$$= \frac{1}{a^2}\left(\frac{1}{s-a} - a\frac{1}{s^2} - \frac{1}{s}\right)$$

$$= \frac{1}{s^2(s-a)} \quad \text{--- ②}$$

From ① & ②

$$L\left[f(t) * g(t)\right] = F(s)\, G(s)$$

is ~~vered~~ verified.

② $F(s) = \frac{1}{s-3}$ $\quad G(s) = \frac{1}{s-4}$

$\Rightarrow$

To verify that

$$L\left[f(t) * g(t)\right] = F(s)\, G(s)$$

$$RHS = F(s)\, G(s) = \frac{1}{(s-3)(s-4)} \quad \text{--- ①}$$

$f(t) = e^{3t}$ $\quad g(t) = e^{4t}$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{3u}\, e^{4(t-u)}\, du$$

$$= e^{4t}\int_0^t e^{3u}\, e^{-4u}\, du$$

$$= e^{4t}\int_0^t e^{-u}\, du$$

$$= e^{4t}\left[-e^{-u}\right]_0^t$$

$$= e^{4t}\left[-e^{-t}-(-1)\right]$$

$$= e^{4t}\left[1-e^{-t}\right]$$

$$LHS = L\left[f(t) * g(t)\right] = L\left[e^{4t} - e^{-3t}\right]$$

$$= \frac{1}{(s-4)} - \frac{1}{(s-3)}$$

$$= \frac{1}{(s-4)(s-3)} \quad \text{--- (2)}$$

LHS = RHS

from (1) & (2)

Convolution theorem is verified

(3) $$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[F(s)\,G(s)\right]$$

$$F(s) = \frac{1}{s} \qquad G(s) = \frac{1}{s^2+a^2}$$

$$f(t) = 1 \qquad g(t) = \frac{\sin at}{a}$$

by convolution theorem

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = f(t) * g(t) = \int_0^t g(u)\, f(t-u)\,du$$

$$= \int_0^t \frac{\sin au}{a}\,du = \left[\frac{-\cos au}{a^2}\right]_0^t$$

$$= \frac{1}{a^2}(1-\cos at)$$

(4) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[F(S)\,G(S)\right]$

$$F(s) = \frac{1}{s^2+a^2} \qquad G(s) = \frac{1}{s^2+a^2}$$

$$f(t) = \frac{\sin at}{a} \qquad g(t) = \frac{\sin at}{a}$$

by convolution theorem

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{\sin au}{a} \cdot \frac{\sin a(t-u)}{a}\, du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au - at) - \cos at\right] du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au - at)}{2a} - \cos at\, u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin at}{2a} - t\cos at\right) - \left(\frac{-\sin(at)}{2a}\right)\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$$

$$= \frac{\sin at - at\cos at}{2a^3}$$

(5) $$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = L^{-1}[F(s)\,G(s)]$$

$$F(s) = \frac{1}{s^2} \qquad G(s) = \frac{1}{(s+1)^2}$$

$$f(t) = t. \qquad g(t) = e^{-t}.\,t$$

By convolution theorem

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = f(t) * g(t) = g(t) * f(t)$$

$$= \int_0^t g(u)\,f(t-u)\,du$$

$$= \int_0^t e^{-u}\,u\,(t-u)\,du$$

$$= \int_0^t \underbrace{e^{-u}}_{\text{II}}\underbrace{(ut-u^2)}_{\text{I}}\,du$$

$$= \left[-(ut-u^2)\,e^{-u} - \int_0^t -\underbrace{(t-2u)}_{\text{I}}\,\underbrace{e^{-u}}_{\text{II}}\,du\right]_0^t$$

$$= \left[-(ut-u^2)\,e^{-u}\right]_0^t + \left[-(t-2u)\,e^{-u} + \int_0^t (-2)\,e^{-u}\right]_0^t$$

$$= 0 + \left[-(t-2u)e^{-u} - 2\left[-e^{-u}\right]_0^t\right]_0^t$$

$$= \left[-(t-2u)\,e^{-u}\right]_0^t - 2\left[-e^{-t}+1\right]$$

$$= \left[-(-t)e^{-t} - (-t)\right] - 2(-e^{-t}+1)$$

$$= te^{-t} + t + 2e^{-t} - 2$$

$$= (t+2)\,e^{-t} + (t-2)$$

(6) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right]$

$= \cos at * \frac{\sin at}{a}$

$= \int_0^t \cos au * \frac{\sin\{a(t-u)}{a}\, du$

$= \frac{t \sin at}{2a}$

(7) $L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = L^{-1}[F(s)\,G(s)]$

$F(s) = \frac{1}{s-2}$ $\quad G(s) = \frac{1}{(s+2)^2}$

$f(t) = e^{2t}$ $\quad g(t) = e^{-2t}t$

By convolution theorem

$L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = g(t) * f(t)$

$= \int_0^t u e^{-2u}\, e^{2(t-u)}\, du$

$= \int_0^t u\, e^{-2u+2t-2u}\, du$

$= \int_0^t u\, e^{2t-4u}\, du$

$= \left[u\left(\frac{e^{2t-4u}}{-4}\right)\right]_0^t - \int_0^t \left(\frac{e^{2t-4u}}{-4}\right) du$

