Formal Geometry and Bordism Operations

Lecture notes

Eric Peterson

Compiled: June 20, 2016
Git hash: 71cd648

Class information

Course ID: MATH 278 (159627).

Meeting times: Spring 2016, MWF 12pm–1pm.

Goals: The primary goal of this class is to teach students to view results in algebraic topology through the lens of (formal) algebraic geometry.

Grading: This class won't have any official assignments. I'll give references as readings for those who would like a deeper understanding, though I'll do my best to ensure that no extra reading is required to follow the arc of the class.

I do want to assemble course notes from this class, but it's unlikely that I will have time to type *all* of them up. Instead, I would like to "crowdsource" this somewhat: I'll type up skeletal notes for each lecture, and then we as a class will try to flesh them out as the semester progresses. As incentive to help, those who contribute to the document will have their name included in the acknowledgements, and those who contribute *substantially* will have their name added as a coauthor. Everyone could use more CV items. (Publication may take a while. I suspect the course won't run perfectly smoothly the first time, so this may takes a second semester pass to become fully workable. But, since topics courses only come around once in a while, this will necessarily mean a delay.)

The source for this document can be found at

`https://github.com/ecpeterson/FormalGeomNotes`.

If you're taking the class or otherwise want to contribute, you can write me at

`ecp@math.harvard.edu`

to request write access.

Acknowledgements

Matthew Ando, Michael Hopkins, Neil Strickland

Constantin Teleman — encouraging me to speak understandably

Haynes Miller, Doug Ravenel, Steve Wilson — the *BP* mafia

Jack Morava

Nat Stapleton

Martin Bendersky — brief consultation about unstable cooperations

Hood Chatham — the spectral sequence package, the picture of $\mathcal{M}_{\mathbf{fg}}$

Hisham Sati — the *Flavors of Cohomology* workshop

Hood Chatham and Geoffrey Lee, Catherine Ray, Kevin Wray — Berkeley students who I talked to a lot

Contributors: Eva Belmont, Hood Chatham (spectral sequence package), Jun Hou Fung, Jeremy Hahn (Yuli Rudyak's alternative proof), Mauro Porta, Krishanu Sankar, Danny Shi, Allen Yuan

Regular participants (whoever I could think of on March 24, again missing people): Eva Belmont, Lukas Brantner, Christian Carrick, Hood Chatham, Jun Hou Fung, Meng Guo, Jeremy Hahn, Erick Knight, Jake McNamara, Akhil Mathew, Krishanu Sankar, Jay Shah, Danny Shi, Allen Yuan

Course registrants (collected March 24, possibly missing some MIT people): Colin Aitken, Adam Al-Natsheh, Eva Belmont, Jason Bland, Dorin Boger, Lukas Brantner, Christian Carrick, Hood Chatham, David Corwin, Jun Hou Fung, Meng Guo, Jeremy Hahn, Changho Han, Chi-Yun Hsu, Erick Knight, Benjamin Landon, Gabriel Long, Yusheng Luo, Jake Marcinek, Jake McNamara, Max Menzies, Morgan Opie, Alexander Perry, Mauro Porta, Kishanu Sankar, Jay Shah, Ananth Shankar, Danny Shi, Koji Shimizu, Geoffrey Smith, Hunter Spinik, Philip Tynan, Yi Xie, David Yang, Zijian Yao, Lynnelle Ye, Chenglong Yu, Allen Yuan, Adrian Zahariuc, Yifei Zhao, Rong Zhou, Yihang Zhu.

People who contributed to the GitHub repository (to be collected later, make sure there's no overlap with the above list):

Other readers: Jon Beardsley, Sune Precht Reeh, Kevin Wray

Contents

1 Unoriented bordism **1**
1.1 Thom spectra and the Thom isomorphism 1
1.2 Cohomology rings and affine schemes 6
1.3 The Steenrod algebra 11
1.4 Hopf algebra cohomology 16
1.5 The unoriented bordism ring 23

A Loose ends **27**
A.1 E_∞ geometry 27
A.2 Rational phenomena: character theory for Lubin–Tate spectra 27
A.3 Knowns and unknowns 31

Bibliography **33**

I would like as few section titles as possible to involve people's names.

A bunch of broken displaymode tombstones can have their positions fixed by using http://tex.stackexchange.com/a/66221/2671.

Compile an index by replacing all the `textit` commands in definition environments with some more fancy macro that tags it for inclusion.

Remember to use $f: A \to B$ everywhere.

Should sections have subsections? Does that help organize the TOC?

Hood wrote a macro called sumfgl (see also sumF and sumG) that will make a bunch of formal group law expressions typeset better. Propagate that change through.

http://tex.stackexchange.com/questions/53513/hyperref-token-not-allowed has information on how to make the PDF Bookmarks warnings go away. You can't use this until after you drop the date prefix macro.

Make sure you use either id or 1 everywhere to denote the identity morphism.

You're not very consistent about using $\widehat{\mathbb{G}}$ or Γ to denote an arbitrary formal group. It seems like you use one or the other based on your preference of whether it has finite height or not.

Case Study 1

Unoriented bordism

This Case Study culminates in the calculation of $MO_*(*)$, the bordism ring of unoriented manifolds, but we mainly take this as an opportunity to introduce several key concepts that will serve us throughout the book. First and foremost, we will require a definition of bordism spectrum that we can manipulate computationally, using just the tools of abstract homotopy theory. Once that is established, we immediately begin to bring algebraic geometry into the mix: the main idea is that the cohomology ring of a space is better viewed as a scheme (with plenty of extra structure), and the homology groups of a spectrum are better viewed as representation for a certain highly elaborate algebraic structure group. This data finds expression in familiar homotopy theory: we show that a form of group cohomology for this representation forms the input to the classical Adams spectral sequence. Finally, we calculate this representation structure for $H\mathbb{F}_{2*}MO$, find that it is suitably free, and thereby gain control of the Adams spectral sequence computing $MO_*(*)$.

Thread Crefs to the relevant theorems below through this introduction.

1.1 Thom spectra and the Thom isomorphism

Our goal is a sequence of theorems about the unoriented bordism spectrum MO. We will begin by recalling a definition of the spectrum MO using just abstract homotopy theory, because it involves ideas that will be useful to us throughout the semester and because we cannot compute effectively with the chain-level definition given in the Introduction.

Definition 1.1.1. For a spherical bundle $S^{n-1} \to \xi \to X$, its Thom space is given by the cofiber

$$\xi \to X \xrightarrow{\text{cofiber}} T_n(\xi).$$

"Proof" of definition. There is a more classical construction of the Thom space: take the associated disk bundle by gluing an n–disk fiberwise, and add a point at infinity by collapsing ξ:

$$T_n(\xi) = (\xi \sqcup'_{S^{n-1}} D^n)^+.$$

To compare this with the cofiber definition, recall that the thickening of ξ to an n–disk bundle is the same thing as taking the mapping cylinder on $\xi \to X$. Since the inclusion into the mapping cylinder is now a cofibration, the quotient by this subspace agrees with both the cofiber of the map and the introduction of a point at infinity. $\square$

Before proceeding, here are two important examples:

Example 1.1.2. If $\xi = S^{n-1} \times X$ is the trivial bundle, then $T_n(\xi) = S^n \wedge (X_+)$. This is supposed to indicate what Thom spaces are "doing": if you feed in the trivial bundle then you get the suspension out, so if you feed in a twisted bundle you should think of it as a *twisted suspension*.

Example 1.1.3. Let ξ be the tautological S^0–bundle over $\mathbb{RP}^\infty = BO(1)$. Because ξ has contractible total space, $EO(1)$, the cofiber degenerates and it follows that $T_1(\xi) = \mathbb{RP}^\infty$. More generally, arguing by cells shows that the Thom space for the tautological bundle over $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$.

Cite me: Give a reference for this general construction of classifying spaces for fibrations..

Now we catalog a bunch of useful properties of the Thom space functor. Firstly, recall that a spherical bundle over X is the same data as a map $X \to BGL_1 S^{n-1}$, where $GL_1 S^{n-1}$ is the subspace of $F(S^{n-1}, S^{n-1})$ expressed by the pullback of spaces

$$\begin{tikzcd}
GL_1 S^{n-1} \arrow[r] \arrow[d] & F(S^{n-1}, S^{n-1}) \arrow[d] & \\
\mathrm{Aut}_{h\mathsf{Spaces}}\, S^{n-1} \arrow[r] & \mathrm{End}_{h\mathsf{Spaces}}\, S^{n-1} \arrow[r, equal] & \pi_0 F(S^{n-1}, S^{n-1}).
\end{tikzcd}$$

We can interpret T_n as a functor off of the slice category over $BGL_1 S^{n-1}$: maps

$$Y \xrightarrow{f} X \xrightarrow{\xi} BGL_1 S^{n-1}$$

induce maps $T_n(f^*\xi) \to T_n(\xi)$, and T_n is suitably homotopy-invariant.

Next, the spherical subbundle of a vector bundle gives a common source of spherical bundles. The action of $O(n)$ on $\mathbb{R}^n$ preserves the unit sphere, and hence gives a map $O(n) \to GL_1 S^{n-1}$. These are maps of topological groups, and the block–inclusion maps $i^n \colon O(n) \to O(n+1)$ commute with the suspension map $GL_1 S^{n-1} \to GL_1 S^n$. In fact, much more can be said:

Lemma 1.1.4. *The block–sum maps $O(n) \times O(m) \to O(n+m)$ are compatible with the join maps $GL_1 S^{n-1} \times GL_1 S^{m-1} \to GL_1 S^{n+m-1}$.*

Again taking a cue from K–theory, we take the colimit as n grows large, using the maps

$$BGL_1 S^{n-1} = BGL_1 S^{n-1} \times * \xrightarrow{\mathrm{id} \times \mathrm{triv}} BGL_1 S^{n-1} \times BGL_1 S^0 \xrightarrow{*} BGL_1 S^n,$$

$$BO(n) = BO(n) \times * \xrightarrow{\mathrm{id} \times \mathrm{triv}} BO(n) \times BO(1) \xrightarrow{\oplus} BO(n+1).$$

Corollary 1.1.5. *The operations of block–sum and topological join imbue the colimiting spaces BO and $BGL_1\mathbb{S}$ with the structure of H–groups. Moreover, the colimiting map*

Should you justify "group" rather than "space"?

$$J_{\mathbb{R}} \colon BO \to BGL_1\mathbb{S},$$

called the stable J–homomorphism, is a morphism of H–groups. $\square$

Finally, we can ask about the compatibility of Thom constructions with all of this. In order to properly phrase the question, we need a version of the construction which operates on stable spherical bundles, i.e., whose source is the slice category over $BGL_1\mathbb{S}$. By calculating

Does this calculation need justification?

$$T_{n+1}(\xi * \mathrm{triv}) \simeq \Sigma T_n(\xi),$$

we are inspired to make the following definition:

Definition 1.1.6. For ξ an S^{n-1}–bundle, we define the *Thom spectrum* of ξ to be

$$T(\xi) := \Sigma^{-n}\Sigma^{\infty}T_n(\xi).$$

By filtering the base space by compact subspaces, this begets a functor

$$T \colon \mathsf{Spaces}_{/BGL_1\mathbb{S}} \to \mathsf{Spectra}.$$

Lemma 1.1.7. *T is monoidal: it carries external fiberwise joins to smash products of Thom spectra. Correspondingly, $T \circ J_{\mathbb{R}}$ carries external direct sums of stable vector bundles to smash products of Thom spectra.* $\square$

Definition 1.1.8. The spectrum MO arises as the universal example of all these constructions, strung together:

$$MO := T(J_{\mathbb{R}}) = \operatorname*{colim}_n T(J^n_{\mathbb{R}}) = \operatorname*{colim}_n \Sigma^{-n}T_n J^n_{\mathbb{R}}.$$

Cite me: There should be a reference here (to Pontryagin, presumably) saying that we recover MO as defined on the first day..

