Module 2 : Relations & Functions

The concept of relations is of primary importance in computer science, especially in the study of data structures such as linked list, array, relational models etc.

If A is the set of all living human males and B is the set of all living human females, then the relation F (Father) can be defined between A and B.

Thus, if $x \in A$ and $y \in B$, then x is related to y by the relation F if x is the father of y, and we write xFy.

The order matters here, we refer to F as a relation from A to B.

We could also consider the relationship S and H from A to B by letting

$x S y$,

which means that x is a son of y, and $x H y$ means that x is the husband of y.

Formal Definition of Relations

Let A and B be two non-empty sets.
A relation R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a,b) \in R$, we say that
"a is related to b" by R, and we also write aRb.

If a is not related to b by R, we write $a \not R b$.

If $A = B$, we say that $R \subseteq A \times A$ is a relation on A, instead from A to A.

Examples:

1. Let $A = \{1, 2, 3\}$, $B = \{x, s\}$

Then $R = \{(1, x), (2, s), (3, x)\}$

is a relation from A to B.

2. Determine which of the following are relations from $A = \{a, b, c\}$ to $B = \{1, 2\}$

a) $R_1 = \{(a, 1), (a, 2), (c, 2)\}$

b) $R_2 = \{(a, 2), (b, 1), (2, a)\}$

c) $R_3 = \{(c, 1), (c, 2), (a, 2)\}$

d) $R_4 = \{(b, 2)\}$

e) $R_5 = \phi$

Sol:

Here $A = \{a, b, c\}$, $B = \{1, 2\}$

$\therefore A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

a) $\because R_1 \subseteq A \times B$

c) $\because R_3 \subseteq A \times B$ It is a relation

d) $\because R_4 \subseteq A \times B$

e) $\because R_5 \subseteq A \times B \rightarrow$ Also a relation called as empty relation

b) $\because R_2 \nsubseteq A \times B \rightarrow$ Not a Relation.

Domain of a Relation

Let A and B be two non-empty sets and R be a relation from A to B.

i.e $R \subseteq A \times B$, then the domain of R is a subset of A such that it is a collection of first elements of all ordered pairs of R.

i.e Domain of $R = \{a \mid (a,b) \in R\}$

Range of a Relation

If R is a relation from A to B, then the range of R is a subset of B such that it is a collection of second elements of all ordered pairs of R

i.e Range of $R = \{b \mid (a,b) \in R\}$

Example

Let $A = \{1, 2, 3\}$, $B = \{x, 5\}$

$R = \{(1, x), (2, 5), (3, x)\}$

$\therefore$ $Dom(R) = \{1, 2, 3\}$

$Ran(R) = \{x, 5\}$

Example

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5\}$

aRb if and only if $a < b$

$\therefore$ $R = \{ \quad \}$

$Dom(R) = \{ \quad \}$

$Ran(R) = \{ \quad \}$

Example

Let $A = \{1,2,3\}$, $B = \{x, 5\}$

$R = \{(1,x), (2,5), (3,x)\}$

$\therefore$ $\text{Dom}(R) = \{1,2,3\}$

$\text{Ran}(R) = \{x, 5\}$

Example

Let $A = \{1,2,3,4,5\}$, $B = \{1,2,3,4,5\}$

aRb if and only if $a < b$

$\therefore$ $R = \{(1,2), (1,3), (1,4), (1,5), (2,3),$
$(2,4), (2,5), (3,4), (3,5), (4,5)\}$

$\text{Dom}(R) = \{1,2,3,4\}$

$\text{Ran}(R) = \{2,3,4,5\}$

Inverse Relation

Let R be a relation from A to B then the inverse relation on R is the collection of all (b,a) such that $(a,b) \in R$.

It is denoted by R^{-1}, defined as

$$R^{-1} = \{ (b,a) \mid (a,b) \in R \}$$

Example

Let R be the relation on $A = \{1,2,3,4\}$ defined by "x is less than y"

i.e 'R' is the relation $<$.

i) Write 'R' as a set of ordered pairs.

ii) Find R^{-1} of the relation R.

iii) Can R^{-1} be described in words?

Sol: i) Here $A = \{1, 2, 3, 4\}$

$R = \{(x, y) \mid xRy \text{ iff } x < y\}$

$= \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

ii) $R^{-1} = \{(y, x) \mid (x, y) \in R\}$

$= \{(2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$

iii) R^{-1} can be described as a statement

" $xR^{-1}y$ iff $x > y$ "

i.e x is greater than y.

Example

Let S be the relation on the set N of the +ve integers defined by an equation $x+3y=13$

i.e $S=\{(x,y) \mid x+3y=13\}$

i) Write S as a set or ordered pairs

ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation

Sol: i) $S=\{ \quad \mid \quad \}$

$\therefore$

$\therefore \qquad \{ \qquad \}$

$\therefore$

Example

Let S be the relation on the set N of the +ve integers defined by an equation $x + 3y = 13$

i.e $S = \{(x, y) \mid x + 3y = 13\}$

i) Write S as a set or ordered pairs

ii) Find the inverse relation S^{-1} of S and describe S^{-1} by an equation

Sol: i) $S = \{(x, y) \mid x + 3y = 13\}$

$\because \quad x + 3y = 13$

$\therefore \quad x = 13 - 3y \qquad \{\because y \text{ cannot exceed } 4\}$

$\therefore \; y = 1, 2, 3, 4$ corresponding to

$x = 10, 7, 4, 1$

$\therefore \quad S = \{(10,1), (7,2), (4,3), (1,4)\}$

ii) $S^{-1} = \{(y,x) \mid (x,y) \in S\}$

$= \{(1,10), (2,7), (3,4), (4,1)\}$

S^{-1} can be described as :-

" x is related to y iff $3x + y = 13$ "

i.e $\quad x S^{-1} y \quad$ iff $\quad 3x + y = 13$

Representation of Relations

1. Graphical & Tabular Form

A relation between two finite sets can be represented in a tabular form as well as in a graphical form

For example

Let $A = \{a, b, c, d\}$

$R = \{\alpha, \beta, \gamma\}$

& R is a relation from A to B

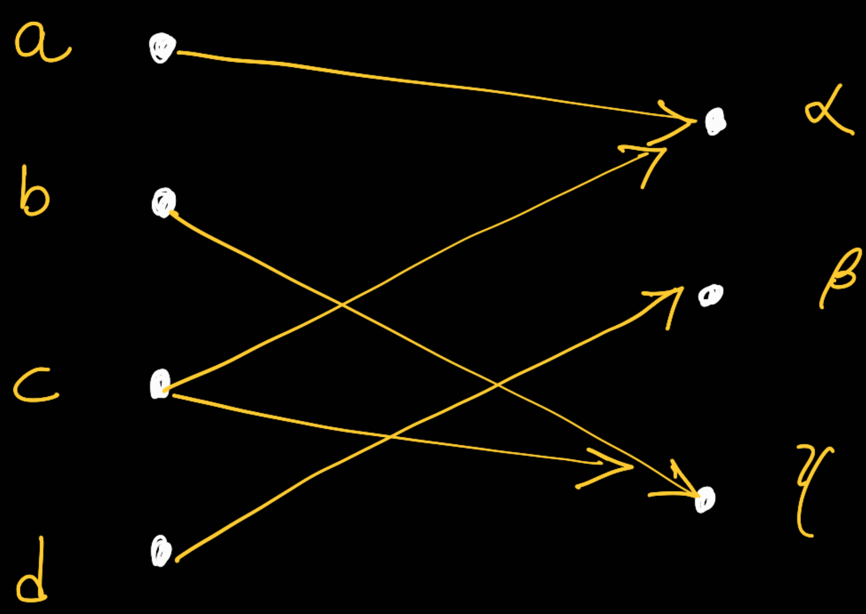
$R = \{(a, \alpha), (b, \gamma), (c, \alpha), (c, \gamma), (d, \beta)\}$

Above relation could be represented in a tabular form & graphical form

Tabular Form:

	α	β	γ
a	✓		
b			✓
c	✓		✓
d		✓	

Graphical Form



2. Matrix Form

We can represent a relation between two finite sets with matrices.

If $A = \{a_1, a_2 \ldots a_m\}$ & $B = \{b_1, b_2, \cdots b_n\}$ are finite sets containing m and n elements, respectively and R is a relation from A to B, we represent R by the $m \times n$ matrix $M_R = [m_{ij}]$, which is defined by,

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the <u>Matrix of R</u>.

It provides an easy way to check whether R has a given property or not.

<u>Examples</u>

Let $A = \{1, 2, 3\}$

$B = \{r, s\}$

Let R be a relation from set A to set B.

$$R = \{(1, r), (2, s), (3, r)\}$$

Then the matrix of R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Conversely, given sets A and B with $|A| = m$ and $|B| = n$, an $m \times n$ matrix whose entries are zeros and ones determines a relation as explained in the next example

Example

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Since M is 3×4, we let

$A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b_4\}$

Then $(a_i, b_j) \in R$ iff $m_{ij} = 1$

Thus $R = \{(a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$

Example

Let $A = \{a, b, c, d\}$,

$B = \{1, 2, 3\}$

$R = \{(a,1), (a,2), (b,1), (c,2), (d,1)\}$

Find the relation matrix

Sol: M_R will have 4 rows & 3 columns

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{bmatrix} \end{array}$$

Example

Let $A = \{1, 2, 3, 4, 8\}$

$B = \{1, 4, 6, 9\}$

Let aRb iff a/b. Find the Relation Matrix.

Example

Let $A = \{1, 2, 3, 4, 8\} = B$

aRb iff $a + b \leq 9$. Find the relation matrix

Example

Let $A = \{a, b, c, d\}$ & let

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ find } R?$$

3. Digraphs

Definition:

If A is a finite set and R is a relation on A, we can also represent R pictorially as follows:-

1) Draw a small circle for each element of A and label the circle with the corresponding element of A. These circles are called vertices.

2) Draw an arrow, called an edge from vertex a_i to vertex a_j iff $a_i \, R \, a_j$.

The resulting pictorial representation of R is called a directed graph or digraph of R.

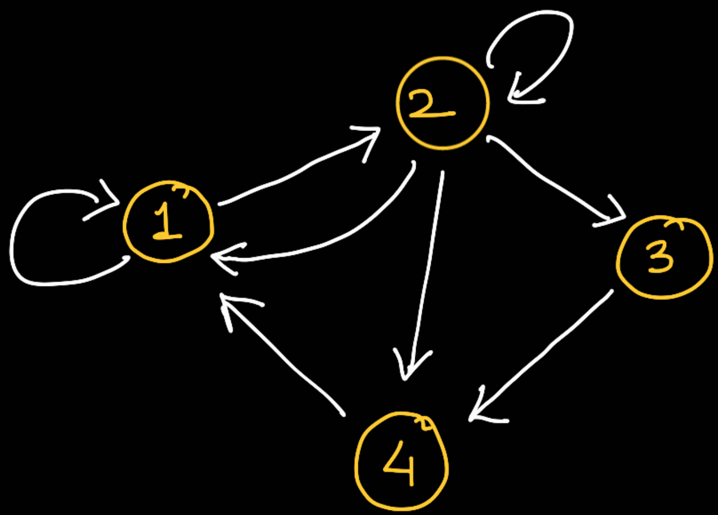
Thus, if R is a relation on A, the edges in the digraph of R correspond exactly to the pairs in R, and the vertices correspond exactly to the elements of set A.

Example

Q.1. Let $A = \{1, 2, 3, 4\}$. Let R is a relation from A to A.

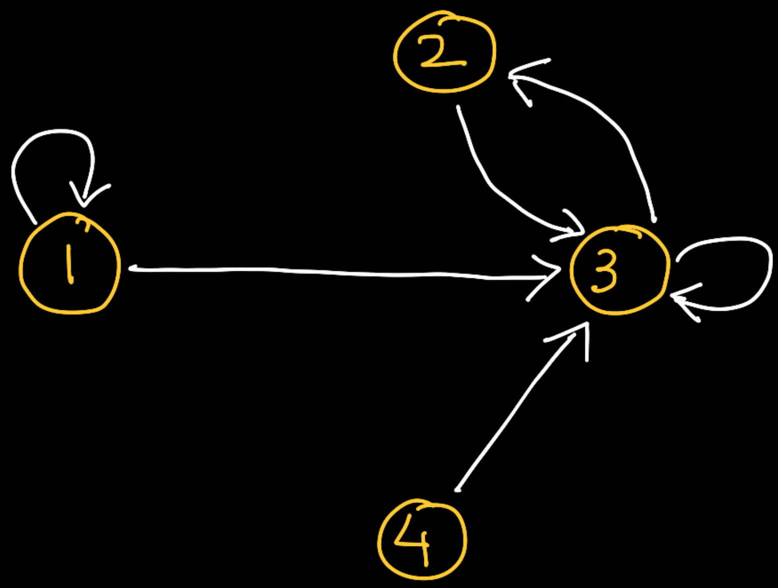
$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$

Then the digraph of R is as follows:-



Example 2

Find the relation determined by the following digraph.



Sol:

Since $a_i R a_j$ if and only if there is an edge from a_i to a_j, we have

$$R = \{(1,1), (1,3), (2,3), (3,2), (3,3), (4,3)\}$$

Degree of Vertex

An important concept for relations is inspired by visual form of digraphs.

If R is a relation on a set A and $a \in A$, then the in-degree of a is the number of $b \in A$ such that $(b, a) \in R$.

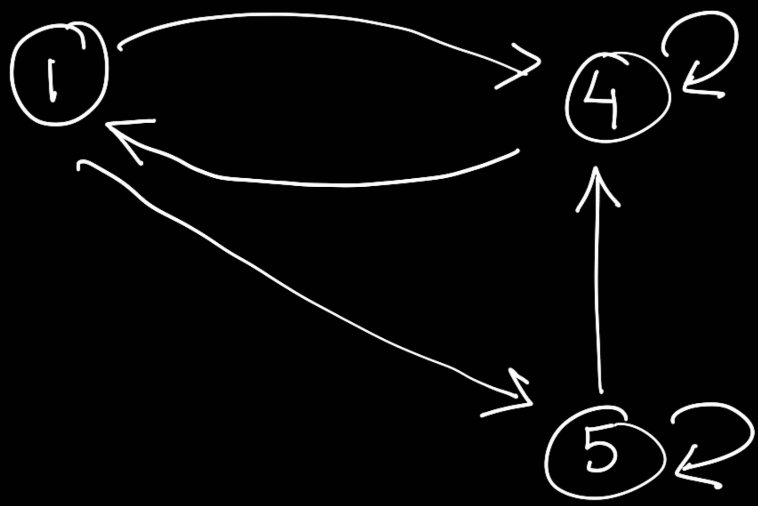
The out-degree of a is the number of $b \in A$ such that $(a, b) \in R$.

What this means, in terms of the digraph of R, is that the in-degree of a vertex is the number of edges terminating at the vertex.

The out-degree of a vertex is the no. of edges leaving the vertex.

Example: 1

Let $A = \{1, 4, 5\}$ & let R be given by the digraph as shown. Find M_R and R.



Sol:

$$M_R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R = \{(1,4), (1,5), (4,1), (4,4), (5,4), (5,5)\}$

Example 2

Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2,3), (3,2), (3,4), (3,5), (4,3), (4,4), (4,5)\}$. Draw its digraph.

Example 3

Let $A = \{a, b, c, d\}$ and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw the digraph of R.

Example 4

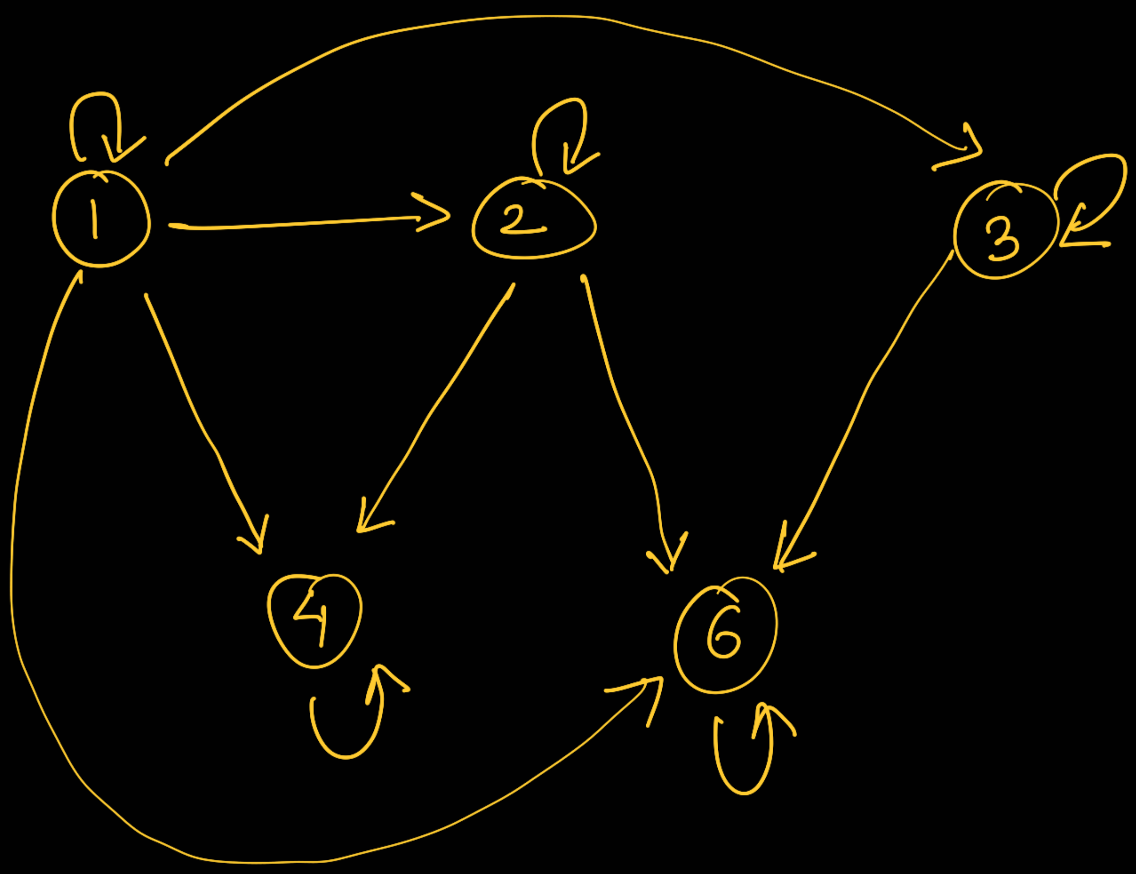
Draw the graphical representation of the relation "<u>less than</u>" on $\{1,2,3,4\}$

Example 5

Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find matrix of R. Find inverse relation of R.

