Bogoliubov Transformations, Particle Creation and Entanglement

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Contents

Abstract

This report describes fundamental results of particle creation from the perspective of quantum field theory, and then applies the mathematics to quantum entanglement. Chapters 2 and 3 give necessary background into quantum field theory by introducing some results that will be referred to throughout the report, including the quantization of the simple and time dependent harmonic oscillators. In chapter 4, analytic solutions of the time dependent oscillator are discussed and interpreted, and Bogoliubov transformations are used to show the resultant particle creation. Some ideas of quantum field theory in curved spacetime are introduced in chapter 5 which will be used in later chapters on entanglement.

Chapters 6 and 7 shifts the focus to quantum entanglement. Chapter 6 introduces the notion of entanglement, as well as some the mathematics used to quantify entanglement. An example of this based on the time dependent oscillator introduced in chapter 4 is given. An interesting consequence of entanglement is that it can be used to estimate cosmological parameters, and the details of this in both the bosonic and fermionic cases are discussed. The report is concluded with a description of entanglement when taking relativity into account in chapter 7. Much of the research in this area has indicated that entanglement is destroyed from the perspective of an accelerated observer, however recent research has found that entanglement may not only be destroyed, but in some cases amplified in this scenario.

1 Introduction

The essence of quantum field theory is that each fundamental particle has its own associated field, and what we think of as particles are, in this theory, energy in the respective fields. This approach rejects the idea of localized particles and instead postulates that each particle has an associated field and the observed properties of a particle is nothing more than the excitation of its respective field. Although this approach is intuitively more difficult, it has proven to explain the observed behaviour of fundamental particles as effectively as quantum mechanics. Furthermore it has explained some of the more unusual results of quantum mechanics. For example, the uncertainty principle in field theory is no longer a paradox but rather is expected. If a particle is seen as an excitation of a field, we would expect its properties, for example, position or momentum to be uncertain/not entirely localized.

Particle fields can interact with one another, and other fields such as the gravitational or electromagnetic field. This causes fluctuations in the energy in the fields. If energy increases it is said that new particles have been created, and conversely, if energy decreases it is said that particles are annihilated. The mathematics of quantum field theory has been used to predict phenomena such as the Casimir effect and the Unruh effect which in part provide some evidence of particle creation. The Casimir effect, named after Hendrick Casimir in 1948 [1], predicted that two uncharged parallel plates in a vacuum would experience an attractive force. This was based on analysis of the quantization of the electromagnetic field, where in these circumstances, the theory predicted small fluctuations in the energy of the field. This could be interpreted as particle creation. In 1996, these predictions were experimentally verified by Steven Lamoreaux [1]. The Unruh effect, which will be discussed further in chapter 5. makes predictions of particle creation based on quantum field theory in curved spacetime. The Unruh effect is the phenomena that an accelerated observer will detect particles in the vacuum of an inertial observer.

In all cases we are interested in how to describe how a quantum field changes with time, and the consequences of the change. A mathematical tool which allows us to do this is a Bogoliubov transformation. The consequences of using a Bogoliubov transformation to describe the change in a field allows us to mathematically describe particle creation, and explicit examples of this will be given in chapter 4.

An application of quantum field theory is that it can be used to describe quantum entanglement. Quantum entanglement is the relationship that can exist between particles, such that if a measurement is made on one, the other feels the effect immediately. In other words if two particles are entangled there is a correlation between measurements made on each particle. The fact that this is the case even if the particles are separated over large distances is a paradox. Information cannot travel faster than light, so how the particles appear to be communicating is a current research topic in itself. Here we will focus on describing the mathematics of quantum entanglement which has many real word applications from quantum computing [2] to quantum cryptography [3].

2 Quantum Field Theory

In this section some fundamental principles of quantum field theory will be discussed that will be referred to throughout the report. As a starting point, we will very briefly look at some concepts of classical field theory, and then show how these ideas can be extended to describe a field on a quantum scale. This will include the quantization of the simple harmonic oscillator and a description of the notion that a infinite set of simple harmonic oscillators can describe a quantum field. An introduction to the notation that will be used throughout the report and various quantum operators will be given. Other important ideas such as the the notion of a vacuum state and a mathematical description of the mode expansion of a scalar field, which are essential background for later chapters, will be described.

2.1 Classical Field Theory

In classical mechanics the motion of a particle is usually described using Newton's second law. This describes the motion of a particle using the forces acting upon the particle and the mass of the particle. Another means of calculating the trajectory of a particle is via the Lagrangian formulation, which is based on the least action principle. Here the dynamics of a system are described using a Lagrangian. An example of a Lagrangian would be a general form of the action of a particle depending upon its trajectory. Variational methods are used to obtain Euler-Lagrange equations which, when applied to the Lagrangian will give the equation of motion of the true trajectory, that is the one with minimum action. These ideas can be extended to find the true equation of a classical field, which is described using a function which depends on both space and time, $\psi(t, \mathbf{x})$. If we know the Lagrangian (or Lagrangian density $\mathcal{L}$) of a field, then we can apply the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi)} = 0, \tag{1}$$

to find the true equation which describes the field.

At this point it is also useful to recall the Hamiltonian, which is related to the Lagrangian is given by

$$H = \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L}, \tag{2}$$

where p_i is the momenta. Furthermore we can relate the Hamiltonian to the Lagrangian through Hamilton's equations given by

$$\frac{\partial H}{\partial p} = \dot{q}, \tag{3}$$

$$\frac{\partial H}{\partial q} = -\dot{p}. \tag{4}$$

2.2 The Klein Gordon Equation

The Lagrangian density for a real scalar field is defined to be

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}m^2\psi^2 - U(\psi), \tag{5}$$

where $\eta^{\mu\nu}$ is the Minowski tensor given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and $\partial_\mu = \partial/\partial x^\mu$,where $x^\mu = (x^0, x^1, x^2, x^3)$ is the usual notation for spacetime with three spatial dimensions. Using the Minkowski tensor and noting that the presence of repeated indices implies a summation, we can rewrite (5) as

$$\mathcal{L} = \frac{1}{2}\dot{\psi}^2 - \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}m^2\psi^2 - U(\psi). \tag{6}$$

We can now apply (1) to obtain

$$-m^2\psi - \frac{\partial U}{\partial \psi} - \frac{\partial}{\partial t}\dot{\psi} + \frac{\partial}{\partial x}\psi_x + \frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z = -m^2\psi - \frac{\partial U}{\partial \psi} - \eta^{\mu\nu}\partial_\mu\partial_\nu\psi = 0. \tag{7}$$

In the case where $U(\psi) = 0$ we obtain the Klein-Gordon equation. We can express the equation in a number of ways:

$$\frac{\partial^2\psi}{\partial t^2} - \sum_{j=1}^{3}\frac{\partial^2\psi}{\partial {x_j}^2} + m^2\psi = 0 \tag{8}$$

or

$$(m^2 + \Box)\psi = 0, \tag{9}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The Klein-Gordon equation is the equation which a real scalar field must satisfy.

2.3 Quantization of the Classical Field

Any field $\psi(\mathbf{x}, t)$ can be expressed by its Fourier spatial decomposition

$$\psi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}(t), \tag{10}$$

where the integral is over $\mathbf{k}$, the set of all three dimensional vectors. Substituting (10) into (8) results in

$$\ddot{\psi}_{\mathbf{k}} + \omega_k^2 \psi_{\mathbf{k}} = 0, \tag{11}$$

where $\dot{\psi} = \partial\psi/\partial t$ and $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$. Therefore any field can be expressed in terms of a set of infinitely many harmonic oscillators, one for each $\mathbf{k}$. For the remainder of this section, we will discuss the process of the quantization of a simple harmonic oscillator.

Changing notation for simplicity, consider the case where we deal with a simple harmonic oscillator with a constant frequency, ω, as described by the equation

$$\ddot{q}(t) + \omega^2 q(t) = 0. \tag{12}$$

To quantize an equation of motion we must consider the energy of the system. The energy of a oscillator is given by

$$E = E_{kinetic} + E_{potential} = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 q^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{13}$$

where $\frac{1}{2}m\omega^2 q^2$ is the potential for a simple harmonic oscillator, and $p = mv$ is the magnitude of the momentum. Raising the energy to an operator gives the Hamiltonian operator of the system. Next we postulate the Heisenberg commutator $[\hat{q}, \hat{p}] = i\hbar$ which follows from the uncertainty principle that both position and momentum cannot be simultaneously measured with exact accuracy. Finally we define the annihilation and creation operators which increase or decrease the energy within the state, and the particle number operator which tells us the number of particles in the state. Note that in quantum field theory, an increase or decrease of energy is interpreted as the increase or decrease in the number of particles present in the system. The annihilation and creation operators respectively are given by

$$\hat{a}^- = \sqrt{\frac{\omega}{2}} \left(\hat{q} + \frac{i}{\omega}\hat{p} \right), \tag{14}$$

$$\hat{a}^\dagger = \sqrt{\frac{\omega}{2}} \left(\hat{q} - \frac{i}{\omega}\hat{p} \right). \tag{15}$$

The annihilation and creation operators are defined as above so that the following relations hold

$$\hat{a}^-|n\rangle = \sqrt{n}|n-1\rangle, \tag{16}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{17}$$

$$\hat{a}^-|0\rangle = 0. \tag{18}$$

That is, the application of the creation or annihilation operators will raise or lower the particle number (eigenvalue) of an arbitrary state (eigenvector) $|n\rangle$. (18) defines the vacuum state, ie the state with minimum energy. Note that by using (14) and (15) we can write the coordinate operator $\hat{q}$ and momenta operator $\hat{p}$ in terms of the annihilation and creation operators.

$$\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a}^- + \hat{a}^\dagger), \tag{19}$$

$$\hat{p} = \frac{\sqrt{\omega}}{i\sqrt{2}}(\hat{a}^- - \hat{a}^\dagger). \tag{20}$$

It is often advantageous to write (19) and (20) in terms of time independent annihilation and creation operators. To do this we begin with Hamilton's equations. Setting m=$\hbar$=1, (3) and (4) applied to (13) gives $p = \dot{q}$ and $\omega^2 q = -\dot{p}$. Differentiating (14) with respect to time, applying these results and comparing with (14) gives a simple first order ordinary differential equation with solution

$$\hat{a}^-(t) = \hat{a}^- e^{-i\omega t}. \tag{21}$$

Similarly we can find

$$\hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t}, \tag{22}$$

where

$$\begin{aligned} [\hat{a}^-_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] &= \delta_{\mathbf{k}\mathbf{k}'} \\ [\hat{a}^-_{\mathbf{k}}, \hat{a}^-_{\mathbf{k}'}] &= [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0. \end{aligned} \tag{23}$$

Substituting (21) and (22) into (19) and (20) gives the following expressions for the coordinate and momenta operators in terms of the time independent annihilation and creation operators $\hat{a}^-$ and $\hat{a}^\dagger$

$$\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a}^{-}e^{-i\omega t} + \hat{a}^{\dagger}e^{i\omega t}), \tag{24}$$

$$\hat{p} = \frac{\sqrt{\omega}}{i\sqrt{2}}(\hat{a}^{-}e^{-i\omega t} - \hat{a}^{\dagger}e^{i\omega t}). \tag{25}$$

Finally, it is worth noting for later use that it can be shown, using (14) and (15), that the following commutation relation is true

$$[\hat{H}, \hat{a}^{\dagger}\hat{a}^{-}] = 0. \tag{26}$$

This means that we can measure the value of both the Hamiltonian operator and $\hat{a}^{\dagger}\hat{a}^{-}$ simultaneously and with exact precision, thus we can set up the following eigenvector equations for an arbitrary state $|n\rangle$.

$$\hat{H}|n\rangle = E_n|n\rangle, \tag{27}$$

$$\hat{a}^{\dagger}\hat{a}^{-}|n\rangle = n|n\rangle, \tag{28}$$

where E_n is interpreted as the energy in the field, and the eigenvalue n is interpreted as the particle number. As a consequence the operator $\hat{a}^{\dagger}\hat{a}^{-}$ is known as the particle number operator. In this subsection we have seen that the process of quantization of a scalar field is equivalent to the quantization of a infinite collection of simple harmonic oscillators. We discussed the process of quantization of the simple harmonic oscillator, including defining creation and annihilation operators, the ground state, and the particle number operator which will be of importance later. Finally we note that for the quantization of a scalar field we must have a set of these defined operators, one for each vector $\mathbf{k}$. For example, the ground state for the scalar field is defined as

$$\hat{a}_{\mathbf{k}}^{-}|0\rangle = 0, \tag{29}$$

for all $\mathbf{k}$.