The spectrum MO has several remarkable properties. The most basic such property is that it is a ring spectrum, and this follows immediately from $J_{\mathbb{R}}$ being a homomorphism of H–spaces. Much more excitingly, we can also deduce the presence of Thom isomorphisms just from the properties stated thus far. That $J_{\mathbb{R}}$ is a homomorphism means that the following square commutes:

$$\begin{tikzcd}
BO \times BO \arrow[r, "\sigma", "\cong"'] \arrow[drr, bend right] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}} \times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

We have extended this square very slightly by a certain shearing map σ defined by $\sigma(x, y) = (xy^{-1}, y)$. It's evident that σ is a homotopy equivalence, since just as we can de-scale the first coordinate by y we can re-scale by it. We can calculate directly the behavior of the long composite:

σ *almost* shows up in giving a categorical definition of a G–torsor. I wish I understood this, but I always get tangled up.

$$J_{\mathbb{R}} \circ \mu \circ \sigma(x, y) = J_{\mathbb{R}} \circ \mu(xy^{-1}, y) = J_{\mathbb{R}}(xy^{-1}y) = J_{\mathbb{R}}(x).$$

It follows that the second coordinate plays no role, and that the bundle classified by the long composite can be written as $J_{\mathbb{R}} \times 0$.[1] We are now in a position to see the Thom isomorphism:

Lemma 1.1.9 (Thom isomorphism, universal example)**.** *As MO–modules,*

$$MO \wedge MO \simeq MO \wedge \Sigma^\infty_+ BO.$$

Proof. Stringing together the naturality properties of the Thom functor outlined above, we can thus make the following calculation:

$$\begin{aligned}
T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}})) &\simeq T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}}) \circ \sigma) && \text{(homotopy invariance)} \\
&\simeq T(J_{\mathbb{R}} \times 0) && \text{(constructed lift)} \\
&\simeq T(J_{\mathbb{R}}) \wedge T(0) && \text{(monoidality)} \\
&\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(Example 1.1.2)} \\
T(J_{\mathbb{R}}) \wedge T(J_{\mathbb{R}}) &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(monoidality)} \\
MO \wedge MO &\simeq MO \wedge \Sigma^\infty_+ BO. && \text{(definition of } MO\text{)}
\end{aligned}$$

The equivalence is one of MO–modules because the MO–module structure of both sides comes from smashing with MO on the left. □

From here, the general version of Thom's theorem follows quickly:

Theorem 1.1.10 (Thom isomorphism)**.** *Let $\xi\colon X \to BO$ classify a vector bundle and let $\varphi\colon MO \to E$ be a map of ring spectra. Then there is an equivalence of E–modules*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X.$$

Modifications to above proof. To accommodate X rather than BO as the base, we redefine $\sigma\colon BO \times X \to BO \times X$ by

$$\sigma(x, y) = \sigma(x\xi(y)^{-1}, y).$$

Follow the same proof as before with the diagram

[1] This factorization does *not* commute with the rest of the diagram, just with the little lifting triangle it forms.

$$
\begin{tikzcd}
BO \times X \arrow[r, "\sigma", "\simeq"'] \arrow[drrr, bend right] & BO \times X \arrow[r, "\mathrm{id}\times\xi", "\simeq"'] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}}\times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}
$$

This gives an equivalence $\theta_{MO} \colon MO \wedge T(\xi) \to MO \wedge \Sigma^\infty_+ X$. To introduce E, note that there is a diagram

$$
\begin{tikzcd}
E \wedge T(\xi) \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}=f"] & E \wedge \Sigma^\infty_+ X \arrow[d, "\eta_{MO}\wedge\mathrm{id}\wedge\mathrm{id}"] \\
MO \wedge E \wedge T(\xi) \arrow[r, "\theta_{MO}\wedge E"] \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=g"] & MO \wedge E \wedge \Sigma^\infty_+ X \arrow[d, "(\mu\circ(\varphi\wedge\mathrm{id}))\wedge\mathrm{id}=h"] \\
E \wedge T(\xi) \arrow[r, "\theta_E"] & E \wedge \Sigma^\infty_+ X
\end{tikzcd}
$$

The bottom arrow θ_E exists by applying the action map to both sides and pushing the map $\theta_{MO} \wedge E$ down. Since θ_{MO} is an equivalence, it has an inverse α_{MO}. Therefore, the middle map has inverse $\alpha_{MO} \wedge E$, and we can similarly push this down to a map α_E, which we now want to show is the inverse to θ_E. From here it is a simple diagram chase: we have renamed three of the maps in the diagram to f, g, and h for brevity. Noting that $g \circ f$ is the identity map because of the unit axiom, we conclude

$$
\begin{aligned}
g \circ f &\simeq g \circ (\alpha_{MO} \wedge E) \circ (\theta_{MO} \wedge E) \circ f \\
&\simeq \alpha_E \circ h \circ (\theta_{MO} \wedge E) \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E \circ g \circ f && \text{(action map)} \\
&\simeq \alpha_E \circ \theta_E.
\end{aligned}
$$

It follows that α_E gives an inverse to θ_E. □

Remark 1.1.11. One of the tentpoles of the theory of Thom spectra is that Theorem 1.1.10 has a kind of converse: if a ring spectrum E has suitably natural and multiplicative Thom isomorphisms for Thom spectra formed from real vector bundles, then one can define an essentially unique ring map $MO \to E$ realizing these isomorphisms via the machinery of Theorem 1.1.10.

Example 1.1.12. We'll close out this section by using this to actually make a calculation. Recall from Example 1.1.3 that $T(\mathcal{L} \downarrow \mathbb{R}\mathrm{P}^n) = \mathbb{R}\mathrm{P}^{n+1}$. Because MO is a connective spectrum, the diagram This is *still* not a proof of this. Ugh.

$$\begin{tikzcd}
MO \wedge MO \arrow[r] \arrow[d] & (MO \wedge MO)(-\infty,0] \arrow[r] \arrow[d, equal] & MO(-\infty,0] \wedge MO(-\infty,0] \arrow[r, equal] \arrow[dl, dotted] & H\pi_0 MO \wedge H\pi_0 MO \arrow[d, dotted] \\
MO \arrow[r] & MO(-\infty,0] \arrow[rr, equal] & & H\pi_0 MO
\end{tikzcd}$$

shows that

$$MO \to MO(-\infty,0] = H\pi_0 MO = H\mathbb{F}_2$$

is a map of ring spectra. Hence, we can apply the Thom isomorphism theorem to the mod–2 homology of Thom complexes coming from real vector bundles:

$$\begin{aligned}
\pi_*(H\mathbb{F}_2 \wedge T(\mathcal{L}-1)) &\cong \pi_*(H\mathbb{F}_2 \wedge T(0)) && \text{(Thom isomorphism)} \\
\pi_*(H\mathbb{F}_2 \wedge \Sigma^{-1}\Sigma^\infty \mathbb{RP}^{n+1}) &\cong \pi_*(H\mathbb{F}_2 \wedge \Sigma^\infty_+ \mathbb{RP}^n) && \text{(Example 1.1.3)} \\
\widetilde{H\mathbb{F}}_{2*+1}\mathbb{RP}^{n+1} &\cong H\mathbb{F}_{2*}\mathbb{RP}^n. && \text{(generalized homology)}
\end{aligned}$$

This powers an induction that shows $H\mathbb{F}_{2*}\mathbb{RP}^\infty$ has a single class in every degree. The cohomological version of the Thom isomorphism, together with the $H\mathbb{F}_2^*\mathbb{RP}^n$–module structure of $H\mathbb{F}_2^* T(\mathcal{L}-1)$, also gives the ring structure:

This could use a reference or a remark or something. Is there a Mahowaldian version of the cohomological Thom isomorphism?

$$H\mathbb{F}_2^*\mathbb{RP}^n = \mathbb{F}_2[x]/x^{n+1}.$$

1.2 Cohomology rings and affine schemes

Make sure you use $\mathbb{F}_2$ everywhere, rather than $\mathbb{Z}/2$.

An abbreviated summary of this semester is that we're going to put "Spec" in front of rings appearing in algebraic topology and see what happens. Before doing any algebraic topology, let me remind you what this means on the level of algebra. The core idea is to replace a ring R by the functor it corepresents, $\operatorname{Spec} R$. For any "test $\mathbb{F}_2$–algebra" T, we set

$$(\operatorname{Spec} R)(T) := \mathsf{Algebras}_{\mathbb{F}_2/}(R, T) \cong \mathsf{Schemes}_{/\mathbb{F}_2}(\operatorname{Spec} T, \operatorname{Spec} R).$$

More generally, we have the following definition:

Definition 1.2.1. An *affine* $\mathbb{F}_2$*–scheme* is a functor $X : \mathsf{Algebras}_{\mathbb{F}_2/} \to \mathsf{Sets}$ which is (non-canonically) isomorphic to $\operatorname{Spec} R$ for some $\mathbb{F}_2$–algebra R. Given such an isomorphism, we will refer to $\operatorname{Spec} R \to X$ as a *parameter* for X and its inverse $X \to \operatorname{Spec} R$ as a *coordinate* for X.

Lemma 1.2.2. *There is an equivalence of categories*

$$\operatorname{Spec} : \mathsf{Algebras}^{op}_{\mathbb{F}_2/} \to \mathsf{AffineSchemes}_{/\mathbb{F}_2}. \quad \square$$

The centerpiece of thinking about rings in this way, for us and for now, is to translate between a presentation of R as a quotient of a free algebra and a presentation of $(\operatorname{Spec} R)(T)$ as selecting tuples of elements in T subject to certain conditions. Consider the following example:

Example 1.2.3. Set $R_1 = \mathbb{F}_2[x]$. Then

$$(\operatorname{Spec} R_1)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], T)$$

is determined by where x is sent — i.e., this Hom–set is naturally isomorphic to T itself. Consider also what happens when we impose a relation by passing to $R_2 = \mathbb{F}_2[x]/(x^{n+1})$. The value

$$(\operatorname{Spec} R_2)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x]/(x^{n+1}), T)$$

of the associated affine scheme is again determined by where x is sent, but now x can only be sent to elements which are nilpotent of order $n+1$. These schemes are both important enough that we give them special names:

$$\mathbb{A}^1 := \operatorname{Spec} \mathbb{F}_2[x], \qquad \mathbb{A}^{1,(n)} := \operatorname{Spec} \mathbb{F}_2[x]/(x^{n+1}).$$

The symbol "$\mathbb{A}^1$" is pronounced "the affine line" — reasonable, since the value $\mathbb{A}^1(T)$ is, indeed, a single T's worth of points. Note that the quotient map $R_1 \to R_2$ induces an inclusion $\mathbb{A}^{1,(n)} \to \mathbb{A}^1$ and that $\mathbb{A}^{1,(0)}$ is a constant functor:

$$\mathbb{A}^{1,(0)}(T) = \{f : \mathbb{F}_2[x] \to T \mid f(x) = 0\}.$$

Accordingly, we pronounce "$\mathbb{A}^{1,(0)}$" as "the origin on the affine line" and "$\mathbb{A}^{1,(n)}$" as "the $(n+1)^{\text{st}}$ order (nilpotent) neighborhood of the origin in the affine line".

We can also express in this language another common object arising from algebraic topology: the Hopf algebra, which appears when taking the mod–2 cohomology of an H–group. In addition to the usual cohomology, the extra pieces of data are those induced by the H–group multiplication, unit, and inversion maps, which on cohomology beget a diagonal map Δ, an augmentation map ε, and an antipode χ respectively. Running through the axioms, one quickly checks the following:

Lemma 1.2.4. *For a Hopf $\mathbb{F}_2$–algebra R, the functor* $\operatorname{Spec} R$ *is naturally valued in groups. Such functors are called group schemes. Conversely, a choice of group structure on* $\operatorname{Spec} R$ *endows R with the structure of a Hopf algebra.*

Proof. The functor $\operatorname{Spec}\colon \mathsf{Algebras}_{\mathbb{F}_2/}^{\mathrm{op}} \to \operatorname{Funct}(\mathsf{Algebras}_{\mathbb{F}_2/}, \mathsf{Sets})$ takes limits into limits. Since tensor products of $\mathbb{F}_2$-algebras compute pushouts in $\mathsf{Algebras}_{\mathbb{F}_2/}$, we see that Hopf algebras are simply cogroup objects in $\mathsf{Algebras}_{\mathbb{F}_2/}$. These remarks imply that Spec takes Hopf algebras into group objects in $\operatorname{Funct}(\mathsf{Algebras}_{\mathbb{F}_2/}, \mathsf{Sets})$. Now, for any small category $\mathcal{C}$, one has

$$\operatorname{Grp}(\operatorname{Funct}(\mathcal{C}, \mathsf{Sets})) \simeq \operatorname{Funct}(\mathcal{C}, \mathsf{Groups}).$$

The conclusion now follows from the fully faithfulness of Spec. $\square$

Example 1.2.5. The functor $\mathbb{A}^1$ introduced above is naturally valued in groups: since $\mathbb{A}^1(T) \cong T$, we can use the addition on T to make it into an abelian group. When considering $\mathbb{A}^1$ with this group scheme structure, we notate it as $\mathbb{G}_a$. Applying the Yoneda lemma, one deduces the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_a \times \mathbb{G}_a &\xrightarrow{\mu} \mathbb{G}_a & x_1 + x_2 &\leftarrow\!\shortmid x, \\ \mathbb{G}_a &\xrightarrow{\chi} \mathbb{G}_a & -x &\leftarrow\!\shortmid x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \mathbb{G}_a & 0 &\leftarrow\!\shortmid x. \end{aligned}$$

I think you explained this differently in class, so I wanted to work out how to use the Yoneda lemma to deduce these. The idea is that we want a map, e.g., $\mu \in \operatorname{Nat}(\mathbb{A}^1 \times \mathbb{A}^1, \mathbb{A}^1)$. The functors involved are corepresented, so this is just $\operatorname{Nat}(h^{\mathbb{F}_2[x_1,x_2]}, h^{\mathbb{F}_2[x]})$. The Yoneda lemma says that this is the same as an algebra map $\mathbb{F}_2[x] \to \mathbb{F}_2[x_1, x_2]$, which is obtained by evaluating the natural transformation at the identity of $\mathbb{F}_2[x_1, x_2]$. The identity map of $\mathbb{F}_2[x_1, x_2]$ corresponds to the pair of elements x_1, x_2 in $\mathbb{F}_2[x_1, x_2]$, which gets sent to $x_1 + x_2$. So the algebra homomorphism we want sends x to $x_1 + x_2$.