Example

Let $A = \{1, 2, 3, 4, 6\}$ and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find matrix of R. Find inverse relation of R.



Example 6

Given $A = \{1,2,3,4\}$ & $B = \{x,y,z\}$.

Let R be the following relation from A to B:

$$R = \{(1,y), (1,z), (3,y), (4,x), (4,z)\}$$

i) Determine the matrix of the relation

ii) Draw the digraph of R

iii) Find R^{-1}

iv) Determine domain & range of R.

Paths

Suppose that R is a relation on a set A. A <u>path of length n</u> in R from a to b is a finite sequence $\pi : a, x_1, x_2, \dots, x_{n-1}, b$, beginning with a and ending with b, such that

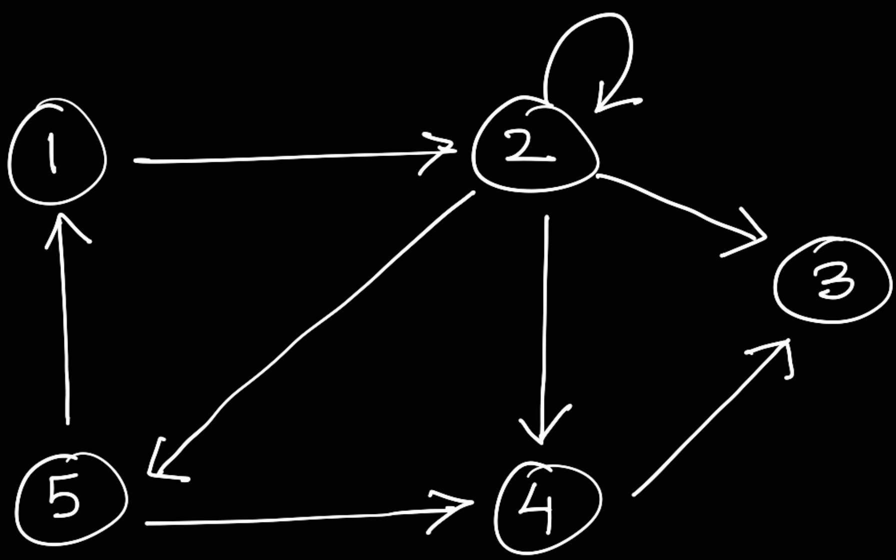
$$a R x_1, \; x_1 R x_2, \; \dots\dots\dots, \; x_{n-1} R b$$

Note that a path of length n involves n+1 elements of A, although they are not necessarily distinct.

A path is most easily visualized with the aid of the digraph of the relation.

It appears as a geometric 'path' or succession of edges in such a digraph, where the indicated directions of the edges are followed, and in fact a path derives its name from this representation.

For example



Then π: 1, 2, 5, 4, 3 is a path of length 4 from vertex 1 to vertex 3.

π_2 : 1, 2, 5, 1 is a path of length 3 from vertex 1 to itself.

π_3 : 2, 2 is a path of length 1 from vertex to itself.

A path that begins & ends at the same vertex is called a cycle.

π_2 & π_3 are cycles of length 3 & 1 respectively.

If $\underline{n}$ is a fixed positive integer, we define a relation R^n on A as follows:-

$\underline{xR^n y}$ means there is a path of length n from x to y in R.

We may also define a relation R^∞ on A, by letting $x R^\infty y$ which means that there is some path in R from x to y.

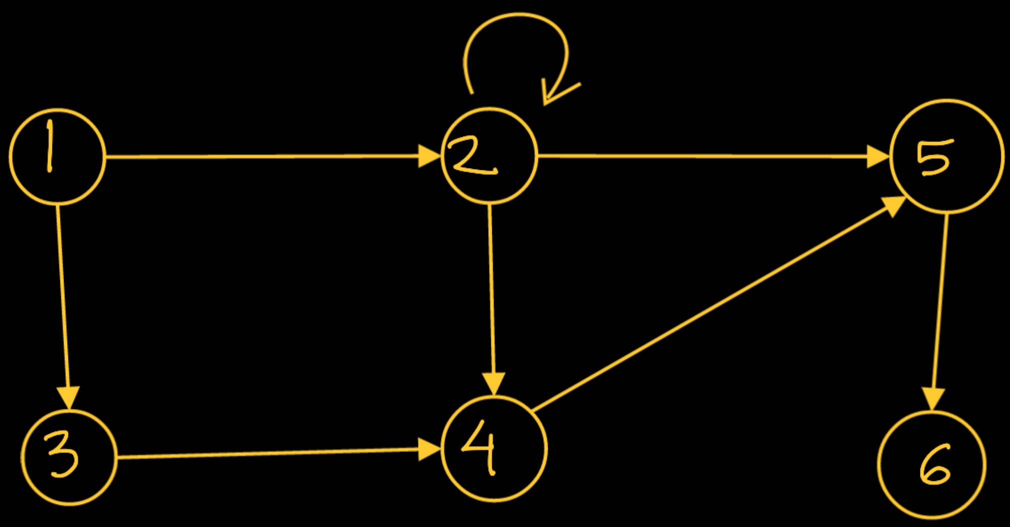
The length of such a path, in general, will depend on x & y.

The relation R^∞ is sometimes called as the "Connectivity" relation for R.

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the relation whose digraph is shown. Find R^2 and draw diagraph of the relation R^2.

Sol: To compute R^2 on A, $x R^2 y$ means that there is a pair of length 2 from x to y in R.



$1 R^2 2$ Since $1R2$ and $2R2$

$1 R^2 4$ Since $1R2$ and $2R4$

$1 R^2 5$ Since $1R2$ and $2R5$

$2 R^2 2$ since $2R2$ & $2R2$

$2 R^2 4$ since $2R2$ & $2R4$

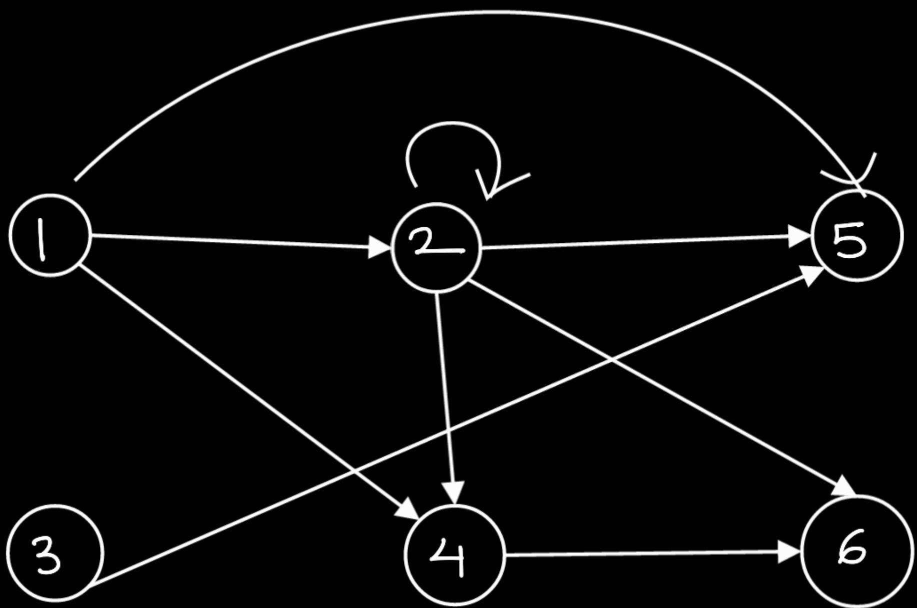
$2 R^2 5$ since $2R2$ & $2R5$

$2 R^2 6$ since $2R5$ & $5R6$

$3 R^2 5$ since $3R4$ & $4R5$

$4 R^2 6$ since $4R5$ & $5R6$

Hence, we can obtain the digraph for R^2 as follows



$\therefore R^2 = \{(1,2), (1,4), (1,5), (2,2), (2,4), (2,5), (2,6), (3,5), (4,6)\}$

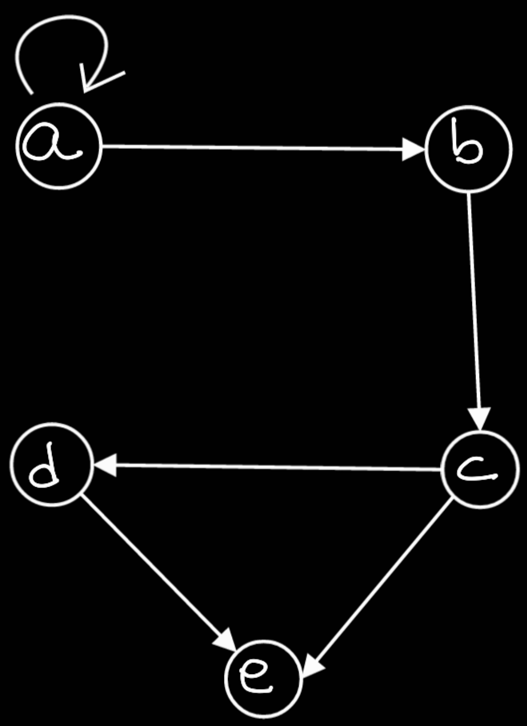
Example

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$

Compute i) R^2

ii) R^∞



i) The digraph is as shown above.

$x R^2 y$ means that there is a path of length n from x to y in R.

$a \, R^2 \, a$ Since aRa & aRa

$a \, R^2 \, b$ Since aRa & aRb

$a \, R^2 \, c$ Since aRb & bRc

$b \, R^2 \, d$ Since bRc & cRd

$b \, R^2 \, e$ Since bRc & cRe

$c \, R^2 \, e$ Since cRd & dRe

Hence $R^2 = \{(a,a), (a,b), (a,c), (b,d), (b,e), (c,e)\}$

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ii) To compute R^∞, we need all ordered pairs of vertices for which there is a path of any length from the first vertex to the second.

$$R^\infty = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)\}$$

If $|R|$ is large, it can be tedious and perhaps difficult to compute R^∞, or even R^2, from the set representation of R.

However, M_R can be used to accomplish these tasks more efficiently.

Let R be a relation on a finite set $A = \{a_1, a_2, \ldots a_n\}$, and let M_R be the $n \times n$ matrix representing R.

Let's see how the matrix M_R of R^2 can be computed from M_R.

$C = [C_{ij}]$ defined by

$$C_{ij} = \begin{cases} 1 & \text{If } a_{ik} = 1 \ \& \ b_{kj} = 1 \text{ for some } k,\ 1 \le k \le p \\ 0 & \text{otherwise} \end{cases}$$

This multiplication is similar to ordinary matrix multiplication.

The preceding formula states that for any i and j, the element C_{ij} of $C = A \odot B$ can be computed in the following way.

1. Select row i of A and column j of B, and arrange them side by side.

2. Compare corresponding entries. If even a single pair of corresponding entries consists of two 1's, then $C_{ij} = 1$.

If this is not the case, then $C_{ij} = 0$

Example

Let $A = \{a, b, c, d, e\}$ and

$R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$

Then

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\because M_R^2 = M_R \cdot M_R$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we compute M_{R^2} from R^2, we obtain the same result.

It is often easier to compute R^2 by computing $M_R \odot M_R$ instead of searching the digraph of R for all vertices that can be joined by a path of length 2.

Similarly, we can show that

$$M_{R^3} = M_R \odot (M_R \odot M_R)$$

Composition of Paths

Let $\pi_1 : a, x_1, x_2, \ldots\ldots x_{n-1}, b$ be a path in a relation R of length n from a to b, and

Let $\pi_2 : b, y_1, y_2, \ldots\ldots, y_{m-1}, c$ be a path in R of length m from b to c.

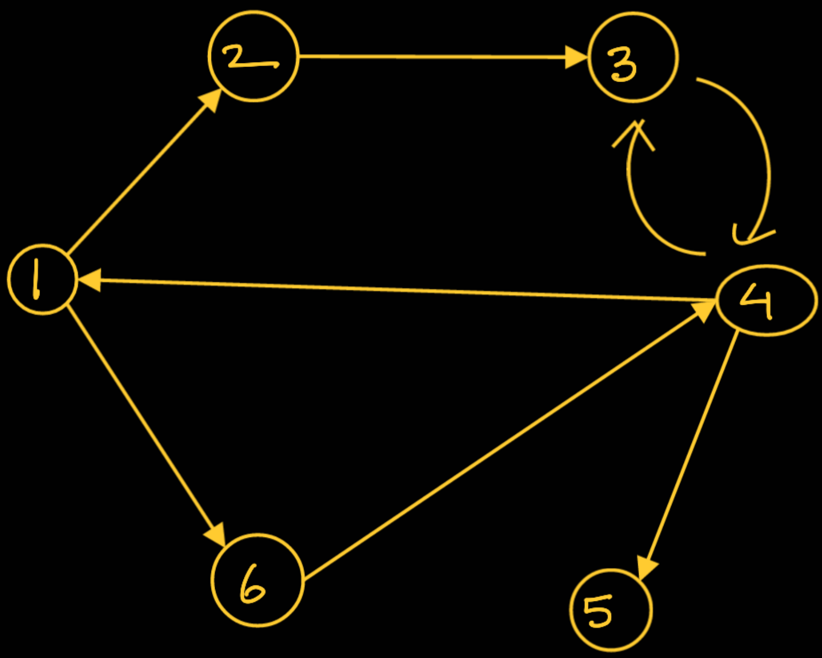
Then, the composition of π_1 and π_2 is the path $a, x_1, x_2 \ldots b, y_1, y_2 \cdots y_{m-1}, c$ of length $n + m$, which is denoted by $\pi_2\, \pi$.

This is a path from a to c.

Example:

Consider the relation whose digraph is given and the paths

$\pi_1 : 1, 2, 3$ & $\pi_2 : 3, 5, 6, 2, 4$.



i) List all path of length 3 starting from vertex 3.

ii) List all paths of length 3

<u>Sol:</u> i) All paths of length 3 starting from vertex 3.

π_1 : 3, 3, 3, 3

π_2 : 3, 3, 4, 3

π_3 : 3, 3, 4, 5

π_4 : 3, 3, 3, 4

π_5 : 3, 4, 3, 4

π_6 : 3, 4, 1, 2

π_7 : 3, 4, 1, 6

π_8 : 3, 4, 3, 3

π_9 : 3, 3, 4, 1

ii) List all paths of length 3

Total 30 paths.

e.g. π_1 : 1, 2, 3, 3

π_2 : 1, 2, 3, 4

π_3 : 1, 6, 4, 5

π_4 : 1, 6, 4, 3

π_5 : 1, 6, 4, 1

Starting from Vertex 1.

Example:

Compute R^2 and draw the digraph of R^2.

Sol: R^2

$1 R^2 3$ Since $1 R 2$ & $2 R 3$

$1 R^2 4$ Since $1 R 6$ & $6 R 4$

$2 R^2 3$ Since $2 R 3$ & $3 R 3$

$2 R^2 4$ Since $2 R 3$ & $3 R 4$

likewise we will also get

$3R^2 1$, $3R^2 3$, $3R^2 3$, $3R^2 4$, $3R^2 5$,
$4R^2 2$, $4R^2 3$, $4R^2 4$, $4R^2 6$, $6R^2 1$,
$6R^2 3$, $6R^2 5$.

<u>Diagram</u>:

Find M_R^2

$$M_R^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Find

i) R^∞

ii) M_R^∞

i) $R^\infty = \{(a,c), (a,b), (a,d), (a,e), (a,f), (b,b), (b,f), (b,d), (b,c), (b,e), (c,b), (c,d), (c,e), (c,f), (c,c), (d,b), (d,c), (d,e), (d,f), (d,d), (e,f), (e,d), (e,b), (e,c), (e,e), (f,b), (f,d), (f,c), (f,e), (f,f)\}$

ii)

$$M_R^{\infty} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example: Let $A = \{1, 2, 3, 4, 5\}$ & R be the relation defined by aRb iff $a < b$, compute R, R^2, R^3 & draw digraphs for the same.

Types of Relations

1) Reflexive Relations

A relation R on a set A is reflexive if for 'every' element $a \in A$, $a R a$

i.e $(a,a) \in R$.

R is not reflective if for 'some' element $a \in A$, $a \not R a$,

i.e $(a,a) \notin R$.

Example:

1. Let $A = \{a, b\}$ and let

   $R = \{(a,a), (a,b), (b,b)\}$

   Then R is reflexive

2. Ler $A = \{1, 2\}$ & let

   $R = \{(1,1), (1,2)\}$. R is not reflexive

   since $(2,2) \notin R$.

We can identify a reflexive relation by its matrix as follows. The matrix of a reflexive relation must have all 1's on its main diagonal.

Similarly, we can characterize the digraph of reflexive relation as follows.

A reflexive relation has a cycle of length 1 at every vertex.

Finally, we may note that if R is reflexive on a set A,

then $\mathrm{Dom}(R) = \mathrm{Ran}(A) = A$

2) Irreflexive Relations

A relation R on a set A is <u>irreflexive</u> if $a \not R a$.

i.e $(a,a) \notin R$ for every $a \in A$.

Thus R is irreflexive if no element is related to itself.

<u>Examples</u>

1) Let $A = \{1,2\}$ and let

$R = \{(1,2),(2,1)\}$

Then R is irreflexive since

$(1,1),(2,2) \notin R$.

2) Let $A = \{1,2\}$ and let

$R = \{(1,2),(2,2)\}$

Then R is not <u>irreflexive</u> since

$(2,2) \in R$

Note that R is not <u>reflexive</u> either since $(1,1) \notin R$

3. Symmetric Relations

A relation R on a set A is <u>symmetric</u> if whever aRb, then bRa.

It then follows that R is <u>not symmetric</u> if we have some a and $b \in A$ with aRb, but $b \not R a$

<u>Examples</u>

1. Let A be set of people. Let <u>aRb</u> if a is a friend of b.

   Then obviously b is related to a.

   Hence the relation of being "<u>friend</u>" is a symmetric relation.

2. Let A be a set of lines in a plane.

For lines $l_1, l_2 \in A$, let $l_1 R l_2$ if l_1 is parallel to l_2.

Then $l_2 R l_1$, since the relation of being "parallel to" is a symmetric relation.

---

3. Let A be a set of people and let aRb if a is brother of b.

Then this is not a symmetric relation since b can be the sister of a.