$= u\left(\cancel{e^{2t-4u}}\right.$

$$= \left[t\left(\frac{e^{-2t}}{-4}\right) - u\left(\frac{e^{2t}}{-4}\right)\right] + \frac{1}{4}\left(\frac{e^{2t-4u}}{-4}\right)_0^t$$

$$= e^{2}\,\frac{1}{4}\left[-te^{-2t} + te^{2t}\right] + \frac{1}{4}\left(\frac{e^{-2t}}{-4} - \frac{e^{2t}}{-4}\right)$$

$$= \frac{1}{4}\left[-te^{-2t} + te^{2t} + \frac{e^{-2t}}{-4} - \frac{e^{2t}}{-4}\right]$$

$$= \frac{1}{16}\left[-4te^{-2t} + 4te^{2t} - e^{-2t} + e^{2t}\right)$$

(8) $$L\left[\frac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2+4}\right]$$

$$= \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2u \,\cos 2(t-u)\, du$$

$$= \frac{1}{2} t\cos 2t + \frac{1}{4}\sin 2t.$$

(9) $$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{s}{s^2+4}\right]$$

$$= \sin t * \cos 2t$$

$$= \int_0^t \sin u \,\cos 2(t-u)\, du$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

(10) $$L^{-1}\left(\frac{(s+2)^2}{(s^2+4s+8)^2}\right) = L^{-1}\left[\frac{(s+2)}{(s^2+4s+8)} \cdot \frac{(s+2)}{(s^2+4s+8)}\right]$$

Application to Solve Initial & Boundary value problem involving ordinary differential equation & simultaneous OI

① Solve $\frac{d^2y}{dt^2} + y = 0$ with $y = 1$ $\frac{dy}{dt} = 0$ at t

$\Rightarrow$

Taking L.T on both the sides

$$L\left[\frac{d^2y}{dt^2} + y\right] = L[0]$$

$$\left(s^2 L[y(t)] - s\,y(0) - y'(0)\right) + L[y(t)] = 0$$

$$\therefore \left(s^2 L[y(t)] - s - 0\right) + L[y(t)] = 0$$

$$\therefore L[y(t)](s^2+1) = s$$

$$\therefore L[y(t)] = \frac{s}{s^2+1}$$

$$y(t) = L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$y(t) = \cos t$$

② Solve

$\frac{d^2y}{dt^2} + y = t\cos 2t$, $y = 0$ $\frac{dy}{dt} = 0$ at $t = 0$

$\Rightarrow$

Taking L.T on both sides

$$L\left[\frac{d^2y}{dt^2} + y\right] = L[t\cos 2t]$$

$$\left(s^2 L[y(t)] - s(y(0) - y'(0)\right) + L[y(t)] = $$

$$(-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$s^2 L[y(t)] - 0 - 0 + L[y(t)] = (-1)\left(\frac{s^2+4-s(2s)}{(s^2+4)^2}\right)$$

$$\therefore (s^2+1) L[y(t)] = -\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$\therefore (s^2+1) L[y(t)] = \frac{s^2-4}{(s^2+4)^2}$$

$$\therefore L[y(t)] = \frac{s^2-4}{(s^2+1)(s^2+4)^2}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^2-4}{(s^2+1)(s^2+4)^2}\right]$$

$$\frac{s^2-4}{(s^2+4)^2(s^2+1)} = \frac{u-4}{(u+1)(u+4)^2}$$

$$= \frac{-5/9}{u+1} + \frac{5/9}{u+4} + \frac{8/3}{(u+4)^2}$$

$u = -1$

$$\frac{-5}{(3)^2}$$

$-5/9$

$u = -4$

$$\frac{-8}{(-3)}$$

$$y(t) = L^{-1}\left[\frac{s^2-4}{(s^2+1)(s^2+4)^2}\right]$$

$$y(t) = \frac{-5}{9} L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{5}{9} L^{-1}\left[\frac{1}{(s^2+4)}\right] + \frac{8}{3} L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$y(t) = -\frac{5}{9}\sin t + \frac{5}{9}\frac{\sin 2t}{2} + \frac{8}{3}\left(\frac{\sin 2t}{2} * \frac{\sin 2t}{2}\right)$$

$$= -\frac{5}{9}\sin t + \frac{5}{9}\frac{\sin 2t}{2} + \frac{8}{3}\left(\frac{1}{16}\sin 2t - \frac{t}{8}\cos 2t\right)$$

$$y(t) = -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t$$

que ③ $\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = e^t t^2$ $y(0)=0$, $y'(0)=0$, $y''(0)=-2$

que ④ Solve $\frac{d^2y}{dt^2} + 9y = \cos 2t$ with $y(0)=1$, $y(\pi/2)=-1$

→ Taking LT on both side

$$L\left[\frac{d^2y}{dt^2} + 9y\right] = L[\cos 2t]$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) + 9L[y(t)] = \frac{s}{s^2+4}$$

$$\therefore\ s^2 L[y(t)] - s - k + 9L[y(t)] = \frac{s}{s^2+4}$$

$$L[y(t)](s^2+9) = \frac{s}{s^2+4} + s + k$$

$$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$y(t) = L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{k}{s^2+9}\right]$$

$$y(t) = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{k}{s^2+9}\right]$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{K \sin 3t}{3}$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{K}{3}\sin 3t$$

$$y(\pi/2) = -1$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{2}\cos 2\left(\frac{\pi}{2}\right) + \frac{4}{5}\cos 3\left(\frac{\pi}{2}\right) + \frac{K}{3}\sin 3\left(\frac{\pi}{2}\right)$$

$$K = \frac{12}{5}$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5(3)}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$

que ⑤ solve $\frac{d^2y}{dt^2} + 9y = 18t$ with $y(0) = 0$ $y(\pi/2) = 0$

$\Rightarrow$ $y(t) = 2t + \pi \sin 3t$

Que (6) $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, $x(0) = 0$, $y(0) = 2$.