Remark 1.2.6. In fact, $\mathbb{A}^1$ is naturally valued in *rings*. It models the inverse functor to Spec in the equivalence of categories above, i.e., the elements of a ring R always form a complete collection of $\mathbb{A}^1$–valued functions on some affine scheme $\operatorname{Spec} R$.

Example 1.2.7. We define the *multiplicative group scheme* by

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_2[x, y]/(xy - 1).$$

Its value $\mathbb{G}_m(T)$ on a test algebra T is the set of pairs (x, y) such that y is a multiplicative inverse to x, and hence $\mathbb{G}_m$ is valued in groups. Applying the Yoneda lemma, we deduce the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_m &\xrightarrow{\mu} \mathbb{G}_m & x_1 \otimes x_2 &\leftarrow\!\shortmid x \\ & & y_1 \otimes y_2 &\leftarrow\!\shortmid y, \\ \mathbb{G}_m &\xrightarrow{\chi} \mathbb{G}_m & (y, x) &\leftarrow\!\shortmid (x, y), \\ \operatorname{Spec} R &\xrightarrow{\eta} \mathbb{G}_m & 1 &\leftarrow\!\shortmid x, y. \end{aligned}$$

Do you want this to be a $\mathbb{Z}$–algebra? Ditto with $\mathbb{A}^1$ and $\mathbb{G}_a$? re: I don't think so, since we're just doing everything over F_2, but Eric could you check? AY

Remark 1.2.8. As presented above, the multiplicative group comes with a natural inclusion $\mathbb{G}_m \to \mathbb{A}^2$. Specifically, the subset $\mathbb{G}_m \subseteq \mathbb{A}^2$ consists of pairs (x, y) in the graph of the hyperbola $y = 1/x$. However, the element x also gives an $\mathbb{A}^1$–valued function $x \colon \mathbb{G}_m \to \mathbb{A}^1$, and because multiplicative inverses in a ring are unique, we see that this map too is an inclusion. These two inclusions have rather different properties relative to their ambient spaces, and we'll think harder about these essential differences later on.

Example 1.2.9 (cf. **??**). The following example shows that it is a bad idea to think of affine group schemes as a scheme-ified version of linear Lie groups. Define the group scheme α_2 to be $\operatorname{Spec}(\mathbb{F}_2[x]/(x^2))$ with group scheme structure given by

$$\begin{aligned} \alpha_2 \times \alpha_2 &\xrightarrow{\mu} \alpha_2 & x_1 + x_2 &\leftarrow\!\shortmid x, \\ \alpha_2 &\xrightarrow{\chi} \alpha_2 & -x &\leftarrow\!\shortmid x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \alpha_2 & 0 &\leftarrow\!\shortmid x. \end{aligned}$$

This group scheme has several interesting properties:

All of these can be proven using Dieudonné theory, which we can write out and stick in a reference to Chapter 4.

1. α_2 has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$but is not isomorphic to it. The easiest way to see this is that $\mathrm{Hom}(\mu_2, \mu_2) = \mathbb{Z}/2\mathbb{Z}$but $\mathrm{Hom}(\alpha_2, \mu_2) = \alpha_2$ (these homs are in the category of affine group schemes and give out an affine group scheme).

You introduced this notation in class, but it hasn't been defined in this document yet. See example 1.2.12.

How is $\mathbb{Z}/2\mathbb{Z}$ an affine group scheme? Is it just $\mathrm{Spec}(\mathbb{F}_2)$?

for jhfung: work this out

2. There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.

3. If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence $0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0$. However, this short exact sequence doesn't split (even after making a base change).

4. The subgroups of $\alpha_2 \times \alpha_2$ of order two are parameterized by $\mathbb{P}^1$. That is to say. if R is an $\mathbb{F}_2$-algebra, then the subgroup schemes of $(\alpha_2 \times \alpha_2)_R$ of order two defined over R are parameterized by $\mathbb{P}^1(R)$.

Additionally, the colimit of the sets $\operatorname{colim}_{n \to \infty} \mathbb{A}^{1,(n)}(T)$ is of use in algebra: it is the collection of nilpotent elements in T. These kinds of conditions which are "unbounded in n" appear frequently enough that we are moved to give these functors a name too:

Definition 1.2.10. An *affine formal scheme* is an ind-system of finiteaffine schemes. The morphisms between such schemes are computed by

$$\mathsf{FormalSchemes}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\alpha \operatorname{colim}_\beta \mathsf{Schemes}(X_\alpha, Y_\beta).$$

How are finite schemes characterized in the functor of points perspective?

Several people had questions about the utility of this. Does it just add certain colimits to the category of *finite* affine schemes?

Question: can we reverse the order of taking lim and colim here?-danny ; inserted answer+explanation in the source, AY

Example 1.2.11. The individual schemes $\mathbb{A}^{1,(n)}$ do not support group structures. After all, the sum of two elements which are nilpotent of order $n + 1$ can only be guaranteed to be nilpotent of order $2n + 1$. It follows that the entire ind-system $\{\mathbb{A}^{1,(n)}\} =: \widehat{\mathbb{A}}^1$ supports a group structure, even though none of its constituent pieces do. We call such an object a *formal group scheme*, and this particular formal group scheme we denote by $\widehat{\mathbb{G}}_a$.

Maybe it would be good at some point to define what Spf means because it's not actually stated explicitly anywhere. AY

Example 1.2.12. Similarly, one can define the scheme $\mathbb{G}_m[n]$ of elements of unipotent order n:

$$\mathbb{G}_m[n] = \operatorname{Spec} \frac{\mathbb{F}_2[x, y]}{(xy - 1, x^n - 1)} \subseteq \mathbb{G}_m.$$

These *are* all group schemes, but there is a second filtration along the lines of the one considered above:

$$\mathbb{G}_m^{(n)} = \operatorname{Spec} \frac{\mathbb{F}_2[x, y]}{(xy - 1, (x - 1)^n)}.$$

These schemes are only occasionally group schemes — specifically, $\mathbb{G}_m^{(2^n)}$ is a group scheme, in which case $\mathbb{G}_m^{(2^n)} \cong \mathbb{G}_m[2^n]$ over $\mathbb{F}_2$. This gives an ind-equivalence between these two subsystems, but $\{\mathbb{G}_m[2^n]\}$ is *not* cofinal in $\{\mathbb{G}_m[n]\}$, and the equivalence does not extend across the larger system.

Mauro was interested in the relationship of this to the punctured formal scheme $\mathbb{F}_p((q))$.

The notion of the punctured disk is also wonky in the category of formal schemes. In particular, one has that $\operatorname{Spf}(\mathbb{F}_2[[t]])$ has only one point in the underlying topological space. This is part of why formal schemes aren't just schemes where the structure sheaf is of topological rings instead of rings: the points correspond to open ideals. The "correct" category to take generic fibers of formal schemes is adic spaces, but that is not a discussion that is worth going into. EK

The reader invitation is as far as I can tell is either incoherent or impossible due to the condition that the maps making up formal schemes need to be infinitesimal thickenings. I think that correcting it goes as follows: requiring the inverse system to be of infintesimal thickenings means that the only m such that $\mathbb{G}_m[m]$ is an infinitesimal thickening of the identity are the m of the form p^n which makes the result "obvious" in some sense. EK

Let's now consider the example that we closed with last time, where we calculated $H\mathbb{F}_2^*(\mathbb{R}P^n) = \mathbb{F}_2[x]/(x^{n+1})$. Putting "Spec" in front of this, we could reinterpret this calculation as

$$\operatorname{Spec} H\mathbb{F}_2^*(\mathbb{R}P^n) \cong \mathbb{A}^{1,(n)}.$$

This is such a useful thing to do that we will give it a notation all of its own:

Definition 1.2.13. Let X be a finite cell complex, so that $H\mathbb{F}_2^*(X)$ is a ring which is finite–dimensional as an $\mathbb{F}_2$–vector space. We will write

$$X_{H\mathbb{F}_2} = \operatorname{Spec} H\mathbb{F}_2^* X$$

for the corresponding finite affine scheme.

Example 1.2.14. Putting together the discussions from this time and last time, in the new notation we have calculated

$$\mathbb{R}P^n_{H\mathbb{F}_2} \cong \mathbb{A}^{1,(n)}.$$

So far, this example just restates things we knew in a mildly different language. Our driving goal for the remainder of today and for tomorrow is to incorporate as much information as we have about these cohomology rings $H\mathbb{F}_2^*(\mathbb{R}P^n)$ into this description, which will result in us giving a more "precise" name for this object. Along the way, we will discover why X had to be a *finite* complex and how to think about more general X. For now, though, let's content ourselves with investigating the Hopf algebra structure on $H\mathbb{F}_2^*\mathbb{R}P^\infty$.

Example 1.2.15. Recall that $\mathbb{R}P^\infty$ is an H–space in two equivalent ways:

1. There is an identification $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$, and the H–space structure is induced by the sum on cohomology.

2. There is an identification $\mathbb{R}P^\infty \simeq BO(1)$, and the H–space structure is induced by the tensor product of real line bundles.

Later on, you need that there's a map on skeleta $\mathbb{R}P^n \times \mathbb{R}P^m \to \mathbb{R}P^{m+n}$. This is made apparent if you inserted another characterization of the H-space structure as the one treating $\mathbb{R}P^\infty$ as the monic polynomials (of all degrees) over $\mathbb{R}$, and then the map is given by multiplication. AY

In either case, this induces a Hopf algebra diagonal

$$H\mathbb{F}_2^*\mathbb{R}P^\infty \otimes H\mathbb{F}_2^*\mathbb{R}P^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{R}P^\infty$$

which we would like to analyze. This map is determined by where it sends the class x, and because it must respect gradings it must be of the form $\Delta x = ax_1 + bx_2$ for some constants $a, b \in \mathbb{F}_2$. Furthermore, because it belongs to a Hopf algebra structure, it must satisfy the unitality axiom

$$\begin{tikzcd}
H\mathbb{F}_2^* \mathbb{RP}^\infty & H\mathbb{F}_2^* \mathbb{RP}^\infty \otimes H\mathbb{F}_2^* \mathbb{RP}^\infty \arrow[l, "\begin{pmatrix} \varepsilon \otimes \mathrm{id} \\ \mathrm{id} \otimes \varepsilon \end{pmatrix}"'] & H\mathbb{F}_2^* \mathbb{RP}^\infty . \arrow[l, "\Delta"'] \arrow[ll, bend left, "\mathrm{id}"]
\end{tikzcd}$$

and hence it takes the form

$$\Delta(x) = x_1 + x_2.$$

Noticing that this is exactly the diagonal map in Example 1.2.5, we tentatively identify "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" with the additive group. This is extremely suggestive but does not take into account the fact that $\mathbb{RP}^\infty$ is an infinite complex, so we haven't allowed ourselves to write "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" just yet. In light of the above discussion, we have left a very particular point open: it's not clear if we should use the name "$\mathbb{G}_a$" or "$\widehat{\mathbb{G}}_a$". We will straighten this out tomorrow.

1.3 The Steenrod algebra

We left off yesterday with an ominous finiteness condition in our definition of $X_{H\mathbb{F}_2}$, and we produced a pair of reasonable guesses as to what "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" could mean. It will turn out that we can answer which of the two guesses is reasonable by rigidifying the target category somewhat. Here are the extra structures we will work toward incorporating:

1. Cohomology rings are *graded*, and maps of spaces respect this grading.
2. Cohomology rings receive an action of the Steenrod algebra, and maps of spaces respect this action.
3. Both of these are complicated further when taking the cohomology of an infinite complex.
4. (Cohomology rings for more elaborate cohomology theories are only skew-commutative, but "Spec" requires a commutative input.)

Today we will fix all these deficiencies of $X_{H\mathbb{F}_2}$ except for #4, which doesn't matter with mod–2 coefficients but which will be something of a bugbear throughout the rest of the semester.

Let's begin by considering the grading on $H\mathbb{F}_2^* X$. In algebraic geometry, the following standard construction is used to track gradings:

Definition 1.3.1 ([3, Definition 2.95]). A $\mathbb{Z}$–*grading* on a ring R is a system of additive subgroups R_k of R satisfying $R = \bigoplus_k R_k$, $1 \in R_0$, and $R_j R_k \subseteq R_{j+k}$. Additionally, a map $f \colon R \to S$ of graded rings is said to *respect the grading* if $f(R_k) \subseteq S_k$. Maybe "$\mathbb{Z}$–filtering" is more appropriate.

