(b) Fourier Sine Integral

When $f(x)$ is an odd function, $f(s)$ will be odd but $f(s)\sin\omega s$ will be **even** and $f(s)\cos\omega s$ will be **odd**. Hence, the first integral will be zero and we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s \, d\omega \, ds} \quad \text{.......... (3)}$$

This is called **Fourier Sine Integral.**

Ex. 1 : Express the function $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as Fourier Integral.

(M.U. 1997, 99, 2002, 03)

Hence, evaluate $\int_0^\infty \frac{\sin\omega \sin\omega x}{\cos\omega} d\omega.$ **(M.U. 1994, 95, 2003)**

Sol. : The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds$$

[By data $f(s) = 0$ from $-\infty$ to -1, $f(s) = 1$ from -1 to 1 and $f(s) = 0$ from 1 to ∞.]

Hence, $$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^1 1 \cdot \cos\omega(s-x)\, d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\omega(s-x)}{\omega}\right]_{-1}^1 d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(1-x) - \sin\omega(-1-x)}{\omega} d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(1+x) + \sin\omega(1-x)}{\omega} d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega$$

$$\therefore \quad \int_0^\infty \frac{\sin\omega \cdot \cos\omega x}{\omega} d\omega = \frac{\pi}{2}\cdot f(x)$$

$$= \begin{cases} \frac{\pi}{2} \text{ for } f(x) = 1 & \text{when } |x| < 1 \\ 0 \text{ for } f(x) = 0 & \text{when } |x| > 1 \end{cases}$$

At $|x| = 1$ *i.e.* $x = \pm 1$, **$f(x)$ is discontinuous and the integral**

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$

$$= \frac{\pi}{4}[1+0] = \frac{\pi}{4}$$

$$\therefore \int_0^{\infty} \frac{\sin\omega \cdot \cos\omega x}{\omega}\, d\omega = \begin{cases} \pi/2 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1 \\ \pi/4 & \text{when } |x| = 1 \end{cases}$$

Cor. 1 : Putting $x = 1$ in the above result, we get

$$\int_0^{\infty} \frac{\sin\omega \cos\omega}{\omega}\, d\omega = \frac{\pi}{4} \quad i.e. \quad \int_0^{\infty} \frac{\sin 2\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

Cor. 2 : Putting $x = 0$, in the above result

$$\int_0^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}$$

Ex. 2 : Express the function $f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x < 0,\ x > \pi \end{cases}$

as Fourier Integral and prove that

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\sin\omega x + \cos[\omega(\pi - x)]}{1 - \omega^2}\, d\omega$$ **(M.U. 2001, 06, 09)**

Hence, deduce that $\int_0^{\infty} \frac{\cos(\omega\pi/2)}{1-\omega^2}\, d\omega = \frac{\pi}{2}$.

Sol. : The Fourier integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\cos\omega(s - x)\, d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi} \sin s \cos\omega(s - x)\, d\omega\, ds$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} 2\sin s \cos\omega(s - x)\, d\omega\, ds$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\pi} [\sin(s + \omega s - \omega x) + \sin(s - \omega s + \omega x)]\, d\omega\, ds$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[-\frac{\cos(s + \omega s - \omega x)}{1+\omega} - \frac{\cos(s - \omega s + \omega x)}{1-\omega}\right]_0^{\pi} d\omega$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[-\frac{\cos(\pi + \pi\omega - \omega x)}{1+\omega} - \frac{\cos(\pi - \pi\omega + \omega x)}{1-\omega} + \frac{\cos\omega x}{1+\omega} + \frac{\cos\omega x}{1-\omega}\right] d\omega$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[\frac{\cos(\pi\omega - \omega x)}{1+\omega} + \frac{\cos(-\pi\omega + \omega x)}{1-\omega} + \frac{\cos\omega x}{1+\omega} + \frac{\cos\omega x}{1-\omega}\right] d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \left[\left(\frac{1}{1+\omega}\right)\{\cos\omega x + \cos\omega(\pi - x)\} + \left(\frac{1}{1-\omega}\right)\{\cos\omega x + \cos\omega(\pi - x)\}\right] d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \left(\frac{1}{1+\omega} + \frac{1}{1-\omega}\right)\{\cos\omega x + \cos\omega(\pi - x)\}\, d\omega$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{2}{(1-\omega^2)}\{\cos\omega x + \cos\omega(\pi - x)\}\, d\omega$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\omega x + \cos\omega(\pi - x)}{(1-\omega^2)}\right] d\omega$$

Putting $x = \pi/2$, we get,

$$\sin\frac{\pi}{2} = \frac{1}{\pi}\int_0^\infty \frac{\left[\cos\frac{\pi\omega}{2} + \cos\frac{\pi\omega}{2}\right]}{(1-\omega^2)} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos \pi\omega/2}{1-\omega^2} d\omega$$

$$\therefore \quad 1 = \frac{2}{\pi}\int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2} d\omega \quad \therefore \quad \frac{\pi}{2} = \int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2} d\omega$$

> **Note ....**
>
> Unfortunately there is no uniformity in the notation of Fourier Integral. Some authors use λ or α in place of ω and t in place of s.