$\Rightarrow$ Apply LT on both the equation

$$L\left[\frac{dx}{dt} + y\right] = L[\sin t] \qquad L\left[\frac{dy}{dt} + x\right] = L[\cos t]$$

$$sL[x(t)] - x(0) + L[y(t)] = \frac{1}{s^2+1} \qquad sL[y(t)] - y(0) + L[x(t)] = \frac{s}{s^2+1}$$

$$\therefore sL[x(t)] - 0 + L[y(t)] = \frac{1}{s^2+1} \qquad sL[y(t)] - 2 + L[x(t)] = \frac{s}{s^2+1}$$

$$sL[x(t)] + L[y(t)] = \frac{1}{s^2+1} \quad \text{— (1)} \qquad sL[y(t)] + L[x(t)] = \frac{2s^2+s+2}{s^2+1}$$

(1) $\times s$

$$s^2 L[x(t)] + sL[y(t)] = \frac{s}{s^2+1} \quad \text{—— (3)}$$

$$L[x(t)] + sL[y(t)] = \frac{2s^2+s+2}{s^2+1} \quad \text{—— (2)}$$

(−) (−) (−)

$$(s^2-1)\, L[x(t)] = \frac{\cancel{s} - 2s^2 - \cancel{s} - 2}{s^2+1}$$

$$(s^2-1)\, L[x(t)] = \frac{-2(s^2+1)}{s^2+1}$$

$$L[x(t)] = \frac{-2}{s^2-1}$$

$$x(t) = L^{-1}\left[\frac{-2}{s^2-1}\right] = -2\sinh t$$

$\frac{1}{s-1}+1$

$1+s-1$

subsitute in ① $L[x(t)]$

$$s\left(\frac{-2}{s^2-1}\right)+L[y(t)]=\frac{1}{s^2+1}$$

$$L[y(t)]=\frac{1}{s^2+1}+\frac{2s}{s^2-1}$$

$$=\cancel{\frac{s^2-1+2s^3+2s}{(s^2+1)(s^2-1)}}$$

$=$

$$L[y(t)]=\frac{1}{s^2+1}+\frac{2s}{s^2-1}$$

$$y(t)=L^{-1}\left[\frac{1}{s^2+1}+\frac{2s}{s^2-1}\right]$$

$$\boxed{y(t)=\sin t+2\cosh t}$$

— ②

que ⑦ Solve $\frac{dx}{dt}-y=e^t$ $\quad \frac{dy}{dt}+x=\sin t$ $\quad x(0)=1$

$y(0)=0$

$\Rightarrow$

Take LT on both the equation

$$L\left[\frac{dx}{dt}-y\right]=L[e^t] \qquad L\left[\frac{dy}{dt}+x\right]=L[\sin t]$$

$$sL[x(t)]-x(0)-L[y(t)]=\frac{1}{s-1} \qquad sL[y(t)]-y(0)+L[x(t)]=\frac{1}{s^2+1}$$

$$sL[x(t)]-L[y(t)]=\frac{s}{s-1} \quad \text{— ①} \qquad sL[y(t)]+L[x(t)]=\frac{1}{s^2+1} \quad \text{— ②}$$

① × s

$$s^2L[x(t)]-sL[y(t)]=\frac{s^2}{s-1}$$

$$s^2 L[x(t)] - sL[y(t)] = \frac{s^2}{s-1} \quad \text{--- } (3)$$

$$s L[x(t)] + sL[y(t)] = \frac{1}{s^2+1}$$

$(+) \qquad (+) \qquad (+)$

$$(s^2+1) L[x(t)] = \frac{s^2}{s-1} + \frac{1}{s^2+1} = \cancel{s^4+s^2+s-1}$$

$$L[x(t)] = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}$$

$$\frac{s^2}{(s-1)(s^2+1)} = \frac{1/2}{s-1} + \frac{s+1/2}{s^2+1}$$

$$\frac{s^2+1}{2} + x(s-1) = s^2$$

$$x(s-1) = s^2 - \frac{(s^2+1)}{2}$$

$$= \frac{2s^2 - s^2 - 1}{2}$$

$$= \frac{s^2-1}{2}$$

$$x = \frac{(s-1)(s+1)}{2}$$

$$x = \frac{s+1}{2}$$

$$x(t) = L^{-1}\left[\frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2}\right]$$

$$x(t) = L^{-1}\left[\frac{s^2}{(s-1)(s^2+1)}\right] + L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$x(t) = L^{-1}\left[\frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)\right] + \frac{1}{2}(\sin t - t\cos t)$$

$$x(t) = \frac{1}{2}\left(e^t + \cos t + \sin(t)\right) + \frac{1}{2}(\sin t - t\cos t)$$

put $L[x(t)]$ in eq^n (1)

$$sL[x(t)] - L[y(t)] = \frac{s}{s-1}$$

$$s\left(\frac{s^2}{(s-1)(s^2+1)}+\frac{1}{(s^2+1)^2}\right)-L[y(t)]=\frac{s}{s-1}$$

$$s\left(\frac{s^2}{(s-1)(s^2+1)}+\frac{1}{(s^2+1)^2}\right)-\frac{s}{s-1}=L[y(t)]$$

