Assignment-3 Solutions

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QT 207

II. Quantum Fourier Transform (QFT) and Unitary Operator

Exercise 2: Find an operator $\hat{F}$ which transforms a state into its DFT and show that it is unitary.

Defining the QFT Operator $\hat{F}$

Consider an N-dimensional Hilbert space with computational basis

$$\{|j\rangle\}_{j=0}^{N-1}.$$

The quantum Fourier transform (QFT) operator $\hat{F}$ is defined by its action on the basis states as

$$\hat{F}\,|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i jk/N}\,|k\rangle, \qquad j = 0, 1, \ldots, N-1. \tag{1}$$

In matrix form,

$$\hat{F} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \langle k|\hat{F}|j\rangle\,|k\rangle\langle j|,$$

with matrix elements

$$\langle k|\hat{F}|j\rangle = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} jk}. \tag{2}$$

Therefore,

$$\hat{F} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{\frac{2\pi i}{N} jk}\,|k\rangle\langle j|. \tag{3}$$

If a general state is

$$|\psi\rangle = \sum_{j=0}^{N-1} x_j |j\rangle,$$

then

$$\hat{F}|\psi\rangle = \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i jk/N} x_j \right) |k\rangle,$$

so the amplitudes are exactly the DFT of $\{x_j\}$.

Showing that $\hat{F}$ is Unitary

An operator is unitary if

$$\hat{F}\hat{F}^\dagger = \hat{I} \quad \text{and} \quad \hat{F}^\dagger\hat{F} = \hat{I}.$$

The matrix elements of $\hat{F}^\dagger$ are

$$\langle j|\hat{F}^\dagger|k\rangle = \overline{\langle k|\hat{F}|j\rangle} = \frac{1}{\sqrt{N}} e^{-2\pi i jk/N}.$$

Consider the matrix element of $\hat{F}\hat{F}^\dagger$:

$$\begin{aligned}
(\hat{F}\hat{F}^\dagger)_{kk'} &= \langle k|\hat{F}\hat{F}^\dagger|k'\rangle \\
&= \sum_{j=0}^{N-1} \langle k|\hat{F}|j\rangle\langle j|\hat{F}^\dagger|k'\rangle \\
&= \sum_{j=0}^{N-1} \left(\frac{1}{\sqrt{N}} e^{2\pi ijk/N}\right)\left(\frac{1}{\sqrt{N}} e^{-2\pi ijk'/N}\right) \\
&= \frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi ij(k-k')/N}.
\end{aligned}$$

The sum over roots of unity gives

$$\sum_{j=0}^{N-1} e^{2\pi ij(k-k')/N} = \begin{cases} N, & k = k', \\ 0, & k \neq k'. \end{cases}$$

Hence

$$(\hat{F}\hat{F}^\dagger)_{kk'} = \delta_{kk'}, \qquad \hat{F}\hat{F}^\dagger = \hat{I}.$$

By the same calculation (or by the fact that $\hat{F}$ is a square matrix with orthonormal columns), one also finds

$$\hat{F}^\dagger\hat{F} = \hat{I}.$$

Therefore $\hat{F}$ is unitary.

III. Discrete Fourier Transform (DFT) on Periodic States

Exercise 3: Performing DFT on the periodic state $|\phi_{r,b}\rangle \to |\tilde{\phi}\rangle$, obtain the resulting state.

The periodic state is

$$|\phi_{r,b}\rangle = \frac{1}{\sqrt{m}}\sum_{z=0}^{m-1} |zr+b\rangle, \tag{4}$$

where r is the period, b is an offset, and m is the number of repetitions.