Lemma 1.3.2 ([3, Proposition 2.96])**.** *A graded ring R is equivalent data to an affine scheme* $\operatorname{Spec} R$ *with an action by* $\mathbb{G}_m$*. Additionally, a map $R \to S$ is homogeneous exactly when the induced map* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is* $\mathbb{G}_m$*–equivariant.*

Proof. A $\mathbb{G}_m$–action on $\operatorname{Spec} R$ is equivalent data to a coaction map

$$\alpha^* : R \to R \otimes \mathbb{F}_2[x^{\pm}].$$

Define R_k to be those points in r satisfying $\alpha^*(r) = r \otimes x^k$. It is clear that we have $1 \in R_0$ and that $R_j R_k \subseteq R_{j+k}$. To see that $R = \bigoplus_k R_k$, note that every tensor can be written as a sum of pure tensors. Conversely, given a graded ring R, define the coaction map on R_k by

$$(r_k \in R_k) \mapsto x^k r_k$$

and extend linearly. $\square$

This notion from algebraic geometry is somewhat different from what we are used to in algebraic topology, as it is designed to deal with things like polynomial rings (where the difference of two polynomials can lie in lower degree), but in classical algebraic topology we only ever encounter sums of terms with homogeneous degree. We can modify our perspective very slightly to arrive at the algebraic geometers': replace $H\mathbb{F}_2$ by the periodified spectrum

$$H\mathbb{F}_2 P = \bigvee_{j=-\infty}^{\infty} \Sigma^j H\mathbb{F}_2.$$

This spectrum has the property that $H\mathbb{F}_2 P^0(X)$ is isomorphic to $H\mathbb{F}_2^*(X)$ as ungraded rings, but now we can make sense of the sum of two classes which used to live in different $H\mathbb{F}_2$–degrees. At this point we can manually craft the desired coaction map α^* so that we are in the situation of Lemma 1.3.2, but we will shortly find that algebraic topology gifts us with it on its own.

You can do a better job of describing where the Steenrod coaction comes from, rather than resting on duality. For instance, you could at least justify *why the Steenrod algebra is a Hopf algebra*. Already, that's kind of unclear.

Our route to finding this internally occurring α^* is by turning to the next supplementary structure: the action of the Steenrod algebra. Naively approached, this does not fit into the framework we've been sketching so far: the Steenrod algebra is a *noncommutative* algebra, and so the action map

$$\mathcal{A}^* \otimes H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X$$

will be difficult to squeeze into any kind of algebro-geometric framework. Milnor was the first person to see a way around this, with two crucial observations. First, the linear-algebraic dual of the Steenrod algebra $\mathcal{A}_*$ is a commutative ring, since the Cartan formula expressing the diagonal on $\mathcal{A}^*$ is evidently symmetric:

$$\operatorname{Sq}^n(xy) = \sum_{i+j=n} \operatorname{Sq}^i(x) \operatorname{Sq}^j(y).$$

Reminder for jhfung: $\mathcal{A}^*$ is a Hopf algebra with comultiplication $\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$, so $\mathcal{A}_*$ is also a Hopf algebra.

I see that if X is a finite complex, it has a Spanier–Whitehead dual, but I don't see how to use this. Is λ^* not just the composition $H\mathbb{F}_2^* X = \mathbb{F}_2 \otimes H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X \otimes \mathcal{A}^* \otimes \mathcal{A}_* \xrightarrow{\lambda \otimes 1} H\mathbb{F}_2^* X \otimes \mathcal{A}_*$?

I think the point is you're using a duality-type thing for $\mathcal{A}_*$ and $\mathcal{A}^*$. Unfortunately, infinite dimensional vector spaces are not dualizable, so strictly what you've written doesn't quite work (for example, $\mathcal{A}^* \otimes \mathcal{A}_*$ doesn't receive a map from $\mathbb{F}_2$). However, on finite complexes, you only get a finite dimensional part of the Steenrod algebra acting non-trivially, so you can do the duality thing. AY

Second, if X is a *finite* complex, then tinkering with Spanier–Whitehead duality gives rise to a coaction map

$$\lambda^* : H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X \otimes \mathcal{A}_*,$$

which we will then re-interpret as an action map

$$\alpha : \operatorname{Spec} \mathcal{A}_* \times X_{H\mathbb{F}_2} \to X_{H\mathbb{F}_2}.$$

Milnor works out the Hopf algebra structure of $\mathcal{A}_*$, by defining elements $\xi_j \in \mathcal{A}_*$ dual to $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} \in \mathcal{A}^*$. Taking $X = \mathbb{R}\mathrm{P}^n$ and $x \in H\mathbb{F}_2^1(\mathbb{R}\mathrm{P}^n)$ the generator, then since $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} x = x^{2^j}$ he deduces the formula

$$\lambda^*(x) = \sum_{j=0}^{\lfloor \log_2 n \rfloor} x^{2^j} \otimes \xi_j.$$

Notice that we can take the limit $n \to \infty$ to get a well-defined infinite sum, provided we permit ourselves to make sense of such a thing. He then makes the following calculation, stable in n:

$$\begin{aligned}
(\lambda^* \otimes \mathrm{id}) \circ \lambda^*(x) &= (\mathrm{id} \otimes \Delta) \circ \lambda^*(x) && \text{(coassociativity)} \\
(\lambda^* \otimes \mathrm{id}) \left(\sum_{j=0}^{\infty} x^{2^j} \otimes \xi_j \right) &= \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^i} \otimes \xi_i \right)^{2^j} \otimes \xi_j &= && \text{(ring homomorphism)} \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= && \text{(characteristic 2).}
\end{aligned}$$

Then, turning to the right-hand side:

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= (\mathrm{id} \otimes \Delta) \left(\sum_{m=0}^{\infty} x^{2^m} \otimes \xi_m \right) \\
\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j &= \sum_{m=0}^{\infty} x^{2^m} \otimes \Delta(\xi_m),
\end{aligned}$$

from which it follows that

$$\Delta \xi_m = \sum_{i+j=m} \xi_i^{2^j} \otimes \xi_j.$$

Finally, Milnor shows that this is the complete story:

Theorem 1.3.3 (Milnor). $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j, \ldots]$.

Cite me: Give a reference in Milnor's paper.

Mosher and Tangora also has a proof of this on pp. 50–52.

Flippant proof. There is at least a map $\mathbb{F}_2[\xi_1, \xi_2, \ldots] \to \mathcal{A}_*$ given by the definition of the elements ξ_j above. This map is injective, since these elements are distinguished by how they coact on $H\mathbb{F}_2^* \mathbb{RP}^\infty$. Then, since these rings are of graded finite type, Milnor can conclude his argument by counting how many elements he has produced, comparing against how many Adem and Cartan found (which we will do ourselves in **??**), and noting that he has exactly enough. $\square$

We are now in a position to uncover the desired map α^* desired earlier. Suppose that we were interested in re-telling Milnor's story with $H\mathbb{F}_2P$ in place of $H\mathbb{F}_2$. The dual Steenrod algebra is defined topologically by

$$\mathcal{A}_* := \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2),$$

which we replace by

$$\mathcal{A}P_0 := \pi_0(H\mathbb{F}_2P \wedge H\mathbb{F}_2P) = H\mathbb{F}_2P_0(H\mathbb{F}_2P) = \mathcal{A}_*[\xi_0^\pm].$$

I think it's a good idea to at least mention that ξ_0 keeps track of the grading, as you explained in class. Also, I see from the wedge axiom that $H\mathbb{F}_2P_0(H\mathbb{F}_2P) \cong \prod_j \mathcal{A}_*$, but how do you get the multiplicative structure of ξ_0?

Lemma 1.3.4 ([1, Formula 3.4, Remark 3.14]). *Projecting to the quotient Hopf algebra $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^\pm]$ gives exactly the coaction map α^*.* $\square$

Include a proof of this. It doesn't seem obvious — Danny and I spent a while talking about it and couldn't get all our algebra straight. It has something to do with trading the invertible element across the smash product in $H\mathbb{F}_2P \wedge_{H\mathbb{F}_2P} H\mathbb{F}_2P \wedge X$, and also with multiplicativity of the action: $\psi(vx) = \psi(v)\psi(x)$ with v the homogenizing generator, or something.

To study the rest of $\mathcal{A}P_0$ in terms of algebraic geometry, we need only identify what the series $\lambda^*(x)$ embodies. Note that this necessarily involves some creativity, and the only justification we can supply will be moral, borne out over time, as our narrative encompasses more and more phenomena. With that caveat in mind, here is one such description. Recall the map induced by the H–space multiplication

$$H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \leftarrow H\mathbb{F}_2^*\mathbb{RP}^\infty.$$

Taking a colimit over finite complexes, we produce an coaction of $\mathcal{A}_*$, and since the map above comes from a map of spaces, it is equivariant for the coaction. Since the action on the left is diagonal, we deduce the formula

$$\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2).$$

The point of this lemma is to say that earlier we traded saying "graded map" for "$\mathbb{G}_m$–equivariant map", which did not seem like a substantial gain. Now we see that saying "Steenrod–equivariant map" already includes saying "graded map", which is a gain in brevity. Try to make this clearer.

Lemma 1.3.5. *The series $\lambda^*(x) = \sum_{j=0}^\infty x^{2^j} \otimes \xi_j$ is the universal example of a series satisfying $\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2)$. The set $(\operatorname{Spec} \mathcal{A}P_0)(T)$ is identified with the set of power series f with coefficients in the $\mathbb{F}_2$–algebra T satisfying*

$$f(x_1 + x_2) = f(x_1) + f(x_2). \quad \square$$

Can you explain this more? In which sense is $\lambda^*(x)$ universal? How do I get from the first statement to the second?

We close our discussion by codifying what Milnor did when he stabilized against n. Each $\mathbb{RP}^n_{H\mathbb{F}_2}$ is a finite affine scheme, and to make sense of the object $\mathbb{RP}^\infty_{H\mathbb{F}_2}$ Milnor's technique was to consider the ind-system $\{\mathbb{RP}^n_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite affine schemes. We will record this as our technique to handle general infinite complexes:

Definition 1.3.6. When X is an infinite complex, filter it by its subskeleta $X^{(n)}$ and define $X_{H\mathbb{F}_2}$ to be the ind-system $\{X^{(n)}_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite schemes.[2]

This choice collapses our uncertainty about the topological example from last time:

Example 1.3.7 (cf. Examples 1.2.11 and 1.2.15). Write $\widehat{\mathbb{G}}_a$ for the ind-system $\mathbb{A}^{1,(n)}$ with the group scheme structure given in Example 1.2.15. That this group scheme structure filters in this way is a simultaneous reflection of two facts:

1. Algebraic: The set $\widehat{\mathbb{G}}_a(T)$ consists of all nilpotent elements in T. The sum of two nilpotent elements of orders n and m is guaranteed to itself be nilpotent with order at most $n + m$.

2. Topological: There is a factorization of the multiplication map on $\mathbb{RP}^\infty$ as $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{n+m}$ purely for dimensional reasons.

Is there an off-by-one here?

As group schemes, we have thus calculated

$$\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a.$$

Example 1.3.8. Additionally, this convention comports with our analysis of $\operatorname{Spec} \mathcal{A}P_0$. Note that the following morphism sets are very different:

We could be clearer about the *two* things that are happening here. First, we are comparing informal schemes to formal schemes (or polynomials to power series). Second, we are comparing external Hom–objects to internal Hom–objects (or sets of power series over $\mathbb{F}_2$ to schemes of power series over $\mathbb{F}_2$–algebras).

$$\begin{aligned}
\mathsf{GroupSchemes}_{/\mathbb{F}_2}(\mathbb{G}_a, \mathbb{G}_a) &\cong \mathsf{HopfAlgebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], \mathbb{F}_2[x]) \\
\mathsf{FormalGroups}_{/\mathbb{F}_2}(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a) &\cong \mathsf{HopfProAlgebras}_{\mathbb{F}_2/}(\mathbb{F}_2[\![x]\!], \mathbb{F}_2[\![x]\!]).
\end{aligned}$$

The former is populated by polynomials satisfying the homomorphism condition and the latter is populated by *power series* satisfying the same, which form a much larger set. Since our description of $\operatorname{Spec} \mathcal{A}P_0$ involves power series, we will favor the latter interpretation. To record this, first amp up this description of maps to a scheme of its own:

$$\underline{\mathsf{FormalSchemes}}(X, Y)(T) = \left\{ (u, f) \,\middle|\, \begin{array}{c} u : \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2, \\ f : u^* X \to u^* Y \end{array} \right\}$$

and conclude that the correct name for $\operatorname{Spec} \mathcal{A}P_0$ is

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

[2]More canonically, when X is "compactly generated", it can be written as the colimit of its compact subspaces $X^{(\alpha)}$, and we define $X_{H\mathbb{F}_2}$ using the ind-system $\{X^{(\alpha)}_{H\mathbb{F}_2}\}_\alpha$.

Finally, the formula $\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a$ is meant to point out that this language of formal schemes has an extremely good compression ratio — you can fit a lot of information into a very tiny space. This formula simultaneously encodes the cohomology ring of $\mathbb{RP}^\infty$ as the formal scheme, its diagonal as the group scheme structure, and the coaction of the dual Steenrod algebra by the identification with $\underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a$.

Include a recursive formula for the antipode map, coming from power series inversion.

1.4 Hopf algebra cohomology

This section is written gradedly and probably shouldn't be, for consistency. (In fact, this is the cause of some of the confusion about the $\mathbb{G}_m$–action used tomorrow to separate out the homotopy degrees...)

Today we'll focus on an important classical tool: the Adams spectral sequence. We're going to study this in greater earnest later on, so I will avoid giving a satisfying construction today. But, even without a construction, it's instructive to see how such a thing comes about. Begin by considering the following three self-maps of the stable sphere:

$$\mathbb{S}^0 \xrightarrow{0} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{1} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{2} \mathbb{S}^0.$$

It's not clear to me that this introduction should be written in terms of cohomology rather than homology. It's true that yesterday we were talking about cohomology, but it's also true that the spectral sequence we're going to build takes in homology. (More generally, contexts take in homology. I still find this a little puzzling, that Strickland's formal schemes don't seem to live in the descent picture.)