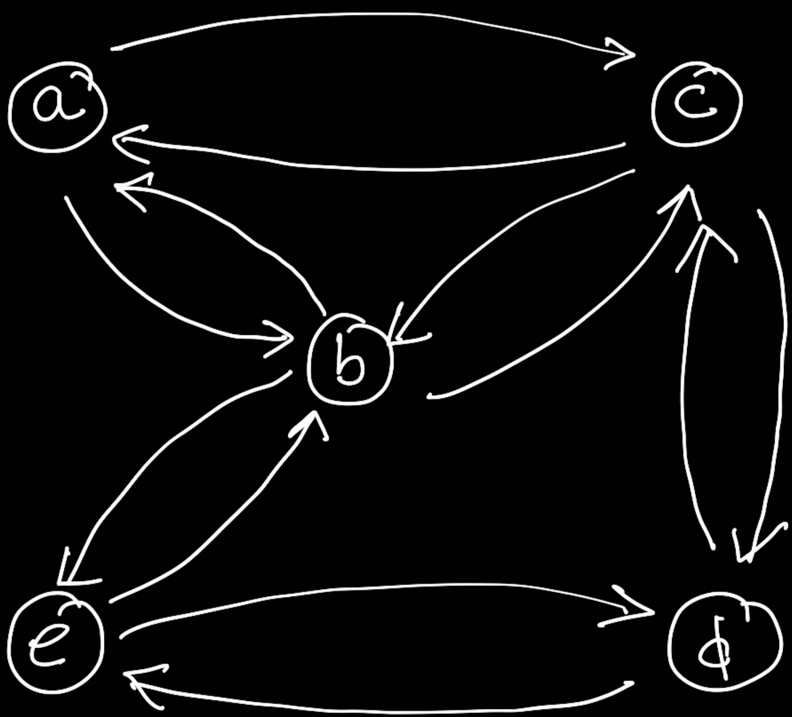
This relation will be symmetric only if A is the set of males

Digraph of Symmetric Relation

Let $A = \{a, b, c, d, e\}$ and let R be the symmetric relation given by,

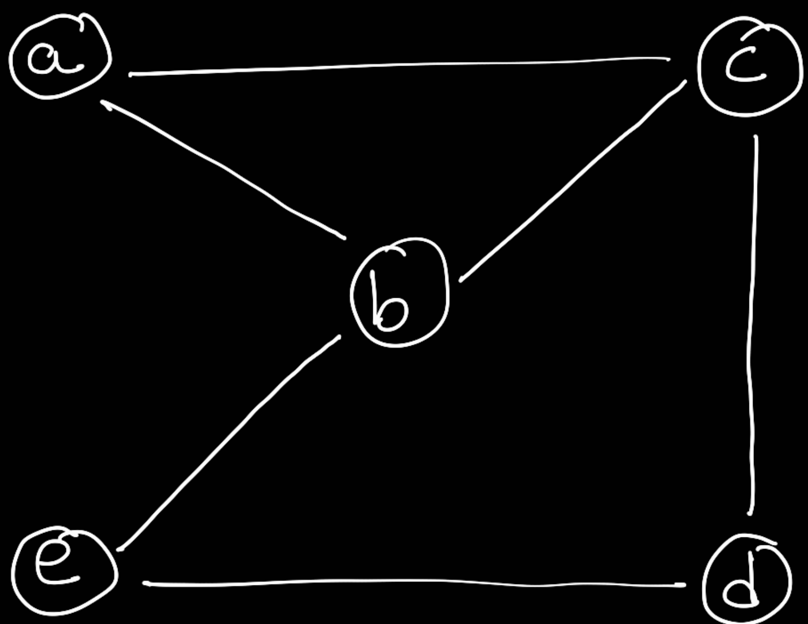
$$R = \{(a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (b,e), (e,b), (e,d), (d,e), (c,d), (d,c)\}$$

The usual digraph of R is shown below. Each undirected edge corresponds to two ordered pairs in the relation R.



An undirected edge between a and b, in the graph of a symmetric relation R, corresponds to a set $\{a,b\}$ such that $(a,b) \in R$ and $(b,a) \in R$.

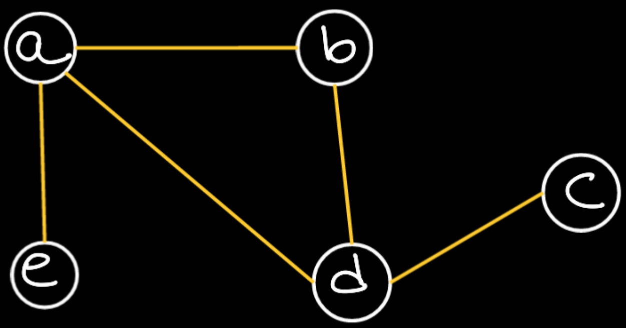
Sometimes we will also refer to such a set $\{a,b\}$ as an 'undirected edge' of the relation R and call a and b 'adjacent vertices'.



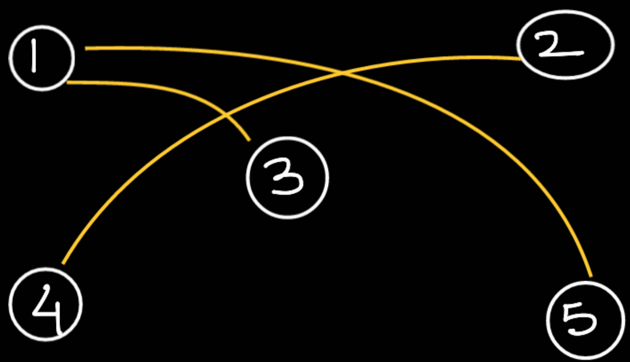
4. Connected Relation

A symmetric relation R on a set A is called connected if there is a path from any element of A to any other element of A.

This simply means that the graph of R is all in one piece.



Connected Symmetric Relation



Unconnected Symmetric Relation

5. Asymmetric Relation

A relation R on a set A is asymmetric if whenever aRb, then $b \not R a$.

It then follows that R is not asymmetric if we have some a and $b \in A$ with both aRb & bRa.

Examples :

1. Let A be the set of real numbers and let R be the relation '$<$'.

If $a < b$, then $b \nless a$,

so '$<$' is asymmetric.

2. Let $A = \{1, 2, 3, 4\}$ and let

$R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$

then R is not asymmetric, since $(2, 2) \in R$

6. Antisymmetric Relations

A relation R on a set A is antisymmetric if whenever $a \neq b$, then either $a \not R b$ or $b \not R a$.

OR

if whenever aRb and bRa, then $a = b$

Contrapositive

Example:

Let $A = \{1,2,3,4\}$ and let $R = \{(1,2),(2,2),(3,4),(4,1)\}$ Is it symmetric, asymmetric or antisymmetric?

Sol: Symmetry: R is not symmetric, since $(1,2) \in R$, but $(2,1) \notin R$.

Asymmetry: R is not asymmetric, since $(2,2) \in R$

Antisymmetry: R is antisymmetric, since if $a \neq b$, either $(a,b) \notin R$ or $(b,a) \notin R$.

7. Transitive Relations

A relation R on a set A is transitive if whenever aRb and bRc, then aRc. It follows that a relation R is not transitive if there exists a, b and c in A so that aRb and bRc but $a \not R c$.

Example

Let $A = \mathbb{Z}$, the set of integers and let R be the relation less than.

Sol: To see whether R is transitive, we assume that aRb & bRc. Thus $a<b$ & $b<c$. It then follows that $a<c$, so aRc.

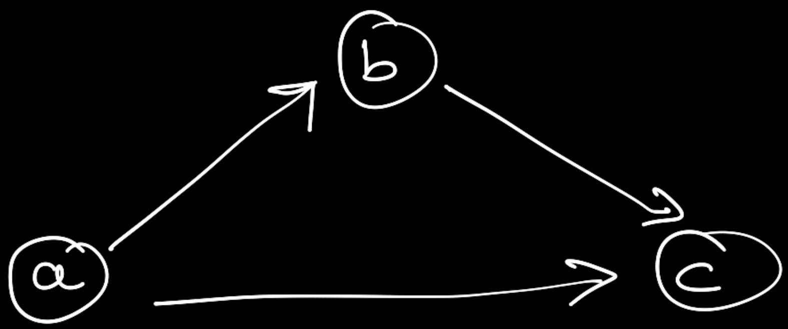
Hence R is transitive.

Digraph of a transitive relation :-

Let $A = \{a, b, c\}$ and let

$R = \{(a,b), (b,c), (a,c)\}$

$\therefore$



8. Identity Relation

Identity relation I on set A is reflexive, transitive and symmetric.

Example

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

9. Void Relation

It is given by $R : A \to B$ such that $R = \phi \; (\subseteq A \times B)$ is a null/void relation.

10. Universal Relation

A relation $R : A \to B$ such that $R = A \times B \; (\subseteq A \times B)$ is a universal relation.

Q.1 Let $A = \{1, 2, 3, 4, 5\}$ and let R and S be the equivalence relation on A whose matrices are given below. Compute the matrix of the smallest equivalence relation containing R and S.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Sol: * Find $M_{R \cup S} = M_R \cup M_S$

* Check if it's an equivalence relation

* If not, find $(R \cup S)_{TC}$

Q.2 Let $A = \{1, 2, 3, 4, 5\}$ & let R and S be the equivalence relations on A whose matrices are given below. Compute the matrix of the smallest equivalence relation containing R & S.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Posets

Partially ordered relation : A relation R on a set A is called partial order if R is reflexive, antisymmetric & transitive.

The set A together with the partial order R is called a **partially ordered set** or simply a poset. It is denoted by (A, R).

Example

1. Let A be a set of positive integers and let R be a binary relation such that (a, b) is in R if a divides b.

Since any integer divides itself. R is reflexive.

Since a divides b means b does not divide a unless $a = b$, R is an antisymmetric relation.

Since a divides b, b divides c, then a divides c, so R is transitive.

Consequently, R is a partial ordered relation.

---

2. Let Z^+ be the set of positive integers. The relation '$\leq$' is a partial order on Z^+ because for any element x.

i) $x \leq x$

ii) if $x \leq y$ & $y \leq x$, then $y = x$.

iii) If $x \leq y$ & $y \leq z$, then $x \leq z$

Dual of Poset

Let R be a partial order on a set A, and let R^{-1} be the inverse relation of R.

Then R^{-1} is also a partial order.

The poset (A, R^{-1}) is called the dual of the poset (A, R) and the partial order R^{-1} is called the dual of the partial order R.

Hasse Diagram

A graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing upward is known as the "Hasse Diagram" of the relation.

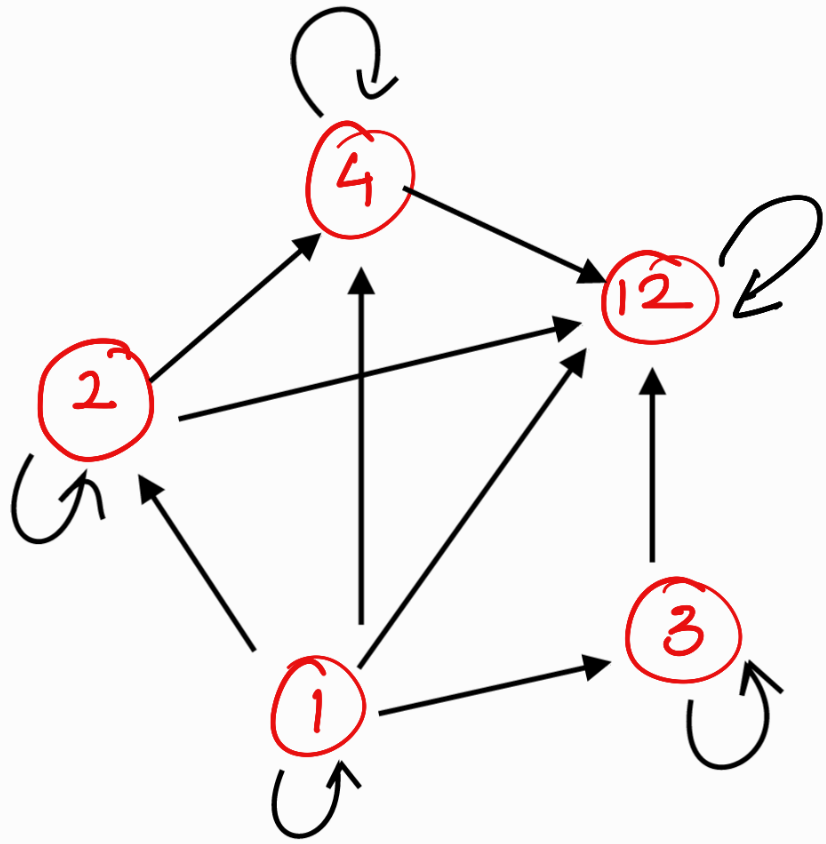
Solved Example

Draw all Hasse Diagrams of posets with three elements.

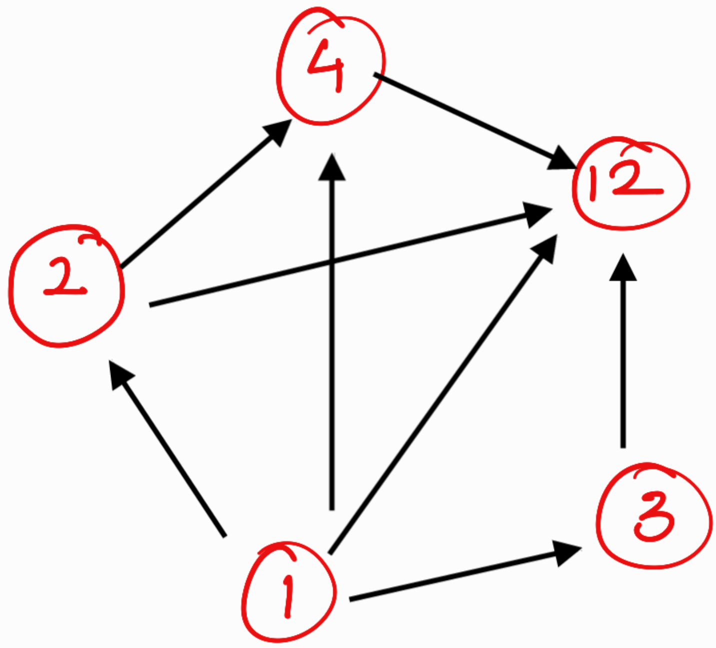
Draw Hasse diagram for the following relations on set $A = \{1, 2, 3, 4, 12\}$

$$R = \{(1,1), (2,2), (3,3), (4,4), (12,12), (1,2), (4,12), (1,3), (1,4), (1,12), (2,4), (2,12), (3,12)\}$$

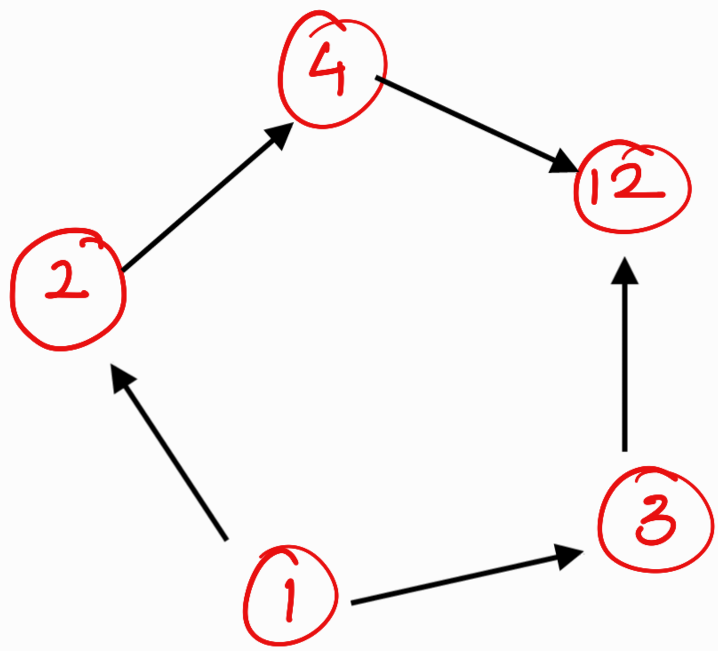
Sol: Digraph



Step 1: Remove Cycle



Step 2: Remove transitive edge

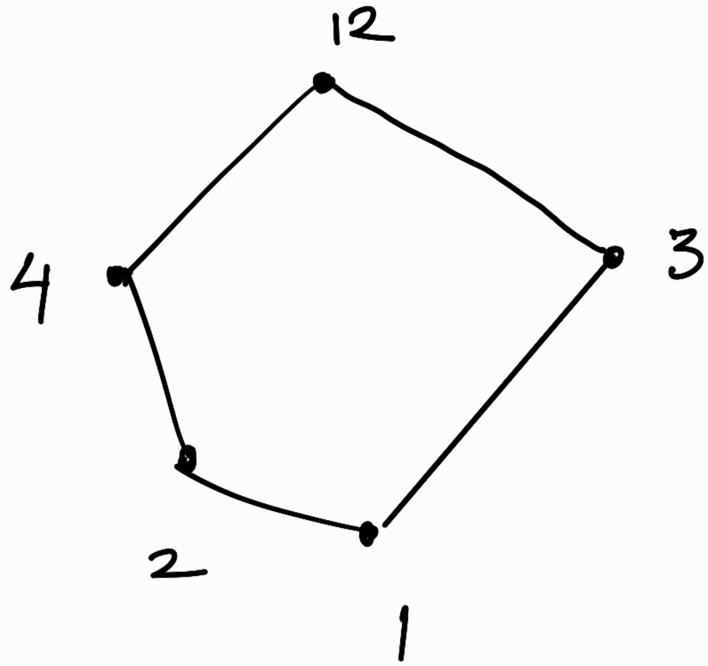


$1R2$, $2R4$ $\therefore$ $1R4$

$2R4$, $4R12$ $\therefore$ $2R12$

$1R4$, $4R12$ $\therefore$ $1R12$

Step 3 Circles are replaced by dots. Arrows are also removed.

