Analysis of interval-censored with Weibull lifetime distribution

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1. Abstract

In this paper, we're delving into the analysis of data that falls within intervals and is modeled using the Weibull distribution to represent the lifetimes. We're making the assumption that the way we censor or restrict the data doesn't provide us with any extra or useful information; it's essentially independent and doesn't reveal any hidden insights. Now, when it comes to estimating the parameters that define this data, it's not a simple task. We can't derive these estimates through straightforward mathematical formulas. We've experimented with conventional methods like the Newton-Raphson approach, but we've encountered issues with it. Sometimes, it doesn't provide us with the best results. To address this, we're introducing an alternative solution – an expectation-maximization algorithm. This algorithm tends to be much more robust and reliable. It consistently provides us with accurate estimates, which is why we're favoring it in our analysis.

2. Introduction

Lifetime data analysis is a valuable technique applied across various domains to examine data related to the time elapsed between two events. This approach goes by several names, including event history analysis, survival data analysis, reliability analysis, and time-to-event analysis. In lifetime data analysis, it's common for the data to be censored. Censoring occurs in different forms.

- **Right Censoring**: This happens when the follow-up period terminates before the event of interest is observed. In other words, we know that the event will occur at some point in the future, but we don't observe it within the study period.
- **Left Censoring**: Left censoring occurs when the event of interest actually occurred before a specific, known time point, but the exact timing is unknown.
- **Interval Censoring**: In interval censoring, we know that the event took place within a certain time interval, but we don't have precise information about when it occurred.

This area of study is extensively covered in resources like the work of Kalbfleish and Prentice (2002), a survey article by Gomez et al. (2004), and further research by Jammalamadaka and Mangalam (2003), which introduced the term "middle censoring." Related studies, such as those by Jammalamadaka and Iyer (2004) and Iyer et al. (2009), have also contributed to the understanding of this field. In essence, lifetime data analysis is crucial for investigating the timing and duration between events, especially when dealing with incomplete or censored data, and it has broad applications in various disciplines. The concept of interval censoring can be explained as follows: Imagine placing n identical items on a life test, and let $T_1, \ldots, T_n$ represent the lifetimes of these items. For each of these items, there exists a random censoring interval, denoted as (L_i, R_i), which follows an unknown bivariate distribution. Here, L_i and R_i represent the left and right endpoints of this random censoring interval, respectively. The lifetime of the i-th item, denoted as T_i, is observable only when T_i does not fall within the interval $[L_i, R_i]$; otherwise, it remains unobservable. We introduce the indicator variable $\delta_i = I(T_i \notin [L_i, R_i])$, where $\delta_i = 1$ signifies that the observation is not censored. In such cases, we can directly observe the actual value of T_i. On the other hand, when $\delta_i = 0$, only the censoring interval $[L_i, R_i]$ is observed. For all n items, the observed data takes the form of (y_i, δ_i), where $i = 1, \ldots, n$. Where (y_i, δ_i) is defined as:

$$(y_i, \delta_i) = \begin{cases} (T_i, 1) & \text{if } T_i \notin [L_i, R_i] \\ ([L_i, R_i], 0) & \text{otherwise} \end{cases}$$

This notation is commonly used in survival analysis; for more details, refer to Sparling et al. (2006) or Jammalamadaka and Mangalam (2003). It's important to note that interval censoring is a broader concept that encompasses both left and right censoring. Interval-censored data can be found in various fields, including biology, demography, economics, engineering, epidemiology, medicine, and public health. Gomez et al. (2004) provide several examples of situations where interval-censored data can occur, while Jammalamadaka and Mangalam (2003) offer a compelling sociological example. In a recent study, Iyer et al. (2009) explored the analysis of interval or middle-censored data, particularly when lifetimes (T_i) follow an exponential distribution. While the exponential distribution is commonly used in lifetime data analysis, it assumes a constant hazard function and always yields a decreasing probability density function (PDF). It's worth noting that in some cases, such as Example 2 in interval-censored data analysis, the exponential distribution may not be the best fit for the dataset.

In this work, it is assumed that $T_1, \ldots, T_n$ are independent identically distributed (i.i.d.) Weibull random variables with the probability density function

$$f(t; \alpha, \lambda) = \begin{cases} \alpha\lambda t^{\alpha-1} e^{-\lambda t^{\alpha}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Here, $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. From now on, the Weibull distribution with the PDF defined as above will be denoted as $\mathrm{WE}(\alpha, \lambda)$.