2.4 Mode Expansion

In 2.3 we saw how to express a scalar field in terms of its Fourier spatial decomposition (10) and by substituting this into the Klein-Gordon equation we deduced that the quantization of a scalar field was equivalent to quantization of a infinite set of harmonic oscillators (11). From (24) we can write ψ_k as

$$\hat{\psi}_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_k}}(\hat{a_{\mathbf{k}}}^{-}e^{-i\omega_k t} + \hat{a}_{\mathbf{k}}^{\dagger}e^{i\omega_k t}), \tag{30}$$

where $\omega_k = \sqrt{|\mathbf{k}|^2 + m^2}$.

If we now substitute this into (10) we obtain

$$\psi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (\hat{a}^-_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)} + \hat{a}^\dagger_{\mathbf{k}} e^{i(-\mathbf{k}\cdot\mathbf{x}+\omega_k t)}). \tag{31}$$

Note the minus sign has appeared in the second exponential to ensure we have a Hermitian expression. (31) is called the mode expansion of the field. Aside: $\psi_{\mathbf{k}}$ satisfies the simple harmonic oscillator equation with constant frequency. By considering the solution of a simple harmonic oscillator we find that $\psi^\star_{\mathbf{k}} = \psi_{-\mathbf{k}}$, where $\psi^\star_{\mathbf{k}}$ is the complex conjugate of $\psi_{\mathbf{k}}$. This is a property that we will refer to later.

3 Quantization of the Time Dependent Oscillator

In quantum field theory, to quantize the classical field one finds that the process involves the quantization of a infinite set of time independent harmonic oscillators 2.3. This is a consequence of considering the spatial Fourier transform applied to the Klein-Gordon equation for a free scalar field. In reality however, most fields which we need to consider are more complex than the ones described using harmonic oscillators with constant frequencies. This is because we need to allow the field to change in time to reflect its interaction with other fields such as the gravitational field. To represent this change in the field we must consider quantizing time dependent harmonic oscillators of the form

$$\ddot{q}(t) + w^2(t)q(t) = 0. \tag{32}$$

In general the time dependent frequency $w(t)$ could take any form, but a useful and common form for this function is one which is almost constant for some time interval $t < t_0$ ('in regime'), which then changes, and returns to a (different) constant $t > t_1$ ('out regime'), see Figure 1. Note this could describe the situation when the field is asymptotically flat in the past and future, but a change occurs in between, which will be the focus of our analysis in later chapters.

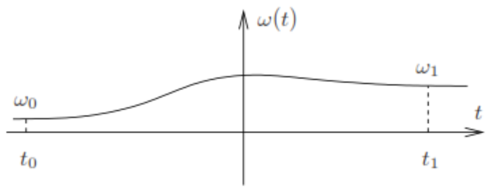

Figure 1: Oscillator with varying frequency

In general the time dependent oscillator (32) cannot be solved analytically, however two examples of when it can be will be calculated in 4 after the preliminary ideas of particle creation are discussed. By considering the relationship between the solution to the time dependent oscillator in the in region and out region, interesting consequences arise in terms of the change in energy within the field, which can be interpreted as particle creation or annihilation. The mathematical tool which describes this change in the field is called a Bogoliubov transformation.

In 2.3 we showed that for a time independent field, the coordinate and momenta operators could be written in terms of the annihilation and creation operators, see (24) and (25). It follows that as the frequency of the oscillator is time dependent, we consider the following for the time dependent field.

$$\hat{q}(t) = \frac{1}{\sqrt{2w(t)}}(\hat{a}^- e^{-iw(t)t} + \hat{a}^\dagger e^{iw(t)t}), \tag{33}$$

or more simply

$$\hat{q}(t) = \frac{1}{\sqrt{2}}(\hat{a}^- v^\star(t) + \hat{a}^\dagger v(t)), \tag{34}$$

where $v(t)$ is a unknown function (known as the mode function) to be determined. A similar expression is found for $\hat{p}$

$$\hat{p}(t) = \frac{1}{\sqrt{2}}(\hat{a}^- \dot{v^\star}(t) + \hat{a}^\dagger \dot{v(t)}). \tag{35}$$

To determine to form of the mode function $v(t)$ we note that it must satisfy the following conditions:

1. $[\hat{q}, \hat{p}] = i$
2. The mean energy at ground state given by $\langle 0|\hat{H}|0\rangle$ is minimised.

Using (34) and (35), and the definition of the commutation relation, it can be shown that

$$[\hat{q}, \hat{p}] = \frac{1}{2}([\hat{a}^-, \hat{a}^\dagger](v^\star \dot{v} - \dot{v}^\star v)). \tag{36}$$

Hence $[\hat{q}, \hat{p}] = i$ implies that

$$[\hat{a}^-, \hat{a}^\dagger] = \frac{2i}{(v^\star \dot{v} - \dot{v}^\star v)}. \tag{37}$$

The term $(v^\star \dot{v} - \dot{v}^\star v)$ is known as the Wronskian of $v(t)$ and must be constant [4], hence we choose

$$(v^\star \dot{v} - \dot{v}^\star v) = 2i \tag{38}$$

to recover the commutation relation $[\hat{a}^-, \hat{a}^\dagger] = 1$ as before.

Using (34) and (35) we can also find expression for the creation and annihilation operators in terms of the mode functions,

$$\hat{a}^{-} = \frac{\dot{v}(t)\hat{q}(t) - v(t)\hat{p}(t)}{i\sqrt{2}}, \tag{39}$$

$$\hat{a}^{\dagger} = -\frac{\dot{v}^{\star}(t)\hat{q}(t) - v^{\star}(t)\hat{p}(t)}{i\sqrt{2}}. \tag{40}$$

The choice of the mode function clearly will effect the creation and annihilation operators. Next we will consider which mode function satisfies the physical condition that ground state has minimal energy. Similar to that as shown in section 2.3, the Hamiltonian for a time dependent system is given by

$$\hat{H}(t) = \frac{\hat{p}^2}{2} + \frac{w^2(t)\hat{q}^2}{2}. \tag{41}$$

To ensure our ground state has minimal energy, we are required to minimize $\langle 0|\hat{H}|0\rangle$. Using (34) and (35) in (41) we obtain

$$E(t) = \langle 0|\hat{H}|0\rangle = \frac{(|\dot{v}|^2 + w^2(t)|v|^2)}{4}. \tag{42}$$

At a fixed time t_0, the instantaneous ground state which arises from minimizing $E(t_0)$ results in the following conditions for the mode functions

$$v(t_0) = \frac{1}{\sqrt{w(t_0)}}, v(\dot{t_0}) = i\sqrt{w(t_0)} = iw(t_0)v(t_0). \tag{43}$$

These conditions on the mode function result in a minimal energy at ground zero at the instantaneous time t_0. Note that if we substitute the conditions (43) into (42) the we get that the minimum energy in the vacuum at an instant t_0 is given by

$$E(t_0) = \frac{1}{2}\omega(t_0). \tag{44}$$

This result should be expected if we compare it with the allowed energy levels of a harmonic oscillator

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \tag{45}$$

with $n = 0$ and $\hbar = 1$.

3.1 Bogoliubov Transformation for the Time Dependent field

We have seen that in a time dependent field, it is often the case that the frequency of the oscillators which describes the field at some time, say $t < t_0$ is almost constant, and at $t > t_1$ the frequency of the oscillators is almost a (different) constant. In this section we will look at a simple case of a Bogoliubov transformation which relates the states of the field at $t < t_0$ and $t > t_1$. From (34) we see that $\hat{q}$ is a linear combination of $v(t)$ and its complex conjugate. (32) shows the oscillator equation which $q(t)$ must satisfy, so it follows that $v(t)$ is also a solution to (32), thus

$$\ddot{v}(t) + w^2(t)v(t) = 0. \tag{46}$$

In the cases where $t < t_0$ and $t > t_1$, $w(t)$ is approximately constants w_0 and w_1 respectively. Hence in these cases, (46) is simply a second order ordinary differential equation with constant coefficients which can be solved using the initial conditions (43). This gives solutions

$$v_{in}(t) \propto e^{iw_0 t} \tag{47}$$

and

$$v_{out}(t) \propto e^{iw_1 t}, \tag{48}$$

where $v_{in}(t)$ is the solution for the in regime where $t < t_0$ and $v_{out}(t)$ is the solution for the out regime where $t > t_1$. As any two solutions to (46) form a basis for all solutions [4] we can write the relationship between the in and out states as

$$v_{in}(t) = \alpha v_{out}(t) + \beta v^{\star}_{out}(t), \tag{49}$$

where α and β are known as Bogoliubov coefficients. This is an example of a Bogoliubov transformation. From (39) and (40) we can see that a relationship between the in and our regime for the creation and annihilation operators is possible:

$$\hat{a}^-_{in} = \alpha \hat{a}^-_{out} - \beta \hat{a}^\dagger_{out}, \tag{50}$$

$$\hat{a}^-_{out} = \alpha^\star \hat{a}^-_{in} + \beta \hat{a}^\dagger_{in}. \tag{51}$$

If β is non-zero we observe from (50) that in this case, the annihilation operator in the in regime is not equal to the annihilation operator in the out regime. This implies that there are different ground states for $t < t_0$ and $t > t_1$. In section 2.3 we saw that the particle number operator is given by $\hat{a}^\dagger \hat{a}^-$. Using the particle number operator we can calculate the mean number of particles in the in regime and out regime. For the in regime,

$$\langle 0_{in} | \hat{a}^\dagger_{in} \hat{a}^-_{in} | 0_{in} \rangle = 0. \tag{52}$$

For the out regime we use (51) to obtain

$$\langle 0_{in} | \hat{a}^\dagger_{out} \hat{a}^-_{out} | 0_{in} \rangle = \langle 0_{in} | (\alpha \hat{a}^\dagger_{in} + \beta^\star \hat{a}^-_{in})(\alpha^\star \hat{a}^-_{in} + \beta \hat{a}^\dagger_{in}) | 0_{in} \rangle = |\beta|^2. \tag{53}$$

Therefore we see that if there is an average of zero particles in the in regime, and as β is non-zero, there is a positive number of particles in the out regime.

3.2 Relationship between in and out states

Now we will consider the relationship between states in the in regime and the states in the out regime. The mathematics in this subsection will be expanded upon in section 6.2 when we consider entanglement. Let $|n_{in}\rangle$ and $|n_{out}\rangle$ form a basis for the in and out regimes respectively. It is intuitively reasonable that if the in regime describes a vacuum state and the out regime is in an excited state, then the out regime must be a superposition of the original vacuum and the excited states $|n_{in}\rangle, n = 1, 2, ...$ However we can also express $|0_{in}\rangle$ as a superposition of the out states, as the out states form a basis for the space, that is we can write

$$|0_{in}\rangle = \sum_{n=0}^{\infty} c_n |n_{out}\rangle. \tag{54}$$

To find an expression for the coefficients c_n we use the definition of a vacuum and (50) to write

$$\begin{aligned} 0 &= \hat{a}_{in}^{-}|0_{in}\rangle \\ &= (\alpha \hat{a}_{out}^{-} - \beta \hat{a}_{out}^{\dagger}) \sum_{n=0}^{\infty} c_n |n_{out}\rangle. \end{aligned} \tag{55}$$

Expanding this and using (16) and (17) we can obtain

$$c_{2n} = c_0 \left(\frac{\beta}{\alpha}\right)^n \frac{\sqrt{(2n-1)!!}}{\sqrt{(2n!!)}}. \tag{56}$$

Using the normalization condition that $\langle 0_{in}|0_{in}\rangle = 1$ and (54) we can deduce that c_0 satisfies

$$|c_0|^2 \sum_{n=0}^{\infty} \left(\frac{|\beta|}{|\alpha|}\right)^{2n} \left(\frac{(2n-1)!!}{2n!!}\right) = 1. \tag{57}$$

After a lot of algebra this equation can be solved to find

$$c_0 = \left(1 - \left(\frac{|\beta|}{|\alpha|}\right)^2\right)^{\frac{1}{4}}. \tag{58}$$