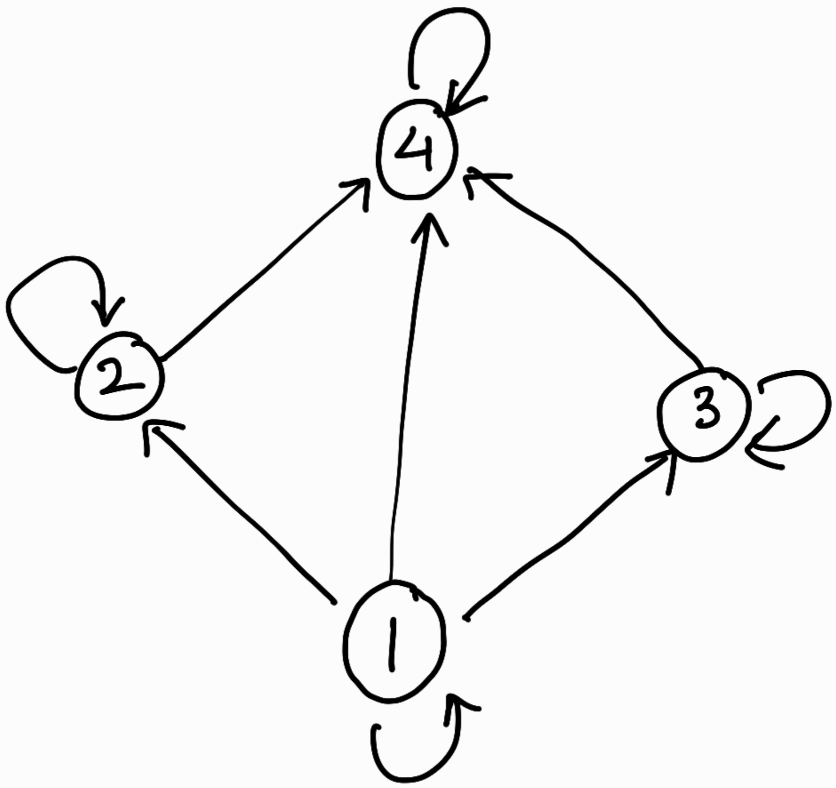


Q.2. Determine the Hasse Diagram of the relation R

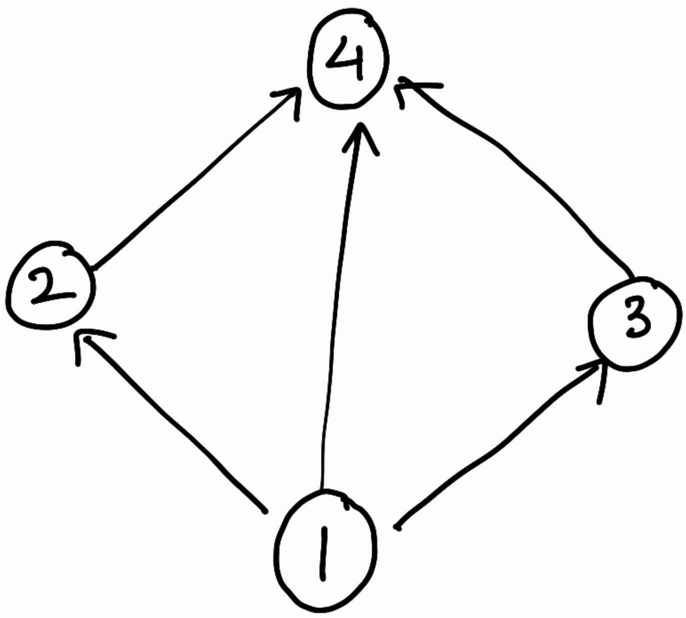
i) $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$

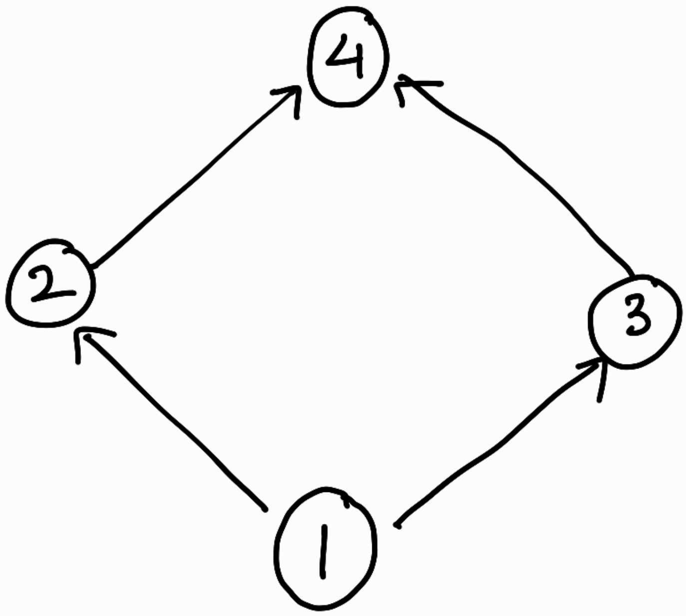
Sol: Digraph for the given relation set is



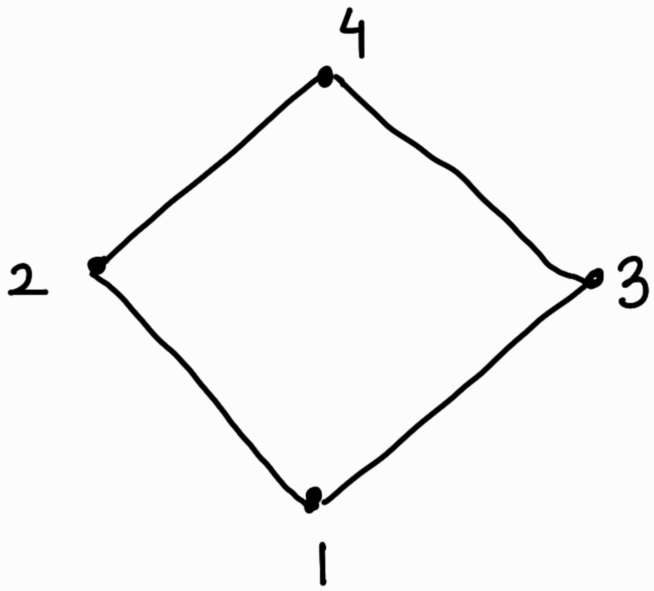
1) Remove Cycles



2) Remove Transitive Edge (1,4)



3.) Make sure that all edges are pointing upwards, then remove arrows from edges, replace circles by dots.

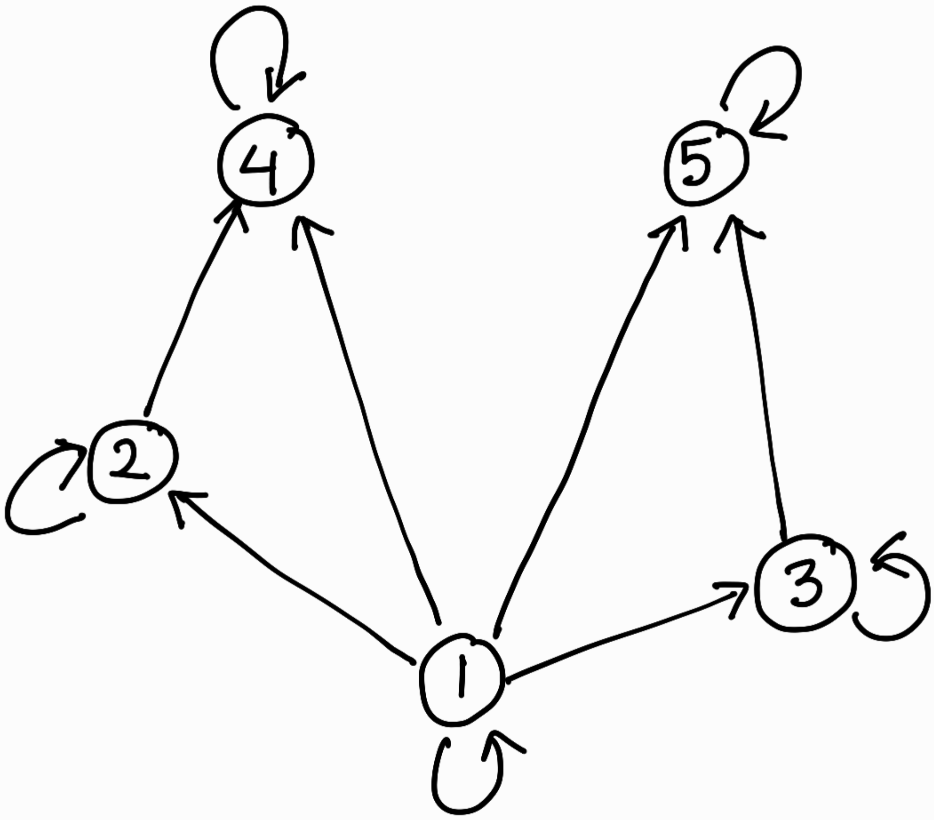


3. Determine the Hasse Diagram of the relation R

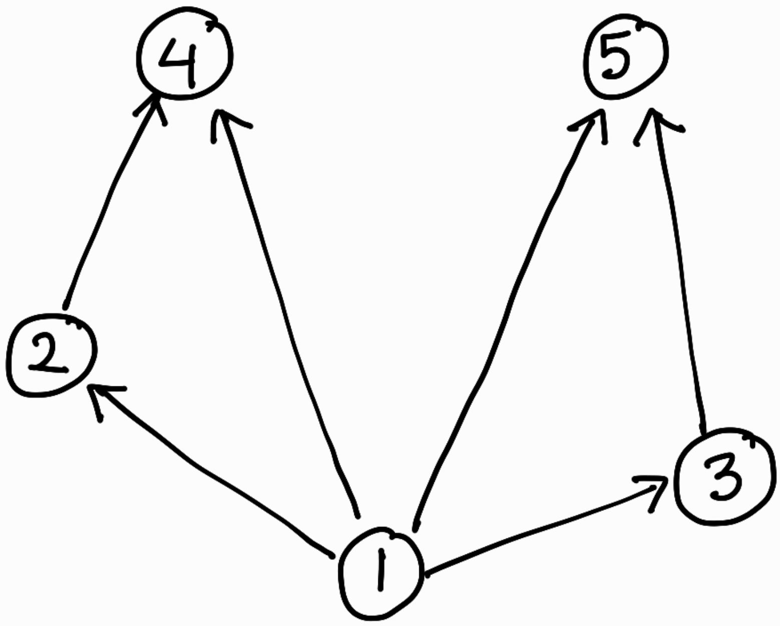
i) $A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,4), (3,5), (2,2), (3,3), (4,4), (5,5)\}$

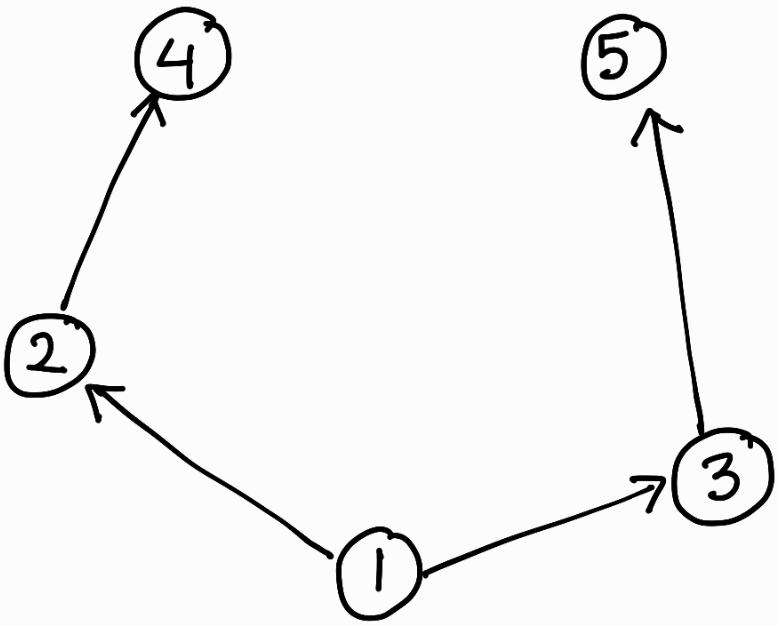
Sol. Digraph of the given relation set is



1. Remove Cycles

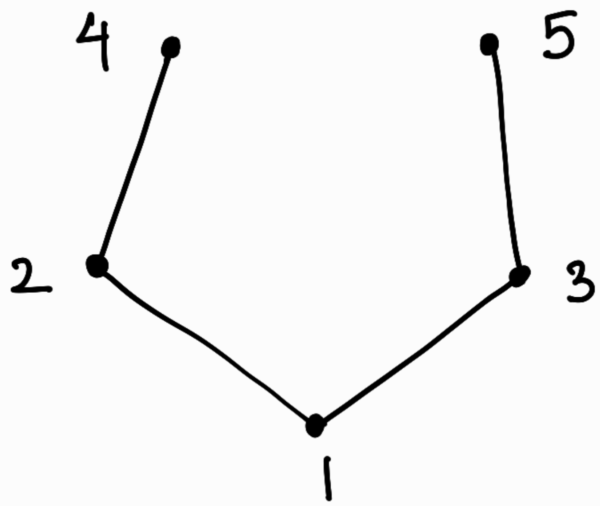


2. Remove Transitive edges (1,4), (1,5).



3. All edges are pointing upwards, remove arrows from edges, replace circles with dots.

Hasse Diagram



Q. 4. Let $A = \{a, b, c, d\}$ & $\times$ be a relation on A, whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i) Prove that R is partial order

ii) Draw Hasse Diagram of R.

Sol: $R = \{(a,a), (a,c), (a,d), (b,b), (b,c), (b,d), (c,c), (c,d), (d,d)\}$

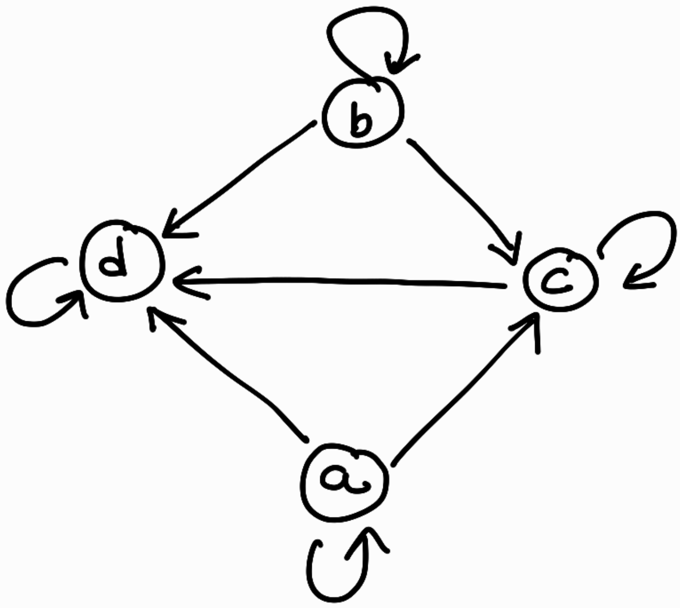
We can see that R is reflexive as $(a,a), (b,b), (c,c), (d,d) \in R$

R is also antisymmetric because it contains a and b such that if $a \neq b$, then $a \not R b$ or $b \not R a$.

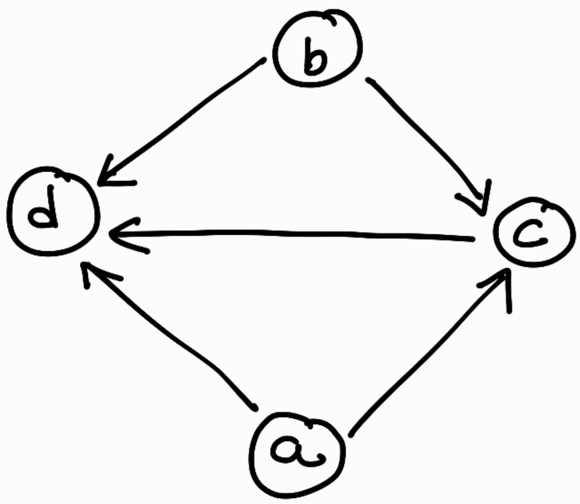
Also, R is transitive since it contains (a,d) & (b,d).

$\therefore$ R is partial order

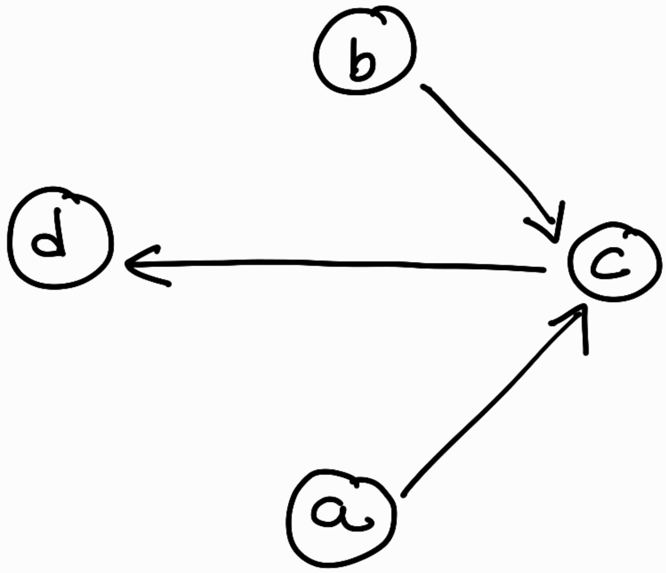
ii) Diagraph of the given relation is given below :-



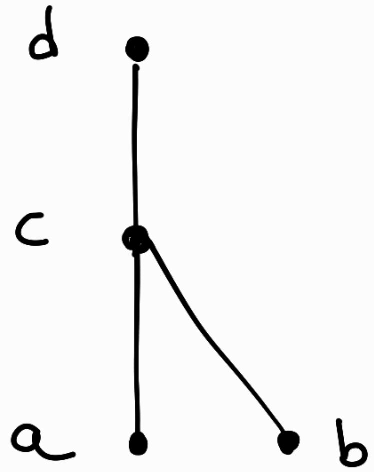
Step 1. Remove the cycles.



