# Geiger Hardware + Some Math

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## 1 Introduction

This document provides documentation for the hardware used in the PHYS-339 Measurements Lab Statistics Experiment.

## 2 Radio Isotopes

There are two radio isotopes which are used in this experiment, cesium-137 ($^{137}_{55}$Cs), which has a half-life of 30.1 years, and barium-133 ($^{133}_{56}$Ba), with a half-life of 10.5 years.

### 2.1 Half-life

Each radio-isotope nucleus has some small probability, $\lambda$ of decay per unit time. Thus the number of atoms decaying per unit time will be the product of this probability with the total number of atoms, since each decay reduces the number of original atoms,

$$\frac{d}{dt}N = -\lambda N \tag{1}$$

This diffential equation will essential solve itself to yield

$$N = N_0 e^{-\lambda t} \tag{2}$$

The half-life is defined as the time required for 50% of the original atoms to have decayed, so

$$e^{-\lambda t_{1/2}} = \frac{1}{2} \tag{3}$$

Since we already know the half-life, we can determine the probability, $\lambda$ of decay per atom per unit time.

$$\lambda = -\frac{\ln \frac{1}{2}}{t_{1/2}} \tag{4}$$

So in the case of the cesium-137, probability of each atom of decaying is $7.46 \times 10^{-10}$ s$^{-1}$.

## 2.2 Activity

Activity is defined as the number of decays per unit time, units for activity include the SI unit, the becquerel (Bq) and the curie (Ci) which everyone really uses. 1 Bq is defined as 1 decay per second, while 1 Ci is defined as $3.7 \times 10^{10}$ decays per second. The sources used in this experiment were, at the time of purchase, either $5\ \mu$Ci or $7.5\ \mu$Ci.

Knowing the probability, $\lambda$, of decay, we can easily calculate how much radio-active material is required to provide a required activity. For the $5\ \mu$Ci cesium-137 sample,

$$N = 5\ \mu\text{Ci}/7.46 \times 10^{-10} = 2.5 \times 10^{14} \tag{5}$$

This is a rather miniscule amount, less than $0.06\ \mu$g of pure cesium-137. For this reason, it is convenient for our purposes, that the cesium is encased in a relatively large plastic disk which is much harder to accidentally inhale.

## 2.3 Nuclear Decay Radiation

Unstable isotopes are those which can reach a lower energy state via reorganizing the constituent components of the nucleus in to a different configuration. Sometimes this requires an external particle, but usually results in the ejection of a particle. The following are the most common types of ejected particle.

- Alpha ($\alpha$) : A helium nucleus, $^4_2$He, is ejected. Due to its large size this radiation interacts very strongly with matter. A common example of this is

$$^{241}_{95}\text{Am} \rightarrow ^{237}_{93}\text{Np} + ^{4}_{2}\text{He} + 5.54\,\text{MeV} \tag{6}$$

- Beta- ($\beta^-$) : When a neutron in the nucleus decays into a proton, an electron and an electron antineutrino are ejected. The antineutrino does not observably interact with matter, but the electron does, although less strongly than $\alpha$ particles. A common example of this radiation is

$$^{137}_{55}\text{Cs} \rightarrow^{137}_{56} \text{Ba} + e^- + \bar{\nu}_e + 1.174\,\text{MeV} \tag{7}$$

- Beta+ ($\beta^+$) : When a proton in the nucleus decays into a neutron, a positron and a neutrino are ejected, the neutrino does not observably interact with matter, but the positron, being the antiparticle of the electron, quickly finds an electron and they annihilate converting all mass into a characteristic 1.2 MeV $\gamma$. A common example of this decay is

$$^{22}_{11}\text{Na} \rightarrow^{22}_{10} \text{Ne} + e^+ + \nu_e + 1.83\,\text{MeV} \tag{8}$$

- Gamma ($\gamma$) : Analogous to the excited states of electrons, the nucleus has excited states, transitions between these states can occur via emission or capture of photons. These photons are usually very energetic and thus act more like particles than waves. Spontaneous $\gamma$ production is usually the result of a previous nuclear decay which has left a nucleus in an excited state.

- Neutron : Emission of a neutron from a nucleus is not a very common mechanism, neutrons are released by spontaneous fission of for example $^{235}_{92}$U, but in this case the original nucleus is broken into several daughter nuclei. A common method of producing neutrons is by activating beryllium

$$^{9}_{4}\text{Be} +^{4}_{2} \text{He} \rightarrow^{13}_{6} \text{C}^* \rightarrow^{12}_{6} \text{C} + n \tag{9}$$

  Where $^{13}_{6}$C$^*$ indicates that the isotope nucleus is in an excited state from the impact of the $\alpha$-particle. $^{13}_{6}$C with a non-excited nucleus is a stable isotope.

## 2.4 Cesium-137 Decay Scheme

$^{137}_{55}$Cs decays via $\beta^-$ emission into $^{137}_{56}$Ba. The energy liberated by this decay is $1.174$ MeV. This energy will be distributed among the fission products, in this case the $^{137}_{56}$Ba nucleus and an electron. The nucleus may be left in an excited state, the balance of energy is divided between the nucleus and the electron as kinetic energy such that momentum is conserved. Conservation of momentum requires

$$\gamma_n m_n v_n = -\gamma_e m_e v_e, \tag{10}$$

where $\gamma_n$ and $\gamma_e$ are the *Lorentz factors* for the nucleus and electron respectively, $m_n$ and $m_e$ the masses and $v_n$ and $v_e$ the velocities. Conservation of energy requires that

$$(\gamma_n - 1)\, m_n c^2 + (\gamma_e - 1)\, m_e c^2 = K, \tag{11}$$

where $K$ is the total kinetic energy of the system after decay. Since the Lorentz factor is defined as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{12}$$

eq(10) can be rewritten

$$\gamma_n^2 m_n^2 \left(1 - \frac{1}{\gamma_n^2}\right) c^2 = \gamma_e^2 m_e^2 \left(1 - \frac{1}{\gamma_e^2}\right) c^2, \tag{13}$$

so long as we remember that the velocity vectors will be in opposite directions. Eq(13) can be solved for $\gamma_n$,

$$\gamma_n = \frac{\sqrt{m_n^2 + m_e^2 \left(\gamma_e^2 - 1\right)}}{m_n} \tag{14}$$

