Vicsek Model
Effect of Size of the Particles

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Abstract

The vicsek model is one of the most important models in flocking. However the model considers the elementsof the system to be point-like particles. The effects of particle size has been investigated in this work along with the verification of previous known results.

Contents

Chapter 1

Introduction

Flocking refers to the collective motion often observed in birds and other specific animal groups. It is an emergent phenomenon in which the particles move in somewhat random direction alone but while in a group tend to move in the same direction as the group. It is the characteristic motion of active particles or self-propelling particles- particles which have internal energy and is capable of using it propel itself. Active systems is of interest in the field of statistical physics since it offers a practical non equilibrium system which has statistical properties similar to spin-lattice systems which have been studied extensively throughout the years.

While considering flocking one of the most effective models till now has been the Vicsek model.The Vicsek model considers all the elements of the system to point-like particles moving in a confined space according to a specific updating rule. The results from this model gives us an idea of how a phase transition from a disordered to an ordered state takes place albeit doesn't give a well enough reason as to why it happens. Particles tend to form clusters and move together. Several attempts have been made to understand why this long range order exists in the first place. They have been highlighted in Chapter 3.

Finally it is essential for us to consider the particle as hard spheres since that would give a much more realistic model for flocking. Point-like particle assumption might hold true if the particle size is small compared to the inter-particle distance . However that is not the case in most of the practical model. Hence it is necessary for us to consider a hard sphere model in which the particles have a finite size. Results from that have been considered in Chapter 4. [1]

1.1 Standard Vicsek Model

In this model we consider a certain number of point-like particles confined to move within a box having dimensions $L \times L$. All the particles have constant velocity amplitude v_0 throughout the motion. The direction at time t is indicated by the angle $\Theta(t)$. At each moment in time the direction of the particle is determined by the direction of the particles within a specific radius r (called the interaction radius). The particle tends to align itself to the average direction of motion of it's neighbours (within interaction radius r). There is also an additional noise which tends to disrupt the direction of the particles. The updating rule is given by,

$$\Theta(t+1) = \langle \Theta(t) \rangle_r + \xi \tag{1.1}$$

$$x(t+1) = x(t) + v_0 \Delta t \tag{1.2}$$

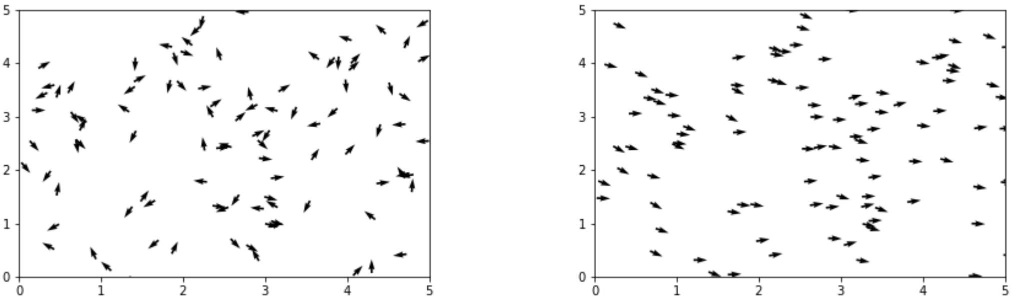
Where ξ is a random variable uniformly distributed in $\left[-\frac{\eta}{2}, \frac{\eta}{2}\right]$ where η is the noise strength. This naming is appropriate since ξ (and hence η) basically controls the deviation of the particle from the average direction of it's neighbours. It has been noted in the original paper by Vicsek[**?**] that the eventual outcome of the results depends on $\rho = \frac{N}{L^2}$ and η as long as $0.03 < v_0 < 0.3$

The particles initially have uniformly distributed positions and angles. As time passes the particles are seen to form clusters and align themselves in the same direction. This cluster forming tendency tends to be greater when the strength of the noise is smaller. This spontaneous alignment of the particles in a specific direction sees a phase transition as we change the strength of the noise.