Cite me: I first saw this presentation from Matt Ando. He must have learned it from someone. I'd like to know who to attribute this to.

If we apply mod–2 cohomology to each line, the induced maps are

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2, \qquad \mathbb{F}_2 \xleftarrow{\mathrm{id}} \mathbb{F}_2, \qquad \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2.$$

We see that mod–2 homology can immediately distinguish between the null map and the identity map just by its behavior on morphisms, but it can't so distinguish between the null map and the multiplication-by-2 map. To try to distinguish these two, we use the only other tool available to us: cohomology theories send cofiber sequences to long exact sequences, and moreover the data of a map f and the data of the inclusion map $\mathbb{S}^0 \to C(f)$ into its cone are equivalent in the stable category. So, we trade our maps 0 and 2 for the following cofiber sequences:

$$\mathbb{S}^0 \longrightarrow C(0) \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^0 \longrightarrow C(2) \longrightarrow \mathbb{S}^1.$$

Applying cohomology, these again appear to be the same:

$$\begin{array}{cccccccccccc}
[1] & & & \bullet & \longleftarrow & \bullet & & & & \bullet & \longleftarrow & \bullet \\
\\
[0] & \bullet & \longleftarrow & \bullet & & & & \bullet & \longleftarrow & \bullet & & \\
\\
& H\mathbb{F}_2^*\mathbb{S}^0 & \leftarrow & H\mathbb{F}_2^*C(0) & \leftarrow & H\mathbb{F}_2^*\mathbb{S}^1, & & H\mathbb{F}_2^*\mathbb{S}^0 & \leftarrow & H\mathbb{F}_2^*C(2) & \leftarrow & H\mathbb{F}_2^*\mathbb{S}^1,
\end{array}$$

It would be nice if the dots aligned directly beneath the spaces in the cofiber sequences above.

where we have drawn a "$\bullet$" for a generator of an $\mathbb{F}_2$–vector space, graded vertically, and arrows indicating the behavior of each map. However, if we enrich our picture with the data we discussed last time, we can finally see the difference. Recall the topological equivalences

$$C(0) \simeq \mathbb{S}^0 \vee \mathbb{S}^1, \quad C(2) \simeq \Sigma^{-1}\mathbb{R}\mathrm{P}^2.$$

In the two cases, the coaction map λ^* is given by

$$\begin{aligned} \lambda^* : H\mathbb{F}_2^* C(0) &\to H\mathbb{F}_2^* C(0) \otimes \mathcal{A}_* \\ \lambda^* : e_0 &\mapsto e_0 \otimes 1 \\ \lambda^* : e_1 &\mapsto e_1 \otimes 1, \end{aligned} \qquad \begin{aligned} \lambda^* : H\mathbb{F}_2^* C(2) &\to H\mathbb{F}_2^* C(2) \otimes \mathcal{A}_* \\ \lambda^* : e_0 &\mapsto e_0 \otimes 1 + e_1 \otimes \xi_1 \\ \lambda^* : e_1 &\mapsto e_1 \otimes 1. \end{aligned}$$

We draw this into the diagram as

Is it possible to work out the coaction on the cones without first identifying their homotopy types?

$$\begin{tikzcd}
{[1]} & & \bullet & \bullet \arrow[l] & & & \bullet \arrow[d, dash, "\xi_1"] & \bullet \arrow[l] \\
{[0]} & \bullet & \bullet \arrow[l] & & & \bullet & \bullet \arrow[l] &
\end{tikzcd}$$

$$H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^* C(0) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1, \qquad H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^* C(2) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1,$$

where the vertical line indicates the nontrivial coaction involving ξ_1. We can now see what trading maps for cofiber sequences has bought us: mod–2 cohomology can distinguish the defining sequences for $C(0)$ and $C(2)$ by considering their induced extensions of comodules over $\mathcal{A}_*$.The Adams spectral sequence bundles this thought process into a single machine:

Can this be phrased so as to indicate how this works for longer extensions? I've never tried to think about even what happens for $C(4)$.

Theorem 1.4.1. *There is a convergent spectral sequence of signature*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow (\pi_*\mathbb{S}^0)^\wedge_2. \quad \square$$

Cite me: Cite this somehow, or at least put a forward reference in.

In effect, this asserts that the above process is *exhaustive*: every element of $(\pi_*\mathbb{S}^0)^\wedge_2$ can be detected and distinguished by some representative class of extensions of comodules for the dual Steenrod algebra. Mildly more generally, if X is a bounded-below spectrum, then there is even a spectral sequence of signature

Mention that there are homological and cohomological $\mathbb{F}_2$–Adams spectral sequences.

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X) \Rightarrow \pi_* X^\wedge_2.$$

Here is where we could divert to talking about the construction of the Adams spectral sequence, but it will fit more nicely into a story later on. Thus, for now we will leave this task for **??**. Before moving on, we will record the following utility lemma about the Adams spectral sequence. It is believable based on the above discussion, and we will need to use before we get around to examining the guts of the spectral sequence.

Lemma 1.4.2. *The* 0*–line of the Adams spectral sequence contains those elements visible to the Hurewicz homomorphism.* □

This feels sloppily stated.

Today we will focus on the algebraic input $\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$, which will require us to grapple with the homological algebra of comodules for a Hopf algebra. To begin, it's both reassuring and instructive to see that homological algebra can, in fact, be done with comodules. In the usual development of homological algebra for modules, the key observations are the existence of projective and injective modules, and there is something similar here.

Remark 1.4.3. Much of the results below do not rely on working with a Hopf algebra over the field $k = \mathbb{F}_2$. In fact, k can usually be taken to be a ring rather than a field. More generally, the theory goes through in the context of comodules over Hopf algebroids.

Cite me: Cite these. A1.1-2 in Ravenel are relevant.

Lemma 1.4.4. *Let A be a Hopf k–algebra, let M be an A–comodule, and let N be a k–module. There is a cofree adjunction:*

$$\mathsf{Comodules}_A(M, N \otimes_k A) \cong \mathsf{Modules}_k(M, N),$$

where $N \otimes_k A$ is given the structure of an A–comodule by the coaction map

$$N \otimes_k A \xrightarrow{\mathrm{id}\otimes\Delta} N \otimes_k (A \otimes_k A) = (N \otimes_k A) \otimes_k A.$$

Proof. Given a map $f\colon M \to N$ of k–modules, we can build the composite

$$M \xrightarrow{\psi_M} M \otimes_k A \xrightarrow{f\otimes\mathrm{id}_A} N \otimes_k A.$$

Alternatively, given a map $g\colon M \to N \otimes_k A$ of A–comodules, we build the composite

$$M \xrightarrow{g} N \otimes_k A \xrightarrow{\mathrm{id}_N\otimes\varepsilon} N \otimes_k k = N.$$ □

Corollary 1.4.5. *The category $\mathsf{Comodules}_A$ has enough injectives. Namely, if M is an A–comodule and $M \to I$ is an inclusion of k–modules into an injective k–module I, then $M \to I \otimes_k A$ is an injective A–comodule under M.* □

Remark 1.4.6. In our case, M itself is always k–injective, so there's already an injective map $\psi_M : M \to M \otimes A$: the coaction map. The assertion that this map is coassociative is identical to saying that it is a map of comodules.

Satisfied that "Ext" at least makes sense, we're free to chase more conceptual pursuits. Recall from algebraic geometry that a module M over a ring R gives rise to quasi-coherent sheaf $\widetilde{M}$ over $\operatorname{Spec} R$. We give a definition that fits with our functorial perspective:

Can this definition be motivated? Is there an easily graspable sense in which it looks like a bundle?

Definition 1.4.7. A presheaf (of modules) over a scheme X is an assignment of maps $\mathcal{F}\colon X(T) \to \mathsf{Modules}_T$, functorially in T. Such a presheaf is said to be *quasicoherent* when a map $\operatorname{Spec} S \to \operatorname{Spec} T \to X$ induces a natural isomorphism $\mathcal{F}(T) \otimes_T S \cong \mathcal{F}(S)$.

Straighten this out, using fibered categories?

Mention how the usual definition of a sheaf of $\mathcal{O}_X$–modules gives rise to such a thing: global sections over $\operatorname{Spec} T$.

There is a todo here that I can't get to compile. See the source.

Cite me: Surely this is in Neil's FSFG somewhere.

Lemma 1.4.8. *An R–module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ by the rule $(\operatorname{Spec} T \to \operatorname{Spec} R) \mapsto M \otimes_R T$. Conversely, every quasicoherent sheaf over an affine scheme arises in this way.* □

Jay was frustrated with which adjoint I put on top (and perhaps which went on which side). Apparently there's some convention, which I should look up and obey.

Definition 1.4.9. A map $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ induces maps $f^* \dashv f_*$ of module sheaf categories, which on the level of quasi-coherent sheaves is given by

$$\begin{tikzcd} \mathsf{QCoh}_{\operatorname{Spec} R} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} S} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\ \mathsf{Modules}_R \arrow[r, shift left, "M \mapsto M \otimes_R S"] & \mathsf{Modules}_S . \arrow[l, shift left, "N \mapsfrom N"] \end{tikzcd}$$

The usual formula for the sheaf cohomology of a sheaf $\mathcal{F}$ over an S–scheme X with structure map $\pi \colon X \to S$ is given by $\operatorname{Ext}(\mathcal{O}_S, \pi_* \mathcal{F})$ which is, indeed, vaguely reminiscent of the formula we were considering above as input to the Adams spectral sequence. Experience in algebraic geometry shows that it is conceptually profitable to consider the *six–functors yoga* more generally and their accompanying base-change formulas. A very basic example of such a formula is

$$\operatorname{Ext}_X(\pi^* \mathcal{O}_S, \mathcal{F}) \cong \operatorname{Ext}_S(\mathcal{O}_S, R\pi_* \mathcal{F}),$$

which describes the functor "Ext" on quasicoherent sheaves of modules over X in terms of the derived functor $R\pi_*$.

Rearrange this middle bit some. "Definition of sheaf, six functors for sheaves on affines, definition of cotensor, adjunction assertion for comodules, remark about cotensoring restricting the image of the coaction" seems too convoluted a narrative structure.

We are thus moved to study derived base-change for comodules, thought of as sheaves equipped with an action by a group scheme. In particular, we want to understand what it means to "tensor" two comodules together. Unsurprisingly, the solution is dual to that for modules: tensor the two comodules together, then restrict to the elements where the coactions on either factor agree.

Definition 1.4.10. Given A–comodules M and N, their cotensor product is the k-module defined by the equalizer

$$M \square_A N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k A \otimes_k N.$$

Typographical suggestion: can you use a slightly small box for the cotensor product? It's very similar to the "qed box"; see lemma 1.4.13.

Lemma 1.4.11. *Given a map $f \colon A \to B$ of Hopf k–algebras, the induced adjunction $f^* \dashv f_*$ is given at the level of comodules by*

$$\begin{tikzcd} \text{“}\mathsf{QCoh}_{(\operatorname{Spec} k, \operatorname{Spec} A)}\text{”} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \text{“}\mathsf{QCoh}_{(\operatorname{Spec} k, \operatorname{Spec} B)}\text{”} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\ \mathsf{Comodules}_A \arrow[r, shift left, "M \mapsto M"] & \mathsf{Comodules}_B . \arrow[l, shift left, "N \square_B A \mapsfrom N"] \end{tikzcd}$$
□

What is this notation Spec k double-slash Spec A? Jeremy mentioned it's a stack, but it hasn't been introduced yet. Without more information, I can't verify this lemma. Perhaps mention that it will be defined more precisely later?

Remark 1.4.12. The formula for f_*N is what one would guess from the formula for push-forward along maps of affine schemes. The comodule f_*N wants to have as its underlying module N, but the coaction map on N needs to be reduced to lie only in A. The equalizer diagram in the definition of the cotensor product enforces this.

As an example application, cotensoring gives rise to a concise description of what it means to be a comodule map:

Lemma 1.4.13 ([2, Lemma A1.1.6b]). *Let M and N be A–comodules with M projective as a k–module. Then there is an equivalence*

$$\mathsf{Comodules}_A(M,N) = \mathsf{Modules}_k(M,k)\Box_A N. \quad \square$$

so what exactly is the six-functors formula? I thought the formalism involved showing some adjunctions between 6 functors, but it's not totally clear to me what they are

From this, we can deduce the six–functors formula described above:

Corollary 1.4.14. *Let $N = N' \otimes_k A$ be a cofree comodule. Then $N\Box_A k = N'$.*

Proof. Picking $M = k$, we have

$$\begin{aligned}\mathsf{Modules}_k(k,N') &= \mathsf{Comodules}_A(k,N) \\ &= \mathsf{Modules}_k(k,k)\Box_A N \\ &= k\Box_A N.\end{aligned}$$

$\square$

Corollary 1.4.15. *There is an isomorphism*

$$\mathsf{Comodules}_A(k,N) = \mathsf{Modules}_k(k,k)\Box_A N = k\Box_A N$$

and hence

$$\mathrm{Ext}_A(k,N) \cong \mathrm{Cotor}_A(k,N).$$

Proof. Resolve N using the cofree modules described above, then apply either functor $\mathsf{Comodules}_A(k,-)$ or $k\Box_A -$. In both cases, you get the same complex. $\square$

Example 1.4.16. Let's contextualize this somewhat. Given a finite group G, we can form a commutative Hopf algebra k^G, the k–valued functions on G. This Hopf algebra is dual to the Hopf algebra $k[G]$, the group–algebra on G. It is classical that a G–module M is equivalent data to a $k[G]$–module structure, and if M is suitably finite, we can dualize the action map to produce a coaction map

$$M^* \to k^G \otimes M^*.$$

Additionally, we have $M^*\Box_{k^G} N^* = (M \otimes_G N)^*$, so that $M^*\Box_{k^G} k = (H^0(G,M))^*$.