Substituting into eq(11) and solving for $\gamma_e$ yields

$$\gamma_e = \frac{2(m_e c^2)^2 + 2(m_n c^2)(m_e c^2) + 2(m_n c^2 + m_e c^2)K + K^2}{2(m_e c^2)(m_n c^2 + m_e c^2 + K)}. \tag{15}$$

The rest mass energy of the electron is well known to be $511$ keV, while the ress mass energy of ${}^{137}_{56}$Ba is $127$ GeV. Assuming the resulting nucleus is in the ground state, that is $K = 1.174$ MeV, then $\gamma_e = 3.297$. Thus the kinetic energy contribution from the electron is

$$K_e = (\gamma_e - 1)\, m_e c^2 = 1.174\,\text{MeV}, \tag{16}$$

while the kinetic energy contribution from the nucleus is

$$K_n = (\gamma_n - 1)\, m_n c^2 = 10.1\ \text{eV}. \tag{17}$$

While $10.1$ eV is large compared with thermal energy ($\frac{3}{2}kT = 39$ meV), it is clear that almost all the energy is carried by the electron.

Analysis of the energy of $\beta$-radiation from ${}^{137}_{55}$Cs actually shows only 5.4% of electrons have $1.174$ MeV, the remaining 94.6% have only $512$ keV, meaning the ${}^{137}_{56}$Ba nucleus has been left in an excited state. Eventually the excited state will decay down to the ground state by emitting a photon, in this case, a $662$ keV $\gamma$-ray. Figure(1) shows this process in the format used by Lederer & Shirley[1]. Gaze in awe at the density of information contained in the figure, nuclear spin, parity, half-life of states, decay mechanism, branching ratios, allowed transitions between excited states and $\gamma$ energies emitted. Then, consider that there are several of these diagrams on almost each of the more than 1500 pages of this reference.

So, according to the activity of the source when purchased, the source should have produced six-thousand seven-hundred $1.174$ MeV $\beta$s, one-hundred and eighty $512$ keV $\beta$s and one-hundred and eighty $662$ keV $\gamma$s per second. It may have shielding to reduce the energies of $\beta$ below the threshold of the geiger counter.

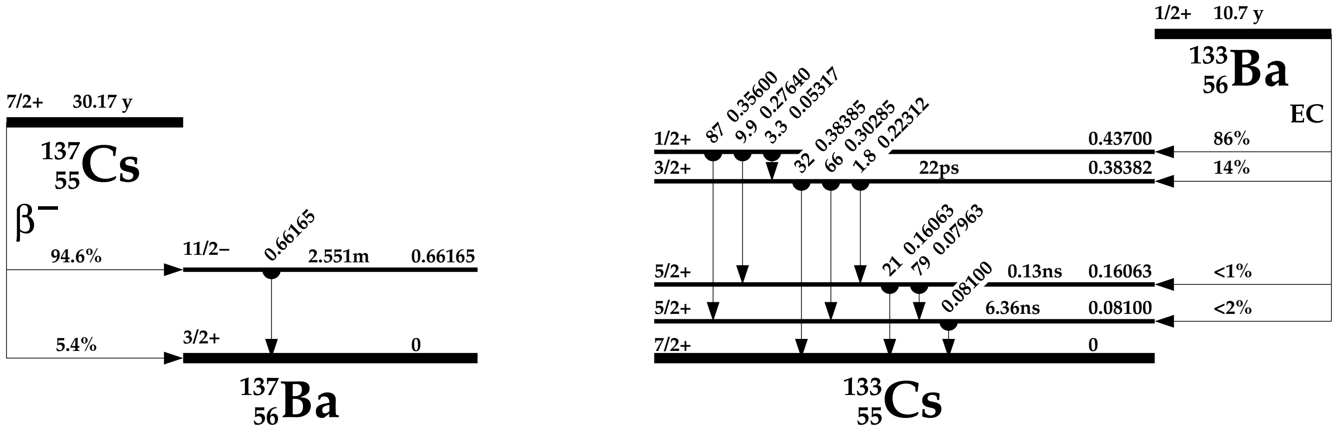


Figure 1: Decay schemes of cesium-137 and barium-133, adapted from Lederer & Shirley[1]

### 2.5 Barium-133 Decay Scheme

As can be seen in figure(1), barium-133 decays into cesium-133 via electron capture. This is where the nucleus captures one of the electrons from an inner shell to convert a proton into a neutron. Other than the neutrino, nothing is ejected from the nucleus, although the nucleus is left in an excited state. The excited states of the cesium-133 nucleus decay to the ground state via a variety of $\gamma$ emissions.