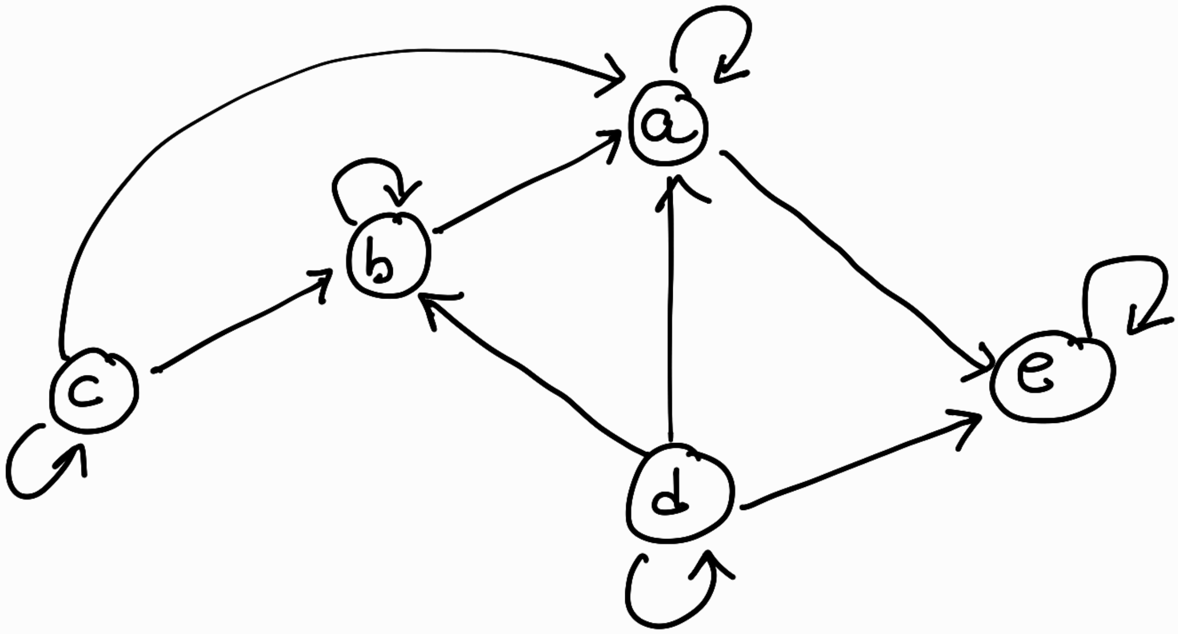
Step 2: Remove transitive edges (a,d) & (b,d)



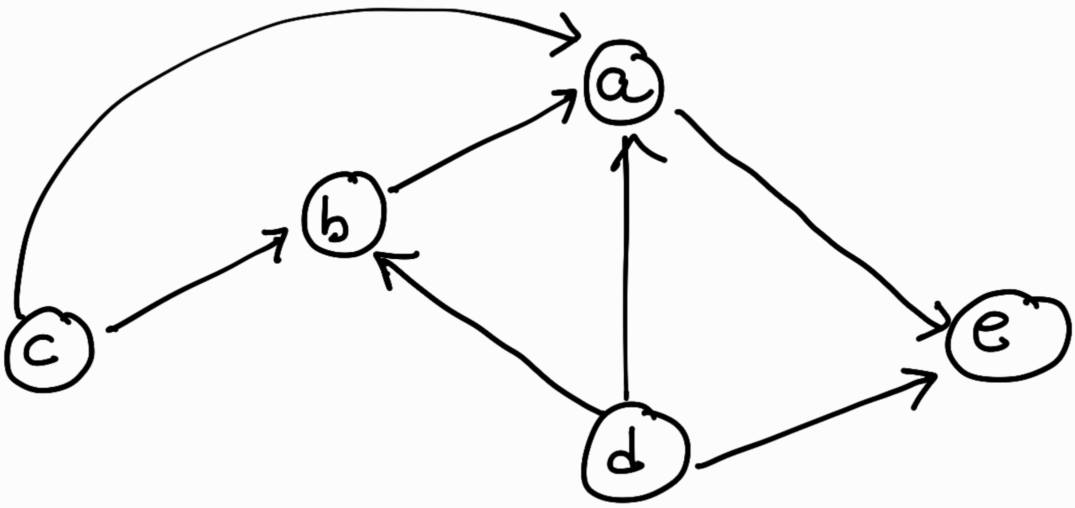
Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Q.5. Determine the Hasse Diagram of partial order having the given digraph.

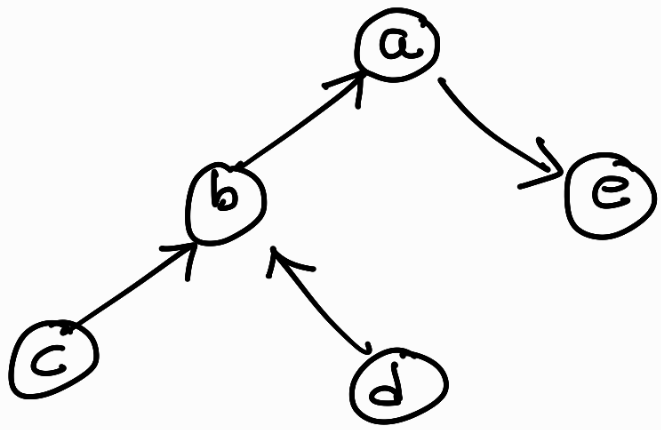


Sol: Step 1. Remove Cycles

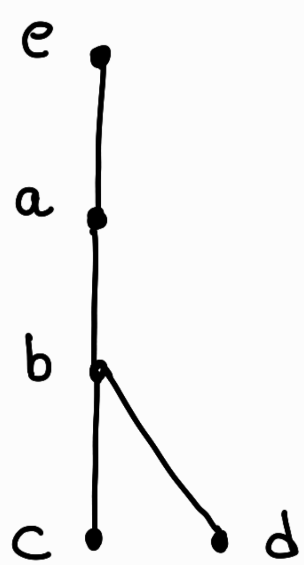


Step 2: Remove transitive edges

(c, a), (d, a), (d, e).



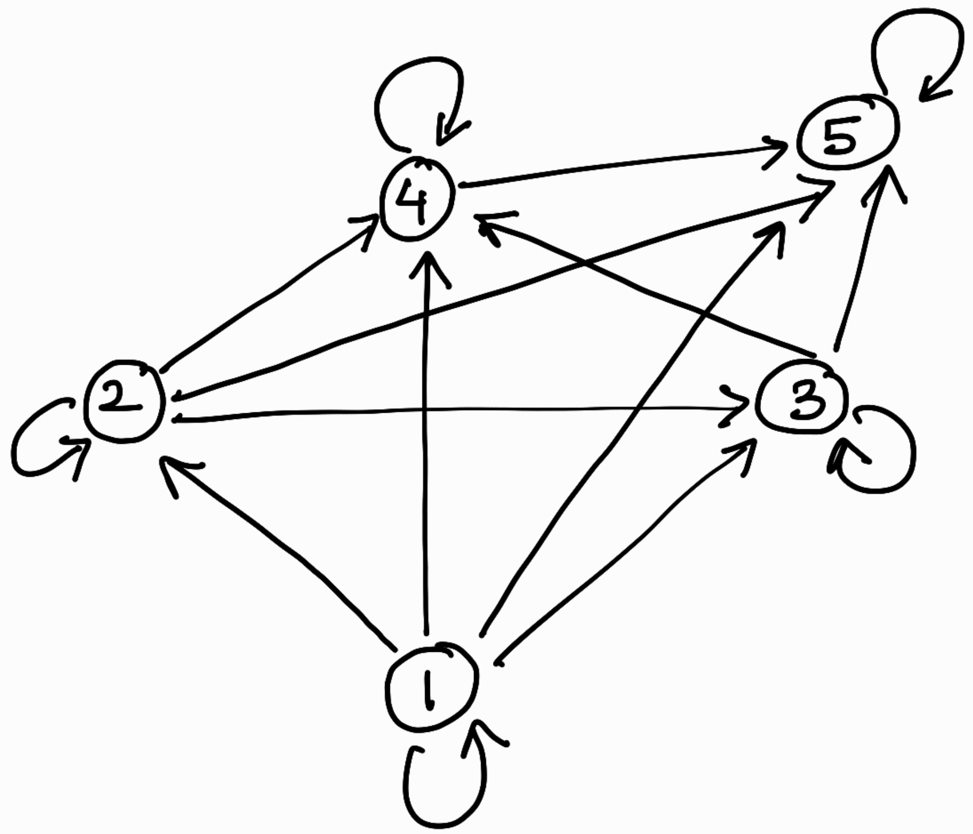
Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Q.6 Determine the Hasse Diagram of the relation on $A = \{1,2,3,4,5\}$, whose matrix is shown:-

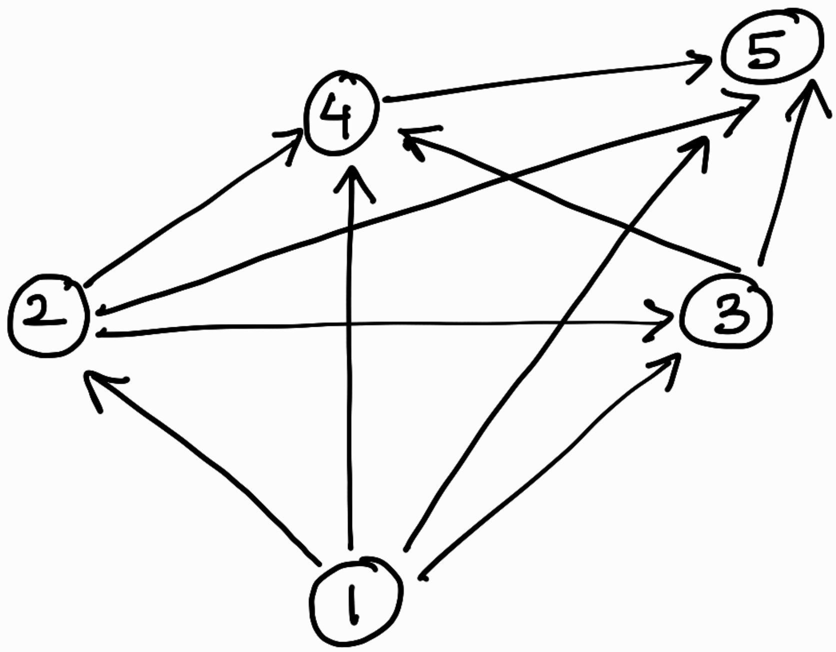
a) $$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: Digraph for the given matrix is



Step 1

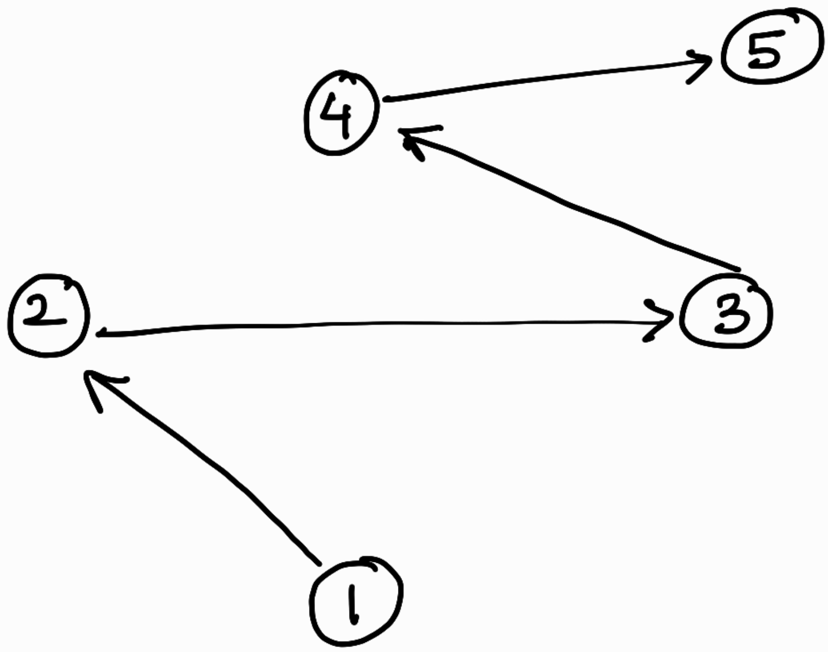
Remove Cycles



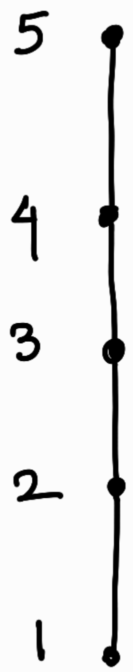
Step 2

Remove Transitive edges

(2,5), (1,3), (2,4), (1,5), (1,4), (3,5)

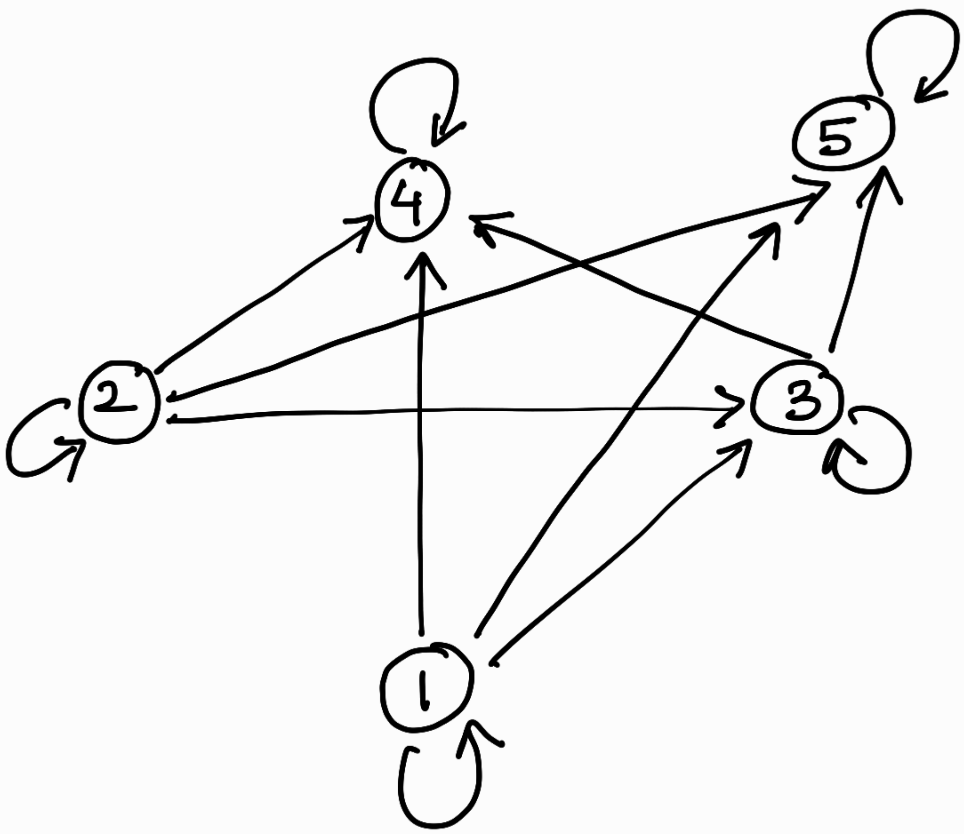


<u>Step 3:</u> Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.

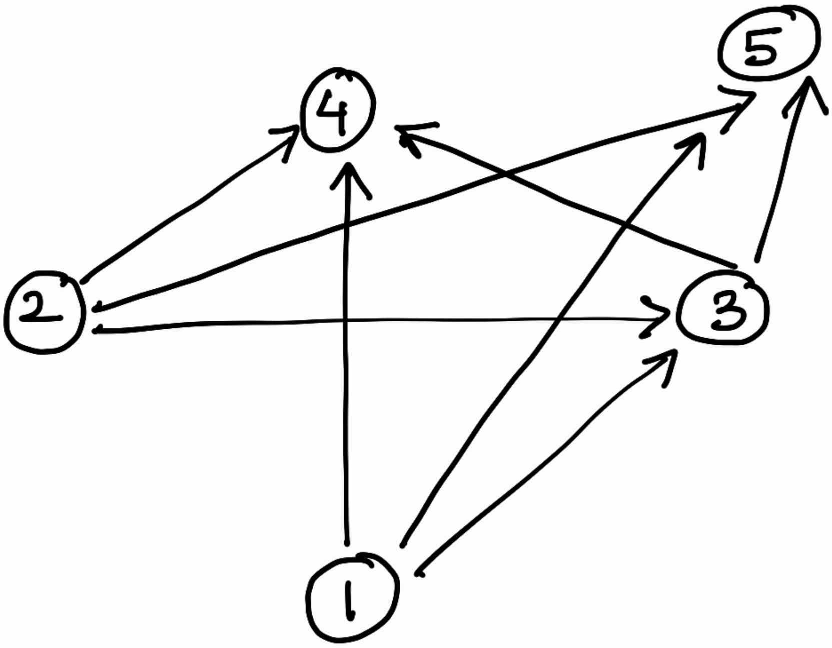


b) $$M_R = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: Digraph for the given matrix is

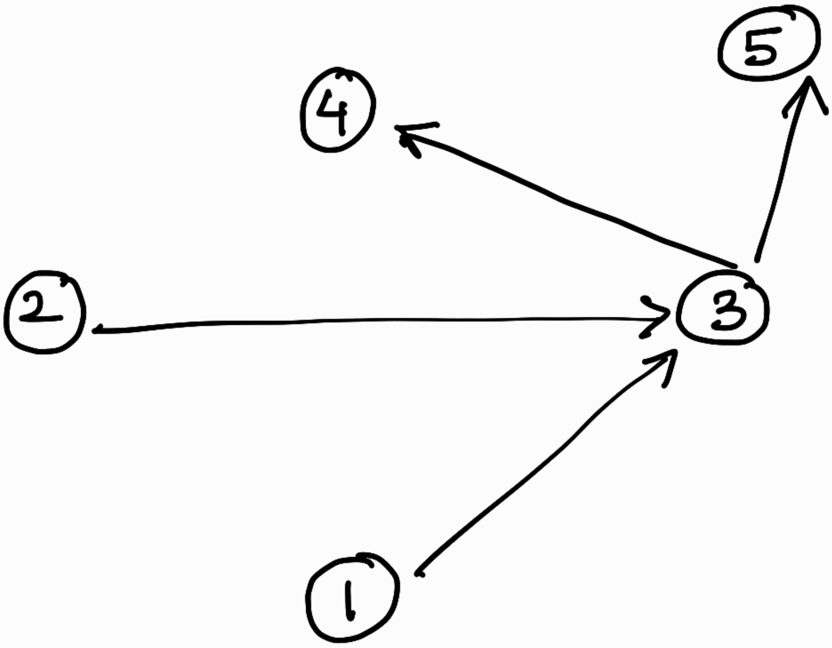


Step 1: Remove Cycles

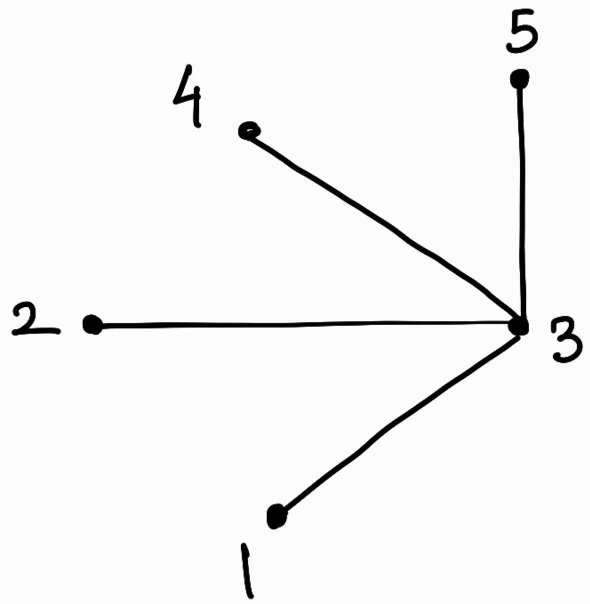


Step2: Remove Transitive edges

(2,4), (1,4), (1,5), (2,5)



Step 3: Make sure, all edges are pointing upwards, Circles are replaced by dots & all arrows removed.



