Eric Peterson

# Formal Geometry and Bordism Operations

Lecture notes

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## *Class information*

*Course ID:* MATH 278 (159627).

*Meeting times:* Spring 2016, MWF 12pm–1pm.

*Goals:* The primary goal of this class is to teach students to view results in algebraic topology through the lens of (formal) algebraic geometry.

*Grading:* This class won't have any official assignments. I'll give references as readings for those who would like a deeper understanding, though I'll do my best to ensure that no extra reading is required to follow the arc of the class.

I do want to assemble course notes from this class, but it's unlikely that I will have time to type *all* of them up. Instead, I would like to "crowdsource" this somewhat: I'll type up skeletal notes for each lecture, and then we as a class will try to flesh them out as the semester progresses. As incentive to help, those who contribute to the document will have their name included in the acknowledgements, and those who contribute *substantially* will have their name added as a coauthor. Everyone could use more CV items. (Publication may take a while. I suspect the course won't run perfectly smoothly the first time, so this may takes a second semester pass to become fully workable. But, since topics courses only come around once in a while, this will necessarily mean a delay.)

The source for this document can be found at

`https://github.com/ecpeterson/FormalGeomNotes`.

If you're taking the class or otherwise want to contribute, you can write me at

`ecp@math.harvard.edu`

to request write access.

## *Acknowledgements*

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Regular participants (whoever I could think of on March 24, again missing people): Eva Belmont, Lukas Brantner, Christian Carrick, Hood Chatham, Jun Hou Fung, Meng Guo, Jeremy Hahn, Erick Knight, Jake McNamara, Akhil Mathew, Krishanu Sankar, Jay Shah, Danny Shi, Allen Yuan

Course registrants (collected March 24, possibly missing some MIT people): Colin Aitken, Adam Al-Natsheh, Eva Belmont, Jason Bland, Dorin Boger, Lukas Brantner, Christian Carrick, Hood Chatham, David Corwin, Jun Hou Fung, Meng Guo, Jeremy Hahn, Changho Han, Chi-Yun Hsu, Erick Knight, Benjamin Landon, Gabriel Long, Yusheng Luo, Jake Marcinek, Jake McNamara, Max Menzies, Morgan Opie, Alexander Perry, Mauro Porta, Kishanu Sankar, Jay Shah, Ananth Shankar, Danny Shi, Koji Shimizu, Geoffrey Smith, Hunter Spinik, Philip Tynan, Yi Xie, David Yang, Zijian Yao, Lynnelle Ye, Chenglong Yu, Allen Yuan, Adrian Zahariuc, Yifei Zhao, Rong Zhou, Yihang Zhu.

People who contributed to the GitHub repository (to be collected later, make sure there's no overlap with the above list):

Other readers: Jon Beardsley, Sune Precht Reeh, Kevin Wray

# Contents

I would like as few section titles as possible to involve people's names.

A bunch of broken displaymode tombstones can have their positions fixed by using http://tex.stackexchange.com/a/66221/2671.

Compile an index by replacing all the `textit` commands in definition environments with some more fancy macro that tags it for inclusion.

Remember to use $f\colon A \to B$ everywhere.

Should sections have subsections? Does that help organize the TOC?

Hood wrote a macro called sumfgl (see also sumF and sumG) that will make a bunch of formal group law expressions typeset better. Propagate that change through.

http://tex.stackexchange.com/questions/53513/hyperref-token-not-allowed has information on how to make the PDF Bookmarks warnings go away. You can't use this until after you drop the date prefix macro.

Make sure you use either id or 1 everywhere to denote the identity morphism.

You're not very consistent about using $\hat{\mathbb{G}}$ or $\Gamma$ to denote an arbitrary formal group. It seems like you use one or the other based on your preference of whether it has finite height or not.

# Chapter 0
# Introduction

## 0.1 Jan 25: Introduction

The goal of this class is to communicate a certain *weltanschauung* uncovered in pieces by many different people working in bordism theory, and the goal just for today is to tell a story about one theorem where it is especially apparent.

To begin, we will define a homology theory called "bordism homology". Recall that the singular homology of a space $X$ is defined by considering the collection of continuous maps $\sigma : \Delta^n \to X$, taking the free $\mathbb{Z}$–module on each of these sets, and constructing a chain complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}\{\Delta^n \to X\} \xrightarrow{\partial} \mathbb{Z}\{\Delta^{n-1} \to X\} \xrightarrow{\partial} \cdots .$$

Bordism homology is constructed analogously, but using manifolds $M$ as the sources instead of simplices:

$$\begin{aligned} \cdots &\xrightarrow{\partial} \{M^n \to X \mid M^n \text{ an } n\text{–manifold}\} \\ &\xrightarrow{\partial} \{M^{n-1} \to X \mid M^{n-1} \text{ an } (n-1)\text{–manifold}\} \\ &\xrightarrow{\partial} \cdots . \end{aligned}$$

**Lemma 1.** *This forms a chain complex of monoids under direct sum of manifolds, and its homology is written $MO_*(X)$. These are naturally abelian groups, and moreover they satisfy the axioms of a generalized homology theory.* □

In fact, we can define a bordism theory $MG$ for any suitable family of structure groups $G(n) \to O(n)$. The coefficient ring of $MG$, or its value $MG_*(*)$ on a point, gives the ring of $G$–bordism classes, and generally $MG_*(Y)$ of some space $Y$ gives a kind of "bordism in families (over $Y$)". There are comparison morphisms for the most ordinary kinds of bordism, dshi:I don't follow here. How does this replacement go explicitly? Somehow I understood it when you explained this to me in person but now I don't see it anymore.

given by replacing a chain of manifolds with an equivalent simplicial chain:

$$MO \to H\mathbb{Z}/2, \quad MSO \to H\mathbb{Z}.$$

In both cases, we can evaluate on a point to get ring maps, called "genera":

$$MO_*(*) \to \mathbb{Z}/2 \quad \text{and} \quad MSO_*(*) \to \mathbb{Z},$$

A comparison of this with the usual spectrum definition of $MX$ appears in Switzer 12.35.

neither of which is very interesting, since they're both zero in positive degrees.

However, having maps of homology theories (rather than just maps of coefficient rings) is considerably more data then just the genus. In fact, we can extract a theory of integration. Consider the following special case of oriented bordism, where we evaluate $MSO_*$ on an infinite loopspace:

$$\begin{aligned} MSO_n K(\mathbb{Z}, n) &= \{\text{oriented } n\text{–manifolds mapping to } K(\mathbb{Z}, n)\} / \sim \\ &= \left\{ \begin{array}{c} \text{oriented } n\text{–manifolds } M \\ \text{with a specified class } \omega \in H^n(M; \mathbb{Z}) \end{array} \right\} / \sim . \end{aligned}$$

Associated to such a representative $(M, \omega)$, the yoga of stable homotopy theory then allows us to build a composite

$$\begin{aligned} \mathbb{S} &\xrightarrow{(M,\omega)} MSO \wedge (\mathbb{S}^{-n} \wedge \Sigma^\infty_+ K(\mathbb{Z}, n)) \\ &\xrightarrow{\text{colim}} MSO \wedge H\mathbb{Z} \\ &\xrightarrow{\varphi \wedge 1} H\mathbb{Z} \wedge H\mathbb{Z} \\ &\xrightarrow{\mu} H\mathbb{Z}, \end{aligned}$$

I changed $\mathbb{S}^0$ to $\mathbb{S}$ here, because that's what you used below, but it seems that the notation for the sphere spectrum has been inconsistent elsewhere too.

where $\varphi$ is the orientation map. Altogether, this composite gives us an element of $\pi_0 H\mathbb{Z}$, i.e., an integer.

I used to think that this gave rise to a Stokes's Theorem, but now I'm not sure. Maybe this comes out of relative homology somehow.

**Lemma 2.** *The integer obtained by the above process is $\int_M \omega$.* □

This definition of $\int_M \omega$ via stable homotopy theory is pretty nice, in the sense that many theorems accompany it for free.

Now take $G = e$ to be the trivial structure group, which is the bordism theory of manifolds with trivialized tangent bundle. In this case, the Pontryagin–Thom construction gives an equivalence $\mathbb{S} \xrightarrow{\simeq} Me$. It is thus possible (and some people have indeed taken up this viewpoint) that stable homotopy theory can be done solely through the lens of "framed bordism". We will prefer to view this the other way: the sphere spectrum $\mathbb{S}$ often appears to us as a natural object, and we will occasionally replace it by $Me$, the framed bordism spectrum. For example, given a ring spectrum $E$ with unit map $\mathbb{S} \to E$, we can reconsider this as a ring map $\mathbb{S} = Me \to E$. Following along the lines of the previous paragraph, we learn that any ring spectrum $E$ is automatically equipped with a theory of integration for framed manifolds.

Sometimes, as in the examples above, this unit map factors:

$$\mathbb{S} = Me \to MO \to H\mathbb{Z}/2.$$

This is a witness to the overdeterminacy of $H\mathbb{Z}/2$'s integral for framed bordism: if the framed manifold is pushed all the way down to an unoriented manifold, there is still enough residual data to define the integral.[1] Given any ring spectrum $E$, we can ask the analogous question: If we filter $O$ by a system of structure groups, at what stage does the unit map $Me \to E$ factor through? For instance, the map

$$\mathbb{S} = Me \to MSO \to H\mathbb{Z}$$

considered above does *not* factor further through $MO$ — an orientation is *required* to define the integral of an integer–valued cohomology class. In the more general case, the map $SO \to O$ is the beginning of the Postnikov filtration of $O$, and we now present a diagram of this filtration and some interesting integration theories related to it:

danny: do you mean postnikov filtration for $MO$? I asked this in class but I think it's good to say how does the filtration go. The classical postnikov filtration for $X$ builds the homotopy groups of $X$ up from the bottom, to get a sequence $\cdots \to X_2 \to X_1 \to X_0$, with $X$ the limit. I think the situation here is the opposite? We'll talk about this.

$$\begin{tikzcd}
Me \arrow[r] \arrow[drr] \arrow[drrr] \arrow[drrrr] & \cdots \arrow[r] & MSpin \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\
 & & kO & H\mathbb{Z} & H\mathbb{Z}/2.
\end{tikzcd}$$

This is the situation homotopy theorists found themselves in some decades ago, when Ochanine and Witten proved the following mysterious theorem using analytical and physical methods:

Given that you mentioned string in the theorem below. Might wanna add string into the diagram

**Theorem 1 (Ochanine, Witten).** *There is a map of rings*

**Cite me:** Ochanine, Witten.

$$\sigma : MSpin_* \to \mathbb{C}((q)).$$

*Moreover, if M is a Spin manifold such that twice its first Pontryagin class vanishes — that is, if M lifts to a String–manifold — then $\sigma(M)$ lands in the subring $MF \subseteq \mathbb{Z}[[q]]$ of modular forms with integral coefficients.* □

However, neither party gave indication that their result should be valid "in families", and no theory of integration was produced. From the perspective of the homotopy theorist, it wasn't even totally clear what such a claim would mean: to give a topological enrichment of these theorems would

[1] It's literally more information than this: even unframeable unoriented manifolds acquire a compatible integral.

mean finding a ring spectrum $E$ such that $E_*(*)$ had something to do with modular forms.

Around the same time, Landweber, Ravenel, and Stong began studying "elliptic cohomology" for independent reasons; sometime much earlier, Morava had constructed an object "$K^{\mathrm{Tate}}$" associated to the Tate elliptic curve; and a decade later Ando, Hopkins, and Strickland put all these things together in the following theorem:

**Cite me:** Landweber–Ravenel–Stong, Morava's *Forms of K–theory*, and Ando–Hopkins–Strickland.

**Theorem 2 (Ando–Hopkins–Strickland).** *If $E$ is an "elliptic cohomology theory", then there is a canonical map $MString \to E$ called the $\sigma$–orientation. In particular, the map $MString_* \to K_*^{\mathrm{Tate}}$ is Witten's genus.* □

We now come to the motivation for this class. The homotopical $\sigma$–orientation was actually first constructed using formal geometry. The original proof of Ando–Hopkins–Strickland begins with a reduction to maps of the form

$$MU[6,\infty) \to E.$$

They then work to show that in especially good cases they can complete the missing arrow in the diagram

$$\begin{tikzcd} MU[6,\infty) \arrow[r] \arrow[dr] & MString \arrow[d, dotted] \\ & E. \end{tikzcd}$$

Leaving aside the extension problem for the moment, their main theorem is the following description of the cohomology ring $E^*MU[6,\infty)$:

**Theorem 3 (Ando–Hopkins–Strickland).** *For $E$ an even–periodic cohomology theory,*

$$\operatorname{Spec} E_*MU[6,\infty) \cong C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0)),$$

*where "$C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0))$" is a certain scheme. When $E$ is taken to be elliptic, so that there is a specified isomorphism $\widehat{\mathbb{G}}_E \cong C_0^\wedge$ for $C$ an elliptic curve, the theory of elliptic curves furnishes the scheme with a canonical point. Hence, there is a preferred class $MU[6,\infty) \to E$, natural in the choice of elliptic $E$.* □

Our real goal is to understand theorems like this last one, where algebraic geometry asserts some real control over something squarely in the domain of homotopy theory.

The structure of the class will be to work through a sequence of case studies where this perspective shines through most brightly. We'll start by working through Thom's calculation of the homotopy of $MO$, which holds the simultaneous attractive features of being approachable while revealing essentially all of the structural complexity. Having seen that through to the end, we'll then venture on to other examples: the complex bordism ring, structure theorems for finite spectra, unstable cooperations, and, finally, the theorem above and its extensions. The overriding theme of the class will be that algebraic geometry is a good organizing principle that gives us one avenue of insight into how homotopy theory functions. In particular, it allows us to organize "operations" of various sorts between spectra derived from bordism theories.

We should also mention that we will specifically *not* discuss the following aspects of this story:

- Analytic techniques will be completely omitted. Much of modern research stemming from the above problem is an attempt to extend index theory across Witten's genus, and this often means heavy analytic work. We will strictly confine ourselves to the domain of homotopy theory.
- As sort of a sub-point (and despite the motivation provided in this Introduction), we will also mostly avoid manifold geometry. (We do give a proof of Quillen's theorem on the structure of $MU_*$ which invokes some mild amount of manifold geometry.)Again, much of the contemporary research about *tmf* is an attempt to find a geometric model, so that geometric techniques can be imported — including equivariance and the geometry of quantum field theories, to name two. If Jeremy's proof changes this, update this sentence.
- In a different direction, our focus will not linger on actually computing bordism rings $MX_*$, nor will we consider geometric constructions on manifolds and their behavior after imaging into the bordism ring. This is also the source of active research: the structure of the symplectic bordism ring remains, to large extent, mysterious, and what we do understand of it comes through a mix of formal geometry and raw manifold geometry. This could be a topic that fits logically into this document, were it not for time limitations and the author's inexpertise.
- The geometry of $E_\infty$ rings will also be avoided, at least to the extent possible. Such objects become inescapable by the conclusion of our story, but there are better resources from which to learn about $E_\infty$ rings, and the pre–$E_\infty$ story is not told so often these days. So, we will focus on the unstructured part and leave $E_\infty$ rings to other authors.
- There will be plenty of places where we will avoid stating things in maximum generality or with maximum thoroughness. The story we are interested in telling draws from a blend of many others from different subfields of mathematics, many of which have their own topics courses. Sometimes this will mean avoiding stating the most beautiful theorem in a subfield in favor of a theorem we will find more useful. Other times this will mean abbreviating someone else's general definition to one more specialized to the task at hand. In any case, we will give references to other sources where you can find these things cast in starring roles.

Finally, we must mention that there are several good companions to these notes. Essentially none of the material here is original — it's almost all cribbed either from published or unpublished sources — but the source documents are quite scattered and individually dense. We will make a point to cite useful references as we go. One document stands out above all others, though: Neil Strickland's *Functorial Philosophy for Formal Phenomena* [41]. These lecture notes can basically be viewed as an attempt to make it through this paper in the span of a semester.

# Case Study 1
# Unoriented bordism

Write an introduction for me.

## 1.1 Jan 27: Thom spectra and the Thom isomorphism

Our first case study is a sequence of theorems about the unoriented bordism spectrum $MO$. I wanted to begin by recalling one definition of the spectrum $MO$, since it involves ideas that will be useful to us throughout the semester.

**Definition 1.** For a spherical bundle $S^{n-1} \to \xi \to X$, its Thom space is given by the cofiber

$$\xi \to X \xrightarrow{\text{cofiber}} T(\xi).$$

*Proof ("Proof" of definition).* There is a more classical construction of the Thom space: take the associated disk bundle by gluing an $n$–disk fiberwise, and add a point at infinity by collapsing $\xi$:

$$T(\xi) = (\xi \sqcup'_{S^{n-1}} D^n)^+.$$

To compare this with the cofiber definition, recall that the thickening of $\xi$ to an $n$–disk bundle is the same thing as taking the mapping cylinder on $\xi \to X$. Since the inclusion into the mapping cylinder is now a cofibration, the quotient by this subspace agrees with both the cofiber of the map and the introduction of a point at infinity.

Before proceeding, here are two important examples:

*Example 1.* If $\xi = S^{n-1} \times X$ is the trivial bundle, then $T(\xi) = S^n \wedge (X_+)$. This is supposed to indicate what Thom spaces are "doing": if you feed in the trivial bundle then you get the suspension out, so if you feed in a twisted bundle you should think of it as a *twisted suspension*.

*Example 2.* Let $\xi$ be the tautological $S^0$–bundle over $\mathbb{R}\mathrm{P}^\infty = BO(1)$. Because $\xi$ has contractible total space, $EO(1)$, the cofiber degenerates and it follows that $T(\xi) = \mathbb{R}\mathrm{P}^\infty$. More generally, arguing by cells shows that the Thom space for the tautological bundle over $\mathbb{R}\mathrm{P}^n$ is $\mathbb{R}\mathrm{P}^{n+1}$.

> Two people (Mauro and someone else) asked in what generality this "$Bh\operatorname{Aut} F$" construction works. This can be clarified in a remark.

> In lecture, you decided to call these $Bh\operatorname{Aut}(S^{n-1})$, which is maybe a healthier choice?

> does this mean we are regarding $h\operatorname{Aut}(S^{n-1})$ as a group? Is $\operatorname{Aut}(S^{n-1})$ a topological group? Would it make sense to say $B\operatorname{Aut}(S^{n-1})$? To be honest I personally like $GL_1 S^n$. It might cause some potential for confusion but it looks more clean. - danny

Now we catalog a bunch of useful properties of the Thom space functor. Firstly, recall that a spherical bundle over $X$ is the same data as a map $X \to BGL_1 S^{n-1}$, where $GL_1 S^{n-1}$ is the subspace of $F(S^{n-1}, S^{n-1})$ expressed by the pullback

$$\begin{tikzcd}
GL_1 S^{n-1} \arrow[r] \arrow[d] & F(S^{n-1}, S^{n-1}) \arrow[d] & \\
\operatorname{Aut}_{h\mathsf{Spaces}} S^{n-1} \arrow[r] & \operatorname{End}_{h\mathsf{Spaces}} S^{n-1} \arrow[r, equal] & \pi_0 F(S^{n-1}, S^{n-1}).
\end{tikzcd}$$

We can interpret $T$ as a functor off of the slice category over $BGL_1 S^{n-1}$: maps

$$Y \xrightarrow{f} X \xrightarrow{\xi} BGL_1 S^{n-1}$$

induce maps $T(f^*\xi) \to T(\xi)$, and $T$ is suitably homotopy-invariant.

> You can make this section clearer by noticing that the map $J^n_{\mathbb{R}}$ is directly induced by a map $O(n) \to h\operatorname{Aut} S^{n-1}$ by the action of $O(n)$ on $\mathbb{R}^n$. This makes the commutativity of the relevant diagrams clear, whereas using the Yoneda lemma does not make it clear that we can take the appropriate homotopy colimits. –Mauro

Next, the spherical subbundle of a vector bundle gives a common source of spherical bundles. Since rank $n$ vector bundles are also classified by an object $BO(n)$, this begets a map $J^n_{\mathbb{R}}\colon BO(n) \to BGL_1 S^{n-1}$ for each $n$. Stable homotopy theorists are very interested in the block–inclusion maps $i^n\colon BO(n) \to BO(n+1)$ and the colimit $BO = BO(\infty)$. The suspension functor induces a map $GL_1 S^{n-1} \to GL_1 S^n$, and we are led to ask about the compatibility of these operations. As a route to answering this, the block–inclusion maps are a special case of a more general direct sum map $\oplus\colon BO(n) \times BO(m) \to BO(n+m)$, given by the precomposition

$$BO(n) = BO(n) \times * \xrightarrow{\mathrm{id} \times \mathrm{triv}} BO(n) \times BO(1) \xrightarrow{\oplus} BO(n+1).$$

The spaces $BGL_1 S^{n-1}$ enjoy a similar "collective monoid" structure, given by taking the fiberwise join two spherical bundles with a common base.

**Lemma 3.** *The fiberwise join is represented by maps*

$$BGL_1 S^{n-1} \times BGL_1 S^{m-1} \to BGL_1 S^{n+m-1},$$

> Split the left vertical arrow into two?

> Index it by $J^n_{\mathbb{R}} \times J^m_{\mathbb{R}}$, and the right vertical arrow by $J^{n+m}_{\mathbb{R}}$? -Danny

*and these maps commute with the block sum maps on the $BO(n)$ family:*

$$\begin{tikzcd}
BO(n) \times BO(m) \arrow[r] \arrow[d] & BO(n+m) \arrow[d] \\
BGL_1 S^{n-1} \times BGL_1 S^{m-1} \arrow[r] & BGL_1 S^{n+m-1}.
\end{tikzcd} \square$$

Again taking a cue from $K$–theory, we take the colimit as $n$ grows large.

**Corollary 1.** *There is a map of $H$–spaces $J_{\mathbb{R}} \colon BO \to BGL_1\mathbb{S}$ called the stable $J$–homomorphism.* □

Finally, we can ask about the compatibility of $T$ with all of this:

The right place to address dimension is here. $T$, as defined above, does not extend to a functor off of the system $\{BhAut(S^{n-1})\}$ unless you reduce each Thom complex by the appropriate dimension shift. So, you should define a stable Thom spectrum functor.

**Lemma 4.** *$T$ is monoidal: it carries external fiberwise joins to smash products of Thom spaces.* □

We are now prepared to define our spectrum $MO$. The unstable $J$–maps $J^n_{\mathbb{R}} \colon BO(n) \to BGL_1 S^{n-1}$ give Thom spaces $T(J^n_{\mathbb{R}})$, equipped with maps

$$\Sigma T(J^n_{\mathbb{R}}) = T(J^n_{\mathbb{R}} \oplus \mathrm{triv}) \to T(J^{n+1}_{\mathbb{R}}).$$

Setting $MO(n) = \Sigma^{-n}\Sigma^\infty T(J^n_{\mathbb{R}})$, we again assemble this data into a single object:

$$MO := \operatorname*{colim}_n MO(n) = \operatorname*{colim}_n \Sigma^{-n} T(J^n_{\mathbb{R}}).$$

Be careful about dimension here: you really mean a reduced tautological bundle, related to how $BO$ has only one connected component.

Fix this equation.

What's wrong with it?-danny

Question: You essentially defined $MO$ here by piecing together the $T(J^n_{\mathbb{R}})$'s. We should mention that another way to pack all this info together is to say $MO = T(J_{\mathbb{R}})$ -danny

There should be a Theorem here saying that we recover $MO$ as defined on the first day.

The spectrum $MO$ has several remarkable properties. The most basic such property is that it is a ring spectrum, and this follows immediately from $J_{\mathbb{R}}$ being a homomorphism of $H$–spaces (from Lemma 3). Much more excitingly, we can also deduce the presence of Thom isomorphisms just from the properties stated thus far. That $J_{\mathbb{R}}$ is a homomorphism means that the following square commutes:

$$\begin{tikzcd}
BO \times BO \arrow[r, "\sigma", "\simeq"'] \arrow[drr, bend right] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}} \times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

We have extended this square very slightly by a certain shearing map $\sigma$ defined by $\sigma(x, y) = (xy^{-1}, y)$. It's evident that $\sigma$ is a homotopy equivalence, since just as we can de-scale the first coordinate by $y$ we can re-scale by it. We can calculate directly the behavior of the long composite:

*$\sigma$ almost* shows up in giving a categorical definition of a $G$–torsor. I wish I understood this, but I always get tangled up.

Have you already mentioned that $BO$ is not just an $H$-space, but an $H$-group?

$$J_{\mathbb{R}} \circ \mu \circ \sigma(x, y) = J_{\mathbb{R}} \circ \mu(xy^{-1}, y) = J_{\mathbb{R}}(xy^{-1}y) = J_{\mathbb{R}}(x).$$

It follows that the second coordinate plays no role, and that the bundle classified by the long composite can be written as $J_{\mathbb{R}} \times 0$.[1] We are now in a position to see the Thom isomorphism:

I'm confused about the commutativity of this factorization with the rest of the diagram.-danny

**Lemma 5 (Thom isomorphism, universal example).** *As $MO$–modules,*

Is it clear that this is an equivalence of $MO$–modules? This should come from the $x$–factor being unmolested, right?

Is it furthermore clear that the cohomological version of this gives an action of $E^*X$ on $E^*T(\xi)$ by the "Thom diagonal"?

[1] This factorization does *not* commute with the rest of the diagram, just with the little triangle it forms.

$$MO \wedge MO \simeq MO \wedge \Sigma^\infty_+ BO.$$

*Proof.* Stringing together the naturality properties of the Thom functor outlined above, we can thus make the following calculation:

$$\begin{aligned}
T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}})) &\simeq T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}}) \circ \sigma) && \text{(homotopy invariance)} \\
&\simeq T(J_{\mathbb{R}} \times 0) && \text{(constructed lift)} \\
&\simeq T(J_{\mathbb{R}}) \wedge T(0) && \text{(monoidality)} \\
&\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(Example 1)} \\
T(J_{\mathbb{R}}) \wedge T(J_{\mathbb{R}}) &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^\infty_+ BO && \text{(monoidality)} \\
MO \wedge MO &\simeq MO \wedge \Sigma^\infty_+ BO. && \text{(definition of } MO\text{)} \; \square
\end{aligned}$$

From here, the general version of Thom's theorem follows quickly:

**Theorem 4 (Thom isomorphism).** *Let $\xi\colon X \to BO$ classify a vector bundle and let $\varphi\colon MO \to E$ be a map of ring spectra. Then there is an equivalence of $E$–modules*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X.$$

Added a little clarification, and gave what I think is a correct proof of the E case. -Krishanu

*Proof (Modifications to above proof).* To accommodate $X$ rather than $BO$ as the base, we redefine $\sigma\colon BO \times X \to BO \times X$ by

$$\sigma(x, y) = \sigma(x\xi(y)^{-1}, y).$$

Follow the same proof as before with the diagram

$$\begin{tikzcd}
BO \times X \arrow[r, "\sigma", "\simeq"'] \arrow[drrr, bend right] & BO \times X \arrow[r, "\xi", "\simeq"'] & BO \times BO \arrow[r, "\mu"] \arrow[d, "J_{\mathbb{R}} \times J_{\mathbb{R}}"] & BO \arrow[d, "J_{\mathbb{R}}"] \\
& & BGL_1\mathbb{S} \times BGL_1\mathbb{S} \arrow[r, "\mu"] & BGL_1\mathbb{S}.
\end{tikzcd}$$

This gives an equivalence $\theta_{MO}\colon MO \wedge T(\xi) \to MO \wedge \Sigma^\infty_+ X$. To introduce $E$, note that there is a diagram

$$\begin{tikzcd}
E \wedge T(\xi) \arrow[d, "\eta_{MO} \wedge \mathrm{id} \wedge \mathrm{id} = f"] & E \wedge \Sigma^\infty_+ X \arrow[d, "\eta_{MO} \wedge \mathrm{id} \wedge \mathrm{id}"] \\
MO \wedge E \wedge T(\xi) \arrow[r, "\theta_{MO} \wedge E"] \arrow[d, "(\mu \circ (\varphi \wedge \mathrm{id})) \wedge \mathrm{id} = g"] & MO \wedge E \wedge \Sigma^\infty_+ X \arrow[d, "(\mu \circ (\varphi \wedge \mathrm{id})) \wedge \mathrm{id} = h"] \\
E \wedge T(\xi) \arrow[r, "\theta_E"] & E \wedge \Sigma^\infty_+ X
\end{tikzcd}$$

The bottom arrow $\theta_E$ exists by applying the action map to both sides and pushing the map $\theta_{MO} \wedge E$ down. Since $\theta_{MO}$ is an equivalence, it has an inverse $\alpha_{MO}$. Therefore, the middle map has inverse $\alpha_{MO} \wedge E$, and we can similarly push this down to a map $\alpha_E$, which we now want to show is the inverse to $\theta_E$. From here it is a simple diagram chase: we have renamed three of the maps in the diagram to $f$, $g$, and $h$ for brevity. Noting that $g \circ f$ is the identity map because of the unit axiom, we conclude

$$\begin{aligned} g \circ f &\simeq g \circ (\alpha_{MO} \wedge E) \circ (\theta_{MO} \wedge E) \circ f \\ &\simeq \alpha_E \circ h \circ (\theta_{MO} \wedge E) \circ f && \text{(action map)} \\ &\simeq \alpha_E \circ \theta_E \circ g \circ f && \text{(action map)} \\ &\simeq \alpha_E \circ \theta_E. && \square \end{aligned}$$

At least hint that there's a converse to this Theorem, to be explored later.

*Example 3.* We'll close out today by using this to actually make a calculation. Recall from Example 2 that $T(\mathcal{L} \downarrow \mathbb{R}P^n) = \mathbb{R}P^{n+1}$. By killing all the homotopy elements in positive degrees, we can also see that the map $MO \to H\mathbb{F}_2$ is a ring map, so that we can apply the Thom isomorphism theorem to the mod–2 homology of Thom complexes coming from real vector bundles:

This requires some justification, like $MO$ being connective .

$$\begin{aligned} \pi_*(H\mathbb{F}_2 \wedge T(\mathcal{L} - 1)) &\cong \pi_*(H\mathbb{F}_2 \wedge T(0)) && \text{(Thom isomorphism)} \\ \pi_*(H\mathbb{F}_2 \wedge \Sigma^{-1}\Sigma^\infty \mathbb{R}P^{n+1}) &\cong \pi_*(H\mathbb{F}_2 \wedge \Sigma^\infty_+ \mathbb{R}P^n) && \text{(Example 2)} \\ \widetilde{H\mathbb{F}}_{2*+1} \mathbb{R}P^{n+1} &\cong H\mathbb{F}_{2*} \mathbb{R}P^n. && \text{(generalized homology)} \end{aligned}$$

This powers an induction that shows $H\mathbb{F}_{2*}\mathbb{R}P^\infty$ has a single class in every degree. The cohomology version of all this, together with the $H\mathbb{F}_2^*\mathbb{R}P^n$–module structure of $H\mathbb{F}_2^* T(\mathcal{L} - 1)$, also gives the ring structure:

Wouldn't hurt to expand on this.

$$H\mathbb{F}_2^* \mathbb{R}P^n = \mathbb{F}_2[x]/x^{n+1}.$$

This part is interesting. I just remembered that the Thom isomorphism theorem I know is actually about cohomology! What's the similar story for cohomology here? We should talk about this. -danny

## 1.2 Jan 29: Cohomology rings and affine schemes

Make sure you use $\mathbb{F}_2$ everywhere, rather than $\mathbb{Z}/2$.

An abbreviated summary of this semester is that we're going to put "Spec" in front of rings appearing in algebraic topology and see what happens. Before doing any algebraic topology, let me remind you what this means on the level of algebra. The core idea is to replace a ring $R$ by the functor it corepresents, Spec $R$. For any "test $\mathbb{F}_2$–algebra" $T$, we set

$$(\operatorname{Spec} R)(T) := \mathsf{Algebras}_{\mathbb{F}_2/}(R, T) \cong \mathsf{Schemes}_{/\mathbb{F}_2}(\operatorname{Spec} T, \operatorname{Spec} R).$$

More generally, we have the following definition:

**Definition 2.** An *affine* $\mathbb{F}_2$*–scheme* is a functor $X : \mathsf{Algebras}_{\mathbb{F}_2/} \to \mathsf{Sets}$ which is (noncanonically) isomorphic to $\operatorname{Spec} R$ for some $\mathbb{F}_2$–algebra $R$. Given such an isomorphism, we will refer to $\operatorname{Spec} R \to X$ as a *parameter* for $X$ and its inverse $X \to \operatorname{Spec} R$ as a *coordinate* for $X$.

**Lemma 6.** *There is an equivalence of categories*

$$\operatorname{Spec} : \mathsf{Algebras}^{\mathrm{op}}_{\mathbb{F}_2/} \to \mathsf{AffineSchemes}_{/\mathbb{F}_2}. \square$$

The centerpiece of thinking about rings in this way, for us and for now, is to translate between a presentation of $R$ as a quotient of a free algebra and a presentation of $(\operatorname{Spec} R)(T)$ as selecting tuples of elements in $T$ subject to certain conditions. Consider the following example:

*Example 4.* Set $R_1 = \mathbb{F}_2[x]$. Then

$$(\operatorname{Spec} R_1)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], T)$$

is determined by where $x$ is sent — i.e., this Hom–set is naturally isomorphic to $T$ itself. Consider also what happens when we impose a relation by passing to $R_2 = \mathbb{F}_2[x]/(x^{n+1})$. The value

$$(\operatorname{Spec} R_2)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x]/(x^{n+1}), T)$$

of the associated affine scheme is again determined by where $x$ is sent, but now $x$ can only be sent to elements which are nilpotent of order $n+1$. These schemes are both important enough that we give them special names:

$$\mathbb{A}^1 := \operatorname{Spec} \mathbb{F}_2[x], \qquad \mathbb{A}^{1,(n)} := \operatorname{Spec} \mathbb{F}_2[x]/(x^{n+1}).$$

The symbol "$\mathbb{A}^1$" is pronounced "the affine line" — reasonable, since the value $\mathbb{A}^1(T)$ is, indeed, a single $T$'s worth of points. Note that the quotient map $R_1 \to R_2$ induces an inclusion $\mathbb{A}^{1,(n)} \to \mathbb{A}^1$ and that $\mathbb{A}^{1,(0)}$ is a constant functor:

$$\mathbb{A}^{1,(0)}(T) = \{f : \mathbb{F}_2[x] \to T \mid f(x) = 0\}.$$

Accordingly, we pronounce "$\mathbb{A}^{1,(0)}$" as "the origin on the affine line" and "$\mathbb{A}^{1,(n)}$" as "the $(n+1)^{\mathrm{st}}$ order (nilpotent) neighborhood of the origin in the affine line".

We can also express in this language another common object arising from algebraic topology: the Hopf algebra, which appears when taking the mod–2 cohomology of an $H$–group. In addition to the usual cohomology, the extra pieces of data are those induced by the $H$–group multiplication, unit, and inversion maps, which on cohomology beget a diagonal map $\Delta$, an augmentation map $\varepsilon$, and an antipode $\chi$ respectively. Running through the axioms, one quickly checks the following:

**Lemma 7.** *For a Hopf $\mathbb{F}_2$–algebra $R$, the functor* $\operatorname{Spec} R$ *is naturally valued in groups. Such functors are called group schemes. Conversely, a choice of group structure on* $\operatorname{Spec} R$ *endows $R$ with the structure of a Hopf algebra.*

*Proof.* The functor $\operatorname{Spec}\colon \mathsf{Algebras}^{\mathrm{op}}_{\mathbb{F}_2/} \to \operatorname{Funct}(\mathsf{Algebras}_{\mathbb{F}_2/}, \mathsf{Sets})$ takes limits into limits. Since tensor products of $\mathbb{F}_2$-algebras compute pushouts in $\mathsf{Algebras}_{\mathbb{F}_2/}$, we see that Hopf algebras are simply cogroup objects in $\mathsf{Algebras}_{\mathbb{F}_2/}$. These remarks imply that Spec takes Hopf algebras into group objects in $\operatorname{Funct}(\mathsf{Algebras}_{\mathbb{F}_2/}, \mathsf{Sets})$. Now, for any small category $\mathcal{C}$, one has

$$\operatorname{Grp}(\operatorname{Funct}(\mathcal{C}, \mathsf{Sets})) \simeq \operatorname{Funct}(\mathcal{C}, \mathsf{Groups}).$$

The conclusion now follows from the fully faithfulness of Spec.

*Example 5.* The functor $\mathbb{A}^1$ introduced above is naturally valued in groups: since $\mathbb{A}^1(T) \cong T$, we can use the addition on $T$ to make it into an abelian group. When considering $\mathbb{A}^1$ with this group scheme structure, we notate it as $\mathbb{G}_a$. Applying the Yoneda lemma, one deduces the following formulas for the Hopf algebra structure maps:

$$\begin{aligned} \mathbb{G}_a \times \mathbb{G}_a &\xrightarrow{\mu} \mathbb{G}_a & x_1 + x_2 &\leftarrow\!\shortmid x, \\ \mathbb{G}_a &\xrightarrow{\chi} \mathbb{G}_a & -x &\leftarrow\!\shortmid x, \\ \operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \mathbb{G}_a & 0 &\leftarrow\!\shortmid x. \end{aligned}$$

I think you explained this differently in class, so I wanted to work out how to use the Yoneda lemma to deduce these. The idea is that we want a map, e.g., $\mu \in \operatorname{Nat}(\mathbb{A}^1 \times \mathbb{A}^1, \mathbb{A}^1)$. The functors involved are corepresented, so this is just $\operatorname{Nat}(h^{\mathbb{F}_2[x_1,x_2]}, h^{\mathbb{F}_2[x]})$. The Yoneda lemma says that this is the same as an algebra map $\mathbb{F}_2[x] \to \mathbb{F}_2[x_1, x_2]$, which is obtained by evaluating the natural transformation at the identity of $\mathbb{F}_2[x_1, x_2]$. The identity map of $\mathbb{F}_2[x_1, x_2]$ corresponds to the pair of elements $x_1, x_2$ in $\mathbb{F}_2[x_1, x_2]$, which gets sent to $x_1 + x_2$. So the algebra homomorphism we want sends $x$ to $x_1 + x_2$.

*Remark 1.* In fact, $\mathbb{A}^1$ is naturally valued in *rings*. It models the inverse functor to Spec in the equivalence of categories above, i.e., the elements of a ring $R$ always form a complete collection of $\widehat{\mathbb{A}}^1$–valued functions on some affine scheme $\operatorname{Spec} R$ .

We haven't defined $\widehat{\mathbb{A}}^1$ at this point yet -danny

re:danny. Should this just be $\mathbb{A}^1$?

*Example 6.* We define the *multiplicative group scheme* by

Do you want this to be a $\mathbb{Z}$–algebra? Ditto with $\mathbb{A}^1$ and $\mathbb{G}_a$?

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_2[x, y]/(xy - 1).$$

Its value $\mathbb{G}_m(T)$ on a test algebra $T$ is the set of pairs $(x, y)$ such that $y$ is a multiplicative inverse to $x$, and hence $\mathbb{G}_m$ is valued in groups. Applying the Yoneda lemma, we deduce the following formulas for the Hopf algebra structure maps:

$$
\begin{aligned}
\mathbb{G}_m \times \mathbb{G}_m &\xrightarrow{\mu} \mathbb{G}_m & x_1 \otimes x_2 &\leftarrow\!\shortmid x \\
& & y_1 \otimes y_2 &\leftarrow\!\shortmid y, \\
\mathbb{G}_m &\xrightarrow{\chi} \mathbb{G}_m & (y, x) &\leftarrow\!\shortmid (x, y), \\
\operatorname{Spec} R &\xrightarrow{\eta} \mathbb{G}_m & 1 &\leftarrow\!\shortmid x, y.
\end{aligned}
$$

*Remark 2.* As presented above, the multiplicative group comes with a natural inclusion $\mathbb{G}_m \to \mathbb{A}^2$. Specifically, the subset $\mathbb{G}_m \subseteq \mathbb{A}^2$ consists of pairs $(x, y)$ in the graph of the hyperbola $y = 1/x$. However, the element $x$ also gives an $\mathbb{A}^1$–valued function $x \colon \mathbb{G}_m \to \mathbb{A}^1$, and because multiplicative inverses in a ring are unique, we see that this map too is an inclusion. These two inclusions have rather different properties relative to their ambient spaces, and we'll think harder about these essential differences later on.

*Example 7 (cf. Example 28).* The following example shows that it is a bad idea to think of affine group schemes as a schemeified version of linear lie groups. Define the group scheme $\alpha_2$ to be $\operatorname{Spec}(\mathbb{F}_2[x]/(x^2))$ with group scheme structure given by

$$
\begin{aligned}
\alpha_2 \times \alpha_2 &\xrightarrow{\mu} \alpha_2 & x_1 + x_2 &\leftarrow\!\shortmid x, \\
\alpha_2 &\xrightarrow{\chi} \alpha_2 & -x &\leftarrow\!\shortmid x, \\
\operatorname{Spec} \mathbb{F}_2 &\xrightarrow{\eta} \alpha_2 & 0 &\leftarrow\!\shortmid x.
\end{aligned}
$$

All of these can be proven using Dieudonné theory, which we can write out and stick in a reference to Chapter 4.

This group scheme has several interesting properties:

You introduced this notation in class, but it hasn't been defined in this document yet. See example 1.2.12.