$$\cancel{\frac{s}{s^2+1}\left(\frac{s^2}{s-1}+\frac{1}{s^2+1}\right)}$$

$$L[y(t)]=\frac{s}{(s^2+1)^2}-\frac{s}{(s-1)(s^2+1)}$$

$$y(t)=L^{-1}\left[\frac{s}{(s^2+1)^2}-\frac{s}{(s-1)(s^2+1)}\right]$$

$$\frac{1}{2}\left(\frac{1}{s-1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right)$$

$$y(t)=\frac{1}{2}t\sin t-\frac{1}{2}\left(e^t-\cos t+\sin t\right)$$

Que. 8 Solve $ty''+y'+4ty=0$ $\quad y(0)=3$, $y'(0)=0$

→ Take LT on both the sides

$$L[ty''+y'+4ty]=L[0]$$

$$L[ty'']+L[y']+L[4ty]=0$$

$$(-1)\frac{d}{ds}\left[L[y'']\right]+L[y']+4(-1)\frac{d}{ds}L[y(t)]=0$$

$$(-1)\frac{d}{ds}\left[s^2L[y(t)]-s\,y(0)-y'(0)\right]+sL[y(t)]-y(0)$$

$$-4\frac{d}{ds}L[y(t)]=0$$

$$(-1)\frac{d}{ds}\left[s^2 L[y(t)] - 3s\right] + \left[sL[y(t)] - 3\right] - 4\frac{d}{ds}L[y(t)] = 0$$

$$(-1)\left[2sL[y(t)] + s^2\frac{d}{ds}L[y(t)] - 3\right] + sL[y(t)] - 3 - 4\frac{d}{ds}L[y(t)] = 0$$

$$-2sL[y(t)] - s^2\frac{d}{ds}L[y(t)] \cancel{+3} + sL[y(t)] \cancel{-3} - 4\frac{d}{ds}L[y(t)] = 0$$

$$(-s^2-4)\frac{d}{ds}L[y(t)] - sL[y(t)] = 0$$

$$\frac{d\,L[y(t)]}{L[y(t)]} + \left(\frac{s}{s^2+4}\right)ds$$

Integrate both side

$$\log\left(L[y(t)]\right) + \frac{1}{2}\log(s^2+4) = \log c$$

$$\log\left(Y(s)\right) + \frac{1}{2}\log(s^2+4) = \log c$$

$$y(s) = \frac{c}{\sqrt{s^2+4}}$$

$$y(t) = c\,L^{-1}\left[\frac{1}{\sqrt{s^2+4}}\right] = c\,J_0(2t)$$

$$y(0) = 3$$

$$3 = c\,J_0(2t)$$

$$3 = c\,J_0(0)$$

$$c = 3$$

$$\boxed{y(t) = 3J_0(2t)}$$

Laplace Transform of special function.

① $f(t+T) = f(t) \quad \forall t$

periodic function

$$f(t+nT) = f(t) \quad \forall t \qquad T \text{ period}$$

Theorem :- Laplace Transform of periodic function $f(t)$ with period T is given by $\cancel{\frac{1}{1-e^{-sT}}}$

$$\frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\, dt \quad \text{i.e}$$

$$\boxed{L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt}$$

proof :- By defn

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[f(t)] = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt$$

In second integral put $t = u+T \quad dt = du$

$t = T, \; u = 0$

$t = \infty, \; u = \infty$

$$L[f(t)] = \int_0^T e^{-st} f(t)\, dt + \int_0^\infty e^{-s(u+T)} f(u+T)\, du$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_0^\infty e^{-su} f(u)\, du$$

e^{-sT}

$f(t)$ is periodic with period T

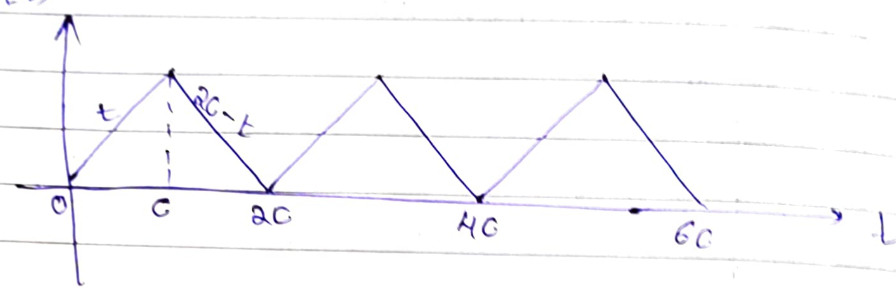
$f(u+T) = f(u) \quad \forall u$

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + e^{-sT} L[f(t)]$$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Que ① Find Laplace Transform of triangular wave function of period 2c given by $f(t) = \begin{cases} t & 0<t<c \\ 2c-t & c<t<2c \end{cases}$

$\Rightarrow$



$\because f(t+2c) = f(t) \ \forall t$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt \quad T=2c$$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \left[\int_0^c e^{-st}\, t\,dt + \int_c^{2c} e^{-st}(2c-t)\,dt \right]$$

$$= \frac{1}{1-e^{-2sc}} \left\{ \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^c + \left[\frac{(2c-t)e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_c^{2c} \right\}$$

$$= \frac{1}{1-e^{-2sc}} \left\{ \cancel{\frac{c e^{-sc}}{-s}} - \frac{e^{-sc}}{s^2} - \left(\frac{-1}{s^2}\right) + \frac{e^{-2sc}}{s^2} - \left(\cancel{\frac{c e^{-sc}}{-s}} + \frac{e^{-sc}}{s^2} \right) \right\}$$

$$= \frac{1}{1-e^{-2cs}} \left\{ -\frac{2e^{-sc}}{s^2} + \frac{1}{s^2} + \frac{e^{-2sc}}{s^2} \right\}$$