Practice Questions

1. Draw the Hasse Diagram for divisibility on the set

i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

ii) $\{1, 2, 3, 4, 5, 7, 11, 13\}$

2. Draw Hasse Diagram for following relation, what the diagram is called as? Justify.

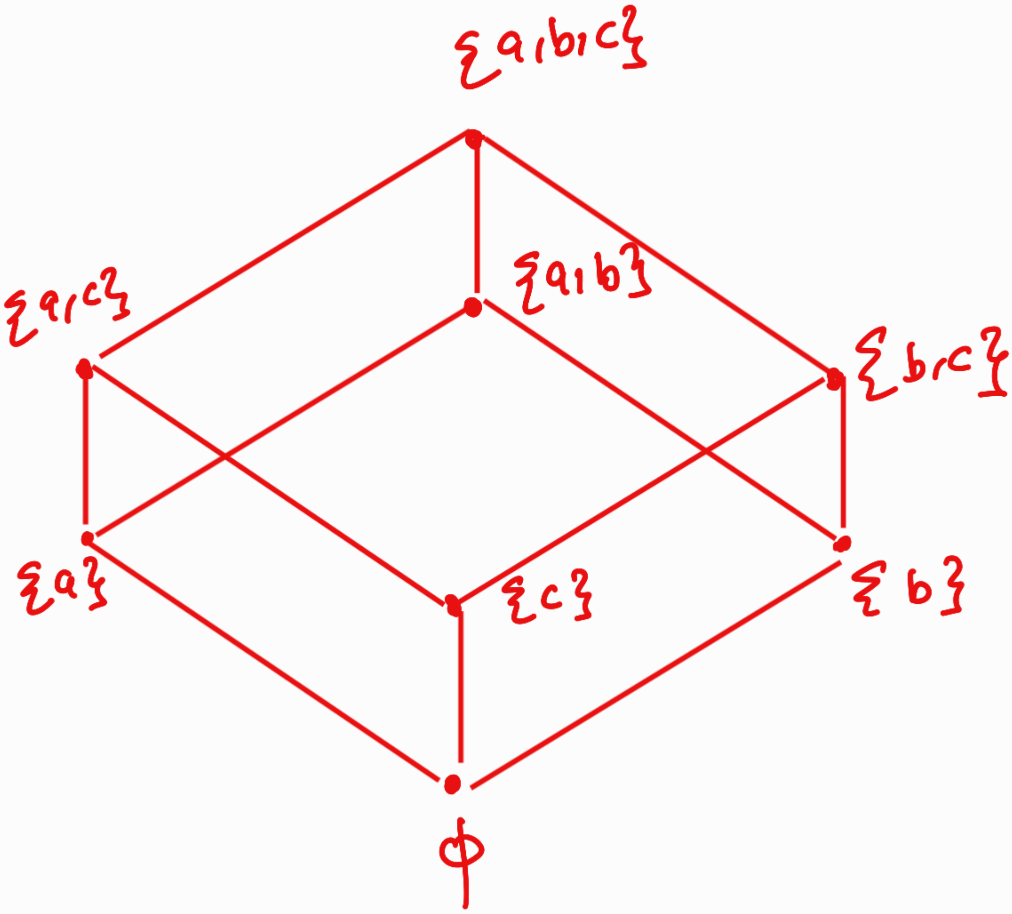
Let $A = \{a, b, c, d, e\}$ &

$R = \{(a,a), (b,b), (c,c), (d,d),$
$(e,e), (a,b), (b,c), (c,d), (d,e), (a,c),$
$(a,d), (a,e), (b,d), (b,e), (c,e)\}$

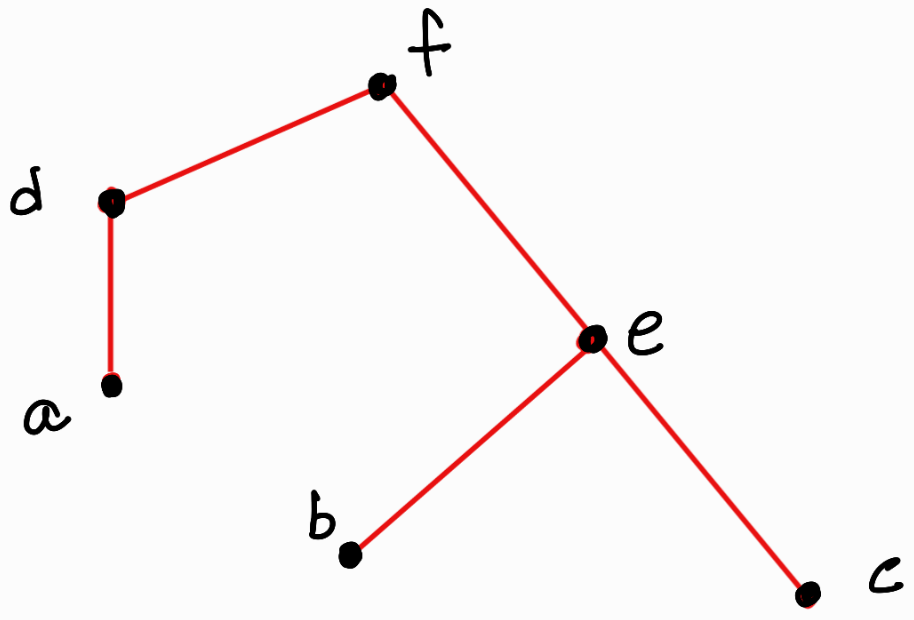
Q. $S = \{a, b, c\}$ and A is a poset under $\subseteq$.

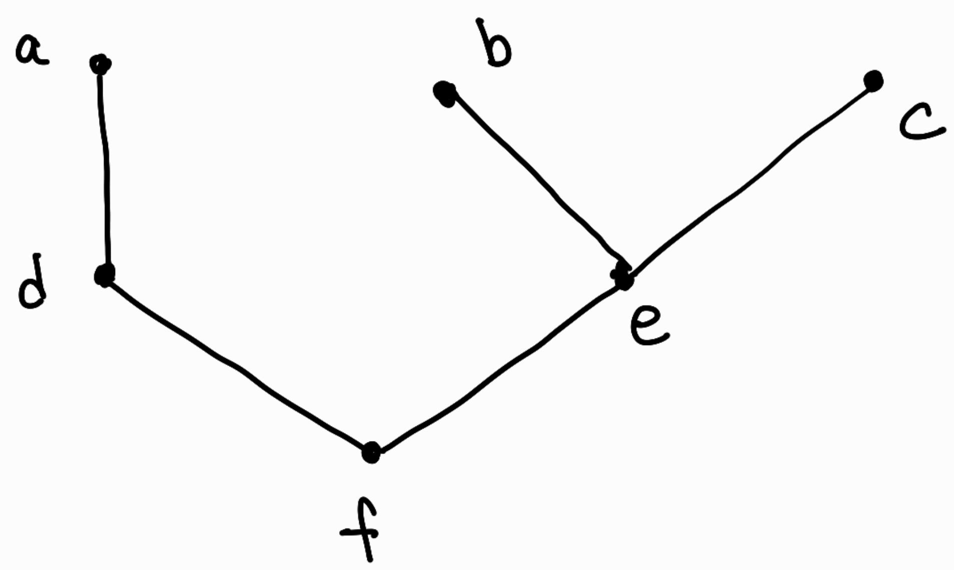
Draw Hasse Diagram of poset $(A, \subseteq)$

Sol: $A = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\})$

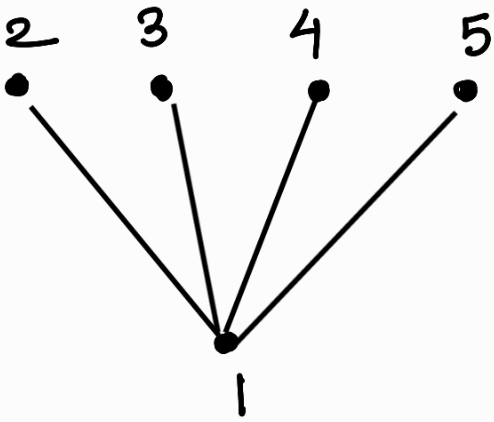


Q. Draw Hasse diagram of dual poset of the poset whose Hasse diagram is given :-

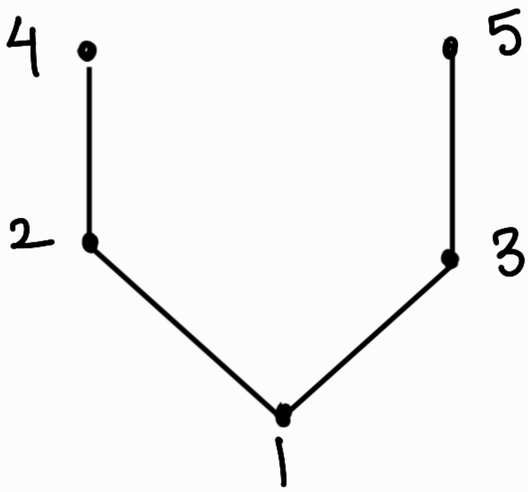




Q. Determine matrix of partial order whose Hasse Diagrams are:-

a) 

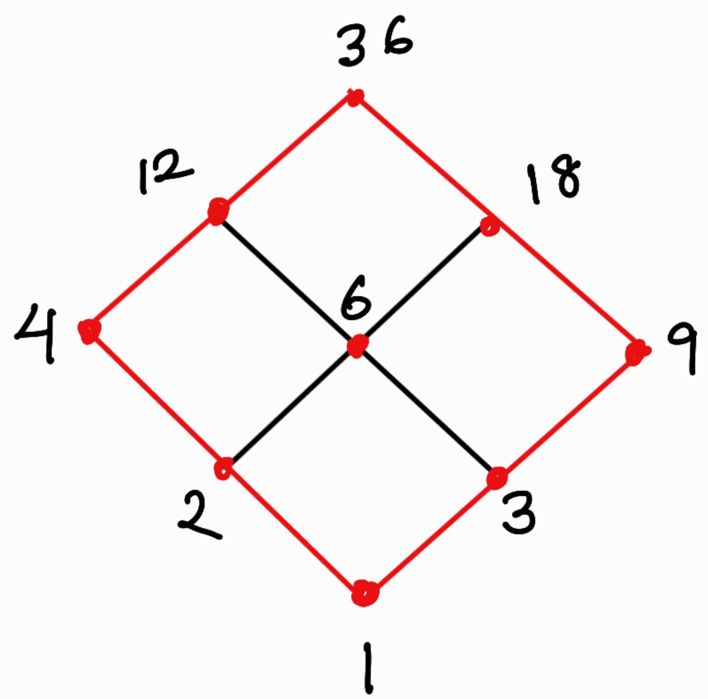
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b) 

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Q. Draw a Hasse Diagram for the set D_{36}.

Sol. $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



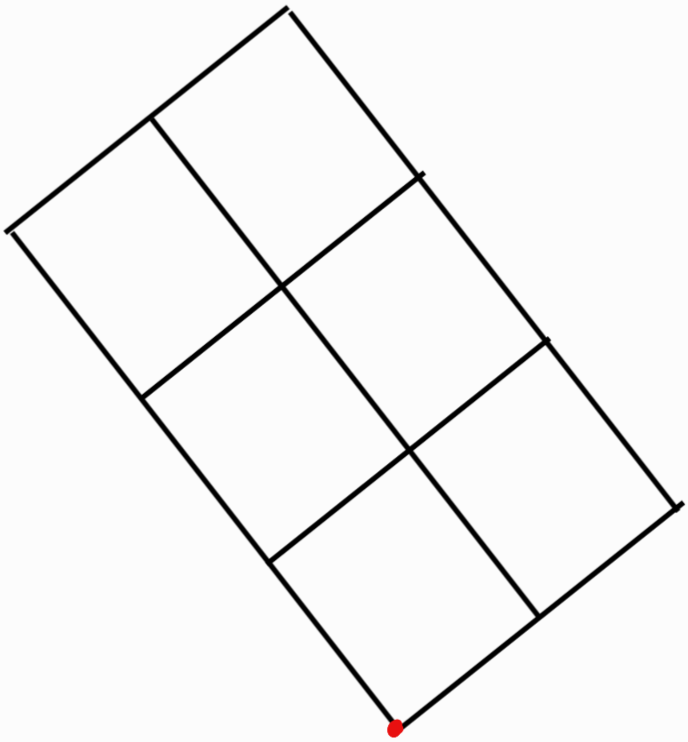
$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (1,18),$
$(1,36), (2,2), (2,4), (2,6), (2,12), (2,18),$
$(2,36), (3,3), (3,6), (3,9), (3,12), (3,18),$
$(3,36), (4,4), (4,12), (4,36), (6,6), (6,12),$
$(6,18), (6,36), (9,9), (9,18), (9,36), (12,12)$
$(12,36), (18,18), (18,36), (36,36)\}$

Q. Draw a Hasse Diagram for the set D_{72}

Sol.

Q. Draw a Hasse Diagram for the set D_{72}

Sol. $D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

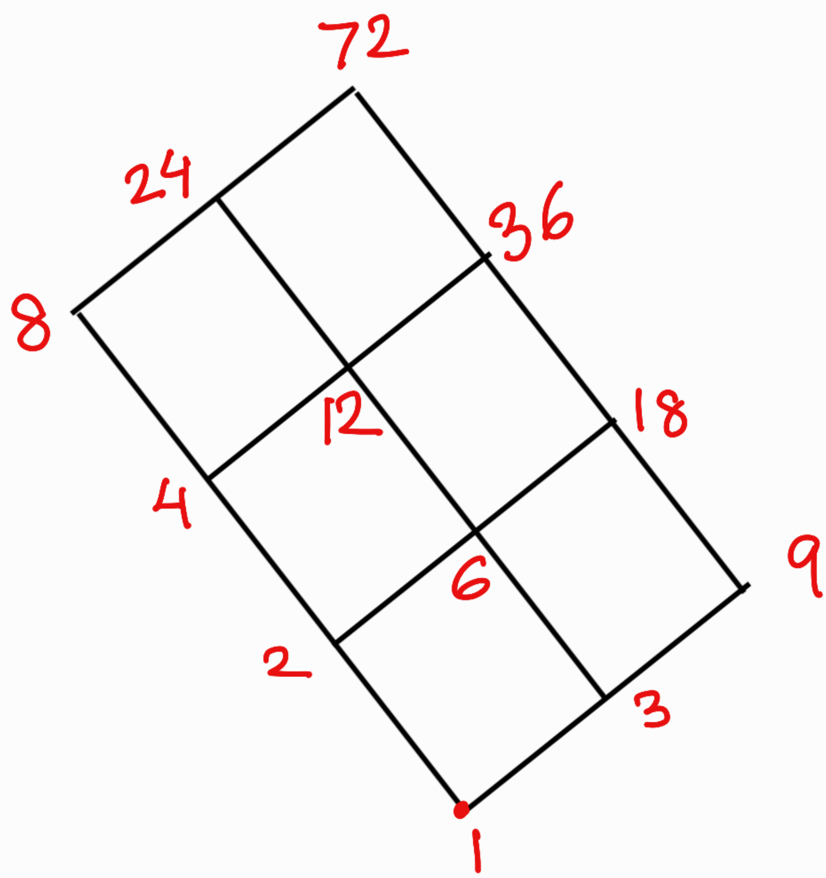


$\Rightarrow \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,9),$
$(1,12), (1,18), (1,24), (1,36), (1,72),$
$(2,2), (2,4), (2,6), (2,8), (2,12), (2,18),$
$(2,24), (2,36), (2,72), (3,3), (3,6),$
$(3,9), (3,12), (3,18), (3,24), (3,36),$
$(3,72), (4,4), (4,8), (4,12), (4,24),$
$(4,36), (4,72), (6,6), (6,12), (6,18),$

(6, 24), (6, 36), (6, 72), (8, 8), (8, 24),
(8, 72), (9, 9), (9, 18), (9, 36), (9, 72),
(12, 12), (12, 24), (12, 36), (12, 72),
(18, 18), (18, 36), (18, 72), (24, 24),
(24, 72), (36, 36), (36, 72), (72, 72)}

Q. Draw a Hasse Diagram for the set D_{72}

Sol. $D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$



$R \Rightarrow \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,9),$
$(1,12), (1,18), (1,24), (1,36), (1,72),$
$(2,2), (2,4), (2,6), (2,8), (2,12), (2,18),$
$(2,24), (2,36), (2,72), (3,3), (3,6),$
$(3,9), (3,12), (3,18), (3,24), (3,36),$
$(3,72), (4,4), (4,8), (4,12), (4,24),$
$(4,36), (4,72), (6,6), (6,12), (6,18),$

$(6,24),\ (6,36),\ (6,72),\ (8,8),\ (8,24),$

$(8,72),\ (9,9),\ (9,18),\ (9,36),\ (9,72),$

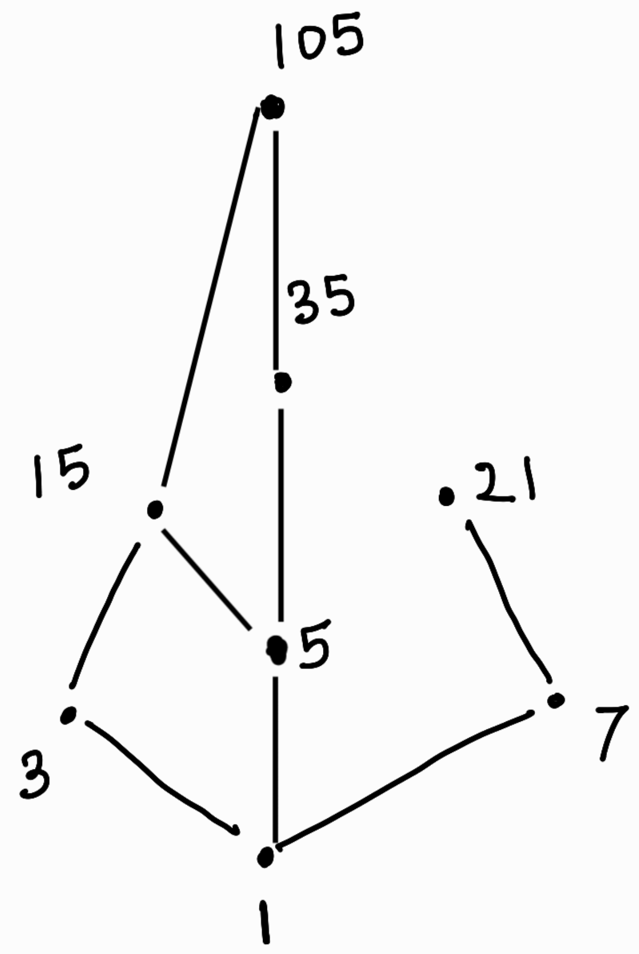
$(12,12),\ (12,24),\ (12,36),\ (12,72),$

$(18,18),\ (18,36),\ (18,72),\ (24,24),$

$(24,72),\ (36,36),\ (36,72),\ (72,72)\}$

Q. Draw Hasse Diagram for D_{105}

Sol: $D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$

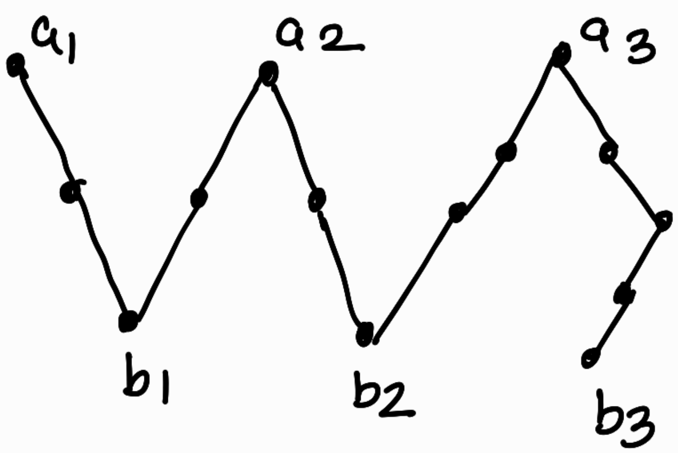


Maximal & Minimal Element.