Therefore we have that

$$c_{2n} = \left(1 - \left(\frac{|\beta|}{|\alpha|}\right)^2\right)^{\frac{1}{4}} \left(\frac{\beta}{\alpha}\right)^n \frac{\sqrt{(2n-1)!!}}{\sqrt{(2n!!)}}. \tag{59}$$

4 Quantum Field Theory for Curved Spacetime

4.1 Generalized coordinates and conformally flat spacetime

Up to now we have looked at quantum field theory in what is known as 'flat' spacetime and this is described using the Minkowski tensor, see (5) where the Minkowski tensor $\eta_{\mu\nu}$ was defined and used explicitly in the expression for the Lagrangian density, which was then used as a basis for our entire analysis up to this point. The Minkowski tensor is a special case of a more generalised tensor which is used to describe spacetime which is not acted upon by external forces such as gravity. In order to analyse a quantum field in the presence of gravity (curved spacetime), we will need to use a more generalized metric denoted $g_{\mu\nu}$ and defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{60}$$

where the notation x^μ is the usual notation used in relativity to denote coordinates. As the metric $g_{\mu\nu}$ could be viewed as a matrix, we define $g^{\mu\nu}$ as it's inverse and g as its determinant. In curved spacetime the action for a free scalar field is given by [4]

$$S = \int \sqrt{-g} d^4x \left[\frac{1}{2} g^{\alpha\beta} (\partial_\alpha \psi)(\partial_\beta \psi) - \frac{1}{2} m^2 \psi^2 \right]. \tag{61}$$

Note the similarity between the integrand and the form of the Lagrangian for a flat spacetime (5). Using the Euler-Lagrange equations, the equation of motion for ψ is given by

$$g^{\mu\nu} \partial_\mu \partial_\nu \psi + \frac{1}{\sqrt{-g}} (\partial^\nu \psi) \partial_\mu (g^{\mu\nu}) + m^2 \psi = 0, \tag{62}$$

which is analogous to the Klein-Gordon equation, but for curved spacetime. A special case of the metric $g_{\mu\nu}$ which describes curved spacetime is given by

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \tag{63}$$

This is almost analogous to the Minkowski tensor, except we now have a time dependent factor $a^2(t)$ which makes the four dimensional spacetime curved. This form of spacetime is called the Friedmann-Robertson-Walker (FRW) spacetime [4] and is consistent with experimental results. In order to quantize the field like we did in sections 2 and 3, we need to introduce a new parameter in order to make the spacetime conformally flat. The new parameter will replace our time coordinate with a conformal time coordinate η defined by

$$\eta = \int_{t_0}^{t} \frac{dt}{a(t)}. \tag{64}$$

Taking the differential of this and substituting our result into (63) we obtain a conformally flat metric.

$$ds^2 = a^2(\eta)[d\eta^2 - d\mathbf{x}^2]. \tag{65}$$

This is the same as the Minkowski tensor multiplied by a factor of $a^2(\eta)$. In order to rewrite the action (61) we need to introduce another substitution of the form $\chi = a(\eta)\psi$. Using this substitution and the equation for the conformal time metric, we can write the conformal action as

$$S = \frac{1}{2}\int [\chi'^2 - (\nabla\chi)^2 - m_{eff}^2(\eta)\chi^2] d^3\mathbf{x} d\eta, \tag{66}$$

where $\chi' = \partial\chi/\partial\eta$ and $m_{eff}^2 = m^2a^2 - (a''/a)$. Noting that $\chi = a(\eta)\psi$ is a function of both η and the spatial coordinates $\mathbf{x}$, we can apply the Euler Lagrange equation to obtain the equation of motion of χ. This results in

$$\chi'' + \left(m^2a^2 - \frac{a''}{a}\right)\chi - \triangle\chi = 0, \tag{67}$$

where $\triangle\chi = \chi_{xx} + \chi_{yy} + \chi_{zz}$ (in three spatial dimensions).

4.2 Quantization of conformally flat spacetime

Exactly analogous to (10), we write the field χ in terms of its spatial Fourier transform,

$$\chi(\mathbf{x}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \chi_{\mathbf{k}}(\eta). \tag{68}$$

Substituting this into (67) results in a infinite collection of (conformal) time dependent harmonic oscillators given by

$$\chi''_{\mathbf{k}} + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0, \tag{69}$$

where $\omega_k^2(\eta) = k^2 + m^2a^2 - \frac{a''}{a}$. We have already seen how to quantize time dependent oscillators and here we will briefly extend this idea to fields. From equation (34) we know that $\chi_{\mathbf{k}}$ can be written in terms of mode functions as follows

$$\chi_{\mathbf{k}} = \frac{1}{\sqrt{2}}[a_{\mathbf{k}}^- v_k^\star(\eta) + a_{\mathbf{k}}^\dagger v_k(\eta)]. \tag{70}$$

Substituting this into (68) and raising everything to operators we obtain the mode expansion of the field

$$\hat{\chi}(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} [\hat{a}^-_{\mathbf{k}} v^\star_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{71}$$

As before, annihilation and creation operators can be defined to describe properties of the quantum field. These will depend on mode functions like we saw in section 3. As before, the mode function is chosen such that equation (38) holds. There are no unique set of mode functions which satisfy this relation [4]. The selected mode functions must minimize the ground state energy, and we will consider this in the next section. However we note here that in a time dependent field the Hamiltonian is time dependent, so naturally the ground state energy is also time dependent, and as such we will require mode functions which describe the minimum vacuum energy at different instants in time. The same mode function will not in general describe the ground state at a moment in time later, instead there will be a new mode function which minimizes that ground state energy at that time. Suppose there are two sets of mode functions $u_k(\eta)$ and $v_k(\eta)$ which satisfy the Wronskian, then we can write down the Bogoliubov transformation of the mode functions by using $u_k(\eta)$ and its complex conjugate as a basis for $v_k(\eta)$,

$$v^\star_k(\eta) = \alpha_k u^\star_k(\eta) + \beta_k u_k(\eta). \tag{72}$$

Under the assumption that $v^\star_k = v_{-k}$ (see section 2.4) we could also write this as

$$v_k(\eta) = \alpha_k u_k(\eta) + \beta^\star_k u_{-k}(\eta). \tag{73}$$

Note that the two sets of mode functions $v_k(\eta)$ and $u_k(\eta)$ define two sets of annihilation operators $\hat{a}^-_{\mathbf{k}}$ and $\hat{b}^-_{\mathbf{k}}$, which therefore define two different vacuum states $|0_{(a)}\rangle$ and $|0_{(b)}\rangle$ respectively. Hence we can also rewrite (71) in terms of the $u_k(\eta)$ mode functions and their annihilation and creation operators,

$$\hat{\chi}(\mathbf{x},\eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} [\hat{b}^-_{\mathbf{k}} u^\star_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}^\dagger_{\mathbf{k}} u_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{74}$$

As (71) and (74) are equivelent, we can equate the right hand sides and using (72) we can deduce

$$\hat{b}^-_{\mathbf{k}} = \alpha_k \hat{a}^-_{\mathbf{k}} + \beta^\star_k \hat{a}^\dagger_{-\mathbf{k}}, \tag{75}$$

$$\hat{b}^\dagger_{\mathbf{k}} = \alpha^\star_k \hat{a}^\dagger_{\mathbf{k}} + \beta_k \hat{a}^-_{-\mathbf{k}}. \tag{76}$$

These represent the Bogoliubov transformations between the annihilation and creation operators that relate to the two different sets of mode functions.

4.3 Interpretation and Mean Particle Number

We have defined two sets of mode functions which describe two sets of vacuum states $|0_{(a)}\rangle$ and $|0_{(b)}\rangle$. As the mode functions at any moment in time must ensure minimum ground state energy, we can derive exactly as in section 3 the conditions which the mode functions must satisfy

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, v_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)} = i\omega_k(\eta_0)v_k(\eta_0). \tag{77}$$

For a time dependent field it is clear that the mode function which satisfies (77) may not satisfy these conditions at a later time. Therefore we have considered two different sets of mode functions by introducing $u_k(\eta)$. Now each of these mode functions will define a different set of annihilation and creation operators and as such different vacuum states. Vacuum states $|0_{(a)}\rangle$ and $|0_{(b)}\rangle$ will have different ground state energy. What we are interested in is how much energy has been created in the transition of the field from the vacuum state $|0_{(a)}\rangle$ to the vacuum state $|0_{(b)}\rangle$ (or vice versa) as this will indicate particle creation. As the vacuum $|0_{(b)}\rangle$ already contains a minimum amount of energy which may be higher or lower that that in $|0_{(a)}\rangle$ we want to quantify on average how much new energy has been created. For this reason it is sometimes easier to think of two sets of particles relating the the two vacuum, ie $|0_{(a)}\rangle$ relates the lack of a particles, and $|0_{(b)}\rangle$ relates the lack of b particles. What we aim to deduce is given a vacuum state of a particles, how many b particles are present, that is if the field changes due to gravity how many new particles are created, taking into account that the new ground state will have a different amount of energy to the previous one.

Mathematically we can define the b particle number operator $\hat{N}_{\mathbf{k}}^{(b)} = \hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}^{-}$ and calculate the expected number of b particles present in the a vacuum,

$$\langle 0_{(a)}|\hat{N}_{\mathbf{k}}^{(b)}|0_{(a)}\rangle = \langle 0_{(a)}|\hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}^{-}|0_{(a)}\rangle = |\beta_k|^2. \tag{78}$$

This result has been calculated using (75) and (76) to express everything in terms of the operators $\hat{a}_{\mathbf{k}}^{-}$ and $\hat{a}_{\mathbf{k}}^{\dagger}$ and the basic results (16) and (17) have been used to simplify the resultant expression. Note that $|\beta_k|^2$ represents the mean density of particles present in the mode $\chi_{\mathbf{k}}$ and so the total number of particles can be found by integrating over all modes.

Up to this point we have shown how the time dependent oscillator is used to describe a quantum field under the influence of gravity. Various Bogoliubov transformations between operators at different points in time have been derived, and we have seen the conditions required for particle creation. We will now apply these ideas to two explicit examples which lead to analytic solutions of the time dependent oscillator. Using the solutions we will calculate the Bogoliubov transformations and deduce the particles are created.

4.4 Calculation of Bogoliubov Coefficients - Simple Case

Consider equation (72). Suppose $v_k(\eta)$ and $u_k(\eta)$ describe the vacuum state at times η_0 and η_1. In order to calculate the Bogoliubov coefficients we need the exact solution of $v_k(\eta)$ and $u_k(\eta)$ subject the conditions (77) at only one instant in time, say at η_0. The following example will illustrate this principle in its simplest form.

Consider the simple case where a time dependent oscillator has frequency given by

$$\omega^2(t) = \begin{cases} \omega_1^2 & \text{if } x < 0 \\ \omega_2^2 & \text{if } x > 0, \end{cases} \tag{79}$$

where ω_1^2 and ω_2^2 are constants. Given that the time dependent oscillator is of the form $\ddot{v} + \omega^2(t)v = 0$ we can solve this equation exactly in both $x < 0$ and $x > 0$ where the frequency is constant. Let $v_{in}(t)$ and $v_{out}(t)$ describe this solution in the regions $x < 0$ and $x > 0$ respectively. Using (43) we deduce the conditions which the mode function which describes the vacuum state at some time t_0 in the in regime and at time t_1 in the out regime to be

$$v_{in}(t_0) = \frac{1}{\sqrt{\omega_1}}, \tag{80}$$

$$\dot{v}_{in}(t_0) = i\sqrt{\omega_1}, \tag{81}$$

with similar expressions for $v_{out}(t_1)$ and $\dot{v}_{out}(t_1)$ in terms of ω_2. Hence applying these conditions gives the following ground state energy solutions

$$v_{in}(t) = \frac{1}{\sqrt{\omega_1}} e^{i\omega_1(t-t_0)}, \tag{82}$$

$$v_{out}(t) = \frac{1}{\sqrt{\omega_2}} e^{i\omega_2(t-t_1)}. \tag{83}$$

Note that in each of the regimes we can recover the ground state energy as in (44), and we could also deduce the annihilation and creation operators in each regime by substituting (80) and (81) into (39) and (40) to obtain (14) and (15) as we would expect for an oscillator with constant frequencies.