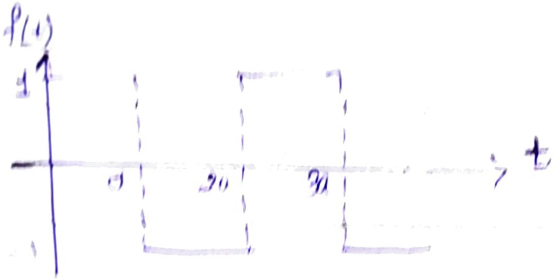
$$= \frac{1}{1-e^{-2as}} \frac{1}{s^2} \left(e^{-2as} - 2e^{-as} + 1 \right)$$

$$= \frac{1}{1-e^{-2as}} \frac{1}{s^2} (1 - e^{-as})^2$$

$$L[f(t)] = \frac{1}{s^2} \left(\frac{1-e^{-as}}{1+e^{-as}} \right) = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Que ② Find Laplace transform of square wave function of period $2a$ $f(t) \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$

→



$\because f(t+2a) = f(t) \ \forall t$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} dt - \int_a^{2a} e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

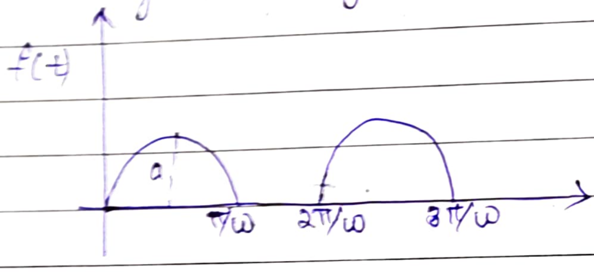
$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-as}}{-s} - \left(\frac{-1}{s}\right) - \left[\frac{e^{-2as}}{-s} - \frac{e^{-as}}{-s} \right] \right]$$

$$= \left(\frac{1}{1-e^{-2as}}\right) \frac{1}{s} \left(1 - 2e^{-as} + e^{-2as}\right)$$

$$= \left(\frac{1}{1-e^{-2as}}\right) \frac{1}{s} \left(1-e^{-as}\right)^2$$

$$= \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

Que ⑧ Find the LT of the half-sine wave function whose graph is given



⇒

$$f(t) \begin{cases} a\sin(\omega t) & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

$$f(t + 2\pi/\omega) = f(t) \quad \forall\, t$$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \qquad T = 2\pi/\omega$$

$$= \frac{1}{1-e^{-s2\pi/\omega}} \int_0^T e^{-s2\pi/\omega} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi/\omega s}} \int_0^{2\pi/\omega} e^{-s2\pi/\omega} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi/\omega s}} \int_0^{\pi/\omega} e^{-s 2\pi/\omega} \, a \sin(\omega t) \, dt$$

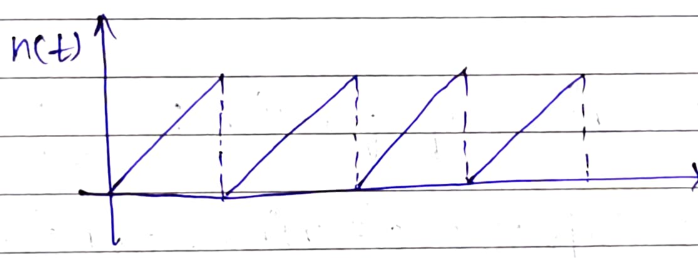
$$= \frac{a}{1-e^{-2\pi/\omega s}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin\omega t - \omega \cos\omega t) \right]_0^{\pi/\omega}$$

$$= \frac{a}{1-e^{-2\pi/\omega s}} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2} (-\omega(-1)) + \frac{\omega}{s^2+\omega^2} \right]$$

$$= \frac{a\omega}{1-e^{-2\pi/\omega s}} \left(\frac{1}{s^2+\omega^2} \right) \left(1 + e^{-\pi s/\omega}\right)$$

$$= \left(\frac{a\omega}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\pi/\omega s}} \right)$$

que ④ Sawtooth wave function $h(t)$



$$h(t) = \frac{k}{P} t \qquad 0 \le t \le P$$

$$h(t+P) = h(t) \;\; \forall t$$

$$L[h(t)] = \frac{1}{(1-e^{-sT})} \int_0^T e^{-st} h(t) \, dt \qquad \underline{\underline{T = P}}$$

$$L[h(t)] = \frac{1}{(1-e^{-sP})} \int_0^P e^{-st} \frac{k t}{P} \, dt$$

$$= \frac{1}{(1-e^{-sp})} \frac{K}{p} \int_0^p e^{-st}\, t\, dt$$

$$= \frac{1}{(1-e^{-sp})} \frac{K}{p} \left[t\frac{e^{-st}}{-s} - \int_0^p \frac{e^{-st}}{-s} \right]_0^p$$

$$= \frac{1}{1-e^{-sp}} \frac{K}{p} \left[\left(\frac{-t}{s} e^{-st}\right)_0^p - \left(\frac{e^{-st}}{s^2}\right)_0^p \right]$$

