PH205: Math Method of Physics

Lecture : Math Method of Physics

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Contents

Chapter 1

Prerequisites

1.1 Linear Operators

A linear operator is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication.

$$A(e_i) = \sum_{j=1}^{n} A_{ji} e_j$$

A linear operator maps from one vector space to another.

$$V \to V'$$

$$e_i \to v'$$

$$i = 1 \ldots n$$

$$j = 1 \ldots n$$

$$\phi(e_i) = \sum_{j=1}^{n} A_{ji} f_j$$

Example: map from n-dimensional vector space to n-dimensional vector space.

$$\phi(e_i) = e_i$$

$$\phi(e_j) = 0, \quad j \neq i$$

This results in a matrix:

$$\begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

1.1.1 Projection Operator

A projection operator is a linear operator P from a vector space to itself such that $P^2 = P$.

$$V = \sum_i c_i e_i$$

$$\phi^2(v) = \phi(\phi(v)) = \phi(\phi(c_i e_i)) = \phi(c_i e_i) = c_i e_i = v$$

$$\phi(c_i v_i) = v_i, \phi(c_i v_i) = v_i$$

Example: mapping 2-dimensional to 3-dimensional v-s.

$$\phi(e_1) = f_1 + f_2 + f_3$$

$$\phi(e_2) = f_1 + f_2 - f_3$$

This gives the matrix representation ϕ:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Matrices occur naturally in representation of linear operator. Matrix algebra are inherited from the rules of linear algebra.

$$A(c_i) = A_{ji}c_j$$

$$B(c_i) = B_{ji}c_j$$

$$(A+B)(e_i) = A(e_i) + B(e_i) = A_{ji}e_j + B_{ji}e_j = (A_{ji} + B_{ji})e_j$$

1.2 Matrix Properties and Operations

1.2.1 Scalar Multiplication and Related Concepts

Scalar multiplication is one of the basic operations defining a vector space.

$$(\lambda A)(e_i) = \lambda A(e_i) = \lambda A_{ji}e_j$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

$$(A^{-1})_{ij} \cdot (A^{-1})_{ij} = \delta_{ij}$$

A^{-1} has to be a square matrix.

$$e^A = I + A + \frac{A^2}{2!} + \dots$$

Similarly there is $\sin(A)$, $\cos(A)$, $\cosh(A)$, $\sinh(A)$.

$$(1-A)^{-1} = \frac{1}{1-A} = I + A + A^2 + \dots$$

Method of computing inverse for matrices close to identity.

1.2.2 Matrices related to a matrix A

$$(A^T)_{ij} = A_{ji} \quad \text{(Transpose)}$$

$$(A^*)_{ij} = (A_{ij})^* \quad \text{(Complex conjugate)}$$

$$(A^\dagger)_{ij} = (A^T_{ij})^* = (A_{ji})^* \quad \text{(Hermitian conjugate)}$$

1.2.3 Special Matrices

- $A^T = A$: Symmetric
- $A^T = -A$: Anti-symmetric
- $A^\dagger = A$: Hermitian
- $A^\dagger = -A$: Anti-Hermitian
- $A^* = A$: Real

1.3 Advanced Matrix Concepts

1.3.1 Diagonal and Block Diagonal Matrices

- Diagonal matrix $A_{ij} = 0$ if $i \neq j$.
- Upper triangular matrix $A_{ij} = 0$ if $i > j$.

Matrices can be decomposed into block diagonal.

Block diagonal matrix

A **block diagonal matrix** is a square matrix that is partitioned into smaller square matrices (blocks) along its diagonal, with all off-diagonal blocks being zero matrices. Formally, a block diagonal matrix M can be written as:

$$M = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix}$$

where each B_i is a square matrix (block) of size $n_i \times n_i$ and the zeros represent zero matrices of appropriate sizes.

Properties:

- The determinant of a block diagonal matrix is the product of the determinants of its blocks:

$$\det(M) = \prod_{i=1}^{k} \det(B_i)$$

- The trace of a block diagonal matrix is the sum of the traces of its blocks:

$$\mathrm{Tr}(M) = \sum_{i=1}^{k} \mathrm{Tr}(B_i)$$

- The inverse of a block diagonal matrix (if all blocks are invertible) is the block diagonal matrix of the inverses:

$$M^{-1} = \begin{pmatrix} B_1^{-1} & 0 & \cdots & 0 \\ 0 & B_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k^{-1} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 5 & 6 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix} & 0 \\ 0 & \begin{matrix} 3 & 4 \\ 5 & 6 \end{matrix} \end{pmatrix}$$

Here, the matrix is block diagonal with two blocks: a 2×2 diagonal block and a 2×2 non-diagonal block.

Block diagonal matrices are useful in linear algebra because they allow us to break down complex problems into smaller, more manageable subproblems, especially when dealing with direct sums of vector spaces or simplifying linear transformations that act independently on subspaces.

1.3.2 Trace and Determinant

Trace of a matrix is sum of diagonal.

$$\mathrm{Tr}(A) = \sum A_{ii}$$

How to find the trace of a matrix

$$a_{11}x_1 + a_{12}x_2 = c_1$$

$$a_{21}x_1 + a_{22}x_2 = c_2$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Determinant of a matrix

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\epsilon^{123\dots n} = 1$$

$$\epsilon^{2134\dots n} = -1$$

$$\epsilon^{i_1 i_2 \dots i_n} = 0$$

$$\det A = \sum_{i_1, i_2, \dots, i_n} \epsilon^{i_1 i_2 \dots i_n} a_{1i_1} a_{2i_2} \dots a_{ni_n}$$

1.3.3 Applications in Quantum Mechanics

Considering 2 particle fermions

The wave function needs to be anti-symmetric.

$$\psi(x_1)\psi(x_2) - \psi(x_2)\psi(x_1)$$

Slater determinant

$$\begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_n(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \dots & \psi_n(x_2) \\ \vdots & & & \\ \psi_1(x_n) & \psi_2(x_n) & \dots & \psi_n(x_n) \end{vmatrix} = \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) \\ \psi_1(x_2) & \psi_2(x_2) \end{vmatrix} \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) \\ \psi_2(x_1) & \psi_1(x_1) \end{vmatrix}$$

1.3.4 Matrix Inversion

Inverse of a Matrix

$$(A^{-1})_{ij} = \frac{\text{Cofactor } A_{ji}}{|A|}$$

Cofactor $A_{ij} = (-1)^{i+j}$

Gauss-Jordan Matrix Inversion

- $A_{ij} \to \lambda A_{ij} \quad \forall j$ (Row operation: multiplication by λ)
- Subtract a row by a multiple of another.
- $A_{ij} \to A_{ij}$