The Bogoliubov transformation is given by (49). Substituting (83) into (49) gives

$$v_{in}(t) = \frac{\alpha}{\sqrt{\omega_2}} e^{i\omega_2(t-t_1)} + \frac{\beta}{\sqrt{\omega_2}} e^{-i\omega_2(t-t_1)}, \tag{84}$$

from which we can calculate its derivative

$$\dot{v}_{in}(t) = \frac{i\alpha\omega_2}{\sqrt{\omega_2}} e^{i\omega_2(t-t_1)} - \frac{i\beta\omega_2}{\sqrt{\omega_2}} e^{-i\omega_2(t-t_1)}. \tag{85}$$

Evaluating (84) and (85) at t_0 and equating to (80) and (81) allows us to compute α and β to be

$$\alpha = \frac{1}{2}\sqrt{\frac{\omega_2}{\omega_1}} \left(\frac{\omega_1 + \omega_2}{\omega_2}\right) e^{i\omega_2(t_1-t_0)}, \tag{86}$$

$$\beta = \frac{1}{2}\sqrt{\frac{\omega_2}{\omega_1}} \left(\frac{\omega_2 - \omega_1}{\omega_2}\right) e^{i\omega_2(t_0-t_1)}. \tag{87}$$

Recall from (53) we found that if β is non zero then we have particle creation. The expressions for α and β show that $\beta = 0$ only if $\omega_1 = \omega_2$ (assuming non-zero frequencies). Hence if $\omega_1 = \omega_2$ there is no particle creation. This is equivalent to a quantum field which has a time independent frequency. In cases where $\omega_1 \neq \omega_2$ we have particle creation.

4.5 Analytic Solution of the Time Dependent Oscillator

The time dependent oscillator, (46) is most often solved numerically, as the presence of the time dependent frequency leads to very complicated, and often non-analytic solutions. In section 4.4 we looked at the most simple case where the frequency was (different) constant in the in and out regimes. In this section we look at a more complex form of the time dependent frequency which gives rise to an analytic solution. Once we have calculated this solution we will find the Bogoliubov transformation. We start with (46) and

$$\omega^2(t) = A + Btanh(\lambda t), \tag{88}$$

where A, B and λ are constants (in section 6.4 we will interpret the meaning of these constants). Note that by considering how ω behaves as t tends to positive (out region) or negative (in region) infinity we can define

$$\omega_{in} = \sqrt{A - B}, \tag{89}$$

$$\omega_{out} = \sqrt{A + B}. \tag{90}$$

In order to make progress we must use some transformations to rewrite our equation in a form which can be solved. Let $z = 1/(1 + e^{-2\lambda t})$. Using the chain rule we can find an expression for $\ddot{v}(z)$ and we can write

$$\omega^2(t) = A + B\left(\frac{e^{\lambda t} - e^{-\lambda t}}{e^{\lambda t} + e^{\lambda t}}\right) = A + B\left(\frac{2}{1 + e^{-2\lambda t}} - 1\right) = A + B(2z - 1) = A - B + 2Z. \tag{91}$$

Using these substitutions our original equation becomes

$$z(1-z)v'' + (1-2z)v' + \left(\frac{A - B + 2BZ}{4\lambda^2 z(1-z)}\right) v = 0, \tag{92}$$

where $v' = dv/dz$. A further substitution is required. Let $v = z^\alpha(1-z)^\beta f(z)$. Then taking the necessary derivatives and substituting into (92) we obtain

$$\begin{aligned} &z(1-z)f'' + [2\alpha + 1 - (2\alpha + 2\beta + 2)z]f' \\ &+ \left(-(\alpha+\beta)(\alpha+\beta+1) + \frac{\alpha^2}{z} + \frac{\beta^2}{(1-z)} + \frac{A-B}{4\lambda^2 z} + \frac{A+B}{4\lambda^2(1-z)}\right) f = 0. \end{aligned} \tag{93}$$

We note that this is almost of the form of a hypergeometric differential equation [5]. To obtain the correct form we now define the values of α and β to be

$$\alpha = \frac{-i\sqrt{A-B}}{2\lambda} = \frac{-i\omega_{in}}{2\lambda}, \tag{94}$$

$$\beta = \frac{i\sqrt{A+B}}{2\lambda} = \frac{i\omega_{out}}{2\lambda}, \tag{95}$$

where the positive or negative roots have been taken for later convenience. With these choices of α and β (93) reduces to

$$z(1-z)f'' + [2\alpha + 1 - (2\alpha + 2\beta + 2)z]f' - (\alpha+\beta)(\alpha+\beta+1)f = 0, \tag{96}$$

which is the exact form of the hypergeometric differential equation where the parameters a, b, c in the general form [5] are related to our parameters α, β by

$$\begin{aligned} a &= \alpha + \beta + 1, \\ b &= \alpha + \beta, \\ c &= 2\alpha + 1. \end{aligned} \tag{97}$$

We now aim to solve the hypergeometric differential equation in the in and out regions. Earlier we defined

$$z = \frac{1}{1 + e^{-2\lambda t}}. \tag{98}$$

Hence as $t \to \infty$, $z \to 1$ and as $t \to -\infty$, $z \to 0$. Therefore to find solutions of the hypergeometric differential equation in the in and out regimes we are required to find solutions about $z = 0$ and $z = 1$ respectively. These solutions are given by [5]

$$f_{in}(z) = F(\alpha + \beta + 1, \alpha + \beta, 2\alpha + 1, z), \tag{99}$$

$$f_{out}(z) = F(\alpha + \beta + 1, \alpha + \beta, 2\beta + 1, 1 - z), \tag{100}$$

where $F(a, b, c, z)$ is the hypergeometric function defined by

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a_n)(b_n)}{(c)_n} \frac{z^n}{n!}, \tag{101}$$

where

$$(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ q(q+1)\cdots(q+n-1) & \text{if } n > 0. \end{cases} \tag{102}$$

The Bogoliubov transformation which gives the relation between the in and out regions will be of the form

$$f_{in} = \delta f_{out} + \gamma f^{\star}_{out}, \tag{103}$$

where δ and γ are Bogoliubov coefficients. Given the form of (99) and (100) there exist defined linear transformation formula which relates (99) and (100) whilst completely specifying the Bogoliubov coefficients δ and γ. This transformation is given by [6]

$$f_{in}(z) = \frac{\Gamma(2\alpha+1)\Gamma(-2\beta)}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta+1)} f_{out}(z) + \frac{\Gamma(2\alpha+1)\Gamma(2\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\beta)} f^{\star}_{out}(z). \tag{104}$$

Noting that $v = z^{\alpha}(1-z)^{\beta} f(z)$ we can obtain the required solutions for $v(z)$ in the in and out regions:

$$v_{in}(z) = z^{\alpha}(1-z)^{\beta} f_{in}(z) = z^{\alpha}(1-z)^{\beta} F(\alpha + \beta + 1, \alpha + \beta, 2\alpha + 1, z), \tag{105}$$

$$v_{out}(z) = z^{\alpha}(1-z)^{\beta} f_{out}(z) = z^{\alpha}(1-z)^{\beta} F(\alpha + \beta + 1, \alpha + \beta, 2\beta + 1, 1 - z). \tag{106}$$

Finally to obtain $v(t)$ first let us introduce the following to simplify the resultant expression:

$$\omega_+ = \frac{\omega_{in} + \omega_{out}}{2} = -i\lambda(\beta - \alpha), \tag{107}$$

$$\omega_- = \frac{\omega_{in} - \omega_{out}}{2} = -i\lambda(\beta + \alpha), \tag{108}$$

where (94) and (95) have been used to express ω_{in} and ω_{out} in terms of α and β. Also note that given (98) we can easily deduce the form of $1 - z$. Using these expressions we can write

$$\begin{aligned}
z^\alpha (1-z)^\beta &= \left(\frac{1}{1+e^{-2\lambda t}}\right)^\alpha \left(\frac{e^{-2\lambda t}}{1+e^{-2\lambda t}}\right)^\beta \\
&= \left(\frac{e^{\lambda t}}{e^{\lambda t}+e^{-\lambda t}}\right)^\alpha \left(\frac{e^{-\lambda t}}{e^{\lambda t}+e^{-\lambda t}}\right)^\beta \\
&= e^{\lambda\alpha t} e^{-\lambda\beta t} \left(\frac{1}{e^{\lambda t}+e^{-\lambda t}}\right)^{\alpha+\beta} \\
&= e^{-it[-i\lambda(\beta-\alpha)]} (2cosh(\lambda t))^{-(\alpha+\beta)} \\
&= e^{-it\omega_+} e^{-(\alpha+\beta) ln(2cosh(\lambda t))} \\
&= e^{i(-\omega_+ t - \frac{\omega_-}{\lambda} ln(2cosh(\lambda t)))}.
\end{aligned} \tag{109}$$

Substituting this into (105) and (106) we obtain

$$v_{in}(t) = e^{i(-\omega_+ t - \frac{\omega_-}{\lambda} ln(2cosh(\lambda t)))} F\left(\alpha+\beta+1, \alpha+\beta, 2\alpha+1, \frac{1+tanh(\lambda t)}{2}\right), \tag{110}$$

$$v_{out}(t) = e^{i(-\omega_+ t - \frac{\omega_-}{\lambda} ln(2cosh(\lambda t)))} F\left(\alpha+\beta+1, \alpha+\beta, 2\beta+1, \frac{1-tanh(\lambda t)}{2}\right), \tag{111}$$

noting that z and $1 - z$ can be wrtten in terms of $tanh(\lambda t)$. Therefore the Bogoliubov transformation relating $v_{in}(t)$ and $v_{out}(t)$ is given by

$$v_{in}(t) = \frac{\Gamma(2\alpha+1)\Gamma(-2\beta)}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta+1)} v_{out}(t) + \frac{\Gamma(2\alpha+1)\Gamma(2\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\beta)} v^\star_{out}(t), \tag{112}$$

where $v_{in}(t)$ and $v_{out}(t)$ are defined by (110) and (111). Note that this can also be written as

$$v_{in}(t) = \frac{\Gamma(1-(i/\lambda)\omega_{in})\Gamma(-(i/\lambda)\omega_{out})}{\Gamma(-(i/\lambda)\omega_+)\Gamma(1-(i/\lambda)\omega_+)} v_{out}(t) + \frac{\Gamma(1-(i/\lambda)\omega_{in})\Gamma((i/\lambda)\omega_{out})}{\Gamma((i/\lambda)\omega_-)\Gamma(1+(i/\lambda)\omega_-)} v^\star_{out}(t), \tag{113}$$

using (107), (108), (94) and (95). This particular form will be referred to later in the report when we contrast these Bogoliubov coefficients with the Dirac field analog.

As noted in section 115, particle creation occurs if $|\beta|^2$ is non-zero. Using properties of the gamma function [11] we can deduce that

$$|\alpha|^2 = \frac{\omega_{in}}{\omega_{out}} \frac{sinh^2(\pi\omega_+/\lambda)}{sinh(\pi\omega_{in}/\lambda)sinh(\pi\omega_{out}/\lambda)}, \tag{114}$$

$$|\beta|^2 = \frac{\omega_{in}}{\omega_{out}} \frac{sinh^2(\pi\omega_-/\lambda)}{sinh(\pi\omega_{in}/\lambda)sinh(\pi\omega_{out}/\lambda)}. \tag{115}$$

Note here that we have labeled the Bogoliubov coefficients by α and β to be consistent with notation used in previous sections; these should not be confused with (94) and (95).

We observe that $|\beta|^2 = 0$ only if $\omega_- = 0$. From (108) this only happens when $\omega_{in} = \omega_{out}$. This corresponds to the case where there is no change in the frequency from the in region to the out region, in other words flat spacetime. In the case where $\omega_{in} \neq \omega_{out}$ we have that $|\beta|^2 \neq 0$ and so we have particle creation.