An element $a \in A$ is called a maximal element of A if there is no element c in A such that $a < c$.

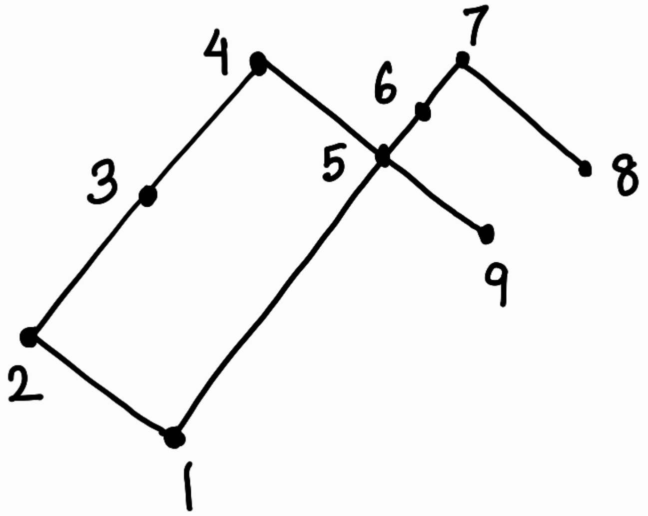
An element $b \in A$ is called a minimal element of A if there is no element c in A such that $c < b$.

Example



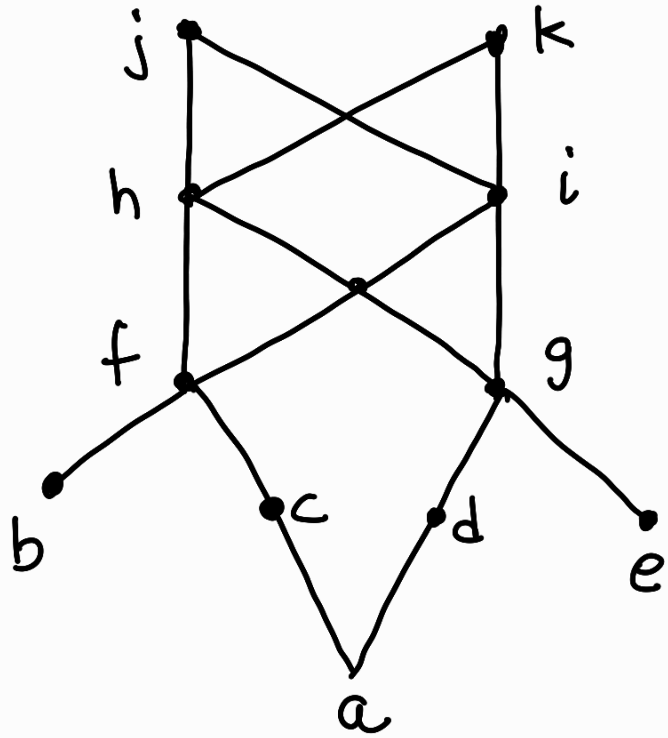
Maximal Elements : a_1, a_2, a_3

Minimal Elements : b_1, b_2, b_3



Maximal : 4, 7

Minimal : 1, 9, 8



Maximal : j, k

Minimal : b, a, e

Greatest & Least Element

An element $a \in A$ is called a greatest element of A if $x \leq a$ for all $x \in A$.

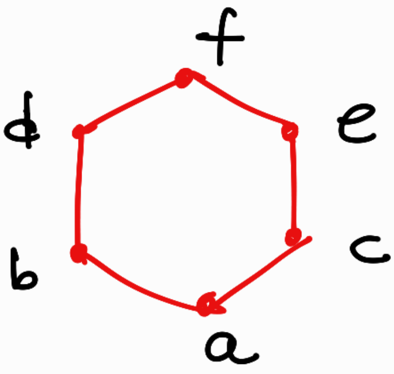
An element $a \in A$ is called a least element of A if $a \leq x$ for all $x \in A$.

A poset has atmost one greatest element and atmost one least element.

The greatest elements of a poset, if it exists, is denoted by I and is often called as the Unit element.

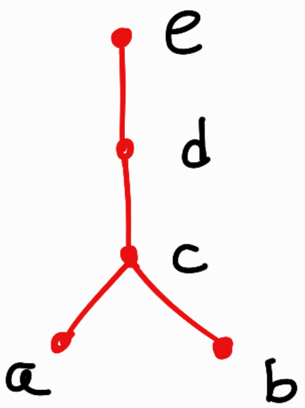
Similarly, the least element of a poset, if it exists, is denoted by '0' and is called as the Zero Element.

Examples



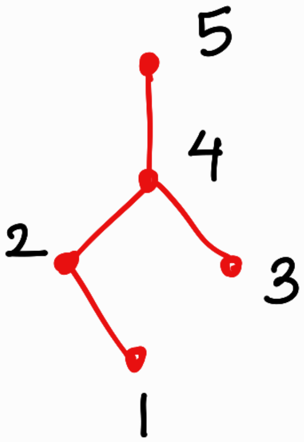
I = f

O = a



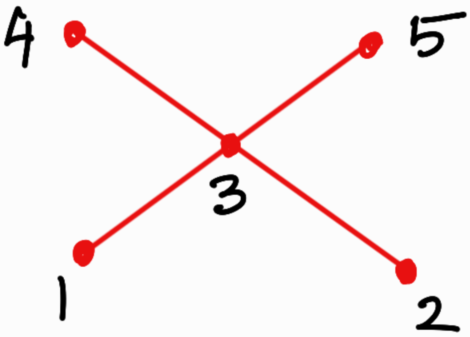
I = e

O = None



I = 5

O = 1



I = None

O = None

Greatest Element is also called as universal upper bound.

Least Element is also called as universal lower bound.

Upper Bound: Consider a poset A and a subset B of A. An element $a \in A$ is called an upper bound of B if $b \leq a$ for all $b \in B$.

Lower Bound: An element $a \in A$ is called a lower bound of B if $a \leq b$ for all $b \in B$.

Least Upper Bound (LUB)

Let A be a poset & B be a subset of A.

An element $a \in A$ is called a Least Upper Bound (LUB) of B if a is an upperbound of B and $a \leq a'$, whenever a' is an upper bound of B.

Thus $a = (LUB)(B)$ if $b \leq a$ for all $b \in B$ and if whenever $a' \in A$ is also an upper bound of B.

Then $a \leq a'$.

Greatest Lower Bound (GLB)

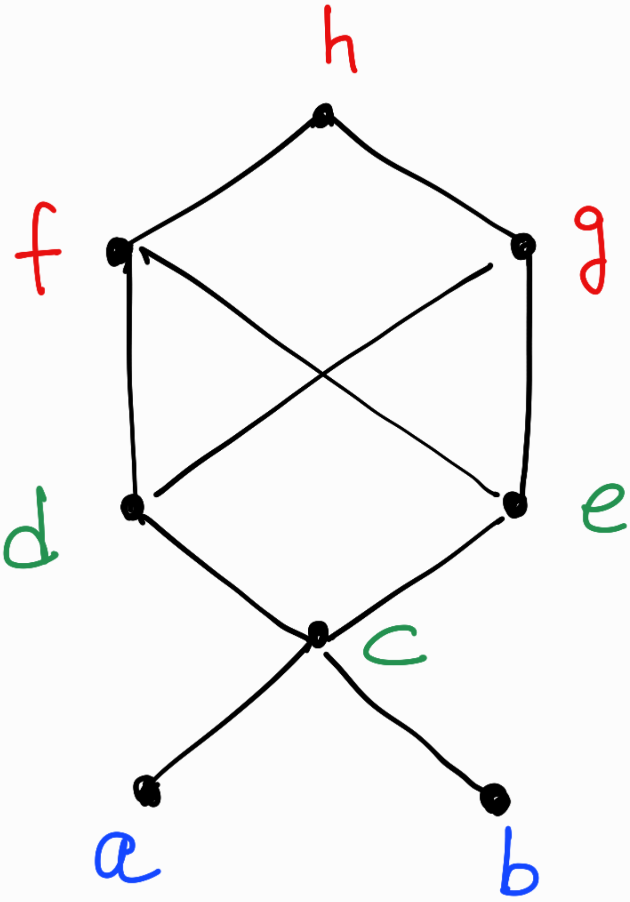
Let A be a poset & B be a subset of A.

An element $a \in A$ is called a Greatest Lower Bound (GLB) of B if a is a lower bound of B

Thus $a = (GLB)(B)$ if $a \leq b$ for all $b \in B$ and if whenever $a' \in A$ is also a lower bound of B.

Then $a' \leq a$.

Example



$A = \{a, b, c, d, e, f, g, h\}$

$B_1 \subseteq A$

$B_2 \subseteq A$

i) $B_1 = \{a, b\}$

ii) $B_2 = \{c, d, e\}$

Find Upper bound, Lower bound, LUB & GLB of B_1 & B_2.

Sol:

i) Upper bounds of set B_1 are
c, d, e, f, g, h

Least Upper Bound is 'c'.

There is no Lower bound of set B_1, so no GLB as well.

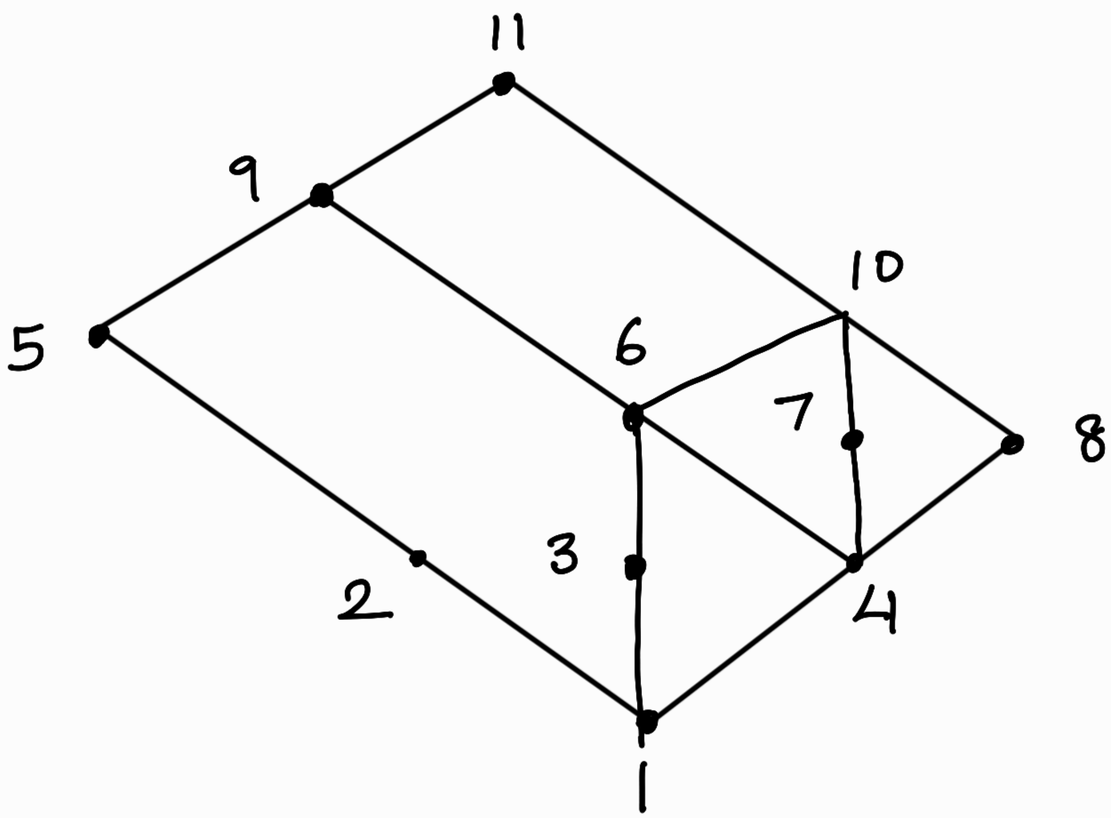
ii) Upper bounds of set B_2 are
f, g & h.

There is no least upper bound in B_2.

Lower bounds are c, a & b.

Greatest lower bound is c.

Q. Let $A = \{1, 2, 3, 4, 5, \ldots, 11\}$ be the poset whose Hasse diagram is shown below. Find LUB & GLB of $B = \{6, 7, 10\}$ if they exist.



Upper Bounds of B $\Rightarrow$ 10, 11

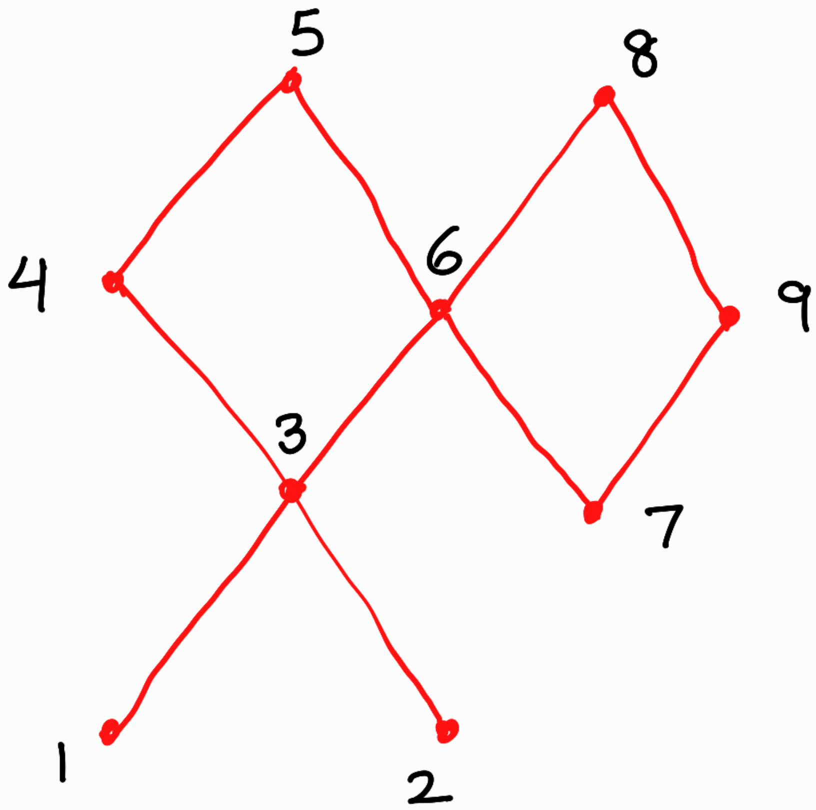
Least Upper Bound $\Rightarrow$ 10

Lower Bounds of B $\Rightarrow$ 1, 4

Greatest Lower Bound $=$ 4

Q. Let A be the poset whose Hasse diagram is as shown.

$A = \{1, 2 \ldots 9\}$. Find GLB, LUB of set $B = \{3, 4, 6\}$



Lower Bounds of B are

GLB is

Upper Bounds of B are

LUB is

Q. Find the GLB & LUB of the set (3, 9, 12) and (1, 2, 4, 5, 10) if they exist in the poset (Z, +, /) where / is a relation of divisibility

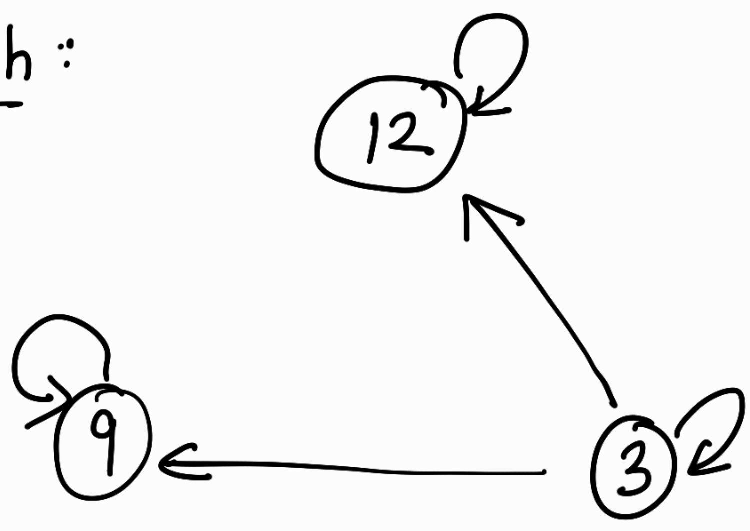
Sol: Set $A = \{3, 9, 12\}$

$$R = \{(3,3), (3,9), (3,12), (9,9), (12,12)\}$$

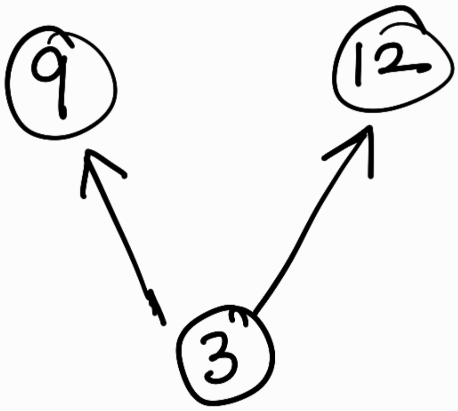
Matrix $M_R =$

$$\begin{array}{c|c} & \begin{array}{ccc} 3 & 9 & 12 \end{array} \\ \begin{array}{c} 3 \\ 9 \\ 12 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

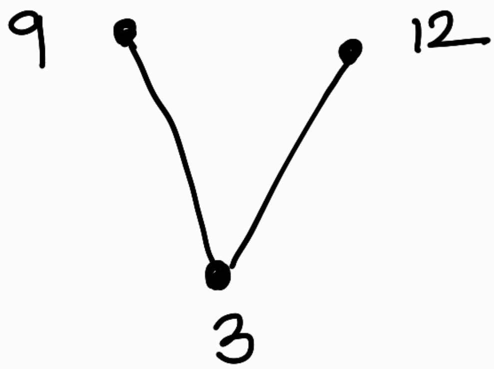
Digraph:



Remove transitive edges & cycles



Hasse Diagram



LUB :

	3	9	12
3	3	9	12
9	9	9	–
12	12	–	12

GLB :

	3	9	12
3	3	3	3
9	3	9	3
12	3	3	12

GLB of $(3, 9, 12) = 3$

LUB of $(3, 9, 12) = 36$ Why??