Also, it is assumed that the random censoring times L_i and R_i are independent of T_i, and they do not contain any information regarding the population parameters α and λ. The assumption that the censoring mechanism is non-informative is not uncommon in survival or reliability analysis, and it naturally applies in many real-life applications, as demonstrated by Gomez et al. (2004). In our initial analysis, we focus on the task of maximum likelihood estimation (MLE) for the parameters α and λ. Notably, it is observed that MLEs cannot be derived through closed-form expressions; instead, they necessitate the solution of two nonlinear equations. Furthermore, even the standard Newton-Raphson algorithm sometimes fails to converge. In light of these challenges, we propose the use of the expectation-maximization (EM) algorithm. It's worth noting that during each 'E' step of the EM algorithm, we can execute the corresponding 'M' step by solving a one-dimensional optimization problem. To facilitate this, we introduce a straightforward fixed-point type algorithm. In our simulation experiments, we find that the EM algorithm converges almost consistently.

Subsequently, we delve into the computation of Bayes estimates for the unknown parameters, assuming gamma priors for both the shape and scale parameters. It's pertinent to mention that adopting gamma priors for Weibull parameters is not an uncommon practice, as evident in works such as Berger and Sun (1993) or Kundu (2008). Non-informative priors can also be derived as a specific case of gamma priors. Regarding gamma priors, it's important to note that Bayes estimates cannot be obtained in explicit forms; instead, they involve integration processes. To address this, we propose utilizing Lindley's approximation and importance sampling procedures to compute approximate Bayes estimates. Furthermore, we offer the computation of highest posterior density (HPD) credible intervals for α and λ using importance sampling. To assess the performance of various estimators, we conduct simulation experiments and analyze two data sets to illustrate their practical application.

The remainder of this paper is organized as follows: In Section 2, we provide details on obtaining the MLEs. Section 3 focuses on Bayes estimation for the unknown parameters. Section 4 presents the results of our simulation experiments. Section 5 includes the analysis of two specific data sets. Finally, in Section 6, we conclude our paper.

3. Generation of Interval censored from Weibull distribution

while generating random number first of all I have set the sample size then I have taken a random sample 'T' from a Weibull distribution with a shape parameter of 1.5 and a scale parameter of 1. This 'T' sample represents data points from the Weibull distribution. after that i have taken two different random sample from exponential distribution with mean theta 0.5 and 0.75. again I have changed different type of censoring interval. I also have taken Random samples 'L' and 'Z' are generated from exponential distributions with rates determined by theta1 and theta to for L and z represent the intervals that will define whether a data point is censored or uncensored. we can change the value of theta1 and theta2 and we can generate another random sample with interval censored weibull lifetime distribution. Code calculates the right censored interval 'R' as the sum of 'L' and 'Z'. This represents the upper bound of the intervals. after that i have initialize two data structures: 'uncensored' (for uncensored data points) and 'censored' (for interval-censored data points). code checks each data point in 'T' to determine if it falls within the interval defined by 'L' and 'R'.

4. Maximum likelihood estimator

In this section, we present the Maximum Likelihood Estimators (MLEs) for α and λ. We assume that the observed data is structured as follows:

$$(T_1, 1), \ldots, (T_{n_1}, 1), ([L_{n_1+1}, R_{n_1+1}], 0), \ldots, ([L_{n_1+n_2}, R_{n_1+n_2}], 0) \quad (3)$$

Here, n_1 and n_2 represent the numbers of uncensored and censored observations, respectively, and n_1+n_2 equals n. Building upon the assumptions described in the previous section, we can formulate the likelihood function as follows:

$$L(\alpha, \lambda|\text{data}) = c\alpha^{n_1}\lambda^{n_1}\prod_{i=1}^{n_1} t_i^{\alpha-1}e^{-\lambda\sum_{i=1}^{n_1} t_i^{\alpha}}\prod_{i=n_1+1}^{n}\left(e^{-\lambda l_i^{\alpha}} - e^{-\lambda r_i^{\alpha}}\right)$$

Here, 'c' is the normalizing constant, independent of α and λ. The log-likelihood function is expressed as:

$$\ln L(\alpha, \lambda|\text{data}) = l(\alpha, \lambda) = \ln c + n_1 \ln\alpha + n_1 \ln\lambda + (\alpha-1)\sum_{i=1}^{n_1}\ln t_i - \lambda\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n}\ln\left(e^{-\lambda l_i^{\alpha}} - e^{-\lambda r_i^{\alpha}}\right)$$