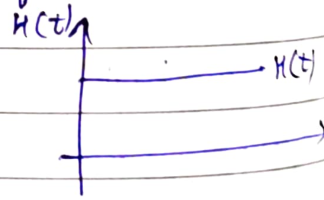
$$= \frac{1}{1-e^{-sp}} \frac{K}{p} \left(\frac{-p}{s} e^{-sp} - \left(\frac{e^{-sp}}{s^2} - \frac{1}{s^2}\right) \right)$$

$$= \frac{1}{1-e^{-sp}} \frac{K}{p} \left(\frac{-p}{s} e^{-sp} - \frac{e^{-sp}}{s^2} + \frac{1}{s^2} \right)$$

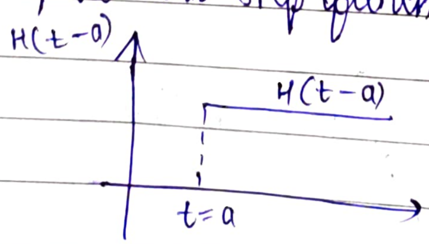
$$= \frac{K}{p s^2} \frac{1}{1-e^{-sp}} \left(1 - e^{-sp} - p s e^{-sp}\right)$$

(2) Heaviside unit step function : $H(t)$
$U(t)$

It is the most important discontinuous function n it is defined as $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$



Displace unit step function :- $H(t-a)$
$U(t-a)$



$$H(t-a) \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

Laplace Transform of function using heaviside unit step function.

If the given function is not defined in the interval $(0, \infty)$ but it is not defined in $(0, a)$ (a, b) & (b, ∞) then we can extend that function to $(0, \infty)$ using a heaviside unit step function.

① Let $f(t)$ is defined in $(0, a)$

$$g(t) = f(t)\,H(t-0) - f(t)\,H(t-a)$$

$$= f(t)\,H(t) - f(t)\,H(t-a)$$

$$= \begin{cases} f(t)\,(1) - f(t)\,(0) = f(t) & 0 < t < a \\ f(t)\,(1) - f(t)\,(1) = 0 & a < t < \infty \end{cases}$$

② Let $f(t)$ be defined in (a, b)

$$g(t) = f(t)\,H(t-a) - f(t)\,H(t-b)$$

$$= \begin{cases} f(t)\,(0) - f(t)\,(0) = 0 & 0 < t < a \\ f(t)\,(1) - f(t)\,(0) = f(t) & a < t < b \\ f(t)\,(1) - f(t)\,(1) = 0 & b < t < \infty \end{cases}$$

③ Let $f(t)$ be defined in (b, ∞)

$$g(t) = f(t)\,H(t-b) - f(t)\,H(t-\infty)$$

$$= \begin{cases} f(t)\,(0) - f(t)\,(0) = 0 & 0 < t < b < \infty \\ f(t)\,(1) - f(t)\,(0) = f(t) & b < t < \infty \end{cases}$$

Formula.

(1) $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\, dt = \int_0^\infty e^{-st}\, dt = [e$$

$$= 1/$$

$$L[H(t)] = 1/s$$

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$

(2) $H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\, dt = \int_0^a e^{-st} (0)\, dt + \int_a^\infty e$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{1}{s} e^{-as}$$

$$L[H(t-a)] = \frac{1}{s} e^{-as}$$

$$L^{-1}\left[\frac{1}{s} e^{-as}\right] = H(t-a)$$

(3) $f(t-a)\, H(t-a) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

$$L[f(t-a) H(t-a)] = \int_0^\infty e^{-st} L[f(t-a) H(t-a)]\, dt$$

$$= \int_a^\infty e^{-st}\, f(t-a)\, dt$$

put $t-a=u$ || $t=a$, $u=0$
$dt = du$ || $t=\infty$, $u=\infty$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-as} F(s)$$

$$\therefore L[f(t-a)H(t-a)] = e^{-as} F(s)$$

$$L^{-1}[e^{-as}F(s)] = f(t-a)H(t-a)$$

(a) put $a=0$

$$L[f(t)H(t)] = F(s)$$
$$L^{-1}[F(s)] = f(t)H(t)$$

(5)

$$f(t)H(t-a) = \begin{cases} f(t) & t>a \\ 0 & t<a \end{cases}$$

$$L[f(t)H(t-a)] = \int_0^\infty e^{-st} f(t)\,H(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t)\,dt$$

put $t = u+a$ || $t=a$ $u=0$
$dt = du$ || $t=\infty$ $u=\infty$

$$= \int_0^\infty e^{-s(u+a)} f(u+a)\,du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u+a)\,du$$

$$= e^{-as} F(s+a)$$

$$= \boxed{e^{-sa} L[f(t+a)]}$$

Problems :-

que ① Express the function in terms of heaviside unit step function and find its Laplace Transform

$$f(t) \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$$

$\Longrightarrow$

$$f(t) = \cos t\, H(t-0) - \cos t\, H(t-\pi) + \sin t\, H(t-\pi)$$

$$\therefore f(t) = \cos t\, H(t) - \cos t\, H(t-\pi) + \sin t\, H(t-\pi)$$

$$L[f(t)] = L[\underbrace{\cos t\, H(t)}_{f(t)H(t)}] - L[\underbrace{\cos t\, H(t-\pi)}_{f(t)H(t-a)}] + L[\underbrace{\sin t\, H(t-\pi)}_{f(t)H(t-\pi)}]$$

$$L[f(t)] = L[\cos t] - e^{-\pi s} L[\cos(t+\pi)] + e^{-\pi s} L[\sin(t+\pi)]$$

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1}$$

$$= \frac{s + e^{-\pi s}(s-1)}{s^2+1}$$

que ② $L[\underbrace{(1+2t-3t^2+4t^3)H(t-2)}_{f(t)H(t-a)}]$

$\Longrightarrow$

$$= e^{-2s} L[1+2(t+2) - 3(t+2)^2 + 4(t+2)^3]$$

$$= e^{-2s} L[25 + 38t + 21t^2 + 4t^3]$$

$$= e^{-2s}\left(25\frac{1}{s} + 38\frac{1}{s^2} + 21\left(\frac{2}{s^3}\right) + 4\left(\frac{6}{s^4}\right)\right)$$

$$= e^{-2s}\left(\frac{2s}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right)$$