Example 1.4.17. In the previous lecture, we identified $\mathcal{A}_*$ with the ring of functions on the group scheme $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$, which is defined by the kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Technically, we've only identified $\operatorname{Spec} \mathcal{A}P_0$, but at least it's plausible that the sub-group scheme in degree 0 (what does this mean??) gives us $\operatorname{Spec} \mathcal{A}_*$???

Today's punchline is that this is analogous to the example above: $\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$ is thought of as "the derived fixed points" of "$G = \mathrm{Aut}_1(\widehat{\mathbb{G}}_a)$" on the "$G$–module" $H\mathbb{F}_{2*}X$.

Say the word "strict" here.

Example 1.4.18. Consider the degenerate case $X = H\mathbb{F}_2$. Then $H\mathbb{F}_{2*}(H\mathbb{F}_2) = \mathcal{A}_*$ is a cofree comodule, and hence Cotor is concentrated on the 0–line:

$$\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}(H\mathbb{F}_2)) = \mathbb{F}_2.$$

How do the duals from the above argument play into this? Do they just cancel out?

The Adams spectral sequence collapses to show the wholly unsurprising equality $\pi_* H\mathbb{F}_2 = \mathbb{F}_2$, and indeed this is the element in the image of the Hurewicz map $\pi_* H\mathbb{F}_2 \to H\mathbb{F}_{2*}H\mathbb{F}_2$.

Example 1.4.19.

The ordering of these examples probably isn't great, but the ASS pictures should appear on consecutive pages.

Example 1.4.20. At the other extreme, we can pick the extremely nondegenerate case $X = \mathbb{S}$, pictured through a range in Figure 1.1.

Mike Hill and Catherine's working of the Adams spectral sequence for kO.

Label elements? Identify some groups?

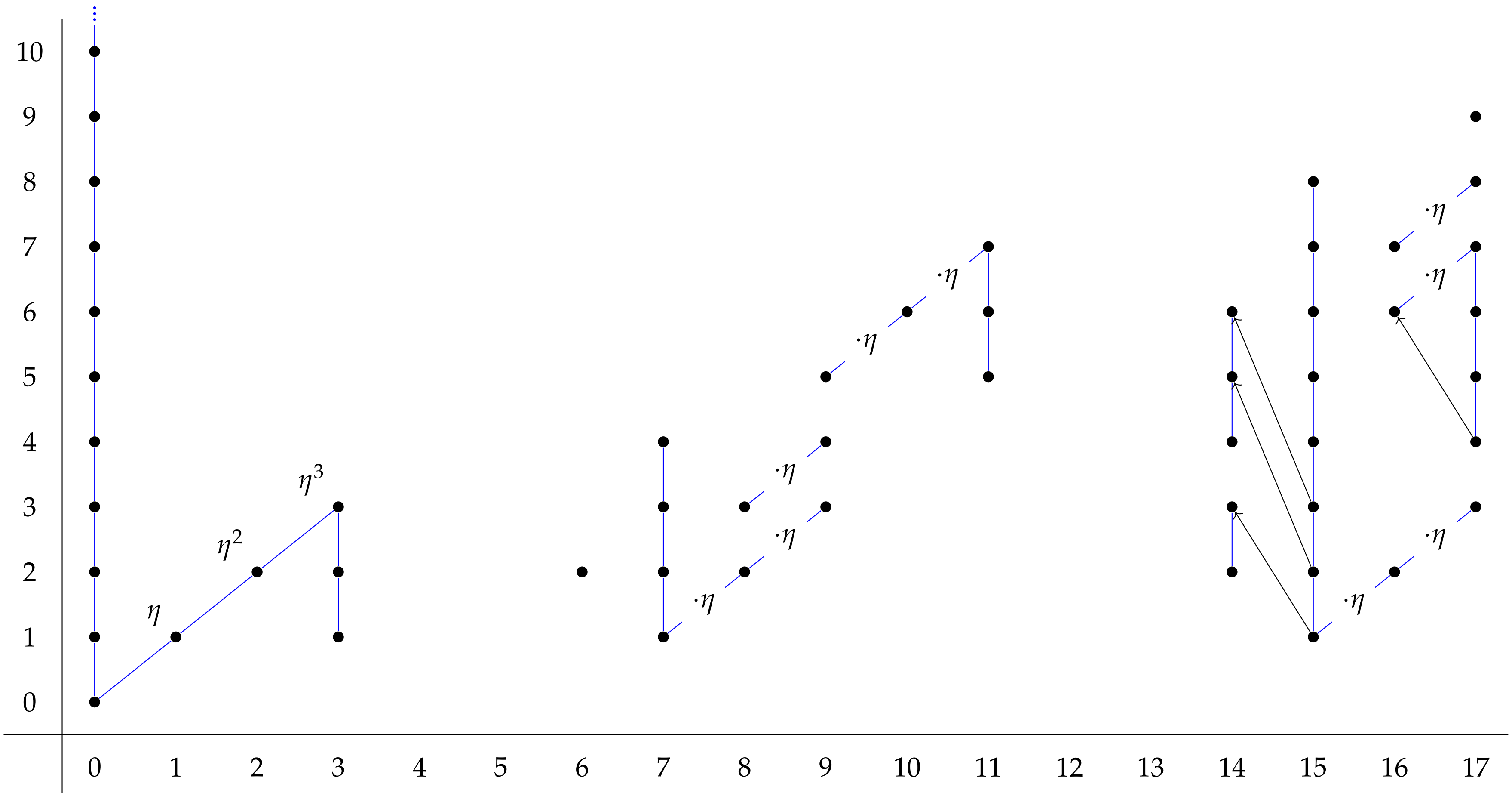

Figure 1.1: A small piece of the $H\mathbb{F}_2$-Adams spectral sequence for the sphere, beginning at the second page. North and north-east lines denote multiplication by 2 and by η, north-west lines denote d_2– and d_3–differentials.

1.5 The unoriented bordism ring

Jon asked: spectral sequences coming from π_* of a Tot tower increase Tot degree. ANSS differentials decrease degree: they run against the multiplicative structure in pictures. What's going on with this? I think this is a duality effect: working with the Steenrod algebra versus its dual.

Our goal today is to use the results of the previous lectures to make a calculation of $\pi_* MO$, the unoriented bordism ring. The Adams spectral sequence converging to this has signature

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 \widetilde{P_0(MO)}) \Rightarrow \pi_* MO,$$

and so we see that we need to understand $H\mathbb{F}_2 P_0(MO)$, together with its comodule structure over the dual Steenrod algebra.

$\mathcal{A}(1)_*$ is the Hopf algebra for a dihedral group. Is this example appropriate somewhere in this section?

Our first step toward this is the following calculation:

Last time you used $\widetilde{H\mathbb{F}_{2*}}$ instead of $H\mathbb{F}_2 P_0$.

Lemma 1.5.1. *$H\mathbb{F}_2 P^0 BO(n) \cong \mathbb{F}_2[\![w_1, \ldots, w_n]\!]$.*

Why do you state this Lemma? What's below isn't exactly a Corollary, I don't feel, without saying a lot more about how the diagonal on cohomology behaves (and then knowing something about Cartier duality or coalgebraic formal schemes).

Proof. The orthogonal groups sit in coset fibration sequences

$$O(n-1) \to O(n) \to S^{n-1},$$

and delooping the groups gives a rotated spherical fibration

$$S^{n-1} \to BO(n-1) \to BO(n).$$

The associated Serre spectral sequence shows that $H\mathbb{F}_2 P^0(BO(n))$ must have one extra free generator in it to receive a differential from the exterior generator of $H\mathbb{F}_2 P^0(S^{n-1})$. Noting $BO(1) \simeq \mathbb{R}P^\infty$, our discussion from previous lectures takes care of the base case. $\square$

Corollary 1.5.2. *There is a triangle*

$$\begin{tikzcd} & \mathrm{Sym}\, H\mathbb{F}_2 P_0(BO(1)) \arrow[d, "equiv"] \\ H\mathbb{F}_2 P_0(BO(1)) \arrow[ur] \arrow[r] & H\mathbb{F}_2 P_0(BO). \end{tikzcd} \quad \square$$

We will defer the proof of this until **??**, since it requires knowing that

$$H\mathbb{F}_2 P^0(BO(k)) \to H\mathbb{F}_2 P^0(BO(1)^{\times k})$$

is injective, which we will revisit later anyhow.

$H\mathbb{F}_2^*(BO(n)) \to H\mathbb{F}_2^*(BO(1)^{\times n})$ lands in the W-invariants, where W is the Weyl group of $BO(1)^{\times n}$ in $BO(n)$ (this is true because the normalizer $N = N_{BO(n)}(BO(1)^{\times n})$ acts on both by conjugation, and it acts trivially on the left term while on the right term it quotients to an action of the Weyl group). You can easily check that N is generated by the permutation matrices and $BO(1)^{\times n}$, so the Weyl group is Σ_n, and it acts by permuting the factors of $BO(1)$. The invariants in cohomology under

With this in hand, however, we can uncover the ring structure on $H\mathbb{F}_2 P_0(MO)$:

Corollary 1.5.3. *There is also a triangle*

$$\begin{tikzcd} & \operatorname{Sym} H\mathbb{F}_2 P_0(MO(1)) \arrow[d, "equiv"] \\ H\mathbb{F}_2 P_0(MO(1)) \arrow[ur] \arrow[r] & H\mathbb{F}_2 P_0(MO). \end{tikzcd}$$

In particular, $H\mathbb{F}_2 P_0(MO) \cong \mathbb{F}_2[b_1, b_2, \ldots]$.

Proof. The block sum maps

$$BO(n) \times BO(m) \to BO(n+m)$$

Thomify to give compatible maps

$$MO(n) \wedge MO(m) \to MO(n+m).$$

Taking the limit in n and m, this gives a ring structure on MO compatible with that on BO. The Corollary then follows from the functoriality of Thom isomorphisms. □

We now seek to understand the scheme $\operatorname{Spec} H\mathbb{F}_2 P_0(MO)$, and in particular its action of $\underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a)$. Our launching-off point for this is a topological version of the "freeness" result in the previous Corollary:

Lemma 1.5.4. *The following square commutes:*

$$\begin{tikzcd} \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2 P_0(MO), \mathbb{F}_2) & \mathsf{Spectra}(MO, H\mathbb{F}_2 P) \arrow[l, "equiv"'] \\ \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2 P_0(MO), \mathbb{F}_2) \arrow[u, hook] & \mathsf{RingSpectra}(MO, H\mathbb{F}_2 P). \arrow[l, "equiv"] \arrow[u, hook] \end{tikzcd}$$

Proof. The top isomorphism asserts only that $\mathbb{F}_2$–cohomology and $\mathbb{F}_2$–homology are linearly dual to one another. The second follows immediately from investigating the effect of the ring homomorphism diagrams in the bottom-right corner in terms of the subset they select in the top-left. □

Corollary 1.5.5. *There is a bijection between homotopy classes of ring maps $MO \to H\mathbb{F}_2 P$ and homotopy classes of factorizations*

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r] \arrow[dr] & MO(1) \arrow[d, dotted] \\ & H\mathbb{F}_2 P. \end{tikzcd}$$

Proof. Given a ring map $MO \to H\mathbb{F}_2P$, we can restrict it along the inclusion $MO(1) \to MO$ to produce a particular cohomology class

$$f \in H\mathbb{F}_2P^0(MO(1)) = \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO(1)), \mathbb{F}_2).$$

Why is this sentence "Interpreting..." needed? Also, the Thom isomorphism should appear somewhere, right?

Interpreting f as such a function, it is determined by its behavior on the basis of vectors in $\widetilde{H\mathbb{F}_2P_0}(MO(1))$ dual to the powers of the usual coordinate $x \in H\mathbb{F}_2^1(\mathbb{RP}^\infty)$. Finally, given *any* module map $\widetilde{H\mathbb{F}_2P_0}(MO(1)) \to \mathbb{F}_2$, we can employ Corollary 1.5.3 and to produce an algebra map $H\mathbb{F}_2P_0(MO) \to \mathbb{F}_2$. Lemma 1.5.4 then gives a ring spectrum map $MO \to H\mathbb{F}_2P$. $\square$

Corollary 1.5.6. *There is an* $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$*–equivariant isomorphism of schemes*

$$\operatorname{Spec} H\mathbb{F}_2P_0(MO) \cong \mathrm{Coord}_1(\mathbb{RP}^\infty_{H\mathbb{F}_2P}),$$

where the latter is the scheme of coordinate functions on $\mathbb{RP}^\infty_{H\mathbb{F}_2P} \to \widehat{\mathbb{A}}^1$ *which restrict to the canonical identification of tangent spaces* $\mathbb{RP}^1_{H\mathbb{F}_2P} = \widehat{\mathbb{A}}^{1,(1)}$.