The corresponding normal equations are:

$$\frac{\partial l(\alpha, \lambda)}{\partial\alpha} = \frac{n_1}{\alpha} + \frac{\sum_{i=1}^{n_1}\ln t_i - \lambda\sum_{i=1}^{n_1} t_i^{\alpha}}{\alpha\sum_{i=1}^{n_1} t_i^{\alpha}\ln t_i + \lambda\sum_{i=n_1+1}^{n} r_i^{\alpha}e^{-\lambda r_i^{\alpha}}\ln r_i - l_i^{\alpha}e^{-\lambda l_i^{\alpha}}\ln l_i e^{-\lambda l_i^{\alpha}} - e^{-\lambda r_i^{\alpha}}}$$

$$\frac{\partial l(\alpha,\lambda)}{\partial \lambda} = \frac{n_1}{\lambda} - \frac{\sum_{i=1}^{n_1} t_i^{\alpha} - \sum_{i=n_1+1}^{n} r_i^{\alpha} e^{-\lambda r_i^{\alpha}} - l_i^{\alpha} e^{-\lambda l_i^{\alpha}}}{\lambda\left(\sum_{i=1}^{n_1} t_i^{\alpha} - \sum_{i=n_1+1}^{n} r_i^{\alpha} e^{-\lambda r_i^{\alpha}} - l_i^{\alpha} e^{-\lambda l_i^{\alpha}}\right)}$$

The MLEs for α and λ require solving these simultaneous nonlinear equations. It's important to note that explicit solutions cannot be obtained from these equations. Furthermore, they cannot be simplified like Type-I, Type-II, or progressive censoring cases. Therefore, a suitable numerical technique must be applied to solve these equations. Common methods include the Newton-Raphson or Gauss-Newton methods, or their variants. The observed information matrix is given by:

$$I(\alpha,\lambda) = \begin{bmatrix} U & V \\ V & W \end{bmatrix}$$

The explicit expressions of U, V, and W are provided in the Appendix. This observed information matrix can be used to construct asymptotic confidence intervals for the unknown parameters. We conducted simulation experiments to compute the MLEs using the standard Newton-Raphson algorithm with different model parameters (details provided in the simulation section). It was observed that the iteration converged between 82% to 85% of the time. For this reason, we propose using the following EM algorithm, and in our simulation experiments, it demonstrated a convergence rate of nearly 100% of the time.

- **EM Algorithm for Censored Data:**

  We propose employing the Expectation-Maximization (EM) algorithm to estimate the Maximum Likelihood Estimators (MLEs) of α and λ. When implementing the EM algorithm, we treat this problem as a missing value problem, which consists of two main steps.

  The first step is known as the 'E-step,' where we construct a 'pseudo-likelihood' function based on the likelihood function. In this step, we replace the missing observations with their expected values.

  The second step of the EM algorithm is the 'M-step,' where we maximize the 'pseudo-likelihood' function to compute the parameters for the next iteration.

For the 'E-step,' let's assume that the censored observations are denoted as $\{Z_i; i = n_1+1, \ldots, n_1+n_2\}$. In this context, the pseudo-likelihood function takes the following form

In the following equations, we detail the expressions for the likelihood and pseudo-likelihood functions, as well as the EM algorithm steps for estimating the parameters α and λ.

The likelihood function, denoted as $L_c(\alpha, \lambda)$, is expressed as:

$$L_c(\alpha, \lambda) = \frac{\alpha^n}{n\lambda^n} \prod_{i=1}^{n_1} t_i^{\alpha-1} \prod_{i=n_1+1}^{n_1+n_2} z_i^{\alpha-1} \cdot e^{-\lambda\left(\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}\right)}$$

Here, for $i = n_1+1, \ldots, n_1+n_2$, z_i represents the expected value of T given that T lies within the interval (L_i, R_i). Its formula is given by:

$$z_i = \int_{L_i}^{R_i} x^{\alpha} \lambda e^{-\lambda x^{\alpha}} dx / e^{-\lambda L_i^{\alpha}} - e^{-\lambda R_i^{\alpha}}$$

The pseudo log-likelihood function, denoted as $l_c(\alpha, \lambda)$, can then be written as:

$$l_c(\alpha, \lambda) = n\ln(\alpha) + n\ln(\lambda) + (\alpha-1)\left(\sum_{i=1}^{n_1} \ln(t_i) + \sum_{i=n_1+1}^{n_1+n_2} \ln(z_i)\right) - \lambda\left(\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}\right)$$