1.2 Simulation Results of the Vicsek Model

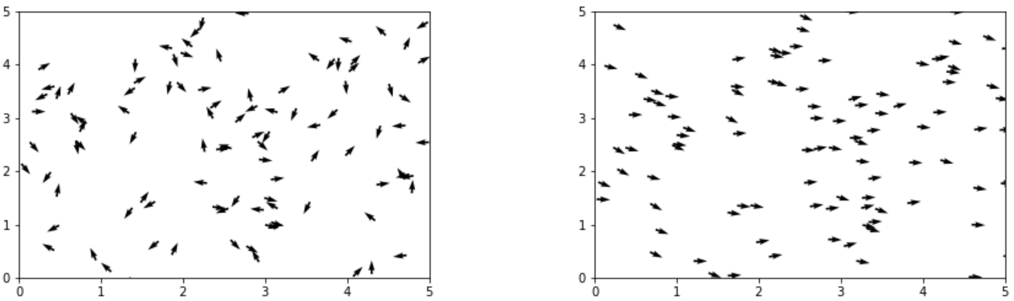
The algorithm for the Vicsek Model has been defined in Appendix A.

We see from the above images that in steady state for low values of η there is a larger tendency for collective motion among the particles than for large values of η. This seemingly intuitive fact has been verified by performing simulations for $N = 100$ particles confined to a box of each side $L = 5$.



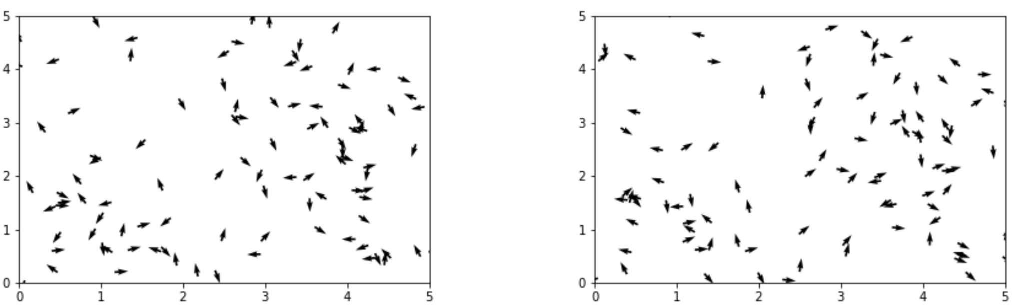
(a) (a)Initially (b) (b)At t=9s

Figure 1.1: $v_0 = 0.03\,, \eta = 0.1$



(a) (a)Initially (b) (b)At t=9s

Figure 1.2: $v_0 = 0.03\,, \eta = 0.3$



(a) (a)Initially (b) (b)At t=9s

Figure 1.3: $v_0 = 0.03\,, \eta = 0.8$

Chapter 2

Order Parameter and it's Related Statistics

Most of the work seems to suggest that this phase transition is of second order. To understand this phase transition better we introduce the order parameter,

$$v_a = \frac{1}{N v_0} \left| \sum_{i=0}^{N} v_i \right| \tag{2.1}$$

Where v_a is the normalized average velocity of the N particles. It can be seen that for complete alignment $v_a = 1$ and for random motion $v_a = 0$. Hence it seems to give a quantitative measure for the alignment of all the particles in a specific direction. Here v_a is the order parameter of the system. The reason for this becomes clear when we plot v_a as a function of the noise for various values of N at a constant density ($\rho = 4$).

As we can see in the image above that for large values of N the plot becomes similar to the magnetization vs temperature curve for a magnet. Here we draw a similarity between the order parameters v_a and magnetization m and the noise strength η with the temperature T.

In the thermodynamic limit we have the following expressions,

$$v_a = (\eta - \eta_c)^{\beta} \qquad \eta < \eta_c \tag{2.2}$$

$$v_a = (\rho - \rho_c)^{\delta} \tag{2.3}$$

where β and δ are the critical exponents corresponding to the noise and density of the system respectively.

2.1 Susceptibility

The susceptibility of the system is defined as,

$$\chi = L^2(\langle\psi^2\rangle - \langle\psi\rangle^2) \tag{2.4}$$

Here $(\langle\psi^2\rangle - \langle\psi\rangle^2)$ is the variance of the order parameter. χ shows a maxima at $\eta = \eta_c$. Hence it provides an alternative for determining η_c. It is also a much more accurate method due to high variance of the order parameter around $\eta = \eta_c$.