### 2.6 $\alpha$ and $\beta-$ Interactions with Matter

Since $\alpha$ and $\beta^-$ radiation are charged particles, they can easily lose energy via Coulomb interations.

$$\text{Stopping Power} = \frac{d}{dx}E(x) \tag{18}$$

where $E(x)$ is the energy of the particle after travelling distance $x$ through the material.[1]

The stopping power of neon for various energies of $\alpha$ and $\beta$ radiation are shown in figure(2). It should be apparent that there will be some characteristic depth at which these particles lose all their energies, this is called the range. Figure(3) shows the energy of these charged particles, each with a starting energy of 1 MeV, as they pass through neon gas at 10 kPa (1/10 atmospheric pressure), which will be of interest when discussing the Geiger-Müller tube. Thus in this gas we can see 1 MeV $\alpha$-particles have a range of 11 cm, while 1 MeV electrons have a range of 5330 cm.

[1]This is a measurement you will make in the PHYS-439 $\alpha$-decay experiment.

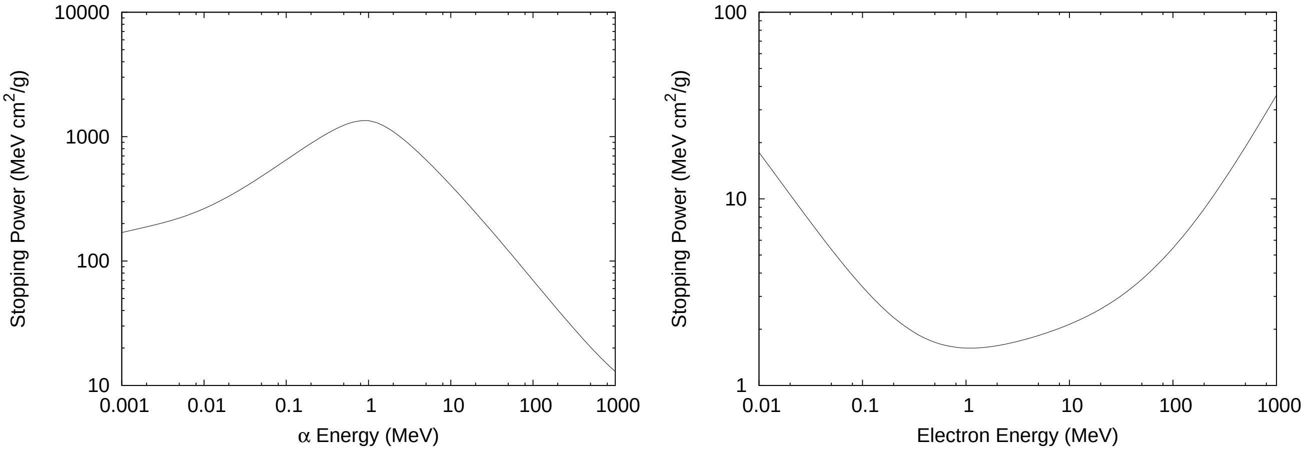


Figure 2: Stopping power of neon for $\alpha$-particles and electrons.[2]

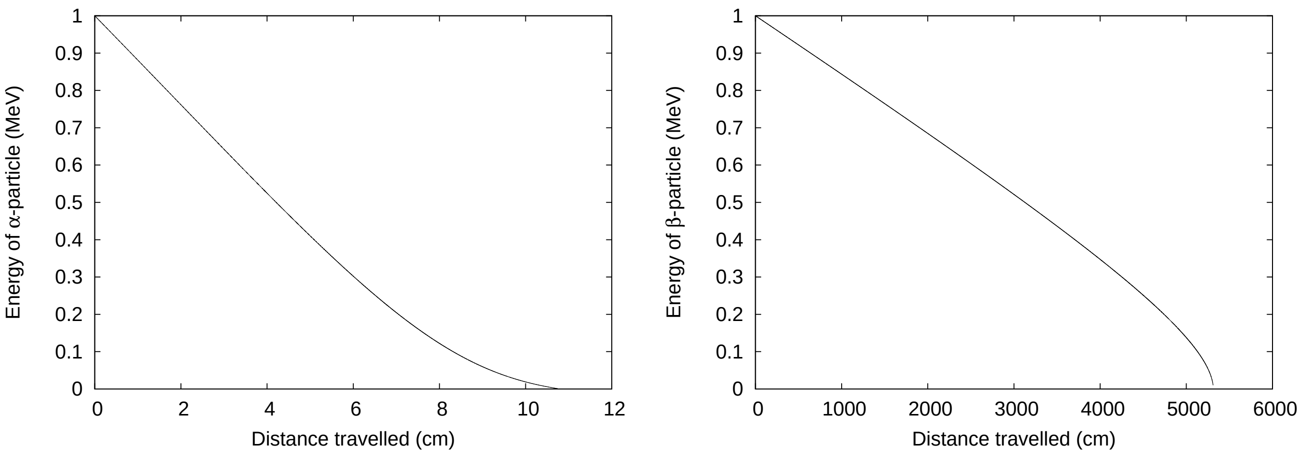


Figure 3: Energy of $\alpha$-particles and electrons as a function of depth of penetration into neon gas at a pressure of 10 kPa.

The property that energy loss for electrons increases as the energy of the electron decreases is make $\beta$ radiation very useful for radiation therapy. Figure(4) shows the stopping power for a150 tissue equivalent plastic, this is a substance designed to be very similar to human tissue for radiological purposes. Using the stopping power data, the energy as a function of depth, $E(x)$ can be calculated. $-\frac{d}{dx}E(x)$ will give the rate at which energy is being depostited into the tissue. As can be seen, the energy of the radiation selects the depth at which energy deposition is maximized.