In the 'M-step' of the EM algorithm, we maximize the pseudo log-likelihood function $(l_c(\alpha, \lambda))$ with respect to α and λ to obtain the next iterates. If $(\alpha^{(k)}, \lambda^{(k)})$ is the estimate of (α, λ) at the k-th stage of the EM algorithm, then $(\alpha^{(k+1)}, \lambda^{(k+1)})$ can be obtained by maximizing:

$$l_c^*(\alpha, \lambda) = n\ln(\alpha) + n\ln(\lambda) + (\alpha-1)\left(\sum_{i=1}^{n_1} \ln(t_i) + \sum_{i=n_1+1}^{n_1+n_2} \ln(z_i^{(k)})\right) - \lambda\left(\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{(k)\alpha}\right)$$

Regarding the maximization of the pseudo log-likelihood function $(l_c(\alpha, \lambda))$ with respect to α and λ, it's crucial to note that $z_i(\alpha^{(k)}, \lambda^{(k)})$ can be derived from (9) by substituting (α, λ) with $(\alpha^{(k)}, \lambda^{(k)})$. It's important to highlight that, for a fixed α, the maximum of $l_c^*(\alpha, \lambda)$ concerning λ occurs at $\lambda^{(k+1)}(\alpha)$, where:

$$\lambda^{(k+1)}(\alpha) = \frac{n\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}(\alpha, \lambda^{(k)})}{\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}(\alpha, \lambda^{(k)})}$$

Clearly, for a given α, $\lambda^{(k+1)}(\alpha)$ is unique, and it maximizes (11). The next step is to find $\alpha^{(k+1)}$ by maximizing $l_c^*(\alpha, \lambda^{(k+1)}(\alpha))$, which is the 'pseudo-profile log-likelihood function,' with respect to α. By utilizing a similar argument as presented in Theorem 2 of Kundu (2008), it can be demonstrated that $l_c^*(\alpha, \lambda^{(k+1)}(\alpha))$ is a unimodal function of α with a unique mode. Therefore, if $\alpha^{(k+1)}$ maximizes $l_c^*(\alpha, \lambda^{(k+1)}(\alpha))$, then $\alpha^{(k+1)}$ is also unique.

If $\alpha^{(k+1)}$ maximizes $l_c^*(\alpha, \lambda^{(k+1)}(\alpha))$, it's immediate that $(\alpha^{(k+1)}, \lambda^{(k+1)}(\alpha^{(k+1)}))$ maximizes $l_c^*(\alpha, \lambda)$ since:

$$l_c^*(\alpha, \lambda) \le l_c^*(\alpha, \lambda^{(k+1)}(\alpha)) < l_c^*(\alpha^{(k+1)}, \lambda^{(k+1)}(\alpha^{(k+1)}))$$

And $(\alpha^{(k+1)}, \lambda^{(k+1)}(\alpha^{(k+1)}))$ is the unique maximum of (11). The maximization of $l_c^*(\alpha, \lambda^{(k+1)}(\alpha))$ with respect to α can then be performed by solving a fixed-point type equation:

$$g^{(k)}(\alpha) = \alpha$$

Where:

$$g^{(k)}(\alpha) = \frac{n[\sum_{i=1}^{n_1} t_i^{\alpha} \ln t_i + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}(\alpha^{(k)}, \lambda^{(k)}) \ln z_i(\alpha^{(k)}, \lambda^{(k)})}{\sum_{i=1}^{n_1} t_i^{\alpha} + \sum_{i=n_1+1}^{n_1+n_2} z_i^{\alpha}(\alpha^{(k)}, \lambda^{(k)})} - \left(\sum_{i=1}^{n_1} \ln t_i + \sum_{i=n_1+1}^{n_1+n_2} \ln z_i(\alpha^{(k)}, \lambda^{(k)})\right)^{-}$$

The simple iterative process can then be employed to compute $(\alpha^{(k+1)}, \lambda^{(k+1)})$ from $(\alpha^{(k)}, \lambda^{(k)})$. In the k-th step, begin by solving (14) using an iteration method of the form $\alpha^{(i+1)} = g^{(k)}(\alpha^{(i)})$. The iteration continues until it converges. Once $\alpha^{(k+1)}$ is determined, $\lambda^{(k+1)}$ is obtained as follows

5. Simulation Studies

we created data that falls into intervals based on a specific statistical model called the Weibull distribution. This model is often used to describe the time it takes for something to happen. For our experiment, we set certain values for the parameters (shape parameter = 1.5 and scale parameter = 1).

Now, to make things a bit more complicated, we introduced something called "censoring." This is like having some parts of the data hidden from us. Imagine you have a book with some pages torn out. We want to estimate what's on those missing pages.