2.2 Binder Cumulant

The fourth-order Binder cumulant is defined as,

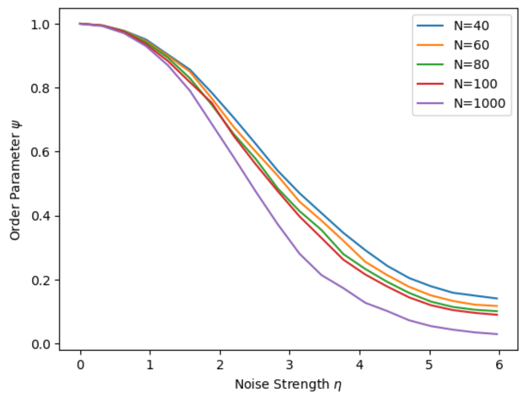
$$C = 1 - \frac{\langle\psi^4\rangle}{\langle\psi^2\rangle^2} \tag{2.5}$$

The Binder cumulant is often used to classify phase transitions. The above expression is similar to the binder cumulant of the ising model in zero field. For $L \to \infty$ we have $C = 0$ for $\eta > \eta_C$ and $C = 2/3$ for $\eta < \eta_C$.

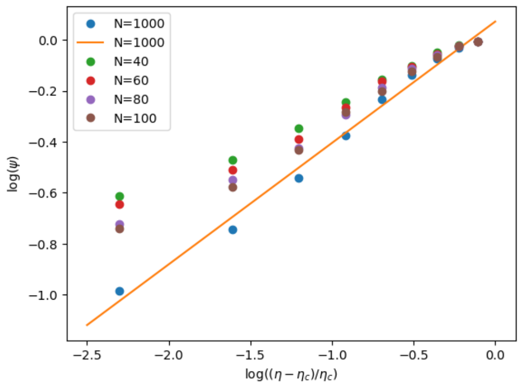
2.3 Determining the Critical Exponents

While determining the critical exponent β we need to make sure that N is sufficiently large since the phase transition is more prominent for larger number of particles. We begin by finding η_c by locating the peak in the susceptibility against η graph. From the plot β we find that $\beta = 0.4768$ which is within the range given by vicsek [?].

Similarly on plotting the order parameter against the density ρ we find a similar graph which shows a phase transition around ρ_c. The value of δ has been computed to be 0.3905 which is again in the required limits.

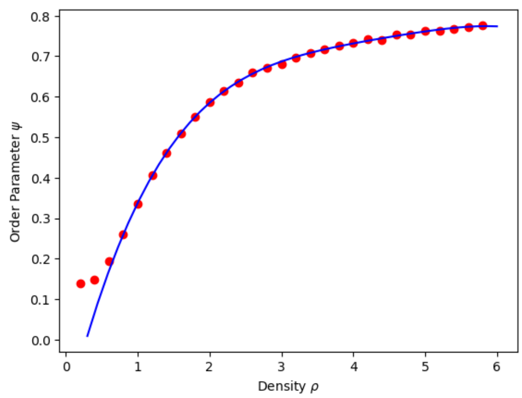


(a) Order Parameter for varying noise strengths

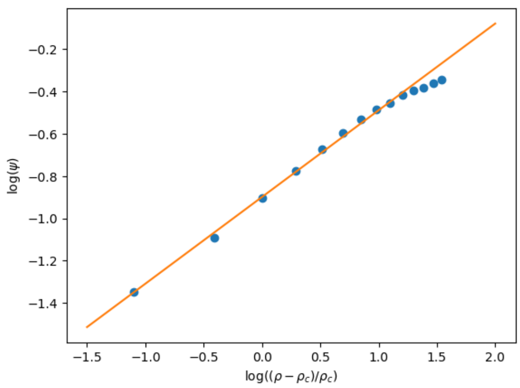


(b) Log-log plot of order parameter against noise

Figure 2.1: Determining β using $N = 1000$



(a) Order Parameter for varying densities



(b) Log-log plot of order parameter against density

Figure 2.2: Determining β using $N = 1000$

Chapter 3

Finite Size Effects

Finite size scaling is one of the most useful methods in determining the critical exponents in a critical phenomenon. From finite size scaling we have the following relations,

$$\psi_L(\eta) = L^{-\beta/\nu}\psi_{\pm}(L^{1/\nu}\epsilon) \tag{3.1}$$

$$\chi_L(\eta) = L^{\gamma/\nu}\chi_{\pm}(L^{1/\nu}\epsilon) \tag{3.2}$$

$$C_L(\eta) = C_{\pm}(L^{1/\nu}\epsilon) \tag{3.3}$$

where $\epsilon = (\eta - \eta_c)/\eta_c$ and the $\pm$ sign refers to whether $\eta > \eta_c$ or $\eta < \eta_c$

From the log-log plots of $C_L(\eta_c), \psi_L(\eta_c)$ and $\chi_L(\eta_c)$ we find the critical exponents β, γ, ν.