1. $\alpha_2$ has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$but is not isomorphic to it. The easiest way to see this is that $\operatorname{Hom}(\mu_2, \mu_2) = \mathbb{Z}/2\mathbb{Z}$but $\operatorname{Hom}(\alpha_2, \mu_2) = \alpha_2$ (these homs are in the category of affine group schemes and give out an affine group scheme).

How is $\mathbb{Z}/2\mathbb{Z}$ an affine group scheme? Is it just $\operatorname{Spec}(\mathbb{F}_2)$?

for jhfung: work this out

2. There is no commutative group scheme $G$ of rank four such that $\alpha_2 = G[2]$.
3. If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence $0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0$. However, this short exact sequence doesn't split (even after making a base change).
4. The subgroups of $\alpha_2 \times \alpha_2$ of order two are parameterized by $\mathbb{P}^1$. That is to say. if $R$ is an $\mathbb{F}_2$-algebra, then the subgroup schemes of $(\alpha_2 \times \alpha_2)_R$ of order two defined over $R$ are parameterized by $\mathbb{P}^1(R)$.

Additionally, the colimit of the sets $\operatorname{colim}_{n \to \infty} \mathbb{A}^{1,(n)}(T)$ is of use in algebra: it is the collection of nilpotent elements in $T$. These kinds of conditions which are "unbounded in $n$" appear frequently enough that we are moved to give these functors a name too:

How are finite schemes characterized in the functor of points perspective?

**Definition 3.** An *affine formal scheme* is an ind-system of finiteaffine schemes. The morphisms between such schemes are computed by

$$\mathsf{FormalSchemes}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\beta \operatorname{colim}_\alpha \mathsf{Schemes}(X_\alpha, Y_\beta).$$

Several people had questions about the utility of this. Does it just add certain colimits to the category of *finite* affine schemes?

Question: can we reverse the order of taking lim and colim here?-danny

Maybe it would be good at some point to define what Spf means because it's not actually stated explicitly anywhere. AY

*Example 8.* The individual schemes $\mathbb{A}^{1,(n)}$ do not support group structures. After all, the sum of two elements which are nilpotent of order $n+1$ can only be guaranteed to be nilpotent of order $2n+1$. It follows that the entire ind-system $\{\mathbb{A}^{1,(n)}\} =: \widehat{\mathbb{A}}^1$ supports a group structure, even though none of its constituent pieces do. We call such an object a *formal group scheme*, and this particular formal group scheme we denote by $\widehat{\mathbb{G}}_a$.

*Example 9.* Similarly, one can define the scheme $\mathbb{G}_m[n]$ of elements of unipotent order $n$:

$$\mathbb{G}_m[n] = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, x^n-1)} \subseteq \mathbb{G}_m.$$

These *are* all group schemes, but there is a second filtration along the lines of the one considered above:

$$\mathbb{G}_m^{(n)} = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy-1, (x-1)^n)}.$$

These schemes are only occasionally group schemes — specifically, $\mathbb{G}_m^{(2^n)}$ is a group scheme, in which case $\mathbb{G}_m^{(2^n)} \cong \mathbb{G}_m[2^n]$ over $\mathbb{F}_2$. This gives an ind-equivalence between these two subsystems, but $\{\mathbb{G}_m[2^n]\}$ is *not* cofinal in $\{\mathbb{G}_m[n]\}$, and the equivalence does not extend across the larger system.

Mauro was interested in the relationship of this to the punctured formal scheme $\mathbb{F}_p((q))$.

The reader invitation is as far as I can tell is either incoherent or impossible due to the condition that the maps making up formal schemes need to be infinitesimal thickenings. I think that correcting it goes as follows: requiring the inverse system to be of infintesimal thickenings means that the only $m$ such that $\mathbb{G}_m[m]$ is an infinitesimal thickening of the identity are the $m$ of the form $p^n$ which makes the result "obvious" in some sense. EK

Let's now consider the example that we closed with last time, where we calculated $H\mathbb{F}_2^*(\mathbb{RP}^n) = \mathbb{F}_2[x]/(x^{n+1})$. Putting "Spec" in front of this, we could reinterpret this calculation as

$$\operatorname{Spec} H\mathbb{F}_2^*(\mathbb{RP}^n) \cong \mathbb{A}^{1,(n)}.$$

This is such a useful thing to do that we will give it a notation all of its own:

**Definition 4.** Let $X$ be a finite cell complex, so that $H\mathbb{F}_2^*(X)$ is a ring which is finite–dimensional as an $\mathbb{F}_2$–vector space. We will write

$$X_{H\mathbb{F}_2} = \operatorname{Spec} H\mathbb{F}_2^* X$$

for the corresponding finite affine scheme.

The notion of the punctured disk is also wonky in the category of formal schemes. In particular, one has that $\mathrm{Spf}(\mathbb{F}_2[[t]])$ has only one point in the underlying topological space. This is part of why formal schemes aren't just schemes where the structure sheaf is of topological rings instead of rings: the points correspond to open ideals. The "correct" category to take generic fibers of formal schemes is adic spaces, but that is not a discussion that is worth going into. EK

*Example 10.* Putting together the discussions from this time and last time, in the new notation we have calculated

$$\mathbb{RP}^n_{H\mathbb{F}_2} \cong \mathbb{A}^{1,(n)}.$$

So far, this example just restates things we knew in a mildly different language. Our driving goal for the remainder of today and for tomorrow is to incorporate as much information as we have about these cohomology rings $H\mathbb{F}_2^*(\mathbb{RP}^n)$ into this description, which will result in us giving a more "precise" name for this object. Along the way, we will discover why $X$ had to be a *finite* complex and how to think about more general $X$. For now, though, let's content ourselves with investigating the Hopf algebra structure on $H\mathbb{F}_2^*\mathbb{RP}^\infty$.

*Example 11.* Recall that $\mathbb{RP}^\infty$ is an $H$–space in two equivalent ways:

1. There is an identification $\mathbb{RP}^\infty \simeq K(\mathbb{Z}/2, 1)$, and the $H$–space structure is induced by the sum on cohomology.
2. There is an identification $\mathbb{RP}^\infty \simeq BO(1)$, and the $H$–space structure is induced by the tensor product of real line bundles.

Later on, you need that there's a map on skeleta $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{m+n}$. This is made apparent if you inserted another characterization of the $H$-space structure as the one treating $\mathbb{RP}^\infty$ as the monic polynomials (of all degrees) over $\mathbb{R}$, and then the map is given by multiplication. AY

In either case, this induces a Hopf algebra diagonal

$$H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{RP}^\infty$$

which we would like to analyze. This map is determined by where it sends the class $x$, and because it must respect gradings it must be of the form $\Delta x = ax_1 + bx_2$ for some constants $a, b \in \mathbb{F}_2$. Furthermore, because it belongs to a Hopf algebra structure, it must satisfy the unitality axiom

$$\begin{tikzcd} H\mathbb{F}_2^*\mathbb{RP}^\infty & H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \arrow[l, "\begin{pmatrix}\varepsilon\otimes\mathrm{id}\\ \mathrm{id}\otimes\varepsilon\end{pmatrix}"'] & H\mathbb{F}_2^*\mathbb{RP}^\infty . \arrow[l, "\Delta"'] \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd}$$

and hence it takes the form

$$\Delta(x) = x_1 + x_2.$$

Noticing that this is exactly the diagonal map in Example 5, we tentatively identify "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" with the additive group. This is extremely suggestive but does not take into account the fact that $\mathbb{RP}^\infty$ is an infinite complex, so we haven't allowed ourselves to write "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" just yet. In light of the above discussion, we have left a very particular point open: it's not clear if we should use the name "$\mathbb{G}_a$" or "$\widehat{\mathbb{G}}_a$". We will straighten this out tomorrow.

## 1.3 Feb 1: The Steenrod algebra

We left off yesterday with an ominous finiteness condition in our definition of $X_{H\mathbb{F}_2}$, and we produced a pair of reasonable guesses as to what "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" could mean. It will turn out that we can answer which of the two guesses is reasonable by rigidifying the target category somewhat. Here are the extra structures we will work toward incorporating:

1. Cohomology rings are *graded*, and maps of spaces respect this grading.
2. Cohomology rings receive an action of the Steenrod algebra, and maps of spaces respect this action.
3. Both of these are complicated further when taking the cohomology of an infinite complex.
4. (Cohomology rings for more elaborate cohomology theories are only skew-commutative, but "Spec" requires a commutative input.)

Today we will fix all these deficiencies of $X_{H\mathbb{F}_2}$ except for #4, which doesn't matter with mod–2 coefficients but which will be something of a bugbear throughout the rest of the semester.

Let's begin by considering the grading on $H\mathbb{F}_2^* X$. In algebraic geometry, the following standard construction is used to track gradings:

**Definition 5 ([42, Definition 2.95]).** A $\mathbb{Z}$–*grading* on a ring $R$ is a system of additive subgroups $R_k$ of $R$ satisfying $R = \bigoplus_k R_k$, $1 \in R_0$, and $R_j R_k \subseteq R_{j+k}$. Additionally, a map $f\colon R \to S$ of graded rings is said to *respect the grading* if $f(R_k) \subseteq S_k$.

Maybe "$\mathbb{Z}$–filtering" is more appropriate.

**Lemma 8 ([42, Proposition 2.96]).** *A graded ring $R$ is equivalent data to an affine scheme* $\operatorname{Spec} R$ *with an action by $\mathbb{G}_m$. Additionally, a map $R \to S$ is homogeneous exactly when the induced map* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is $\mathbb{G}_m$–equivariant.*

*Proof.* A $\mathbb{G}_m$–action on $\operatorname{Spec} R$ is equivalent data to a coaction map

$$\alpha^* : R \to R \otimes \mathbb{F}_2[x^{\pm}].$$

Define $R_k$ to be those points in $r$ satisfying $\alpha^*(r) = r \otimes x^k$. It is clear that we have $1 \in R_0$ and that $R_j R_k \subseteq R_{j+k}$. To see that $R = \bigoplus_k R_k$, note that every tensor can be written as a sum of pure tensors. Conversely, given a graded ring $R$, define the coaction map on $R_k$ by

$$(r_k \in R_k) \mapsto x^k r_k$$

and extend linearly.

This notion from algebraic geometry is somewhat different from what we are used to in algebraic topology, as it is designed to deal with things like polynomial rings (where the difference of two polynomials can lie in lower degree), but in classical algebraic topology we only ever encounter sums of terms with homogeneous degree. We can modify our perspective very slightly to arrive at the algebraic geometers': replace $H\mathbb{F}_2$ by the periodified spectrum

$$H\mathbb{F}_2 P = \bigvee_{j=-\infty}^{\infty} \Sigma^j H\mathbb{F}_2.$$

This spectrum has the property that $H\mathbb{F}_2 P^0(X)$ is isomorphic to $H\mathbb{F}_2^*(X)$ as ungraded rings, but now we can make sense of the sum of two classes which used to live in different $H\mathbb{F}_2$–degrees. At this point we can manually craft the desired coaction map $\alpha^*$ so that we are in the situation of Lemma 8, but we will shortly find that algebraic topology gifts us with it on its own.

You can do a better job of describing where the Steenrod coaction comes from, rather than resting on duality. For instance, you could at least justify *why the Steenrod algebra is a Hopf algebra.* Already, that's kind of unclear.

Our route to finding this internally occurring $\alpha^*$ is by turning to the next supplementary structure: the action of the Steenrod algebra. Naively approached, this does not fit into the framework we've been sketching so far: the Steenrod algebra is a *noncommutative* algebra, and so the action map

$$\mathcal{A}^* \otimes H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X$$

will be difficult to squeeze into any kind of algebro-geometric framework. Milnor was the first person to see a way around this, with two crucial observations. First, the linear-algebraic dual of the Steenrod algebra $\mathcal{A}_*$ is a commutative ring, since the Cartan formula expressing the diagonal on $\mathcal{A}^*$ is evidently symmetric:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y).$$

Reminder for jhfung: $\mathcal{A}^*$ is a Hopf algebra with comultiplication $\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$, so $\mathcal{A}_*$ is also a Hopf algebra.

I see that if $X$ is a finite complex, it has a Spanier–Whitehead dual, but I don't see how to use this. Is $\lambda^*$ not just the composition $H\mathbb{F}_2^* X = \mathbb{F}_2 \otimes H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X \otimes \mathcal{A}^* \otimes \mathcal{A}_* \xrightarrow{\lambda \otimes 1} H\mathbb{F}_2^* X \otimes \mathcal{A}_*$?

I think the point is you're using a duality-type thing for $\mathcal{A}_*$ and $\mathcal{A}^*$. Unfortunately, infinite dimensional vector spaces are not dualizable, so strictly what you've written doesn't quite work (for example, $\mathcal{A}^* \otimes \mathcal{A}_*$ doesn't receive a map from $\mathbb{F}_2$). However, on finite complexes, you only get a finite dimensional part of the Steenrod algebra acting non-trivially, so you can do the duality thing. AY

Second, if $X$ is a *finite* complex, then tinkering with Spanier–Whitehead duality gives rise to a coaction map

$$\lambda^* : H\mathbb{F}_2^* X \to H\mathbb{F}_2^* X \otimes \mathcal{A}_*,$$

which we will then re-interpret as an action map

$$\alpha : \operatorname{Spec} \mathcal{A}_* \times X_{H\mathbb{F}_2} \to X_{H\mathbb{F}_2}.$$

Milnor works out the Hopf algebra structure of $\mathcal{A}_*$, by defining elements $\xi_j \in \mathcal{A}_*$ dual to $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} \in \mathcal{A}^*$. Taking $X = \mathbb{RP}^n$ and $x \in H\mathbb{F}_2^1(\mathbb{RP}^n)$ the generator, then since $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} x = x^{2^j}$ he deduces the formula

$$\lambda^*(x) = \sum_{j=0}^{\lfloor \log_2 n \rfloor} x^{2^j} \otimes \xi_j.$$

Notice that we can take the limit $n \to \infty$ to get a well-defined infinite sum, provided we permit ourselves to make sense of such a thing. He then makes the following calculation, stable in $n$:

$$(\lambda^* \otimes \mathrm{id}) \circ \lambda^*(x) = (\mathrm{id} \otimes \Delta) \circ \lambda^*(x) \qquad \text{(coassociativity)}$$

$$(\lambda^* \otimes \mathrm{id}) \left( \sum_{j=0}^{\infty} x^{2^j} \otimes \xi_j \right) =$$

$$\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} x^{2^i} \otimes \xi_i \right)^{2^j} \otimes \xi_j = \qquad \text{(ring homomorphism)}$$

$$\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j = \qquad \text{(characteristic 2).}$$

Then, turning to the right-hand side:

$$\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j = (\mathrm{id} \otimes \Delta) \left( \sum_{m=0}^{\infty} x^{2^m} \otimes \xi_m \right)$$

$$\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j} \right) \otimes \xi_j = \sum_{m=0}^{\infty} x^{2^m} \otimes \Delta(\xi_m),$$

from which it follows that

$$\Delta \xi_m = \sum_{i+j=m} \xi_i^{2^j} \otimes \xi_j.$$

Finally, Milnor shows that this is the complete story:

**Theorem 5 (Milnor).** $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j, \ldots]$.

**Cite me:** Give a reference in Milnor's paper.

Mosher and Tangora also has a proof of this on pp. 50–52.

*Proof (Flippant proof).* There is at least a map $\mathbb{F}_2[\xi_1, \xi_2, \ldots] \to \mathcal{A}_*$ given by the definition of the elements $\xi_j$ above. This map is injective, since these elements are distinguished by how they coact on $H\mathbb{F}_2^* \mathbb{R}\mathrm{P}^\infty$. Then, since these rings are of graded finite type, Milnor can conclude his argument by counting how many elements he has produced, comparing against how many Adem and Cartan found (which we will do ourselves in Lecture 4.2), and noting that he has exactly enough.

We are now in a position to uncover the desired map $\alpha^*$ desired earlier. Suppose that we were interested in re-telling Milnor's story with $H\mathbb{F}_2 P$ in place of $H\mathbb{F}_2$. The dual Steenrod algebra is defined topologically by

$$\mathcal{A}_* := \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2),$$

which we replace by

$$\mathcal{A}P_0 := \pi_0(H\mathbb{F}_2P \wedge H\mathbb{F}_2P) = H\mathbb{F}_2P_0(H\mathbb{F}_2P) = \mathcal{A}_*[\xi_0^{\pm}].$$

I think it's a good idea to at least mention that $\xi_0$ keeps track of the grading, as you explained in class. Also, I see from the wedge axiom that $H\mathbb{F}_2P_0(H\mathbb{F}_2P) \cong \prod_j \mathcal{A}_*$, but how do you get the multiplicative structure of $\xi_0$?

**Lemma 9 ([9, Formula 3.4, Remark 3.14]).** *Projecting to the quotient Hopf algebra $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ gives exactly the coaction map $\alpha^*$.* □

Include a proof of this. It doesn't seem obvious — Danny and I spent a while talking about it and couldn't get all our algebra straight. It has something to do with trading the invertible element across the smash product in $H\mathbb{F}_2P \wedge H\mathbb{F}_2P \wedge_{H\mathbb{F}_2P} H\mathbb{F}_2P \wedge X$, and also with multiplicativity of the action: $\psi(vx) = \psi(v)\psi(x)$ with $v$ the homogenizing generator, or something.

The point of this lemma is to say that earlier we traded saying "graded map" for "$\mathbb{G}_m$–equivariant map", which did not seem like a substantial gain. Now we see that saying "Steenrod–equivariant map" already includes saying "graded map", which is a gain in brevity. Try to make this clearer.

To study the rest of $\mathcal{A}P_0$ in terms of algebraic geometry, we need only identify what the series $\lambda^*(x)$ embodies. Note that this necessarily involves some creativity, and the only justification we can supply will be moral, borne out over time, as our narrative encompasses more and more phenomena. With that caveat in mind, here is one such description. Recall the map induced by the $H$–space multiplication

$$H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty \otimes H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty \leftarrow H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^\infty.$$

Taking a colimit over finite complexes, we produce an coaction of $\mathcal{A}_*$, and since the map above comes from a map of spaces, it is equivariant for the coaction. Since the action on the left is diagonal, we deduce the formula

$$\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2).$$

**Lemma 10.** *The series $\lambda^*(x) = \sum_{j=0}^\infty x^{2^j} \otimes \xi_j$ is the universal example of a series satisfying $\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2)$. The set $(\operatorname{Spec} \mathcal{A}P_0)(T)$ is identified with the set of power series $f$ with coefficients in the $\mathbb{F}_2$–algebra $T$ satisfying*

$$f(x_1 + x_2) = f(x_1) + f(x_2). \square$$

Can you explain this more? In which sense is $\lambda^*(x)$ universal? How do I get from the first statement to the second?

We close our discussion by codifying what Milnor did when he stabilized against $n$. Each $\mathbb{R}\mathrm{P}^n_{H\mathbb{F}_2}$ is a finite affine scheme, and to make sense of the object $\mathbb{R}\mathrm{P}^\infty_{H\mathbb{F}_2}$ Milnor's technique was to consider the ind-system $\{\mathbb{R}\mathrm{P}^n_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite affine schemes. We will record this as our technique to handle general infinite complexes:

**Definition 6.** When $X$ is an infinite complex, filter it by its subskeleta $X^{(n)}$ and define $X_{H\mathbb{F}_2}$ to be the ind-system $\{X^{(n)}_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite schemes.[2]

This choice collapses our uncertainty about the topological example from last time:

*Example 12 (cf. Examples 8 and 11).* Write $\widehat{\mathbb{G}}_a$ for the ind-system $\mathbb{A}^{1,(n)}$ with the group scheme structure given in Example 11. That this group scheme structure filters in this way is a simultaneous reflection of two facts:

[2] More canonically, when $X$ is "compactly generated", it can be written as the colimit of its compact subspaces $X^{(\alpha)}$, and we define $X_{H\mathbb{F}_2}$ using the ind-system $\{X^{(\alpha)}_{H\mathbb{F}_2}\}_\alpha$.

1. Algebraic: The set $\widehat{\mathbb{G}}_a(T)$ consists of all nilpotent elements in $T$. The sum of two nilpotent elements of orders $n$ and $m$ is guaranteed to itself be nilpotent with order at most $n+m$.
2. Topological: There is a factorization of the multiplication map on $\mathbb{RP}^\infty$ as $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{n+m}$ purely for dimensional reasons.

Is there an off-by-one here?

As group schemes, we have thus calculated

$$\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a.$$

*Example 13.* Additionally, this convention comports with our analysis of $\operatorname{Spec} \mathcal{A}P_0$. Note that the following morphism sets are very different:

We could be clearer about the *two* things that are happening here. First, we are comparing informal schemes to formal schemes (or polynomials to power series). Second, we are comparing external Hom–objects to internal Hom–objects (or sets of power series over $\mathbb{F}_2$ to schemes of power series over $\mathbb{F}_2$–algebras).

$$\begin{aligned} \mathsf{GroupSchemes}_{/\mathbb{F}_2}(\mathbb{G}_a, \mathbb{G}_a) &\cong \mathsf{HopfAlgebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], \mathbb{F}_2[x]) \\ \mathsf{FormalGroups}_{/\mathbb{F}_2}(\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a) &\cong \mathsf{HopfProAlgebras}_{\mathbb{F}_2/}(\mathbb{F}_2[\![x]\!], \mathbb{F}_2[\![x]\!]). \end{aligned}$$

The former is populated by polynomials satisfying the homomorphism condition and the latter is populated by *power series* satisfying the same, which form a much larger set. Since our description of $\operatorname{Spec} \mathcal{A}P_0$ involves power series, we will favor the latter interpretation. To record this, first amp up this description of maps to a scheme of its own:

$$\underline{\mathsf{FormalSchemes}}(X, Y)(T) = \left\{ (u, f) \,\middle|\, \begin{array}{c} u : \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2, \\ f : u^*X \to u^*Y \end{array} \right\}$$

and conclude that the correct name for $\operatorname{Spec} \mathcal{A}P_0$ is

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

Finally, the formula $\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a$ is meant to point out that this language of formal schemes has an extremely good compression ratio — you can fit a lot of information into a very tiny space. This formula simultaneously encodes the cohomology ring of $\mathbb{RP}^\infty$ as the formal scheme, its diagonal as the group scheme structure, and the coaction of the dual Steenrod algebra by the identification with $\underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a$.

Include a recursive formula for the antipode map, coming from power series inversion.

## 1.4 Feb 3: Hopf algebra cohomology

This section is written gradedly and probably shouldn't be, for consistency. (In fact, this is the cause of some of the confusion about the $\mathbb{G}_m$–action used tomorrow to separate out the homotopy degrees...)

Today we'll focus on an important classical tool: the Adams spectral sequence. We're going to study this in greater earnest later on, so I will avoid giving a satisfying construction today. But, even without a construction, it's

It's not clear to me that this introduction should be written in terms of cohomology rather than homology. It's true that yesterday we were talking about cohomology, but it's also true that the spectral sequence we're going to build takes in homology. (More generally, contexts take in homology. I still find this a little puzzling, that Strickland's formal schemes don't seem to live in the descent picture.)

**Cite me:** I first saw this presentation from Matt Ando. He must have learned it from someone. I'd like to know who to attribute this to.

instructive to see how such a thing comes about. Begin by considering the following three self-maps of the stable sphere:

$$\mathbb{S}^0 \xrightarrow{0} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{1} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{2} \mathbb{S}^0.$$

If we apply mod–2 cohomology to each line, the induced maps are

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2, \qquad \mathbb{F}_2 \xleftarrow{\mathrm{id}} \mathbb{F}_2, \qquad \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2.$$

We see that mod–2 homology can immediately distinguish between the null map and the identity map just by its behavior on morphisms, but it can't so distinguish between the null map and the multiplication-by-2 map. To try to distinguish these two, we use the only other tool available to us: cohomology theories send cofiber sequences to long exact sequences, and moreover the data of a map $f$ and the data of the inclusion map $\mathbb{S}^0 \to C(f)$ into its cone are equivalent in the stable category. So, we trade our maps 0 and 2 for the following cofiber sequences:

$$\mathbb{S}^0 \longrightarrow C(0) \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^0 \longrightarrow C(2) \longrightarrow \mathbb{S}^1.$$

Applying cohomology, these again appear to be the same:

$$\begin{tikzcd}
{[1]} & & \bullet & \bullet \arrow[l] & & \bullet & \bullet \arrow[l] \\
{[0]} & \bullet & \bullet \arrow[l] & & \bullet & \bullet \arrow[l] &
\end{tikzcd}$$

$$H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^*C(0) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1, \qquad H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^*C(2) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1,$$

It would be nice if the dots aligned directly beneath the spaces in the cofiber sequences above.

where we have drawn a "$\bullet$" for a generator of an $\mathbb{F}_2$–vector space, graded vertically, and arrows indicating the behavior of each map. However, if we enrich our picture with the data we discussed last time, we can finally see the difference. Recall the topological equivalences

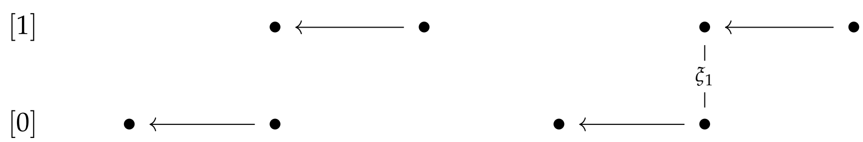
$$C(0) \simeq \mathbb{S}^0 \vee \mathbb{S}^1, \quad C(2) \simeq \Sigma^{-1}\mathbb{R}P^2.$$

In the two cases, the coaction map $\lambda^*$ is given by

$$\begin{aligned}
&\lambda^* : H\mathbb{F}_2^*C(0) \to H\mathbb{F}_2^*C(0) \otimes \mathcal{A}_* && \lambda^* : H\mathbb{F}_2^*C(2) \to H\mathbb{F}_2^*C(2) \otimes \mathcal{A}_* \\
&\qquad \lambda^* : e_0 \mapsto e_0 \otimes 1 && \qquad \lambda^* : e_0 \mapsto e_0 \otimes 1 + e_1 \otimes \xi_1 \\
&\qquad \lambda^* : e_1 \mapsto e_1 \otimes 1, && \qquad \lambda^* : e_1 \mapsto e_1 \otimes 1.
\end{aligned}$$

Is it possible to work out the coaction on the cones without first identifying their homotopy types?

We draw this into the diagram as




$$H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^*C(0) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1, \qquad H\mathbb{F}_2^*\mathbb{S}^0 \leftarrow H\mathbb{F}_2^*C(2) \leftarrow H\mathbb{F}_2^*\mathbb{S}^1,$$

where the vertical line indicates the nontrivial coaction involving $\xi_1$. We can now see what trading maps for cofiber sequences has bought us: mod–2 cohomology can distinguish the defining sequences for $C(0)$ and $C(2)$ by considering their induced extensions of comodules over $\mathcal{A}_*$. The Adams spectral sequence bundles this thought process into a single machine:

Can this be phrased so as to indicate how this works for longer extensions? I've never tried to think about even what happens for $C(4)$.

**Theorem 6.** *There is a convergent spectral sequence of signature*

$$\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow (\pi_*\mathbb{S}^0)^\wedge_2. \square$$

**Cite me:** Cite this somehow, or at least put a forward reference in.

In effect, this asserts that the above process is *exhaustive*: every element of $(\pi_*\mathbb{S}^0)^\wedge_2$ can be detected and distinguished by some representative class of extensions of comodules for the dual Steenrod algebra. Mildly more generally, if $X$ is a bounded-below spectrum, then there is even a spectral sequence of signature

Mention that there are homological and cohomological $\mathbb{F}_2$–Adams spectral sequences.

$$\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X) \Rightarrow \pi_*X^\wedge_2.$$

Here is where we could divert to talking about the construction of the Adams spectral sequence, but it will fit more nicely into a story later on. Thus, for now we will leave this task for Lecture 3.1. Before moving on, we will record the following utility lemma about the Adams spectral sequence. It is believable based on the above discussion, and we will need to use before we get around to examining the guts of the spectral sequence.

**Lemma 11.** *The 0–line of the Adams spectral sequence contains those elements visible to the Hurewicz homomorphism.* $\square$

This feels sloppily stated.

Today we will focus on the algebraic input $\operatorname{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$, which will require us to grapple with the homological algebra of comodules for a Hopf algebra. To begin, it's both reassuring and instructive to see that homological algebra can, in fact, be done with comodules. In the usual development of homological algebra for modules, the key observations are the existence of projective and injective modules, and there is something similar here.

*Remark 3.* Much of the results below do not rely on working with a Hopf algebra over the field $k = \mathbb{F}_2$. In fact, $k$ can usually be taken to be a ring rather than a field.

**Cite me:** Cite these. A1.1-2 in Ravenel are relevant.

**Lemma 12.** *Let $A$ be a Hopf $k$–algebra, let $M$ be an $A$–comodule, and let $N$ be a $k$–module. There is a cofree adjunction:*

$$\mathsf{Comodules}_A(M, N \otimes_k A) \cong \mathsf{Modules}_k(M, N),$$

*where $N \otimes_k A$ is given the structure of an $A$–comodule by the coaction map*

$$N \otimes_k A \xrightarrow{\mathrm{id} \otimes \Delta} N \otimes_k (A \otimes_k A) = (N \otimes_k A) \otimes_k A.$$

*Proof.* Given a map $f \colon M \to N$ of $k$–modules, we can build the composite

$$M \xrightarrow{\psi_M} M \otimes_k A \xrightarrow{f \otimes \mathrm{id}_A} N \otimes_k A.$$

Alternatively, given a map $g \colon M \to N \otimes_k A$ of $A$–comodules, we build the composite

$$M \xrightarrow{g} N \otimes_k A \xrightarrow{\mathrm{id}_N \otimes \varepsilon} N \otimes_k k = N. \square$$

**Corollary 2.** *The category $\mathsf{Comodules}_A$ has enough injectives. Namely, if $M$ is an $A$–comodule and $M \to I$ is an inclusion of $k$–modules into an injective $k$–module $I$, then $M \to I \otimes_k A$ is an injective $A$–comodule under $M$.* □

*Remark 4.* In our case, $M$ itself is always $k$–injective, so there's already an injective map $\psi_M : M \to M \otimes A$: the coaction map. The assertion that this map is coassociative is identical to saying that it is a map of comodules.

Satisfied that "Ext" at least makes sense, we're free to chase more conceptual pursuits. Recall from algebraic geometry that a module $M$ over a ring $R$ gives rise to quasi-coherent sheaf $\widetilde{M}$ over $\operatorname{Spec} R$. We give a definition that fits with our functorial perspective:

Can this definition be motivated? Is there an easily graspable sense in which it looks like a bundle?

**Definition 7.** A presheaf (of modules) over a scheme $X$ is an assignment of maps $\mathcal{F} \colon X(T) \to \mathsf{Modules}_T$, functorially in $T$. Such a presheaf is said to be *quasicoherent* when a map $\operatorname{Spec} S \to \operatorname{Spec} T \to X$ induces a natural isomorphism $\mathcal{F}(T) \otimes_T S \cong \mathcal{F}(S)$.

Straighten this out, using fibered categories?

Mention how the usual definition of a sheaf of $\mathcal{O}_X$–modules gives rise to such a thing: global sections over $\operatorname{Spec} T$.

There is a todo here that I can't get to compile. See the source.

**Lemma 13.** *An $R$–module $M$ gives rise to a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ by the rule $(\operatorname{Spec} T \to \operatorname{Spec} R) \mapsto M \otimes_R T$. Conversely, every quasicoherent sheaf over an affine scheme arises in this way.* □

**Cite me:** Surely this is in Neil's FSFG somewhere.

Jay was frustrated with which adjoint I put on top (and perhaps which went on which side). Apparently there's some convention, which I should look up and obey.

**Definition 8.** A map $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ induces maps $f^* \dashv f_*$ of module sheaf categories, which on the level of quasi-coherent sheaves is given by

$$\begin{tikzcd}
\mathsf{QCoh}_{\operatorname{Spec} R} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \mathsf{QCoh}_{\operatorname{Spec} S} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\
\mathsf{Modules}_R \arrow[r, shift left, "M \mapsto M \otimes_R S"] & \mathsf{Modules}_S. \arrow[l, shift left, "N \leftarrow\!\shortmid N"]
\end{tikzcd}$$

The usual formula for the sheaf cohomology of a sheaf $\mathcal{F}$ over an $S$–scheme $X$ with structure map $\pi\colon X \to S$ is given by $\operatorname{Ext}(\mathcal{O}_S, \pi_*\mathcal{F})$ which is, indeed, vaguely reminiscent of the formula we were considering above as input to the Adams spectral sequence. Experience in algebraic geometry shows that it is conceptually profitable to consider the *six–functors yoga* more generally and their accompanying base-change formulas. A very basic example of such a formula is

$$\operatorname{Ext}_X(\pi^*\mathcal{O}_S, \mathcal{F}) \cong \operatorname{Ext}_S(\mathcal{O}_S, R\pi_*\mathcal{F}),$$

which describes the functor "Ext" on quasicoherent sheaves of modules over $X$ in terms of the derived functor $R\pi_*$.

We are thus moved to study derived base-change for comodules, thought of as sheaves equipped with an action by a group scheme. In particular, we want to understand what it means to "tensor" two comodules together. Unsurprisingly, the solution is dual to that for modules: tensor the two comodules together, then restrict to the elements where the coactions on either factor agree.

Rearrange this middle bit some. "Definition of sheaf, six functors for sheaves on affines, definition of cotensor, adjunction assertion for comodules, remark about cotensoring restricting the image of the coaction" seems too convoluted a narrative structure.

**Definition 9.** Given $A$–comodules $M$ and $N$, their cotensor product is defined by the coequalizer

$$M \square_A N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k A \otimes_k N.$$

Typographical suggestion: can you use a slightly small box for the cotensor product? It's very similar to the "qed box"; see lemma 1.4.13.

**Lemma 14.** *Given a map $f\colon A \to B$ of Hopf $k$–algebras, the induced adjunction $f^* \dashv f_*$ is given at the level of comodules by*

$$\begin{tikzcd}
\text{“}\mathsf{QCoh}_{(\operatorname{Spec} k, \operatorname{Spec} A)}\text{”} \arrow[r, shift left, "f^*"] \arrow[d, equal] & \text{“}\mathsf{QCoh}_{(\operatorname{Spec} k, \operatorname{Spec} B)}\text{”} \arrow[l, shift left, "f_*"] \arrow[d, equal] \\
\mathsf{Comodules}_A \arrow[r, shift left, "M \mapsto M"] & \mathsf{Comodules}_B. \arrow[l, shift left, "N \square_B A \leftarrow\!\shortmid N"]
\end{tikzcd} \square$$

What is this notation Spec $k$ double-slash Spec $A$? Jeremy mentioned it's a stack, but it hasn't been introduced yet. Without more information, I can't verify this lemma. Perhaps mention that it will be defined more precisely later?

*Remark 5.* The formula for $f_*N$ is what one would guess from the formula for pushforward along maps of affine schemes. The comodule $f_*N$ wants to have as its underlying module $N$, but the coaction map on $N$ needs to be reduced to lie only in $A$. The equalizer diagram in the definition of the cotensor product enforces this.

As an example application, cotensoring gives rise to a concise description of what it means to be a comodule map:

**Lemma 15 ([32, Lemma A1.1.6b]).** *Let $M$ and $N$ be $A$–comodules with $M$ projective as a $k$–module. Then there is an equivalence*

$$\mathsf{Comodules}_A(M, N) = \mathsf{Modules}_k(M, k)\square_A N. \square$$

From this, we can deduce the six–functors formula described above:

**Corollary 3.** *Let $N = N' \otimes_k A$ be a cofree comodule. Then $N \square_A k = N'$.*

*Proof.* Picking $M = k$, we have

$$\begin{aligned}\mathsf{Modules}_k(k, N') &= \mathsf{Comodules}_A(k, N) \\ &= \mathsf{Modules}_k(k, k)\square_A N \\ &= k\square_A N. \square\end{aligned}$$

**Corollary 4.** *There is an isomorphism*

$$\mathsf{Comodules}_A(k, N) = \mathsf{Modules}_k(k, k)\square_A N = k\square_A N$$

*and hence*

$$\mathrm{Ext}_A(k, N) \cong \mathrm{Cotor}_A(k, N).$$

*Proof.* Resolve $N$ using the cofree modules described above, then apply either functor $\mathsf{Comodules}_A(k, -)$ or $k\square_A -$. In both cases, you get the same complex.

*Example 14.* Let's contextualize this somewhat. Given a finite group $G$, we can form a commutative Hopf algebra $k^G$, the $k$–valued functions on $G$. This Hopf algebra is dual to the Hopf algebra $k[G]$, the group–algebra on $G$. It is classical that a $G$–module $M$ is equivalent data to a $k[G]$–module structure, and if $M$ is suitably finite, we can dualize the action map to produce a coaction map

$$M^* \to k^G \otimes M^*.$$

Additionally, we have $M^* \square_{k^G} N^* = (M \otimes_G N)^*$, so that $M^* \square_{k^G} k = (H^0(G, M))^*$.

*Example 15.* In the previous lecture, we identified $\mathcal{A}_*$ with the ring of functions on the group scheme $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$, which is defined by the kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Technically, we've only identified Spec $\mathcal{AP}_0$, but at least it's plausible that the subgroup scheme in degree 0 (what does this mean??) gives us Spec $\mathcal{A}_*$???

Today's punchline is that this is analogous to the example above: $\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$ is thought of as "the derived fixed points" of "$G = \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$" on the "$G$–module" $H\mathbb{F}_{2*}X$.

Say the word "strict" here.

How do the duals from the above argument play into this? Do they just cancel out?

*Example 16.* Consider the degenerate case $X = H\mathbb{F}_2$. Then $H\mathbb{F}_{2*}(H\mathbb{F}_2) = \mathcal{A}_*$ is a cofree comodule, and hence Cotor is concentrated on the 0–line:

$$\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}(H\mathbb{F}_2)) = \mathbb{F}_2.$$

The Adams spectral sequence collapses to show the wholly unsurprising equality $\pi_* H\mathbb{F}_2 = \mathbb{F}_2$, and indeed this is the element in the image of the Hurewicz map $\pi_* H\mathbb{F}_2 \to H\mathbb{F}_{2*} H\mathbb{F}_2$.

*Example 17.* At the other extreme, we can pick the extremely nondegenerate case $X = \mathbb{S}$, pictured through a range in Figure 1.1.

Label elements? Identify some groups?

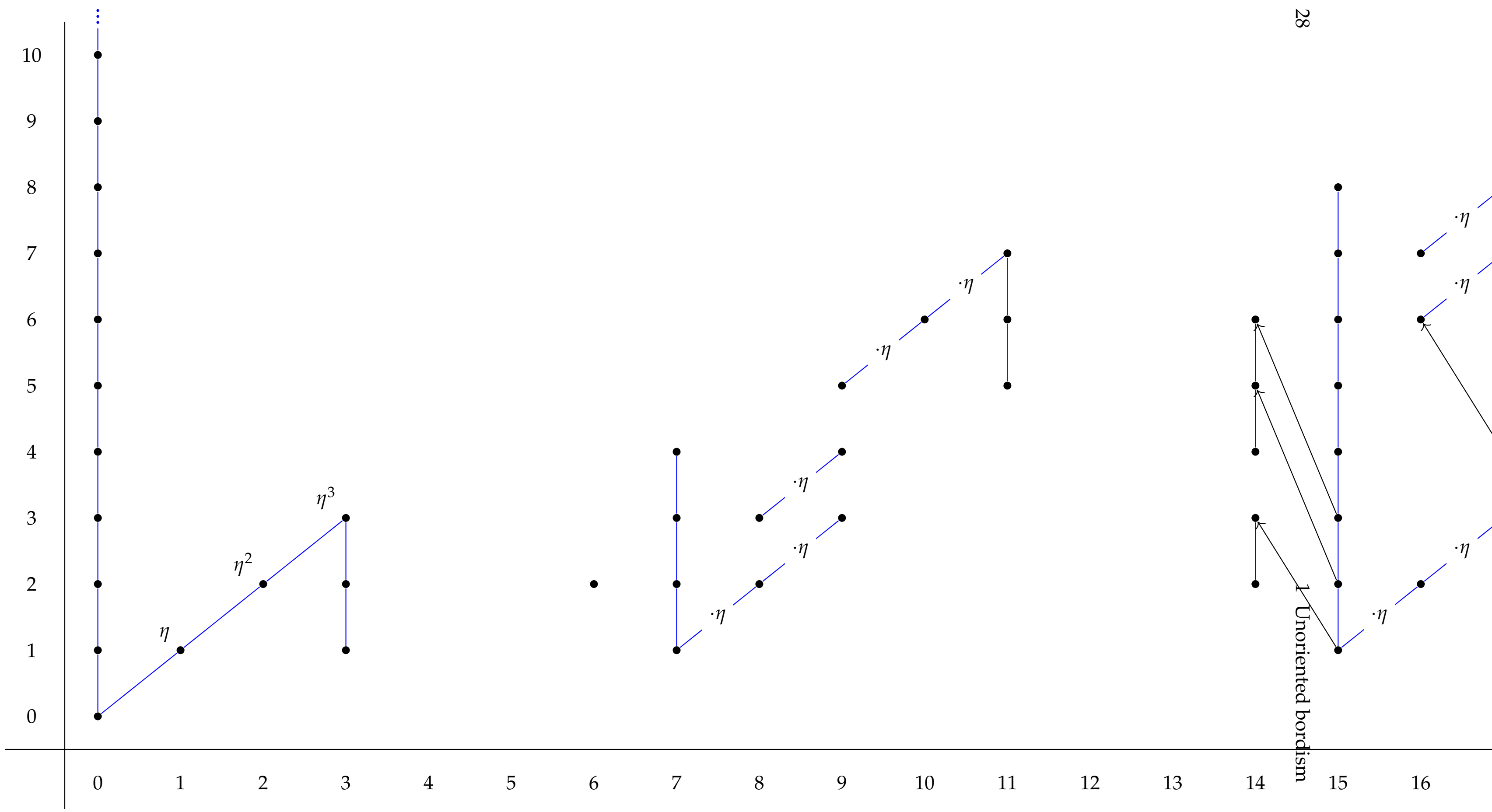



**Fig. 1.1** A small piece of the $H\mathbb{F}_2$-Adams spectral sequence for the sphere, beginning at the second page. North and north-east lines denote multiplication by 2 and by $\eta$, north-west lines denote $d_2$– and $d_3$–differentials.