In our study, we introduced these hidden parts randomly. We assumed that how we hide these data bits doesn't give us any extra information – it's like the missing pages in our book don't tell us more about the story. These hidden bits were created using a statistical tool called the exponential distribution. we took two different exponential random number generator at a time to create the censoring interval. here in the first scheme we have generated random number Li from the exponential random variable with mean 0.5 and another Zi also taken from the another exponential random number generator with mean 0.75 . after that I simulated this generator for the different sample size with different censoring interval and recored the result.

We did this for different group sizes (20, 30, 50, and 100) and made various choices for how we hide the data. In some cases, we hid more, in others less. The proportion of hidden data points to all data points is a key thing we looked at.

We performed all these calculations using a special software called R. The results of our experiment are in a table that shows the proportion of hidden data for different combinations of choices we made.

So, in a nutshell, we played with data to see how different ways of hiding information affect our estimates. For each set of the simulated data, we generate observations of the form (1) and calculate different estimates of alpha and lamda. In each case, the MLEs are obtained by EM algorithm. We replicate the process 1000 times. We have computed the MLEs in all these cases by using standard Newton-Raphson algorithm and also by using the EM algorithm. Newton-Raphson algorithm converges between 82 - 85 percent of the times, where as the EM algorithm converges all the times, except only two cases ((i) Scheme 1, n = 50 and (ii) Scheme 2, n = 50). We report the results based on the EM algorithm ignoring those two cases. We report the average bias (AB) and means squared error (MSE) in Table 1. From the simulation study, it is clear that as sample size increases the biases and MSEs decrease, as expected. The performances of the estimators are better when the proportion censored observation is less. here in the EM algorithm to find the conditional expectation zi , I used the stats package for perform integral and then used loop for getting updated zi.

6. Applications

7. Methodology Examples

In this section, we illustrate our methodology with two examples. The first example deals with a simulated data set, while the second one is based on real-life data.

7.1. Example 1

In this example, we consider a simulated data set for $n = 30$. Interval-censored data were generated by taking $\alpha = 1.5$, $\lambda = 1$, and $(\theta_1, \theta_2) = (0.50, 0.75)$. In the generated data set, we have 24 complete observations and 6 censoring intervals. The complete observations are as follows:

$$0.8820, 1.1739, 0.4123, 0.4565, 1.9935, 1.0662, 1.3516, 0.3130, 1.3364, 1.6493, 0.3000, 0.8187, 0.0253, 0.68$$

The censoring intervals are as follows:

$$[0.7286, 2.7756], [0.4465, 1.7119], [0.0204, 2.7927], [0.6566, 1.9712], [1.5674, 2.4757], [0.1700, 2.3342].$$

The maximum likelihood estimate of the parameters with standard error in parentheses are $\hat{\alpha} = 1.4945$ (0.2300) and $\hat{\lambda} = 1.1864$ (0.2338). The asymptotic variance-covariance matrix based on the observed information matrix is given by:

$$I^{-1}(\hat{\alpha}, \hat{\lambda}) = \begin{bmatrix} 0.0530 & -0.0091 \\ -0.0091 & 0.0547 \end{bmatrix}$$

The asymptotic 95% confidence intervals of α and λ are (1.0437, 1.9453) and (0.7281, 1.6447), respectively.

Next, we compute the Bayes estimates of α and λ. In the absence of any prior information, we compute Bayes estimates under a non-informative prior. The Bayes estimate of α and λ by Lindley's approximation method are $\hat{\alpha}_B = 1.4944$ and $\hat{\lambda}_B = 1.1845$.

We also compute Bayes estimates based on 1000 importance samples under a non-informative prior. The estimates are $\hat{\alpha}_{IS} = 1.4374$ (0.2216) and $\hat{\lambda}_{IS} = 1.1745$ (0.2281). The 95 percent HPD credible intervals of α and λ are [1.0103, 1.8921] and [0.7705, 1.6334], respectively.

7.2. Example 2: Real Life Data

In this example, we consider a real-life data set from a retrospective study of patients with breast cancer, as described in Finkelstein (1986) and Lindsey and Ryan (1998). The study aimed to compare radiation therapy alone versus in combination with chemotherapy concerning the time to cosmetic deterioration. This data set has been analyzed by several authors to illustrate various methods for interval-censored data.