From the plots we find that $\beta/\nu = 3.12$

Chapter 4

Long Range Order

One of the most interesting features of the vicsek model is the existence of long range order (due to clustering) in spite of breaking continuous symmetry is able to display long range range order which contradicts Mermin Wagner theorem. Although the theorem merely holds true for equilibrium systems while the vicsek model deals with a non equilibrium system. This is due to fact that the particles are allowed to move. If it weren't then it would be analogous to the XY model in which there is no long range order. In the case of clustering we deal with effective dimension of the cluster.

To find the effective dimension of a cluster we compute the average path length(APL) - the average distance between all the pairwise particle path lengths. The effective dimension of the vicsek model was found out using the formula,

$$APL = n^{1/D} \tag{4.1}$$

in [?] to be around 4 where n is the number of particles in the cluster. This gives us a much better perspective of the model. It is in a way analogous to an equilibrium system in 4 dimensions. Hence the Mermin Wagner theorem does indeed hold true.

4.1 Frozen Clusters

Once the movement of particles is restricted the resulting clusters are known as frozen cluster. This is the case in which the non equilibrium system has reached steady state. Hence further rearrangement of the cluster disappears.

We start the motion from origin and start moving along the positive x direction. The updating rule is given by,

$$v_i(t+1) = \eta\xi_i(t) \tag{4.2}$$

The probability of the particle to have direction v is

$$P(v) = \frac{1}{2\pi\eta} U(\eta) \tag{4.3}$$

where $U(\eta)$ is random variable uniformly distributed in $[-\eta\pi, \eta\pi]$. Here we assume that the ne direction does not change with time.

$$x_i(t_n) = \sum_{k=0}^{n} \cos v_i(t_k) v_0 \Delta t \tag{4.4}$$

$$y_i(t_n) = \sum_{k=0}^{n} \sin v_i(t_k) v_0 \Delta t \tag{4.5}$$

We have the following results for the average displacement,

$$\begin{aligned} \langle x(t_n) \rangle &= \int_{-\pi}^{\pi} P(v) x(t_n) dv \\ = \frac{v_0 \Delta t}{2\pi\eta} \sum_{k=0}^{n} &\int_{-\eta\pi}^{\eta\pi} \cos(v(t_k))\, dv \\ &= \frac{v_0 n \Delta t}{\pi\eta} \sin(\eta\pi) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \langle x(t_n) \rangle &= \int_{-\pi}^{\pi} P(v) x(t_n) dv \\ = \frac{v_0 \Delta t}{2\pi\eta} \sum_{k=0}^{n} &\int_{-\eta\pi}^{\eta\pi} \sin(v(t_k))\, dv \\ &= 0 \end{aligned} \tag{4.7}$$

Assuming that in the mean field limit the displacements were equal in each time interval we get,

$$\psi = \frac{x(t_n)}{n \Delta t v_0} = \frac{\sin(\pi\eta)}{\pi\eta} \tag{4.8}$$

4.2 Continuum Theory

The polar ordered phase was explained by Toner and Tu by describing the hydrodynamic behaviour of the particles with the help of a continuum model which helped explain this behaviour. By modeling the continuum equations of motion they were able to calculate the scaling exponents of the order parameter. This model also predicted broken continuous symmetry for 2-dimensional Vicsek model.

However we can apply this model only in the large scale model (where N is very large). This is appropriate since the phase transition is mostly visible for this case as well. In hydrodynamics rather than computing the position of all the particles based on the intermolecular forces (which would be very hard analytically) we deal with the Navier-Stokes equations. In this limit we consider the coarse grained density ρ and average velocity $\vec{v}$.

Chapter 5

Effect of Particle Size

Till now we have derived the results for point like particles. We now consider particles with finite size. Collective motion of finite-size objects have already been analyzed by *Peruani*. In the work it has already mentioned the phase transition from rotational symmetry to polar order in the presence of active rods. They showed that in the case that the rods are squares the particles tend not to form clusters.