Que (3) $L\left[\sin t\left(H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right)\right]$

$$L\left[\sin t\, H\left(t-\frac{\pi}{2}\right) - \sin t\, H\left(t-\frac{3\pi}{2}\right)\right]$$

$$e^{-\pi/2\, s} L\left[\sin\left(t+\frac{\pi}{2}\right)\right] - e^{-\frac{3\pi}{2}s} L\left[\sin\left(t+\frac{3\pi}{2}\right)\right]$$

$$e^{-\pi/2\, s} L[\cos t] + e^{-\frac{3\pi}{2}s} L[\cos t]$$

$$= \frac{s}{s^2+1}\left(e^{-\pi/2\, s} + e^{-3\pi/2\, s}\right)$$

Que (4)

$$f(t) \begin{cases} t+1 & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}$$

In term of Heaviside usf.

$$f(t) = (t+1)\, H(t-0) - (t+1)\, H(t-2) + 3\, H(t-2)$$

$$L[f(t)] = L\left[\underbrace{(t+1) H(t)}_{f(t)\,H(t)} - \underbrace{(t+1) H(t-2)}_{f(t)\,H(t-a)} + 3\, \underbrace{H(t-2)}_{H(t-a)}\right]$$

$$= L[t+1] - e^{-2s} L[t+2+1] + 3 e^{-2s} \frac{1}{s}$$

$$= \left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + 3e^{-2s}\frac{1}{s}$$

$$= \left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2}\right)$$

ex ⑤ $L^{-1}\left[\frac{s\,e^{-2\pi s}}{s^2-4s+29}\right]$

$L^{-1}\left[\frac{s}{s^2-4s+29}\right] = L^{-1}\left[\frac{s}{(s-2)^2+25}\right]$

using FST

$= e^{+2t}\, L^{-1}\left[\frac{s+2}{s^2+25}\right]$

$= e^{2t}\left(\cos(5t) + 2\,\frac{\sin(5t)}{5}\right)$

$= f(t)$

$L^{-1}\left[\frac{s\,e^{-2\pi s}}{s^2-4s+29}\right] = e^{2(t-2\pi)}\left[\cos(5(t-2\pi)) + \frac{2}{5}\sin(5(t-2\pi))\right]$

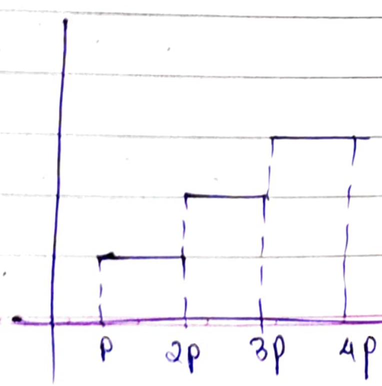
$\times H(t-2\pi)$

ex ⑥ $L^{-1}\left[\frac{(1-\sqrt{s})^2}{s^4}\, e^{-3s}\right]$

ex ⑦ Find the LT of staircase function $f(t)$ defined as $f(t) = Kn$ for $np < t < (n+1)p$, where $n = 0, 1, 2 \ldots$

→

$$f(t)\begin{cases} 0 & 0 < t < p \\ K & p < t < 2p \\ 2K & 2p < t < 3p \\ \vdots & \vdots \end{cases}$$

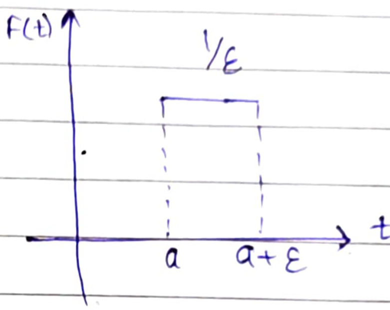


(3) Dirac - Delta function (or unit Impulse function)

Consider a function F(t) defined by

$$F(t) \begin{cases} 0 & 0 < t < a \\ 1/\varepsilon & a \le t \le a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$$

The limiting value of F(t) as $\varepsilon \to 0$ is called a unit Impulse function or a Dirac - Delta Function & it is denoted by $\delta(t-a)$



$$\delta(t-a) = \lim_{\varepsilon \to 0} F(t)$$

★ Laplace Transform of Dirac - Delta Function

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_a^{a+\varepsilon} \frac{1}{\varepsilon} e^{-st}\, dt$$

$$= \frac{1}{\varepsilon} \int_a^{a+\varepsilon} e^{-st}\, dt$$

$$= \frac{1}{\varepsilon} \left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$

$$= \frac{1}{\varepsilon} \left[\frac{e^{-s(a+\varepsilon)}}{-s} - \frac{e^{-sa}}{-s}\right]$$

$$= \frac{-1}{s\varepsilon} \left[e^{-sa-s\varepsilon} - e^{-sa}\right]$$

$$= -\frac{e^{-sa}}{s\varepsilon} \left[e^{-s\varepsilon} - 1\right] = \frac{e^{-sa}}{s\varepsilon}\left(1 - e^{-s\varepsilon}\right)$$