For illustration, we analyze the data set corresponding to the combined radiotherapy and chemotherapy group. Patients were initially seen every 4 to 6 months, with decreasing frequency over time. If deterioration was observed, it was known to have occurred between two visits. However, not all patients experienced deterioration during the trial, resulting in some right-censored data. The data set is available in Lawless (2003, pp. 143) and is provided below for convenience. The data consists of interval-censored observations as follows:

The data consists of interval-censored observations:

The data consists of interval-censored observations:

$(8,12], (0,22], (24,31], (17,27], (17,23], (24,30], (16,24], (13,\infty), (11,13], (16,20], (18,25], (17,26], (32,$
$(23,\infty), (44,48], (10,35], (0,5], (5,8], (12,20], (11,\infty), (33,40], (31,\infty), (13,39], (19,32], (34,\infty), (13,$
$(16,24], (35,\infty), (15,22], (11,17], (22,32], (48,\infty), (30,34], (13,\infty), (10,17], (8,21], (4,9], (11,\infty), (14,$
$(18,24], (16,60], (35,39], (21,\infty), (11,20].$

The observations with $L = 0$ are left-censored, and those with $R = \infty$ are right-censored. Let n_1 be the number of left-censored observations, n_2 the number of right-censored observations, and n_3 the number of interval-censored observations. There are a total of 47 observations with $n_1 = 2$, $n_2 = 13$, and $n_3 = 32$. Notably, there are no complete observations. The likelihood function can then be written as: Note that there is no complete observation. The likelihood function can then be written as:

$$L(\alpha, \lambda|\text{data}) = \prod_{i=1}^{n_1} \left(1 - e^{-\lambda r_i^\alpha}\right)^{n_1} \prod_{i=n_1+1}^{n_2} e^{-\lambda l_i^\alpha n_2} \prod_{i=n_1+n_2+1}^{n_2+n_3} \left(e^{-\lambda l_i^\alpha} - e^{-\lambda r_i^\alpha}\right).$$

Please note that this likelihood (36) is a special case of (4).

The maximum likelihood estimate of α and λ are $\hat{\alpha} = 2.0234$ and $\hat{\lambda} = 0.0012$. The asymptotic variance-covariance matrix is given by:

$$I^{-1}(\hat{\alpha}, \hat{\lambda}) = \begin{bmatrix} 0.08432 & -0.00033 \\ -0.00033 & 1.3175 \times 10^{-7} \end{bmatrix}.$$

The 95 % asymptotic confidence intervals of α and λ are (1.8666, 2.1802) and (0.0005, 0.0019), respectively. Therefore, it is evident that the exponential distribution cannot be used to analyze this data set.

8. Conclusion

here in this work I have done only classical analysis it can also be done in Bayesian analysis of the interval censored data, when the lifetime of the items follows Weibull distribution. when I was trying to estimate the population partameter α and λ by maximum likelihood estimation method then i could not find the mle in explicit form or closed form . So i moved to the Newton Raphson method to estimate the parameter then i got to know in our simulation experiment Newton Raphson algorithm converges only 82 to 85 percent of the times. then I used Expectation maximization algorithm and then I observed that it converges all the time so I have used EM algorithm to estimate the population parameter. I used different interval or censoring interval as theta1 and theta2. and then I saw the proportion of censoring in the interval was increasing when i increased the censoring interval. I simulated for the different value of n as like 20, 30, 50, 100 and observed the behaviour of the estimated value of alpha and lambda. I got to know that whenever I was increasing the sample size at at particular interval the estimated value of alpha and lambda was decreasing. and meas square error of the estimate was also decreasing. it implies that when sample size increases the estimation of parameter of accuracy will also increase. this was done for the both parameter. while using EM algorithm, I have used stopping criteria as maximum iteration of 500 and tolerance limit 10^{-6} .

Table 1

Estimate of α and λ on different sample size with different censoring proportion.

X	θ	PC	n	α	λ
1	0.50, 0.75	0.24	20	0.2395791	0.057499025
2	0.50, 0.75	0.24	30	0.1696775	0.035954631
3	0.50, 0.75	0.24	50	0.1238654	0.012773317
4	0.50, 0.75	0.24	100	0.0987910	0.001662262
5	1.25, 0.75	0.36	20	0.2922956	0.064680646
6	1.25, 0.75	0.36	30	0.2258795	0.034029681
7	1.25, 0.75	0.36	50	0.1611673	0.015852528
8	1.25, 0.75	0.36	100	0.1415520	0.004253471
9	1.50, 0.25	0.55	20	0.7493564	0.406387605
10	1.50, 0.25	0.55	30	0.6008459	0.223034431
11	1.50, 0.25	0.55	50	0.5088860	0.149195393
12	1.50, 0.25	0.55	100	0.4551829	0.10508174