Let the Hilbert space dimension be N, and assume $N = mr$ so that $m = N/r$ is an integer. The QFT (DFT) modulo N acts as

$$\hat{F}_N|x\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi ixk/N}\,|k\rangle. \tag{5}$$

Apply $\hat{F}_N$ to $|\phi_{r,b}\rangle$:

$$\begin{aligned}
\hat{F}_N|\phi_{r,b}\rangle &= \frac{1}{\sqrt{m}}\sum_{z=0}^{m-1} \hat{F}_N|zr+b\rangle \\
&= \frac{1}{\sqrt{m}}\sum_{z=0}^{m-1}\left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i(zr+b)k/N}|k\rangle\right) \\
&= \frac{1}{\sqrt{mN}}\sum_{k=0}^{N-1} e^{2\pi ibk/N}\left(\sum_{z=0}^{m-1} e^{2\pi izrk/N}\right)|k\rangle.
\end{aligned}$$

Define the inner sum

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi izrk/N}.$$

Using $N = mr$, this becomes

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi i z k/m}.$$

This is a geometric series with common ratio $e^{2\pi i k/m}$. Hence

$$S(k) = \begin{cases} m, & k \equiv 0 \pmod{m}, \\ 0, & k \not\equiv 0 \pmod{m}. \end{cases}$$

Thus only k that are multiples of m survive. Write $k = tm$ with $t = 0, 1, \ldots, r-1$:

$$\begin{aligned} \hat{F}_N |\phi_{r,b}\rangle &= \frac{1}{\sqrt{mN}} \sum_{t=0}^{r-1} e^{2\pi i b(tm)/N} \, m \, |tm\rangle \\ &= \frac{\sqrt{m}}{\sqrt{N}} \sum_{t=0}^{r-1} e^{2\pi i bt/r} \, |tm\rangle. \end{aligned}$$

Since $m = N/r$,

$$\frac{\sqrt{m}}{\sqrt{N}} = \sqrt{\frac{m}{mr}} = \frac{1}{\sqrt{r}},$$

and $tm = tN/r$. Therefore the Fourier-transformed state is

$$|\tilde{\phi}\rangle = \hat{F}_N |\phi_{r,b}\rangle = \frac{1}{\sqrt{r}} \sum_{t=0}^{r-1} e^{2\pi i bt/r} \left| \frac{tN}{r} \right\rangle. \tag{6}$$

IV. Relation Between Hadamard and QFT

Exercise 4: Show that $\frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} |y\rangle$ can be obtained by applying Hadamard and also by applying Fourier transform. What does this tell you about the relation between Hadamard and Fourier transform?

Let $q = 2^k$. The first register has k qubits, initialized to $|0\rangle^{\otimes k}$.

Using the Hadamard Gate

For a single qubit,

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Therefore,

$$\begin{aligned} \hat{H}^{\otimes k} |0\rangle^{\otimes k} &= \bigotimes_{j=1}^{k} \hat{H}_j |0\rangle_j \\ &= \bigotimes_{j=1}^{k} \frac{1}{\sqrt{2}}(|0\rangle_j + |1\rangle_j) \\ &= \frac{1}{\sqrt{2^k}} \sum_{y_1, \ldots, y_k \in \{0,1\}} |y_1 y_2 \ldots y_k\rangle. \end{aligned}$$

Identifying $y = y_1 y_2 \ldots y_k$ as the binary representation of $y \in \{0, \ldots, q-1\}$, we get

$$\hat{H}^{\otimes k} |0\rangle^{\otimes k} = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} |y\rangle. \tag{7}$$

Using the QFT

The QFT on $q = 2^k$ basis states is defined as

$$\hat{F}_q|j\rangle = \frac{1}{\sqrt{q}}\sum_{y=0}^{q-1} e^{2\pi i j y/q}|y\rangle.$$

Apply it to $|0\rangle$:

$$\hat{F}_q|0\rangle = \frac{1}{\sqrt{q}}\sum_{y=0}^{q-1} e^{2\pi i\cdot 0\cdot y/q}|y\rangle = \frac{1}{\sqrt{q}}\sum_{y=0}^{q-1}|y\rangle. \tag{8}$$

Conclusion

Both operations produce the same equal superposition from the initial state $|0\rangle^{\otimes k}$:

$$\hat{H}^{\otimes k}|0\rangle^{\otimes k} = \hat{F}_q|0\rangle.$$

This shows:

- For $N = 2$, the single-qubit Hadamard gate is exactly the QFT on $\mathbb{Z}_2$.
- For $N = 2^k$, $\hat{H}^{\otimes k}$ and $\hat{F}_q$ coincide on $|0\rangle^{\otimes k}$ and are closely related. The QFT over $\mathbb{Z}_{2^k}$ can be built from Hadamard gates plus controlled phase rotations.