We have found an analytic solution of the oscillator equation with a time dependent frequency, and the Bogoliubov transformation which describes the transformation in the field from the asymptotically flat in and out regions at times in the distant past and future respectively. The Bogoliubov coefficients show that as we have a non-zero β coefficient, we have particle creation. As the quantization process of a field leads to an infinite set of harmonic oscillators, one for each vector $\mathbf{k}$, when considering the quantization of the field we note that the above calculations will include a dependence on k, i.e

$$v_{k_{in}}(t) = \frac{\Gamma(2\alpha+1)\Gamma(-2\beta)}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta+1)} v_{k_{out}}(t) + \frac{\Gamma(2\alpha+1)\Gamma(2\beta)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha+\beta)} v^{\star}_{-k_{out}}(t), \tag{116}$$

where the k dependence arises from [7]

$$\omega_{in} = \sqrt{k^2 + m^2(A-B)}, \tag{117}$$

$$\omega_{out} = \sqrt{k^2 + m^2(A+B)}, \tag{118}$$

where m is the mass of the field.

5 Quantum Field Theory and Relativity

5.1 Notation and Preliminary Ideas

In this section we will consider how accelerated observers perceive a quantum field and the implications of this on particle creation. We will consider two frames of reference; the inertial frame of one observer, and the frame of an observer who is moving with constant acceleration (proper frame) $\mathbf{a} = (a, 0, 0)$ with respect to the inertial observer. Each observer has their own set of coordinates. Here we will limit the case to the one spatial dimension case, so that the inertial frame has coordinate (t, x) and the proper frame uses coordinates (τ, ζ). The relationship between such coordinate systems are given by

$$t(\tau, \zeta) = \left(\frac{1 + a\zeta}{a}\right) sinh(a\tau), \tag{119}$$

$$x(\tau, \zeta) = \left(\frac{1 + a\zeta}{a}\right) cosh(a\tau), \tag{120}$$

$$\tau(t, x) = \frac{1}{2a} ln\left(\frac{x + t}{x - t}\right), \tag{121}$$

$$\zeta(t, x) = -\frac{1}{a} + \sqrt{x^2 - t^2}. \tag{122}$$

The spacetime diagram 2 shows the relationship between these two sets of coordinates. The hyperbolic lines are the coordinate system of the accelerated observer from the perspective of the inertial observer. The hyperbolic worldline with directional arrows indicates the worldline of the accelerated observer, and events P,Q and R are not covered by the proper coordinates [4]. As such this type of coordinate system is known as incomplete. We will see later an alternative to this is section 7 where a different relationship between the coordinate systems is required to analyze entanglement between particles in Minkowski spacetime, from the perspective of Rindler spacetime.

As we are working in two dimensional spacetime the Minkowski metric is given by $ds^2 = dt^2 - dx^2$ and so using the relationship between the coordinate systems we can derive the analogous expression for the proper frame

$$ds^2 = dt^2 - dx^2 = (1 + a\zeta)^2 d\tau^2 - d\zeta^2. \tag{123}$$

The spacetime with this metric is called Rindler spacetime. In order to proceed with the quantization process, we must first use a change of variable to simplify things. Let $\hat{\zeta} = \frac{1}{a} ln(1 + a\zeta)$ then $d\zeta = (1 + a\zeta) d\hat{\zeta}$ hence

$$ds^2 = (1 + a\zeta)^2 d\tau^2 - (1 + a\zeta)^2 d\hat{\zeta}^2. \tag{124}$$

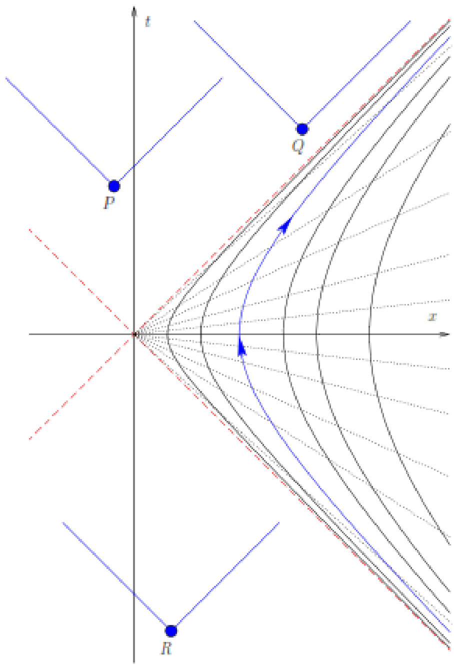

Figure 2: The accelerated observer frame from the Minkowski perspective

If $\hat{\zeta} = \frac{1}{a} ln(1 + a\zeta)$ then rearranging gives $(1 + a\zeta)^2 = e^{2a\hat{\zeta}}$ and so

$$ds^2 = e^{2a\hat{\zeta}}(d\tau^2 - d\hat{\zeta}^2). \tag{125}$$

Note here that the invariant line element is simply the Minkowski line element multiplied by a factor of $e^{2a\hat{\zeta}}$. We can now also rewrite (119) and (120) in terms of this new variable

$$t(\tau, \hat{\zeta}) = \frac{1}{a} e^{a\hat{\zeta}} sinh(a\tau), \tag{126}$$

$$x(\tau, \hat{\zeta}) = \frac{1}{a} e^{a\hat{\zeta}} cosh(a\tau). \tag{127}$$

5.2 Quantum Field in Rindler Spacetime

Consider a massless scalar field in a two dimensional spacetime denoted by $\phi(t, x)$. The action is given by

$$s[\phi] = \frac{1}{2} \int g^{\alpha\beta} \phi_{,\alpha}\, \phi_{,\beta} \sqrt{-g} d^2x. \tag{128}$$

As we are only considering two dimensional spacetime, it can be shown that the action is invariant to transformations. To show this consider a transformation such that

$$g_{\alpha\beta} \rightarrow \hat{g}^{\alpha\beta} = \Omega^2(t,x) g_{\alpha\beta}, \tag{129}$$

and

$$\sqrt{-g} \rightarrow \Omega^{-2}\sqrt{-g}. \tag{130}$$

Using these transformations we can see that

$$g^{\alpha\beta} \rightarrow \Omega^{-2}(t,x) g^{\alpha\beta}. \tag{131}$$

Hence substituting (130) and (131) into (128) shows that the action is invariant to transformations of these types. In the laboratory coordinates of the inertial observer we know that we are in Minkowski spacetime so $ds^2 = dt^2 - dx^2$. Hence the metric can be explicitly calculated and (128) reduces to

$$s[\phi] = \frac{1}{2}\int (\partial_t \phi)^2 - (\partial_x \phi)^2 dt dx. \tag{132}$$

As we have shown in (125), the metric for the proper frame is the same as the Minkowski frame multiplied by a factor of $e^{2a\hat{\zeta}}$. We have shown that a transformation of this type leaves the action functional invariant hence the action in the proper frame is given by

$$s[\phi] = \frac{1}{2}\int (\partial_\tau \phi)^2 - (\partial_{\hat{\zeta}} \phi)^2 d\tau d\hat{\zeta}. \tag{133}$$

Applying the Euler Lagrange equations (1) to the action functionals we can find the equation of motions are given by

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{134}$$

with an analogous expression for the field $\phi(\tau, \hat{\zeta})$. These equations of motion are the Klein-Gordon equation in the case where we have one spatial variable and a massless field. Therefore we can refer back to 2.4 and immediately write the mode expansions of the fields (with $\omega_k = |k|$),

$$\begin{aligned}
\hat{\phi}(t,x) &= \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} (\hat{a}_k^- e^{i(kx-|k|t)} + \hat{a}_k^\dagger e^{i(-kx+|k|t)}) \\
&= \int_{-\infty}^{0} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} (\hat{a}_k^- e^{i(kx+kt)} + \hat{a}_k^\dagger e^{i(-kx-kt)}) \\
&+ \int_{0}^{\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2k}} (\hat{a}_k^- e^{i(kx-kt)} + \hat{a}_k^\dagger e^{i(-kx+kt)}).
\end{aligned} \tag{135}$$

Similarly

$$\begin{aligned}\hat{\phi}(\tau,\hat{\zeta}) &= \int \frac{dk}{(2\pi)^{1/2}}\frac{1}{\sqrt{2|k|}}(\hat{b}_k^- e^{i(kx-|k|t)} + \hat{b}_k^\dagger e^{i(-kx+|k|t)}) \\ &= \int_{-\infty}^{0} \frac{dk}{(2\pi)^{1/2}}\frac{1}{\sqrt{2|k|}}(\hat{b}_k^- e^{i(kx+kt)} + \hat{b}_k^\dagger e^{i(-kx-kt)}) \\ &= \int_{0}^{\infty} \frac{dk}{(2\pi)^{1/2}}\frac{1}{\sqrt{2k}}(\hat{b}_k^- e^{i(kx-kt)} + \hat{b}_k^\dagger e^{i(-kx+kt)}),\end{aligned} \tag{136}$$

where the operators $\hat{a}_k^{\pm}$ and $\hat{b}_k^{\pm}$ define the vacuum and excited states of the field in the inertial (laboratory) and proper frame respectively. The vacuum in each frame are

$$\hat{a}_k^-|0_M\rangle = 0, \tag{137}$$

$$\hat{b}_k^-|0_R\rangle = 0, \tag{138}$$

for all k where $|0_M\rangle$ and $|0_R\rangle$ denote the Minkowski vacuum in the inertial frame and the Rindler vacuum in the proper frame respectively. Note that from the perspective of the proper frame in vacuum state, the energy in the inertial frame is higher, and so an accelerating observer may observe particles in the inertial frame. This phenomena is called the Unruh effect. We will now look at a Bogoliubov transformation which links the two fields, and aim to find an expression for the number of particles created. To further simplify things we will introduce another transformation of the form

$$\begin{aligned}\bar{u} &= t - x, \\ \bar{v} &= t + x, \\ u &= \tau - \hat{\zeta}, \\ v &= \tau + \hat{\zeta}.\end{aligned} \tag{139}$$

Note that based on these transformations, the coordinates $(\bar{u},\bar{v})$ are coordinates in the inertial laboratory frame, ie Minkowski spacetime, and (u,v) are coordinates in the Rindler spacetime. Using these transformations and (126) and (127) we can deduce that

$$\begin{aligned}\bar{u} &= t - x = -\frac{1}{a}e^{-au}, \\ \bar{v} &= t + x = \frac{1}{a}e^{av}.\end{aligned} \tag{140}$$

Using these change of coordinates we can use the chain rule and rewrite (134) as

$$\begin{aligned} \frac{\partial^2\phi}{\partial\bar{u}\partial\bar{v}} &= 0, \\ \frac{\partial^2\phi}{\partial u\partial v} &= 0. \end{aligned} \tag{141}$$

Now we can obtain the mode expansions in the new variables by letting $\omega = |k|$ and using (139) and (135) to get

$$\hat{\phi}(\bar{u},\bar{v}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{2\omega}(\hat{a}_\omega^- e^{-i\omega\bar{u}} + \hat{a}_\omega^\dagger e^{i\omega\bar{u}} + \hat{a}_{-\omega}^- e^{-i\omega\bar{v}} + \hat{a}_{-\omega}^\dagger e^{i\omega\bar{v}}). \tag{142}$$

Similarly we obtain

$$\hat{\phi}(u,v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}}\frac{1}{2\Omega}(\hat{b}_\Omega^- e^{-i\Omega u} + \hat{b}_\Omega^\dagger e^{i\Omega u} + \hat{b}_{-\Omega}^- e^{-i\Omega v} + \hat{b}_{-\Omega}^\dagger e^{i\Omega v}), \tag{143}$$

where Ω has been used instead of ω to differentiate between the different coordinate frames. Note that we could write

$$\begin{aligned} \hat{\phi}(\bar{u},\bar{v}) &= \hat{A}(\bar{u}) + \hat{B}(\bar{v}), \\ \hat{\phi}(u,v) &= \hat{P}(u) + \hat{Q}(v), \end{aligned} \tag{144}$$

where

$$\hat{A}(\bar{u}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{2\omega}(\hat{a}_\omega^- e^{-i\omega\bar{u}} + \hat{a}_\omega^\dagger e^{i\omega\bar{u}}), \tag{145}$$

with corresponding expressions for $\hat{B}, \hat{P}, \hat{Q}$. (140) shows that $\bar{u} = \bar{u}(u)$, hence we must have that

$$\begin{aligned} \hat{A}(\bar{u}) &= \hat{P}(u), \\ \hat{B}(\bar{v}) &= \hat{Q}(v). \end{aligned} \tag{146}$$

Note from these expressions we can see that the fields in both coordinate frames can be decomposed into parts with positive and parts with negative momentum. Furthermore, there is no mixing between operators of positive and negative momentum, ie (146) shows that the Bogoliubov transformations can only involve relations between operators with positive momentum or operators with negative momentum.