Jon asked: spectral sequences coming from $\pi_*$ of a Tot tower increase Tot degree. ANSS differentials decrease degree: they run against the multiplicative structure in pictures. What's going on with this? I think this is a duality effect: working with the Steenrod algebra versus its dual.

## 1.5 Feb 5: The unoriented bordism ring

Our goal today is to use the results of the previous lectures to make a calculation of $\pi_* MO$, the unoriented bordism ring. The Adams spectral sequence converging to this has signature

$\mathcal{A}(1)_*$ is the Hopf algebra for a dihedral group. Is this example appropriate somewhere in this section?

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); \widetilde{H\mathbb{F}_2 P_0(MO)}) \Rightarrow \pi_* MO,$$

and so we see that we need to understand $H\mathbb{F}_2 P_0(MO)$, together with its comodule structure over the dual Steenrod algebra.

Last time you used $H\mathbb{F}_{2*}$ instead of $\widetilde{H\mathbb{F}_2 P_0}$.

Our first step toward this is the following calculation:

**Lemma 16.** $H\mathbb{F}_2 P^0 BO(n) \cong \mathbb{F}_2[\![w_1, \ldots, w_n]\!]$.

Why do you state this Lemma? What's below isn't exactly a Corollary, I don't feel, without saying a lot more about how the diagonal on cohomology behaves (and then knowing something about Cartier duality or coalgebraic formal schemes).

*Proof.* The orthogonal groups sit in coset fibration sequences

$$O(n-1) \to O(n) \to S^{n-1},$$

and delooping the groups gives a rotated spherical fibration

$$S^{n-1} \to BO(n-1) \to BO(n).$$

The associated Serre spectral sequence shows that $H\mathbb{F}_2 P^0(BO(n))$ must have one extra free generator in it to receive a differential from the exterior generator of $H\mathbb{F}_2 P^0(S^{n-1})$. Noting $BO(1) \simeq \mathbb{RP}^\infty$, our discussion from previous lectures takes care of the base case.

**Corollary 5.** *There is a triangle*

$$\begin{tikzcd} & \mathrm{Sym}\, H\mathbb{F}_2 P_0(BO(1)) \arrow[d, "equiv"] \\ H\mathbb{F}_2 P_0(BO(1)) \arrow[ur] \arrow[r] & H\mathbb{F}_2 P_0(BO). \end{tikzcd}$$ □

We will defer the proof of this until Case Study 2, since it requires knowing that

$$H\mathbb{F}_2 P^0(BO(k)) \to H\mathbb{F}_2 P^0(BO(1)^{\times k})$$

is injective, which we will revisit later anyhow.

$H\mathbb{F}_2^*(BO(n)) \to H\mathbb{F}_2^*(BO(1)^{\times n})$ lands in the W-invariants, where W is the Weyl group of $BO(1)^{\times n}$ in $BO(n)$ (this is true because the normalizer $N = N_{BO(n)}(BO(1)^{\times n})$ acts on both by conjugation, and it acts trivially on the left term while on the right term it quotients to an action of the Weyl group). You can easily check that $N$ is generated by the permutation matrices and $BO(1)^{\times n}$, so the Weyl group is $\Sigma_n$, and it acts by permuting the factors of $BO(1)$. The invariants in cohomology under $\Sigma_n$ are the symmetric polynomials, and these are generated by the elementary symmetric polynomials. Take the limit in $n$, and there you have it. Now we just need to know, as you say, that the map on cohomology is injective...

With this in hand, however, we can uncover the ring structure on $H\mathbb{F}_2 P_0(MO)$:

**Corollary 6.** *There is also a triangle*

**You owe a proof of:**

$$\begin{tikzcd} & \operatorname{Sym} H\mathbb{F}_2P_0(MO(1)) \arrow[d, "equiv"] \\ H\mathbb{F}_2P_0(MO(1)) \arrow[ur] \arrow[r] & H\mathbb{F}_2P_0(MO). \end{tikzcd}$$

*In particular,* $H\mathbb{F}_2P_0(MO) \cong \mathbb{F}_2[b_1, b_2, \ldots]$.

*Proof.* The block sum maps

$$BO(n) \times BO(m) \to BO(n+m)$$

Thomify to give compatible maps

$$MO(n) \wedge MO(m) \to MO(n+m).$$

Taking the limit in $n$ and $m$, this gives a ring structure on $MO$ compatible with that on $BO$. The Corollary then follows from the functoriality of Thom isomorphisms.

We now seek to understand the scheme Spec $H\mathbb{F}_2P_0(MO)$, and in particular its action of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$. Our launching-off point for this is a topological version of the "freeness" result in the previous Corollary:

**Lemma 17.** *The following square commutes:*

$$\begin{tikzcd} \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) & \mathsf{Spectra}(MO, H\mathbb{F}_2P) \arrow[l, "equiv"'] \\ \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \arrow[u, hook] & \mathsf{RingSpectra}(MO, H\mathbb{F}_2P). \arrow[l, "equiv"] \arrow[u, hook] \end{tikzcd}$$

*Proof.* The top isomorphism asserts only that $\mathbb{F}_2$–cohomology and $\mathbb{F}_2$–homology are linearly dual to one another. The second follows immediately from investigating the effect of the ring homomorphism diagrams in the bottom-right corner in terms of the subset they select in the top-left.

**Corollary 7.** *There is a bijection between homotopy classes of ring maps $MO \to H\mathbb{F}_2P$ and homotopy classes of factorizations*

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r] \arrow[dr] & MO(1) \arrow[d, dotted] \\ & H\mathbb{F}_2P. \end{tikzcd}$$

*Proof.* Given a ring map $MO \to H\mathbb{F}_2P$, we can restrict it along the inclusion $MO(1) \to MO$ to produce a particular cohomology class

$$f \in H\mathbb{F}_2P^0(MO(1)) = \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO(1)), \mathbb{F}_2).$$

Interpreting $f$ as such a function, it is determined by its behavior on the basis of vectors in $\widetilde{H\mathbb{F}_2P_0}(MO(1))$ dual to the powers of the usual coordinate $x \in H\mathbb{F}_2^1(\mathbb{RP}^\infty)$. Finally, given *any* module map $\widetilde{H\mathbb{F}_2P_0}(MO(1)) \to \mathbb{F}_2$, we can employ Corollary 6 and to produce an algebra map $H\mathbb{F}_2P_0(MO) \to \mathbb{F}_2$. Lemma 17 then gives a ring spectrum map $MO \to H\mathbb{F}_2P$.

Why is this sentence "Interpreting…" needed? Also, the Thom isomorphism should appear somewhere, right?

**Corollary 8.** *There is an* $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$*–equivariant isomorphism of schemes*

$$\operatorname{Spec} H\mathbb{F}_2P_0(MO) \cong \mathrm{Coord}_1(\mathbb{RP}^\infty_{H\mathbb{F}_2P}),$$

*where the latter is the scheme of coordinate functions on* $\mathbb{RP}^\infty_{H\mathbb{F}_2P} \to \widehat{\mathbb{A}}^1$ *which restrict to the canonical identification of tangent spaces* $\mathbb{RP}^1_{H\mathbb{F}_2P} = \widehat{\mathbb{A}}^{1,(1)}$.

Here's the unitality assumption again.

*Proof.* The method of the previous proof is to exhibit a isomorphism between these schemes. To learn that this isomorphism is equivariant for $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$, you need only know that the image of the map $MO(1) \to MO$ on mod–2 homology generates $H\mathbb{F}_2P_0(MO)$ as an algebra.

I don't really see what argument is being made about $\mathrm{Aut}\,\widehat{\mathbb{G}}_a$–equivariance. I don't think that the topological map is surjective...?

This is clumsily stated.

We are now ready to analyze the group cohomology of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ with coefficients in the comodule $H\mathbb{F}_2P_0(MO)$.

I'd like more details for this argument too.

**Theorem 7.** *The action of* $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ *on* $\mathrm{Coord}_1(\widehat{\mathbb{G}}_a)$ *is free.*

So it seems like what you end up using about this theorem for the purposes of the calculation at hand is not just that the action is free, but you have such a nice identification of the quotient. Maybe it would be clearer to include this as part of the statement?

*Proof.* Recall that $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ is defined by the (split) kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Consider a point $f \in \mathrm{Coord}_1(\widehat{\mathbb{G}}_a)(R)$, which in terms of the standard coordinate can be expressed as

$$f(x) = \sum_{j=1}^{\infty} b_{j-1}x^j,$$

where $b_0 = 1$. Decompose this series as $f(x) = f_2(x) + f_r(x)$, with

$$f_2(x) = \sum_{k=0}^{\infty} b_{2^k-1}x^{2^k}, \qquad f_r(x) = \sum_{j \neq 2^k} b_{j-1}x^j.$$

Why is this? I think I have a somewhat incorrect idea of what Coord is. In addition to being a set map that sends the set of nilpotent elements in $R$ to itself, what other conditions does $f$ have to satisfy?

Note that $f_2$ gives a point $f_2 \in \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$, so we can de-scale by it to give a new coordinate $g(x) = f_2^{-1}(f(x))$ with analogous series $g_2(x)$ and $g_r(x)$. Note that $g_2(x) = x$ and that $f_2$ is the unique point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$ that has this property.

Right, because raising to powers of two satisfies "Freshman's dream", so $f_2(x+y) = f_2(x) + f_2(y)$, so $f_2$ is a automorphism of group schemes. What condition does the subscript 1 imply in this case?

**Corollary 9.** $\pi_*MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1, j \geq 1]$ *with* $|b_j| = j$.

Compare this with the base change theorems from the previous day.

Amplitude just means row, right?

*Proof.* Set $M = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]$. It follows from the above that the $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$–cohomology of $H\mathbb{F}_2P_0(MO)$ has amplitude 0:

Maybe you should justify the step that looks like reassociation of the tensor and cotensor products. You almost definitely proved this on the previous day.

$$\begin{aligned} \mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_2P_0(MO)) &= \mathrm{Cotor}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \square_{\mathcal{A}_*} (\mathcal{A}_* \otimes_{\mathbb{F}_2} M) \\ &= \mathbb{F}_2 \otimes_{\mathbb{F}_2} M = M. \end{aligned}$$

Here you used $H\mathbb{F}_2P_0(MO) \simeq \mathcal{A}_* \otimes M$, but what you seem to have proved in the previous theorem has to do with $\mathcal{A}P_0$ instead. Does it make a difference?

Since the Adams spectral sequence

$$H^*_{\mathrm{gp}}(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2P_0(MO)) \Rightarrow \pi_* MO$$

How?

is concentrated on the 0–line, it collapses. Using the residual $\mathbb{G}_m$–action to infer the grading, we thus deduce

$$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]. \square$$

Is there a reason we should expect this, even in hindsight?

This is pretty remarkable: some big statement about manifold geometry came down to understanding how we could reparametrize a certain formal group, itself a (fairly simple) purely algebraic problem. We could close here, but there's an easy homotopical consequence of this fact that is worth recording before we leave:

**Lemma 18.** *$MO$ splits as a wedge of shifts of $H\mathbb{F}_2$.*

Does this not just follow from the fact that $\pi_*MO$ is a $\mathbb{F}_2$ vector space?

*Proof.* «««< HEAD Referring to Lemma 11, we have a $\pi_*$–injection $MO \to H\mathbb{F}_2 \wedge MO$. Pick an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha$ for $\pi_* MO$ and extend it to a $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha \cup \{w_\beta\}_\beta$ for $\pi_* H\mathbb{F}_2 \wedge MO$. Altogether, this larger basis can be represented as a single map

$$\bigvee_\alpha \Sigma^{m_\alpha} \mathbb{S} \vee \bigvee_\beta \Sigma^{n_\beta} \mathbb{S} \xrightarrow{\bigvee_\alpha v_\alpha \vee \bigvee_\beta w_\beta} H\mathbb{F}_2 \wedge MO.$$

Smashing through with $H\mathbb{F}_2$ gives an equivalence

$$\bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{n_\beta} H\mathbb{F}_2 \xrightarrow{\simeq} H\mathbb{F}_2 \wedge MO.$$

It's clear, but you haven't specified that $m_\alpha$ and $n_\beta$ are the degrees of $v_\alpha$ and $w_\beta$.

The composite map

$$MO \to H\mathbb{F}_2 \wedge MO \xleftarrow{\simeq} \bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{n_\beta} H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{m_\alpha} H\mathbb{F}_2$$

is a weak equivalence.

*Remark 6.* Just using that $\pi_* MO$ is connective and $\pi_0 MO = \mathbb{F}_2$, we can produce a ring spectrum map $MO \to H\mathbb{F}_2$. What we've learned is that this map has a splitting: $MO$ is also an $H\mathbb{F}_2$–algebra.

Should you eventually mention the stable cooperations $MO^{MO}$? Rather than coming with a specified logarithm, it's an isomorphism between any pair of additive formal groups — or, I suppose, a pair of logarithms.

# Case Study 2
# Complex bordism

Write an introduction for me.

## 2.1 Feb 8: Formal varieties

I think this lecture may be too long. On the other hand, the rational stuff at the end will go rather quickly — which I know from experience in the Pittsburgh talks.

Having totally dissected unoriented bordism, we can now turn our attention to other sorts of bordism theories, and there are many available: oriented, *Spin*, *String*, complex, .... We would like to replicate the results from Case Study 1 for these other contexts, but we quickly see that only one of the listed bordism theories supports this program. The space $\mathbb{R}P^\infty = BO(1)$ was a key player in the unoriented bordism story, and the only other bordism theory with a similar ground object is complex bordism, with $\mathbb{C}P^\infty = BU(1)$. So, we will focus on it.

The contents of Lecture 1.1 can be replicated essentially *mutatis mutandis*, resulting in the following theorems:

**Theorem 8.** *There is a complex J–homomorphism*

Cite me: Give a reference from Lecture 1.1.

$$J_{\mathbb{C}} : BU \to BGL_1\mathbb{S}. \square$$

**Definition 10.** The associated Thom spectrum is written "$MU$" and called *complex bordism*. A map $MU \to E$ of ring spectra is said to be a *complex orientation of $E$*.

Cite me: Give a reference from Lecture 1.1.

Something I've seen more than once is an equivalence $MU(k) \simeq BU(k)/BU(k-1)$. It's not immediately obvious to me where this comes from. Where does it come from? Is it helpful to think about?

**Theorem 9.** *For a complex vector bundle $\xi$ on a space $X$ and a complex-orientedring spectrum $E$, there is a natural equivalence*

Cite me: Give a reference from Lecture 1.1.

I don't remember discussing orientations in the last chapter (probably because it's not needed), but maybe you can say something about where the complex orientation is used in this theorem?

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X. \square$$

Cite me: Give a reference from Lecture 1.1.

**Corollary 10.** *In particular, for a complex-oriented ring spectrum E it follows that $E^*\mathbb{C}P^\infty$ is isomorphic to a one–dimensional power series ring.* ☐

Maybe I'm confused about grading issues, but I thought $E^*\mathbb{C}P^\infty$ was a polynomial ring and $EP^0\mathbb{C}P^\infty$ is the power series ring?

Also, this is a nice argument. Usually this computation proceeds through the AHSS. Can this method be adapted to spaces other than $\mathbb{C}P^\infty$?

In light of these results, it seems prudent to develop some of the theory of formal schemes and formal varieties outside of the context of $\mathbb{F}_2$–algebras.

Erick has been complaining about this definition for a while, and I think he's right. His suggestion is for each scheme to be a nilpotent thickening over its reduction, which is of finite type (over whatever base). I kept compulsively writing "Artinian", but he pointed out that $\mathbb{Z}[x]/x^n$ is *not* Artinian, and so this can't be the right assumption. I'm not sure why I was so stuck on this word... am I forgetting some important case?

**Definition 11.** Fix a scheme $S$. A formal $S$–scheme $X = \{X_\alpha\}_\alpha$ is an ind-system of Artinian $S$–schemes $X_\alpha$.$S$–schemes $X_\alpha$.

*Remark 7.* In the case $S = \operatorname{Spec} k$ for a field $k$, "Artinian" means that $\mathcal{O}_{X_\alpha}$ is a finite–dimensional $k$–vector space.

These ind-systems arise when studying completions of rings. To address the geometric situation, we first owe ourselves a definition of a closed subscheme:

Why Artinian? Also, you used to say finite instead of Artinian, so maybe you can standardize the terminology across chapters.

**Definition 12.** Let $X$ be an affine formal scheme, and pick a chart $\operatorname{Spec} R \to X$. A subscheme $Y \subseteq X$ is called *closed* when it has the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, dotted, no head] & X \arrow[d, equal] \\ \operatorname{Spec}(R/I) \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

I'm guessing this means isomorphic to the constant ind-system $\{\operatorname{Spec} R\}$?

I don't understand your decorations for the vertical arrows. Shouldn't they be arrows? And why is the left one densely dotted?

There's a complementary notion of an open subscheme, which we will continue to avoid for now. These definitions are both best stated in a coordinate–free way, but the open subscheme version really *requires* it, so we will postpone it until later.For now, we will proceed with the geometry:

**You owe a proof of:** Definition of open scheme.

**You owe a proof of:** Definition of closed subscheme without chart.

**Definition 13.** Consider such a closed subscheme $Y$ of an affine $S$–scheme $X$, modeled by a map $\operatorname{Spec} R/I \to \operatorname{Spec} R$. We define the *$n^{th}$ order neighborhood of $Y$ in $X$* to be the scheme $\operatorname{Spec} R/I^{n+1}$. The *formal neighborhood of $Y$ in $X$* is then the ind-system

$$X^\wedge_Y := \left\{\operatorname{Spec} R/I \to \operatorname{Spec} R/I^2 \to \operatorname{Spec} R/I^3 \to \cdots\right\}.$$

When is this system one of Artinian schemes? The condition we came to in class was that $I$ is its own radical and $R$ is Noetherian. I'm very mildly uncomfortable with this condition on $I$.

I thought in class you said that $\sqrt{I}$ is maximal instead of $I$ is radical?

So, formal schemes arise naturally when studying the local geometry of $X$ near a subscheme $Y$. An exceedingly common situation is for $X$ to be a variety and $Y$ to be a smooth point, so that $X^\wedge_Y$ looks like "a small piece of affine space". We pin this important case down with a definition:

**Definition 14.** In the case that $S = \operatorname{Spec} R$ is affine, formal affine $n$–space over $S$ is defined by

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!].$$

A *formal affine variety* is a formal scheme $V$ which is (noncanonically) isomorphic to $\widehat{\mathbb{A}}^n$ for some $n$. The two maps in an isomorphism pair

I have very little intuition for Spf $R$. Could you define this more precisely, especially as an ind-system from the functors of points perspective?

$$V \to \widehat{\mathbb{A}}^n, \quad V \leftarrow \widehat{\mathbb{A}}^n$$

are called a *coordinate (system)* and a *parameter (system)* respectively.

**Lemma 19.** *A pointed map $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ is identical to an $m$–tuple of $n$–variate power series with no constant term.* □

*Remark 8.* In some sense, Lemma 19 is a full explanation for why anyone would even think to involve formal geometry in algebraic topology (nevermind how useful the program has been in the long run). Calculations in algebraic topology are frequently expressed in terms of power series rings, and with this Lemma we are provided geometric interpretations for such statements.

Lemma 19 shows how formal varieties are especially nice, because maps between them can be boiled down to statements about power series. In particular, this allows local theorems from analytic differential geometry to be imported, including a version of the inverse function theorem, which we will now work towards.

Work an example first of $\widehat{\mathbb{A}}^1$, rather than an abstract formal variety.

**Definition 15.** Let $V$ be a formal variety and let $I_V = \widehat{\mathbb{A}}^1(V)$ be the ideal of functions vanishing at the origin. Then, we define the *cotangent space of $V$* at the origin by

Strictly speaking, I don't think this notation has been introduced.

$$T^*V = I_V / I_V^2.$$

**Lemma 20.** *There is an isomorphism*

$$TV \cong \mathsf{Modules}_R(T^*V, R).$$

*Proof.* A point $f \in V(R[\varepsilon]/\varepsilon^2)$ is given by a map $f\colon \mathcal{O}_V \to R[\varepsilon]/\varepsilon^2$. If $f$ is pointed, then it carries the ideal $I(0)$ of functions vanishing at zero to the ideal $(\varepsilon)$, and hence also carries $I(0)^2$ to $(\varepsilon)^2 = 0$. Hence, $f$ induces a map

$$\begin{tikzcd}
I(0)/I(0)^2 \arrow[r, "f"] \arrow[d, equal] & (\varepsilon)/(\varepsilon)^2 \arrow[d, equal] \\
T^*V \arrow[r, "f"] & R,
\end{tikzcd}$$

hence a point in $\mathsf{Modules}_R(T^*V, R)$. This assignment is visibly bijective.

**Theorem 10.** *A map $f\colon V \to W$ of finite–dimensional formal varieties is an isomorphism if and only if the induced map $Tf\colon TV \to TW$ is an isomorphism of $R$–modules.*

**Cite me:** This is 3.1.8 in the Crystals notes..

*Proof.* First, reduce to the case where $V \cong \widehat{\mathbb{A}}^n$ and $W \cong \widehat{\mathbb{A}}^n$ have the same dimension , and select charts for both. Then, $Tf$ is a matrix of dimension $n \times n$. If $Tf$ fails to be invertible, we are done, and if it is invertible, we replace $f$ by $f \circ (Tf)^{-1}$ so that $Tf$ is the identity matrix.

I don't know how you are defining Spf, but there may be something to show here, since you're not actually "reducing" to the case but showing that it 'is" the case.

We now construct the inverse function by induction on degree. Set $g^{(1)}$ to be the identity function, so that $f$ and $g^{(1)}$ are mutual inverses when restricted to the first-order neighborhood. So, suppose that $g^{(r-1)}$ has been constructed, and consider its interaction with $f$ on the $r^{\text{th}}$ order neighborhood:

$$g_i^{(r-1)}(f(x)) = x_i + \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n} + o(r+1).$$

By adding in the correction term

$$g_i^{(r)} = g_i^{(r-1)} - \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n},$$

we have $g_i^{(r)}(f(x)) = x_i + o(r)$.

We now return to our motivating example of $\mathbb{C}\mathrm{P}^\infty_E$ for $E$ a complex-oriented cohomology theory, where we saw that the complex-orientation determines an isomorphism $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$. However, the object "$E^*\mathbb{C}\mathrm{P}^\infty$" is something that exists independent of the orientation map $MU \to E$, and we now have the language to tease apart this situation:

**Lemma 21.** *A cohomology theory $E$ is complex orientable (i.e., it is able to receive a ring map from $MU$) precisely when $\mathbb{C}\mathrm{P}^\infty_E$ is a formal curve. A choice of map $MU \to E$ determines a coordinate $\mathbb{C}\mathrm{P}^\infty_E \cong \widehat{\mathbb{A}}^1$.* □

Is this just a formal variety of dimension 1?

As we saw in the first case study, $\mathbb{C}\mathrm{P}^\infty_E$ has more structure than just a formal scheme: it also carries the structure of a group. We close today with some remarks about such objects.

**Definition 16.** A formal group is a formal variety endowed with an abelian group structure.[1]

**Cite me:** Theorem 2.2.6 of the Crystals notes.

*Remark 9.* As with formal schemes, formal groups can arise as formal completions of an algebraic group at its identity point. It turns out that there are many more formal groups than come from this procedure, a phenomenon that is of keen interest to stable homotopy theorists.

Add some kind of reference to a complaint about this? It's not like we're going to talk about *TMF* much.

[1] Formal groups in dimension 1 are automatically commutative if and only if the ground ring has no elements which are simultaneously nilpotent and torsion.

**Corollary 11.** *As with physical groups, the formal group addition map on $\widehat{\mathbb{G}}$ determines the inverse law.*

*Proof.* Consider the shearing map

$$\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \xrightarrow{\sigma} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}},$$
$$(x, y) \mapsto (x, x + y).$$

The induced map $T\sigma$ on tangent spaces is evidently invertible, so by Theorem 10 there is an inverse map $(x, y) \mapsto (x, y - x)$. Setting $y = 0$ and projecting to the second factor gives the inversion map.

**Definition 17.** Let $\widehat{\mathbb{G}}$ be a formal group. In the presence of a coordinate $\varphi\colon \widehat{\mathbb{G}} \cong \widehat{\mathbb{A}}^n$, the addition law on $\widehat{\mathbb{G}}$ begets a map

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[r] \arrow[d, equal] & \widehat{\mathbb{G}} \arrow[d, equal] \\ \widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \arrow[r] & \widehat{\mathbb{A}}^n, \end{tikzcd}$$

Again, the vertical arrows should be arrows, not equal signs?

and hence a $n$-tuple of $(2n)$-variate power series "$+_\varphi$", satisfying

$$\underline{x} +_\varphi \underline{y} = \underline{y} +_\varphi \underline{x}, \qquad \text{(commutativity)}$$
$$\underline{x} +_\varphi \underline{0} = \underline{x}, \qquad \text{(unitality)}$$
$$\underline{x} +_\varphi (\underline{y} +_\varphi \underline{z}) = (\underline{x} +_\varphi \underline{y}) +_\varphi \underline{z}. \qquad \text{(associativity)}$$

Such a tuple $+_\varphi$ is called a *formal group law*.

Let's now consider two examples of $E$ which are complex-orientable and describe $\mathbb{C}\mathrm{P}^\infty_E$ for them.

*Example 18.* There is an isomorphism $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{Z}P} \cong \widehat{\mathbb{G}}_a$. This follows from reasoning identical to that given in Example 12.

*Example 19.* There is also an isomorphism $\mathbb{C}\mathrm{P}^\infty_{KU} \cong \widehat{\mathbb{G}}_m$. Given a complex line bundle $\mathcal{L}$ over a space $X$, we use the complex orientation of $KU$ where the total Chern class of $\mathcal{L}$ is given by

This calculation ignores the grading, which isn't great. If you're careful and distinguish $c$ from $c_1$, things should fall into place better.

Make it clear that we're using the coordinate $1 - t$ on $\mathbb{G}_m$.

$$c(\mathcal{L}) = 1 - [\mathcal{L}].$$

Given this definition, we perform a manual computation:

$$\begin{aligned} c(\mathcal{L}_1 \otimes \mathcal{L}_2) &= 1 - [\mathcal{L}_1 \otimes \mathcal{L}_2] = 1 - [\mathcal{L}_1][\mathcal{L}_2] \\ &= -1 + [\mathcal{L}_1] + [\mathcal{L}_2] - [\mathcal{L}_1][\mathcal{L}_2] + 1 - [\mathcal{L}_1] + 1 - [\mathcal{L}_2] \\ &= (1 - [\mathcal{L}_1]) + (1 - [\mathcal{L}_2]) - (1 - [\mathcal{L}_1])(1 - [\mathcal{L}_2]) \\ &= c(\mathcal{L}_1) + c(\mathcal{L}_2) - c(\mathcal{L}_1)c(\mathcal{L}_2). \end{aligned}$$

Why is the notation $+_!$?

In this coordinate on $\mathbb{C}P^\infty_{KU}$, the group law is then $x +_! y = x + y - xy$.

We will close today by showing that the rational theory of formal groups is highly degenerate, similar to the rational theory of spectra.

**Definition 18.** The module of *Kähler differentials* on a $k$–algebra $R$ is an $R$–module $\Omega^1_{R/k}$. It is generated by symbols $dr$ for each element $r \in R$, subject to the two families of relations

$$\begin{aligned} ds &= 0,\, s \in k && \text{(differentiation is linear for "scalars")} \\ d(rr') &= rdr' + r'dr. && \text{($d$ is a derivation)} \end{aligned}$$

Elements of $\Omega^1_{R/k}$ are referred to as 1–*forms*.

Do we ever use this? Does it provide intuition? Hmph.

**Lemma 22.** *The module $\Omega^1_{R/k}$ is universal for derivations into R–modules:*

$$\mathsf{Derivations}_k(R, M) = \mathsf{Modules}_R(\Omega^1_{R/k}, M).\square$$

These definitions are interesting in this level of generality, but suppose now that $k$ is a $\mathbb{Q}$–algebra and that $R = k[\![x]\!]$ is the coordinatized ring of functions on a formal line over $k$. What's special about this rational curve case is that differentiation gives an isomorphism between $\Omega^1_{R/k}$ and the ideal $(x)$ of functions vanishing at the origin, i.e., the ideal sheaf selecting the closed subscheme $\operatorname{Spec} k \to \operatorname{Spf} R$. Its inverse is formal integration:

$$\int : \left(\sum_{j=0}^{\infty} c_j x^j\right) dx \mapsto \sum_{j=0}^{\infty} \frac{c_j}{j+1} x^{j+1}.$$

We should show here that the submodule of invariant 1–forms is equivalent to the module of tangent vectors.

The condition $\omega = T_y^* \omega$ looks more reasonable.

Also define the *sheaf* of invariant differentials, since you use that in a couple of days.

Taking a cue from classical Lie theory, we attempt to define exponential and logarithm functions for a given formal group law $F$. This is typically accomplished by studying left–invariant differentials: a 1–form $f(x)dx$ is said to be left–invariant under $F$when

$$f(x)dx = f(y +_F x) d(y +_F x) = f(y +_F x) \frac{\partial(y +_F x)}{\partial x} dx.$$

Restricting to the origin by setting $x = 0$, we deduce the condition

$$f(0) = f(y) \cdot \left. \frac{\partial(y +_F x)}{\partial x} \right|_{x=0}.$$

If $R$ is a $\mathbb{Q}$–algebra, then setting the boundary condition $f(0) = 1$ and integrating against $y$ yields

$$\log_F(y) = \int f(y)\, dy = \int \left( \left. \frac{\partial(y +_F x)}{\partial x} \right|_{x=0} \right)^{-1} dy.$$

To see that the series $\log_F$ has the claimed homomorphism property, note that

While the notation log is suggestive, I don't think you claimed anything yet at this point.

$$\frac{\partial \log_F(y +_F x)}{\partial x} dx = f(y +_F x) d(y +_F x) = f(x) dx = \frac{\partial \log_F(x)}{\partial x} dx,$$

so $\log_F(y +_F x)$ and $\log_F(x)$ differ by a constant. Checking at $y = 0$ shows that the constant is $\log_F(x)$, hence

$$\log_F(x +_F y) = \log_F(x) + \log_F(y).$$

In all, this argument bundles into the following coordinate-free theorem:

**Theorem 11.** *There is a unique isomorphism*

This feels like a bit of a jump. "$\otimes$", for instance?

$$\widehat{\mathbb{G}} \xrightarrow{\log} \operatorname{Lie} \widehat{\mathbb{G}} \otimes \widehat{\mathbb{G}}_a. \square$$

First you have map $\widehat{\mathbb{G}} \otimes (\Omega^1)^F \to \widehat{\mathbb{G}}_a$, and then you move $(\Omega^1)^F$ to the right, changing it to $\operatorname{Lie} \widehat{\mathbb{G}}$.

It would also be good to put the example of the standard logarithm for $\widehat{\mathbb{G}}_m$ here.

## 2.2 Feb 10: Divisors on formal curves

**Cite me:** Section 2.4.2 of the AHS preprint.

We now have a solid foundation for the most important case of the complex-oriented cohomology of a space: $E^* \mathbb{CP}^\infty$. We turn next to an algebro-geometric model for the other topological operation complex-oriented cohomology theories are well-suited for: the formation of Thom complexes. Recall the theorem from the beginning of last time:

**Theorem 12 (Theorem 9).** *For a complex vector bundle $\xi$ on a space $X$ and a complex-oriented ring spectrum $E$, there is a natural equivalence*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X. \square$$

Recalling also the perspective on modules as quasicoherent sheaves from Lecture 1.4, we are thus moved to study sheaves of modules on $X_E$ which are 1–dimensional — i.e., line bundles. Having said all that, we will leave the topology for tomorrow and focus on the algebra today. We fix the following three pieces of data:

- $S$ is our "base" formal scheme.
- $C$ is a formal curve over $S$.
- $\zeta : S \to C$ is a distinguished point on $C$.

Recall that yesterday we defined what it meant for a subscheme to be closed. The notion of a divisor on a formal curve is a particular sort of closed subscheme:

**Cite me:** Def 2.33 of AHS preprint.

**Definition 19.** An *effective Weil divisor* $D$ on $C$ is a closed $S$–subscheme of $C$ whose structure map $D \to S$ is flat and whose ideal sheaf $\mathcal{I}_D$ is free of rank 1 as an $\mathcal{O}_S$–module. We say that the *rank* of $D$ is $n$ when its ring of functions $\mathcal{O}_D$ is free of rank $n$ over $\mathcal{O}_S$.

Is there a distinction between free and locally free? I guess not because everything is affine.

Consider the case of interest to us, where we have selected a coordinate $x$ on $C$. In that case, there are isomorphisms $S = \operatorname{Spec} E_*$ and $C \cong \operatorname{Spf} E^*[\![x]\!]$, so that a divisor $D$ must be of the form $D \cong \operatorname{Spf} E^*[\![x]\!]/f$ for some $f$ not a zero-divisor. We see then that $\mathcal{I}_D$ corresponds to the principal ideal $E^*[\![x]\!] \cdot f \cong E^*[\![x]\!]$, and $D$ is a divisor exactly when $E^*[\![x]\!]/f$ is a flat $E^*$–module.

Is this last condition easy to unpack? I'd hope it means something about monicity. Maybe see Lemma 17.1 of FPFP?

Before considering their connection to line bundles, we will study the concept of a divisor in isolation.

**Lemma 23.** *The scheme of such effective Weil divisors of rank $n$ exists:* $\operatorname{Div}_n^+ C$*. It is a formal variety of dimension $n$. In fact, a coordinate $x$ on $C$ determines an isomorphism* $\operatorname{Div}_n^+ C \cong \widehat{\mathbb{A}}^n$.

How do you even show that $f$ can be chosen to be a polynomial, and not just a power series? Weierstrass?

I wonder if it's possible to frame this argument with Theorem 84. The proof given here is Prop 5.2 of FSFG.

*Proof.* Begin with the definition

$$\operatorname{Div}_n^+(C)(R) = \left\{ (a, D) \middle| \begin{array}{c} a : \operatorname{Spec} R \to S, \\ D \text{ is an effective divisor on } C \times_S \operatorname{Spec} R \end{array} \right\}.$$

To show that it is a formal variety, we pick a coordinate $x$ on $C$ and consider a point $(a, D) \in \operatorname{Div}_n^+(C)(R)$. In this case, $C \times \operatorname{Spec} R$ is presented as

$$C \times_X \operatorname{Spec} R = \operatorname{Spf} R[\![x]\!]$$

and hence $D$ can be presented as the closed subscheme

$$D = \operatorname{Spf} R[\![x]\!]/(x^n - g(x)), \quad g(x) = \sum_{j=0}^{n-1} a_j(D) x^j.$$

One checks that $a_j(D)$ is a nilpotent element of $R$ for all $j$, and hence determines a map $\operatorname{Spec} R \to \widehat{\mathbb{A}}^n$. Conversely, given such a map, we can form the polynomial $g(x)$ and hence the divisor $D$.

This statement has real content! If $a_j(D)$ were not nilpotent, then Weierstrass factorization would strip off a smaller monic polynomial. But, we haven't talked about Weierstrass preparation yet... and we were intending to leave it for much later. Maybe we should have done this today.

Actually, Jeremy and Jun Hou point out that Weierstrass preparation requires hypotheses on the ground scheme (like: it's Spf of a complete and local ring) that aren't necessarily satisfied here. So, what geometric thing do we *really* mean?

This proof lays bare the moral value of this scheme: it parametrizes collections of points on $C$ which arise as zero loci of polynomials. It's well-known how basic operations on polynomials affect their zero loci, and these operations are also reflected on the level of divisor schemes. For instance, there is a unioning map:

**Lemma 24.** *There is a map*

$$\begin{aligned} \mathrm{Div}_n^+ C \times \mathrm{Div}_m^+ C &\to \mathrm{Div}_{n+m}^+ C, \\ (D_n, D_m) &\mapsto D_n \sqcup D_m. \square \end{aligned}$$

*Remark 10.* On the level of the polynomials $g_n$, $g_m$, and $g_{n+m}$, this map is given by

$$(g_n, g_m) \mapsto x^{n+m} - (x^n - g_n(x)) \cdot (x^m - g_m(x)) =: g_{n+m}(x).$$

What was wrong with $g_{n+m} := g_n g_m$ from class? The current $g_{n+m}$ has degree $< n+m \ldots$

Note that there is a canonical isomorphism $C \to \mathrm{Div}_1^+ C$. Iterating the above addition map gives the vertical map in the following triangle:

$$\begin{tikzcd} & C^{\times n} \arrow[dl] \arrow[d] \\ C^{\times n}_{\Sigma_n} \arrow[r, dotted, "\cong"] & \mathrm{Div}_n^+ C. \end{tikzcd}$$

**Lemma 25.** *The object $C^{\times n}_{\Sigma_n}$ exists, it factors the iterated addition map, and the dotted arrow is an isomorphism.*

Make a point that the other arrow is not surjective.

*Proof.* The first assertion is a consequence of Newton's theorem on symmetric polynomials: the subring of symmetric polynomials in $R[x_1, \ldots, x_n]$ is itself polynomial on generators

$$\sigma_j(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = j}} x_{S_1} \cdots x_{S_j},$$

and hence

$$R[\sigma_1, \ldots, \sigma_n] \subseteq R[x_1, \ldots, x_n].$$

Picking a coordinate on $C$ allows us to import this fact into formal geometry to deduce the existence of $C^{\times n}_{\Sigma_n}$. The factorization then follows by noting that the iterated $\sqcup$ map is symmetric. Finally, Remark 10 shows that the horizontal map pulls the coordinate $a_j$ back to $\sigma_j$, so the third assertion follows.

We now consider the effects of maps $q \colon C \to C'$ between curves.

**Lemma 26.** *Let $q \colon C \to C'$ be a map of formal curves over $S$, and let $D \subseteq C$ be a divisor on $C$. Then the composite $D \to C \to C'$, denoted $q_* D$, is also a divisor.*

*Proof.* The structure map map $D \to S$ is unchanged and hence still flat, and the ideal sheaf $\mathcal{I}_{q_* D} \subseteq \mathcal{O}_{C'}$ is given by tensoring up the original ideal sheaf:

$$\mathcal{I}_{q_* D} = \mathcal{I}_D \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}.$$

Hence, it is still free of rank 1.

*Remark 11.* For a general map $q$, the pullback $D \times_{C'} C$ of a divisor $D \subseteq C'$ will not be a divisor on $C$. However, conditions on $q$ can be imposed so that this is so, and in this case $q$ is called an *isogeny*. We will return to this in the future.

Now we use the pointing $\zeta : S \to C$. Together with the $\sqcup$ map, this gives a composite

$$\begin{array}{ccccccc} \mathrm{Div}_n^+ C & \longrightarrow & C \times \mathrm{Div}_n^+ C & \longrightarrow & \mathrm{Div}_1^+ C \times \mathrm{Div}_n^+ C & \longrightarrow & \mathrm{Div}_{n+1}^+ C, \\ D & \longmapsto & (\zeta, D) & \longmapsto & ([\zeta], D) & \longmapsto & [\zeta] \sqcup D. \end{array}$$

**Definition 20.** We define the following variants of "stable divisor schemes":

$$\begin{aligned} \mathrm{Div}^+ C &= \coprod_{n \geq 0} \mathrm{Div}_n^+ C, \\ \mathrm{Div}_n C &= \mathrm{colim}\left(\mathrm{Div}_n^+ C \xrightarrow{[\zeta]+-} \mathrm{Div}_{n+1}^+ C \xrightarrow{[\zeta]+-} \cdots\right), \\ \mathrm{Div}\, C &= \mathrm{colim}\left(\mathrm{Div}^+ C \xrightarrow{[\zeta]+-} \mathrm{Div}^+ C \xrightarrow{[\zeta]+-} \cdots\right) \\ &\cong \coprod_{n \in \mathbb{Z}} \mathrm{Div}_n C. \end{aligned}$$

**Theorem 13.** *The scheme* $\mathrm{Div}^+ C$ *models the free formal monoid on the formal curve* $C$*. The scheme* $\mathrm{Div}\, C$ *models the free formal group on the formal curve* $C$.[2] *The scheme* $\mathrm{Div}_0 C$ simultaneously *models the free formal monoid and the free formal group on the* pointed *formal curve* $C$. □

Can you at least see where the negation maps would come from w/o reference to coalgebras? (See FPFP Prop 18.1?)

We will postpone the proof of this theorem until later, once we've developed a theory of coalgebraic formal schemes.