Ex. 3 : Express the function

$$f(x) = \begin{cases} -e^{kx} & \text{for } x < 0 \\ e^{-kx} & \text{for } x > 0 \end{cases}$$

as Fourier Integral and hence, prove that

$$\int_0^\infty \frac{\omega \sin\omega x}{\omega^2 + k^2} d\omega = \frac{\pi}{2} e^{-kx} \text{ if } x > 0,\ k > 0 \qquad \textbf{(M.U. 2002)}$$

Sol. : Since, the given function $f(x)$ is an odd function we use (3)

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s\, d\omega\, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty e^{-ks}\sin\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \left[\frac{1}{k^2+\omega^2} e^{-ks}(-k\sin\omega s - \omega\cos\omega s)\right]_0^\infty d\omega$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{\omega}{k^2+\omega^2}\,d\omega$$

$$\therefore\quad \int_0^\infty \frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-kx}\ \text{ if } x>0$$

Ex. 4 : Using Fourier Cosine Integral prove that

$$e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{(\omega^4+4)}\cdot\cos\omega x\,d\omega$$ **(M.U. 2002, 05, 07)**

Sol. : By Fourier cosine integral formula

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty e^{-s}\cos s\cdot\cos\omega s\,d\omega\,ds$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\int_0^\infty e^{-s}\left[\cos(\omega+1)s+\cos(\omega-1)s\right]d\omega\,ds$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{1+(\omega+1)^2}\cdot e^{-s}\{-\cos(\omega+1)s+(\omega+1)\sin(\omega+1)s\}_0^\infty\right.$$
$$\left.+\frac{1}{1+(\omega-1)^2}\cdot e^{-s}\{-\cos(\omega-1)s+(\omega-1)\sin(\omega-1)s\}_0^\infty\right]$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{1+(\omega+1)^2}+\frac{1}{1+(\omega-1)^2}\right]d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{\omega^2-2\omega+2+\omega^2+2\omega+2}{\{(\omega^2+2)+2\omega\}\{(\omega^2+2)+2\omega\}}\right]d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \cos\omega x\cdot\frac{2(\omega^2+2)}{(\omega^2+2)^2-4\omega^2}\cdot d\omega$$

$$\therefore\quad f(x)=\frac{1}{\pi}\int_0^\infty \cos\omega x\cdot\frac{(\omega^2+2)}{(\omega^4+4)}\,d\omega.$$

Ex. 5 : Find Fourier Integral representation for

$$f(x)=\begin{cases}1-x^2 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$$ **(M.U. 1998, 2003, 08)**

Sol. : By data $f(s)=0$ from $-\infty$ to -1, $f(s)=1-s^2$ from -1 to 1 and $f(s)=0$ from 1 to ∞.

Also $f(-s)=1-(-s)^2=1-s^2$
$= f(s)$ from -1 to 1

Hence, $f(s)$ is an even function and we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\int_0^1 (1-s^2)\cos\omega s \, ds\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[(1-s^2)\left(\frac{\sin\omega s}{\omega}\right) - \left(-\frac{\cos\omega s}{\omega^2}\right)(-2s) + \left(-\frac{\sin\omega s}{\omega^3}\right)(-2)\right]_0^1 d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[0 - \frac{2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right] d\omega$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cdot \cos\omega x \, d\omega$$

Ex. 6 : Find Fourier integral representation of

$$f(x) = \begin{cases} e^{ax} & x \le 0,\ a > 0 \\ e^{-ax} & x \ge 0,\ a > 0 \end{cases}$$ **(M.U. 1996, 97, 2002, 09)**

Hence, show that $\int_0^\infty \frac{\cos\omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}$, $x > 0$, $a > 0$.