5.3 Bogoliubov Transformations

In this subsection we will derive the Bogoliubov transformations between the annihilation and creation operators in the two frames. Using (146) we can explicitly write

$$\int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{2\omega} (\hat{a}_\omega^- e^{-i\omega\bar{u}} + \hat{a}_\omega^\dagger e^{i\omega\bar{u}}) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{2\Omega} (\hat{b}_\Omega^- e^{-i\Omega u} + \hat{b}_\Omega^\dagger e^{i\Omega u}). \tag{147}$$

Applying a Fourier transform to both sides of this equation, then after some algebra and the use of the Dirac delta distribution we obtain Bogoliubov transformations of the form

$$\hat{b}_\Omega^- = \int_0^\infty \alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^\dagger d\omega, \tag{148}$$

where

$$\begin{aligned} \alpha_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega), \\ \beta_{\omega\Omega} &= \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega), \end{aligned} \tag{149}$$

and

$$F(\omega, \Omega) = \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u - i\omega\bar{u}} = \int_{-\infty}^\infty \frac{du}{2\pi} e^{i\Omega u - i\omega \frac{e^{au}}{a}}. \tag{150}$$

Again note that the Bogoliubov transformations imply a relationship only exists between operators with positive momentum and operators with negative momentum. For example there is no Bogoliubov transformation between $\hat{b}_{-\Omega}^-$ and $\hat{a}_\omega^-, \hat{a}_\omega^\dagger$.

6 Entanglement

6.1 Background

Particles or sets of particles sometimes behave in a way such that a measurement on one particle is correlated to the measurement of another. This is known as quantum entanglement. A interesting consequence of entanglement is that if two entangled particles are separated, measuring a property on one will give us information about the state of the other instantaneously. As information cannot travel faster than light, there is current research into how entangled particles appear to communicate instantaneously over large distances.

A standard measure of the entanglement present in a bipartite pure state system is the Von Neumann entropy [8]. In this section we will review some concepts of quantum information including the notion of pure and mixed states, the density operator, and finally give a definition of the Von Neumann entropy.

A quantum state is called pure if it can be represented by the state vector $|\psi\rangle$. $|\psi\rangle$ will normally be represented as a superposition of eigenstates of the system with coefficients related to the probabilities of measuring an particular eigenstate. However their may be uncertainty about the nature of $|\psi\rangle$ itself and as such there may be more than one state vector $|\psi_1\rangle$ and $|\psi_2\rangle$, for example with related probabilities of finding the system in each. In such cases we have a mixed state. The density operator, usually denoted ρ is a matrix which describes the probabilities associated with a mixed state. It is defined by

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|, \tag{151}$$

where p_α is the probability of finding the system in state $|\psi_\alpha\rangle$. It can be shown that a state is pure if and only if $\rho^2 = \rho$. If we have a bipartite system, i.e one where $\alpha = [1, 2]$, we may wish to find the density operator for the individual states rather than for the whole system. These are called reduced density operators and are found by tracing out one of the particles, for example if we wished to find the reduced density operator for particle 1, denoted ρ_1 we would trace out over particle 2. This is denoted

$$\rho_1 = tr_2\rho. \tag{152}$$

The Von Neumann entropy is defined as [9]

$$S(\rho) = -tr(\rho ln\rho). \tag{153}$$

Note that the entropy is zero valued for a pure state, hence the Von Neumann entropy is used to measure the entropy in the out regime given the in regime has non (it is a pure state). This is done as the out regime is a mixed state, so we trace out over the particle to obtain the reduced density operator, then we apply this to the entropy equation. This is a measure of the entanglement.

6.2 Entropy

In this subsection we will discuss some general results for the value of entropy which is a measure of entanglement between particles. Entanglement can only be calculated for a bipartite system [10]. We can use the fact the the modes can be separated into ones of positive and negative momentum k (see section 5.2). This defines a bipartition of the system. We have calculated in section 3.2 the relationship between the vacuum state on the in regime and the excited states in the out regime. Taking into account the bipartition of the system, we can in a similar calculation to section 3.2 deduce that

$$|0_{in}\rangle = \sum_{n=0}^{\infty} c_n |n_{k_{out}}\rangle |n_{-k_{out}}\rangle, \tag{154}$$

where

$$c_n = \left(\frac{\beta_k^\star}{\alpha_k^\star}\right) \sqrt{1 - \left(\frac{|\beta_k|}{|\alpha_k|}\right)^2}, \tag{155}$$

and $|n_{k_{out}}\rangle$ describes the excited states corresponding to the mode k. As we are assuming that the in region is a vacuum state, there will only be ground state modes of frequency k and $-k$, hence using the definition of the density operator (151) we simply have

$$\rho = |0_{-k} 0_k\rangle_{in} \langle 0_k 0_{-k}|. \tag{156}$$

This defines the density operator on the in region which is a vacuum state. Note that although the density operator represents a pure state, this is only from the perspective from an observer in the in region. As we can write (154), from the perspective of an observer in the out region, the in vacuum is a mixed state. We can therefore trace out modes with momentum $-k$ by calculating the reduced density operator. Applying the definition of the Von Neumann entropy to this reduced matrix will give a measure of the entanglement between the modes with momentum k and modes with momentum $-k$. We calculate the reduced density operator using the formula

$$\rho_k = \sum_{m=0}^{\infty} \langle m_{-k}|\rho|m_{-k}\rangle. \tag{157}$$

Using our expression for the density operator (156) and (154) we can explicitly calculate the reduced density operator

$$\rho_k = \left(1 - \left|\frac{\alpha_k}{\beta_k}\right|^2\right) \sum_{n=0}^{\infty} \left|\frac{\alpha_k}{\beta_k}\right|^{2n} |n_k\rangle_{out} \langle n_k|. \tag{158}$$

Applying (153) we can calculate the entropy

$$S = -tr(\rho_k log \rho_k) = log\left(\frac{\gamma^{\gamma/\gamma-1}}{1-\gamma}\right), \tag{159}$$

where $\gamma = \left|\frac{\beta_k}{\alpha_k}\right|^2$ and α_k and β_k are the appropriate Bogoliubov coefficents.

6.3 Example

Consider a quantum field which is flat at some distant point in the past and again at a distant point in the future, where in the period between there is curved spacetime. An example of this can be described by a time dependent harmonic oscillator with a specific time dependent frequency, see for example 4.5. Suppose the in region is a vacuum state with respect to an inertial observer in the in region. We have shown through Bogoliubov transformations, that in the out region there will be a presence of particles caused by a curved spacetime in the time lapsed between the in and out regimes. In the example 4.5 we found Bogoliubov transformations of the mode functions of the form (116). Furthermore, we have shown that in curved spacetime we can make some mathematical substitutions to make the analysis conformally flat so that dealing with curved spacetime reduces to dealing with time dependent harmonic oscillators. In this case we found that the Bogoliubov transformations between annihilation and creation operators of various mode functions are of the form (74) and (75). That is the Bogoliubov transformations of the mode functions, and annihilation and creation operators involve a mixing between modes with momentum k and $-k$. Using (114) and (115) we can hence calculate

$$\gamma = \frac{sinh^2(\pi\omega_-/\lambda)}{sinh^2(\pi\omega_+/\lambda)}. \tag{160}$$

Therefore entanglement between modes with postive momentum k and negative momentum $-k$ is given by (159) with γ specified above.

We observe that if $\omega_- = 0$ then $\gamma = 0$, which means the entanglement $S = 0$. This is clearly the case whenever $\omega_{in} = \omega_{out}$, thus there is no entanglement in flat spacetime. From (108), (117) and (118) we observe that $\omega_- = 0$ also when $m = 0$, which again corresponds to the case where there is no gravitational field. Hence we deduce that the presence of mass, and so the gravitational field results in the creation of entangled particles in this example 4.5.

6.4 Entropy to determine cosmological parameters

In this section we will review some results from [12] which allows us to use (159) to find an expression for γ in terms of the entanglement S, and as such derive an expression for the parameter λ (known as a cosmological parameter) which was introduced in 4.5. Although the majority of calculation in 4.5 is completely analogous to that used in [12], there are some small changes in the exact form of the problem and as such some notation will be initially introduced. Following this we will explain how to find an expression for the cosmological parameter and interpret what this means, as well as noting some limitations. Finally we will also look at how the parameter changes when considering Dirac fields in a simple 1+1 dimensional universe in contrast to the scalar field in a 1+1 dimensional universe which is what we have looked at so far.

Consider the two dimensional conformally flat metric (65). If we choose the scale factor such that

$$a^2(\eta) = 1 + \epsilon tanh(\lambda\eta), \tag{161}$$

where η is the conformal time, and ϵ and λ are parameters which control the total volume and how fast the universe expands [12], then we obtain the same expression for the entanglement entropy (γ is given by (160)) between modes of positive and negative momentum as we did for the case of the time dependent oscillator in 4.5. However there are small changes in the analysis made in 4.5. We now define

$$\begin{aligned} \omega_{in} &= \sqrt{k^2 + m^2}, \\ \omega_{out} &= \sqrt{k^2 + m^2(1 + 2\epsilon)}, \end{aligned} \tag{162}$$

where m is mass. Consider what happens if $m \ll 2\lambda\epsilon^{-\frac{1}{2}}$. [12] found that in this limit, the parameter ϵ which is related to the total volume can be approximated by

$$\epsilon \approx \frac{2E_k^2}{m^2}\sqrt{\gamma(s)}, \tag{163}$$

where $\gamma(s)$ can be found from (159) (if the entanglement is known), and $E_k = \sqrt{k^2 + m^2}$ is the energy of a mode with momentum k. Similarly in this limit is can be found by considering the entanglement between two modes of similar energy E that

$$\lambda \approx \frac{\pi E}{2}\left(\frac{1 + \gamma(s)}{-\frac{E}{4}\frac{d}{dE}ln\gamma(s) - 1}\right). \tag{164}$$

Equations (163) and (164) show that if we calculate the entanglement between the modes, then we can estimate the cosmological parameters ϵ and λ. A limitation of these calculations is that all approximations are based on the limit that $m \ll 2\lambda\epsilon^{-\frac{1}{2}}$. In section 6.3 we showed that when the mass is zero then so is the entanglement, hence our approximations hold in the case where the entanglement is almost zero, which is not necessarily true when describing the observed universe, thus the estimations of the cosmological parameters may not be very precise. To obtain better parameter estimates we will now consider a brief overview of an example of entanglement in a Dirac field.