**You owe a proof of:** The various Div constructions give free monoids / groups.

*Remark 12.* This gives another way to interpret Lemma 26. A map $q\colon C \to C'$ postcomposes to give a map $C \to C' \to \mathrm{Div}\, C'$. Since the target of this map is a formal group scheme, universality induces a map $q_*\colon \mathrm{Div}\, C \to \mathrm{Div}\, C'$.

To close today, we finally link divisors to the study of line bundles.

**Cite me:** Def 2.38 of AHS preprint.

**Definition 21.** Suppose that $\mathcal{L}$ is a line bundle on $C$ and select a section $u$ of $\mathcal{L}$. There is a largest closed subscheme $D \subseteq C$ where the condition $u|_D = 0$ is satisfied. If $D$ is a divisor, $u$ is said to be *divisorial* and $D = \mathrm{div}\, u$.

Is being divisorial much of a condition at all? The zero section is not divisorial — are there other sections of other line bundles which are not divisorial? (Maybe if the base scheme is sufficiently nasty, then you can build things that literally aren't this counterexample, but which are close enough and ubiquitous enough as to be hard to avoid by naming properties, so we invent a new name...)

Cite me: Def 2.38 of AHS preprint.

**Lemma 27.** *Let $u$ be a divisorial section of $\mathcal{L}$. Then, $u$ gives a trivialization of $\mathcal{L} \otimes \mathcal{I}_D$, so that $\mathcal{L} \cong \mathcal{I}_D^{-1}$.* □

**Lemma 28.** *This construction is suitably monoidal: if $u$ and $v$ are divisorial sections of $\mathcal{L}$ and $\mathcal{M}$ respectively, then $u \otimes v$ is a divisorial section of $\mathcal{L} \otimes \mathcal{M}$ and* $\operatorname{div}(u \otimes v) = \operatorname{div} u + \operatorname{div} v$. □

This Lemma induces us to consider the extension of this concept to meromorphic functions:

**Definition 22.** A *meromorphic divisorial section* of a line bundle $\mathcal{L}$ is a decompositon $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ together with an expression of the form $u_+/u_-$, where $u_+$ and $u_-$ are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We set $\operatorname{div}(u_+/u_-) = \operatorname{div} u_+ - \operatorname{div} u_-$.

The fundamental theorem is that, in the case of a curve $C$, meromorphic functions (sometimes called "Cartier divisors") and Weil divisors essentially agree.

**Definition 23.** The ring of meromorphic functions on $C$, $\mathcal{M}_C$, is obtained by inverting all coordinates in $\mathcal{O}_C$.[3]

A particular meromorphic function spans a 1–dimensional $\mathcal{O}_C$–submodule sheaf of $\mathcal{M}_C$, and hence it determines a line bundle. Conversely, a line bundle is determined by local gluing data, which is exactly the data of a meromorphic function. However, it is clear that there is some overdeterminacy in this first operation: scaling a meromorphic function by a nowhere vanishing entire function will not modify the submodule sheaf. This suggests the following operation: to a meromorphic function $f$, we assign the difference of its zero locus and its infinite locus, considered as a divisor. This determines a map

$$\mathcal{M}_C^\times \to (\operatorname{Div} C)(S).$$

**Definition 24.** We then augment this to a scheme $\operatorname{Mer}(C, \mathbb{G}_m)$ of meromorphic functions on $C$ by

$$\operatorname{Mer}(C, \mathbb{G}_m)(R) := \left\{ (u, f) \,\middle|\, \begin{array}{l} u : \operatorname{Spec} R \to S, \\ f \in \mathcal{M}^\times_{C \times_S \operatorname{Spec} R} \end{array} \right\}.$$

**Theorem 14.** *In the case of a formal curve $C$, there is a short exact sequence of formal groups*

Cite me: This is Prop 5.26 in FSFG..

$$0 \to \underline{\mathsf{FormalSchemes}}(C, \mathbb{G}_m) \to \operatorname{Mer}(C, \mathbb{G}_m) \to \operatorname{Div}(C) \to 0. \square$$

[2] That is, the group-completion of $\operatorname{Div}^+ C$ gives $\operatorname{Div} C$, even in absence of a pointing on $C$.

[3] In fact, it suffices to invert any single one.

## 2.3 Feb 12: Projectivization and Thom spaces

**Cite me:** Section 8 of the $H_\infty$ AHS paper..

This title needs improvement.

You are not consistent about calling vector bundles $V$ or $\xi$.

Today we will exploit all of the algebraic geometry we set up yesterday to deduce a load of topological results.

**Definition 25.** Let $E$ be a complex-orientable theory and let $V \to X$ be a complex vector bundle over a space $X$. According to Theorem 9, the cohomology of the Thom space $E^*T(V)$ forms a 1–dimensional $E^*X$–module. We denote the associated line bundle over $X_E$ by $\mathbb{L}(V)$.

This construction enjoys many properties already established.

**Cite me:** Make back-references..

this corollary seems to be stated in the wrong order, as the first half uses virtual bundles which are defined in the second half. AY

**Corollary 12.** *If $f\colon X \to Y$ is a map and $V$ is a virtual bundle over $Y$, then there is an isomorphism*

$$\mathbb{L}(f^*V) \cong (f_E)^*\mathbb{L}(V).$$

*Using Lemma 4, there is also is a canonical isomorphism*

$$\mathbb{L}(V \oplus W) = \mathbb{L}(V) \otimes \mathbb{L}(W).$$

*Finally, this property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles:*

$$\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}. \square$$

*Remark 13.* One of the main utilities of this definition is that it only uses the *property* that $E$ is complex-orientable, and it begets only the *property* that $\mathbb{L}(V)$ is a line bundle.

The following example connects this topic with that of Lecture 2.1:

Make this clearer: point out that the cofiber definition of the Thom space helps you make the first identification (that's what the "zero section" refers to), and then point out that $\zeta^*(M) = M/\mathcal{I}(0) \cdot M$ helps you over the last hump.

In class you wrote $\mathcal{L} - 1$ instead of $\mathcal{L}$ almost everywhere in this example.

I think you should mention that the tilde means the sheaf associated to the module. Also, in class you wrote $\pi_{-2}E$ instead of $\pi_2 E$

It might be confusing using $\varepsilon$ for two different things in the same paragraph.

*Example 20.* If $\mathcal{L}$ denotes the canonical line bundle over $\mathbb{C}\mathrm{P}^\infty$, then the zero section identifies $E^0(\mathbb{C}\mathrm{P}^\infty)^{\mathcal{L}}$ [4] with the augmentation ideal in $E^0\mathbb{C}\mathrm{P}^\infty$, and so we have an isomorphism $\mathbb{L}(\mathcal{L}) \cong \mathcal{I}(0)$. Then, consider the map $\varepsilon : * \to \mathbb{C}\mathrm{P}^\infty$, which classifies a line bundle that Thomifies to $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$. Using naturality, we see

$$\widetilde{\pi_2 E} \cong \mathbb{L}(* \to \mathbb{C}\mathrm{P}^\infty) \cong 0^*\mathcal{I}(0) \cong \omega_{\widehat{\mathbb{G}}_E},$$

where $\widehat{\mathbb{G}}_E = \mathbb{C}\mathrm{P}^\infty_E$ is the formal group associated to $E$–theory and $\omega_{\widehat{\mathbb{G}}_E}$ is its sheaf of invariant differentials[5]. More generally, if $k\varepsilon$ is the trivial bundle of

[4] What does this notation mean? I would guess it's something like maps $\mathcal{L} \to E^0(\mathbb{C}\mathrm{P}^\infty)$, but this doesn't seem to make sense.

[5] The identification of this with the sheaf of invariant differentials is something of a choice. Certainly it is naturally isomorphic to $T_0^*\mathbb{C}\mathrm{P}^\infty_E$, and this in turn is naturally isomorphic to $\omega_{\widehat{\mathbb{G}}_E}$, but deciding which of these two to write is a decision to be borne out as "correct".

dimension $k$ over a point, then $\mathbb{L}(k\varepsilon) \cong \omega_{\widehat{\mathbb{G}}_E}^{\otimes k}$. If $f\colon E \to F$ is an $E$–algebra (e.g., $F = E^{X_+}$), then this gives an interpretation of $\pi_{2k}F$ as $f_E^*\omega_{\widehat{\mathbb{G}}_E}^{\otimes k}$.

Aside from this example, though, this construction on its own does not allow for the ready manipulation of line bundles. However, our discussion yesterday centered on an equivalent presentation of line bundles on a formal curve: their corresponding divisors. Following that cue, we now seek out a topological construction on vector bundles $V \to X$ which produces finite schemes over $X_E$. A quick browse through the literature will lead one to the following:

**Definition 26.** Let $\xi$ be a complex vector bundle of rank $n$ over a base $X$. Define $\mathbb{P}(\xi)$, the *projectivization of* $\xi$, to be the $\mathbb{CP}^{n-1}$–bundle over $X$ whose fiber of $x \in X$ is the space of complex lines in the original fiber $\xi|_x$.

**Theorem 15.** *Take $E$ to be complex-oriented. The $E$–cohomology of $\mathbb{P}(\xi)$ is given by the formula*

$$E^*\mathbb{P}(\xi) \cong E^*(X)[\![t]\!]/c(\xi)$$

*for a certain monic polynomial*

$$c(\xi) = t^n - c_1(\xi)t^{n-1} + c_2(\xi)t^{n-2} - \cdots + (-1)^n c_n(\xi).$$

*Proof.* We fit all of the fibrations we have into a single diagram:

$$\begin{tikzcd}
& \mathbb{C}^\times \arrow[dd, equal] \arrow[dr] & & & \\
\mathbb{C}^n \arrow[dd] & & \mathbb{C}^n \setminus \{0\} \arrow[ll] \arrow[r] \arrow[dd] & \mathbb{CP}^{n-1} \arrow[r] \arrow[dd] & \mathbb{CP}^\infty \arrow[dd, equal] \\
& \mathbb{C}^\times \arrow[dr] & & & \\
\xi \arrow[d] & & \xi \setminus \zeta \arrow[ll] \arrow[r] \arrow[d] & \mathbb{P}(\xi) \arrow[r] \arrow[d, "\pi"] & \mathbb{CP}^\infty \arrow[d] \\
X \arrow[u, bend left, dotted, "\zeta"] \arrow[rr, equal] & & X \arrow[r, equal] & X \arrow[r] & *.
\end{tikzcd}$$

We read this diagram as follows: on the far left, there's the vector bundle we began with, as well as its zero-section $\zeta$. Deleting the zero-section gives the second bundle, a $\mathbb{C}^n \setminus \{0\}$–bundle over $X$. Its quotient by the scaling $\mathbb{C}^\times$–action gives the third bundle, a $\mathbb{CP}^{n-1}$–bundle over $X$. Additionally, the quotient map $\mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is itself a $\mathbb{C}^\times$–bundle, and this induces the structure of a $\mathbb{C}^\times$–bundle on the quotient map $\xi \setminus \zeta \to \mathbb{P}(\xi)$. Thinking of these as complex line bundles, they are classified by a map to $\mathbb{CP}^\infty$, which can itself be thought of as the last vertical fibration, fibering over a point.

Note that the map between these two last fibers is surjective on $E$–cohomology. It follows that the Serre spectral sequence for the third vertical fibration is degenerate, since all the classes in the fiber must survive.[6] We thus conclude that $E^*\mathbb{P}(\xi)$ is a free $E^*(X)$–module on the classes $\{1, t, t^2, \dots, t^{n-1}\}$ spanning $E^*\mathbb{C}\mathrm{P}^{n-1}$. To understand the ring structure, we need only compute $t^{n-1} \cdot t$, which must be able to be written in terms of the classes which are lower in $t$–degree:

$$t^n = c_1(\xi)t^{n-1} - c_2(\xi)t^{n-2} + \cdots + (-1)^{n-1}c_n(\xi)$$

for some classes $c_i(\xi) \in E^*X$. The theorem follows.

In coordinate-free language, we have the following Corollary:

**Corollary 13 (Theorem 15 redux).** *Take $E$ to be* complex-orientable. *The map*

$$\mathbb{P}(\xi)_E \to X_E \times \mathbb{C}\mathrm{P}^\infty_E$$

*is a closed inclusion of $X_E$–schemes, and the structure map $\mathbb{P}(\xi)_E \to X_E$ is free and finite of rank $n$. It follows that $\mathbb{P}(\xi)_E$ is a divisor on $\mathbb{C}\mathrm{P}^\infty_E$ (considered over $X_E$).* □

Be more careful about this "over $X_E$" thing. Maybe just emphasize that having a Chern polynomial with coefficients in $E^*X$ really forces you to take this perspective to make things typecheck.

The next major theorems concerning projectivization are the following:

**Corollary 14.** *The sub-bundle of $\pi^*(\xi)$ consisting of vectors $(v, (\ell, x))$ such that $v$ lies along the line $\ell$ splits off a canonical line bundle.* □

**Corollary 15 ("Splitting principle" / "Complex–oriented descent").** *Associated to any $n$–dimensional complex vector bundle $\xi$ over a base $X$, there is a canonical map $i_\xi \colon Y_\xi \to X$ such that $(i_\xi)_E \colon (Y_\xi)_E \to X_E$ is finite and faithfully flat, and there is a canonical splitting into complex line bundles:*

$$i_\xi^*(\xi) \cong \bigoplus_{i=1}^n \mathcal{L}_i. \square$$

This last Corollary is extremely important. Its essential contents is to say that any question about characteristic classes can be checked for sums of line bundles. Specifically, because of the injectivity of $i_\xi^*$, any relationship among the characteristic classes deduced in $E^*Y_\xi$ must already be true in the ring $E^*X$. The following theorem is a consequence of this principle:

Proposition 8.31 in FSFG shows that the isomorphism $BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{C}\mathrm{P}^\infty_E$ is independent of coordinate. Read it.

**Theorem 16.** *Again take $E$ to be complex-oriented. The coset fibration*

$$U(n-1) \to U(n) \to S^{2n-1}$$

[6] This is called the Leray–Hirsch theorem.

*deloops to a spherical fibration*

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

*The associated Serre spectral sequence*

$$E_2^{*,*} = H^*(BU(n); E^*S^{2n-1}) \Rightarrow E^*BU(n-1)$$

*degenerates at $E_{2n}$ and induces an isomorphism*

$$E^*BU(n) \cong E^*[\![\sigma_1, \ldots, \sigma_n]\!].$$

*Now, let $\xi\colon X \to BU(n)$ classify a vector bundle $\xi$. Then the coefficient $c_j$ in the polynomial $c(\xi)$ is selected by $\sigma_j$:*

Shouldn't this be a polynomial ring instead of a power series ring? Or are you considering the periodified version $EP^0BU(n)$?

$$c_j(\xi) = \xi^*(\sigma_j).$$

*Proof (Proof sketch).* The first part is a standard calculation. To prove the relation between the Chern classes and the $\sigma_j$, the splitting principle states that we can factor complete the map $\xi\colon X \to BU(n)$ to a square

$$\begin{tikzcd} Y_\xi \arrow[r, dotted, "\bigoplus_{i=1}^n \mathcal{L}_i"] \arrow[d, "f_\xi"] & BU(1)^{\times n} \arrow[d, "\oplus"] \\ X \arrow[r, "\xi"] & BU(n). \end{tikzcd}$$

The equation $c_j(f_\xi^*\xi) = \xi^*(\sigma_j)$ can be checked in $E^*Y_\xi$.

We now see that not only does $\mathbb{P}(\xi)_E$ produce a point of $\operatorname{Div}_n^+(\widehat{\mathbb{G}}_E)$, but actually the scheme $\operatorname{Div}_n^+(\widehat{\mathbb{G}}_E)$ itself appears internally to topology:

**Corollary 16.** *For a complex orientable cohomology theory $E$, there is an isomorphism*

$$BU(n)_E \cong \operatorname{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty,$$

*so that maps $\xi\colon X \to BU(n)$ are transported to divisors $\mathbb{P}(\xi)_E \subseteq \mathbb{C}\mathrm{P}_E^\infty \times X_E$. Selecting a particular complex orientation of $E$ begets two isomorphisms*

$$BU(n)_E \cong \widehat{\mathbb{A}}^n, \qquad \operatorname{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty \cong \widehat{\mathbb{A}}^n,$$

*and these are compatible with the centered isomorphism above.*[7] □

[7] Something to take away from this Theorem is the *faithfulness* of this interpretation of the $E$–cohomology of vector bundles. That this map is an isomorphism means that $\operatorname{Div}_n^+$ captures everything that $E$–cohomology can see. There's nothing left in the theory of characteristic classes that is left untouched.

What's most remarkable about the description in this theorem is its coherence with topological facts we know about $BU(n)$. The theorem follows from the projectivization construction, but there are natural operations on both sides of the isomorphism that continue to match up. For instance, the Whitney sum map $BU(n) \times BU(m) \to BU(n+m)$ has the following behavior:

**Lemma 29.** *The sum map*

$$BU(n) \times BU(m) \xrightarrow{\oplus} BU(n+m)$$

*induces on Chern polynomials the identity*

$$c(\xi \oplus \zeta) = c(\xi) \cdot c(\zeta).$$

*In terms of divisors,*

$$\mathbb{P}(\xi \oplus \zeta)_E = \mathbb{P}(\xi)_E \sqcup \mathbb{P}(\zeta)_E,$$

*and hence there is an induced square*

$$\begin{array}{ccc} BU(n)_E \times BU(m)_E & \xrightarrow{\quad\oplus\quad} & BU(n+m) \\ \| & & \| \\ \operatorname{Div}_n^+ \mathbb{CP}_E^\infty \times \operatorname{Div}_m^+ \mathbb{CP}_E^\infty & \xrightarrow{\sqcup} & \operatorname{Div}_{n+m}^+ \mathbb{CP}_E^\infty. \end{array} \square$$

The following is a consequence of combining this Lemma with the splitting principle:

**Corollary 17.** *The map $Y_E \xrightarrow{f_\xi} X_E$ pulls $\mathbb{P}(\xi)_E$ back to give*

$$Y_E \times_{X_E} \mathbb{P}(\xi)_E \cong \bigoplus_{i=1}^{n} \{c_1(\mathcal{L}_i)\}. \square$$

This says that the splitting principle is a topological enhancement of the claim that a divisor can be base-changed along a finite flat map where it splits as a sum of points. The other theorems from yesterday are also easily matched up with topological counterparts:

Draw a table comparing the different notions of vector bundles (stable vs unstable, rank $n$ vs virtual rank $n$) to the different notions of Weil divisors.

**Corollary 18.** *There are natural isomorphisms $BU_E \cong \operatorname{Div}_0 \mathbb{CP}_E^\infty$ and $(BU \times \mathbb{Z})_E \cong \operatorname{Div} \mathbb{CP}_E^\infty$. Additionally, $(BU \times \mathbb{Z})_E$ is the free formal group on the curve $\mathbb{CP}_E^\infty$.* $\square$

**Corollary 19.** *There is a commutative diagram*

$$\begin{array}{ccc} BU(n)_E \times BU(m)_E & \xrightarrow{\quad\otimes\quad} & BU(nm)_E \\ \| & & \| \\ \operatorname{Div}_n^+ \mathbb{CP}_E^\infty \times \operatorname{Div}_m^+ \mathbb{CP}_E^\infty & \xrightarrow{\cdot} & \operatorname{Div}_{nm}^+ \mathbb{CP}_E^\infty, \end{array}$$

*where the bottom map acts by*

$$(D_1, D_2 \subseteq \mathbb{C}\mathrm{P}^\infty_E \times X_E) \mapsto (D_1 \times D_2 \subseteq \mathbb{C}\mathrm{P}^\infty_E \times \mathbb{C}\mathrm{P}^\infty_E \xrightarrow{\mu} \mathbb{C}\mathrm{P}^\infty_E),$$

*and $\mu$ is the map induced by the H-space multiplication $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$.*

*Proof.* By the splitting principle, it is enough to check this on sums of line bundles. A sum of line bundles corresponds to a totally decomposed divisor, and on a pair of such divisors $\bigsqcup_{i=1}^n [a_i]$ and $\bigsqcup_{j=1}^m [b_j]$, the map acts by

$$\left(\bigsqcup_{i=1}^n [a_i]\right)\left(\bigsqcup_{j=1}^m [b_j]\right) = \bigsqcup_{i,j} [\mu_{\mathbb{C}\mathrm{P}^\infty_E}(a_i, b_j)].\square$$

Finally, we can connect our analysis of the divisors coming from topological vector bundles with the line bundles studied at the start of the section.

**Lemma 30.** *Let $\zeta : X_E \to X_E \times \mathbb{C}\mathrm{P}^\infty_E$ denote the pointing of the formal curve $\mathbb{C}\mathrm{P}^\infty_E$, and let $\mathcal{I}(\mathbb{P}(\xi)_E)$ denote the ideal sheaf on $X_E \times \mathbb{C}\mathrm{P}^\infty_E$ associated to the divisor subscheme $\mathbb{P}(\xi)_E$. There is a natural isomorphism of sheaves over $X_E$:*

$$\zeta^* \mathcal{I}(\mathbb{P}(\xi)_E) \cong \mathbb{L}(\xi).\square$$

*Remark 14.* In terms of a complex-oriented $E$ and Theorem 15, the effect of pulling back along the zero section is to set $t = 0$, which collapses the Chern polynomial to just the top class $c_n(\xi)$. This element, called *the Euler class of $\xi$*, provides the $E^*X$–module generator of $E^*T(\xi)$ — or, equivalently, the trivializing section of $\mathbb{L}(\xi)$.

**Theorem 17.** *A trivialization $t\colon \mathbb{L}(\mathcal{L} - 1) \cong \mathcal{O}_{\mathbb{C}\mathrm{P}^\infty_E}$ of the Thom sheaf associated to the canonical bundle induces a ring map $MU \to E$.*

*Proof.* Suppose that $\xi$ is a rank $n$ vector bundle over $X$, and let $f\colon Y \to X$ be the space guaranteed by the splitting principle to provide an isomorphism $f^*\xi \cong \bigoplus_{j=1}^n \mathcal{L}_j$. The chosen trivialization $t$ then pulls back to give a trivialization of $\mathcal{I}(\mathbb{P}(f^*\xi)_E)$, and by finite flatness this descends to also give a trivialization of $\mathcal{I}(\mathbb{P}(\xi)_E)$. Pulling back along the zero section gives a trivialization of $\mathbb{L}(\xi)$. Then note that the system of trivializations produced this way is multiplicative, as a consequence of $\mathbb{P}(\xi \oplus \zeta)_E \cong \mathbb{P}(\xi)_E \sqcup \mathbb{P}(\zeta)_E$. In the universal examples, this gives a sequence of compatible maps $MU(n) \to E$ which assemble on the colimit $n \to \infty$ to give the desired map of ring spectra.

I think another definition of the Thom space is as the cofiber of $\mathbb{P}(V) \to \mathbb{P}(V \oplus \mathbb{C})$. This might come in handy.

Say that the top Chern class is the Euler class / the Thom class.

## 2.4 Feb 17: Operations and a model for cobordism

Our eventual goal, like in Case Study 1, is to give an algebro-geometric description of $MU_*(*)$ and of the cooperations $MU_*MU$. There is such a description that passes through the Adams spectral sequence, also like last time, but $MU_*(*)$ is an integral algebra and so we cannot make do with working out the mod–2 Adams spectral sequence alone. We would have to at least work out the mod–$p$ Adams spectral sequence for every $p$, but there is the following unfortunate theorem:

**Theorem 18.** *There is an isomorphism*

$$H\mathbb{F}_p P_0 H\mathbb{F}_p P \cong \mathbb{F}_p[\xi_0^{\pm}, \xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots]$$

*with* $|\xi_j| = 2p^j - 2$ *and* $|\tau_j| = 2p^j - 1$. □

There are odd–dimension classes in this algebra, and because we are no longer working in characteristic 2 we see that the dual mod–$p$ Steenrod algebra is *graded-commutative*. This is the first time we have encountered Hindrance #4 from Lecture 1.3 in the wild, and for now we will simply avoid these methods and find another approach.

There is such an alternative proof, due to Quillen, that bypasses the Adams spectral sequence. This approach has some deficiencies of its own: it requires studying the algebra of operations $MU^*MU$, which we do not expect to be at all commutative, and it requires studying *power operations*, which are in general very technical creatures. However, we will eventually want to talk about power operations anyway, and because this is the road less traveled we will elect to take it. Our job today is to define these two kinds of cohomology operations, as well as revisit the model of complex cobordism Quillen uses.

The description of the first class of operations follows immediately from our discussion of complex cobordism up to this point, so we will begin there. We learned in Corollary 16 that for any complex-oriented cohomology theory $E$ we have the calculation

$$E^*BU \cong E^*[\![\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots]\!],$$

and we gave a rich interpretation of this in terms of divisor schemes:

$$BU_E \cong \operatorname{Div}_0 \mathbb{C}\mathrm{P}^\infty_E.$$

Two lectures ago, we learned that the stable divisor scheme has a universal property: it is the free formal group on the formal curve $\mathbb{C}\mathrm{P}^\infty_E$. Another avatar of this same fact is a description of the *homology ring*, using the maps

$$E_* BU(n) \otimes E_* BU(m) \to E_* BU(n+m)$$

to induce a multiplicative structure on $E_* BU$:

**Corollary 20.** *Let E be a complex-orientable cohomology theory. Then:*

$$E_* BU \cong \mathrm{Sym}_{E_*} \widetilde{E}_* \mathbb{C}\mathrm{P}^\infty.$$

*A specific complex orientation of E begets*

What is this a corollary of? Have you proven this?

$$E_* \mathbb{C}\mathrm{P}^\infty \cong E_*\{\beta_0, \beta_1, \dots, \beta_n, \dots\}$$

*and hence*

$$E_* BU \cong \mathrm{Sym}_{E_*} E_*\{\beta_1, \beta_2, \dots\} = E_*[b_1, b_2, \dots]. \square$$

**You owe a proof of:** Free formal schemes agree with symmetric Hopf algebras on comodules.

Thomifying these "$\oplus$" maps gives maps

$$E_* MU(n) \otimes E_* MU(m) \to E_* MU(n+m),$$

and the naturality of the $E$–Thom isomorphism produces an additional corollary:

**Corollary 21.** *The Thom isomorphism $E_* BU \cong E_* MU$ respects both the $E_*$–module structure and the ring structure. Hence,*

$$E_* MU \cong E_*[c_1, c_2, \dots, c_n, \dots],$$

*where $c_j$ is the image of $b_j$ under the Thom map.* $\square$

This compact description of $E_* MU$ as an algebra will be useful to us later, but right now we are interested in $E^* MU$ and especially in $MU^* MU$. The former is *not* a ring, and although the latter is a ring its multiplication is exceedingly complicated. Instead, we will content ourselves with an $E_*$–module basis:

A corollary of the splitting principle is supposed to be that a Thom isomorphism for $\mathbb{C}\mathrm{P}^\infty$ begets Thom isomorphisms for everything, and hence a ring spectrum map $MU \to E$. We should produce that corollary now. This is Lemma II.4.6 in Adams's blue book.

Does this totally prohibit us from giving a formal group re-exposition of Quillen's proof? I wonder...

Why is it not a ring? Isn't $E$ a ring spectrum?

**Definition 27.** Let $\alpha = (\alpha_1, \dots, \alpha_n, \dots)$ denote a multi-index where every entry is nonnegative and almost every entry is zero, and let $c_\alpha$ denote the corresponding monomial

$$c_\alpha = \prod_{j=1}^{\infty} c_j^{\alpha_j}.$$

Additionally, we let $s_\alpha \in E^* MU$ denote the image of $c_\alpha$ under the duality isomorphism

$$E^* MU = \mathsf{Modules}_{E_*}(E_* MU, E_*).$$

It is called the $\alpha^{th}$ *Landweber–Novikov operation* (from $MU$ to $E$).

**Cite me:** I.5.1 in Adams's blue book.

Why is this duality isomorphism an isomorphism? You must be using corollary 2.4.3 somehow, but I can't see the argument. Do we already know $E^*MU$? Also, some of the grading is bugging me, but I guess all these issues go away because everything is in degree 0 by periodification?

*Remark 15.* Let $E = MU$. The Landweber–Novikov operations are the *stable* operations acting on $MU$–cohomology, analogous to the Steenrod operations we started the semester talking about. They satisfy the following properties:

- $s_0$ is the identity.
- $s_\alpha$ is natural: $s_\alpha(f^*x) = f^*(s_\alpha x)$.
- $s_\alpha$ is stable: $s_\alpha(\sigma x) = \sigma(s_\alpha x)$.
- $s_\alpha$ is additive: $s_\alpha(x+y) = s_\alpha(x) + s_\alpha(y)$.
- $s_\alpha$ satisfies a Cartan formula. Define

  $$s_{\mathbf{t}}(x) := \sum_\alpha s_\alpha(x)\mathbf{t}^\alpha := \sum_\alpha s_\alpha(x) \cdot t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \cdots$$

  for an infinite sequence of indeterminates $t_1, t_2, \ldots$. Then:

  $$s_{\mathbf{t}}(xy) = s_{\mathbf{t}}(x) \cdot s_{\mathbf{t}}(y).$$

- Let $\xi\colon X \to BU(n)$ classify a vector bundle and let $\varphi$ denote the Thom isomorphism

  $$\varphi\colon MU^*X \to MU^*T(\xi).$$

  Then the Chern classes of $\xi$ are related to the Landweber–Novikov operations on the Thom spectrum by the formula

  $$\sum_\alpha \varphi c_\alpha(\xi)\mathbf{t}^\alpha = \sum_\alpha s_\alpha \varphi(1)\mathbf{t}^\alpha.$$

I don't understand where everything is landing. What is 1 on the right hand side? It seems $s_\alpha\varphi(1) \in E^*T(\xi)$, but $c_\alpha(\xi) \in E^*X$, so how do I apply $\varphi$ to get…Wait, nvm. The $\varphi$ on the left is not the $\varphi$ in the display above, but rather the $E$-Thom isomorphism $E^*X \to E^*T(\xi)$. Maybe you can decorate the $\varphi$ to distinguish them?

Jeremy Hahn, following Rudyak, produced a proof of the incidence relation which doesn't rely on this (particular) geometric model of complex bordism. His write-up of the $p = 2$ case is elsewhere in the repository. The end of this lecture and all of the next one should be reworked to use this other perspective! Manifolds are gross.

Here's a copy-paste of what Rudyak has to say about power operations.

**Cite me:** Rudyak Definition 7.4.

Let $EC_n^{(m)}$ denote the $m$–skeleton of the bar-complex model of the contractible space with a free $C_n$–action. For a point space $X$, we set $\Gamma_n^+(X) := (EC_n^{(m)})^+ \wedge_{C_n} X^{\wedge n}$, so that $\operatorname{colim}_n \Gamma_n^+(X) = X^{\wedge n}_{hC_n}$. If $\xi\colon X \to BU(k)$ selects a complex vector bundle on $X$, then we get a complex vector bundle $p^*\xi^{\times n} \to EC_n^{(m)} \wedge X^{\times n}$, and there is a quotient map

$$(p^*\xi^{\times n})/C_n \to \Gamma_m(X)$$

which is also a complex vector bundle $\xi(m)$ of dimension $n \cdot \dim \xi$. This intertwines with Thomification by

$$T(\xi(n)) = \Gamma_m^+(T\xi).$$

I don't think I got all the $m$s and $n$s straight here.

The external Steenrod–tom Dieck operation has signature

$$EP_n^{2r} \colon \widetilde{MU}^{2r}(X) \to \widetilde{MU}^{4r}(\Gamma_n^+(X)).$$

Beginning with a class

$$f \colon S^{2l}X \to MU(l+r),$$

which induces a map

$$h \colon \widetilde{MU}^{2n(l+r)}(\Gamma_m^+(MU(l+r))) \to \widetilde{MU}^{2n(l+r)}\Gamma_m^+(S^{2l}X) \cong \widetilde{MU}^{2nr}\Gamma_m^+(X)$$

which images the Thom–Dold class

$$U_{\gamma^{l+r}} \in \widetilde{MU}^{2\cdot n\cdot(l+r)}(\Gamma_m^+(MU(l+r)))$$

to

$$EP_m^{2r}(a) := h(U_{\gamma^{l+r}(n)}).$$

A different way of thinking of this is to assume that $MU$ has a sufficiently commutative multiplication so that there is a factorization

$$MU^{\wedge n} \to MU^{\wedge n}_{hC_n} \xrightarrow{\mu} MU.$$

Given a class $x \in MU^*X$, you then get a map

$$\begin{tikzcd}
 & X \arrow[r, "\Delta"] \arrow[d] & X^{\wedge n} \arrow[r, "x^{\wedge n}"] \arrow[d] & MU^{\wedge n} \arrow[d] & \\
X \times BC_n \arrow[r, equal] & X_{hC_n} \arrow[r] & X^{\wedge n}_{hC_n} \arrow[r] & MU^{\wedge n}_{hC_n} \arrow[r, "\mu"] & MU
\end{tikzcd}$$

by taking the composite on the bottom row. This is the same map. It has the behavior of raising $x$ to the $n^{\text{th}}$ power if you restrict to the basepoint inside of $BC_n$, but there is more data present beyond that.

What follows is the original content of the lecture.

We now turn to the construction of the other cohomology operations we will be interested in: the power operations. Power operations get their name from their *multiplicative* properties, and correspondingly we do not (*a priori*) expect them to be additive operations, so they are quite distinct from the Landweber–Novikov operations. Power operations arise from "$E_\infty$" structures on ring spectra[8], but most such structures arise in nature from geo-

[8] Or, by some accounts, "$H_\infty$" structures.

metric models of cohomology theories. To produce them for complex cobordism, we will use a particular model, alluded to in Lecture 0.1.

It is very annoying that you tend to switch $f$, $i$, and $j$; $X$, $Y$, and $Z$; what is attached to what; and what is drawn in what direction. You'd do well to standardize this.

**Definition 28.** Let $f : Y \to X$ be a map of manifolds. A *complex-orientation on the map $f$* is the data of a factorization

$$\begin{tikzcd} & E \arrow[d] \\ Y \arrow[ur, dotted, "i"] \arrow[r, "f"] & X \end{tikzcd}$$

through a complex vector bundle $E$ on $X$ such that $i$ is an embedding and its normal bundle $\nu_i$ has a complex structure.Two such factorizations are *equivalent* when they appear as subbundles of a larger bundle and the embeddings are isotopic, compatibly with the structures on their normal bundles.

Account for the odd-dimensional case and the dimension-jumping case.

**Lemma 31.** *For* $\dim E \gg 0$*, this equivalence class is unique, if it exists.* □

I think you can at least give a heuristic argument here. You haven't spelled out the precise definition above, but if I 'm not mistaken this just boils down to the fact that if the rank of $E$ is large relative to the dimension of $Y$ (say at least twice as big), then any two embeddings are isotopic.

**Definition 29.** Two complex-oriented maps $f_0 \colon Y_0 \to X$ and $f_1 \colon Y_1 \to X$ are called *cobordant* when there is a complex-oriented map $W \to X \times \mathbb{R}$ and elements $b_0, b_1 \in \mathbb{R}$ such that

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & X \times \{b_0\} \arrow[d] \\ W \arrow[r] & X \times \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[r] \arrow[d] & X \times \{b_1\} \arrow[d] \\ W \arrow[r] & X \times \mathbb{R} \end{tikzcd}$$

become pull-back squares of complex-oriented maps of manifolds.

**Cite me:** This is cited as [Tho51] in Matt's thesis.

**Theorem 19 (Thom).** *For a manifold $X$, $MU^{-q}(X)$ is canonically isomorphic to the cobordism classes of complex-oriented maps of dimension $q$.* □

Remark that this, as expected, puts the cobordism ring into negative degrees.

*Remark 16.* This model has a variety of nice features. For instance, its two variances are visible from the construction. For a map $g \colon X' \to X$, there is an induced map $g^* \colon MU^*X \to MU^*X'$ given by selecting a class $f \colon Y \to X$, perturbing $g$ so that it is transversal to $f$, and taking the pullback

$$\begin{tikzcd} Y \times_X X' \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X. \end{tikzcd}$$

But, also, if $g$ is additionally proper and complex-orientable, then it induces a wrong way map
$$g_*\colon MU^{-q}X' \to MU^{-q-d}X,$$
where $d$ is the dimension of $g$. This is simply by postcomposition: a representative $f'\colon Y' \to X'$ begets a new representative $g_* f' = g \circ f'$. This construction goes by various names: the *Gysin map*, the *complex-oriented pushforward*, the *shriek map*, ....

Additionally, these push and pull maps are related:

**Lemma 32.** *Consider a Cartesian square of manifolds*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X, \end{array}$$

*where $g$ is transversal to $f$, $f$ is proper and complex-oriented, and $f'$ is endowed with the pull-back of the complex orientation of $f$. Then*

$$g^* f_* = f'_*(g')^*\colon MU^{-q}(Z) \to MU^{-q-d}(Y).\square$$

We are now in a position to describe the power operations.

**Definition 30.** Consider a class in $MU^{-2q}(X)$ represented by a proper complex-oriented map $f\colon Y \to X$. Its $n$–fold Cartesian product determines a class $f^{\times n}\colon Y^{\times n} \to X^{\times n}$, and taking the homotopy quotient by a group $G$ acting transitively on $\{1, \ldots, n\}$ gives a class

$$Y^{\times n} \to X^{\times n} \to EG \times_G X^{\times n}$$

and hence an *external power operation*

$$P^{\text{ext}}\colon MU^{-2q}(X) \to MU^{-2qn}(EG \times_G X^{\times n}).$$

Pulling back along the diagonal $\Delta : X \to X^{\times n}$ gives the the *internal power operation*
$$P\colon MU^{-2q}(X) \to MU^{-2qn}(BG \times X).$$

Its action on the class represented by a proper complex-oriented even-dimensional map $f : Z \to X$ can also be written as

$$P(f_* 1) = \Delta^* f^{\times n}_{hG\,*} 1.$$

I think Jay pointed out this after class, but the two 1's on either side mean slightly different things. Also, by 1 do you mean the unit element in the ring $MU^*Z$? It took me a while to realize that $MU^*Z$ was a ring...

*Remark 17.* It's apparent that we've really needed this geometric model to accomplish this construction: we needed to understand how to take Cartesian powers of maps in a way that inherited a $G$–action. This is not data that a ring spectrum is naturally equipped with, and if we were to tease out exactly what extra information we need to encode this operation, we would eventually arrive at the notion of an $E_\infty$–ring spectrum.

*Remark 18.* A picky reader will (rightly) point out that $BG$ is not a manifold, and so we shouldn't be mixing it with out geometric model for $MU$. This is a fair point, but since $BG$ can be approximated through any cellular dimension by a manifold, we won't worry about it.

*Remark 19.* The chain model for ordinary homology is actually rigid enough to define power operations there, too. Curiously, they are all generated by the quadratic power operations (i.e., the "squares"), and all the quadratic power operations turn out to be *additive* — that is, you just get the Steenrod squares again! This appears to be a lucky degeneracy, but tomorrow we will exploit something very similar with a particular power operation in complex cobordism.

Can we name some of the formal properties of power operations? Multiplicativity, say?

## 2.5 Feb 19: An incidence relation among operations

Danny pointed out that this is a little confused about fixed points versus orbits and homotopy vs genuine. Make sure this is straightened out.

It would be nice if all the Cartesian diagrams in this section were typeset with the little pullback corners.

Our goal today is to apply a version of Lemma 32 to the push-pull definition of the power operation for $MU$ given in Definition 30. The relevant Cartesian square in that case has the form

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"] & EG \times_G Y^{\times k} \arrow[d, "f_{hG}^{\times k}"] \\ BG \times X \arrow[r, "\Delta"] & EG \times_G X^{\times k}. \end{tikzcd}$$

However, since we have so little control over vertical map $f_{hG}^{\times k}$, we can't rely on the other hypotheses of Lemma 32 to be satisfied. So, we investigate the following slightly more general situation.

**Definition 31.** Let $X$ be a manifold. Two closed submanifolds $Y$ and $Z$ are said to *intersect cleanly* when $W = Y \cap Z$ is a submanifold and for each $w \in W$, the tangent space of $W$ at $w$ is given by $T_w W = T_w Y \cap T_w Z$. In this case, we draw a Cartesian square

$$\begin{tikzcd} W \arrow[r, "j'"] \arrow[d, "i'"] & Z \arrow[d, "i"] \\ Y \arrow[r, "j"] & X. \end{tikzcd}$$

The *excess bundle* of the intersection, $F$, is defined by the exact sequence

We had to stare at this in class to decide that it was reasonable.

$$\begin{tikzcd} & & (i')^*TY \arrow[dr] & & & \\ 0 \arrow[r] & \nu_{i'} \arrow[ur] \arrow[rr] & & (j')^*\nu_i \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

What is the map $(i')^*TY \to (j')^*\nu_i$?

*Remark 20.* The submanifolds $Y$ and $Z$ intersect transversally exactly when $F = 0$.

The proof of the following Lemma is fairly easy, but geometric, so we omit it.

**Lemma 33 ([30, Proposition 3.3]).** *Suppose that $\nu_{i'}$, $\nu_i$, and $F$ are endowed with complex structures compatible with this exact sequence. For $z \in MU^*(Z)$,*

$$j^* i_* z = i'_*(e(F) \cdot (j')^* z)$$

*in $MU^{*+a}(Y, Y \setminus W)$, where $a = \dim \nu_i$.* □

Mention what $e(F)$ is (the Euler class of $F$, right?)