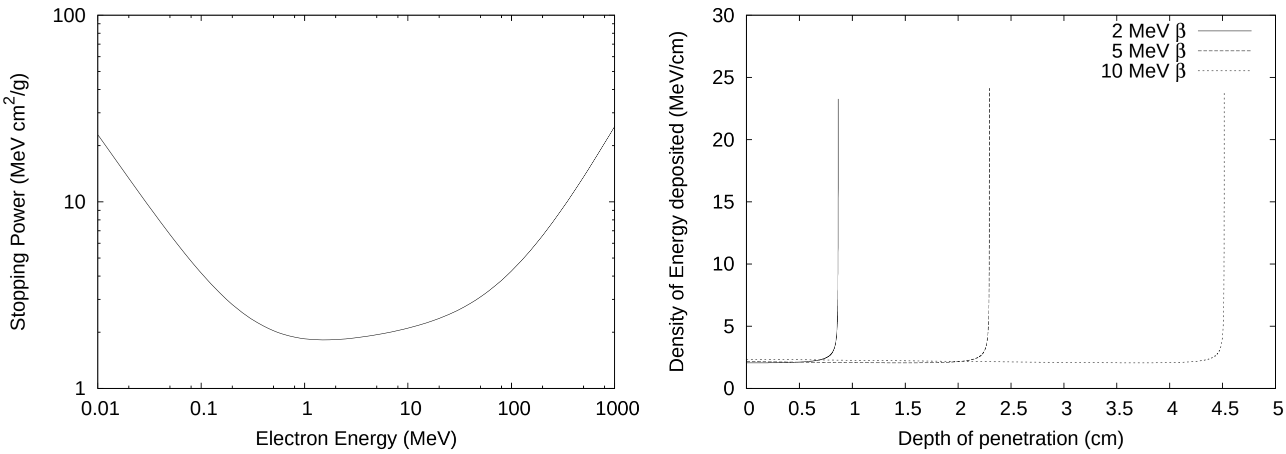


Figure 4: Stopping power of tissue equivalent a150 plastic for electrons, and the resulting energy deposition profiles for different energy electrons.

#### 2.6.1 $\gamma$ Interactions with Matter

Although a high energy photon has properties similar to a particle, it can not lose energy through action at a distance, although it is possible to bounce photons off free or lightly bound electrons thus yielding small energy loss to the photon which is a function of the angle of deflection (Compton Effect), most interaction happen with more tightly bound electrons which effectively make the photon disappear.

For a beam having intensity $I$ of monoenergetic $\gamma$ photons which pass through a layer of material having thickness $t$ and density $\rho$, the intensity of the resulting beam will be[3]

$$I = I_0 e^{-\frac{\mu}{\rho}\rho t}, \tag{19}$$

where $\frac{\mu}{\rho}$ is called the *mass attenuation coefficient*. Figure(5) shows the energy dependence of $\frac{\mu}{\rho}$ values for neon. We can use the mass attenuation coefficient and eq(19) to calculate the probability of a photon being captured while travelling through a given thickness of material,

$$P = 1 - e^{-\frac{\mu}{\rho}\rho t}. \tag{20}$$

An example of this probability is shown in figure(5) where the material is a layer of neon gas, 1 cm thick, at a pressure of 10 kPa at room temperature. Notice how the probability of capture increases with decreasing photon energy.

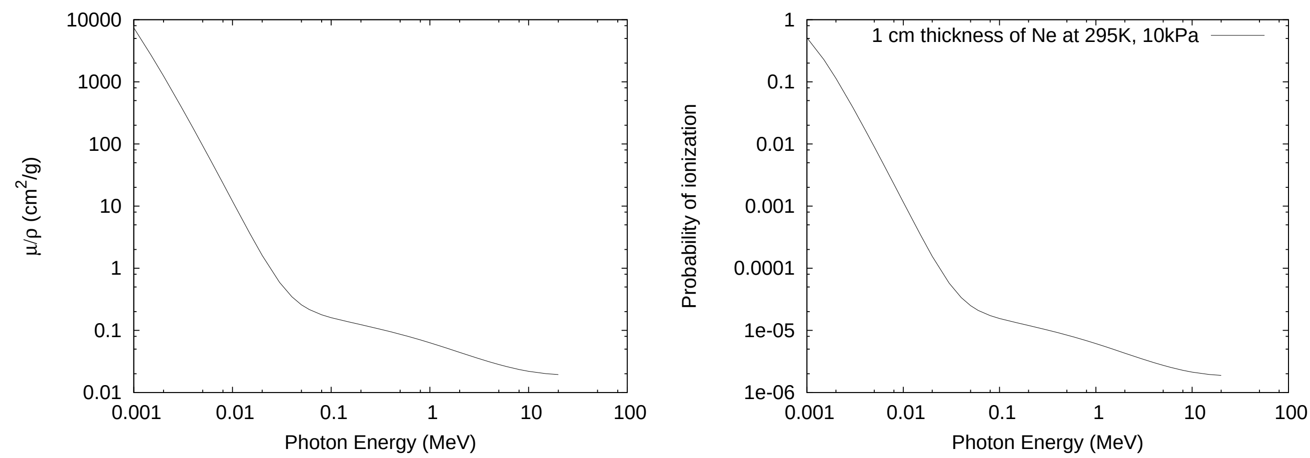


Figure 5: Mass Attenuation Coefficient, $\mu/\rho$, of neon as a function of photon energy[4] and the probability of ionization for a photon passing through a 1 cm thick sample of neon at room temperature at a pressure of 10 kPa (1/10 atm.)

# 3 Geiger-Müller Tube

A Geiger-Müller tube is effectively a large cylindrical capacitor, with a low pressure inert gas as the dielectric. The capacitor is charged so that an ionization of a gas molecule is likely to create and avalanche breakdown which will result in a rapid discharge in the capacitor. Once the breakdown stops, the capacitor will quickly recharge, the resulting current pulse indicates that an ionizing event has occured in the tube. A low pressure is required to increase the probability of avalanche[5], although it also decreases the probability of ionization. A simple diagram of this is shown in figure(6).

