Vicsek Model
Effect of Size of the Particles

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Abstract

The vicsek model is one of the most important models in flocking. However the model considers the elementsof the system to be point-like particles. The effects of particle size has been investigated in this work along with the verification of previous known results.

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Chapter 1

Introduction

Flocking refers to the collective motion often observed in birds and other specific animal groups. While considering flocking one of the most effective models till now has been the vicsek model.[1]

In this model we consider a certain number of point-like particles confined to move within a box having dimensions $L \times L$. All the particles have constant velocity amplitude v_0 throughout the motion. The direction at time t is indicated by the angle $\Theta(t)$. At each moment in time the direction of the particle is determined by the direction of the particles within a specific radius r (called the interaction radius). The updating rule is given,

$$\Theta(t+1) = \langle \Theta(t) \rangle_r + \xi \tag{1.1}$$

$$x(t+1) = x(t) + v_0 \Delta t \tag{1.2}$$

Where ξ is a random variable uniformly distributed in $\left[-\frac{\eta}{2}, \frac{\eta}{2}\right]$ where η is the noise strength. This naming is appropriate since ξ (and hence η) basically controls the deviation of the particle from the average direction of it's neighbours. It has been noted in the original paper by Vicsek[**?**] that the eventual outcome of the results depends on $\rho = \frac{N}{L^2}$ and η as long as $0.03 < v_0 < 0.3$

The particles initially have uniformly distributed positions and angles. As time passes the particles are seen to form clusters and align themselves in the same direction. This cluster forming tendency tends to be greater when the strength of the noise is smaller. This spontaneous alignment of the particles in a specific direction sees a phase transition as we change the strength of the noise.

Chapter 2

Order Parameter and it's Related Statistics

Most of the work seems to suggest that this phase transition is of second order. To understand this phase transition better we introduce the order parameter,

$$v_a = \frac{1}{N v_0} \left| \sum_{i=0}^{N} v_i \right| \tag{2.1}$$

Where v_a is the normalized average velocity of the N particles. It can be seen that for complete alignment $v_a = 1$ and for random motion $v_a = 0$. Hence it seems to give a quantitative measure for the alignment of all the particles in a specific direction. Here v_a is the order parameter of the system. The reason for this becomes clear when we plot v_a as a function of the noise for various values of N at a constant density ($\rho = 4$).

As we can see in the image above that for large values of N the plot becomes similar to the magnetization vs temperature curve for a magnet. Here we draw a similarity between the order parameters v_a and magnetization m and the noise strength η with the temperature T.

In the thermodynamic limit we have the following expressions,

$$v_a = (\eta - \eta_c)^{\beta} \qquad \eta < \eta_c \tag{2.2}$$

$$v_a = (\rho - \rho_c)^{\delta} \tag{2.3}$$

where β and δ are the critical exponents corresponding to the noise and density of the system respectively.

2.1 Susceptibility

The susceptibility of the system is defined as,

$$\chi = L^2(\langle\psi^2\rangle - \langle\psi\rangle^2) \tag{2.4}$$

Here $(\langle\psi^2\rangle - \langle\psi\rangle^2)$ is the variance of the order parameter. χ shows a maxima at $\eta = \eta_c$. Hence it provides an alternative for determining η_c. It is also a much more accurate method due to high variance of the order parameter around $\eta = \eta_c$.

2.2 Binder Cumulant

The fourth-order Binder cumulant is defined as,

$$C = 1 - \frac{\langle\psi^4\rangle}{\langle\psi^2\rangle^2} \tag{2.5}$$

The Binder cumulant is often used to classify phase transitions. The above expression is similar to the binder cumulant of the ising model in zero field. For $L \to \infty$ we have $C = 0$ for $\eta > \eta_C$ and $C = 2/3$ for $\eta < \eta_C$.

Chapter 3

Long Range Order

One of the most interesting features of the vicsek model is the existence of long range order (due to clustering) in spite of breaking continuous symmetry is able to display long range range order which contradicts Mermin Wagner theorem. Although the theorem merely holds true for equilibrium systems while the vicsek model deals with a non equilibrium system. This is due to fact that the particles are allowed to move. If it weren't then it would be analogous to the XY model in which there is no long range order. In the case of clustering we deal with effective dimension of the cluster.