Now let $G$ be a finite group and let $i\colon Z \to X$ be an embedding of $G$–manifolds. Then the $G$–fixed submanifold $X^G$ and $Z$ intersect cleanlyin the diagram

Why?

$$\begin{tikzcd} Z^G \arrow[r, "r_Z"] \arrow[d, "i^G"] & Z \arrow[d, "i"] \\ X^G \arrow[r, "r_X"] & X. \end{tikzcd}$$

Since $r_Z^*(\nu_i)$ is a $G$–bundle over a trivial $G$ space, there is a decomposition $r_Z^*(\nu_i) = \nu_{i^G} \oplus \mu_i$, where $\nu_{i^G}$ has no $G$–action and $\mu_i = F$, the excess bundle, carries all of the nontrivial $G$–action. Applying $EG \times_G (-)$ to the diagram and picking $z \in MU^*(EG \times_G Z)$, Lemma 33 then gives

$$r_X^* i_* z = i_*^G(e(\mu_i) \cdot r_Z^* z) \in MU^*(BG \times X^G, (BG \times X^G) \setminus (BG \times Z^G)).$$

Replacing the embedding condition with orientability, this gives the following:

**Lemma 34 ([30, Proposition 3.8]).** *Let $f\colon Z \to X$ be a proper complex-oriented $G$-map, represented by a factorization*

$$Z \xrightarrow{i} E \xrightarrow{p} X.$$

*Let $\mu(E)$ be excess summand of $r_X^* E$ corresponding to the part of $E$ on which $G$ acts nontrivially, where, as before, $r_X$ is the inclusion of the fixpoint submanifold $X^G \subseteq X$. Then, for $z \in MU^*(EG \times_G Z)$, we have:*

$$e(\mu(E)) \cdot r_X^* f_* z = f_*^G(e(\mu_i) \cdot r_Z^* z) \in MU^*(BG \times X^G).\square$$

We are now in a position to apply Lemma 34 to our power operation square.

There's no reason to use $m$, then $r$, then change $r$'s name to $n$ in Lecture 2.6. Straighten out this terrible naming scheme.

Make it clearer what you mean here. You want the witness to the complex–orientability of $f$ to be homotopically independent of choice.

**Lemma 35 ([30, Proposition 3.12]).** *Suppose $G$ acts transitively on $\{1, \ldots, k\}$ and let $\rho$ denote the induced reduced regular $G$-representation. Suppose that $f : Z \to X$ is a proper complex-oriented map of dimension $2q$ and that $m$ is an integer largerthan the dimension of $Z$, so that $m\varepsilon + \nu_f$ is a vector bundle over $Z$, well-defined up to isomorphism, where $\varepsilon$ is the trivial complex line bundle. Then*

$$e(\rho)^m P(f_* 1) = f_* e(\rho \otimes (m\varepsilon + \nu_f)) \in MU^{2m(k-1)-2qk}(BG \times X).$$

*Proof.* We can take $m$ large enough that the complex-orientation on $f$ can be represented by a factorization

$$Z \xrightarrow{i} m\varepsilon \xrightarrow{p} X,$$

and consider its $k^{\text{th}}$ power

$$Z^{\times k} \xrightarrow{i^{\times k}} (m\varepsilon)^{\times k} \xrightarrow{p^{\times k}} X^{\times k}.$$

We calculate the excess bundles to be

$$\mu_{i^{\times k}} = \rho \otimes \nu_i, \qquad \mu((m\varepsilon)^{\times k}) = \rho \otimes m\varepsilon.$$

Since $G$ acts transitively, $\Delta : X \to X^{\times k}$ represents the inclusion of the $G$–fixed points. Packaging all this into Lemma 34 gives

$$e(\rho \otimes m\varepsilon) \cdot \Delta^* f_{hG*}^{\times k}(1) = f_*(e(\rho \otimes \nu_i) \cdot r^*_{W \to Z^{\times k}}(1)).$$

The claim about $\nu_i = m\varepsilon + \nu_f$ is a little mysterious. We had to stare at it too before it became believable.

We then investigate each part separately:

$$e(\rho \otimes m\varepsilon) = e(\rho^{\oplus m}) = e(\rho)^m, \qquad \Delta^* f_{hG*}^{\times k}(1) = P(f_* 1),$$
$$e(\rho \otimes \nu_i) = e(\rho \otimes (m\varepsilon + \nu_f)), \qquad r^*_{W \to Z^{\times k}}(1) = 1$$

from which the claim follows.

The utility of this theorem comes from our ability to compute just a little bit about the Euler classes involved in its statement.

**Corollary 22 ([30, Proposition 3.17]).** *Specialize to $G = C_k$, and let $\eta$ denote the line bundle on $BG$ owing to the inclusion $C_k \subseteq U(1)$. Set $e(\eta) = v$ and $e(\rho) = w$. Then, the Steenrod operation and Landweber operations are related by the formula*

Should we use $a_\alpha(v)$ as the notation?

In previous sections, you've been using $\xi$ to denote arbitrary vector bundles, not $E$.

$$w^{r+q} Px = \sum_{|\alpha| \leq r} w^{r-|\alpha|} a(v)^\alpha s_\alpha(x)$$

*for $x \in MU^{-2q}(X)$ and $r$ is any integer sufficiently large with respect to* $\dim X$ *and $q$, where $a_j(T)$ are power series with coefficients in the subring $C$ generated by the coefficients of the tautological formal group law on $MU^*(*)$.*

*Proof.* The bundle $\rho$ splits as $\bigoplus_{i=1}^{k-1} \eta^{\otimes i}$. Then, if $\mathcal{L}$ is any other line bundle with a trivial $G$–action,

$$\begin{aligned} e(\rho \otimes \mathcal{L}) &= e\left(\bigoplus_{i=1}^{k-1} \eta^i \otimes \mathcal{L}\right) = \prod_{i=1}^{k-1} e(\eta^i \otimes \mathcal{L}) \\ &= \prod_{i=1}^{k-1} F([i]_F(v), e(\mathcal{L})) = w + \sum_{j=1}^{\infty} a_j(v) e(\mathcal{L})^j, \end{aligned}$$

where

$$w = e(\rho) = (k-1)! v^{k-1} + \sum_{j \geq k} b_j v^j$$

You could justify this part. The point is to look at the product of all the factor summands which don't involve $e(\mathcal{L})$ at all.

for $b_j \in C$. In general, the splitting principle shows that $e(\rho \otimes E)$ has

$$e(\rho \otimes E) = \sum_{|\alpha| \leq r} w^{r-|\alpha|} a(v)^\alpha c_\alpha(E).$$

Setting $E = m\varepsilon + \nu_f$, we calculate $r = \dim(m\varepsilon + \nu_f) = m - q$. Inserting this into Lemma 35 then gives

$m\varepsilon$ already has rank $m$. Why does $m\varepsilon + \nu_f$ have rank $m - q$ instead of something like $m + 2q$?

$$\begin{aligned} w^m P(f_* 1) &= f_* \left( \sum_{|\alpha| \leq r} w^{r-|\alpha|} a(v)^\alpha c_\alpha(m\varepsilon + \nu_f) \right) \\ w^{r+q} P(f_* 1) &= \sum_{|\alpha| \leq r} w^{r-|\alpha|} a(v)^\alpha f_* c_\alpha(m\varepsilon + \nu_f) \\ &= \sum_{|\alpha| \leq r} w^{r-|\alpha|} a(v)^\alpha s_\alpha(f_* 1). \square \end{aligned}$$

Check that this last line is right. Can you pull Gysin maps past Euler classes? What happened to $m\varepsilon$? — are you using the definition of Landweber–Novikov operations for $\nu_i$ instead of $\nu_f$? Why?

This formula is quite remarkable — it says that a certain power operation defined for $MU$ is, in fact, additive and stable (after multiplying by $w$ some)! This is certainly not the case in general, and I'm not aware of an *a priori* reason to expect this to have happened all along. Tomorrow, we will use it to power an induction to say something about the coefficient ring $MU_*$.

In Rudyak's / Jeremy's approach, the main point is that $C_n$–equivariant vector bundles can be traded for ordinary vector bundles with a $BC_n$ factor in the base. So, a decomposition of the $n$–fold sum of the tautological vector bundle, considered as a $C_n$–bundle, induces a decomposition of the associated bundle over $BC_n$, which computes the effect of the $n^{\text{th}}$ power operation. Then, using the splitting principle, one recovers Quillen's incidence Theorem. (I'm too sleepy to really work my way through this. Both Jeremy and Rudyak specialize to the case $n = 2$, and so we'll need to rewrite what they do at an arbitrary prime. This will be easy, but it will require a clear head.)

## 2.6 Feb 22: Quillen's theorem

With Corollary 22 in hand, we deduce Quillen's major structural theorem about $MU_*$. We will continue to use the following notations:

- $C$ is the subring of $MU_*$ generated by the coefficients of the formal group law associated to the identity complex–orientation.
- $G = C_k$ acts by cyclic permutation on $\{1, \dots, k\}$. In particular, the action is transitive.
- $\rho$ is the associated reduced regular representation of rank $k-1$, and $w = e(\rho)$ its Euler class.
- $\eta \colon BC_k \to BU(1)$ is the associated line bundle, and $v = e(\eta)$ its Euler class.

**Theorem 20 ([30, Theorem 5.1]).** *If $X$ has the homotopy type of a finite complex, then*

$$MU^*(X) = C \cdot \sum_{q \geq 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X).$$

*Proof.* We can focus on the claim

$$\widetilde{MU}^{2*}(X) \stackrel{?}{=} C \cdot \sum_{q>0} MU^{2q}(X) =: R^{2*},$$

since $MU^{2*+1}(*) = 0$ and $\widetilde{MU}^{2*+1}(X)$ can be handled by suspending $X$ once, and then the unreduced case follows directly. We will show this by working $p$–locally and inducting on the value of "$*$". Suppose that

Remark on the base case: in all the negative dimensions, the claim is trivial.

$$R^{-2j}_{(p)} = \widetilde{MU}^{-2j}(X)_{(p)}$$

for $j < q$ and consider $x \in \widetilde{MU}^{-2q}(X)$. Then, for $n \gg 0$, we have

$$w^{n+q}Px = \sum_{|\alpha| \leq n} w^{n-|\alpha|} a(v)^\alpha s_\alpha x.$$

Recall that $w$ is a power series in $v$ with coefficients in $C$ and leading term $(p-1)!v^{p-1}$, so that $v^{p-1} = w \cdot \theta(v)$ for some invertible series $\theta$ with coefficients in $C$. Since $s_\alpha$ lowers degree, we have $s_\alpha x \in R$ by the inductive hypothesis, so we may write

Mention that you are fixing a prime $p$ and looking at the $p$th power operation.

$$v^m(w^q Px - x) = \psi_x(v)$$

with $\psi_x(T) \in R_{(p)}[\![T]\!]$.

Suppose $m \geq 1$ is the least integer for which we can write such an equation — we will show $m = 1$ in a moment. Applying the inclusion $i \colon X \to X \times B\mathbb{Z}/p$ to this equation sets $v = 0$ and yields $\psi_x(0) = 0$, hence $\psi_x(T) = T\varphi_x(T)$ and

$$v(v^{m-1}(w^q Px - x) - \varphi_x(v)) = 0.$$

Since $v$ annihilates this equation, we can use the Gysin sequence associated to the spherical bundle

$$S^1 \to S(\eta) \to BC_p$$

to produce a class $y \in \widetilde{MU}^{2(m-1)-2q}(X)$ with

It's not clear (from this presentation) why $\langle p \rangle(v)$ is involved in this sequence or where the shift by $-1$ in the dimension went. I'm a little confused about Quillen's presentation of the total space as "$S^\infty \times_{C_p} S^1$", too.

When you figure this out, can you write down the Gysin sequence?

$$v^{m-1}(w^q Px - x) = \varphi_x(x) + y\langle p \rangle(v).$$

If $m > 1$, then $y \in R_{(p)}$ for degree reasons and hence the right-hand side gives an equation contradicting our minimality hypothesis. So, $m = 1$, and the outer factor of $v^{m-1}$ is not present in the last expression. Restricting along $i$ again, we obtain the equation

$$\left.\begin{array}{r} -x \text{ if } q > 0 \\ x^p - x \text{ if } q = 0 \end{array}\right\} = \varphi_x(0) + py.$$

In the first case, where $q > 0$, it follows that $MU^{-2q}(X) \subseteq R^{-2q} + pMU^{-2q}(X)$, and since $MU^{-2q}(X)$ has finite order torsion, it follows that $MU^{-2q}(X) = R^{-2q}$. In the other case, $x$ can be rewritten as a sum of things in $R^0$, things in $pMU^0(X)$, and things in $(MU^0)^p$. Since the ideal $\widetilde{MU}^0(X)$ is nilpotent, it follows that $\widetilde{MU}^0(X) = R^0$, and induction proves the theorem.

**Corollary 23.** *The coefficients of the formal group law span $MU_*$.* □

*Remark 21.* This proof actually also goes through for $MO$ as well. In that case, it's even easier, since the equation $2 = 0$ in $\pi_0 MO$ causes much of the algebra to collapse. One can try to further perturb this proof in two ways:

1. One can try to replace the identity complex–orientation $MU \xrightarrow{\mathrm{id}} MU$ with a nontrivial complex–orientation $MU \xrightarrow{\varphi} E$ which is suitably compatible with power operations. It would be nice to understand why this doesn't give more information about $E$ than what's visible in the Hurewicz image of $\varphi$. Or, conversely, it would be nice to understand a proof of Mahowald's theorem that the free $E_2$–algebra with $p = 0$ is $H\mathbb{F}_p$, which this proof portends to give information about.

Straighten this out.

2. One can also try to replace $MO$ and $MU$ with $MSp$ or $MSO$. These, too, have presentations in terms of bordism theories and hence similar power operations to the ones we used above. On the other hand, the Euler classes in $MSp$–theory, while simple, are not so well-behaved, because they are not controlled by a formal group law. Characteristic classes in $MSO$–theory are not even simple.

This isn't well-stated either.

We now have a foothold on $\pi_* MU$, and this alone is enough to move us to study $\mathcal{M}_{\mathbf{fgl}}$, the moduli scheme of formal group laws. However, while we're here, it's possible for us to prove the rest of Quillen's theorem, if we get just slightly ahead of ourselves and assume one algebraic fact about $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. The place to start is with the following topological observation about mixing complex–orientations:

**Lemma 36 ([1, Lemma 6.3 and Corollary 6.5]).** *Let $\varphi\colon MU \to E$ be complex–oriented and consider the two orientations*

$$\mathbb{S} \wedge MU \xrightarrow{\eta_E \wedge 1} E \wedge MU, \qquad MU \wedge \mathbb{S} \xrightarrow{\varphi \wedge \eta_{MU}} E \wedge MU.$$

*The two induced coordinates $x^E$ and $x^{MU}$ on $\mathbb{C}\mathrm{P}^\infty_{E \wedge MU}$ are related by the formulas*

$$x^{MU} = \sum_{j=0}^{\infty} b_j^E (x^E)^{j+1} = g(x^E),$$
$$g^{-1}(x^{MU} +_{MU} y^{MU}) = g^{-1}(x^E) +_E g^{-1}(y^E).$$

*where $E_* MU \cong E_*[b_1, b_2, \ldots]$.*

*Proof.* The second formula is a direct consequence of the first. The first formula comes from taking the module generators $\beta_{j+1} \in E_{2(j+1)}\mathbb{C}\mathrm{P}^\infty = E_{2j}MU(1)$ and pushing them forward to get the algebra generators $b_j \in E_{2j}MU$. Then, the triangle

$$\begin{tikzcd}
{[\mathbb{C}\mathrm{P}^\infty, MU]} \arrow[rr] \arrow[dr] & & {[\mathbb{C}\mathrm{P}^\infty, E \wedge MU]} \arrow[dl, "\cong"] \\
& \mathsf{Modules}_{E_*}(E_*\mathbb{C}\mathrm{P}^\infty, E_*MU) &
\end{tikzcd}$$

allows us to pair $x^{MU}$ with $(x^E)^{j+1}$ to determine the coefficients of the series.

**Corollary 24 ([1, Corollary 6.6]).** *In particular, for the orientation $MU \to H\mathbb{Z}$ we have*

$$x_1 +_{MU} x_2 = \exp^H(\log^H(x_1) + \log^H(x_2)),$$

*where* $\exp^H(x) = \sum_{j=0}^{\infty} b_j x^{j+1}$. □

However, one also notes that $H\mathbb{Z}_* MU = \mathbb{Z}[b_1, b_2, \ldots]$ carries the universal example of a formal group law with a logarithm — this observation is independent of any knowledge about $MU_*$. It turns out that this brings us one step away from understanding $MU_*$:

**Theorem 21 (To be proven as Theorem 25).** *There is a ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ carrying the universal formal group law, and it is free: it is a polynomial ring over $\mathbb{Z}$ in countably many generators.* □

**Corollary 25.** *The map $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ classifying the formal group law on $MU_*$ is an isomorphism.*

*Proof.* We proved in Corollary 23 that this map is surjective. We also proved in Theorem 11 that every rational formal group law has a logarithm, i.e., the long composite

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} \to MU_* \otimes \mathbb{Q} \xrightarrow{\cong} (H\mathbb{Z}_* MU) \otimes \mathbb{Q}$$

is an isomorphism. Using Theorem 21, it follows that the map is also injective, hence an isomorphism.

**Corollary 26.** *The ring $\pi_*(MU \wedge MU)$ carries the universal example of two strictly isomorphic formal group laws. Additionally, the ring $\pi_0(MUP \wedge MUP)$ carries the universal example of two isomorphic formal group laws.*

*Proof.* Combine Lemma 36 and Corollary 25.

There's buzz about a "Frobenius map" for structured rings going around these days. I guess the point is that an $E_2$–algebra structure is enough to get a multiplicative map $E^0 X \to E^0 X \otimes E^0 BC_p$. This isn't additive, so it can't come from an infinite loop map, but it becomes additive when passing to the Tate construction: $E^X \to (E^X)^{tC_p}$, using the fact that the genuine $C_p$ fixed points of $E^{X \times p}$ is $E^X$, and the square relating genuine, homotopy, and geometric fixed points. Mike has been claiming that these results of Quillen's can be interpreted in this way, but I'm not sure what the interpretation is. He says it has something to do with inverting the Euler class and the part of Quillen's argument that involves walking down the multiples of Euler classes on both sides of the equation.

Make a point about the difference between the two "moduli problems" here (or in the context lecture, Lecture 3.1): the natural map $\mathsf{RingSpectra}(MU /\!/ MU \wedge MU, E) \to \mathcal{M}_{\mathbf{fg}}(E_*)$ given by passing to homotopy groups hits *at most one* connected component. See also the beginning of the next Case Study for a relevant todo.

# Case Study 3
# Finite spectra

Write an introduction for me.

still need to talk about closed and open subschemes, their basic properties

Do you need to come to grips with ind-coherent sheaves? Hovey has a paper called Homotopy theory of comodules over a Hopf algebroid where he defines the stable category of a Hopf algebroid, which is sort of about this perspective — making $BP_*$ into a compact object in a reasonable category of $BP_*BP$–comodules.

## 3.1 Feb 24: The context of a spectrum

I don't think I mention Hopf algebroids during this lecture! This is a miserable oversight that *must* be corrected. Also, I should mention the cotensor product for Hopf algebroids. Update by d.s.: fixed. My reference was appendix I of Ravenel. Feel free to delete anything you think is unnecessary for our purposes. I think some of the homological algebra near the end can be left out.

Callan wrote something at http://chat.stackexchange.com/transcript/message/28980 which addresses the unicity of the formal group attached to a local number field. I thought I drew an analogy to that somewhere in here, but now I can't find it. I would like to.

**Cite me:** Pridham's article *Presenting higher stacks as simplicial schemes* seems like a good reference? Maybe some Toen things are appropriate? I don't really know where this simplicial scheme stuff is written down..

**Definition 32.** A *Hopf Algebroid* over a commutative ring $K$ is a pair $(A, \Gamma)$ of commutative $K$-algebras with structure maps such that for any other commutative $K$-algebra $B$, the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid. The structure maps are

1. $\eta_L : A \to \Gamma$ (source)
2. $\eta_R : A \to \Gamma$ (target)
3. $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ (composition)
4. $\varepsilon : \Gamma \to A$ (identity)
5. $c : \Gamma \to \Gamma$ (inverse)

There are some relations among the structure maps that mimics the defining properties of a groupoid. I won't mention them here but they can be found in Ravenel's green book, appendix I. A graded Hopf algebroid is *connected* if the left and right sub $A$-modules generated by $\Gamma_0$ are both isomorphic to $A$. If $\eta_R = \eta_L$, then $\Gamma$ is a commutative Hopf algebra over $A$.

**Definition 33.** A left $\Gamma$-*comodule* $M$ is a left $A$-module $M$ together with a left $A$-linear map $\psi : M \to \Gamma \otimes_A M$ that is both counitary and coassociative.

From now on, we assume that $\Gamma$ is flat over $A$.

**Definition 34.** Let $M$ be a right $\Gamma$-comodule, and $N$ a left $\Gamma$-comodule. Their *cotensor product* over $\Gamma$ is the $K$-module defined by the exact sequence $0 \to M\Box_\Gamma N \to M \otimes_A N \xrightarrow{\psi\otimes N - M\otimes\psi} M \otimes_A \Gamma \otimes_A N$, where $\psi$ are the comodule structure maps for $M$ and $N$.

Notice that if $M$ is a left comodule, then it can be given the structure of a right comodule by the composition

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M\otimes c} M \otimes \Gamma,$$

where $T$ swaps the two factors and $c$ is the conjugation map. From this, it is easy to deduce that $M\Box_\Gamma N = N\Box_\Gamma M$. The following lemma relates cotensor products to Hom.

**Lemma 37.** *Let $M$ and $N$ be left $\Gamma$-comodules with $M$ projective over $A$. Then*

1. *$Hom_A(M, A)$ is a right $\Gamma$-module.*
2. *$Hom_\Gamma(M, N) = Hom_A(M, A)\Box_\Gamma N$. In particular, when $M = A$, we have $Hom_\Gamma(A, N) = A\Box_\Gamma N$.*

*Proof.* There exist maps $\psi_M^*, \psi_N^* : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, \Gamma \otimes_A N)$, defined by

$$M \xrightarrow{\psi_M} \Gamma \otimes M \xrightarrow{\Gamma\otimes f} \Gamma \otimes_A N,$$

$$M \xrightarrow{f} N \xrightarrow{\psi_N} \Gamma \otimes_A N.$$

Since $M$ is projective over $A$, there is a canonical isomorphism

$$\mathrm{Hom}_A(M, A) \otimes_A N \simeq \mathrm{Hom}_A(M, N).$$

When $N = A$, we obtain the map

$$\psi_M^* : \mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_A(M, A) \otimes_A \Gamma.$$

It is easy to check that this map satisfies the coassociativity axiom.

For the second part, note that by definition, we have

$$\mathrm{Hom}(M, N) = \ker(\psi_M^* - \psi_N^*) \subset \mathrm{Hom}_A(M, N),$$
$$\mathrm{Hom}_A(M, A)\Box_\Gamma N = \ker(\psi_M^* \otimes N - \mathrm{Hom}_A(M, A) \otimes \psi_N).$$

The claim then follows from the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(M,A)\otimes N & \xrightarrow{\simeq} & \mathrm{Hom}_A(M,N) \\ {\scriptstyle \psi_M^*\otimes N}\downarrow\downarrow{\scriptstyle \mathrm{Hom}(M,A)\otimes\psi_N} & & {\scriptstyle \psi_M^*}\downarrow\downarrow{\scriptstyle \psi_N^*} \\ \mathrm{Hom}(M,A)\otimes\Gamma\otimes N & \xrightarrow{\simeq} & \mathrm{Hom}_A(M,\Gamma\otimes_A N) \end{array}$$

**Definition 35.** A *map of Hopf algebroids* $f : (A,\Gamma) \to (B,\Sigma)$ is a pair of $K$-algebra maps $f_1 : A \to B, f_2 : \Gamma \to \Sigma$ such that $f_1\varepsilon = \varepsilon f_2$, $f_2\eta_R = \eta_R f_1$, $f_2\eta_L = \eta_L f_1$, $f_2 c = c f_2$, and $\Delta f_2 = (f_2 \otimes f_2)\Delta$.

Now we will discuss some homological algebra of Hopf algebroids. It turns out that the category of $\Gamma$-comodules has enough injectives, and so we can make the following definition:

**Definition 36.** For left $\Gamma$-comodules $M$ and $N$, $\mathrm{Ext}^i_\Gamma(M,N)$ is the $i$th right derived functor of $\mathrm{Hom}_\Gamma(M,N)$, regarded as a functor of $N$. For $M$ a right $\Gamma$-module, $\mathrm{Cotor}^i_\Gamma(M,N)$ is the $i$th right derived functor of $M\square_\Gamma N$, also regarded as a functor of $N$. The corresponding graded groups are denoted $\mathrm{Ext}_\Gamma(M,N)$ and $\mathrm{Cotor}_\Gamma(M,N)$, respectively.

The next lemma shows that the resolution can satisfy a weaker condition than being injective.

**Lemma 38.** *Let*

$$0 \to N \to R^0 \to R^1 \to \cdots$$

*be a long exact sequence of left* $\Gamma$*-comodules such that* $\mathit{Cotor}^n_\Gamma(M,R^i) = 0$ *for all* $n > 0$*. Then* $\mathit{Cotor}_\Gamma(M,N)$ *is the cohomology of the complex*

$$\mathit{Cotor}^0_\Gamma(M,R^0) \xrightarrow{\delta_0} \mathit{Cotor}^0_\Gamma(M,R^1) \xrightarrow{\delta_1} \cdots .$$

There are many candidates that satisfy the condition of 38. If $M$ is a projective $A$-module and $N$ is an $A$-module, then $\mathrm{Cotor}^i_\Gamma(M,\Gamma\otimes_A N) = 0$ for $i > 0$ and $\mathrm{Cotor}^0_\Gamma(M,\Gamma\otimes_A N) = M\otimes_A N$. A relative injective $\Gamma$-comodule is a direct summand of comodules of the form $\Gamma\otimes_A N$.

**Definition 37.** (Cobar Resolution) Let $M$ be a left $\Gamma$-comodule. For a right $\Gamma$-comodule $L$ that is projective over $A$, the cobar complex $C^*_\Gamma(L,M)$ is $C^s_\Gamma(L,M) = L\otimes_A \overline{\Gamma}^{\otimes s}\otimes_A M$ (with the obvious differential). When $L = \Gamma$, $D_\Gamma(M) = C_\Gamma(\Gamma,M)$ is called the cobar resolution of $M$.

It turns out that $D_\Gamma(M)$ is a resolution of $M$ by relative injectives, and we have the following proposition:

**Proposition 1.** *If L is projective over A, then* $H(C^*_\Gamma(L,M)) = \mathit{Cotor}_\Gamma(L,M)$*. In particular, if* $L = A$*, then* $H(C^*_\Gamma(A,M)) = \mathit{Ext}_\Gamma(A,M)$*.*

end of Hopf algebroid. Actual class starts

Today we will make good on our promise, made during our investigation of the unoriented bordism ring, to explain where the Adams spectral sequence comes from. This story neatly divides into two parts, and the first half is just an investigation of how rich of an algebraic category C we can find that supports a factorization

$$\begin{tikzcd} \mathsf{Spectra} \arrow[rr, "E_*"] \arrow[dr] & & \mathsf{Modules}_{E_*} \\ & \mathsf{C}. \arrow[ur] & \end{tikzcd}$$

Our answer to this question will come out of considering Grothendieck's framework of descent. Classically, descent concerns itself with a map $f\colon R \to S$ of a rings and an $S$–module $N$, and it asks questions like:

- When is there an $R$–module $M$ such that $N \cong M \otimes_R S = f^*M$?
- What extra data can be placed on $N$, called *descent data*, so that the category of descent data for $N$ is equivalent to the category of $R$–modules under the map $f^*$?
- What conditions can be placed on $f$ so that the category of descent data for any given module is always contractible, called *effectivity*?

The essential structure of these answers is easy to guess if we proceed by example, using the few tools available to us. Suppose that we begin instead with an $R$–module $M$ and we set $N = M \otimes_R S$. By tensoring up, we have two $R$–algebra maps $S \to S \otimes_R S$, given by including along either factor, and we can further tensor $N$ up to $N \otimes_R S$ or $S \otimes_R N$. Since $N$ came from the $R$–module $M$, these are canonically isomorphic:

$$\varphi\colon ((f \otimes 1) \circ f)^*M \cong ((1 \otimes f) \circ f)^*M.$$

The notation $f^*$ confused me briefly: you push-forward modules but pull-back quasicoherent sheaves.

Repeating this process produces more isomorphisms which compose according to the triangle

$$\begin{tikzcd} N \otimes_R S \otimes_R S \arrow[rr, "\varphi_{13}", "\simeq"'] \arrow[dr, "\varphi_{12}", "\simeq"'] & & S \otimes_R S \otimes_R N \\ & S \otimes_R N \otimes_R S, \arrow[ur, "\simeq", "\varphi_{23}"'] & \end{tikzcd}$$

**Cite me:** Allen said he knew a good reference for this descent picture.

where $\varphi_{ij}$ denotes applying $\varphi$ to the $i^{\text{th}}$ and $j^{\text{th}}$ coordinates.

**Definition 38.** Let $f\colon R \to S$ be a map of rings as above. An $S$–module $N$ equipped with an isomorphism $S \otimes_R N \cong N \otimes_R S$ of $S \otimes_R S$–modules which causes the above triangle to commute is called a *descent datum* for the map $f$.

*Remark 22.* There are other ways to view this data. For example, later on we will revisit it from the categorical perspective of *comonads*. However, there is another perspective which we have already encountered earlier on: that of the *canonical coalgebra* or *Amitsur complex*. Associated to the map $f\colon R \to S$, we can form the ring $S \otimes_R S$, which supports a map

$$S \otimes_R S \simeq S \otimes_R R \otimes_R S \to S \otimes_R S \otimes_R S \simeq (S \otimes_R S) \otimes_S (S \otimes_R S).$$

One can check that descent data on a $S$–module is the same as the data of a coaction against $S \otimes_R S$. As a first step, notice the similarity of function signatures:

$$N \xrightarrow{\psi} N \otimes_S (S \otimes_R S) \simeq N \otimes_R S.$$

The following theorem is the usual culmination of an initial investigation into descent:

**Theorem 22 (Grothendieck).** *If $f\colon R \to S$ is faithfully flat, then there is an equivalence of $R$–modules and $S$–modules equipped with descent data.*

*Proof (Jumping off point).* The basic observation is that $0 \to R \to S \to S \otimes_R S$ is an exact sequence of $R$–modules. This makes much of the homological algebra involved work out.

For details and additional context, see Section 4.2.1 of [43]; the story in the context of Hopf algebroids is also spelled out in detail in [24].

I added the citation requested above, but make a pass over this since I may have put it in an awkward place - AY.

In our situation, this hypothesis will essentially never be satisfied, so we will pursue a less dramatic statement of the properties of descent. To see what kind of theorem one might expect, consider the example of $f\colon \mathbb{Z} \to \mathbb{F}_p$, which is neither faithful nor flat. Then, consider the following list of problems (and their partial solutions):

- The tensor functor $f^*$ cannot distinguish even between the $\mathbb{Z}$–modules $\mathbb{Z}$ and $\mathbb{Z}/p$. However, if we use $Lf^*$ and resolve $\mathbb{Z}/p$ as $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$, the complexes $Lf^*(\mathbb{Z})$ and $Lf^*(\mathbb{Z}/p)$ do look distinct.
- Once we pass to the derived category, then we are no longer in a situation where we can expect the single cocycle condition from the descent data above to suffice. Instead, we can form a simplicial scheme, called the *descent object*, by the formula

$$\mathcal{D}_{\mathbb{Z}\to\mathbb{F}_p} := \left\{ \begin{array}{ccccccc} & & & & & \longleftarrow & \\ & & & \longleftarrow & \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \\ & \longleftarrow & \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \times_{\operatorname{Spec}\mathbb{Z}} & \longleftarrow & \\ \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \times_{\operatorname{Spec}\mathbb{Z}} & \longleftarrow & \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \cdots \\ & \longleftarrow & \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \times_{\operatorname{Spec}\mathbb{Z}} & \longleftarrow & \\ & & & \longleftarrow & \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \\ & & & & & \longleftarrow & \end{array} \right\}.$$

This is meant to look like the Čech nerve for the "cover" $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$.

- Accordingly, we need to update our notion of quasicoherent sheaf to live over a simplicial scheme [38, Tag 09VK]. Given a simplicial scheme $X$, a sheaf $\mathcal{F}$ on $X$ will be a sequence of sheaves $\mathcal{F}[n]$ on $X[n]$ as well as, for each map $\varphi : [m] \to [n]$ in the simplicial indexing category inducing a map $X(\varphi) : X[n] \to X[m]$, a choice of map of sheaves

$$\mathcal{F}(\varphi)_* \colon \mathcal{F}[m] \to X(\varphi)_* \mathcal{F}[n].$$

  Such a sheaf will be called *quasicoherent* when it is levelwise quasicoherent.

- Finally, we can characterize the structure a quasicoherent sheaf over $\mathcal{D}_{\mathbb{Z} \to \mathbb{F}_p}$ receives when it is tensored down from $\mathbb{Z}$. Such a sheaf enjoys that the adjoint map

$$\mathcal{F}(\varphi)^* \colon X(\varphi)^* \mathcal{F}[m] \to \mathcal{F}[n]$$

  is an isomorphism, and in this case we say that $\mathcal{F}$ is *Cartesian*.

**Cite me:** Hovey's *Morita theory for Hopf algebroids and presheaves of groupoids.*.

**Lemma 39.** *Without passing to the derived category, there is an equivalence of categories between Cartesian quasicoherent sheaves on the descent object and quasicoherent sheaves equipped with descent data.* □

you haven't mentioned quasi. coh. sheaves equipped with descent data. But I take it that it's obvious from the previous paragraphs that it's the same thing as modules equipped with descent data? A concrete definition about quasi-coherent sheaves equipped with descent data would be nice here - d.s.

The real utility of this framework is that it pulls apart the question of descent into two distinct pieces, summarized in the following theorem:

**Theorem 23.** *Let $i \colon A \to X$ be a closed subscheme, and consider the formal completion*

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "j"'] & & X \\ & X^\wedge_A . \arrow[ur, "k"'] & \end{tikzcd}$$

You haven't defined $X^\wedge_A$ at this point.

Surely you're supposed to be saying "bounded" sometimes when you talk about the derived category.

*If $X$ is Noetherian, then $k^*$ is flat as a functor of sheaves, $j^*$ is conservative as a functor in the derived category of sheaves, and there is an equivalence of derived categories of sheaves over $X^\wedge_A$ and sheaves over the descent object $\mathcal{D}_{A \to X}$.* □

Is this true? Ha, well, I hope so.

*Remark 23.* The usual theorem about faithfully flat descent then follows by using the hypotheses on $i$ to deduce that, e.g., if $i^*$ and $j^*$ are both conservative, then so must $k^*$ be.

We now transfer what we've learned to the situation of homotopical algebra. Recalling that spectra are equivalent to $\mathbb{S}$–modules, $\mathbb{S}$ the usual sphere spectrum, then any other ring spectrum comes equipped with a unit map $\eta : \mathbb{S} \to E$ and hence push and pull functors

$$\eta_* \colon M \mapsto M, \qquad\qquad \eta^* \colon X \mapsto E \wedge X.$$

Correspondingly, to any spectrum $X$ we can define the following cosimplicial spectrum:

**Definition 39.** Let $\mathcal{D}_E(X)$ be the cosimplicial spectrum determined by the formula

$$\mathcal{D}_E(X) := \left\{ \begin{tikzcd} \begin{matrix} E \\ \wedge \\ X \end{matrix} \arrow[r, shift left=3, "\eta_L"] \arrow[r, shift right=3, "\eta_R"'] & \begin{matrix} E \\ \wedge \\ E \\ \wedge \\ X \end{matrix} \arrow[l, "\mu"'] \arrow[r, shift left=4] \arrow[r, "\Delta"] \arrow[r, shift right=4] & \begin{matrix} E \\ \wedge \\ E \\ \wedge \\ E \\ \wedge \\ X \end{matrix} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \end{tikzcd} \right\}.$$

It is called *the descent object for $X$ from $E$ to $\mathbb{S}$.*

**Lemma 40.** *When $E$ is an $A_\infty$–ring spectrum, the descent object $\mathcal{D}_E(X)$ can be naturally considered as a cosimplicial object in the $\infty$–category of spectra.* □

I don't intend to prove this, but maybe we could say some mealy words about why it's true. At worst, we could give reference to the relevant part of Higher Algebra.

**Definition 40.** Let $E$ be an $A_\infty$–ring spectrum. Then $X^\wedge_E := \operatorname{Tot} \mathcal{D}_E(X)$ is called the *$E$–nilpotent completion of $X$*. The spectral sequence resulting from the coskeletal filtration is called the *$E$–Adams spectral sequence* (for $X$).

It is not always the case that $X^\wedge_E$ can be lifted from a cosimplicial object in the homotopy category to a sufficiently structured cosimplicial object that we could take its totalization or homotopy colimit.

In general, it's quite rare that the $E$–nilpotent completion of a spectrum $X$ recovers $X$, but in the nice cases we typically work in, it has been known to happen. In particular, there is the following theorem:

**Lemma 41.** *Let $E$ be a connective $A_\infty$ ring spectrum and let $X$ be any connective spectrum. Then $X^\wedge_E$ is equivalent to the "$\pi_0E$–localization" of $X$, i.e., for a prime $p$ the spectrum $X^\wedge_E$ is $p$–local if $\pi_0 E$ is $p$–local, it is $p$–complete if $\pi_0 E$ is $p$–torsion, and otherwise it is just $X$.* □

**Cite me:** Ravenel's *Localizations w/r/t …* paper.

*Proof (Proof sketch).*

You really can just look at the Adams tower...

Finally, we can compare the topological situation with the algebraic situation. To have any hope of applying algebra and algebraic geometry, we must impose some nicety properties. Here is the first:

**Definition 41.** $E$ satisfies **CH**, the **C**ommutativity **H**ypothesis, when $\pi_* E^{\wedge j}$ is commutative for all $j \geq 1$.

**Definition 42.** Suppose that $E$ is a ring spectrum satisfying **CH**. We define a simplicial scheme associated to $E$, called its *context*, to be

$$\begin{aligned}\mathcal{M}_E &:= \operatorname{Spec}\pi_*\mathcal{D}_E(\mathbb{S})\\ &= \left\{ \operatorname{Spec}\pi_* E \begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array} \operatorname{Spec}\pi_*\begin{pmatrix}E\\ \wedge\\ E\end{pmatrix} \begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array} \operatorname{Spec}\pi_*\begin{pmatrix}E\\ \wedge\\ E\\ \wedge\\ E\end{pmatrix} \begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array} \cdots \right\}.\end{aligned}$$

The context is the wellspring of the algebraic category $\mathsf{C}$ dreamed of in the introduction to this lecture.

**Definition 43.** For a ring spectrum $E$ satisfying **CH** and input spectrum $X$, we define the following diagram of abelian groups:

$$\Gamma\mathcal{M}_E(X) := \left\{ \pi_*\begin{pmatrix}E\\ \wedge\\ X\end{pmatrix} \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} \pi_*\begin{pmatrix}E\\ \wedge\\ E\\ \wedge\\ X\end{pmatrix} \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} \pi_*\begin{pmatrix}E\\ \wedge\\ E\\ \wedge\\ E\\ \wedge\\ X\end{pmatrix} \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} \cdots \right\},$$

The $j^{\text{th}}$object is a module for $\mathcal{O}(\mathcal{M}_E[j])$, and hence determines a quasicoherent sheaf over the scheme $\mathcal{M}_E[j]$. Suitably interpreted, the maps of abelian groups determine maps of pushforwards so that $\mathcal{M}_E(X)$ is a quasicoherent sheaf over the simplicial scheme $\mathcal{M}_E$.

There is also a common hypothesis on $E$ that brings us back into the world of coalgebra, down from simplicial schemes.

**Definition 44.** Take $E_*E$ to be an $E_*$–module using the left-unit map. We will say that $E$ satisfies **FH**, the **F**latness **H**ypothesis, when the right-unit map $E_* \to E_*E$ is a flat map of $E_*$–modules.[1]

*Remark 24.* The main utility of this is that it obviates us from working through the homological algebra of sheaves over simplicial schemes. Instead, since

[1] The essential point of this is that it causes $E_*E \otimes_{E_*} E_*X$ to become a homology theory and $E_*E \otimes_{E_*} E_*X \to (E \wedge E)_*X$ to become an isomorphism on a point. Alternatively, this can be viewed as a degeneration condition on the Künneth spectral sequence for $E_*(E \wedge E)$.

**FH** causes $\mathcal{M}_E$ to become 1–truncated, we can refer to Remark 22 and simply refer back to the homological algebra of comodules. In light of the discussion in Examples 14 and 15, we also see an interpretation of these groupoid–valued simplicial schemes: they are valued in sets equipped with an action by $\operatorname{Spec} E_* E$, which acts also on the base $\operatorname{Spec} E_*$. To denote this "homotopical quotient" or "action groupoid", we will write

$$\operatorname{Spec} E_* // \operatorname{Spec} E_* E.$$

Such affine groupoid–valued schemes are themselves quite tangible: their rings of functions form *Hopf algebroids*, and Cartesian quasicoherent sheaves on the groupoid scheme correspond to comodules for the Hopf algebroid.