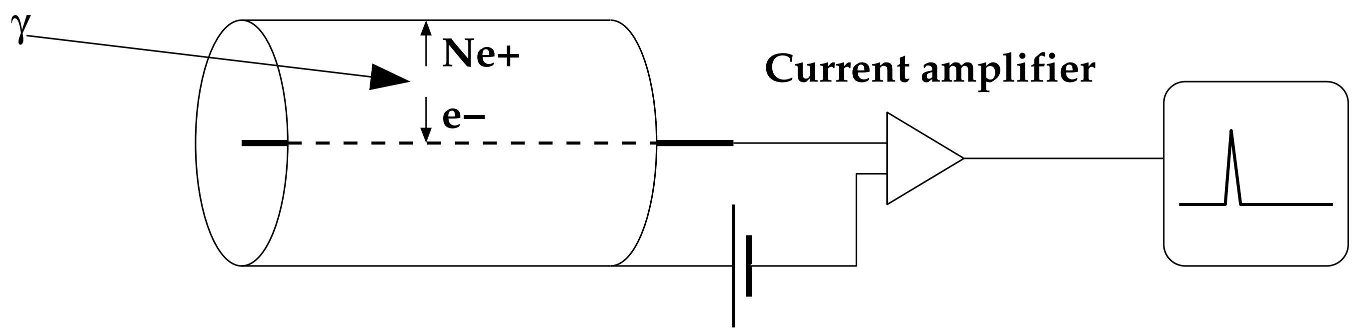


Figure 6: Simple Geiger-Müller tube operation. $\gamma$ radiation ionizes neon gas, electric field accelerates neon ion to the wall, accelerates electron to central conductor causing further ionization which causes a measurable current.

There are several different ways in which the photon may be stopped, but one of the most probable is that the photon kicks an inner K-shell electron out via photoelectric effect. The ionization energies for K-shell electrons in neon are of the order 900 eV[6], the remaining energy is kinetic energy of the ejected which if we consider the 662 keV photon from cesium-137 will result in a 661 keV electron. Examination of figure(2) shows that at the given pressure, energy loss from the electron will be of order 150 eV/cm so it is unlikely that this will induce much ionization. However, the hole in the neon K-shell will quickly be filled resulting in one or more photons to be emitted, none of which will have an energy of more than 900 eV. Examination of figure(5) shows that these photons are extremely likely to cause further ionization. Thus, it is probably a pretty good assumption that very few ionization caused by 662 keV $\gamma$ do not result in avalanche.

Our Geiger-Müller tubes are the DX-1 model manufactured by RDX Nuclear, there are two variants, differing mainly in the physical dimensions of the actual tube, these are shown in figure(7).

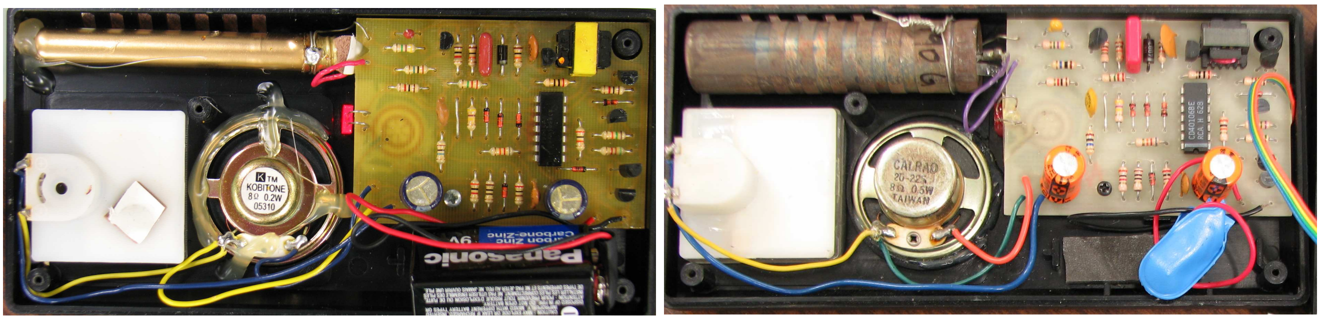


Figure 7: Two variants of RDX Nuclear DX-1 geiger counters.

My estimate of the voltage on the DX-1 Geiger-Müller tube is 2.5 kV.[2]

## 4 Counting Hardware

The **i9513** is a versatile programmable timer/counter chip which is part of the **Labmaster** data acquisition system. It features five independent 16 bit counters, each counter can be programmed to operate in one of nineteen different modes. A block diagram of the device is shown in figure(8).

The `CONTROL` register, the `LOAD` and `HOLD` registers can be accessed at addesses on the computer bus. It is actually slightly more complicated than this, but that stuff is hidden in the kernel driver for the **Labmaster**.

[2]Measuring high voltages is difficult because usually the output impedance of the voltage source is much greater than the input impedance of the voltmeter. In the case where the output impedance is low, the probability of death is become surprisingly high.

A rough overview is that `COUNTER 2` is programmed to provide a square wave which is used to alternately enable and disable `COUNTER 1` and `COUNTER 3` via their gates. When a counter is enabled, each positive edge applied to `SRC 1` will increment the value of the counter. `COUNTER 1` is programmed to count when its gate is high, while `COUNTER 3` is programmed to count when its gate is low. Thus at any time, either `COUNTER 1` or `COUNTER 3` is counting, while the other is inactive. While a counter is inactive, it is safe to retrieve the contents of the counter, which is achieved via the `HOLD` register. The period of the square wave at the output of `COUNTER 2` is twice, $2T$, the period specified by the user, $T$, thus each counter is enabled for time $T$. `COUNTER 4` and `COUNTER 5` are not used.