Because 36 is a number which is divisible by all the 3 numbers, i.e 3, 9 & 12.

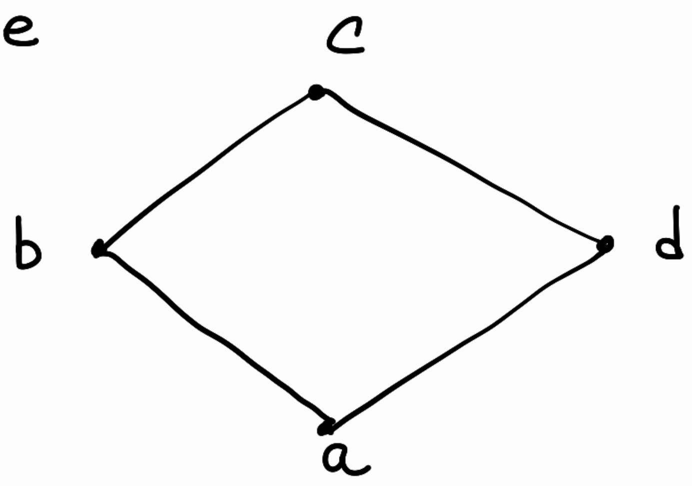
Lattices

A lattice is a poset in which every subset $\{a, b\}$ consisting of two elements, has a least upper bound and a greatest lower bound.

We denote LUB by $a \vee b$ and call it the join of a & b.

Similarly, we denote GLB by $a \wedge b$ and we call it as the meet of a and b.

Example



Determine whether the above Hasse diagram represents a lattice or not.

Sol. LUB:

$\vee$	a	b	c	d
a	a	b	c	d
b	a	b	c	c
c	c	c	c	c
d	d	c	c	d

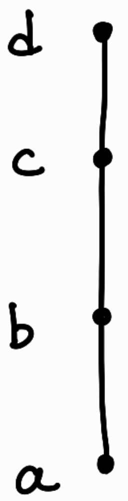
GLB:

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

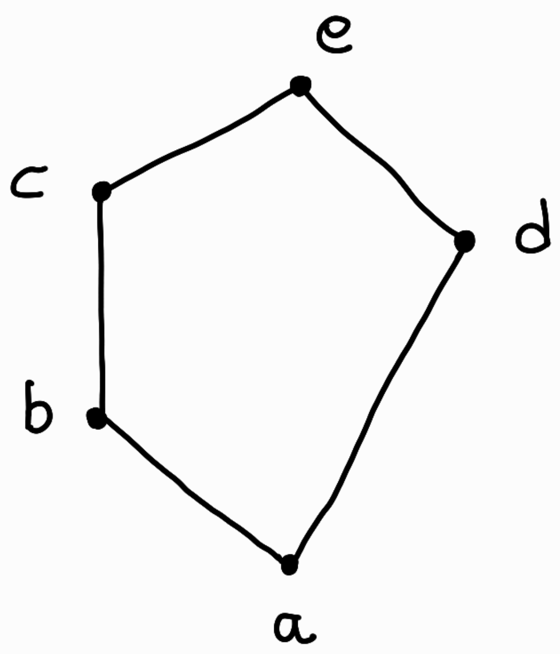
Here, every subset has a least upper bound and greatest lower bound, hence it is a lattice.

Practice

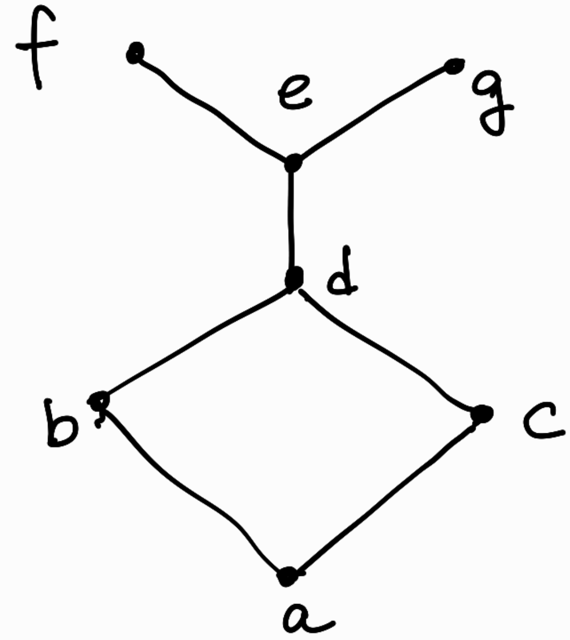
1.



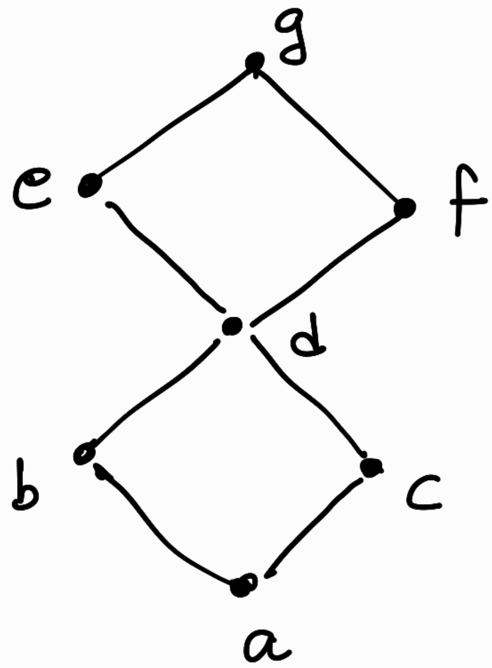
2.



3.



4.



Properties of Lattices

Let L be a lattice then,

1. Idempotent Property

   a) $a \vee a = a$

   b) $a \wedge a = a$

2. Commutative Property

   a) $a \vee b = b \vee a$

   b) $a \wedge b = b \wedge a$

3. Associative Property

   a) $a \vee (b \vee c) = (a \vee b) \vee c$

   b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Property

   a) $a \vee (a \vee b) = a$

   b) $a \wedge (a \vee b) = a$

Types of Lattices

1. Bounded Lattice

A lattice L is said to be bounded if it has a greatest element I and a least element O. If L is a bounded lattice, then for all $a \in A$.

$$O \leq a \leq I$$

$$a \vee O = a \quad , \quad a \wedge O = O$$

$$a \vee I = I \quad , \quad a \wedge I = a$$

2. Distributive Lattice

A lattice L is called distributive if for any elements a, b and c in L, we have the following distributive properties.

a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

b) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

3. Complemented Lattice

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$.

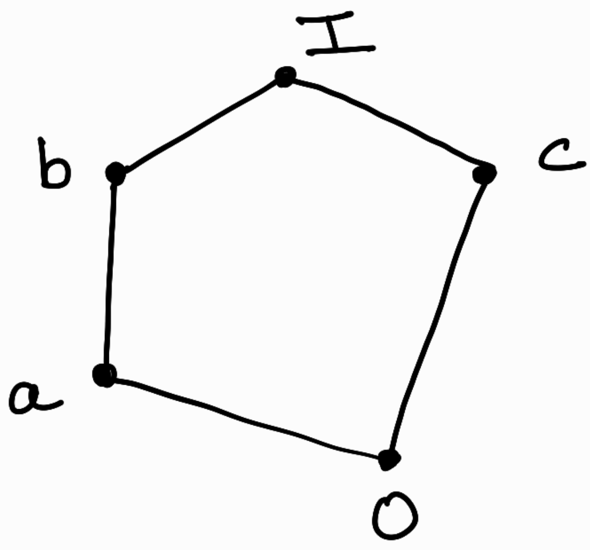
An element $a' \in L$ is called a complement of a if

$$a \vee a' = I \quad \text{and} \quad a \wedge a' = 0$$

A lattice L is called complemented if it is bounded and if every element in L has a complement.

Example:

Show that this is a complemented lattice



Sol.

$$a \vee c = I$$

$$a \wedge c = 0$$

$$b \vee c = I$$

$$b \wedge c = 0$$

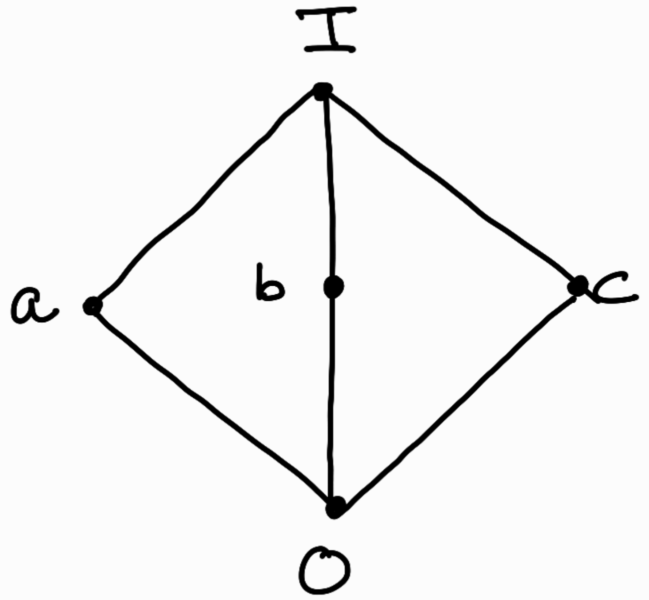
$\therefore$ Complement of a & b are c.

$\therefore$ Complement of c are b & a.

$\therefore$ Every element has a complement

$\therefore$ Above lattice is a complemented lattice.

Q. Check if the following is a complemented lattice or not.



<u>Sol.</u> $a \vee b = I$, $a \wedge b = 0$

$\therefore$ Complement of a is b

$\therefore$ Complement of b is a

Also $a \vee c = I$, $a \wedge c = 0$

$\therefore$ Complement of a is C

$\therefore$ Complement of c is a.

$\therefore$ Every element has a complement.

So above lattice is complemented

Functions

Definition:

Let A and B be non-empty sets. A function f from A to B, denoted as $f : A \to B$, is a relation from A to B such that for every $a \in A$, there exists a <u>unique</u> $b \in B$ such that $(a,b) \in f$.

Normally if $(a,b) \in f$, we write $f(a) = b$

If $f(a) = b$ & $f(a) = c$

then $b = c$

Examples:

1. Let $A = \{1, 2, 3\}$ & $B = \{p, q, r\}$ & $f = \{(1, p), (2, q), (3, r)\}$.

Determine whether the given relation is a function.

Sol:

$$f(1) = \{p\}$$
$$f(2) = \{q\}$$
$$f(3) = \{r\}$$

'f' is a function, since each set $f(n)$ consists of a single element.

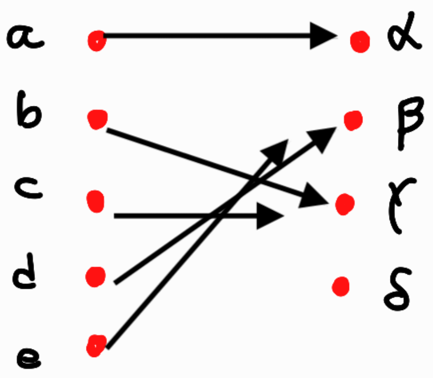
A function can be represented in graphical & tabular form.

For example :-

Let A be a set of houses and B be a set of colors. Then a function from A to B is an assignment of colours for painting the houses. i.e

$f(a) = \alpha \qquad f(b) = \gamma \qquad f(c) = \gamma$

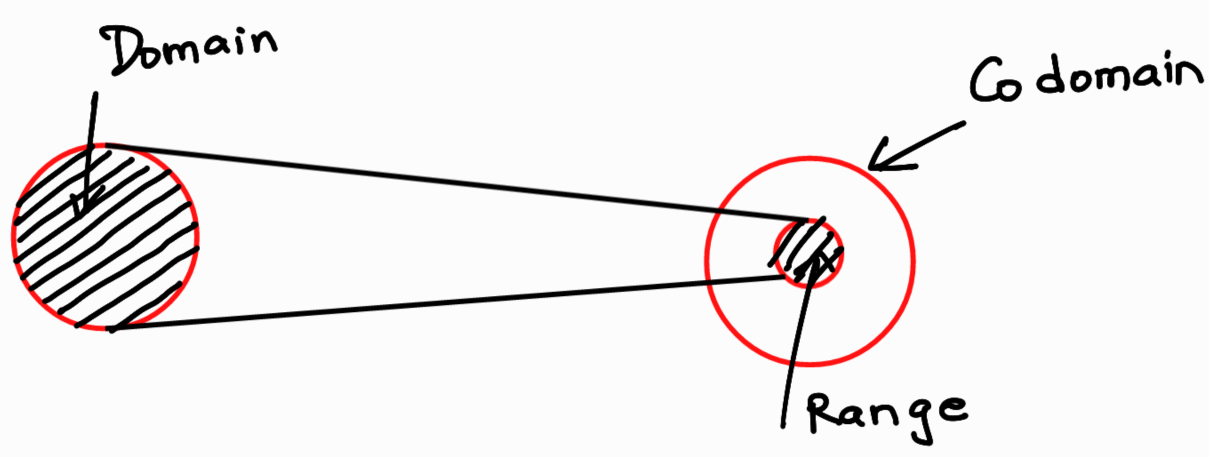
$f(d) = \beta \qquad f(e) = \beta$



Graphical

	α	β	γ	δ
a	✓			
b			✓	
c			✓	
d		✓		
e		✓		

Tabular



The set A is called as the <u>domain</u> of f, denoted as <u>Dom (f)</u>.

The set B is called as the <u>codomain</u> and the set $\{f(a) \mid a \in A\}$, which is the subset of B, is called as the <u>range</u> of f, denoted by <u>Ran (f)</u>

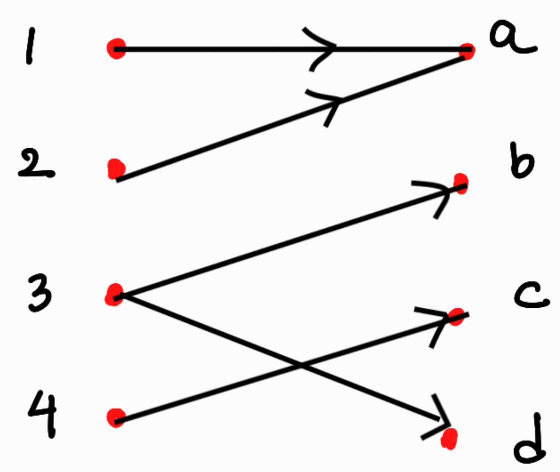
The element a is called an <u>argument</u> of the function f and f(a) is called the <u>value</u> of the function for the argument a.

Types of Functions

1. Onto or Surjective Functions

A function from A to B is said to be an onto function if every element of B is the image of one or more elements of A.

Onto function is also called surjective or 'f' is ONTO if $Ran(f) = B$.



Example : Let $A = \{1,2,3,4\}$ & $B = \{a,b,c,d\}$

and $f = \{(1,a), (2,a), (3,d), (4,c), (3,b)\}$

$f(1) = a$, $f(2) = a$, $f(3) = d$,

$f(4) = c$, $f(3) = b$

$\therefore$ $Ran(f) = \{a, b, c, d\} = B$

So this function is onto or surjective function.

2. One to One or Injective Function

A function from A to B is said to be a <u>one to one</u> function if no two elements of A have the same image

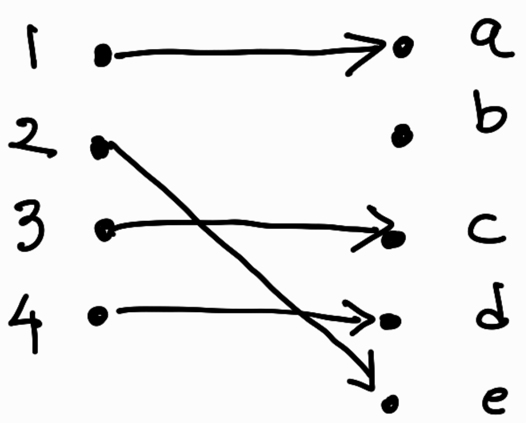
<u>Example</u>

Let $A = \{1, 2, 3, 4\}$ & $B = \{a, b, c, d, e\}$

& $f = \{(1, a), (2, e), (3, c), (4, d)\}$

$\therefore$ $f(1) = a$, $f(2) = e$

$f(3) = c$, $f(4) = d$



3. One to One Onto Function or Bijective Function.

A function from A to B is said to be a <u>one to one onto</u> function if it is both an onto and one to one function.

One to One Onto function is also called as <u>bijective</u> function.

<u>Example.</u>

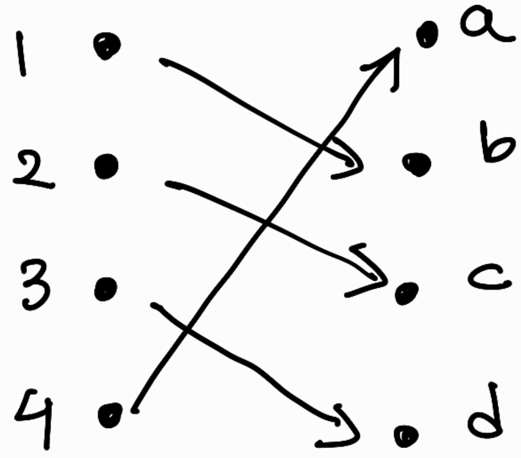
Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$f = \{(1, b), (2, c), (3, d), (4, a)\}$

$f(1) = b$, $f(2) = c$,

$f(3) = d$, $f(4) = a$

Given function f is one to one onto or bijective function



4. <u>Everywhere Defined Function</u>

A function from A to B is said to be everywhere defined if

$Dom(f) = A$.

<u>Example</u>: Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $f = \{(1, c), (2, b), (3, a)\}$

$f(1) = c$, $f(2) = b$, $f(3) = a$

$\therefore$ $Dom(f) = \{1, 2, 3\}$.

Thus given function f is everywhere defined function