Up to this point in the report, we have considered quantization of scalar fields. This process and most of the theory of basic quantum field theory is based on bosonic fields. In contrast a Dirac field is an example of a fermionic field. Bosons and fermions are essentially two different classes of particles, one of which (bosons) have integer spin, and the other (fermions) have half-integer spin. A Dirac field, ψ obeys the Dirac equation

$$(i\gamma^\mu D_\mu + m)\psi = 0, \tag{165}$$

where γ^μ are the Dirac matrices in general spacetime, D_μ is the covariant derivative of the fermionic field and m is mass. A feature of a fermionic field is that they obey anticommutation relations as opposed to the commutation relations that bosonic fields obey. For example, in flat spacetime the Dirac matrices $\hat{\gamma}^{\mu}$ satisfy

$$[\hat{\gamma}^{\alpha}, \hat{\gamma}^{\beta}] = \hat{\gamma}^{\alpha}\hat{\gamma}^{\beta} + \hat{\gamma}^{\beta}\hat{\gamma}^{\alpha}. \tag{166}$$

It can be shown that the relationship between the Dirac matrices in curved spacetime and flat spacetime is given by

$$\gamma^{\mu} = e^{\mu}{}_{\alpha}\hat{\gamma}^{\alpha}, \tag{167}$$

where $e^{\mu\alpha}$ are an orthonormal set of vector fields, known as a vierbein field [13]. In a flat, Robinson-Walker universe, the vierbein reduces to [14]

$$e^{\alpha\mu} = a(\eta)\eta^{\alpha\mu}, \tag{168}$$

where η is the conformal time, and $\eta^{\alpha\mu}$ is the Minkowski metric. Now to apply a similar problem to 4.5 in the context of a Dirac field we choose the conformal factor to be

$$a(\eta) = 1 + \epsilon tanh(\lambda\eta). \tag{169}$$

Note the slight difference to the conformal factor use in (161). This is because to obtain an analytic solution in the Dirac field, we are solving for the vierbein field, hence as the form of the vierbein field in this conformally flat spacetime is given by (168), we need to make the adjustment so that the vierbein field is proportional to $1+\epsilon tanh(\lambda\eta)$. This is exactly analogous to the earlier case where it was the metric (65) which was proportional to $1+\epsilon tanh(\lambda\eta)$ also. In a process similar to that in 4.5 the relevant Bogoliubov coefficients can be found, and have a only slightly more complex form [15] to those in (113)

$$\begin{aligned} \alpha_k^{\pm} &= \frac{\Gamma(1-(i/\lambda)\omega_{in})\Gamma(-(i/\lambda)\omega_{out})}{\Gamma(1-(i/\lambda)\omega_{+} \pm im\epsilon/\lambda)\Gamma(-(i/\lambda)\omega_{+} \mp im\epsilon/\lambda)}, \\ \beta_k^{\pm} &= \frac{\Gamma(1-(i/\lambda)\omega_{in})\Gamma((i/\lambda)\omega_{out})}{\Gamma(1+(i/\lambda)\omega_{-} \pm im\epsilon/\lambda)\Gamma((i/\lambda)\omega_{-} \mp im\epsilon/\lambda)}, \end{aligned} \tag{170}$$

In this case of the Dirac field the expression for the entropy also differs. It was found by [16] that the entropy is given by

$$S = -tr(\rho_k log \rho_k) = log\left(\frac{1+\gamma}{\gamma^{\frac{2\gamma}{\gamma+1}}}\right), \tag{171}$$

where

$$\gamma = \frac{(\omega_{-}+m\epsilon)(\omega_{+}+m\epsilon)sinh(\pi/\lambda[\omega_{-}-m\epsilon])sinh(\pi/\lambda[\omega_{-}+m\epsilon])}{(\omega_{-}-m\epsilon)(\omega_{+}-m\epsilon)sinh(\pi/\lambda[\omega_{+}+m\epsilon])sinh(\pi/\lambda[\omega_{+}-m\epsilon])}. \tag{172}$$

This expression has some similarities to what we found for the entanglement of a scalar field. Most notably, again this shows that there is no entanglement when the mass is zero. However there clearly is a difference in entanglement between modes in a bosonic and fermionic field, and these differences can be most clearly seen when we plot the entanglement entropy against the mode k. Note that the entropy, S depends on the cosmological parameter λ as well as k, hence Figures 3 and 4 show the entanglement entropy against k for varying values of λ, [16].

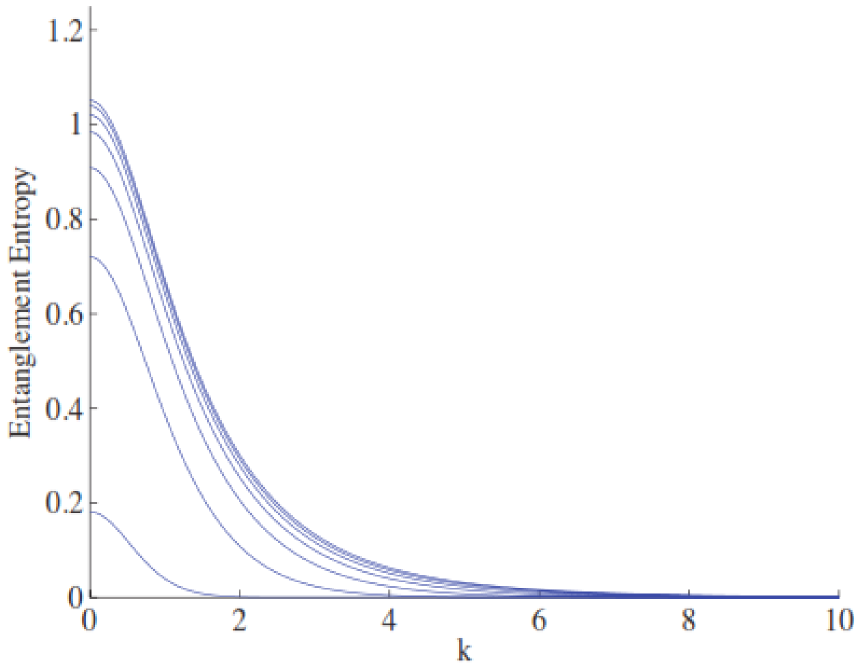

Figure 3: Scalar field entanglement with $m = 1$, $\epsilon = 1$, for varying λ

From Figures 3 and 4 we can see the difference in entanglement entropy between the scalar field and Dirac field cases. Figure 3 shows that for a scalar field, the mode with maximum entanglement is the one with momentum $k = 0$, and entanglement entropy decreases as k increases. This is in contrast to the entanglement in a Dirac field, where a mode with non zero momentum k has the maximum entanglement entropy. [16] notes that the value of k for which the entanglement is maximized in the Dirac field case is very sensitive to the cosmological parameter λ but is almost unaffected by ϵ, whereas the value of the maximum entanglement is sensitive to ϵ but almost unaffected by λ. Hence if the value of the maximum entropy is known, as well as the corresponding mode, this will tell us information about the cosmological parameters. Finally we note that in the Dirac field case our deductions are not based on limits of small mass, hence studying the entanglement in a Dirac field may give more information about the cosmological parameters than in the case of entanglement in a scalar field.

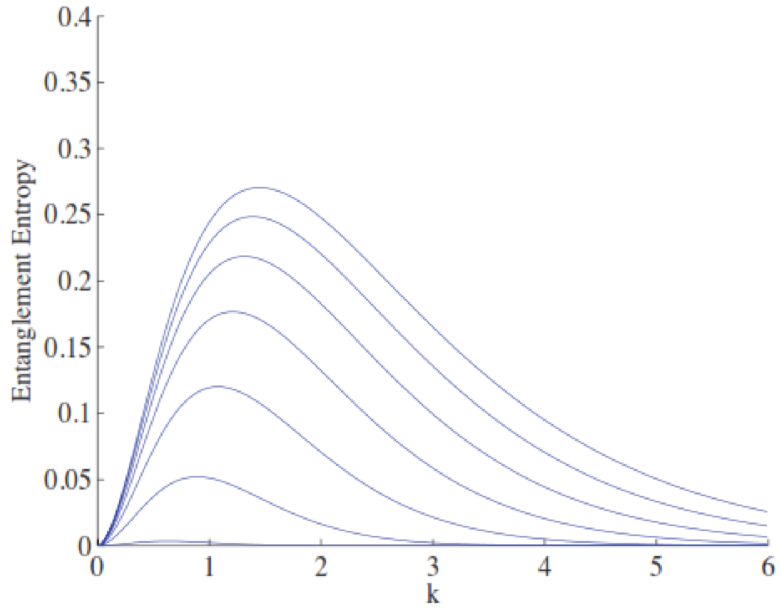

Figure 4: Dirac field entanglement with $m = 1$, $\epsilon = 1$, for varying λ

7 Entanglement and relativity

In this section we will discuss some results of quantum entanglement in non-inertial frames. In section 5.2 we looked at the Unruh effect; the effect where an observer in an accelerating frame (Rindler spacetime) could observe particles in the vacuum state of a inertial frame (Minkowski spacetime). Here we review some results of the entanglement of these particles, and the effect of acceleration on entanglement. We consider the scenario where, in the inertial frame, all modes are in ground state, except two which are entangled. We have seen in section 6.4 that entanglement varies between modes. In this case we suppose that the two modes in the inertial frame have maximum entanglement, and consider the measure of this entanglement from the perspective of the accelerated observer.

In order to describe the entanglement, we will first introduce Unruh modes, which are a special type of mode which can be used as a link between the modes in Rindler and Minkowski spacetime. Transformations between the various modes will be stated and Bogoliubov transformations between various annihilation and creation operators will be discussed. Next I will describe an alternative measure of entanglement that will be used in this case: Negativity. This is an alternative to entanglement entropy as discussed in 6.2, and we will see that entanglement reduces to zero as acceleration increases.

7.1 Unruh Modes

We start by considering the Klein Gordon equation in Minkowski spacetime. The positive energy solutions with respect to the timelike Killing vector field [17] is given by

$$u_{\omega,M}(x,t) = \frac{1}{4\pi\omega} e^{-i\omega(t-\epsilon x)}, \tag{173}$$

where a Killing vector field is a vector field which preserves the metric $g_{\mu\nu}$, ω is the frequency in Minkowski spacetime, and $\epsilon = 1$ for modes with positive momentum, and $\epsilon = -1$ for modes with negative momentum. In order to represent the accelerated observer, we introduce Rindler coordinates (η, χ) such that

$$\begin{aligned} \eta &= atanh\left(\frac{t}{x}\right), \\ \chi &= \sqrt{x^2 - t^2}. \end{aligned} \tag{174}$$

In contrast to the results introduced in 5.1, these coordinates are used to describe an accelerated observer in both regions $x > |t|$ and $x < -|t|$ which we denote region I and region II respectively. The spacetime diagram can be seen in Figure 5, where regions I and II highlight worldlines of accelerated observers in Rindler spacetime, denoted Rob and anti-Rob respectively. The worldline of Alice is a representation of the worldline of an inertial observer.

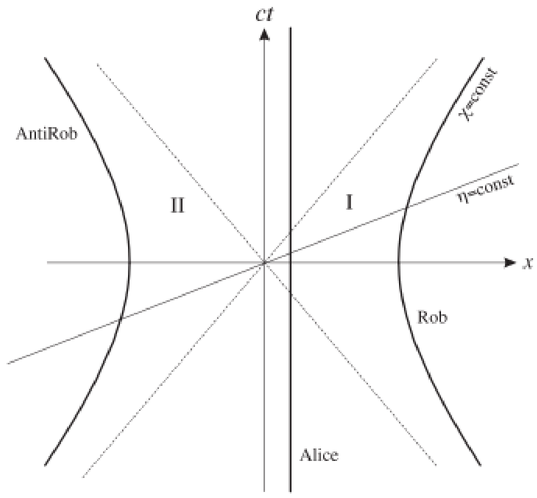

Figure 5: Worldlines of accelerated observers from perspective of inertial observer coordinate frame

The solutions to the Klein-Gordon equation in regions I and II are given by [17]

$$\begin{aligned} u_{\Omega,I}(t,x) &= \frac{1}{\sqrt{4\pi\Omega}} \left(\frac{x - \epsilon t}{l_\Omega}\right)^{i\epsilon\Omega}, \\ u_{\Omega,II}(t,x) &= \frac{1}{\sqrt{4\pi\Omega}} \left(\frac{\epsilon t - x}{l_\Omega}\right)^{-i\epsilon\Omega}, \end{aligned} \tag{175}$$

where Ω is a positive dimensionless constant, and l_Ω is a positive constant with dimension of length related to the phase of the modes. Note that Ω is related to the frequency of the Rindler modes, and the dimensional equivalent is

$$\Omega_a = a\Omega, \tag{176}$$

where a is the acceleration.

Using (175), a new set of modes can be introduced

$$\begin{aligned} u_{\Omega,R} &= cosh(r_\Omega)u_{\Omega,I} + sinh(r_\Omega)u^\star_{\Omega,II}, \\ u_{\Omega,L} &= cosh(r_\Omega)u_{\Omega,II} + sinh(r_\Omega)u^\star_{\Omega,I}, \end{aligned} \tag{177}$$

where $tanh(r_\Omega) = e^{-\pi\Omega}$. These are called Unruh modes. Note that from (176) we observe that if $a = 0$ then $u_{\Omega,R} = u_{\Omega,I}$ and $u_{\Omega,L} = u_{\Omega,II}$.

Unruh modes are useful as they allow the analysis of entanglement between modes in Minkowski and Rindler spacetime easier to compute. This can be done as it is known that the Unruh modes have a high Rindler frequency and so correspond to solutions in the Rindler frame, as well as being a linear combination of (positive frequency) solutions in the Minkowski frame [18]. Hence analyzing the relationship between the Rindler and Unruh modes with tell us something about the relationship between the Rindler and Minkowski modes.