Sol. : Since, $f(x)$ is an even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-as}\cos\omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{a^2+\omega^2} \cdot e^{-as}(-a\cos\omega s + \omega\sin\omega s)\right]_0^\infty d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \cdot \frac{a}{a^2+\omega^2} d\omega$$

$$\therefore \quad \int_0^\infty \frac{\cos\omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax},\ x > 0,\ a > 0$$

Ex. 7 : Find Fourier Integral representation of

$f(x) = x, \quad 0 < x < a$

$= 0, \quad x > a$

$f(-x) = f(x)$ **(M.U. 1995)**

Sol. : Since, $f(x)$ is even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^a s \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} - \int \frac{\sin \omega s}{\omega}(1) \cdot ds\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} + \frac{\cos \omega s}{\omega^2}\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{a \sin a\omega}{\omega} + \frac{\cos a\omega}{\omega^2} - \frac{1}{\omega^2}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left(\frac{a\omega \sin a\omega + \cos a\omega - 1}{\omega^2}\right) d\omega$$

Ex. 8 : Express the function

$$f(x) = \begin{cases} \pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

as Fourier Sine Integral. **(M.U. 1998)**

Hence, show that

$$\int_0^\infty \frac{1 - \cos \pi\omega}{\omega} \sin \omega x \, d\omega = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Sol. : Fourier Sine Integral by (3), page 10-25 is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\pi \frac{\pi}{2} \cdot \sin \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin \omega x \cdot \frac{\pi}{2}\left[-\frac{\cos \omega s}{\omega}\right]_0^\pi d\omega$$

$$= \int_0^\infty \sin \omega x \left[\frac{-\cos \pi \omega + 1}{\omega}\right] d\omega$$

$$= \int_0^\infty \frac{1 - \cos \pi \omega}{\omega} \cdot \sin \omega x \, d\omega$$

$$\therefore \quad \int_0^\infty \frac{1 - \cos \pi \omega}{\omega} \cdot \sin \omega x \cdot d\omega = f(x) = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Ex. 9 : Find Fourier Sine integral representation for $f(x) = \dfrac{e^{-ax}}{x}$. **(M.U. 2004, 09)**

Sol. : By (3) Fourier Sine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty f(s) \sin \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty \frac{e^{-as}}{s} \sin \omega s \, ds \, d\omega$$

To evaluate $\int_0^\infty \frac{e^{-as}}{s} \sin \omega s \, ds$

we use the rule of differentiation under the integral sign

Let $I = \int_0^\infty \frac{e^{-as}}{s} \cdot \sin \omega s \, ds$

$$\therefore \quad \frac{dI}{d\omega} = \int_0^\infty \frac{\partial}{\partial \omega}\left(\frac{e^{-as}}{s} \cdot \sin \omega s\right) ds$$

$$= \int_0^\infty \frac{e^{-as}}{s} \cdot (\cos \omega s) \cdot s \, ds = \int_0^\infty e^{-as} \cos \omega s \, ds$$

$$= \frac{1}{a^2 + \omega^2}\left[e^{-as}(-a \cos \omega s + \omega \sin \omega s)\right]_0^\infty$$

$$= \frac{1}{a^2 + \omega^2}(a)$$

$$\therefore \quad \frac{dI}{d\omega} = \frac{a}{a^2 + \omega^2}$$

Integrating w.r.t. ω,

$$I = a \cdot \frac{1}{a} \tan^{-1} \frac{\omega}{a} + C$$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s} \cdot \sin \omega s \, ds = \tan^{-1} \frac{\omega}{a} + C$$

To find C, we put $\omega = 0$.

$$\therefore \quad 0 = 0 + C \quad \therefore \quad C = 0$$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s} \cdot \sin \omega s \, ds = \tan^{-1} \frac{\omega}{a}$$

Hence, $f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \cdot \tan^{-1} \frac{\omega}{a} \, d\omega$

Ex. 10 : Find Fourier Sine integral of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

(M.U. 1999)

Sol. : Fourier Sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s \cdot d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\int_0^1 s\sin\omega s\, ds + \int_1^2 (2-s)\sin\omega s\, ds + \int_2^\infty 0\cdot\sin\omega s\, ds\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left\{\left[s\left(-\frac{\cos\omega s}{\omega}\right)-\left(-\frac{\sin\omega s}{\omega^2}\right)(1)\right]_0^1\right.$$

$$\left.+\left[(2-s)\left(-\frac{\cos\omega s}{\omega}\right)-\left(-\frac{\sin\omega s}{\omega^2}\right)(-1)\right]_1^2\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left\{\left[-\frac{\cos\omega}{\omega}+\frac{\sin\omega}{\omega^2}\right]\right.$$

$$\left.+\left[0-\frac{\sin 2\omega}{\omega^2}+\frac{\cos\omega}{\omega}+\frac{\sin\omega}{\omega^2}\right]\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{(2\sin\omega-\sin 2\omega)}{\omega^2}\,d\omega$$

Ex. 11 : Find Fourier cosine integral for

$$f(x) = \begin{cases} 1-x^2, & 0\le x\le 1 \\ 0, & x>1 \end{cases}$$

Hence, evaluate $\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\cdot dx$. **(M.U. 2003)**