**Cite me:** A lot of this could use citation. Most of it is in Ravenel's appendix or Hovey's paper..

*Remark 25.* This homotopical perspective is quite useful — for instance, a map of groupoid–schemes which induces on points a natural weak equivalence of groupoids also induces an equivalence of comodule categories. In fact, the *derived* comodule category depends only upon the stack associated to the groupoid–scheme, which allows still more contexts to be identified. We won't need this observation in what's to come, though, and it introduces substantial technical distractions. **However, we *may* got sloppy and say "stack" from time to time.**

I'm still hazy over these two remarks. I understand what they are trying to say but I'm feeling hazy on the details. You should explain it to me sometime - d.s.

The standard Johnson–Wilson chart for $\mathcal{M}_{\mathbf{fg}}^{\leq d}$ is a standard example of a place where stackiness is actually relevant. Admit to this here and put a forward reference.

Could also explain the difference: levelwise sheaves of 0–types vs sheaves of $\infty$–types.

I think there is a notion of quasicoherent sheaf directly over $\mathcal{D}_E$ and an interpretation of Cartesian sheaves in that setting. I think that a different view on **FH** is that it causes the functor $\pi_*$ to preserve Cartesianness.

*Example 21.* Most of the homology theories we will discuss have this property. For an easy example, $H\mathbb{F}_2P$ certainly has this property: there is only one possible algebraic map $\mathbb{F}_2 \to \mathcal{A}_*$, so **FH** is necessarily satisfied. This grants us access to a description of the context for $H\mathbb{F}_2$:

$$\mathcal{M}_{H\mathbb{F}_2P} = \operatorname{Spec} F_2 // \underline{\operatorname{Aut}}\, \widehat{\mathbb{G}}_a.$$

*Example 22.* The context for $MUP$ is considerably more complicated, but Quillen's theorem can be equivalently stated as giving a description of it. It is isomorphic to the moduli of formal groups:

$$\mathcal{M}_{MUP} \simeq \mathcal{M}_{\mathbf{fg}} := \mathcal{M}_{\mathbf{fgl}} // \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}},$$

where $\mathcal{M}_{\mathbf{ps}} = \underline{\operatorname{End}}(\widehat{\mathbb{A}}^1)$ is the moduli of self-maps of the affine line (i.e., of power series) and $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is the multiplicative subgroup of invertible such maps.

*Remark 26.* If $E$ is a complex-oriented ring spectrum, then the simplicial sheaf $\mathcal{M}_{MU}(E)$ has an extra degeneracy, cause the $MU$–based Adams spectral sequence for $E$ to degenerate. In this sense, the "stackiness" of $\mathcal{M}_{MU}(E)$ is a measure of the failure of $E$ to be orientable.

**Cite me:** Mike's Talbot talk, which is in the TMF volume..

Say what open, closed, flat maps of simplicial schemes are?

Jon thinks that this picture can be instructively recast in terms of the cotangent complex. I'm not sure how, but it's something to keep in mind for later.

## 3.2 Feb 26: Fiberwise analysis and chromatic homotopy theory

Andy Senger correctly points out that "stalkwise" is the wrong word to use in all this (if we mean to be working in the Zariski topology, which surely we must). The stalks are selected by maps from certain local rings; $E_\Gamma$ selects the formal neighborhood of the special point inside of this; and $K_\Gamma$ selects the special point itself. Is "fiberwise" enough of a weasel word to get out of this?

Our first goal for today is to outline a program for the rest of this Case Study. Yesterday, we developed a rich target for $E$–homology: sheaves over an algebro-geometric object $\mathcal{M}_E$. Furthermore, we have explored in Example 22 an identification $\mathcal{M}_{MUP} \simeq \mathcal{M}_{\mathbf{fg}}$, where $\mathcal{M}_{\mathbf{fg}}$ is the "moduli of formal groups". Our initial goal for today is to outline a program by which we can leverage this to study $MUP$. Abstractly, one can hope to study any sheaf, including $\mathcal{M}_{MUP}(X)$, by analyzing its stalks. The main utility of Quillen's theorem is that it gives us access to a concrete model of $\mathcal{M}_{MUP}$, so that we can determine where to even look for those stalks.

The bottom of COCTALOS page 68 has a better interpretation of what picking a formal group law has to do with anything, I think.

With this in mind, given a map $f$ in the diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "f"] \arrow[dr] & \mathcal{M}_{\mathbf{fgl}} \arrow[r, equal] \arrow[d] & \mathcal{M}_{MUP}[0] \arrow[r, equal] \arrow[d] & \operatorname{Spec} MUP_0 \\ & \mathcal{M}_{\mathbf{fg}} \arrow[r, equal] & \mathcal{M}_{MUP}, \end{tikzcd}$$

please check the P's, 0's, and labels of arrows? I tried to fudge around with it since it looked funny but I want to make sure I didn't mess anything up, AY.

should this (and the one later) be $R(X)$ and not $R_0$? For example when $R = MUP_0$

life would be easiest if the $R$–module determined by $f^*\mathcal{M}_{MUP}(X)$ were itself the value of a homology theory $R_0(X) = MUP_0 X \otimes_{MUP_0} R$. After all, the pullback of some arbitrary sheaf along some arbitrary map has no special behavior, but homology functors do have familiar special behaviors which we could hope to exploit. Generally, this is unreasonable to expect: homology theories are functors which convert cofiber sequences of spectra to long exact sequences of groups, but base–change from $\mathcal{M}_{\mathbf{fg}}$ to $\operatorname{Spec} R$ preserves exact sequences exactly when the diagonal arrow is *flat*. In that case, this gives the following theorem:

**Theorem 24 (Landweber).** *Given such a diagram where the diagonal arrow is flat, the functor*

$$R_0(X) := MUP_0(X) \otimes_{MUP_0} R$$

*is a homology theory.*

In the course of proving this theorem, Landweber devised a method to recognize flat maps. Recall that a map $f$ is flat exactly when for any closed substack $i\colon A \to \mathcal{M}_{\mathbf{fg}}$ with ideal sheaf $\mathcal{I}$ there is an exact sequence

$$0 \to f^*\mathcal{I} \to f^*\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}} \to f^* i_* \mathcal{O}_A \to 0.$$

Landweber classified the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, thereby giving a method to check maps for flatness.

This appears to be a moot point, however, as it is unreasonable to expect this idea to apply to computing stalks: the inclusion of a closed substack (and so, in particular, a closed point $\Gamma$) is flat only in highly degenerate cases. We saw in Theorem 23 that this can be repaired: the inclusion of the formal completion of a closed substack of a Noetherian[2] stack is flat, and so we naturally become interested in the infinitesimal deformation spaces of the closed points $\Gamma$ on $\mathcal{M}_{\mathbf{fg}}$. If we can analyze those, then Landweber's theorem will produce homology theories called $E_\Gamma$. Moreover, if we find that these deformation spaces are *smooth*, it will follow that their deformation rings support regular sequences. In this excellent case, by taking the regular quotient we will be able to recover a *homology theory* $K_\Gamma$ which plays the role of computing the stalk of $\mathcal{M}_{MUP}(X)$ at $\Gamma$.[3]

We have thus assembled a task list:

- Describe the open and closed substacks of $\mathcal{M}_{\mathbf{fg}}$.
- Describe the geometric points of $\mathcal{M}_{\mathbf{fg}}$.
- Analyze their infinitesimal deformation spaces.

This will occupy us for the next few lectures. Today, we will embark on this analysis by studying the scheme $\mathcal{M}_{\mathbf{fgl}}$ which naturally covers the stack $\mathcal{M}_{\mathbf{fg}}$.

**Definition 45.** There is an affine scheme $\mathcal{M}_{\mathbf{fgl}}$ classifying formal group laws. Begin with the scheme classifying all bivariate power series:

$$\operatorname{Spec} \mathbb{Z}[a_{ij} \mid i, j \geq 0] \leftrightarrow \{\text{bivariate power series}\},$$
$$f \in \operatorname{Spec} \mathbb{Z}[a_{ij} \mid i, j \geq 0](R) \leftrightarrow \sum_{i,j \geq 0} f(a_{ij}) x^i y^j.$$

Then, set $\mathcal{M}_{\mathbf{fgl}}$ to be the closed subscheme selected by the formal group law axioms in Definition 17.

This presentation of $\mathcal{M}_{\mathbf{fgl}}$ as a subscheme appears to be extremely complicated in that its ideal is generated by many hard-to-describe elements, but $\mathcal{M}_{\mathbf{fgl}}$ itself is actually not complicated at all. We will prove the following theorem:

---

[2] $\mathcal{M}_{\mathbf{fg}}$ is not Noetherian, but we will find that each closed point except $\widehat{\mathbb{G}}_a$ lives in an open substack that happens to be Noetherian.

[3] Incidentally, this program has no content when applied to $\mathcal{M}_{H\mathbb{F}_2}$, as $\operatorname{Spec} \mathbb{F}_2$ is simply too small.

**Theorem 25 ([20, Théorème II]).** *There is a noncanonical isomorphism*

$$L_\infty := \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \cong \mathbb{Z}[t_n \mid 1 \leq n < \infty]. \square$$

The most important consequence of this is *smoothness*:

**Corollary 27.** *Given a formal group law $\varphi$ over a ring $R$ and a surjective ring map $f\colon S \to R$, there exists a formal group law $\widetilde{\varphi}$ over $S$ with*

$$\varphi = f^* \widetilde{\varphi}. \square$$

What filtration?

*Remark 27.* One might hope that the filtrationabove has an immediate geometric realization. After all, one can consider the $n^{\text{th}}$ order formal neighborhood $\widehat{\mathbb{A}}^{1,(n)}$ of Example 4. The appropriate analogue of Lemma 19 shows that a map

$$\widehat{\mathbb{A}}^{1,(n)} \times \widehat{\mathbb{A}}^{1,(n)} \to \widehat{\mathbb{A}}^{1,(n)}$$

is represented by a bivariate power series, *modulo the ideal* $(x^{n+1}, y^{n+1})$. This ideal is distinct from $(x, y)^{n+1}$, and so the source scheme of a formal $n$–bud is not the square of $\widehat{\mathbb{A}}^{1,(n)}$, and a formal $n$–bud does *not* determine a group object on some finite scheme. This is actually a good thing: there are structure theorems preventing many of these intermediate group structures on finite schemes from existing.

**Cite me:** Akhil is who reminded me of this, back in Berkeley..

There's some hidden text here about $n$–buds, but I don't think we ever care about it.

*Proof (Proof of Theorem 25).* Let $U = \mathbb{Z}[b_0, b_1, b_2, \ldots]/(b_0 - 1)$ be the universal ring supporting a "strict" exponential

$$\exp(x) := \sum_{j=0}^{\infty} b_j x^{j+1}$$

with compositional inverse

$$\log(x) := \sum_{j=0}^{\infty} m_j x^{j+1}.$$

They induce a formal group law on $U$ by the formula

$$x +_U y = \exp(\log(x) + \log(y)),$$

Why are you writing $L_\infty$ instead of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$?

classified by a map $u\colon L_\infty \to U$. Modulo decomposables, this map can be computed as

$$u(a_{i(n-i)}) = \binom{n}{i} b_{n-1} \pmod{\text{decomposables}}.$$

Writing $d_n = \gcd\left(\binom{n}{i} \middle| 0 < k < n\right)$, the map $Qu$ on degree $2n$ has image the subgroup generated by $d_{n+1}b_n$. We write $T_{2n}$ for this subgroup. Using the splitting of $Qu$ from Lemma 42.4 below, we use the freeness of $U$ to *choose* an algebra splitting

$$U \xrightarrow{v} L_\infty \xrightarrow{u} U.$$

The map $v$ is an isomorphism because $uv$ is injective and because we have checked that $v$ is surjective on indecomposables.

This last sentence was a little quick for me. For example, I don't think you're "checking" that $v$ is surjective but more observing that it is so by construction.

**Definition 46.** In order to prove the missing Lemma 42, it will be useful to study the series $+_\varphi$ "up to degree $n$", i.e., modulo $(x, y)^{n+1}$. Such a truncated series satisfying the analogues of the formal group law axioms is called a *formal n–bud*. Additionally, a *symmetric* 2*–cocycle* is a symmetric polynomial $f(x, y)$ satisfying the equation

Do the intermediate rings matter here?

$$f(x, y) - f(t + x, y) + f(t, x + y) - f(t, x) = 0.$$

**Lemma 42 (Symmetric 2–cocycle lemma (Part 1)).** *The following are equivalent:*

1. *Symmetric* 2*–cocycles that are homogeneous polynomials of degree n are spanned by*
$$c_n = \frac{1}{d_n} \cdot ((x + y)^n - x^n - y^n).$$
2. *For F is an r–bud, the set of* $(r + 1)$*–buds extending F form a torsor under addition for* $R_{2n-2} \otimes c_r$.
3. *Any homomorphism* $(QL)_{2n} \to A$ *factors through the map* $(QL)_{2n} \to T_{2n}$.
4. *There is a canonical splitting* $T_{2n} \to (QL)_{2n}$.

Things suddenly become graded here — and you really make use of this. Explain yourself.

*Proof (Equivalences).* Verifying that Claims 1 and 2 are equivalent is a matter of writing out the purported $(r + 1)$–buds and taking their difference. To see that Claim 2 is equivalent to Claim 3, follow the chain

Do you mean write out a $(r+1)$-bud and applying the associativity axiom?

$$\mathsf{Groups}((QL)_{2n}, A) \cong \mathsf{Rings}(\mathbb{Z} \oplus (QL)_{2n}, \mathbb{Z} \oplus \Sigma^{2n} A) \cong \mathsf{Rings}(L, \mathbb{Z} \oplus \Sigma^{2n} A).$$

This shows that such a homomorphism of groups determines an extension of the $n$–bud $\widehat{\mathbb{G}}_a$ to an $(n + 1)$–bud, which takes the form of a 2–cocycle with coefficients in $A$, and hence factors through $T_{2n}$. Finally, Claim 4 is the universal case of Claim 3.

We will prove Claim 1 tomorrow.

## 3.3 Feb 29: The structure of $\mathcal{M}_{\mathrm{fg}}$ I: Distance from $\widehat{\mathbb{G}}_a$

We begin by finishing up our proof of Lazard's theorem (and, specifically, the Symmetric 2–cocycle Lemma).

**Lemma 43 (Symmetric 2–cocycle lemma (Part 2): Claim 1 of Lemma 42).** *Symmetric 2–cocycles that are homogeneous polynomials of degree $n$ are spanned by*

$$c_n = \frac{1}{d_n} \cdot ((x+y)^n - x^n - y^n).$$

Cite me: This follows Chapter 3 of COCTALOS..

*Proof.* It suffices to show the Lemma over a finitely generated ring. In fact, the Lemma is true for $A \oplus B$ if and only if it's true for $A$ and for $B$, so the structure theorem for finitely generated abelian groups reduces to the cases of $\mathbb{Z}$ and $\mathbb{Z}/p^r$. If $A \subseteq B$ and the Lemma is true for $B$, it's true for $A$, so we can further reduce the $\mathbb{Z}$ case to $\mathbb{Q}$.We can also reduce from $\mathbb{Z}/p^r$ to $\mathbb{Z}/p$ using an inductive, Bockstein-style argument. Hence, we can now freely assume that our ground object is a prime field.

Rephrase this in terms of localizations.

For a formal group scheme $\widehat{\mathbb{G}}$, we can form a simplicial scheme $B\widehat{\mathbb{G}}$ in the usual way:

$$B\widehat{\mathbb{G}} := \left\{ \begin{array}{ccccccc} & & & & * & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \times & \longrightarrow & \cdots \\ * & \longleftarrow & \times & \longrightarrow & \widehat{\mathbb{G}} & \longleftarrow & \\ & & * & \longleftarrow & \times & \longrightarrow & \\ & & & & * & \longleftarrow & \end{array} \right\}.$$

By applying the functor $\underline{\mathsf{FormalGroups}}(-, \widehat{\mathbb{G}}_a)(k)$, we get a cosimplicial abelian group, hence a cochain complex, of which we can take the cohomology. In the case $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$, the 2–cocycles in this cochain complex are *precisely* the things we've been calling 2–cocycles[4], so we are interested in computing $H^2$. The first observation in this direction is that $d^1(x^k) = d_k c_k$. Secondly, one may check that this complex also computes

$$\mathrm{Cotor}_{\mathcal{O}^*_{\widehat{\mathbb{G}}}}(k,k) \cong \mathrm{Ext}_{\mathcal{O}^*_{\widehat{\mathbb{G}}}}(k,k),$$

which we're now going to compute using a more efficient complex.

$\mathbb{Q}$: There is a resolution

$$0 \to \mathbb{Q}[t] \xrightarrow{\cdot t} \mathbb{Q}[t] \to \mathbb{Q} \to 0,$$

[4] They aren't obligated to be symmetric, though.

from which this follows:

$$H^* \underline{\mathsf{FormalGroups}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{when } * = 0, \\ \mathbb{Q} & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that every 2–cocycle is a coboundary, symmetric or not.

$\mathbb{F}_p$: Again, we switch to working with Ext over a free divided power algebra. Such an algebra splits as a tensor of truncated polynomial algebras, and again computing a minimal free resolution results in the calculation

$$H^* \underline{\mathsf{FormalGroups}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{F}_p) = \Lambda[\alpha_k \mid k \geq 0] \otimes \mathbb{F}_p[\beta_k \mid k \geq 0],$$

with $\alpha_k \in \mathrm{Ext}^1$ and $\beta_k \in \mathrm{Ext}^2$. In fact, $\alpha_k$ is represented by $x^{p^k}$ and $\beta_k$ is represented by $c_{p^k}(x, y)$, and in the case $p = 2$ the exceptional class $\alpha_{k-1}^2$ is represented by $C_{2^k}(x, y)$. Since we have representatives for the surviving homology classes and we know where the bounding class lives, it follows that $c_n(x, y)$ and $x^{p^a} y^{p^b}$ give a basis for *all* of the 2–cocycles. It's easy to select the symmetric ones, and it agrees with the prediction of the statement of the Lemma. I'm not sure how to do any of these calculations! Ha.

This finally concludes the proof of Theorem 25.

Section 12 of Neil's FG notes talk about the infinite height subscheme of $\mathcal{M}_{\mathbf{fgl}}$. He compares it to $H_* MO$ and to the Hurewicz image $\pi_* MU \to H_*(MU; \mathbb{F}_p)$.

Also, people seem to say things about the Mischenko logarithm rather than the invariant differential, but I wonder if we should phrase things in those terms.

Having described the structure of $\mathcal{M}_{\mathbf{fgl}}$, we turn to understanding the quotient stack $\mathcal{M}_{\mathbf{fg}}$. Earlier, we proved the following theorem:

**Theorem 26.** *Let $k$ be any field of characteristic 0. Then there is a unique map* Be careful about $\star$-isomorphisms versus isomorphisms.

$$\operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}. \square$$

*Proof.* This is a rephrasing of Theorem 11 in the language of stacks.

We would like to have a similar classification of the closed points in positive characteristic. We proved the theorem above by solving a certain differential equation, which necessitated integrating a power series. Integration is what we expect to fail in positive characteristic. The following definition tracks *where* it fails:

**Definition 47.** Let $+_\varphi$ be a formal group law. Let $n$ be the largest degree such that there exists a formal power series $\ell$ with

$$\ell(x +_\varphi y) = \ell(x) + \ell(y) \pmod{(x, y)^n},$$

i.e., $\ell$ is a logarithm for the $n$–bud determined by $+_\varphi$. The *p–height of* $+_\varphi$ is defined to be $\log_p(n)$. Should that be a $n+1$?

We will show that this definition is well-behaved, in the following sense:

**Lemma 44.** *Over a field of positive characteristic $p$, the $p$–height of a formal group law is always an integer. (That is, $n = p^d$ for some natural number $d$.)*

We will have to develop some machinery to get there. First, notice that this definition of height really depends on the formal group rather than the formal group law.

**Lemma 45.** *The height of a formal group law is an isomorphism invariant, i.e., it descends to a function on $\mathcal{M}_{\mathbf{fg}}$.*

*Proof.* The series $\ell$ is a partial logarithm for the formal group law $\varphi$, i.e., an isomorphism between the formal group defined by $\varphi$ and the additive group. Since isomorphisms compose, this statement follows.

With this in mind, we look for a more standard form for formal group laws, where Lemma 44 will hopefully be obvious. In light of our goal, the most obvious standard form is as follows:

**Definition 48.** Suppose that a formal group law $+_\varphi$ does have a logarithm. We say that $+_\varphi$ has a *p–typical logarithm* in the case that its logarithm has the form
$$\log_\varphi(x) = \sum_{j=0}^{\infty} \ell_j x^{p^j}.$$

**Cite me:** This is due to Hazewinkel.

**Lemma 46.** *Every formal group law $+_\varphi$ with a logarithm $\log_\varphi$ is naturally isomorphic to one whose logarithm is p–typical, called the p–typification of $+_\varphi$.*

*Proof.* Let $\widehat{\mathbb{G}}$ denote the formal group associated to $+_\varphi$, and consider its inherited coordinate
$$g_0 \colon \widehat{\mathbb{A}}^1 \xrightarrow{\cong} \widehat{\mathbb{G}},$$
so that
$$\log_\varphi = \log \circ g_0 = \sum_{n=1}^{\infty} a_n x^n.$$
Our goal is to perturb this coordinate to a new coordinate $g$ which has the property that it couples with the logarithm
$$\widehat{\mathbb{A}}^1 \xrightarrow{g} \widehat{\mathbb{G}} \xrightarrow{\log} \widehat{\mathbb{G}}_a$$
to give a power series expansion
$$\log(g(x)) = \sum_{n=0}^{\infty} a_{p^n} x^{p^n}.$$

To do this, we introduce four operators on *curves*:

- Given $r \in R$, we can define a *homothety* by rescaling the coordinate by $r$:

$$\log(g(rx)) = \sum_{n=1}^{\infty} (a_n r^n) x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a shift operator (or *Verschiebung*) by

$$\log(V_\ell g(x)) = \log(g(x^\ell)) = \sum_{n=1}^{\infty} a_n x^{n\ell}.$$

- Given an $\ell \in \mathbb{Z}_{(p)}$, we define the *$\ell$–series* by

$$\log([\ell](g(x))) = \ell \log(g(x)) = \sum_{n=1}^{\infty} \ell a_n x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a *Frobenius operator* by

$$\log(F_\ell g(x)) = \log\left(\sum_{j=1}^{\ell}{}_{\widehat{\mathbb{G}}}\, g(\zeta_\ell^j x^{1/\ell})\right) = \sum_{n=1}^{\infty} \ell a_{n\ell} x^n,$$

  where $\zeta_\ell$ is a primitive $\ell^{\text{th}}$ root of unity. Because this formula is Galois–invariant in choice of primitive root, it actually expands to a series which lies over the ground ring (without requiring an extension by $\zeta_\ell$). But, by pulling the logarithm through and noting

$$\sum_{j=1}^{\ell} \zeta_\ell^{jn} = \begin{cases} \ell & \text{if } \ell \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

  we can explicitly compute the behavior of $F_\ell$.[5]

Stringing these together, for $p \nmid \ell$ we have

$$\log([1/\ell] V_\ell F_\ell g(x)) = \sum_{n=1}^{\infty} a_{n\ell} x^{n\ell}.$$

Hence, we can consider the curve $g -_{\widehat{\mathbb{G}}} \sum_{p \nmid \ell} [1/\ell] V_\ell F_\ell g$, which is another coordinate on the same formal group $\widehat{\mathbb{G}}$, but with a $p$–typical logarithm.

The sum is taken over all $\ell \in \mathbb{Z}$ not a multiple of $p$, using the formal group addition, right? In that case, aren't you subtracting too much? If $n$ is not a power of $p$, then the logarithm of this expression is $a_n$ times $1-$ the number of factors of $n$ coprime to $p$. Also, why is this still an isomorphism?

Some of these $g$s should be $g_0$s.

Of course, not every formal group law supports a logarithm — after all, this is the point of "height". There are two ways around this: one is to pick a

[5] The definition of Frobenius comes from applying the Verschiebung to the character group (or "Cartier dual") of $\widehat{\mathbb{G}}$.

surjection $S \to R$ from a torsion-free ring $S$, *choose* a lift of the formal group law to $S$, then pass to $S \otimes \mathbb{Q}$ and study how much of the resulting logarithm descends to $R$. However, it is not clear that this procedure is independent of choice. We therefore pursue an alternative approach: an intermediate definition that applies to all formal group laws and which specializes to the one above in the presence of a logarithm. To do this, we consider what computations are made easier with this sort of formula for a logarithm, and we arrive at the following:

**Definition 49.** The *p–series* of a formal group law $+_\varphi$ is given by the formula

$$[p]_\varphi(x) := \overbrace{x +_\varphi \cdots +_\varphi x}^{p \text{ times}}.$$

**Lemma 47.** *If $+_\varphi$ is a formal group law with p–typical logarithm, then there are elements $v_n$ with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots .$$

*Proof (Proof sketch).* This comes from comparing the two series

$$\begin{aligned} \log_\varphi(px) &= px + \cdots , \\ \log_\varphi([p]_\varphi(x)) &= p \log_\varphi(x) = px + \cdots . \end{aligned}$$

The difference is concentrated in degrees of the form $p^d$, beginning in degree $p$, so one can find an element $v_1$ so that

$$p \log_\varphi(x) - (\log_\varphi(px) + \log_\varphi(v_1 x^p))$$

starts in degree $p^2$, and so on. In all, this gives the equation

$$p \log_\varphi(x) = \log_\varphi(px) + \log_\varphi(v_1 x^p) + \log_\varphi(v_2 x^{p^2}) + \cdots$$

at which point we can use formal properties of the logarithm to deduce

$$\begin{aligned} \log_\varphi [p]_\varphi(x) &= \log_\varphi \left( px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots \right) \\ [p]_\varphi(x) &= px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots \square \end{aligned}$$

**Definition 50.** A formal group law is itself said to be *p–typical* when its $p$–series has the above form. (In particular, the logarithm of a $p$–typical formal group law is a $p$–typical logarithm.)

**Corollary 28 (Lemma 46).** *Every formal group law is naturally isomorphic to a p–typical one.*

*Proof.* The procedure applied to the formal group law $+_\varphi$ in the proof of Lemma 46 applies equally well to an arbitrary formal group law, even without a logarithm — it just wasn't clear what was being gained. Now, it is clear: we are gaining the conclusion of this Corollary.

I'm still not sure about this. Lemma 3.3.7 doesn't say anything about $p$-series, and the previous two lemmas use the existence of a logarithm rather substantially.

*Remark 28.* There is an inclusion of groupoid–valued sheaves from $p$–typical formal group laws with isomorphisms to all formal group laws with isomorphisms. Corollary 28 can be viewed as presenting this inclusion as a deformation retraction, and in particular the inclusion is a natural *equivalence* of groupoids. It follows that they both present the same stack: $\mathcal{M}_{\mathbf{fg}}$. In fact, the moduli of $p$–typical formal group laws is isomorphic to $\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ — every possible $p$–series is realized by a unique $p$–typical formal group law.

I don't think we've shown this? For what it's worth, the statements around Lemma 11.9 and Application 13.10 of COCTALOS prove it.

*Remark 29.* In fact, the rational logarithm coefficients can be recursively recovered from the coefficients $v_d$, using a similar manipulation:

$$\begin{aligned}
p \log_\varphi(x) &= \log_\varphi\left([p]_\varphi(x)\right) \\
p \sum_{n=0}^{\infty} m_n x^{p^n} &= \log_\varphi\left(\sum_{d=0}^{\infty}{}_\varphi v_d x^{p^d}\right) = \sum_{d=0}^{\infty} \log_\varphi\left(v_d x^{p^d}\right) \\
\sum_{n=0}^{\infty} p m_n x^{p^n} &= \sum_{d=0}^{\infty}\sum_{j=0}^{\infty} m_j v_d^{p^j} x^{p^{d+j}} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} m_k v_{n-k}^{p^k}\right) x^{p^n},
\end{aligned}$$

implicitly taking $m_0 = 1$ and $v_0 = p$.

*Proof (Proof of Lemma 44).* Replace the formal group law by its $p$–typification. Based on Remark 29, we see that the height of a $p$–typical formal group law over a field of characteristic $p$ is coincides with the appearance of the first nonzero coefficient in its $p$–series.

You could be clearer about the varying assumptions on the ground rings in these different theorems. Some need to work over $k$, others work over any $\mathbb{Z}_{(p)}$–algebra.

## 3.4 Mar 2: The structure of $\mathcal{M}_{\mathbf{fg}}$ II: Large scales

With the notion of "height" firmly in hand, we are now in a position to classify the geometric points of $\mathcal{M}_{\mathbf{fg}}$.

**Theorem 27 ([20, Théorème IV]).** *Let $\bar{k}$ be an algebraically closed field of positive characteristic $p$. There is a bijection between maps*

$$\Gamma : \operatorname{Spec} \bar{k} \to \mathcal{M}_{\mathbf{fg}}$$

*and numbers $1 \le d \le \infty$ given by $\Gamma \mapsto \operatorname{ht}(\Gamma)$.*

You haven't used the notation $\operatorname{ht}(\Gamma)$ before.

*Proof.* The easy part of the proof is surjectivity: recalling Remark 28, take the $p$–typical formal group law over $\mathbb{F}_p$ determined by the $p$–series $[p]_{\varphi_d}(x) = x^{p^d}$, sometimes called the *Honda formal group law*.

To show injectivity, we must show that every $p$–typical formal group law $\varphi$ over $\bar{k}$ is isomorphic to the appropriate Honda group law. Suppose that the $p$–series for $\varphi$ begins

$$[p]_\varphi(x) = x^{p^d} + ax^{p^{d+k}} + \cdots .$$

These +s should be $+_\varphi$?

Then, we will construct a coordinate transformation $g(x) = \sum_{j=1}^{\infty} b_j x^j$ satisfying

$$g(x^{p^d}) \equiv [p]_\varphi(g(x)) \pmod{x^{p^{d+k}+1}}$$
$$\sum_{j=1}^{\infty} b_j x^{jp^d} \equiv \sum_{j=1}^{\infty} b_j^{p^d} x^{jp^d} + \sum_{j=1}^{\infty} ab_j^{p^{d+k}} x^{jp^{d+k}} \pmod{x^{p^{d+k}+1}}.$$

For $g$ to be a coordinate transformation, we must have $b_1 = 1$, which in the critical degree $x^{p^{d+k}}$ forces the relation

$$b_{p^k} = b_{p^k}^{p^d} + a.$$

Since $\bar{k}$ is algebraically closed, this relation is solvable, and the coordinate can be perturbed so that the term $x^{p^{d+k}}$ does not appear in the $p$–series. If we set the earlier terms in the series to be 0, then we can induct on $d$.

**Cite me:** Remark 11.2 in Neil's FG class notes.

*Remark 30.* From this, it follows that the "coarse moduli of formal groups" — i.e., the functor from rings to isomorphism classes of formal groups over that ring — is not representable by a scheme. The infinitely many isomorphism classes over $\operatorname{Spec} \mathbb{F}_p$ produce infinitely many over $\operatorname{Spec} \mathbb{Z}$ as well. On the other hand, there is a single $\mathbb{Q}$–valued point of the coarse moduli, whereas the $\mathbb{Z}$–points of a representable functor would inject into its $\mathbb{Q}$–points.

We now turn to the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, which also admit a reasonable presentation in terms of height.

**Lemma 48 ([44, Theorem 4.6]).** *Recall that the moduli scheme of $p$–typical formal group laws is presented as*

$$\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} = \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots].$$

What is this universal group law? Is it the one with $p$-series $px + v_1 x^p + \cdots$?

*Suppose $g(x) = \sum_{j=0}^{\infty} {}_L t_j x^{p^j}$ is the universal $p$-typical coordinate transformation, which we can use to conjugate the universal group law "$+_L$" to a second $p$–typical group law "$+_R$", whose $p$–series has the form*

$$[p]_R(x) = \sum_{d=0}^{\infty} {}_R\eta_R(v_d)x^{p^d}.$$

*Modulo p, there is the relation:*

$$\sum_{\substack{i\geq 0\\ j>0}} {}_L t_i \eta_R(v_j)^{p^i} \equiv \sum_{\substack{i>0\\ j\geq 0}} {}_L v_i t_j^{p^i} \pmod{p}.$$

*Proof (Proof sketch).* Work modulo $p$, one can Freshman's Dream the identity $[p]_L(g(x)) = g([p]_R(x))$ to death.

**Corollary 29 ([44, Lemmas 4.7-8]).** *Write $I_d$ for the ideal $I_d = (p, v_1, \ldots, v_{d-1})$. Then*

$$\eta_R(v_d) \equiv v_d \pmod{I_d}.$$

*It follows that the ideals $I_d$ are invariant for all d.* □

What is *much* harder to prove is the following:

**Theorem 28 ([44, Theorem 4.9]).** *If $I$ is an invariant prime ideal, then $I = I_d$ for some d.*

*Proof (Proof sketch).* Inductively assume that $I_d \subseteq I$. If this is not an equality, we want to show that $I_{d+1} \subseteq I$ is forced. Take $y \in I \setminus I_d$; if we could show

$$\eta_R(y) = av_d^j t^K + \cdots,$$

we could proceed by primality to show that $v_d \in I$ and hence $I_{d+1} \subseteq I$. This is possible (and, indeed, this is how the proof goes), but it requires serious bookkeeping.

The equivalent statement in terms of stacks is:

**Theorem 29 (Landweber).** *The unique closed substack of $\mathcal{M}_{\mathbf{fg},(p)} := \mathcal{M}_{\mathbf{fg}} \times$ Spec $\mathbb{Z}_{(p)}$ of codimension d is selected by $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}}/(p, v_1, \ldots, v_{d-1})$.* □

*Remark 31.* The complementary open substack of dimension $d$ is harder to describe. From first principles, we can say only that it is the locus where the coordinate functions $p, v_1, \ldots, v_d$ do not *all simultaneously vanish*. It turns out that:

1. On a cover, at least one of these coordinates can be taken to be invertible.

What is $\eta_R$? Why does the coefficient of $x^{p^d}$ depend only on $v_d$?

To understand where the $\eta_R$ comes from in the formal below it's the best to mention that the two formal group laws coming from $\eta_L, \eta_R : BP_* \to BP_*BP$ are isomorphic. And the formula for this isomorphism is the one in the lemma below. I don't think we've mentioned $BP$ yet, though. But I guess $\mathcal{M}_{BPP}$ is the moduli of $p$-typical formal group laws mentioned above, so we are very close to introducing $BP$. I have part of this written up about the Hopf algebroid $(BP_*, BP_*BP)$. I am wary of including it and risk botching up the big picture. -d.s.

Under $\eta_R$?

I'm gonna try tidying this proof up a bit. -d.s.

How does the base case go?

I can't extrapolate this series just from one term, but I'm guessing the rest obviously lives in $I_d$? Also, what is $a$ and $t$ and why can't they be in $I$ instead of $v_d$?

**Cite me:** Landweber must have a paper?.

This actually uses the Zariski topology on the affine site, and hence may really use stackiness instead of levelwise schemeiness. This is a problem, since you just told your students that stackiness won't come up...

2. Once one of them is invertible, a coordinate change on the formal group law can be used to make $v_d$ (and perhaps others in the list) invertible. Hence, we can use $v_d^{-1}\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}}$ as a coordinate chart.
3. Over a further base extension and a further coordinate change, the higher coefficients $v_{d+k}$ can be taken to be zero. Hence, we can also use $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$ as a coordinate chart.

Hood wanted to know: What, exactly, is required here?

We can now rephrase Theorem 24 in terms of algebraic conditions.

Cite me: Find Landweber's original paper.

**Theorem 30 (Landweber, cf. Theorem 24, see also [13, Theorem 21.4 and Proposition 21.5]).** *Let $M$ be a module over*

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}} \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_d, \ldots].$$

*If $(p, v_1, \ldots, v_d, \ldots)$ forms an infinite regular sequence on $M$, then*

$$X \mapsto M \otimes_{\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}}} MU_0(X)$$

*determines a homology theory* on finite spectra $X$. *Moreover, if $M/I_d = 0$ for some $d \gg 0$, then the same formula determines a homology theory* on all spectra $X$.

*Proof.* This is a direct consequence of the classification of closed substacks of $\mathcal{M}_{\mathbf{fg},(p)}$ in Theorem 29. Specifically, $M$ determines a flat quasicoherent sheaf on $\mathcal{M}_{\mathbf{fg},(p)}$ when $\mathrm{Tor}_1(M, N) = 0$ for any other comodule $N$. Using the classification of closed substacks and the regularity condition, one can iteratively use the short exact sequences

Why comodule and not module? Can you even take Tor of a module with a comodule?

$$0 \to M/I_d \xrightarrow{\cdot v_d} M/I_d \to M/I_{d+1} \to 0$$

to trade this condition for the list of conditions

- $\mathrm{Tor}_1(p^{-1}M, N) = 0$.
- $\mathrm{Tor}_2(v_1^{-1}M/p, N) = 0$.
- $\ldots$
- $\mathrm{Tor}_d(v_{d-1}^{-1}M/I_{d-1}, N) = 0$.
- $\mathrm{Tor}_{d+1}(M/I_d, N) = 0$.

for any $d$. If $N$ is coherent, as in the case $N = MU_*(X)$ for a finite spectrum $X$, then this final condition is satisfied automatically for $d \gg 0$. (Alternatively, we can assume that $M$ eventually satisfies this condition on its own.) By observing the length of the Koszul resolution associated to the cover $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$, one finally sees that

Can you explain this more? How is the classification of closed substacks used? Where are these conditions coming from?

Is this true? There's some going-around / extension-problem stuff that I've never understood, and it's gotten me in trouble before.

$$\mathrm{Tor}_d(v_{d-1}^{-1}M/I_{d-1}, N) = 0$$

is satisfied for *any* quasicoherent sheaf.

*Remark 32.* It's worth pointing out how strange all of this is. In Euclidean geometry, open subspaces are always top-dimensional, and closed subspaces can drop dimension. Here, proper open substacks of every dimension appear, and every nonempty closed substack is $\infty$–dimensional (albeit of positive codimension).

## 3.5 Mar 4: The structure of $\mathcal{M}_{\mathbf{fg}}$ III: Small scales

This is the most confusing section in this chapter.

We now turn to the deformation theory of formal groups, which is about the appearance of formal groups in families. Specifically, following Lecture 3.2 we will be interested in infinitesimal deformations of formal groups over fields of positive characteristic.

**Definition 51.** Given a formal group $\Gamma$ classified by a map $\operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$, then a *deformation of* $\Gamma$ *to a scheme* $X$ is a factorization

$$\operatorname{Spec} k \to X \to \mathcal{M}_{\mathbf{fg}}.$$

If $X$ is a nilpotent thickening of $\operatorname{Spec} k$ (or an ind-system of such), then the deformation is said to be *infinitesimal*.

The study of all possible infinitesimal deformations of a particular map $\operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ has a geometric interpretation, embodied by the following Lemma:

**Lemma 49.** *Let* $\operatorname{Spec} k \to Y$ *be any map, and let* $\operatorname{Spec} k \to X \to Y$ *be a factorization through a nilpotent thickening* $X$ *of* $\operatorname{Spec} k$*. Then there is a natural further factorization*

Actually, maybe this came up at the beginning of the previous day?

$$\operatorname{Spec} k \to X \dashrightarrow Y^{\wedge}_{X} \to Y. \square$$

The spirit of the Lemma, then, is that the study of infinitesimal deformations of $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ is equivalent to the study of $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$ itself. So, this fits into our program of analyzing the (local) structure of $\mathcal{M}_{\mathbf{fg}}$.

*Example 23.* It's also helpful to expand what an infinitesimal deformation is in our case of interest. Set $Y = \mathcal{M}_{\mathbf{fg}}$, and fix a map $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ classifying a formal group $\Gamma$ over $\operatorname{Spec} k$. Let $S$ be a local ring with maximal ideal $\mathfrak{m}$ so that $S$ is a nilpotent thickening of $S/\mathfrak{m}$. A deformation of $\Gamma$ to $S$ is the data of a formal group $\widehat{\mathbb{G}}$ over $\operatorname{Spec} S$, an identification $i\colon \operatorname{Spec} k \to \operatorname{Spec} S/\mathfrak{m}$, and a choice of an isomorphism $f$ fitting together into the following diagram:

$$\begin{tikzcd}
\Gamma \arrow[r, "f", "\simeq"'] \arrow[dr] & i^* j^* \widehat{\mathbb{G}} \arrow[rr] \arrow[d] & & \widehat{\mathbb{G}} \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "i"] & \operatorname{Spec} S/\mathfrak{m} \arrow[r, "j"] & \operatorname{Spec} S.
\end{tikzcd}$$

*Example 24.* Consider the case of an infinitesimal parameter space $X = \widehat{\mathbb{A}}^1$. A map $\widehat{\mathbb{A}}^1 \to \mathcal{M}_{\mathbf{fg}}$ can be presented by a map $\widehat{\mathbb{A}}^1 \to \mathcal{M}_{\mathbf{fgl}}$, which corresponds to a "family" of formal group laws $+_{\varphi_h}$ of the form

$$x +_{\varphi_h} y = (x +_{\varphi} y) + h(x +_{\varphi(1)} y) + h^2(x +_{\varphi(2)} y) + \cdots$$

for some series $+_{\varphi(n)}$. In particular, $+_{\varphi(0)}$ is a formal group law over $k$.

The analysis of $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$ is due to Lubin and Tate, but we first follow a more structured approach written down by Lazarev.

**Cite me:** Cite both of these.

Can this be phrased geometrically?