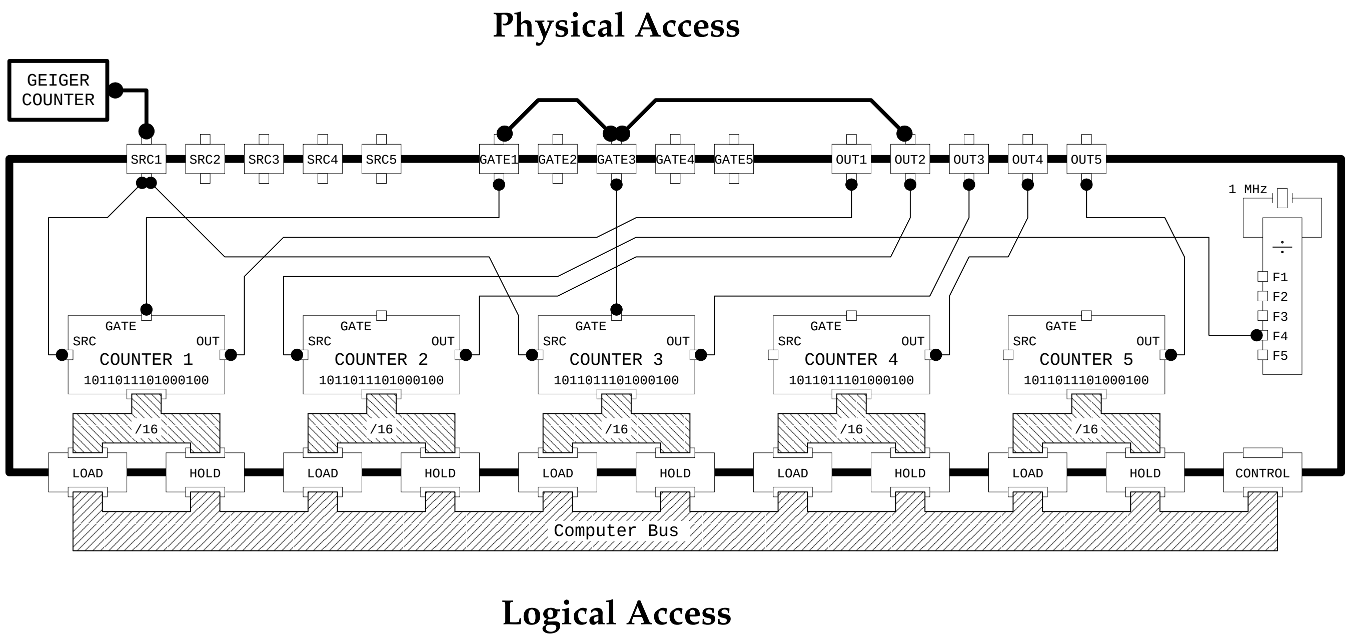


Figure 8: Block diagram illustrating key components of Intel i9513 chip as programmed to measure events from external input SRC1.

The external connections are from the geiger counter output to `SRC 1`, and jumper cables which connect `GATE 1` and `GATE 3` to `OUT 2`. The other connections show are either hardwired or achieved by programming via the `CONTROL` register.

## 4.1 `COUNTER 2` Squarewave Generation

In the top right corner of the **i9513** shown in figure(8) there is a 1 MHz crystal oscillator, this is connected to a divider which is programmed to provide decade division, thus `F1` provides a 1 MHz squarewave, `F2` 100 kHz, `F3` 10 kHz, `F4` 1 kHz and `F5` 100 Hz. `COUNTER 2` is programmed to use `F4` as its counting source. It is programmed to operate in **Mode D Rate Generator with no Hardware Gating**, as defined by the chip documentation. It is programmed to count down from its starting value until the value reaches zero, this state is called *Terminal Count*. Upon *Terminal Count*, the counter loads its value from the `LOAD` register and the output is toggled to the opposite logic level. Thus the value placed into the `LOAD` register of `COUNTER 2` by the `geiger` program is the desired period in units of milliseconds.

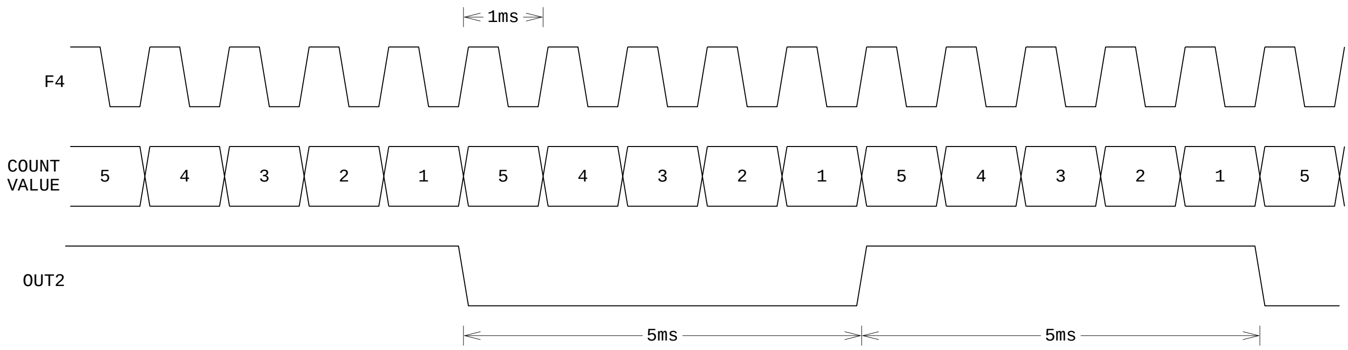


Figure 9: Signals related to operation of `COUNTER 2` when the value stored in `HOLD 2` is 5.