The mode expansion of the field can be written as

$$\phi = \int_0^\infty (a^-_{\omega,M}u_{\omega,M} + a^\dagger_{\omega,M}u^\star_{\omega,M})d\omega, \tag{178}$$

where $a^-_{\omega,M}$ and $a^\dagger_{\omega,M}$ are the annihilation and creation operators in the Minkowski coordinates. As the Unruh modes are a linear combination of the Minkowski modes [18], this could also be written as

$$\phi = \int_0^\infty (A^-_{\Omega,R}u_{\Omega,R} + A^\dagger_{\Omega,R}u^\star_{\Omega,R} + A^-_{\Omega,L}u_{\Omega,L} + A^\dagger_{\Omega,L}u^\star_{\Omega,L})d\Omega, \tag{179}$$

where $A^-_{\Omega,R}$,$A^\dagger_{\Omega,R}$,$A^-_{\Omega,L}$ and $A^\dagger_{\Omega,L}$ are the Unruh annihilation and creation operators. Finally we could then use the relationship between the Rindler and Unruh modes to write

$$\phi = \int_0^\infty (a^-_{\Omega,I}u_{\Omega,I} + a^\dagger_{\Omega,I}u^\star_{\Omega,I} + a^-_{\Omega,II}u_{\Omega,II} + a^\dagger_{\Omega,II}u^\star_{\Omega,II})d\Omega, \tag{180}$$

where $a^-_{\Omega,I}$,$a^\dagger_{\Omega,I}$,$a^-_{\Omega,II}$ and $a^\dagger_{\Omega,II}$ are the Rindler annihilation and creation operators.

The transformation between the Minkowski and Rindler modes can be found by taking appropriate inner products and appealing to orthornormality between the operators.

$$
\begin{aligned}
u_{\omega,M} &= \int_0^\infty (\alpha^R_{\omega\Omega} u_{\Omega,R} + \alpha^L_{\omega\Omega} u_{\Omega,L}) d\Omega, \\
u_{\Omega,R} &= \int_0^\infty (\alpha^R_{\omega\Omega})^\star u_{\omega,M} d\omega, \\
u_{\Omega,L} &= \int_0^\infty (\alpha^L_{\omega\Omega})^\star u_{\omega,M} d\omega,
\end{aligned} \tag{181}
$$

where

$$
\begin{aligned}
\alpha^R_{\omega\Omega} &= \frac{1}{\sqrt{2\pi\omega}} (\omega l)^{i\epsilon\Omega}, \\
\alpha^L_{\omega\Omega} &= \frac{1}{\sqrt{2\pi\omega}} (\omega l)^{-i\epsilon\Omega}.
\end{aligned} \tag{182}
$$

Using similar methods to those in 5.3 we can also deduce the Bogoliubov transformations between the annihilation and creation operators

$$
\begin{aligned}
a^-_{\omega,M} &= \int_0^\infty [(\alpha^R_{\omega\Omega})^\star A_{\Omega,R} + (\alpha^L_{\omega\Omega})^\star A_{\Omega,L}] d\Omega, \\
A_{\Omega,R} &= \int_0^\infty \alpha^R_{\omega\Omega} a_{\omega,M} d\omega, \\
A_{\Omega,L} &= \int_0^\infty \alpha^L_{\omega\Omega} a_{\omega,M} d\omega.
\end{aligned} \tag{183}
$$

Since the transformation between the Unruh and Minkowski modes do not mix annihilation and creation operators [17], the vacuum states are equal, that is

$$
|0\rangle_U = |0\rangle_M. \tag{184}
$$

The Rindler vacuum can be written in terms of the mixed excited states in regions I and II as [17]

$$
|0_\Omega\rangle_U = \sum_n \frac{(tanhr_\Omega)^n}{cosh_\Omega} |n_\Omega\rangle_I |n_\Omega\rangle_{II}. \tag{185}
$$

Consider the Rindler operators $a^-_{\Omega,I}$,$a^\dagger_{\Omega,I}$,$a^-_{\Omega,II}$ and $a^\dagger_{\Omega,II}$. As the Unruh modes are Rindler modes of high frequency, we could write a creation operator of the general form

$$
a^\dagger_{\Omega,U} = q_L A^\dagger_{\Omega,L} + q_R A^\dagger_{\Omega,R}, \tag{186}
$$

where $|q_L|^2 + |q_R|^2 = 1$ are complex constants.

7.2 Measure of entanglement

In this subsection we will look at the entanglement between the Minkowski and Unruh modes. As opposed to the bipartite system we considered in section 6.2, we are now dealing with a tripartite system consisting of the Unruh modes (both L and R) with frequency Ω and the Minkowski modes of frequency ω. These can be separated into two bipartitions by considering the Minkowski modes and the R Unruh modes, and the Minkowski modes and the L Unruh modes. These correspond to a partition between Alice and Rob, and Alice and Anti-Rob respectively, see Figure 5. An example of such a bipartite state is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_\omega\rangle_M|0_\Omega\rangle_U + |1_\omega\rangle_M|1_\Omega\rangle_U). \tag{187}$$

As opposed to entanglement entropy that was used in 6.2, an alternative, Negativity, can be used to give a measure of entanglement. Negativity is defined as

$$N(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \tag{188}$$

where λ_i are eigenvalues of the partially transposed density matrix [17]. The following results are presented in [17], where the density matrices and negativity are calculated to measure the entanglement between Alice and Rob, and Alice and Anti-Rob (see Figure 5).

The Alice-Rob density matrix is obtained by tracing out over region II to obtain

$$\rho_{AR} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{(tanhr_\Omega)^n}{coshr_\Omega}\right)^2 \rho_{AR}^n, \tag{189}$$

where

$$\begin{aligned}\rho_{AR}^n = {} & |0_n\rangle\langle 0_n| + \frac{n+1}{cosh^2 r_\Omega}(|q_R|^2|1_n+1\rangle\langle 1_n+1| + |q_L|^2|1_n\rangle\langle 1_n|) \\ & + \frac{\sqrt{n+1}}{coshr_\Omega}(q_R|1_n+1\rangle\langle 0_n| + q_L tanhr_\Omega|1_n\rangle\langle 0_n+1|) \\ & + \frac{\sqrt{(n+1)(n+2)}}{cosh^2 r_\Omega} q_R q_L^\star tanhr_\Omega|1_n+2\rangle\langle 1_n| + (H.c)_{non-diag},\end{aligned} \tag{190}$$

and $(H.c)_{non-diag}$ means the Hermitian conjugate of the non-diagonal elements only. A similar expression can be found for the Alice-AntiRob density matrix.

To calculate the negativity, we require the eigenvalues of the density matrices. Due to the complex nature of the the density matrices, these calculations can only be obtained

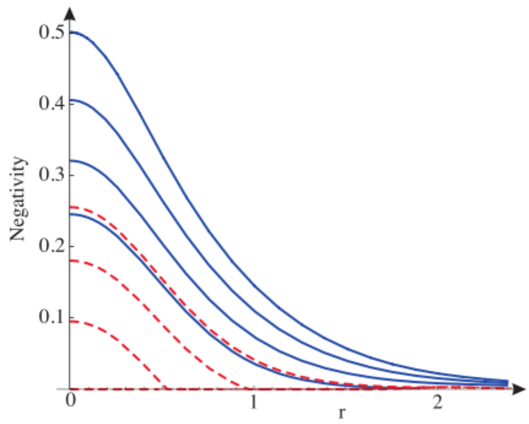

Figure 6: Negativity against r for varying values of $|q_R|$. Blue lines correspond to Alice-Rob bipartition, and red to Alice-AntiRob

numerically in all but some special cases. Figure 6 is a plot that of the negativity in both the Alice-Rob, and Alice-Anti-Rob cases, for varying values of $|q_R|$, [17].

Note that $r = arctanh(e^{-(\pi\Omega_a)/a})$, hence as a increases, so does r, and from Figure 6 we see that if r increases then the entanglement reduces. Thus if if acceleration tends to infinity the entanglement reduces to zero. It can also be shown that in the fermionic case the amount of entanglement reduces as acceleration increases, however instead of reducing to zero, it reduces to a finite limit.

In this chapter we have seen that when considering two observers; one inertial and one accelerating, not only will the accelerating observer observe particles in the vacuum of the inertial observer, but there is also entanglement. However this entanglement is dependent on the acceleration, and in increasing limits, the entanglement reduces to zero. These results have been reported in many papers in the last decade, however there are some more recent papers where different results were obtained. [18] showed that entanglement exists, and even increases as the acceleration of an observer increases.

In [18], instead of considering the bipartite state (187), a family of bipartite states were considered

$$\begin{aligned} |\psi\rangle &= P|0_\omega\rangle_M[\alpha|1_\Omega\rangle_U + \sqrt{1-\alpha^2}|0_\Omega\rangle_U] \\ &\quad + \sqrt{1-P^2}|1_\omega\rangle_M[\beta|1_\Omega\rangle_U + \sqrt{1-\beta^2}|0_\Omega\rangle_U]. \end{aligned} \tag{191}$$

Using this family of states, [18] used negativity to show that for the case where $P = 0.4$, $\alpha = 0$ and $\beta = 1$ then there is actually an increase in entanglement as acceleration increases. Figure 7 shows that in the fermionic case, there is an increase in entanglement between the modes in the Alice and Anti-Rob case.

Entanglement is very sensitive to interactions with the environment. As such experiments which may attempt to capture quantum entanglement and its degradation due to acceleration is problematic, as the question may be asked of the underlying cause of the reduction in entanglement. However the creation or increase of quantum entanglement,

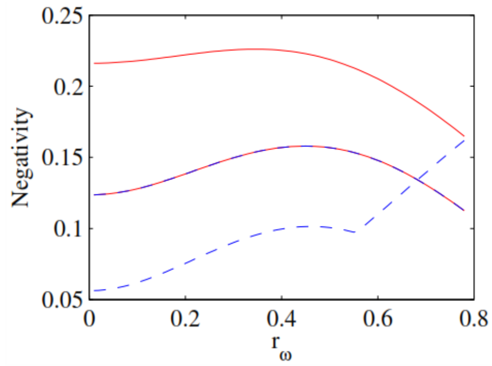

Figure 7: Negativity against r where $P = 0.4$, $\alpha = 0$ and $\beta = 1$ in fermionic field. Red lines correspond to Alice-Rob bipartition, and blue to Alice-AntiRob

if detected would confirm the theory, as there is nothing in the environment that could create the entanglement, hence we would have to accept it was due to the acceleration [18].

8 Conclusion

In this report some key aspects of quantum field theory have been discussed. In particular, the quantization of a scalar field, which in chapter 2 we saw is equivalent to the quantization of a infinite set of simple harmonic oscillators. This was then extended to the time dependent oscillator in section 3 which allowed us to describe a field which is changing in time, and hence we introduced the Bogoliubov transformations. In chapter 4, quantum field theory in the presence of gravity was introduced, and it was shown that by using a change of variable allowed for techniques developed in 2 to be employed. Explicit examples of using Bogoliubov transformations to show particle creation were then discussed in two cases. The first was a case of a harmonic oscillator of constant frequency which changed to a new constant frequency at a particular point, and the second was a more complex case where the time dependence of the oscillator was a hyperbolic tangent function. Section 5 introduced concepts of relativity into the analysis. The notion of Rindler spacetime and the Unruh effect were explained through the use of quantum field theory and Bogoliubov transformations. These concepts were particularly important as grounding for the final sections on entanglement.

Sections 6 and 7 focused on applying quantum field theory to describe entanglement. 6 introduced some concepts of entanglement, as well as the Von-Neumann entropy which is a measure of entanglement. This was used to calculate entanglement between modes in the example in 4. The analogous problem in a fermionic field was also computed and comparisons were drawn between the two cases. Figures 3 and 4 show the fundamental differences between entanglement in the bosonic and fermionic fields where there is a specific mode in the fermionic case for which entanglement is maximized. As this mode is very sensitive to cosmological parameters, we concluded that entanglement in the fermionic case is a more useful measure if we are interested in estimating the values of particular cosmological parameters.

Finally in section 7, some of the mathematics of how to measure entanglement between particles in an inertial frame from the perspective of an accelerating frame was reviewed. This allowed us to use some ideas from 5 as well as to explore an alternative measure of entanglement; negativity. Figure 6 shows that as acceleration increases, then the entanglement reduces to zero. This is a result that agrees with much of the literature on the subject, however we have also seen from recent research that acceleration may not only destroy entanglement, but it can also amplify entanglement, and these results are very briefly reviewed in the final part of the report.

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