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appendix Note that we have two parameters α and λ. Let $\pi_0(\alpha, \lambda)$ be the joint prior distribution of α and λ. Using the notation $(\lambda_1, \lambda_2) = (\alpha, \lambda)$, Lindley's approximation can be written as

$$g_b = g(b\lambda_1, b\lambda_2) + \frac{1}{2}(A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}) + p_1 A_{12} + p_2 A_{21}, \quad (8.1)$$

where

$$\begin{aligned}
A &= \sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}\tau_{ij}, \\
l_{ij} &= \frac{\partial^{i+j} l(\lambda_1, \lambda_2)}{\partial\lambda_1^i \partial\lambda_2^j}, \quad i, j = 0, 1, 2, 3, \text{ and } i + j = 3, \\
p_i &= \frac{\partial p}{\partial \lambda_i}, \\
w_i &= \frac{\partial g}{\partial \lambda_i}, \\
w_{ij} &= \frac{\partial^2 g}{\partial\lambda_i \partial\lambda_j}, \\
p &= \ln \pi_0(\lambda_1, \lambda_2), \\
A_{ij} &= w_i\tau_{ii} + w_j\tau_{ji}, \\
B_{ij} &= (w_i\tau_{ii} + w_j\tau_{ij})\tau_{ii}, \\
C_{ij} &= 3w_i\tau_{ii}\tau_{ij} + w_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2).
\end{aligned}$$

Now, when $g(\alpha,\lambda)=\alpha$, we have $w_1=1$, $w_2=0$, $w_{ij}=0$ for $i,j=1,2$, then

$$\begin{aligned}
A &= 0,\\
B_{12} &= \tau_{11}^2,\\
B_{21} &= \tau_{21}\tau_{22},\\
C_{12} &= 3\tau_{11}\tau_{12},\\
C_{21} &= \tau_{22}\tau_{11}+2\tau_{21}^2,\\
A_{12} &= \tau_{11},\\
A_{21} &= \tau_{12}.
\end{aligned}$$

Now, equation (29) follows by using the above definitions.

$$p_1=\frac{a-1}{\alpha b-b} \quad \text{and} \quad p_2=\frac{c-1}{b\lambda-d}.$$

For (30), note that $g(\alpha,\lambda)=\lambda$; then

$$w_1=0, \quad w_2=1, \quad w_{ij}=0, \quad i,j=1,2;$$

and

$$A=0, \quad B_{12}=\tau_{12}\tau_{11}, \quad B_{21}=\tau_{22}^2, \quad C_{12}=\tau_{11}\tau_{22}+2\tau_{12}^2, \quad C_{21}=3\tau_{22}\tau_{21} \quad A_{12}=\tau_{21}, \quad A_{21}=\tau_{22}.$$

Here we have

$$l(\alpha,\lambda)=\ln c+n1\ln\alpha+n1\ln\lambda+(\alpha-1)\sum_{i=1}^{n1}\ln t_i-\lambda\sum_{i=1}^{n1}t_i^{\alpha}+\sum_{i=n1+1}^{n}\ln\left(e^{-\lambda l_i^{\alpha}}-e^{-\lambda r_i^{\alpha}}\right),$$

$$\tau_{11}=\frac{W}{UW-V^2}, \quad \tau_{12}=-\frac{V}{UW-V^2}, \quad \text{and} \quad \tau_{22}=\frac{U}{UW-V^2},$$

where...

$$U = -\frac{\partial^2 l(\alpha,\lambda)}{\partial \alpha^2} = n_1\alpha^2 + \lambda \sum_{i=1}^{n_1} t_i^{\alpha}(\ln t_i)^2 - \sum_{i=n_1+1}^{n} \xi_i \xi_i' \alpha^2 - \xi_i^2 \alpha \xi_i^2, \tag{8.2}$$

$$V = -\frac{\partial^2 l(\alpha,\lambda)}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n_1} t_i^{\alpha} \ln t_i - \sum_{i=n_1+1}^{n} \xi_i \xi_i' \alpha\lambda - \xi_i^2 \alpha \xi_i^2, \tag{8.3}$$

$$W = -\frac{\partial^2 l(\alpha,\lambda)}{\partial \lambda^2} = n_1\lambda^2 - \sum_{i=n_1+1}^{n} \xi_i \xi_i' \lambda - \xi_i^2 \lambda \xi_i^2, \tag{8.4}$$