**Definition 52.** Let $+_{\varphi}$ be a formal group law over $R$, and let $M$ be an $R$–module. The deformation complex $\widehat{C}^*(\varphi; M)$ is defined by

$$M \to M[\![x_1]\!] \to M[\![x_1, x_2]\!] \to M[\![x_1, x_2, x_3]\!] \to \cdots$$

with differential

$$\begin{aligned}(df)(x_1, \ldots, x_{n+1}) = \varphi_1 &\left( \sum_{i=1}^{n} {}_{\varphi} x_i, x_{n+1} \right) \cdot f(x_1, \ldots, x_n) \\ &+ \sum (-1)^i f(x_1, \ldots, x_i +_{\varphi} x_{i+1}, \ldots, x_{n+1}) \\ &+ (-1)^{n+1} \left( \varphi_2 \left( x_1, \sum_{i=2}^{n+1} {}_{\varphi} x_i \right) \cdot f(x_2, \ldots, x_{n+1}) \right),\end{aligned}$$

where we have written

$$\varphi_1(x, y) = \frac{\partial(x +_{\varphi} y)}{\partial x}, \qquad \varphi_2(x, y) = \frac{\partial(x +_{\varphi} y)}{\partial y}.$$

This complex tracks the data of infinitesimal deformations. For instance, consider a deformed automorphism $f$ of $+_{\varphi}$, expressed as

$$f(x) = f_0(x) + h f_1(x) + h^2 f_2(x) + \cdots,$$

and satisfying

$$f(x +_{\varphi} y) = f(x) +_{\varphi} f(y).$$

Applying $\left.\frac{\partial}{\partial h}\right|_{h=0}$ to this equality gives

$$f_1(x +_{\varphi} y) = \varphi_1(x, y) f_1(x) + \varphi_2(x, y) f_1(y)$$

and thus $f_1$ is a 1–cocycle in the deformation complex. A similar sequence of observations culminates in the following theorem:

Why is this $\varphi_1(x, y)$ and not $\varphi_1(f_0(x), f_0(y))$? Is it the point that $f(x) \equiv x$ mod $h$?

**Theorem 31 ([21, p. 1320]).** *Let $+_\varphi$ be a formal group law over a ring $R$ and let $S \to R$ be a square–zero extension with kernel $M$.*

1. *Automorphisms of $+_\varphi$ over $S$ covering the identity on $R$ correspond to elements in $\widehat{Z}^1(\varphi; M)$.*
2. *Extensions of $+_\varphi$ to $S$ correspond to elements in $\widehat{Z}^2(\varphi; M)$.*
3. *Two such extensions are isomorphic as formal group laws over $S$ if their cocycles differ by an element in $\widehat{B}^2(\varphi; M)$.* □

So, this complex contains all the information we're interested in. Miraculously, we actually already studied the main input to computing this complex yesterday:

**Lemma 50 ([21, p. 1320]).** *This is quasi-isomorphic to the usual bar complex:*

$$\widehat{C}^*(\varphi; M) \to \mathsf{FormalSchemes}(B\widehat{\mathbb{G}}_\varphi, M \otimes \widehat{\mathbb{G}}_a)$$
$$f \mapsto \varphi_1\left(0, \sum_{i=1}^{n} {}_\varphi x_i\right)^{-1} f(x_1, \ldots, x_n). \square$$

What is $\widehat{\mathbb{G}}_\varphi$? Also, you stop using $\widehat{\mathbb{G}}_\varphi$ below.

Of course, yesterday we computed the specific case of $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$. However, by filtering the multiplication on $\widehat{\mathbb{G}}$ by degree, we can use this specific calculation to get up to the general one.

**Lemma 51.** *Let $\widehat{\mathbb{G}}$ be a formal group of finite height $d$ over a field $k$. Then the group $H^2(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$ classifying isomorphism classes of deformations is a free $k$–vector space of dimension $(d-1)$.*

$d$ or $(d-1)$? There's $\beta_0$ through $\beta_{d-1}\cdots$

*Proof (Proof (after Hopkins)).* Using $p$–typification, we select a coordinate on $\widehat{\mathbb{G}}$ of the form

$$x +_\varphi y = x + y + (\text{unit})c_{p^d} + \cdots.$$

Then, filter $\widehat{\mathbb{G}}$ by degree and consider the resulting spectral sequence of signature

$$H^*(\widehat{\mathbb{G}}_a; M \otimes \widehat{\mathbb{G}}_a) \cong M \otimes (\Lambda_k[\alpha_j \mid j \geq 0] \otimes k[\beta_j \mid j \geq 0]) \Rightarrow H^*(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a).$$

To compute the differentials in this spectral sequence, one computes by hand the formula for the differential in the bar complex, working up to lowest visible degree. In order to compute

$$(x +_\varphi y)^{p^r} - (x^{p^r} + y^{p^r}) = (\text{unit}) \cdot c_{p^{d+r}}(x, y) + \cdots,$$

where we used $c_{p^d}^{p^r} = c_{p^{r+d}}$. So, we see that there are $d-1$ things at the bottom of the spectral sequence which are not coboundaries, and we need to check that they are indeed permanent cocycles. To do this, we need only show that they are realized by deformations, which Lubin and Tate accomplish in Lemma 52.

This last "sentence" is missing a few words. Also, maybe you should remind us what $\alpha_j$ and $\beta_j$ are in terms of $x$ and $y$.

**Lemma 52 ([22, Proposition 1.1]).** *Let $W$ be a local ring with residue field $k$, and let $\varphi$ be a group law of height $d$ on $k$. There is a group law $\tilde{\varphi}$ over $W[\![u_1, \ldots, u_{d-1}]\!]$ restricting to $\varphi$ on $k$ such that for some $j \geq 1$,*

$$x +_{\tilde{\varphi}} y \equiv x + y + u_j c_{p^j}(x, y) \pmod{u_1, \ldots, u_{j-1}, (x, y)^{p^j+1}}. \square$$

Picking $W = \mathbb{W}_p(k)$ to be the ring of Witt vectors, Lemma 52 produces the universal example of a deformation of a group law $\varphi$ to $\tilde{\varphi}$.

**Cite me:** You got this from 7.5.1 of the Crystals notes..

**Theorem 32.** *Let $\operatorname{Spf} R$ be an infinitesimal deformation of its residue field $\operatorname{Spec} k$. For each lift of $\varphi$ to $\psi$ over $\operatorname{Spf} R$, there is a unique homomorphism*

$$\alpha \in \mathsf{FormalSchemes}(\operatorname{Spf} R, \operatorname{Spf} \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!])$$

*with $\alpha^* \tilde{\varphi}$ uniquely strictly isomorphic to $\psi$.*

*Proof.* We will prove this inductively on the neighborhoods of $\operatorname{Spec} k = \operatorname{Spec} R/I$ in $\operatorname{Spf} R$. Suppose that we have demonstrated the Theorem for $\psi_{r-1} = R/I^{r-1} \otimes \psi$, so that there is a map $\alpha_{r-1} \colon \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!] \to R/I^{r-1}$ and a strict isomorphism $g_{r-1} \colon \psi_{r-1} \to \alpha_{r-1}^* \tilde{\varphi}$. The exact sequence

$$0 \to I^{r-1}/I^r \to R/I^r \to R/I^{r-1} \to 0$$

exhibits $R/I^r$ as a square–zero extension of $R/I^{r-1}$ by $M = I^{r-1}/I^r$.

My source material wants $R$ to be Noetherian so that $M$ is finite dimensional. This is important?

Let $\beta$ be *any* lift of $\alpha_{r-1}$ and $h$ be *any* lift of $g_{r-1}$ to $R/I^r$, and let $A$ and $B$ be the induced group laws

$$x +_A y = \beta^* \tilde{\varphi}, \qquad x +_B y = h\left(h^{-1}(x) +_{\psi_r} h^{-1}(y)\right).$$

Since these both deform the group law $\psi_{r-1}$, by Lemma 51 there exist $m_j \in M$ and $f(x) \in M[\![x]\!]$ satisfying

$$(x +_B y) - (x +_A y) = (df)(x, y) + \sum_{j=1}^{d-1} m_j v_j(x, y),$$

where $v_j(x, y)$ is the 2–cocycle associated to the cohomology 2–class $\beta_j$. The following definitions complete the induction:

$$g_r(x) = h(x) - f(x), \qquad \alpha_r(u_j) = \beta(u_j) + m_j. \square$$

*Remark 33.* Our calculation $H^1(\widehat{\mathbb{G}}_\varphi; M \otimes \widehat{\mathbb{G}}_a)$ also shows that there are *no* automorphisms of the formal group $\Gamma$ over the special fiber which induce automorphisms of the universal deformation. Specifically, *any* deformation of a nontrivial automorphism of $\Gamma$ acts nontrivially on Lubin–Tate space by permuting the deformations living over the various fibers. A consequence of this observation is that the deformation space produced in Theorem 32 is a *formal scheme*, carrying only the previously-known inertial group of $\operatorname{Aut}\Gamma$ at the special fiber, rather than a full-on stack.

*Remark 34.* We also see that our analysis fails wildly for the case $\Gamma = \widehat{\mathbb{G}}_a$. The differential calculation in Lemma 51 are meant to give us an upper bound on the dimensions of $H^1(\Gamma; \widehat{\mathbb{G}}_a)$ and $H^2(\Gamma; \widehat{\mathbb{G}}_a)$, but this family of differentials is zero in the additive case. Accordingly, both of these vector spaces are infinite dimensional — the infinitesimal

**Cite me:** Neil's FG notes in the first half of section 18 talk about additive extensions and their relation to infinitesimal deformations. In the second half, he (more or less) talks about the de Rham crystal and shows that $\mathrm{Ext}_{\mathrm{rigid}}(G, \widehat{\mathbb{G}}_a) \cong \mathrm{Prim}(H^1_{dR}(G/X))$ in 18.37..

Having accomplished all our major goals, we close our algebraic analysis of $\mathcal{M}_{\mathbf{fg}}$ with a diagram summarizing our results.

I still have some confusion about the formal similarity between deforming formal group laws over square-zero extensions of the base and deforming formal $n$–buds over the finite order nilpotent neighborhoods of a point. This would be a good place to sort that out.

PICTURE GOES HERE.

## 3.6 Mar 7: Spectra detecting nilpotence

We have now arrived at the conclusion of our program from Lecture 3.2 for manufacturing interesting homology theories from Quillen's theorem: we have an ample supply of open and closed substacks of $\mathcal{M}_{\mathbf{fg}}$, and we have analyzed its geometric points as well as their deformation neighborhoods.

**Definition 53.** We define the following "chromatic" homology theories:

- Recall that the moduli of $p$–typical group laws is affine, presented by the scheme $\operatorname{Spec} BPP_0$, $BPP_0 := \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$. Since the inclusion of $p$–typical group laws into all group laws induces an equivalence of stacks, it follows that this formula determines a homology theory on finite spectra, called *Brown–Peterson homology*:
$$BPP_0(X) := MUP_0(X) \otimes_{MUP_0} BPP_0.$$
- A chart for the open substack $\mathcal{M}_{\mathbf{fg}}^{\leq d}$ in terms of $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} \cong \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ is $\operatorname{Spec} E(d)P_0 := \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}]$. It follows that there is a homology theory $E(d)P$, called *the $d^{th}$ Johnson–Wilson homology*, defined on all spectra by
$$E(d)P_0(X) := MUP_0(X) \otimes_{MUP_0} E(d)P_0.$$
- Similarly, for a formal group $\Gamma$ of height $d < \infty$, there is a chart $\operatorname{Spf} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!]$ for its deformation neighborhood. Correspondingly, there is a homology theory $E_\Gamma$, called *the (discontinuous) Morava E–theory for* $\Gamma$, determined by
$$E_{\Gamma 0}(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!].$$

Why is this not written as $E_\Gamma P_0$?

- Since $(p, u_1, \ldots, u_{d-1})$ forms a regular sequence on $E_{\Gamma *}$, we can form the regular quotient $K_\Gamma$ in the homotopy category. This determines a spectrum, and hence determines a homology theory called *the Morava K–theory for* $\Gamma$. In the case where $\Gamma$ comes from the Honda $p$–typical formal group law (of height $d$), this spectrum is often written as $K(d)$. As an edge case, we also set $K(\infty) = H\mathbb{F}_p$.[6]
- More delicately, there is a version of Morava $E$–theory which takes into account the formal topology on $(\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma}$, called *continuous Morava E–theory*. It is defined by the pro-system $\{E_\Gamma(X)/u^I\}$, where $I$ ranges over multi-indices and the quotient is taken in the "homotopical sense", i.e.,

Can you be more precise about this?

[6] By Theorem 27, it often suffices to consider just these spectra to make statements about all $K_\Gamma$. With more care, it even often suffices to consider formal groups $\Gamma$ of finite height.

$u_j$–torsion elements contribute to the odd-degree homotopy of the quotient.

- There is also a homology theory associated to the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$. Since $I_d = (p, v_1, \ldots, v_{d-1})$ is generated by a regular sequence on $BPP_0$, we can directly define the spectrum $P(d)P$ by a regular quotient:

$$P(d)P = BP/(p, v_1, \ldots, v_{d-1}).$$

This spectrum does have the property $P(d)P_0 = BPP_0/I_d$, but it is *only* the case that $P(d)P_0 = BPP_0(X)/I_d$ when $I_d$ forms a regular sequence on $BPP_0(X)$ — which is reasonably rare among the cases of interest.

Ravenel's Localization W/R/T, Corollaries 2.14 and 2.16, is another reference for this. He, in turn, cites Yosimura's *Universal coefficient sequences for cohomology theories of CW-spectra*.

*Remark 35 ([19, Section 5.2], [39, Theorem 2.13]).* Morava $K$–theory at the even prime is not commutative. Instead, there is a derivation $Q_d : K(d) \to \Sigma K(d)$ satisfying

$$ab - ba = uQ_d(a)Q_d(b).$$

In particular, $K(d)^*X$ is a commutative ring whenever $K(d)^1X = 0$.

Having constructed these "stalk" homology theories, I want to show that you can actually perform stalkwise analyses of the sheaves coming from bordism theory. Our example case is a famous theorem: the solution of Ravenel's nilpotence conjectures by Devinatz, Hopkins, and Smith. Their theorem concerns spectra which "detect nilpotence" in the following sense:

All the stuff after this point is written graded-ly. I guess we still haven't decided whether this is the right presentation.

**Cite me:** Hopkins–Smith, or maybe the intro to D–H–S.

**Definition 54.** A ring spectrum $E$ *detects nilpotence* if, for any ring spectrum $R$, the kernel of the Hurewicz homomorphism $E_* : \pi_*R \to E_*R$ consists of nilpotent elements.

First, a word about why one would care about such a condition. The following theorem is classical:

**Cite me:** Nishida.

**Theorem 33 (Nishida).** *Every homotopy class $\alpha \in \pi_{\geq 1}\mathbb{S}$ is nilpotent.* $\square$

However, people studying $K$–theory in the '70s discovered the following phenomenon:

**Cite me:** Adams.

**Theorem 34 (Adams).** *Let $M_{2n}(p)$ denote the mod–$p$ Moore spectrum with bottom cell in degree $2n$. Then there is an index $n$ and a map $v : M_{2n}(p) \to M_0(p)$ such that $KU_*v$ acts by multiplication by the $n^{th}$ power of the Bott class. The minimal such $n$ is given by the formula*

$$n = \begin{cases} p-1 & \text{when } p \geq 3, \\ 4 & \text{when } p = 2. \square \end{cases}$$

In particular, this means that $v$ cannot be nilpotent, since a null-homotopic map induces the zero map in any homology theory. Just as we took the non-nilpotent endomorphism $p$ in $\pi_0 \operatorname{End} \mathbb{S}$ and coned it off, we can take the endomorphism $v$ in $\pi_{2p-2} \operatorname{End} M_0(p)$ and cone it off to form a new spectrum called $V(1)$.[7]

One can ask, then, whether the pattern continues: does $V(1)$ have a non-nilpotent self-map, and can we cone it off to form a new such spectrum with a new such map? Can we then do that again, indefinitely? In order to study this question, we are motivated to find spectra $E$ as above, since an $E$ that detects nilpotence cannot send such a nontrivial self-map to zero. In fact, we found one such $E$ already:

**Theorem 35 (Devinatz–Hopkins–Smith).** *Complex cobordism MU detects nilpotence.* □

Cite me: Devinatz, Hopkins, Smith.

They also show that the $MU$ is the universal object which detects nilpotence, in the sense that any other ring spectrum can have this property checked stalkwise on $\mathcal{M}_{MU}$:

**Corollary 30 ([15, Theorem 3]).** *A ring spectrum $E$ detects nilpotence if and only if for all $0 \leq d \leq \infty$ and for all primes $p$, $K(d)_*E \neq 0$.*

*Proof.* If $K(d)_*E = 0$ for some $d$, then the non-nilpotent unit map $\mathbb{S} \to K(d)$ lies in the kernel of the Hurewicz homomorphism for $E$, so $E$ fails to detect nilpotence.

Hence, for any $d$ we must have $K(d)_*E \neq 0$. Because $K(d)_*$ is a field, it follows by picking a basis of $K(d)_*E$ that $K(d) \wedge E$ is a nonempty wedge of suspensions of $K(d)$. So, for $\alpha \in \pi_*R$, if $E_*\alpha = 0$ then $(K(d) \wedge E)_*\alpha = 0$ and hence $K(d)_*\alpha = 0$. So, we need to show that if $K(d)_*\alpha = 0$ for all $n$ and all $p$ then $\alpha$ is nilpotent. Taking Theorem 35 as given, it would suffice to show merely that $MU_*\alpha$ is nilpotent. This is equivalent to showing that the ring spectrum $MU \wedge R[\alpha^{-1}]$ is contractible or that the unit map is null:

$$\mathbb{S} \to MU \wedge R[\alpha^{-1}].$$

A nontrivial result of Johnson and Wilson shows that if $MU_*X = 0$ for any $X$, then for any $d$ we have $K([0,d])_*X = 0$ and $P(d+1)_*X = 0$. (Specifically, it is immediate that $MU_*X = 0$ forces $P(d+1)_*X = 0$ and $v_{d'}^{-1}P(d')_*(X) = 0$ for all $d' < d$. What's nontrivial is showing that $v_{d'}^{-1}P(d')_*(X) = 0$ if and only if $K(d')_*(X) = 0$ [31, Theorem 2.1.a], [18, Section 3].) Taking $X = R[\alpha^{-1}]$, we have assumed all of these are zero except for

[7] The spectrum $V(1)$ is actually defined to be a finite spectrum with $BP_*V(1) \cong BP_*/(p, v_1)$. At $p = 2$ this spectrum doesn't exist and this is a misnomer. More generally, at odd primes $p$ Nave shows that $V((p+1)/2)$ doesn't exist [29, Theorem 1.3].

$P(d+1)$. But $\operatorname{colim}_d P(d+1) \simeq H\mathbb{F}_p \simeq K(\infty)$, and $\mathbb{S} \to K(\infty) \wedge R[\alpha^{-1}]$ is assumed to be null as well. By compactness of $\mathbb{S}$, that null-homotopy factors through some finite stage $P(d+1) \wedge R[\alpha^{-1}]$ with $d \gg 0$.

As another example of the primacy of these methods, we can show the following interesting result. Say that $R$ is a field spectrum when every $R$–module (in the homotopy category) splits as a wedge of suspensions of $R$. It is easy to check (as mentioned in the proof above) that $K(d)$ is an example of such a spectrum.

**Corollary 31.** *Every field spectrum $R$ splits as a wedge of Morava's $K(d)$ theories.*

*Proof.* Set $E = \bigvee_{\text{primes } p} \bigvee_{d \in [0,\infty]} K(d)$, so that $E$ detects nilpotence. The class 1 in the field spectrum $R$ is non-nilpotent, so it survives when paired with some $K$–theory $K(d)$, and hence $R \wedge K(d)$ is not contractible. Because both $R$ and $K(d)$ are field spectra, the smash product of the two simultaneously decomposes into a wedge of $K(d)$s and a wedge of $R$s. So, $R$ is a retract of a wedge of $K(d)$s, and picking a basis for its image on homotopy shows that it is a sub-wedge of $K(d)$s.

*Remark 36.* This is interesting in its own right, because field spectra are exactly those spectra which have Künneth isomorphisms. So, even if you weren't neck-deep in algebraic geometry, you might still have struck across these homology theories just if you like to compute things, since Künneth formulas make things computable.

Jake asked if there was a geometric interpretation of these cohomology theories $K_\Gamma$. At present, there isn't one. Maybe remark on this.

## 3.7 Mar 9: Periodicity in finite spectra

We're now well-situated to address Ravenel's question about finite spectra and periodic self-maps. The solution to this problem passes through some now-standard machinery for triangulated $\otimes$–categories.

**Definition 55.** A subcategory of the category of a triangulated category (e.g., $p$–local finite spectra) is *thick* if...

Do triangulated categories come with weak equivalences?

Lurie's definition (lecture 26) doesn't talk about weak equivalences.

- ...it is closed under weak equivalences.
- ...it is closed under retracts.
- ...it has a 2-out-of-3 property for cofiber sequences.

Examples of thick subcategories include:

- The category $\mathsf{C}_d$ of $p$–local finite spectra which are $K(d-1)$–acyclic. (For instance, if $d = 1$, the condition of $K(0)$–acyclicity is that the spectrum have purely torsion homotopy groups.) These are called "finite spectra of type at least $d$".
- The category $\mathsf{D}_d$ of $p$–local finite spectra $F$ which have a self-map $v : \Sigma^N F \to F$, $N \gg 0$, inducing multiplication by a unit in $K(d)$–homology. These are called "$v_d$–self–maps".

Ravenel shows the following useful result interrelating the $\mathsf{C}_d$:

**Lemma 53 ([31, Theorem 2.11]).** *For $X$ a finite complex, there is a bound*

$$\dim K(d-2)_* X \leq \dim K(d-1)_* X.$$

*In particular, there is an inclusion $\mathsf{C}_{d-1} \subseteq \mathsf{C}_d$.* □

Hopkins and Smith show the following classification theorem:

**Theorem 36 ([15, Theorem 7]).** *Any thick subcategory $\mathsf{C}$ of $p$–local finite spectra must be $\mathsf{C}_d$ for some finite $d$.*

*Proof.* Since $\mathsf{C}_d$ are nested by Lemma 53 and they form an exhaustive filtration, it is thus sufficient to show that any object $X \in \mathsf{C}$ with $X \in \mathsf{C}_d$ induces an inclusion $\mathsf{C}_d \subseteq \mathsf{C}$. Write $R$ for the endomorphism ring spectrum $R = F(X, X)$, and write $F$ for the fiber of its unit map:

Why do you not need $\mathsf{C}_\infty$ for this?

$$F \xrightarrow{f} \mathbb{S} \xrightarrow{\eta_R} R.$$

Finally, let $Y$ be *any* finite spectrum of type at least $d$.

Now consider applying $K(n)$–homology (for *arbitrary* $n$) to the map

$$1 \wedge f \colon Y \wedge F \to Y \wedge \mathbb{S}.$$

The induced map is always zero:

- In the case that $K(n)_* X$ is nonzero, then $K(n)_* \eta_R$ is injectiveand $K(n)_* f$ is zero.

  Recall, $K(n)_* \mathbb{S}$ has rank 1.
- In the case that $K(n)_* X$ is zero, then $n \leq d$and, because of the bound on type, $K(n)_* Y$ is zero as well.

  In class you eventually decided on $n < d$.

  Do we need $X$ to be exactly of type $d$ to make this conclusion?

By a small variant of local nilpotence detection (Corollary 30, [15, Corollary 2.5]), it follows for $j \gg 0$ that

$$Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}$$

is null-homotopic. Hence, one can calculate the cofiber to be

$$\operatorname{cofib}\left(Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}\right) \simeq Y \wedge \operatorname{cofib} f^{\wedge j} \simeq Y \vee (Y \wedge \Sigma F^{\wedge j}),$$

so that $Y$ is a retract of this cofiber.

We now work to show that this smash product lies in the thick subcategory $\mathsf{C}$ of interest. First, note that $X \wedge Z$ lies in $\mathsf{C}$ for any finite complex $Z$, since $Z$ can be expressed as a finite gluing diagram of spheres and smashing this through with $X$ expresses $X \wedge Z$ as the iterated cofiber of maps with source and target in $\mathsf{C}$. Next, consider the following smash version of the octahedral axiom: the factorization

$$F \wedge F^{\wedge(j-1)} \xrightarrow{f \wedge 1} \mathbb{S} \wedge F^{\wedge(j-1)} \xrightarrow{1 \wedge f^{\wedge(j-1)}} \mathbb{S} \wedge \mathbb{S}^{\wedge(j-1)}$$

Is this sequence backwards? (Note that it doesn't matter: you can just write the factorization in the other order...)

begets a cofiber sequence

$$F \wedge \operatorname{cofib} f^{\wedge(j-1)} \to \operatorname{cofib} f^{\wedge j} \to \operatorname{cofib} f \wedge \mathbb{S}^{\wedge(j-1)}.$$

Recall, $f$ is the fiber of the $\eta_R$.

Now turn an eye toward induction. Noting that $\operatorname{cofib}(f) = R = X \wedge DX$ lies in $\mathsf{C}$, we can use the 2-out-of-3 property on the octahedral sequence to see that $\operatorname{cofib}(f^{\wedge j})$ lies in $\mathsf{C}$. It follows that $Y \wedge \operatorname{cofib}(f^{\wedge j})$ also lies in $\mathsf{C}$, and using the retraction $Y$ belongs to $\mathsf{C}$ as well.

As an application of this classification, they also show the following considerably harder theorem:

It's also a corollary of these same methods that the inclusion $\mathsf{C}_d \subsetneq \mathsf{C}_{d-1}$ is proper.

**Theorem 37 ([15, Theorem 9]).** *A $p$–local finite spectrum is $K(d-1)$–acyclic exactly when it admits a $v_d$–self–map.*

*Proof (Executive summary of proof).* Given the classification of thick subcategories, if a property is closed under thickness then one need only exhibit a single spectrum with the property to know that all the spectra in the thick subcategory it generates also all have that property. Inductively, they manually construct finite spectra $M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}})$ for sufficiently large indices $i_*$ which admit a self-map $v$ governed by a commuting square

$$\begin{array}{ccc} BP_* M_{|v_d| i_d}(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{v} & BP_* M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) \\ \| & & \| \\ \Sigma^{|v_d| i_d} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{- \cdot v_d^{i_d}} & BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}). \end{array}$$

You wrote $BPP_0$ yesterday.

These maps are guaranteed by very careful study of Adams spectral sequences.

*Remark 37.* We ran into the asymptotic condition $I \gg 0$ yesterday, when we asserted that there is no root of the 2–local $v_1$–self–map $v\colon M_8(2) \to M_0(2)$.

There is a second interesting application of these ideas, investigated by Paul Balmer as part of a broad attempt to analyze a geometric object through its modules.

**Definition 56.** Given a triangulated $\otimes$–category $\mathsf{C}$, define a thick subcategory $\mathsf{C}' \subseteq \mathsf{C}$ to be...

**Cite me:** Reference Balmer's SSS paper throughout this tail.

- ... a $\otimes$–*ideal* when it has the additional property that $x \in \mathsf{C}'$ forces $x \otimes y \in \mathsf{C}'$ for any $y \in \mathsf{C}$.
- ... a *prime* $\otimes$–*ideal* when $x \otimes y \in \mathsf{C}'$ also forces at least one of $x \in \mathsf{C}'$ or $y \in \mathsf{C}'$.

Finally, define the *spectrum* of $\mathsf{C}$ to be its collection of prime $\otimes$–ideals, topologized so that $U(x) = \{\mathsf{C}' \mid x \in \mathsf{C}'\}$ form a basis of opens.

Double check that you have the directionality of this right. Is $U$ a basic open or a basic closed? It it full of things that contain $x$ or that don't contain in $x$?

**Theorem 38 (Balmer).** *The spectrum of $D^{\mathrm{perf}}(\mathsf{Mod}_R)$ is naturally homeomorphic to the Zariski spectrum of $R$.* □

Balmer's construction applies much more generally. The category $\mathsf{Spectra}$ can be identified with $\mathsf{Modules}_{\mathbb{S}}$, and so one is moved to attempt to compute the Balmer spectrum of $\mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}} = \mathsf{Spectra}^{\mathrm{fin}}$. In fact, we just finished this.

**Theorem 39.** *The Balmer spectrum of $\mathsf{Spectra}^{\mathrm{fin}}_{(p)}$ consists of the thick subcategories $\mathsf{C}_d$, and $\{\mathsf{C}_n\}_{n=0}^d$ are its open sets.*

*Proof.* Using the characterization of $\mathsf{C}_d$ as the kernel of $K(d-1)_*$, we see that it is a prime $\otimes$–ideal:

$$K(d-1)_*(X \wedge Y) \cong K(d-1)_* X \otimes_{K(d-1)_*} K(d-1)_* Y$$

is zero exactly when at least one of $X$ and $Y$ is $K(d-1)$–acyclic.

*Remark 38.* In fact, our favorite functor $\mathcal{M}_{MU}(-)\colon \mathsf{Spectra} \to \mathsf{QCoh}(\mathcal{M}_{MU})$ induces a homeomorphism of the Balmer spectrum of $\mathsf{Spectra}^{\mathrm{fin}}$ to that of $\mathcal{M}_{\mathbf{fg}}$. However, this functor does *not* exist on the level triangulated categories, so this remark has to be interpreted somewhat lightly.

Be careful about what the latter half of this means. Do you mean again to form something like $D^{\mathrm{perf}}(\mathcal{M}_{\mathbf{fg}})$?

## 3.8 Mar 11: Chromatic localization

Balmer's construction is remarkably successful at describing the most salient features of the stable category, but it falls a ways short of the rich "spectrum" object we've come to know from algebraic geometry. In particular, we have only a topological space, and not anything like a locally ringed space (or a space otherwise equipped locally with algebraic data). It's also totally unclear why $MU$ plays such an important mediating role between geometry (i.e., the stable category) and algebra (i.e., the moduli of formal groups). Nonetheless, taking that as granted, we can use Bousfield's theory of homological localization to access "local" categories of spectra of the sort that a sheaf of local rings would supply us with.

Is this just motivation or is there something specific you can say relating the following to Balmer spectra?

**Theorem 40 ([6], [23, Theorem 7.7]).** *Let $R_*$ denote the homology theory associated by Landweber's Theorem 24 to a flat map $j\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$. There is then a diagram*

$$\begin{tikzcd} \mathsf{Spectra}_R \arrow[r, "R_*", "\text{conservative}"'] \arrow[d, shift right, "i"'] \arrow[d, phantom, "\dashv"] & \mathsf{QCoh}(\operatorname{Spec} R) \arrow[d, shift right, "j_*"'] \arrow[d, phantom, "\dashv"] \\ \mathsf{Spectra} \arrow[u, shift right, "L_R"'] \arrow[ur, "R_*"] \arrow[r, "MU_*"] & \mathsf{QCoh}(\mathcal{M}_{MU}), \arrow[u, shift right, "j^*"'] \end{tikzcd}$$

*such that $i$ is fully faithful, $i$ is left-adjoint to $L_R$, $j^*$ is left-adjoint to $j_*$, $i$ and $j_*$ are inclusions of full subcategories, the red composites are all equal, and $R_*$ is conservative on $\mathsf{Spectra}_R$.*[8] □

Jay was rightfully fussy about the difference between, e.g., the open submoduli and its affine cover. Write this more carefully.

In the case when $R$ models the inclusion of the deformation space around the point $\Gamma_d$, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{\widehat{L}_d} \mathsf{Spectra}_{\Gamma_d}.$$

In the case when $R$ models the inclusion of the open complement of the unique closed substack of codimension $d$, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{L_d} \mathsf{Spectra}_d = \mathsf{Spectra}_{\mathcal{M}_{\mathbf{fg}}^{\le d}}.$$

We have set up our situation so that the following properties of these localizations either have easy proofs or are intuitive from the algebraic analogue of $j^* \vdash j_*$:

1. There is an equivalence

$$L_d X \simeq (L_d \mathbb{S}) \wedge X,$$

[8] The meat of this theorem is in overcoming set-theoretic difficulties in the construction of $\mathsf{Spectra}_R$. Bousfield accomplished this by describing a model structure on $\mathsf{Spectra}$ for which $R$–equivalences create the weak–equivalences.

analogous to $j^*M \simeq R \otimes M$ in the algebraic setting [33, Theorem 7.5.6]. Because $\widehat{L}_d$ is associated to the inclusion of a formal scheme (i.e., an ind-finite scheme), it has the formula

$$\widehat{L}_d X \simeq \lim_I \left( M_0(v^I) \wedge L_d X \right)$$

analogous to $j^*M \simeq \lim_j(R/I^j \otimes M)$ in the complete algebraic setting [16, Proof of Lemma 2.3].

**Cite me:** Ravenel (and Hopkins).

2. Because the open substack of dimension $d$ properly contains both the open substack of dimension $(d-1)$ and the infinitesimal deformation neighborhood of the closed point of height $d$, there are natural factorizations

Also mention that there are results about thinking of this thing as a pro-spectrum rather than a spectrum? For instance, there's the Davis–Lawson result on $\{M_0(v^I)\}$ forming an $E_\infty$–ring in the pro-category.

$$\mathrm{id} \to L_d \to L_{d-1}, \qquad \mathrm{id} \to L_d \to \widehat{L}_d.$$

In particular, $L_d X = 0$ implies both $L_{d-1}X = 0$ and $\widehat{L}_d X = 0$.

Also, idempotence?

3. The inclusion of the open substack of dimension $d-1$ into the one of dimension $d$ has relatively closed complement the point of height $d$. Algebraically, this gives a gluing square (or Mayer-Vietoris square), and this is reflected in homotopy theory by a homotopy pullback square (or chromatic fracture square):

This deserves a proof or a reference. (I spent a moment looking, and I can't actually find a nice "old" reference for chromatic fracture squares in the literature...)

$$\begin{tikzcd} L_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{L}_d \arrow[d] \\ L_{d-1} \arrow[r] & L_{d-1}\widehat{L}_d. \end{tikzcd}$$

*Remark 39.* More generally, whenever $L_B L_A = 0$, there is a fracture square

$$\begin{tikzcd} L_{A\vee B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_B \arrow[d] \\ L_A \arrow[r] & L_A L_B. \end{tikzcd}$$

So, this last fact follows from $L_d \simeq L_{K(0)\vee\cdots\vee K(d)}$ and $L_{K(d)}L_{K(d-1)} = 0$. Similarly, there is an "arithmetic fracture square"

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}}, \end{tikzcd}$$

which is a topological instantiation of the adèlic decomposition of a $\mathbb{Z}$–module.

There are also considerably more complicated facts known about these functors:

**Theorem 41 ([33, Theorem 7.5.7]).** *The homotopy limit of the tower*

$$\cdots \to L_d F \to L_{d-1} F \to \cdots \to L_1 F \to L_0 F$$

*recovers the p–local homotopy type of any finite spectrum F.*[9]

This suggests a productive method for analyzing the homotopy groups of spheres: study the homotopy groups of each $L_d\mathbb{S}$ and perform the reassembly process encoded by this inverse limit. Writing $M_d$ for the fiber in the sequence

$$M_d \to L_d \to L_{d-1},$$

the "geometric chromatic spectral sequence" associated to this tower takes the form

$$\pi_* M_* \mathbb{S} \Rightarrow \pi_* \mathbb{S}_{(p)}.$$

So, $M_d$ means the difference between the assembled layers $L_d$ and $L_{d-1}$ — but this was also the heuristic job of $\widehat{L}_d$ above. It turns out that these are interrelated by the following two theorems:

**Cite me:** Gross–Hopkins?.

**Theorem 42.** *There is a pair of natural equivalences*

$$\widehat{L}_d M_d \simeq \widehat{L}_d, \qquad\qquad M_d \widehat{L}_d \simeq M_d. \square$$

**Cite me:** I forget who this is due to.

Referring to "1." is clumsy. Make the previous things into Lemmas or something, rather than a bulleted list.

**Theorem 43.** *Analogous to "1." above, there is a natural equivalence*

$$M_d X \simeq \operatorname*{colim}_I \left( M^0(v^I) \wedge L_d X \right),$$

*where $M^0(v^I)$ denotes a generalized Moore spectrum with* top *cell in dimension* 0*.*

*Remark 40.* It is possible to draw the chromatic fracture square and the definition of $M_d$ in the same diagram:

[9] Spectra satisfying this limit property are said to be *chromatically complete*, which is closely related to being *harmonic*, i.e., being local with respect to $\bigvee_{d=0}^{\infty} K(d)$. (I believe this a joke about "music of the spheres".) It is known that nice Thom spectra are harmonic (so, in particular, every suspension and finite spectrum), that every finite spectrum is chromatically complete, and that there exist some harmonic spectra which are not chromatically complete.

$$\begin{tikzcd}
M_d X \arrow[r, equal] \arrow[d] & M_d X \arrow[d] \\
L_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{L}_d X \arrow[d] \\
L_{d-1} X \arrow[r] & L_{d-1} \widehat{L}_d X.
\end{tikzcd}$$

From this, we see that there is a fiber sequence

$$M_d X \to \widehat{L}_d X \to L_{d-1} \widehat{L}_d X.$$

The case $d = 1$ gives the prototypical example of the difference between these two presentations of the "exact height $d$ data", where the sequence becomes:

$$\operatorname*{colim}_j (M^0(p^j) \wedge L_1 X) \to \lim_j (M_0(p^j) \wedge L_1 X) \to \left( \lim_j (M_0(p^j) \wedge L_1 X) \right)_{\mathbb{Q}}.$$

If, for instance, $\pi_0 L_1 X = \mathbb{Z}_{(p)}$, then the long exact sequence of homotopy groups associated to this fiber sequence gives

$$\begin{tikzcd}
\pi_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_0 L_0 \widehat{L}_1 X \arrow[r] \arrow[d, equal] & \pi_{-1} M_1 X \arrow[d, equal] \\
\mathbb{Z}_p^\wedge \arrow[r] & \mathbb{Q}_p \arrow[r] & \mathbb{Z}/p^\infty.
\end{tikzcd}$$

This is a model for what happens generally: the $v_j$–torsion–free groups get converted to infinitely $v_j$–divisible groups, with some dimension shifts. (*Exactly* what happens is often hard to work out, and I'm not aware of a totally general statement.)

In any case, one sees that it is also profitable to consider the homotopy groups of $\widehat{L}_d \mathbb{S}$. The spectral version $\mathcal{D}_{\mathbb{S} \to E}(F)$ of $\mathcal{M}_E(F)$ considered on the first day furnishes us with a tool by which we can approach this:

**Theorem 44 (Example 20, Definition 40, and Remark 24).** *The $E_\Gamma$–based Adams spectral sequence for the sphere converges strongly to $\pi_* \widehat{L}_d \mathbb{S}$. Writing $\omega$ for the line bundle on $\mathcal{M}_{E_\Gamma}$ of invariant differentials, we have*

$$E_2^{*,*} = H^*_{\text{stack}}(\mathcal{M}_{E_\Gamma}; \omega^{\otimes *}) \Rightarrow \pi_* \widehat{L}_d \mathbb{S}. \square$$

The utility of this theorem is in the identification with stack cohomology. Recalling the discussion in Examples 14 and 15, as well as the identification

Show that the action of the stabilizer group lifts to an action on Lubin–Tate space. This is relevant for what you're about to write.

$$\mathcal{M}_{E_{\Gamma_d}} = (\mathcal{M}_{\mathbf{fg}})^{\wedge}_{\Gamma_d} \simeq \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)} // \underline{\mathrm{Aut}}(\Gamma_d)$$

in Remark 33, we become interested in the action of $\operatorname{Aut}\Gamma_d$ on $LT_d$. We will deduce the following description of $\operatorname{Aut}\Gamma_d$ later on:

**Theorem 45 (Corollary 39).** *For $\Gamma_d$ the Honda formal group law of height $d$ over a perfect field $k$ of positive characteristic $p$, we compute*

$$\operatorname{Aut}\Gamma_d \cong \mathbb{W}_p(k)\langle S\rangle \Big/ \begin{pmatrix} Sw = w^{\varphi}S, \\ S^d = p \end{pmatrix},$$

*where $\varphi$ denotes a lift of the Frobenius from $k$ to $\mathbb{W}_p(k)$.* □

As a matter of emphasis, this Theorem does not give a description of the *representation* of $\operatorname{Aut}\Gamma_d$, which is very complicated. Nonetheless, we have reduced the computation of all of the stable homotopy groups of spheres to an arithmetically–founded problem in profinite group cohomology, so that arithmetic geometry might lend a hand.

*Example 25 (Adams).* In the case $d = 1$, $\operatorname{Aut}(\Gamma_1) = \mathbb{Z}_p^\times$ and it acts on $\pi_* E_1 = \mathbb{Z}_p[u^{\pm}]$ by $\gamma \cdot u^n \mapsto \gamma^n u^n$. At odd primes $p$ (so that $p$ is coprime to the torsion part of $\mathbb{Z}_p^\times$), one computes

$$H^s(\operatorname{Aut}(\Gamma_1); \pi_* E_1) = \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \bigoplus_{j=2(p-1)k} \mathbb{Z}_p\{u^j\}/(pku^j) & \text{when } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This, in turn, gives the calculation

$$\pi_t \widehat{L}_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = t|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

These groups are familiar to homotopy theorists: the $J$–homomorphism $J : BU \to BF$described on the first day selects exactly these elements (for nonnegative $t$).

I don't remember what $F$ is.