Now, $\delta(t-a) = \lim_{\varepsilon \to 0} f(t)$

$$\therefore L[\delta(t-a)] = L\left[\lim_{\varepsilon \to 0} F(t)\right]$$

$$= \lim_{\varepsilon \to 0} L[F(t)]$$

$$= \lim_{\varepsilon \to 0} \frac{e^{-sa}}{s\varepsilon}(1-e^{-s\varepsilon})$$

$$= \lim_{\varepsilon \to 0} e^{-sa}\left(\frac{e^{s\varepsilon}-1}{s\varepsilon}\right)\frac{1}{e^{s\varepsilon}}$$

$$= \left(\lim_{\varepsilon \to 0} e^{-sa}\right)\underbrace{\left(\lim_{\varepsilon \to 0} \frac{(e^{s\varepsilon}-1)}{s\varepsilon}\right)}_{1}\underbrace{\left(\lim_{\varepsilon \to 0} \frac{1}{e^{s\varepsilon}}\right)}_{1}$$

$$= \lim_{\varepsilon \to 0} e^{-sa}$$

$$\boxed{L[\delta(t-a)] = e^{-sa}}$$

put $a = 0$

$$L[\delta(t)] = 1$$

$$\therefore L^{-1}[1] = \delta(t) = \lim_{\varepsilon \to 0} F(t)$$

$$F(t) \begin{cases} 1/\varepsilon & 0 \le t \le \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

★ A Relation Between Heaviside & Dirac-Delta Function

we have $F(t) = \begin{cases} 0 & 0 < t < a \\ 1/\varepsilon & a \le t \le a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$

we express $f(t)$ in terms of Heaviside unit function

$$f(t) = \frac{1}{\varepsilon} H(t-a) - \frac{1}{\varepsilon} H(t-a-\varepsilon)$$

$$\delta(t-a) = \lim_{\varepsilon \to 0} f(t) = \lim_{\varepsilon \to 0} \frac{H(t-a) - H(t-a-\varepsilon)}{\varepsilon}$$

$$\boxed{\delta(t-a) = H'(t-a)}$$

$$\lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x)$$

★ First Shifting property of $\delta(t-a)$

$$\int_0^\infty f(t)\, \delta(t-a)\, dt = f(a)$$

we have

$$\delta(t-a) = \lim_{\varepsilon \to 0} F(t), \quad F(t) = \begin{cases} 0 & 0 < t \le a \\ 1/\varepsilon & a \le t \le a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases}$$

$$\therefore \int_0^\infty f(t)\, \delta(t-a)\, dt = \int_0^\infty f(t) \left[\lim_{\varepsilon \to 0} F(t)\right] dt$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^\infty f(t)\, F(t)\, dt \right)$$

$$= \lim_{\varepsilon \to 0} \left(\int_a^{a+\varepsilon} \frac{1}{\varepsilon} f(t)\, dt \right)$$

$$= \cancel{\frac{1}{\varepsilon}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_a^{a+\varepsilon} f(t)\, dt$$

(using mean value theorem of integration)

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\cancel{a} + \varepsilon - \cancel{a})\, f(c)$$

where $a \le c \le a+\varepsilon$

$$= \lim_{\varepsilon \to 0} \cancel{\frac{\varepsilon}{\varepsilon}} f(c)$$

$$= f(a)$$

Corallary $L[f(t)\,\delta(t-a)] = e^{-as} f(a)$.

Proof $L[f(t)\,\delta(t-a)] = \int_0^\infty \underbrace{e^{-st} f(t)}\, \underbrace{\delta(t-a)}\, dt$

$$= e^{-as} f(a)$$

que ① $L[\sin 2t\ \delta(t-\pi/4)] = \cancel{\sin}\ e^{-\pi/4\, s} \sin 2\left(\frac{\pi}{4}\right)$

$\quad\quad f(t) \quad \delta(t-a)$

$$= e^{-\pi/4\, s}$$

que ② $L[\sin 2t\ \delta(t-2)] = e^{-2s} \sin 4$

que ③ $L[t e^{-2t} \delta(t-2)] = e^{-2s}\, 2e^{-4} = 2e^{-(2s+4)}$

que ④ $L[t H(t-4) + t^2 \delta(t-2)] = e^{-4s} L[t+4] + e^{-2s} (2)^2$

$\quad\quad f(t) H(t-a) \quad f(t)\delta(t-a)$

$$= e^{-4s}\left[\frac{1}{s^2} + \frac{4}{s}\right] + e^{-2s}(4)$$

Que ⑤ Evaluate $\int_0^\infty \cos 2t \; \delta(t - \pi/3) \, dt$

$$\Rightarrow \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

Que ⑥ $L\left[t^2 u(t-2) - \cosh t \; \delta(t-4)\right]$

$f(t) \; u(t-a) \qquad f(t) \; \delta(t-a)$

$$= e^{-2s} L\left[(t+2)^2\right] - e^{-4s} \cosh 4$$

$$= e^{-2s} L\left[t^2 + 4t + 4\right] - e^{-4s} \cosh 4$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s} \cosh 4$$

Que ⑦ Evaluate $\int_0^\infty t^m (\log t)^n \; \delta(t-3) \, dt$

$$= 3^m (\log 3)^n$$