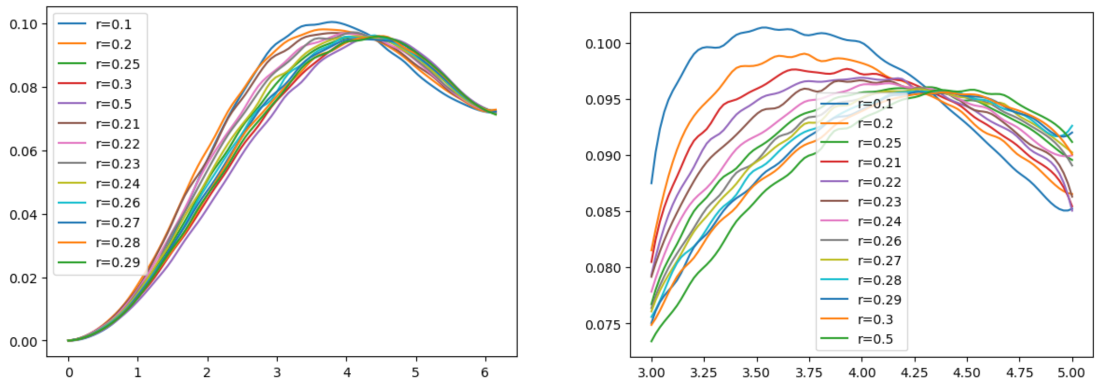
In this work we look at the case in which the updating rule is that given by Vicsek. We perform simulations to derive the results for this case. Care must be taken when defining the particle size since we do not wish it to be larger than the interaction radius.

Within the algorithm we follow a specific sequence of updating in which the i^{th} particle observes the position of particle j at time $t+\Delta t$ (where $j < i$) before deciding to make make the move. If they happen to collide at that time then the move is not made. Similarly for $j > i$, the position of the particle j is considered at time t since it is yet to make a move. The updating rule becomes,

$$x_i(t+\Delta t) = x_i(t) \qquad \text{if} \qquad \begin{cases} |x_i(t+\Delta t) - x_j(t+\Delta t)| < 2r, & \text{for } j < i \\ |x_i(t+\Delta t) - x_j(t)| < 2r, & \text{for } j > i \end{cases} \tag{5.1}$$

Previous works have shown that increasing the density of the particles has the effect of increasing the value of critical noise (η_c) In the case if such an updating rule we see an interesting result for the susceptibility. We find that the peak of χ increases with increasing radius. At around $r = 0.25$ this increase is remarkably larger than any other value of radius. In other words the change in η_c is large around $r_c = 0.25$

The fit on the second plot was made to a hyperbolic tangent curve. However there is no specific reason for doing except that it highlights the rapid change at $r_c = 0.25$.