Sol. : Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty f(t)\cos\omega t\, dt\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty (1-t^2)\cos\omega t\, dt\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\left[(1-t^2)\frac{\sin\omega t}{\omega}-(-2t)\left(-\frac{\cos\omega t}{\omega^2}\right)\right.\right.$$

$$\left.\left.+(-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right]_0^1\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{-2\cdot\frac{\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right\}d\omega$$

$$1-x^2 = \frac{4}{\pi}\int_0^\infty \cos\omega x\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)d\omega$$

Now put $x = 1/2$,

$$\therefore \quad \frac{3\pi}{16} = \int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\cdot d\omega$$

EXERCISE - III

1. Express the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{for } x > 1 \end{cases}$$

as a Fourier Cosine Integral and hence, show that

$$\int_0^\infty \frac{\sin\omega\cdot\cos\omega x}{\omega}\,d\omega = \frac{\pi}{2} \quad \text{if } 0 \le x < 1$$

Also show that the integral is equal to $\pi/4$ for $x = 1$ and zero for $x > 1$.

$\left(\textbf{Hint :}\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^1 1\cdot\cos\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\cdot\sin\omega}{\omega}\,d\omega\right)$

2. Find the Fourier Integral representation of

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

$\left(\textbf{Hint :}\ f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty 0\,d\omega\,ds + \int_0^\infty\int_0^\infty e^{-s}\cos\omega(s-x)\,d\omega\,ds\right.$

$$= \frac{1}{\pi}\int_0^\infty\left\{\cos\omega x\int_0^\infty e^{-s}\cos\omega s\,ds + \sin\omega s\int_0^\infty e^{-s}\sin\omega s\,ds\right\}d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega$$

$\left.\text{when } x = 0,\ f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\,d\omega = \frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty = \frac{1}{2}\right)$

3. Express the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

as Fourier Sine Integral and evaluate

$$\int_0^\infty \frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\,d\omega$$

(M.U. 2000)

$$\left(\textbf{Hint : } f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty \sin s \sin \omega s \, d\omega \, ds \right.$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[-\frac{1}{2}\right]\int_0^\infty [\cos s(1+\omega) - \cos s(1-\omega)]\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left(-\frac{1}{2}\right)\cdot\left(-2\cdot\frac{\sin\pi\omega}{1-\omega^2}\right)d\omega$$

$$\left. = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\, d\omega \right)$$

4. Express the function

$$f(x) = e^{-x} - e^{-2x}, \quad x \ge 0$$

as Fourier Sine Integral and evaluate

$$\int_0^\infty \frac{\omega \sin\omega x}{(1+\omega^2)(4+\omega^2)}\, d\omega . \qquad \left[\textbf{Ans. : } \frac{\pi}{6}(e^{-x} - e^{-2x})\right]$$

5. Express the function $f(x) = e^{-x}$ as Fourier Sine integral ($x \ge 0$)

and show that $\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}\cdot e^{-x}$. **(M.U. 2006)**

6. Express $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x} & \text{for } x \ge 0 \end{cases}$

as a Fourier Integral and show that

$$\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{for } x < 0 \\ 1/2 & \text{for } x = 0 \\ \pi e^{-x} & \text{for } x > 0 \end{cases}$$

(**Hint :** Foe second result put $x = 0$ in the integral, then

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\, d\omega = \frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty = \frac{1}{2}.)$$

7. Express $f(x) = \frac{\pi}{2} e^{-x}\cos x$ for $x > 0$

as Fourier Sine integral and show that

$$\int_0^\infty \frac{\omega^3 \sin\omega x}{\omega^4 + 4}\, d\omega = \frac{\pi}{2} e^{-x}\cos x$$ **(M.U. 2002)**

(**Hint :** Use $2\sin\omega x\cos x = [\sin(\omega+1)x + \sin(\omega-1)x]$ and

$$\int e^{ax}\sin bx\, dx = \frac{1}{a^2+b^2} e^{ax}[a\sin bx - b\cos bx])$$

8. Express $f(x) = e^{-kx}$ $(k > 0)$

as Fourier Sine and Cosine Integral and show respectively that

(i) $\int_0^{\infty} \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-kx}$

(ii) $\int_0^{\infty} \frac{\omega \cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2k} e^{-kx}$ **(M.U. 2003, 08, 09)**

9. Express the following function as Fourier Integral

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{2}{\pi} \int_0^{\infty} \left(1 - \frac{2}{\omega^2} \sin \omega + \frac{2}{\omega} \cos \omega \right) \frac{\cos \omega x}{\omega} d\omega \right]$$

Theory

1. State Fourier Integral Theorem.
2. Define Fourier Sine and Cosine Integral . **(M.U. 1998, 2005)**
3. State Complex Fourier Series. **(M.U. 2004)**
4. Define orthogonal and orthonormal set of functions. **(M.U. 2008)**

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