So the Hadamard transform is a special (very simple) case of a quantum Fourier transform.

V. Destructive Interference in QFT

Exercise 5: The sum changed from q/r terms to r terms during QFT. This was a result of destructive interference in the QFT on the state. Show how it happened.

From Shor's order-finding procedure, after measuring the second register we obtain in the first register the state

$$|\phi_{a_0}\rangle = \frac{1}{\sqrt{q/r}}\sum_{z=0}^{q/r-1}|zr + a_0\rangle, \tag{9}$$

where $q = mr$ and hence $q/r = m$.

Apply the QFT modulo q:

$$\hat{F}_q|x\rangle = \frac{1}{\sqrt{q}}\sum_{k=0}^{q-1} e^{2\pi i x k/q}|k\rangle.$$

Then

$$\begin{aligned}
\hat{F}_q|\phi_{a_0}\rangle &= \frac{1}{\sqrt{m}}\sum_{z=0}^{m-1}\hat{F}_q|zr + a_0\rangle \\
&= \frac{1}{\sqrt{m}}\sum_{z=0}^{m-1}\left(\frac{1}{\sqrt{q}}\sum_{k=0}^{q-1} e^{2\pi i(zr+a_0)k/q}|k\rangle\right) \\
&= \frac{1}{\sqrt{mq}}\sum_{k=0}^{q-1} e^{2\pi i a_0 k/q}\left(\sum_{z=0}^{m-1} e^{2\pi i z r k/q}\right)|k\rangle.
\end{aligned}$$

Again define

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi i z r k/q}.$$

Using $q = mr$, we get

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi i z k/m}.$$

This is the same geometric series as before:

$$S(k) = \begin{cases} m, & k \equiv 0 \pmod{m}, \\ 0, & k \not\equiv 0 \pmod{m}. \end{cases}$$

Destructive interference. If k is not a multiple of $m = q/r$, then $e^{2\pi i k/m} \neq 1$, and the terms

$$e^{2\pi i z k/m}, \quad z = 0, \dots, m-1$$

sum to zero. That is,

$$S(k) = 0 \quad \Rightarrow \quad \text{amplitude of } |k\rangle = 0.$$

All such components are removed by destructive interference.

Constructive interference. If k *is* a multiple of m, say $k = tm$ with $t = 0, 1, \dots, r-1$, then

$$e^{2\pi i z k/m} = e^{2\pi i z t} = 1$$

for all z, so

$$S(k) = \sum_{z=0}^{m-1} 1 = m.$$

Thus for $k = tm$,

$$\begin{aligned} \hat{F}_q |\phi_{a_0}\rangle &= \frac{1}{\sqrt{mq}} \sum_{t=0}^{r-1} e^{2\pi i a_0 (tm)/q}\, m\, |tm\rangle \\ &= \frac{\sqrt{m}}{\sqrt{q}} \sum_{t=0}^{r-1} e^{2\pi i a_0 t/r}\, |tm\rangle. \end{aligned}$$

Since $m = q/r$, we have $\sqrt{m/q} = 1/\sqrt{r}$, and $tm = tq/r$, so

$$\hat{F}_q |\phi_{a_0}\rangle = \frac{1}{\sqrt{r}} \sum_{t=0}^{r-1} e^{2\pi i a_0 t/r} \left| \frac{tq}{r} \right\rangle. \tag{10}$$

Originally, $|\phi_{a_0}\rangle$ was a superposition of $q/r = m$ basis states. After the QFT, only r basis states (those with $k = tq/r$) have non-zero amplitude. The reduction from q/r terms to r terms is exactly due to the destructive interference of all other components in the geometric sum $S(k)$.