Assignment-3 Solutions

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QT 207

II. Quantum Fourier Transform (QFT) and the Unitary QFT Operator

Exercise 2: Find an operator $\hat{F}$ which implements the discrete Fourier transform and show that it is unitary.

Definition of the QFT operator

The quantum Fourier transform (QFT) operator $\hat{F}$ acts on the computational basis $\{|j\rangle\}_{j=0}^{N-1}$ as

$$\hat{F}|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle. \tag{1}$$

In matrix form the elements of $\hat{F}$ are

$$F_{kj} = \langle k|\hat{F}|j\rangle = \frac{1}{\sqrt{N}} e^{2\pi ijk/N}, \tag{2}$$

so that

$$\hat{F} = \frac{1}{\sqrt{N}} \sum_{k,j=0}^{N-1} e^{2\pi ijk/N} |k\rangle\langle j|. \tag{3}$$

Unitarity of $\hat{F}$

An operator is unitary if $\hat{F}\hat{F}^\dagger = \hat{F}^\dagger\hat{F} = I$. The adjoint acts on basis states as

$$\hat{F}^\dagger|k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-2\pi ijk/N} |j\rangle. \tag{4}$$

Compute the matrix product elements:

$$(\hat{F}\hat{F}^\dagger)_{kk'} = \sum_{j=0}^{N-1} F_{kj} F^*_{k'j} \tag{5}$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi ij(k-k')/N}. \tag{6}$$

The sum on the right is a finite geometric series which equals N when $k = k'$ and 0 otherwise, so

$$(\hat{F}\hat{F}^\dagger)_{kk'} = \delta_{kk'}, \tag{7}$$

and therefore $\hat{F}\hat{F}^\dagger = I$. Similarly one shows $\hat{F}^\dagger\hat{F} = I$, proving $\hat{F}$ is unitary.

III. Discrete Fourier Transform (DFT) on Periodic States

Exercise 3: Apply the QFT to the periodic state $|\phi_{r,b}\rangle$ and obtain the resulting state.

Consider the periodic state

$$|\phi_{r,b}\rangle = \frac{1}{\sqrt{m}} \sum_{z=0}^{m-1} |zr + b\rangle, \tag{8}$$

where r is the period, b is an offset, and m is the number of repetitions. Applying the QFT (dimension N) gives

$$\hat{F}|\phi_{r,b}\rangle = \frac{1}{\sqrt{m}} \sum_{z=0}^{m-1} \hat{F}|zr+b\rangle \tag{9}$$

$$= \frac{1}{\sqrt{mN}} \sum_{z=0}^{m-1} \sum_{k=0}^{N-1} e^{2\pi i(zr+b)k/N}|k\rangle. \tag{10}$$

Interchanging the sums and factoring out the z-independent factor yields

$$\hat{F}|\phi_{r,b}\rangle = \sum_{k=0}^{N-1} \left[\frac{1}{\sqrt{mN}} e^{2\pi ibk/N} \sum_{z=0}^{m-1} e^{2\pi izrk/N} \right] |k\rangle. \tag{11}$$

Define the geometric sum

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi izrk/N}. \tag{12}$$

If rk/N is not an integer then the phases in the sum cancel and $S(k) = 0$ (destructive interference). If rk/N is an integer, write $k = t(N/r)$ with integer t, then each term equals 1 and

$$S(k) = \sum_{z=0}^{m-1} 1 = m. \tag{13}$$

Using $N = mr$, the amplitude for those k values becomes

$$A_k = \frac{1}{\sqrt{mN}} e^{2\pi ibk/N} \times m = \sqrt{\frac{m}{N}} e^{2\pi ibk/N} = \frac{1}{\sqrt{r}} e^{2\pi ibk/N}. \tag{14}$$

Thus the transformed state is a uniform superposition over the r values of k satisfying $k = t(N/r)$:

$$\hat{F}|\phi_{r,b}\rangle = \frac{1}{\sqrt{r}} \sum_{t=0}^{r-1} e^{2\pi ibt/r}\, |tN/r\rangle. \tag{15}$$

This is the standard result used in period-finding: constructive interference picks out frequencies that are integer multiples of N/r while other frequencies vanish by destructive interference.

IV. Relation Between Hadamard and QFT

Exercise 4: Show that the state $\frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} |y\rangle$ can be prepared both by Hadamards and by the QFT.

Let $q = 2^k$. Starting from the register $|0\rangle^{\otimes k}$, applying k single-qubit Hadamard gates gives

$$H^{\otimes k}|0\rangle^{\otimes k} = \frac{1}{\sqrt{2^k}} \sum_{y=0}^{2^k-1} |y\rangle = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} |y\rangle. \tag{16}$$

On the other hand, the QFT on $N = q = 2^k$ satisfies

$$\hat{F}|0\rangle = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} e^{2\pi i(0)y/q}|y\rangle = \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} |y\rangle. \tag{17}$$

Hence both operations prepare the same equally weighted superposition. In particular, for $k = 1$ (single qubit) the Hadamard gate is the QFT for $N = 2$.

V. Destructive Interference in QFT

Exercise 5: Explain how the number of terms reduces from q/r to r after the QFT due to destructive interference.

Starting with the periodic state

$$|\phi_{a_0}\rangle = \frac{1}{\sqrt{m}} \sum_{z=0}^{m-1} |zr + a_0\rangle, \qquad m = q/r, \tag{18}$$

the QFT produces amplitudes proportional to the geometric sum

$$S(k) = \sum_{z=0}^{m-1} e^{2\pi i z r k/q}. \tag{19}$$

If rk/q is not an integer then $S(k) = 0$ because the phases cancel (destructive interference). If $k = t(q/r)$ then $S(k) = m = q/r$ and the amplitude for such k is

$$A_k = \frac{1}{\sqrt{mq}} e^{2\pi i a_0 k/q} S(k) = \frac{1}{\sqrt{r}} e^{2\pi i a_0 k/q}. \tag{20}$$

Only r values of k (those proportional to q/r) have nonzero amplitude, so the original superposition over $m = q/r$ periodic positions is mapped to a superposition over r frequency modes. This is the mechanism behind the period-finding step in Shor's algorithm.