Here's the unitality assumption again.

Proof. The method of the previous proof is to exhibit a isomorphism between these schemes. To learn that this isomorphism is equivariant for $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$, you need only know that the image of the map $MO(1) \to MO$ on mod–2 homology generates $H\mathbb{F}_2P_0(MO)$ as an algebra. $\square$

I don't really see what argument is being made about $\mathrm{Aut}\,\widehat{\mathbb{G}}_a$–equivariance. I don't think that the topological map is surjective...?

We are now ready to analyze the group cohomology of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ with coefficients in the comodule $H\mathbb{F}_2P_0(MO)$.

This is clumsily stated.

Theorem 1.5.7. *The action of* $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ *on* $\mathrm{Coord}_1(\widehat{\mathbb{G}}_a)$ *is* free.

I'd like more details for this argument too.

So it seems like what you end up using about this theorem for the purposes of the calculation at hand is not just that the action is free, but you have such a nice identification of the quotient. Maybe it would be clearer to include this as part of the statement?

Proof. Recall that $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ is defined by the (split) kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Consider a point $f \in \mathrm{Coord}_1(\widehat{\mathbb{G}}_a)(R)$, which in terms of the standard coordinate can be expressed as

$$f(x) = \sum_{j=1}^{\infty} b_{j-1}x^j,$$

where $b_0 = 1$. Decompose this series as $f(x) = f_2(x) + f_r(x)$, with

$$f_2(x) = \sum_{k=0}^{\infty} b_{2^k-1}x^{2^k}, \qquad f_r(x) = \sum_{j \neq 2^k} b_{j-1}x^j.$$

Why is this? I think I have a somewhat incorrect idea of what Coord is. In addition to being a set map that sends the set of nilpotent elements in R to itself, what other conditions does f have to satisfy?

Note that f_2 gives a point $f_2 \in \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$, so we can de-scale by it to give a new coordinate $g(x) = f_2^{-1}(f(x))$ with analogous series $g_2(x)$ and $g_r(x)$. Note that $g_2(x) = x$ and that f_2 is the unique point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$ that has this property. $\square$

Right, because raising to powers of two satisfies "Freshman's dream", so $f_2(x+y) = f_2(x) + f_2(y)$, so f_2 is a automorphism of group schemes. What condition does the subscript 1 imply in this case?

Corollary 1.5.8. *$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1, j \geq 1]$ with $|b_j| = j$.*

Compare this with the base change theorems from the previous day.

Amplitude just means row, right?

Proof. Set $M = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]$. It follows from the above that the $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$–cohomology of $H\mathbb{F}_2 P_0(MO)$ has amplitude 0:

$$\begin{aligned}\mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_2 P_0(MO)) &= \mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \square_{\mathcal{A}_*} (\mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \otimes_{\mathbb{F}_2} M = M.\end{aligned}$$

Maybe you should justify the step that looks like reassociation of the tensor and cotensor products. You almost definitely proved this on the previous day.

Here you used $H\mathbb{F}_2 P_0(MO) \cong \mathcal{A}_* \otimes M$, but what you seem to have proved in the previous theorem has to do with $\mathcal{A}P_0$ instead. Does it make a difference?

How?

Since the Adams spectral sequence

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)) \Rightarrow \pi_* MO$$

is concentrated on the 0–line, it collapses. Using the residual $\mathbb{G}_m$–action to infer the grading, we thus deduce

$$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]. \qquad \square$$

Is there a reason we should expect this, even in hindsight? like is MO an HF_2 module spectrum a priori?

This is pretty remarkable: some big statement about manifold geometry came down to understanding how we could reparametrize a certain formal group, itself a (fairly simple) purely algebraic problem. We could close here, but there's an easy homotopical consequence of this fact that is worth recording before we leave:

Lemma 1.5.9. *MO splits as a wedge of shifts of $H\mathbb{F}_2$.*

Does this not just follow from the fact that $\pi_* MO$ is a $\mathbb{F}_2$ vector space?

Proof. «««< HEAD Referring to Lemma 1.4.2, we have a π_*–injection $MO \to H\mathbb{F}_2 \wedge MO$. Pick an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha$ for $\pi_* MO$ and extend it to a $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha \cup \{w_\beta\}_\beta$ for $\pi_* H\mathbb{F}_2 \wedge MO$. Altogether, this larger basis can be represented as a single map

$$\bigvee_\alpha \Sigma^{m_\alpha}\mathbb{S} \vee \bigvee_\beta \Sigma^{n_\beta}\mathbb{S} \xrightarrow{\bigvee_\alpha v_\alpha \vee \bigvee_\beta w_\beta} H\mathbb{F}_2 \wedge MO.$$

Smashing through with $H\mathbb{F}_2$ gives an equivalence

$$\bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{n_\beta} H\mathbb{F}_2 \xrightarrow{\simeq} H\mathbb{F}_2 \wedge MO.$$

It's clear, but you haven't specified that m_α and n_β are the degrees of v_α and w_β.

The composite map

$$MO \to H\mathbb{F}_2 \wedge MO \xleftarrow{\simeq} \bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{n_\beta} H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2$$

is a weak equivalence. $\square$

Remark 1.5.10. Just using that $\pi_* MO$ is connective and $\pi_0 MO = \mathbb{F}_2$, we can produce a ring spectrum map $MO \to H\mathbb{F}_2$. What we've learned is that this map has a splitting: MO is also an $H\mathbb{F}_2$–algebra.

Should you eventually mention the stable cooperations MO^{MO}? Rather than coming with a specified logarithm, it's an isomorphism between any pair of additive formal groups — or, I suppose, a pair of logarithms.

Appendix A

Loose ends

I'd like to spend a couple of days talking about ways the picture in this class can be extended, finally, some actually unanswered questions that naturally arise. The following two section titles are totally made up and probably won't last.

A.1 E_∞ geometry

?? is an inspiration for considering *tmf* as well.

The modularity of the *MString* orientation

E_∞ orientations by *MString*

tmf, *TMF*, and *Tmf* in terms of $\mathcal{M}_{\mathbf{ell}}$

Thom spectra and ∞–categories

The Bousfield–Kuhn functor and the Rezk logarithm

A.2 Rational phenomena: character theory for Lubin–Tate spectra

There's a sufficient amount of reliance on character theory in Matt's thesis that we should talk about it. You should write that action and then backtrack here to see what you need for it.

See Morava's *Local fields* paper

Remark A.2.1. Theorem 2.6 of Greenlees–Strickland for a nice transchromatic perspective. See also work of Stapleton and Schlank–Stapleton, of course. Flesh this out.

Theorem A.2.2. *Let E be any complex-oriented cohomology theory. Take G to be a finite group and let Ab_G be the full subcategory of the orbit category of G built out of abelian subgroups of G. Finally, let X be a finite G–CW complex. Then, each of the natural maps*

Cite me: Theorem A.

$$E^*(EG \times_G X) \to \lim_{A \in \mathsf{Ab}_G} E^*(EG \times_A X) \to \int_{A \in \mathsf{Ab}_G} E^*(BA \times X^A)$$

becomes an isomorphism after inverting the order of G. In particular, there is an isomorphism

$$\frac{1}{|G|} E^* BG \to \lim_{A \in \mathsf{Ab}_G} \frac{1}{|G|} E^* BA. \quad \square$$

This is an analogue of Artin's theorem:

Theorem A.2.3. *There is an isomorphism*

$$\frac{1}{|G|} R(G) \to \lim_{C \in \mathsf{Cyclic}_G} \frac{1}{|G|} R(C). \quad \square$$

HKR intro material connecting Theorem A to character theory:

Recall that classical characters for finite groups are defined in the following situation: take $L = \mathbb{Q}^{\mathrm{ab}}$ to be the smallest characteristic 0 field containing all roots of unity, and for a finite group G let $Cl(G; L)$ be the ring of class functions on G with values in L. The units in the profinite integers $\widehat{\mathbb{Z}}$ act on L as the Galois group over $\mathbb{Q}$, and since $G = \mathsf{Groups}(\widehat{\mathbb{Z}}, G)$ they also act naturally on G. Together, this gives a conjugation action on $Cl(G; L)$: for $\varphi \in \widehat{\mathbb{Z}}$, $g \in G$, and $\chi \in Cl(G; L)$, one sets

$$(\varphi \cdot \chi)(g) = \varphi(\chi(\varphi^{-1}(g))).$$

The character map is a ring homomorphism

$$\chi : R(G) \to Cl(G; L)^{\widehat{\mathbb{Z}}},$$

and this induces isomorphisms

$$\chi : L \otimes R(G) \xrightarrow{\simeq} Cl(G; L)$$

and even

$$\chi : \mathbb{Q} \otimes R(G) \xrightarrow{\simeq} Cl(G; L)^{\widehat{\mathbb{Z}}}.$$

Now take $E = E_\Gamma$ to be a Morava E–theory of finite height $d = \mathrm{ht}(\Gamma)$. Take $E^*(B\mathbb{Z}_p^d)$ to be topologized by $B(\mathbb{Z}/p^j)^d$. A character $\alpha : \mathbb{Z}_p^d \to S^1$ will induce a map $\alpha^* : E^*\mathbb{CP}^\infty \to E^*B\mathbb{Z}_p^d$. We define $L(E^*) = S^{-1}E^*(B\mathbb{Z}_p^d)$, where S is the set of images of a coordinate on $\mathbb{CP}_E^\infty$ under α^* for nonzero characters α. Note that this ring inherits an $\mathrm{Aut}(\mathbb{Z}_p^d)$ action by E^*–algebra maps.

The analogue of $Cl(G;L)$ will be $Cl_{d,p}(G;L(E^*))$, defined to be the ring of functions $\chi : G_{d,p} \to L(E^*)$ stable under G–orbits. Noting that

$$G_{d,p} = \mathrm{Hom}(\mathbb{Z}_p^d, G),$$

one sees that $\mathrm{Aut}(\mathbb{Z}_p^d)$ acts on $G_{d,p}$ and thus on $Cl_{d,p}(G;L(E^*))$ as a ring of E^*–algebra maps: given $\varphi \in \mathrm{Aut}(\mathbb{Z}_p^d)$, $\alpha \in G_{d,p}$, and $\chi \in Cl_{d,p}(G;L(E^*))$ one lets

$$(\varphi \cdot \chi)(\alpha) = \varphi(\chi(\varphi^{-1}(\alpha))).$$

Now we introduce a finite G–CW complex X. Let

$$\mathrm{Fix}_{d,p}(G,X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)} X^{\mathrm{im}\,\alpha}.$$

This space has commuting actions of G and $\mathrm{Aut}(\mathbb{Z}_p^d)$. We set

$$Cl_{d,p}(G,X;L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{d,p}(G,X))^G,$$

which is again an E^*–algebra acted on by $\mathrm{Aut}(Z_p^d)$. We define the character map "componentwise": a homomorphism $\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)$ induces

$$E^*(EG \times_G X) \to E^*(B\mathbb{Z}_p^d) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}) \to L(E^*) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}).$$

Taking the direct sum over α, this assembles into a map

$$\chi_{d,p}^G : E^*(EG \times_G X) \to Cl_{d,p}(G,X;L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}.$$

Theorem A.2.4. *The invariant ring is $L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^d)} = p^{-1}E^*$, and $L(E^*)$ is faithfully flat over $p^{-1}E^*$. The character map $\chi_{d,p}^G$ induces isomorphisms*

$$\begin{aligned} \chi_{d,p}^G \colon\ L(E^*) \otimes_{E^*} E^*(EG \times_G X) &\xrightarrow{\simeq} Cl_{d,p}(G,X;L(E^*)), \\ \chi_{d,p}^G \colon\ p^{-1}E^*(EG \times_G X) &\xrightarrow{\simeq} Cl_{d,p}(G,X;L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}. \end{aligned}$$

In particular, when $X = *$, there are isomorphisms

$$\begin{aligned} \chi_{d,p}^G \colon\ L(E^*) \otimes_{E^*} E^*(BG) &\xrightarrow{\simeq} Cl_{d,p}(G;L(E^*)), \\ \chi_{d,p}^G \colon\ p^{-1}E^*(BG) &\xrightarrow{\simeq} Cl_{d,p}(G;L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}. \quad \square \end{aligned}$$

Nat taught you how to say all these things with p–adic tori, which was *much* clearer.

Cite me: Theorem C.

Checking this invariant ring claim is easiest done by comparing the functors the two things corepresent.

—–

Jack gives an interpretation of this in terms of formal $\mathcal{O}_L$–modules.

—–

I also have this summary of Nat's of the classical case:

It's not easy to decipher if you weren't there for the conversation, but here's my take on it. First, the map we wrote down today was the non-equivariant chern character: it mapped non-equivariant $KU \otimes \mathbb{Q}$ to non-equivariant $H\mathbb{Q}$, periodified. The first line on Nat's board points out that if you use this map on Borel-equivariant cohomology, you get nothing interesting: $K^0(BG)$ is interesting, but $H\mathbb{Q}^*(BG) = H\mathbb{Q}^*(*)$ collapses for finite G. So, you have to do something more impressive than just directly marry these two constructions to get something interesting.