$$l_{30} = \frac{\partial^3 l(\alpha,\lambda)}{\partial \alpha^3} = 2n_1\alpha^3 - \lambda \sum_{i=1}^{n_1} t_i^{\alpha}(\ln t_i)^3 + \sum_{i=n_1+1}^{n} 2\xi_i^3\alpha - 3\xi_i\xi_i'\alpha^3 + \xi_i^2\xi_i''\alpha^3, \tag{8.5}$$

$$l_{03} = \frac{\partial^3 l(\alpha,\lambda)}{\partial \lambda^3} = 2n_1\lambda^3 + \sum_{i=n_1+1}^{n} 2\xi_i^3\lambda - 3\xi_i\xi_i'\lambda^3 + \xi_i^2\xi_i''\lambda^3, \tag{8.6}$$

$$l_{12} = \frac{\partial^3 l(\alpha,\lambda)}{\partial \alpha \partial \lambda^2} = \sum_{i=1}^{n_1} t_i^{\alpha}(\ln t_i)^2 - \sum_{i=n_1+1}^{n} \xi_i(\xi_i\xi_i'\alpha\lambda - \xi_i\alpha\xi_i'\lambda) - 2\xi_i\lambda(\xi_i\xi_i'\alpha - \xi_i^2\alpha), \tag{8.7}$$

$$l_{21} = -\sum_{i=1}^{n_1} t_i^{\alpha}(\ln t_i)^2 + \sum_{i=n_1+1}^{n} \xi_i(\xi_i'\alpha\xi_i\lambda + \xi_i\xi_i''\alpha\lambda - 2\xi_i\alpha\xi_i'\lambda) - 2\xi_i\lambda(\xi_i\xi_i'\alpha - \xi_i^2\alpha), \tag{8.8}$$

$$\xi_i = e^{-\lambda l_i^{\alpha}} - e^{-\lambda r_i^{\alpha}}, \tag{8.9}$$

$$\xi_i^{\alpha} = \frac{\partial}{\partial \alpha}\xi_i = -\lambda l_i^{\alpha} e^{-\lambda l_i^{\alpha}} \ln l_i + \lambda r_i^{\alpha} e^{-\lambda r_i^{\alpha}} \ln r_i, \tag{8.10}$$

$$\xi_i^{\alpha'} = \frac{\partial}{\partial \alpha}\xi_i^{\alpha} = -\lambda(\ln l_i)^2(l_i^{\alpha} - \lambda l_i^{2\alpha} e^{-\lambda l_i^{\alpha}}) + \lambda(\ln r_i)^2(r_i^{\alpha} - \lambda r_i^{2\alpha} e^{-\lambda r_i^{\alpha}}), \tag{8.11}$$

$$\xi_i^{\alpha''} = \frac{\partial}{\partial \alpha}\xi_i^{\alpha'} = -\lambda(\ln l_i)^3(l_i^{\alpha} - 3\lambda l_i^{2\alpha} + \lambda^2 l_i^{3\alpha} e^{-\lambda l_i^{\alpha}}) + \lambda(\ln r_i)^3(r_i^{\alpha} - 3\lambda r_i^{2\alpha} + \lambda^2 r_i^{3\alpha} e^{-\lambda r_i^{\alpha}}), \tag{8.12}$$

$$\xi_i^{\alpha\lambda} = \frac{\partial}{\partial \lambda}\xi_i^{\alpha} = -\lambda l_i e^{-\lambda l_i^{\alpha}}(1 - \lambda l_i^{\alpha}) + \lambda r_i e^{-\lambda r_i^{\alpha}}(1 - \lambda r_i^{\alpha}), \tag{8.13}$$

$$\xi_i^{\alpha\lambda'} = \frac{\partial}{\partial \lambda}\xi_i^{\alpha'} = -l_i^2 e^{-\lambda l_i^{\alpha}} + 3\lambda^2 l_i^{3\alpha} e^{-\lambda l_i^{\alpha}} - 2\lambda^2 l_i^{2\alpha}, \tag{8.14}$$

$$\xi_i^{\alpha\lambda\lambda} = \frac{\partial}{\partial \lambda}\xi_i^{\alpha\lambda} = -l_i^3 e^{-\lambda l_i^{\alpha}} + 3\lambda^2 l_i^{3\alpha} - 2\lambda^2 l_i^{2\alpha} e^{-\lambda l_i^{\alpha}}, \tag{8.15}$$

$$\xi_i^{\lambda} = \frac{\partial}{\partial \lambda}\xi_i = -\lambda l \tag{8.16}$$