Figure(9) shows the signals relevant to the operation of `COUNTER 2`. When the value of the `HOLD` register has been set to 5 for the purpose of this example. Notice that the count value never reaches zero, rather loading the `HOLD` register when zero would be reached. Since the counters are have sixteen bits, the maximum value which can be loaded is 65535, corresponding to 65.535 s.

## 4.2 Event Counting

Figure(10) shows the operation of `COUNTER 1` and `COUNTER 3`. Notice that `COUNTER 1` only increments when `OUT 2` is high, while `COUNTER 3` only increments when `OUT 2` is low. The command `LOAD` transfers the value in the `LOAD` register in to the counter, in this case the values stored in the `LOAD` registers for `COUNTER 1` and `COUNTER 3` are both zero. The command `SAVE` transfers the contents of the counter into the `HOLD` register.

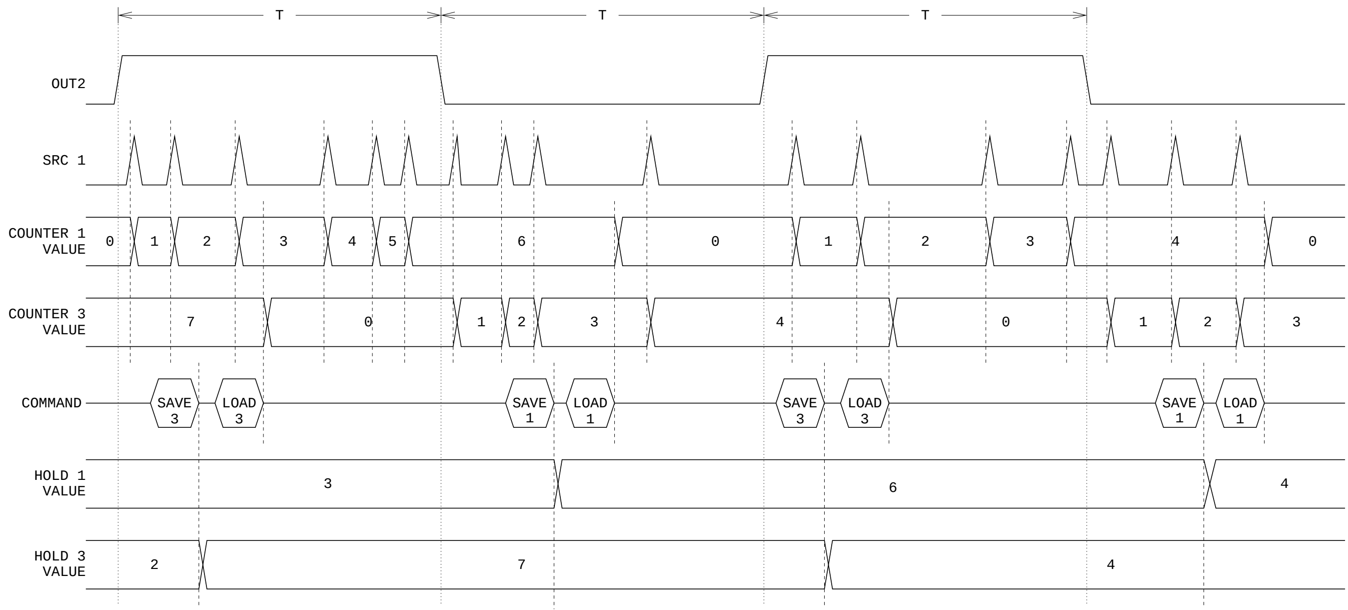


Figure 10: Signals related to operation of `COUNTER 1` and `COUNTER 3`.

For the purpose of the example shown in figure(10) the commands to load and save the counter contents just happen to arrive at the right time. Getting this to happen is a software problem. The way this is done is to watch the `COMMAND` register, it contains a bitmap of all counter outputs so by polling the register and checking the status of the bit corresponding to the output of `COUNTER 2` we can wait until the appropriate time to issue the commands.

```
for i = 1 to replicas {
  for j = 1 to intervals/2 {
    wait until counter 2 output is low
    send_command (SAVE 1)
    send_command (LOAD 1)
    events = get_counter_hold (1)
    increment histogram[i][events]
    wait until counter 2 output is high
    send_command (SAVE 3)
    send_command (LOAD 3)
    events = get_counter_hold (3)
    increment histogram[i][events]
  }
}
```

Figure 11: Pseudo-code of data collection algorithm used by `geiger` software.

# 5 Some Math

## 5.1 Gaussian Normalization

The gaussian probability distribution can be defined as

$$G(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}. \tag{21}$$

The integral over all possible values of the range is unity,

$$\int_{-\infty}^{\infty} G(x,\mu,\sigma)dx = 1. \tag{22}$$

Figure(12) shows a gaussian distribution with the following properties $\mu = 2$ and $\sigma = \sqrt{2}$. The region for $x > 0$ is shaded. The problem is that given eq(22) it is clear that