To find the effective dimension of a cluster we compute the average path length(APL) - the average distance between all the pairwise particle path lengths. The effective dimension of the vicsek model was found out using the formula,

$$APL = n^{1/D} \tag{3.1}$$

in [**?**] to be around 4 where n is the number of particles in the cluster. This gives us a much better perspective of the model. It is in a way analogous to an equilibrium system in 4 dimensions. Hence the Mermin Wagner theorem does indeed hold true.

3.1 Frozen Clusters

Once the movement of particles is restricted the resulting clusters are known as frozen cluster. This is the case in which the non equilibrium system has reached steady state. Hence further rearrangement of the cluster disappears.

We start the motion from origin and start moving along the positive x direction. The updating rule is given by,

$$v_i(t+1) = \eta\xi_i(t) \tag{3.2}$$

The probability of the particle to have direction v is

$$P(v) = \frac{1}{2\pi\eta}U(\eta) \tag{3.3}$$

where $U(\eta)$ is random variable uniformly distributed in $[-\eta\pi, \eta\pi]$. Here we assume that the ne direction does not change with time.

$$x_i(t_n) = \sum_{k=0}^{n} \cos v_i(t_k) v_0 \Delta t \tag{3.4}$$

$$y_i(t_n) = \sum_{k=0}^{n} \sin v_i(t_k) v_0 \Delta t \tag{3.5}$$

We have the following results for the average diplacement,

$$\begin{aligned} \langle x(t_n)\rangle &= \int_{-\pi}^{\pi} P(v)x(t_n)dv \\ = \frac{v_0\Delta t}{2\pi\eta}\sum_{k=0}^{n}\int_{-\eta\pi}^{\eta\pi}&\cos(v(t_k))\,dv \\ &= \frac{v_0 n\Delta t}{\pi\eta}\sin(\eta\pi) \end{aligned} \tag{3.6}$$

$$\begin{aligned} \langle x(t_n)\rangle &= \int_{-\pi}^{\pi} P(v)x(t_n)dv \\ = \frac{v_0\Delta t}{2\pi\eta}\sum_{k=0}^{n}\int_{-\eta\pi}^{\eta\pi}&\sin(v(t_k))\,dv \\ &= 0 \end{aligned} \tag{3.7}$$

Assuming that in the mean field limit the displacements were equal in each time interval we get,

$$\psi = \frac{x(t_n)}{n\Delta t v_0} = \frac{\sin(\pi\eta)}{\pi\eta} \tag{3.8}$$

Chapter 4

Effect of Particle Size

Till now we have derived the results for point like particles. We now consider particles with finite size. We perform simulations to derive the results for this case. Care must be taken when defining the particle size since we do not wish it to be larger than the interaction radius.

Within the algorithm we follow a specific sequence of updating in which the i^{th} particle observes the position of particle j at time $t+\Delta t$ (where $j < i$) before deciding to make make the move. If they happen to collide at that time then the move is not made. Similarly for $j > i$, the position of the particle j is considered at time t since it is yet to make a move. The updating rule becomes,

$$x_i(t+\Delta t) = x_i(t) \qquad \text{if} \qquad \begin{cases} |x_i(t+\Delta t) - x_j(t+\Delta t)| < 2r, & \text{for } j < i \\ |x_i(t+\Delta t) - x_j(t)| < 2r, & \text{for } j > i \end{cases} \tag{4.1}$$

In the case if such an updating rule we see an interesting result for the susceptibility. We find that the peak of χ increases with increasing radius. At around $r = 0.25$ this increase is remarkably larger than any other value of radius. In other words the change in η_c is large around $r_c = 0.25$

Bibliography

[1] Albert Einstein. Zur Elektrodynamik bewegter Körper. (German) [On the electrodynamics of moving bodies]. *Annalen der Physik*, 322(10):891–921, 1905.