That bottom row is Nat's suggestion of what "more interesting" could mean. (Not really his, of course, but I don't know who did this first. Chern, I suppose.) For an integer n, there's an evaluation map of (forgive me) topological stacks

$$*//(\mathbb{Z}/n) \times \mathrm{Hom}(*//(\mathbb{Z}/n), *//G) \xrightarrow{\mathrm{ev}} *//G$$

which upon applying a global-equivariant theory like K_G gives

$$K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod_{\text{conjugacy classes of } g \text{ in } G} *) \xleftarrow{ev^*} K_G(*).$$

Now, apply the genuine G-equivariant Chern character to the K_G factor to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) \leftarrow K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod *),$$

where the coproduct is again taken over conjugacy classes in G. Now, compute $K_{\mathbb{Z}/n}(*) = R(\mathbb{Z}/n) = \mathbb{Z}[x]/(x^n - 1)$, and insert this calculation to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) = \mathbb{Q}(\zeta_n) \otimes (\bigoplus_{\text{conjugacy classes}} \mathbb{Q}),$$

where ζ_n is an n^{th} root of unity. As n grows large, this selects sort of the part of the complex numbers $\mathbb{C}$ that the character theory of finite groups cares about, and so following all the composites we've built a map

$$K_G(*) \to \mathbb{C} \otimes (\bigoplus_{\text{conjugacy classes}} \mathbb{C}).$$

The claim, finally, is that this map sends a G-representation (thought of as a point in $K_G(*)$) to its class function decomposition.

—–

A.3 Knowns and unknowns

Higher orientations

TAF and friends

The $\alpha_{1/1}$ argument: Prop 2.3.2 of Hovey's v_n–elements of ring spectra

Equivariance

This is tied up with the theory of power operations in a way I've never really thought about. Seems complicated.

Index theorems

Connections with analysis

The Stolz–Teichner program

—

Contexts for structured ring spectra

Difficulty in computing $\mathbb{S}_d \circlearrowleft E_d^*$. (Gross–Hopkins and the period map.)

Barry's p–adic measures

Fixed point spectra and e.g. $L_{K(2)} tmf$.

Blueshift, A–M–S, and the relationship to A–F–G?

Does E_n receive an E_∞ orientation? Does BP?

Remark 12.13 of published H_∞ AHS says their obstruction framework agrees with the E_∞ obstruction framework (if you take everything in sight to have E_∞ structures). This is almost certainly related to the discussion at the end of Matt's thesis about the MU–orientation of E_d.

Section 12.4 compares doing H_∞ descent with doing E_∞ descent and shows that they're the same (in the case of interest?).

Hovey's paper on v_n–periodic elements in ring spectra. He has a nice (and thorough!) exposition on why one should be interested in bordism spectra and their splittings: for instance, a careful analysis of *MSpin* will inexorably lead one toward studying *KO*. It would be nice if studying *MString* (and potentially higher analogues) would lead one toward non-completed, non-connective versions of EO_n. Talk about *BoP*, for instance.

Matt's short resolutions of chromatically localized *MU*.

Bibliography

[1] Paul G. Goerss. Quasi-coherent sheaves on the moduli stack of formal groups. 2008.

[2] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

[3] Neil P. Strickland. Formal schemes and formal groups. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 263–352. Amer. Math. Soc., Providence, RI, 1999.

Material for lecture

Mike's 1995 announcement is a nice read. There are many snippets you could pull out of it for use here. "$H\mathbb{Q}$ serves as the target for the Todd genus, but actually the Todd genus of a manifold is an integer and it turns out that KU refines the Todd genus." The end of section 3, with $\tau \mapsto 1/\tau$, is mysterious. In section 4, Mike claims that there's a $BU[6,\infty)$–structured splitting principle into sums of things of the form $(1-\mathcal{L}_1)(1-\mathcal{L}_2)(1-\mathcal{L}_3)$. He then says that one expects the characteristic series of a $BU[6,\infty)$–genus to be a series of 3 variables, which is nice intuition. Could mention that Θ^k is a kind of k^{th} difference operator, so that things in the kernel of Θ^k are "k^{th} order polynomials". (More than this, the theorem of the cube is reasonable from this perspective, since Θ^3 kills "quadratic things" and the topological object $H^2(-;\mathbb{Z})$ classifying line bundles is indeed "quadratic".) If the bundle admits a symmetry operation, then the fiber over $(x, y, -x-y)$ is canonically trivialized, so a Σ–structure on a symmetric line bundle is a Θ^3–structure that restricts to the identity on these canonical parts. Mike claims (Theorem 6.2) that if $1/2 \in E^0(*)$ or if E is $K(n)$–local, $n \leq 2$, then $BString^E$ is the parameter space of Σ–structures on the sheaf of functions vanishing at the identity on G_E. The map $MString \to KO_{\text{Tate}}$ actually factors through $MSpin$, so even though this produces the right q–series, you really need to know that *MString* factors through *tmf* and *MSpin doesn't* to deduce the modularity for *String*–manifolds. (You can prove modularity separately for $BU[6, infty)$–manifolds, though, by essentially the same technique: refer to the rest of the (complex!) moduli of elliptic curves, which exist as $MU[6, infty)$–spectra.)

Generally: if X is a space, then $X_{H\mathbb{F}_2}$ is a scheme with an $\operatorname{Aut}\hat{\hat{\mathbb{G}}}_a$–action. If X is a spectrum (so it fails to have a diagonal map) then $(H\mathbb{F}_2)_*X$ is just an $\mathbb{F}_2$–module, also with an $\operatorname{Aut}\hat{\hat{\mathbb{G}}}_a$–action.

The cohomology of a qc sheaf pushed forward from a scheme to a stack along a cover agrees with just the cohomology over the scheme. (In the case of $* \to *//G$, this probably uses the cospan $* \to *//G \leftarrow *$ with pullback G...)

Akhil Mathew has notes from an algebraic geometry class (https://math.berkeley.edu/~amathew/
) where lectures 3–5 address the theorem of the cube.

Equivalences of various sorts of cohomologies: Ext in modules and quasicoherent cohomology (goodness. Hartshorne, I suppose); Ext in comodules and quasicoherent cohomology on stacks (COCTALOS Lemma 12.4); quasicoherent cohomology on simplicial schemes (Stacks project 09VK).

Make clear the distinction between E_n and $\widehat{E(n)}$. Maybe explain the Devinatz–Hopkins remark that $r : \widehat{E(n)} \to E_n$ is an inclusion of fixed points and as such does not classify the versal formal group law.

when describing Quillen's model, he makes a lot of use of Gysin maps and Thom / Euler classes. at this point, maybe you can introduce what a Thom sheaf / Thom class is for a pointed formal curve?

I think this theorem is motivated by Artin–Mazur formal groups, and the Crystals notes use it to extract a formal group from a Dieudonné module. Some motivation could go here.

Theorem A.3.1. *Let A be a Noetherian ring and $G : \mathsf{AdicAlgebras}_A \to \mathsf{AbelianGroups}$ be a functor such that*

1. $G(A) = 0$.
2. *G takes surjective maps to surjective maps.*
3. *There is a finite, free A–module M and a functorial isomorphism*

$$I \otimes_A M \to G(B) \to G(B')$$

whenever I belongs to a square-zero extension of adic A–algebras

$$I \to B \to B'.$$

Then, $G \cong \widehat{\mathbb{A}}^n$ as a functor to sets, where $n = \dim M$.

Proof. This is 9.6.4 in the Crystals notes. □

MUP happens to be the Thom spectrum of $BU \times \mathbb{Z}$.

–Formal groups in algebraic topology

—Day 1

+ Warning: noncontinuous maps of high-dimensional formal affine spaces.

—Day 2

+ Three definitions of complex orientable / oriented cohomology theories. + Some proofs: the splitting principle, Chern roots, diagrammatic Adam's condition, ...

—Day 3

+ Lemma and proof: homomorphisms $F \to G$ of $\mathbb{F}_p$-FGLs factor as $F \to G' \to G$, where $G' \to G$ is a Frobenius isogeny and $F \to G'$ is invertible. + Definition of height. Examples: $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$. + Redefinition of height as the log-p rank of the p-torsion. + Logarithms for FGLs over torsion-free rings. The integral equation. Height as radius.

—Day 4

+ A picture of $\mathcal{M}_{\mathbf{fg}} \times \mathbb{Z}_{(p)}$ + Definition of "deformation" + Plausibility argument for square-zero deformations being classified by "$\operatorname{Ext}^1(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$" + Theorem statement: $\operatorname{Ext}^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is computed by $H^* \operatorname{Hom}(B\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)(R)$. + Theorem statement: That cochain complex is quasi-isomorphic to Lazarev's infinitesimal complex. + Proofs: Infinitesimal homomorphisms gives 1-cocycles, infinitesimal deformations give 2-cocycles. + Theorem statement (LubinâĂŤTate): H^0, H^1, H^2 calculations. + Implications for Bockstein spectral sequence computing infinitesimal deformations. + Clarification about relative deformations and what "arithmetic deformation" means

—Day 5

+ The rational complex bordism ring. + QuillenâĂŹs theorem as refining rational complex genera to integral ones. + HondaâĂŹs theorem about ζ–functions as manufacturing integral genera. + A statement of Landweber's theorem about regularity, stacky interpretation, no proof. + Definition of forms of a module, map to Galois cohomology + Computation of the Galois cohomology for: $H\mathbb{F}_p$, MU/p, KU/p + Computation of the Galois cohomology for $\widehat{\mathbb{G}}_m$, explicit description of the invariant via the ζ–function + MoravaâĂŹs sheaf over $L_1(\mathbb{Z}_p^{nr})$, Gamma-equivariance and transitivity, ConnerâĂŤFloyd + Identification of L_1/Γ with $\mathcal{M}_{\mathbf{fg}}^{\leq 1}$, connection to LEFT.

—Day 6

+ The invariant differential. + de Rham cohomology in positive characteristic. The de Rham cohomology of $\widehat{\mathbb{A}}^1/\mathbb{F}_p$. + Cohomologically invariant differentials and the functor D. + Crystalline properties of D. The map F. + The Dieudonne functor, the main theorem.

—Future topics

+ The main theorem of class field theory + Lubin and Tate's construction of abelian extensions of local number fields + Description of the Lubin–Tate tower and the local Langlands correspondence + Lazard's theorem + Uniqueness of $\mathcal{O}_K$-module structure in characteristic zero + p-typification + Construction of the spectra MU, BP, $E(n)$, $K(n)$, E_n + Goerss–Hopkins–Miller and Devinatz–Hopkins + Gross–Hopkins period map and the calculation of the Verdier dualizing sheaf on LT_n + The Ravenel–Wilson calculation, exterior powers of p-divisible groups + Kohlhaase's Iwasawa theory + Lubin's dynamical results on formal power series + Classification of field spectra

Ideas

1. Singer–Stong calculation of $H^*BU[2k,\infty)$. [ASIDE: $H\mathbb{F}_2^* ko$ and the Hopf algebra quotient of $\widehat{\mathbb{A}}_*$.]

2. Ando, Hopkins, Strickland on H_∞–orientations and the norm condition

3. The rigid, real σ–orientation: AHR. Its effect in homology.

4. The Rezk logarithm and the Bousfield–Kuhn functor

5. Statement of Lurie's characterization of *TMF*, using this to determine a map from *MString* by AHR

6. Dylan's paper on String orientations

7. Matt's calculation of E_∞–orientations of $K(1)$–local spectra using the short free resolution of MU in the $K(1)$–local category

---

8. Cartier duality

9. Subschemes and divisors

10. Coalgebraic formal schemes

11. *Forms of K–theory*, Elliptic spectra, Tate K–theory, *TMF*

12. What are Weil pairings for geometers?

13. The Atiyah–Bott–Shapiro orientation (Is there a complex version of this? I understand it as a splitting of *MSpin*...)

14. Sinkinson's calculation and $MBP\langle m\rangle$–orientations

15. Hovey–Ravenel on nonorientations of E_n by $MO[k,\infty)$. Other things in H–R?

16. Wood's cofiber sequence and $KO_{(p\geq 3)}$

17. The Serre–Tate theorem

18. The fundamental domain of π_{GH}

19. Orientations and the functor gl_1.

Resources

Ando, Hopkins, Strickland (Theorem of the Cube)

Ando, Hopkins, Strickland (H_∞ map)

Ando, Strickland

Ando, Hopkins, Rezk

Barry Walker's thesis

Bill Singer's thesis, Bob Stong's *Determination*

Morava's *Forms of K–theory*

Neil's Functorial Philosophy for Formal Phenomena

Ravenel, Wilson

Kitchloo, Laures, Wilson

Akhil wrote a couple of blog posts about Ochanine's theorem: `https://amathew.wordpress.co` and `https://amathew.wordpress.com/2012/05/31/the-other-direction-of-ochaines-` Mentioning a more precise result might lend to a more beefy introduction.

What follows are notes from other talks I've given about quasi-relevant material which can probably be cannibalized for this class.