$$\int_{0}^{\infty} G(x,\mu,\sigma)dx < 1. \tag{23}$$

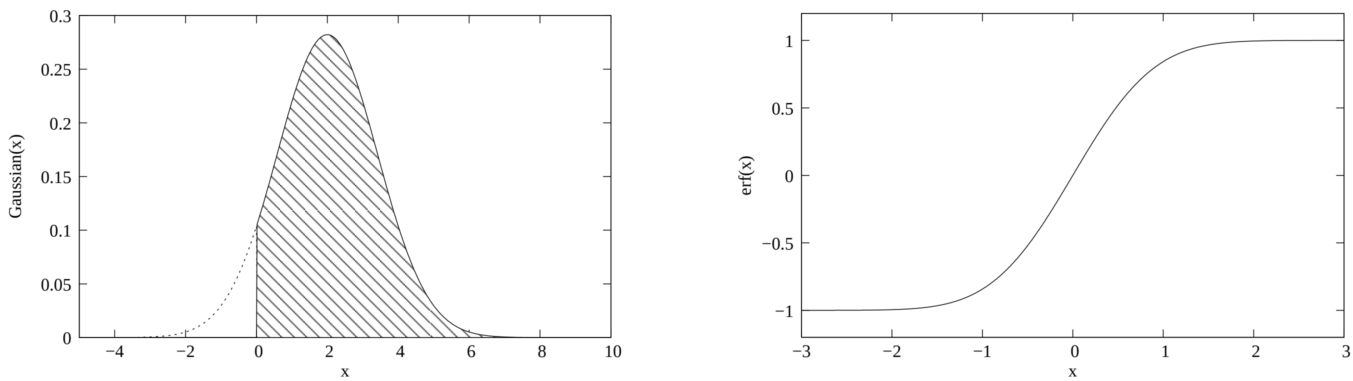


Figure 12: Gaussian.

What we need is to define a new distribution, $F(x,\mu,\sigma)$, which has the same shape as a gaussian, but for which

$$\int_{0}^{\infty} F(x,\mu,\sigma)dx = 1. \tag{24}$$

A good candidate for $F(x,\mu,\sigma)$ is

$$F(x,\mu,\sigma) = \gamma G(x,\mu,\sigma) \tag{25}$$

Integrating both sides,

$$\int_{0}^{\infty} F(x,\mu,\sigma) = \gamma \int_{0}^{\infty} G(x,\mu,\sigma) \tag{26}$$

Which can be solved for $\gamma$

$$\gamma = \frac{\int_0^\infty F(x,\mu,\sigma)}{\int_0^\infty G(x,\mu,\sigma)} = \frac{1}{\int_0^\infty G(x,\mu,\sigma)} \tag{27}$$

In order to solve the integral in eq(27), I introduce the *error-function*,[7]

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dx \tag{28}$$

It is *just* a change of variables off from what we want,

$$\begin{aligned}
\int_0^\infty G(x,\mu,\sigma)dx &= \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx &(29)\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{\frac{-\mu}{\sigma\sqrt{2}}}^\infty e^{-t^2}\sqrt{2}\sigma dt &(30)\\
&= \frac{1}{\sqrt{\pi}}\int_{\frac{-\mu}{\sigma\sqrt{2}}}^\infty e^{-t^2}dt &(31)\\
&= \frac{1}{\sqrt{\pi}}\left(\int_0^\infty e^{-t^2}dt - \int_0^{\frac{-\mu}{\sigma\sqrt{2}}} e^{-t^2}dt\right) &(32)\\
&= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt - \frac{2}{\sqrt{\pi}}\int_0^{\frac{-\mu}{\sigma\sqrt{2}}} e^{-t^2}dt\right) &(33)\\
&= \frac{1}{2}\left(\mathrm{erf}(\infty) - \mathrm{erf}(-\frac{\mu}{\sigma\sqrt{2}})\right) &(34)\\
&= \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{\mu}{\sigma\sqrt{2}}) &(35)
\end{aligned}$$

where the following substitution has been made at eq(30):

$$t = \frac{x - mu}{\sigma\sqrt{2}} \tag{36}$$

## 5.2 $\chi^2$ Distribution

It may be useful to graphically compare your distributions of $\chi^2$ values with the expected distribution. You can make histograms of values in vectors in MATLAB with the `hist` function. The expected $\chi^2$ distribution depends only upon the number of degrees of freedom, $\nu$.

$$f(\chi^2|\nu) = \frac{(\chi^2)^{\frac{\nu-2}{2}}e^{-\frac{\nu}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \tag{37}$$

or is log-space where overflows happen less frequently,

$$\log f(\chi^2|\nu) = \frac{\nu-2}{2}\log\chi^2 - \frac{\nu}{2}(\log 2 + 1) - \log\Gamma(\frac{\nu}{2}) \tag{38}$$

$\log\Gamma()$ is provided in MATLAB as the function `gammaln`.

# References


[1] Virginia S. Shirley C. Michael Lederer, editor. *Table of Isotopes*. John Wiley & Sons, Inc., 7th edition, 1978.

[2] NIST. Stopping-power and range tables for electrons, protons, and helium ions. `http://www.physics.nist.gov/PhysRefData/Star/Text/contents.html`.

[3] NIST. X-ray mass attenuation coefficients. `http://www.physics.nist.gov/PhysRefData/XrayMassCoef/chap2.html`.

[4] NIST. X-ray mass attenuation coefficients. `http://www.physics.nist.gov/PhysRefData/XrayMassCoef/tab3.html`.

[5] F. Paschen Weid. Ueber die zum funkenübergang in luft, wasserstoff und kohlensäure bie verschiedenen drucken erforderliche potentialdifferenz. *Annalen der Physik*, 37:pp. 69–96, 1889.

[6] NIST. X-ray transition energies. `http://www.physics.nist.gov/PhysRefData/XrayTrans/index.html`.

[7] George Arfken. *Mathematical Methods for Physicists*. Academic Press, 3rd edition, 1985.