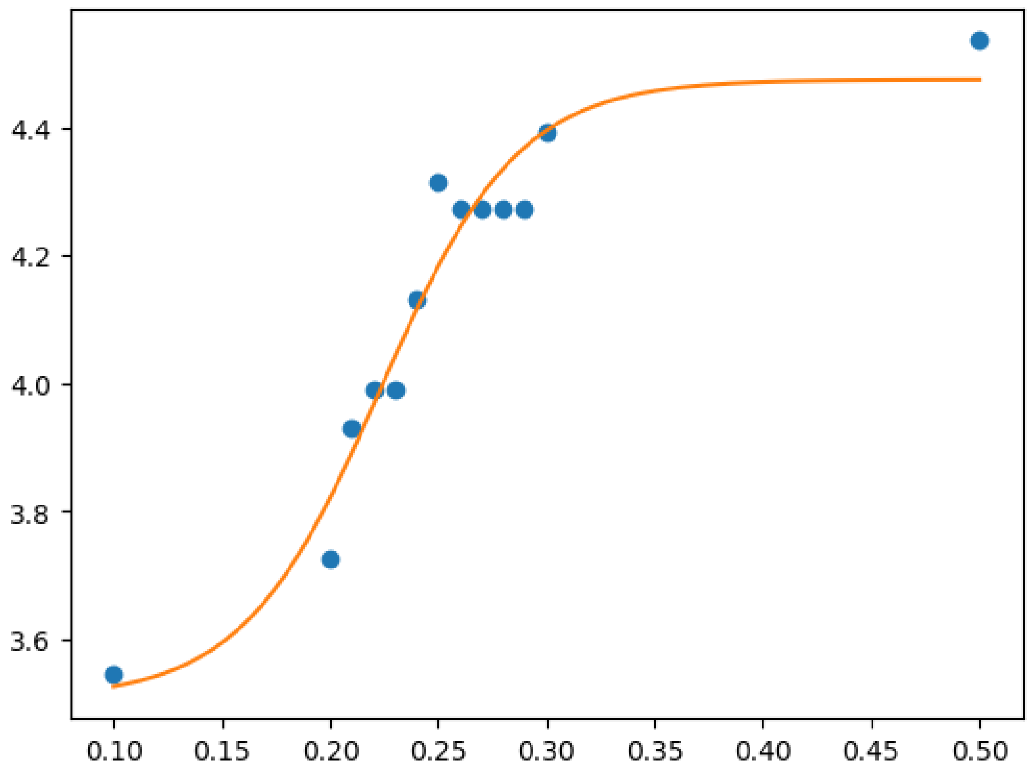


(a) Susceptibility for various densities

(b) Susceptibility zoomed in

Figure 5.1: Effect of radius on the susceptibility

Figure 5.2: Changes in η_c for varying radius



Appendix A

Algorithmic Implementation

Simulating the vicsek model can be a computationally demanding task and care must be taken to write the code so as to reduce to time complexity as far as possible. Here are a couple of ways that have been/can be implemented to make the code run faster:

1. One of the most important things to remember is that once non-equilibrium steady state is reached the probability distribution of the order parameter becomes fixed. Hence rather than creating multiple instances of the system initial condition and waiting for it to reach steady state to extract one data point we can extract multiple data points after this instance of time.

2. One performing the simulations we find that the time taken to reach steady state usually varies as we change the noise strength. Statistical properties of the order parameter like it's expected value at steady state for small noise strengths can be estimated using a much lesser number of runs as compared to larger noise strengths. Hence there seems to be no apparent reason why we should expect each simulation to run the same number of times irrespective of of noise strength. For this reason the code has been so that a base number of runs is always performed. On top of that we check the average result of a group of runs (say r runs) and compare it with the previous group of r runs. If the difference happens to be less than ϵ then we say that steady state has been achieved and the average result of the previous r runs is good enough to give us an estimate of the desired statistical property within specific error bounds.

3. Mersenne Twister Pseudo Random Number Generator happens to be one of the best methods when it comes to generating random variables for the Vicsek model. This is due to the fact that it is faster than most of the other random number generators and it has a very large period implying that the chances for

generating statistically correlated random numbers happens to be very less as compared to other methods.

4. Another way to reduce the time complexity is to divide the box into grids of size $r \times r$ each and associate the particles to each box. So whenever we are to look up all the particles within an interaction radius r we simply need to consider all the particles in the current box and all the surrounding boxes. Hence we need to consider all the particles only in these 9 boxes. This reduces the time taken by an around $LxL/9$. Depending on the box dimensions this could be a drastic improvement on the speed of the code.

5. While finding the peak of χ we use the golden section method. This makes it computationally much less intensive since at each step we are getting exponentially closer to the maxima while at the same time having to compute minimal number of points. The golden section method has been described in [**?**]

Appendix B

Golden Section Search

This method is optimal for finding the extremum of an unimodal function in the case where no special property about the function apart from that it is unimodal and continuous is known about it.

We start with 3 points x_1, x_2 and x_3 ($x_1 < x_2 < x_3$) such that $f_2 > f_1$ and $f_2 > f_3$ and consequently find the function at x_4 which lies in this range. Depending on whether f_4 is greater or lesser than f_2 we have the following 2 cases:

1. $f_{4a} > f_2$: In this case we want to change our set of points to x_1, x_2 and x_4.
2. $f_{4a} < f_2$: In this case we want to change our set of points to x_2, x_4 and x_3.

We consider,

$$\frac{c}{a} = \frac{a}{b} \rightarrow \frac{c}{b-c} = \frac{a}{b} \rightarrow \frac{a}{b} = \phi \tag{B.1}$$

Where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Bibliography

[1] Albert Einstein. Zur Elektrodynamik bewegter Körper. (German) [On the electrodynamics of moving bodies]. *Annalen der Physik*, 322(10):891–921, 1905.