It is a good exercise to work out what this calculation means in terms of the rest of the fracture square and for $M_1\mathbb{S}$.

# Case Study 4
# Unstable cooperations

Write an introduction for me.

## 4.1 Mar 21: Unstable contexts

Today we will take the framework of contexts discussed in Lecture 3.1 and augment it in twoimportant (and very distinct) ways. First, we will assume that $X$ is a *space* rather than a spectrum, and try to encode the extra structure appearing on $E_*X$ from this assumption. Toward that end, recall that the levels of $\mathcal{M}_E(X)$ are defined by repeatedly smashing $X$ with $E$, and that we had arrived at this by considering descent for the adjunction

Fix this intro. Don't name "two" things, for instance.

$$\mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}} \underset{\longleftarrow}{\xrightarrow{-\wedge E}} \mathsf{Modules}_E$$

induced by the algebra map $\mathbb{S} \to E$. Given a spectrum $X$, our framework was set up to give its best possible approximation $X^\wedge_E$ within $E$–module spectra.

We will extend this to spaces by sewing this adjunction together with another:

$$\mathsf{Spaces} \underset{\xleftarrow[\Omega^\infty]{}}{\xrightarrow{\Sigma^\infty}} \mathsf{Modules}_{\mathbb{S}} \underset{\longleftarrow}{\xrightarrow{-\wedge E}} \mathsf{Modules}_E.$$

We will write $E(-)$ for the induced monad on Spaces, given by the formula

You avoided talking about *monadic* descent in the previous lectures, and instead you were vague about it. Maybe you have to spell that out now.

$$E(X) = \operatorname*{colim}_{j \to \infty} \Omega^j(\underline{E}_j \wedge X) = \Omega^\infty(E \wedge \Sigma^\infty X),$$

where $\underline{E}_*$ are the constituent spaces in the $\Omega$–spectrum of $E$. This space has the property that $\pi_* E(X) = \widetilde{E}_* X$ (in nonnegative dimensions). The monadic structure comes from the two evident natural transformations:

Danny didn't like the colimit definition. We also don't need it; everything can be phrased stably. Maybe remove it.

$$\begin{aligned}
\eta\colon\ X &\simeq S^0 \wedge X \\
&\to \underline{E}_0 \wedge X \\
&\to \operatorname*{colim}_{j\to\infty} \Omega^j(\underline{E}_j \wedge X) = E(X), \\
\mu\colon\ E(E(X)) &= \operatorname*{colim}_{j\to\infty} \Omega^j \left( \underline{E}_j \wedge \operatorname*{colim}_{k\to\infty} \Omega^k(\underline{E}_k \wedge X) \right) \\
&\to \operatorname*{colim}_{\substack{j\to\infty \\ k\to\infty}} \Omega^{j+k}(\underline{E}_j \wedge \underline{E}_k \wedge X) \\
&\xrightarrow{\mu} \operatorname*{colim}_{\substack{j\to\infty \\ k\to\infty}} \Omega^{j+k}(\underline{E}_{j+k} \wedge X) \xleftarrow{\simeq} E(X).
\end{aligned}$$

Just as in the stable situation, we can extract from this a cosimplicial space:

**Definition 57.** Consider the descent cosimplicial object

$$\mathcal{UD}_E(X) := \left\{ \begin{matrix} E \\ \circ \\ X \end{matrix} \; \begin{matrix} \xrightarrow{\eta_L} \\ \xleftarrow{\mu} \\ \xrightarrow{\eta_R} \end{matrix} \; \begin{matrix} E \\ \circ \\ E \\ \circ \\ X \end{matrix} \; \begin{matrix} \longrightarrow \\ \longleftarrow \\ \xrightarrow{\Delta} \\ \longleftarrow \\ \longrightarrow \end{matrix} \; \begin{matrix} E \\ \circ \\ E \\ \circ \\ E \\ \circ \\ X \end{matrix} \; \begin{matrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{matrix} \; \cdots \right\}.$$

Its totalization gives the *unstable $E$–completion of $X$*.

Under suitable hypotheses, we can extract from this an unstable analog of $\mathcal{M}_E$. Recall that our goal in Lecture 3.1 was to associate to $E_*X$ a quasi-coherent sheaf over $\mathcal{M}_E$, a fixed object, dependent on $E$ but independent of $X$. In the presence of further hypotheses called "**FH**", we saw in Remark 24 that this same data could be expressed as an $E_*E$–comodule structure on $E_*X$. In particular, **FH** caused the marked map in

$$E_*X \xrightarrow{\eta_R} E_*(E \wedge X) \xleftarrow{\star} E_*E \otimes_{E_*} E_*X$$

to become invertible.

In the present setting, consider the analogous composite

$$\begin{aligned}
\pi_m E(X) &\xrightarrow{\eta_R} \pi_m E(E(X)) \\
&\xleftarrow{\mu \circ 1} \pi_m E(E(E(X))) \\
&\xleftarrow{\text{compose}} \pi_m E(E(S^n)) \times \pi_n E(X).
\end{aligned}$$

**Definition 58.** The *unstable context of E* is the collection of cosimplicial abelian groups $\pi_* \mathcal{UD}_E(S^n)$. In the case $n = 0$, this is a cosimplicial ring, and in the case $n \neq 0$ the 0–simplices merely form a module over $\pi_* \mathcal{UD}_E(S^0)[0]$.

This is just $E_*$, right?

Sort out exactly what structure lives here.

*Remark 41.* In the case that $E$ has Künneth isomorphisms, the "backwards" maps above become invertible, which is a kind of unstable analogue of the condition **FH**. This is the situation in which most of the classical work on this topic was done.

**Cite me:** BCM, BJW, ....

Talk about how this motivates us to consider algebraically the 0– and 1–simplices along, hoping that an eventual analogue of **FH** will keep us from having to consider anything further.

I *don't* really understand what sort of algebraic structure this gives us. It would be nice to have an unstable scheme-theoretic analogue of the stable context, so that the homology of spaces gave us "quasi-coherent sheaves" over this unstable object (and, in good cases, the unstable Adams spectral sequence had its $E_2$–page computed by some homological algebra over this object; see BCM Section 6).

Ignoring for the moment what the correct scheme-theoretic analogue of this might be, we will press onward and record the algebraic objects appearing in the presence of the unstable analogue of **FH**.

**Definition 59.** A Hopf ring $A_{*,*}$ over a graded ring $R_*$ is itself a graded ring object in the category $\mathsf{Coalgebras}_{R_*}$, sometimes called an $R_*$–coalgebraic graded ring object. It has the following structure maps:

Can this definition be made without specifying the grading as such and instead using a $\hat{\mathbb{G}}_m$–action?

$$
\begin{aligned}
+&\colon A_{s,t} \times A_{s,t} \to A_{s,t} && (A_{s,t} \text{ is an abelian group})\\
\cdot&\colon R_{s'} \otimes_{R_*} A_{s,t} \to A_{s+s',t} && (A_{*,t} \text{ is a } R_*\text{–module})\\
\Delta&\colon A_{s,t} \to \bigoplus_{s'+s''=s} A_{s',t} \otimes_{R_*} A_{s'',t} && (A_{*,t} \text{ is a } R_*\text{–coalgebra})\\
*&\colon A_{s,t} \otimes_{R_*} A_{s',t} \to A_{s+s',t} && (\text{addition for the ring in } R_*\text{–coalgebras})\\
\eta_*&\colon R_* \to A_{*,0} && (\text{null element for ring addition})\\
\chi&\colon A_{s,t} \to A_{s,t} && (\text{negation for the ring in } R_*\text{–coalgebras})\\
\circ&\colon A_{s,t} \otimes_{R_*} A_{s',t'} \to A_{s+s',t+t'} && (\text{multiplication map for the ring in } R_*\text{–coalgebras})\\
\eta_\circ&\colon R_* \to A_{*,0} && (\text{null element for ring multiplication}).
\end{aligned}
$$

These are required to satisfy various commutative diagrams. The least obvious is displayed in Figure 4.1, encoding the distributivity of ∘–"multiplication" over ∗–"addition".

I think that the "skew–commutativity" of ∘–multiplication is also worth mentioning. This confused me for a good while, being most familiar with the material in Lecture 4.7.

*Remark 42.* A ring spectrum $E$ with Künneth isomorphisms

$$E_*(\underline{E}_m \times \underline{E}_n) \cong E_*(\underline{E}_m) \otimes_{E_*} E_*(\underline{E}_n)$$

gives rise to a Hopf ring $E_*\underline{E}_n = \pi_* \mathcal{UD}_E(S^n)[1]$. For a space $X$, the homology groups $E_* X$ form a comodule for this Hopf ring.

One can modify this story in a number of minor ways.

*Remark 43.* One can restrict to the *additive* unstable cooperations by passing to the quotient $Q^* E_* \underline{E}_*$. These corepresent the morphisms in a cocategory

$$\begin{tikzcd}
A_{s,t} \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[r, "1\otimes *"] \arrow[d, "\Delta\otimes(1\otimes 1)"] & A_{s,t} \otimes_{R_*} A_{s'+s'',t'} \arrow[dddd, "\circ"] \\
\left(\bigoplus_{s_1+s_2=s} A_{s_1,t} \otimes_{R_*} A_{s_2,t}\right) \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[d, "\simeq"] & \\
\bigoplus_{s_1+s_2=s} (A_{s_1,t} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[d, "1\otimes\tau\otimes 1"] & \\
\bigoplus_{s_1+s_2=s} (A_{s_1,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s'',t'}) \arrow[d, "\circ\otimes\circ"] & \\
\bigoplus_{s_1+s_2=s} (A_{s_1+s',t+t'} \otimes_{R_*} A_{s_2+s'',t+t'}) \arrow[r, "*"] & A_{s+s'+s'',t+t'},
\end{tikzcd}$$

**Fig. 4.1** The distributivity axiom for $*$ over $\circ$ in a Hopf algebra.

object in Rings (using the $\circ$–product for multiplication, which descends to $*$–indecomposables). The ring $E_*$ corepresents the objects in this cocategory object.

A lot of the homological algebra of unstable comodules exists only after passing to this quotient. Try to explain why.

Explain this Remark, really. (1) Why does passing to the indecomposables project onto the additive cooperations? This should be thought of as an operation dual to restricting to the primitive unstable *operations*. (2) Why a cocategory object?

*Remark 44.* The procedure in Remark 42 can be generalized to the case of *two* ring spectra, $E$ and $F$, equipped with Künneth isomorphisms

$$E_*(\underline{F}_m \times \underline{F}_n) \cong E_*(\underline{F}_m) \otimes_{E_*} E_*(\underline{F}_n).$$

Again, the bigraded object $E_*\underline{F}_*$ forms a Hopf ring. These "mixed cooperations" appear as part of the cooperations for the ring spectrum $E \vee F$ — or, from the perspective of spectral shemes, for the joint cover $\{\mathbb{S} \to E, \mathbb{S} \to F\}$. The role of the mixed cooperations in this setting is to prevent the $(E \vee F)$–based unstable Adams spectral sequence from double-counting homotopy elements visible to both the unstable $E$– and $F$–completions.

Maybe put in the unstable $\subseteq$ unstable additive $\subseteq$ stable diagram? Talk about examples of these three things in, say, mod–2 homology? Talk about homology suspension on this day, and make clear the comparison to the stable situation.

I feel that this can be used to take an unstable comodule for $E$–theory and produce from it an unstable comodule for $F$–theory (up to a wrong-way map). Martin Bendersky thought this was strange, but I don't think it's so odd, and I would like to understand how to straighten it out.

Does "Cartesian" mean anything in this setting?

Section III.11 of Wilson's *Primer* has a synopsis of how additive unstable operations should be treated.

**Cite me:** Bendersky Curtis Miller's [4] *The unstable Adams spectral sequence for generalized homology*.

**Cite me:** Boardman Johnson Wilson's [5] *Unstable operations in generalized cohomology*.

## 4.2 Mar 23: Unstable cooperations in ordinary homology

The objects discussed in the previous Lecture appear to be almost bottomlessly complicated: there are so many groups and so many structure maps. At first glance, it might seem like it's a hopeless enterprise to actually try to compute $\mathcal{UM}_E^*$ for any spectrum $E$, but in fact the plenty of structure maps give enough footholds that this is often feasible, provided we have sufficiently strong stomachs. Today we will treat the case $E = H\mathbb{F}_2$, which requires us to introduce all of the relevant tools but whose computations turn out to be very straightforward.

The place to start is with a very old lemma:

**Lemma 54.** *If $E$ is a spectrum with $\pi_{-1}E = 0$, then $\underline{E}_1 \simeq B\underline{E}_0$.* □

clarification: should we specify $E$ to be an $\Omega$-spectrum so the condition gives us that it's connective?

The essential point is that $B$ gives the connective delooping of $\underline{E}_0$, so if $E$ is connective then this will yield the spaces in the $\Omega$–spectrum of $E$. This is useful to us because $B\underline{E}_0$ comes with a natural skeletal filtration, and this gives rise to a spectral sequence:

**Corollary 32 ([35, Theorem 2.1]).** *There is a convergent spectral sequence of Hopf algebras of signature*

$$E^1_{*,j} = F_*(\Sigma\underline{E}_0)^{\wedge j} \Rightarrow F_*\underline{E}_1.$$

*In the case that F has Künneth isomorphisms of the form*

In class you didn't write any $\Sigma$.

$$F((\Sigma\underline{E}_0)^{\wedge j}) \cong F(\Sigma\underline{E}_0)^{\otimes j},$$

*the $E^2$–page is identifiable as*

$$E^2_{*,*} \cong \operatorname{Tor}^{F_*\underline{E}_0}_{*,*}(F_*, F_*).\square$$

In general, if $E$ is a connective spectrum, we get a family of spectral sequences of signature

$$E^2_{*,*} \cong \operatorname{Tor}^{F_*\underline{E}_j}_{*,*}(F_*, F_*) \Rightarrow F_*\underline{E}_{j+1}.$$

For this you'll want an analogue of the lemma, something like $\underline{E}_j \simeq B\underline{E}_{j-1}$. What is the connective hypothesis for this? It can't be something like $\pi_{j-2}E = 0$, because that won't be satisfied?

That this spectral sequence is multiplicative for the $*$–product is useful enough, but the situation is actually much, much better than this:

**Lemma 55 ([35, Theorem 2.2]).** *Denote by $E^r_{*,*}(F_*\underline{E}_j)$ the spectral sequence considered above whose $E^2$–term is constructed from* Tor *over $F_*\underline{E}_j$. There are maps*

**Cite me:** This isn't the right citation. They blame this generality on a Thomason–Wilson article..

$$E^r_{*,*}(F_*\underline{E}_j) \otimes_{F_*} F_*\underline{E}_m \to E^r_{*,*}(F_*\underline{E}_{j+m})$$

*which agree with the map*

$$F_*\underline{E}_{j+1} \otimes_{F_*} F_*\underline{E}_m \xrightarrow{\circ} F_*\underline{E}_{j+m+1}$$

*on the $E^\infty$–page and which satisfy*

$$d^r(x \circ y) = (d^r x) \circ y.\square$$

This Lemma is obscenely useful: it means that differentials can be transported *between spectral sequences* for classes which can be decomposed as $\circ$–

products. This means that the bottom spectral sequence (i.e., the case $j = 0$) exerts a large amount of control over the others — and this spectral sequence often turns out to be very computable.

We now turn to our example of $E = H\mathbb{F}_2$ and $F = H\mathbb{F}_2$. To ground our induction, we will consider the first spectral sequence

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}B\mathbb{F}_2.$$

Using that $\mathbb{R}\mathrm{P}^\infty$ gives a model for $B\mathbb{F}_2$, we use Example 3 to analyze this spectral sequence: that Example states that as an $\mathbb{F}_2$–module, there is an isomorphism

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \mathbb{F}_2\{a_j \mid j \geq 0\}.$$

Using our further computation in Example 11, we can also give a presentation of the Hopf algebra structure on $H\mathbb{F}_{2*}B\mathbb{F}_2$: it is dual to the primitively-generated polynomial algebra on a single class, so forms a divided power algebra on a single class $a_\varnothing$. In characteristic 2, this decomposes as

I don't understand why the notation is $\varnothing$.

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \Gamma[a_\varnothing] \cong \bigotimes_{j=0}^{\infty} \mathbb{F}_2[a_{(j)}]/a_{(j)}^2,$$

where we have written $a_{(j)}$ for $a_\varnothing^{[2^j]}$ in the divided power structure.

**Corollary 33.** *This* Tor *spectral sequence collapses at the $E^2$–page.*

*Proof.* As an algebra, the homology $H\mathbb{F}_{2*}(\mathbb{F}_2)$ of the discrete space $\mathbb{F}_2$ is presented by the truncated polynomial algebra

$$H\mathbb{F}_{2*}(\mathbb{F}_2) \cong \mathbb{F}_2[\mathbb{F}_2] = \mathbb{F}_2[[1] - [0]]/([1] - [0])^{*2}.$$

Danny found this line confusing. You explain it more elaborately in person, by carefully moving the $(-)^{*2}$ around.

The Tor–algebra of this is then divided power on a single class:

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \Gamma[a_\varnothing].$$

In order for the two computations to agree, there can therefore be no differentials in the spectral sequence.

Now we turn to the rest of the induction:

**Theorem 46.** *$H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_t$ is the exterior $*$–algebra on the $t$–fold $\circ$–products of the generators $a_{(j)} \in H\mathbb{F}_{2*}B\mathbb{F}_2$.*

*Proof.* Make the inductive assumption that this is true for some fixed value of $t$. It follows that the Tor groups of the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_{t+1}$$

form a divided power algebra generated by the same $t$–fold $\circ$–products. An analogue of another Ravenel–Wilson lemma [35, Lemma 9.5] gives a congruence

$$(a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]} \equiv a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})} \pmod{\text{decomposables}}.$$

It's conceivable that this congruence can be repaired to an equality, since the 2–series for $\widehat{\mathbb{G}}_a$ is so abbreviated. I have not worked this out.

It follows from Lemma 55 that the differentials vanish:

$$\begin{aligned} d((a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]}) &\equiv d(a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})}) \pmod{\text{decomposables}} \\ &= a_{(j_1)} \circ d(a_{(j_2)} \circ \cdots \circ a_{(j_{t+1})}) = 0. \end{aligned}$$

I'm guessing you mean $*$-decomposables? In the calculation of the differential below, you can then use multiplicativity to deal with the decomposable terms, right?

Hence, the spectral sequence collapses. To see that there are no multiplicative extensions, note that the only potentially undetermined multiplications occur as $*$–squares of exterior classes. However, the $*$–squaring map is induced by the topological map

$$\underline{H\mathbb{F}_2}_t \xrightarrow{\cdot 2} \underline{H\mathbb{F}_2}_t,$$

which is already null on the level of spaces. It follows that there are no extensions and the induction holds.

**Corollary 34.** *It follows that*

$$H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* \xleftarrow{\cong} \bigoplus_{t=0}^{\infty} (H_*(\mathbb{RP}^\infty; \mathbb{F}_2))^{\wedge t},$$

*where $(-)^{\wedge t}$ denotes the $t^{th}$ exterior power in the category of Hopf algebras.*

*Proof.* The leftward direction of this isomorphism is realized by the $\circ$–product.

*Remark 45.* Our computation of the full Hopf ring of unstable cooperations can be winnowed down to give information about particular classes of cooperations. For instance, the *additive* unstable cooperations are given by passing to the $*$–indecomposable quotient

$$\begin{aligned} Q_* H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* &\cong \mathbb{F}_2\left\{a_{(I_0)} \circ \cdots \circ a_{(I_t)}\right\} \\ &\cong \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots]. \end{aligned}$$

Later you use $Q^*$ instead of $Q_*$ to denote $*$–indecomposables. Settle on one of the two. See also remark 4.16.

In terms of Lemma 10, we have

What's $I_0, I_1, \ldots$?

$$\operatorname{Spec} Q_* H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_* \cong \underline{\operatorname{End}}(\widehat{\mathbb{G}}_a).$$

One passes to the *stable* cooperations by taking the colimit along the homology suspension element $a_{(0)} = \xi_0$.This has the effect of adjoining a $\circ$–

Define what the homology suspension element $e$ is. The point is that the equivalence $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$ is adjoint to a map $\Sigma \underline{E}_n \to \underline{E}_{n+1}$, and the effect of this map on $F$–homology is $\circ$–ing with $e$.

Is this right? What happened to $\mathcal{A}_*$ versus $\mathcal{A}P_0$?

product inverse to $a_{(0)}$, i.e.,

$$(Q_* H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_*)[a_{(0)}^{\circ(-1)}] \cong \mathbb{F}_2[\xi_0^{\pm}, \xi_1, \xi_2, \ldots],$$

which is exactly the ring of functions on $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ considered in Lemma 10.

Explain this. You messed it up in class.

*Remark 46 ([44, Theorems 8.5 and 8.11]).* The odd–primary analogue of this result appears in Wilson's book. In that situation, the bar spectral sequences do not degeneratebut rather have a single family of differentials, and the result imposes a single relation on the free Hopf ring. The end result is

$$H\mathbb{F}_{p*}\underline{H\mathbb{F}_p}_* \cong \frac{\bigotimes_{I,J} \mathbb{F}_p[e_1 \circ \alpha_I \circ \beta^J, \alpha_I \circ \beta^J]}{(e_1 \circ \alpha_I \circ \beta^J)^{*2} = 0, (\alpha_I \circ \beta^J)^{*p} = 0, e_1 \circ e_1 = \beta_1},$$

where $e_1 \in (H\mathbb{F}_p)_1\underline{H\mathbb{F}_p}_1$ is the homology suspension element, $\alpha_{(j)} \in (H\mathbb{F}_p)_{2p^j}\underline{H\mathbb{F}_p}_1$ are the analogues of the elements considered above, and $\beta_{(j)} \in (H\mathbb{F}_p)_{2p^j}\mathbb{C}\mathrm{P}^\infty$ are the algebra generators of the Hopf algebra dual of the ring of functions on the formal group $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{F}_p}$ associated to $H\mathbb{F}_p$ by its natural complex orientation. (In particular, the Hopf ring is *free* on these Hopf algebras, subject to the single interesting relation $e_1 \circ e_1 = \beta_{(0)}$.)

I think this relation is supposed to be analogous to $S^1 \wedge S^1 \simeq S^2 = \mathbb{C}\mathrm{P}^1$.

You wrote $\beta^J$ and $\beta_1$ in the display. What is the relation between $\beta^J$ and $\beta_1$ and $\beta_{(j)}$? Is $\beta_{(0)} = \beta_1$?

Neil's MO answer about $H_*K(\mathbb{Z}, 3)$: http://mathoverflow.net/a/216041/1094

## 4.3 Mar 25: Algebraic unstable cooperations

One of our goals for this Case Study is to study the mixed unstable cooperations $E_*\underline{G}_{2*}$ for complex-orientable cohomology theories $E$ and $G$. These turn out to behave more regularly than one might expect, in the sense that there is a uniform algebraic model and a comparison map which is often an isomorphism. In order to formulate what will become our main result, we will need to begin with some algebraic definitions.

**Definition 60.** Let $R$ and $S$ be graded rings. We can form a Hopf ring over $R$ by forming the "ring–ring" $R[S]$: as an $R$–module, this is free and generated by symbols $[s]$ for $s \in S$. The Hopf ring maps $*$, $\circ$, and $\Delta$ are determined by the formulas

$$\begin{aligned} R[S] \otimes_R R[S] &\xrightarrow{*} R[S] & [s] * [s'] &= [s + s'], \\ R[S] \otimes_R R[S] &\xrightarrow{\circ} R[S] & [s] \circ [s'] &= [s \cdot s'], \\ R[S] &\xrightarrow{\Delta} R[S] \otimes_R R[S] & \Delta[s] &= [s] \otimes [s]. \end{aligned}$$

For instance, the distributivity axiom is checked in the calculation

$$\begin{aligned}[][s''] \circ ([s] * [s']) &= ([s''] \circ [s]) * ([s''] \circ [s']) \\ [s''] \circ [s + s'] &= \\ [s''(s + s')] &= \\ &= [s''s] * [s''s'] \\ &= [s''s + s''s'].\end{aligned}$$

**Definition 61.** Let $C$ be an $R$–coalgebra, and let $S$ be an auxiliary ring. We can form a free Hopf ring $R[S][C]$ on $C$ under $R[S]$, which has the property

$$\mathsf{HopfRings}_{R[S]/}(R[S][C], T) \cong \mathsf{Coalgebras}_{R/}(C, T).$$

In terms of elements, it is an $R$–module spanned by $R[S]$ and $C$, as well as free $*$– and $\circ$–products of elements of $C$, altogether subjected to the Hopf ring relations.

*Remark 47.* Given an $R$–coalgebra $C$, we can form the free commutative Hopf algebra on $C$ by taking its associated symmetric algebra. This is a degenerate case of a free Hopf ring construction, where $S$ is taken to be the zero ring.

Now we turn our eyes to topology. Let $E$ and $F$ be two complex-orientable cohomology theories where $F$ has enough Künneth isomorphisms. Set $R = F_*$, $S = E_*$, and $C = F_*\mathbb{C}\mathrm{P}^\infty$ to form the free Hopf ring $R[S][C] = F_*[E_*][F_*\mathbb{C}\mathrm{P}^\infty]$. What does this mean specifically, again?

**Lemma 56.** *Orientations of $E$ induce maps $F_*[E^*][F_*\mathbb{C}\mathrm{P}^\infty] \to F_*\underline{E}_*$.* You're not very consistent with $F_*[E_*]$ vs. $F_*[E^*]$.

*Proof.* To construct this map using universal properties, we need to check that $F_*\underline{E}_*$ is a Hopf ring under $F_*[E^*]$, and then we need to produce a map $F_*\mathbb{C}\mathrm{P}^\infty \to F_*[E^*]$. For the first task, $F_*\underline{E}_*$ is already an $F_*$–module. An element $v \in E^n$ corresponds to a path component $[v] \in \pi_0\underline{E}_n$, which pushes forward along This means $\pi_{-n}E$, right?

$$\pi_0\underline{E}_n \to F_0\underline{E}_n$$

to give an element $[v] \in F_0\underline{E}_n$. One can check that this determines a map of Hopf rings $F_*[E^*] \to F_*\underline{E}_*$.

Next, we will use our assumed data of orientations. The complex-orientation of $E$ gives a preferred class $\mathbb{C}\mathrm{P}^\infty \to \underline{E}_2$, representing the coordinate $x \in E^2\mathbb{C}\mathrm{P}^\infty$. By applying $F$–homology to this representing map, we get a map of $F_*$–coalgebras

$$F_*\mathbb{C}\mathrm{P}^\infty \to F_*\underline{E}_2 \subseteq F_*\underline{E}_*.$$

Universality gives the desired map of Hopf rings.

There is no reason to expect $F_*\underline{E}_*$ to be a free Hopf ring, and so it would be naive to expect this map to be an equivalence. Indeed, Ravenel and Wilson show that orientations of $E$ and $F$ together beget an interesting relation. An orientation on $E$ gives us a comparison map as above, and an orientation on $F$ gives a collection of preferred elements $\beta_j \in F_{2j}\mathbb{C}P^\infty$. Their result is to show that these elements are subject to the formal group laws *both* of $F$ and of $E$:

**Theorem 47 ([34, Theorem 3.8], [44, Theorem 9.7]).** *Write $\beta(s)$ for the formal sum $\beta(s) = \sum_j \beta_j x^j$. Then, in $F_*\underline{E}_*[\![s,t]\!]$, there is an equation*

$$\beta(s +_F t) = \beta(s) +_{[E]} \beta(t),$$

*where*

$$\beta(s +_F t) = \sum_n \beta_n \left(\sum_{i,j} a_{ij}^F s^i t^j\right)^n,$$
$$\beta(s) +_{[E]} \beta(t) = \mathop{\ast}_{i,j} \left([a_{ij}^E] \circ \left(\sum_k \beta_k s^k\right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell\right)^{\circ j}\right).$$

*Proof (Proof sketch).* This is a matter of calculating the behavior of

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty \xrightarrow{x} \underline{E}_2$$

in two different ways: using the effect of $\mu$ in $F$–homology and pushing forward in $x$, or using the effect of $\mu$ in $E$–cohomology and pushing forward along the Hurewicz map $\mathbb{S} \to F$.

Altogether, this motivates our algebraic model for the Hopf ring of unstable cooperations:

There is probably a natural map to the scheme of homomorphisms that doesn't require picking a coordinate.

I don't like the upper-$R$ notation. Having a scheme theoretic description of this object should let us pick a better name. I'm also unhappy that "mixed unstable cooperations" is an achiral name, meaning it doesn't indicate which object is the spectrum and which is the infinite loopspace.

**Definition 62.** Define $F_*^R\underline{E}_*$ to be the quotient of $F_*[E^*][F_*\mathbb{C}P^\infty]$ by the relation above. There is a natural *comparison map*

$$F_*^R\underline{E}_* \to F_*\underline{E}_*.$$

We will show that for many such $E$ and $G$ this map is an isomorphism. Before embarking on this, however, we would like to explore the connection to formal groups suggested by the formula in Theorem 47. Note that the Hopf ring-ring $R[S]$ has a natural augmentation given by $[s] \mapsto 1$, so that $\langle s\rangle = [s] - [0]$ form a generating set of the augmentation ideal.

I'm confused about the $*$–indecomposable quotient. For example, is the expression $[s] * [s']$ you write below equal to zero? In fact, why isn't everything zero: $[x] = [x-y] * [y] \equiv 0$? Normally you look at products of elements of positive degree, but what does this mean in this case?

Is it useful to say that passing to $Q^*$ "sends $*$ to $*$" in the sense described below? And that this degenerates to "sends $*$ to $+$" in the case of a ring-ring?

what do you mean by sending $*$ to $*$? This doesn't seem to happen below. AY

**Lemma 57.** *In the $*$–indecomposable quotient $Q^*R[S]$, there are the formulas*

$$\langle s\rangle + \langle s'\rangle = \langle s+s'\rangle, \qquad \langle s\rangle \circ \langle s'\rangle = \langle ss'\rangle.$$

*Proof.* Modulo $*$–decomposables, we can write

$$0 \equiv \langle s \rangle * \langle s' \rangle = [s] * [s'] - [s] - [s'] + [0] = \langle s + s' \rangle - \langle s \rangle - \langle s' \rangle.$$

We can also directly calculate

$$\langle s \rangle \circ \langle s' \rangle = [ss'] - [0] - [0] + [0] = \langle ss' \rangle. \square$$

**Corollary 35.** *Orientations of E and F induce isomorphisms*

$$\operatorname{Spec} Q^* F_*^R \underline{E}_* \cong \underline{\mathsf{FormalGroups}}(\mathbb{C}P_E^\infty, \mathbb{C}P_F^\infty).$$

This is a little sloppy. Where are the coefficients of $E$ being sent? Is $Q^*R[S]$ really $R \otimes S$ like Hood calculated? Hm.

*Proof.* This is a matter of calculating $Q^* F_*^R \underline{E}_*$. Using Lemma 57, we have

$$\mathop{\ast}_{i,j} \left( [a_{ij}^E] \circ \left( \sum_k \beta_k s^k \right)^{\circ i} \circ \left( \sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right) \equiv \sum_{i,j} a_{ij}^E \left( \sum_k \beta_k s^k \right)^i \left( \sum_\ell \beta_\ell t^\ell \right)^j \text{ (in } Q^*).$$

It follows that

$$Q^* F_*^R \underline{E}_* = F_*[\beta_0, \beta_1, \beta_2, \ldots] / (\beta(s +_F t) = \beta(s) +_E \beta(t)) . \square$$

What happened to the $E_*$ in $F_*[E_*]$ in this expression?

You could also include the odd part of the approximation, with $e \circ e = \beta_1$, and from that calculate the algebraic model of the stabilization.

Next time, we will investigate $F_* \underline{E}_*$ in the more modest and concrete setting of $F = H\mathbb{F}_p$ and $E = BP$. One might think that this is merely a first guess at a topological computation that seems accomplishable after Lecture 4.2, but we will quickly show that it plays the role of a universal example of this sort of calculation.

## 4.4 Mar 28: Complex-orientable cooperations

**Convention: We will write $H$ for $H\mathbb{F}_p$ for the duration of the lecture.**

Today we are aiming for a proof of the following Theorem:

Jeremy found a paper (Chan's *A simple proof that the unstable (co-)homology of the Brown–Peterson spectrum is torsion-free*, see also Wilson's Primer's Section 10) where $H_* \underline{BP}_{2*}$ is proven to be bipolynomial (and even!) without any Hopf ring rigamarole. It looks like the method of proof is not very different from the Hopf ring one, but it's much shorter... and maybe the result will fall out of the Dieudonne module calculations anyhow? Consider it as an option after you break this lecture in two.

**Theorem 48 ([34, Theorem 4.2]).** *The natural homomorphism*

$$H_*^R \underline{BP}_{2*} \to H_* \underline{BP}_{2*}$$

*is an isomorphism. (In particular, $H_* \underline{BP}_{2*}$ is even–concentrated.)*

This is proved by a fairly elaborate counting argument, and as such our first move will be to produce an upper bound for the size of the source Hopf ring. To begin, consider the following consequence of Lemma 57:

**Cite me:** Pages 266–270 of Ravenel–Wilson, especially the bottom of 268..

**Corollary 36.** *As a $\circ$–algebra,*

$$Q^*H_0^R\underline{BP}_{2*} \cong \mathbb{F}_p[[v_n] - [0_{-|v_n|}] \mid n \geq 1],$$

*where $0_{-|v_n|}$ denotes the null element of $BP^{|v_n|}(*)$.* □

Is $b_j = \beta_j$?

Directly from the definition of $H_*^R\underline{BP}_{2*}$, we now know that $Q^*H_*^R\underline{BP}_{2*}$ is generated by $[v_n] - [0_{-|v_n|}]$ for $n \geq 1$ and $b_j$, $j \geq 0$. In fact, $p$–typicality shows [34, Lemma 4.14] that it suffices to consider $b_{p^d} = b_{(d)}$ for $i \geq 0$. Altogether, this gives a secondary comparison map

$$A := \mathbb{F}_p[[v_n], b_{(d)} \mid n > 0, d \geq 0] \twoheadrightarrow Q^*H_*^R\underline{BP}_{2*}.$$

I don't remember how $A$ is graded.

This map is not an isomorphism, as these elements are subject to the following relation:

Should this be $Q^*H_*\underline{BP}_{2*}/I^{\circ 2} \circ Q^*H_*\underline{BP}_{2*}$ instead?

**Lemma 58 ([34, Lemma 3.14], [44, Theorem 9.13]).** *Write $I = ([p], [v_1], [v_2], \ldots)$, and work in $Q^*H_*\underline{BP}_2/I^{\circ 2} \circ Q^*H_*\underline{BP}_2$. For any $n$ we have*

$$\sum_{i=1}^{n} [v_i] \circ b_{(n-i)}^{\circ p^i} \equiv 0.$$

*Proof.* Consider the series expansion of $\beta_0 = \beta(ps) = [p]_{[BP]}(\beta(s))$.

Let $r_n$, the $n^{\text{th}}$ relation, denote the same sum taken in $A$ instead:

$$r_n := \sum_{i=1}^{n} [v_i] \circ b_{(n-i)}^{\circ p^i}.$$

You write $\circ$. Is $A$ a Hopf ring as well?

There's a missing thought here (which Hood caught in class): why does the death of this element under $I^{\circ 2}$ say anything about killing $r_n$ in the original algebra?

I wonder if there is a better version of this argument where formal geometry gets involved.

The Lemma then shows that the pushforward of $r_n$ into $Q^*H_*\underline{BP}_{2*}$ is in the ideal generated by $I^{\circ 2}$.Ravenel and Wilson show the following well-behavedness result about these relators, by a fairly tedious argument:

**Lemma 59 ([34, Lemma 4.15.b]).** *The sequence $(r_1, r_2, \ldots)$ is regular in $A$.* □

This is exactly what we need to get our size bound.

**Lemma 60.** *Set*

$$c_{i,j} = \dim_{\mathbb{F}_p} Q^*H_i^R\underline{BP}_{2j}, \qquad d_{i,j} = \dim_{\mathbb{F}_p} \mathbb{F}_p[[v_n], b_{(0)}]_{i,j}.$$

*Then $c_{i,j} \leq d_{i,j}$ and $d_{i,j} = d_{i+2,j+1}$.*

*Proof.* We have seen that $c_{i,j}$ is bounded by the $\mathbb{F}_p$–dimension of

$$\mathbb{F}_p[[v_n], b_{(d)} \mid d \geq 0]_{i,j}/(r_1, r_2, \ldots).$$

But, since this ideal is regular and $|r_j| = |b_{(j)}|$, this is the same count as $d_{i,j}$. The other relation among the $d_{i,j}$ follows from multiplication by $b_{(0)}$, with $|b_{(0)}| = (2, 1)$.

We also need that one of the bidegrees of $[v_n]$ is zero, right?

We now turn to showing that this estimate is *sharp* and that the secondary comparison map is *onto*, and hence an isomorphism, using the bar spectral sequence. Recalling that the bar spectral sequence converges to a the homology of the *connective* delooping, let $\underline{BP}'_{2*}$ denote the connected component of $\underline{BP}_{2*}$ containing $[0_{2*}]$. We will then demonstrate the following theorem inductively:

**Theorem 49 ([34, Induction 4.18]).** *The following hold for all values of the induction index k:*

1. *$Q^* H_{\leq 2(k-1)} \underline{BP}'_{2*}$ is generated by ∘–products of the $[v_n]$ and $b_{(j)}$.*
2. *$H_{\leq 2(k-1)} \underline{BP}'_{2*}$ is isomorphic to a polynomial algebra in this range.*
3. *For $0 < i \leq 2(k-1)$, we have $d_{i,j} = \dim_{\mathbb{F}_p} Q^* H_i \underline{BP}_{2j}$.*

Before addressing the theorem, we show that this finishes our calculation:

*Proof (Proof of Theorem 48, assuming Theorem 49 for all k).* Recall that we are considering the natural map

This could be typeset better, by numbering the parts of Theorem 48 and just referring to numbered claims.

$$H^R_* \underline{BP}_{2*} \to H_* \underline{BP}_{2*}.$$

The first part of Theorem 49 shows that this map is a surjection. The third part of Theorem 49 together with our counting estimate shows that the induced map

$$Q^* H^R_* \underline{BP}_{2*} \to Q^* H_* \underline{BP}_{2*}$$

is an isomorphism. Finally, the second part of Theorem 49 says that the original map, before passing to $*$–indecomposables, must be an isomorphism as well.

*Proof (Proof of Theorem 49).* The infinite loopspaces in $\underline{BP}_{2*}$ are related by $\Omega^2 \underline{BP}'_{2(*+1)} = \underline{BP}_{2*}$, so we will use two bar spectral sequences to extract information about $\underline{BP}'_{2(*+1)}$ from $\underline{BP}_{2*}$. Since we have assumed that $H_{\leq 2(k-1)} \underline{BP}_{2*}$ is polynomial in the indicated range, we know that in the first spectral sequence

$$E^2_{*,*} = \operatorname{Tor}^{H_* \underline{BP}_{2*}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_* \underline{BP}_{2*+1}$$

the $E^2$–page is, in the same range, exterior on generators in Tor–degree 1 and topological degree one higher than the generators in the polynomial algebra. Since differentials lower Tor–degree, the spectral sequence is multiplicative, and there are no classes on the 0–line, it collapses in the range $[0, 2k-1]$. Additionally, since all the classes are in odd topological degree, there are no algebra extension problems, and we conclude that $H_*\underline{BP}_{2*+1}$ is indeed exterior up through degree $(2k-1)$.

We now consider the second bar spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{H_*\underline{BP}_{2*+1}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*\underline{BP}_{2(*+1)}.$$

The Tor algebra of an exterior algebra is divided power on a class of topological dimension one higher. Since these classes are now all in even degrees, the spectral sequence collapses in the range $[0, 2k]$. Additionally, these primitive classes are related to the original generating classes by double suspension, i.e., by circling with $b_{(0)}$. This shows the first inductive claim on the *primitive classes* through degree $2k$, and we must argue further to deduce our generation result for $x^{[p^j]}$ of degree $2k$ with $j > 0$. By inductive assumption, we can write

$$x = [y] \circ b_{(0)}^{\circ I_0} \circ b_{(1)}^{\circ I_1} \circ \cdots,$$

and one may as well consider the element

$$z := [y] \circ b_{(j)}^{\circ I_0} \circ b_{(j+1)}^{\circ I_1} \circ \cdots.$$

This element isn't $x^{[p^j]}$ on the nose, but the diagonal of $z - x^{[p^j]}$ lies in lower filtration degree — i.e., it is primitive as far as the filtration is concerned — and so we are again done.

The remaining thing to do is to use the size bounds: the only way that the map

$$H^R_*\underline{BP}_{2*} \to H_*\underline{BP}_{2*}$$

could be surjective is if there were multiplicative extensions in the spectral sequence joining $x^{[p]}$ to $x^p$. Granting this, we see that the module ranks of the algebra itself and of its indecomposables are exactly the right size to be a free (i.e., polynomial) algebra, and hence this must be the case.

Having accomplished Theorem 48, we reduce a general computation to it:

**Corollary 37 ([34, Corollary 4.7]).** *For a complex-orientable cohomology theory $E$, the natural maps*

$$E^R_*\underline{MU}_{2*} \to E_*\underline{MU}_{2*}, \qquad E^R_*\underline{BP}_{2*} \to E_*\underline{BP}_{2*}$$

*are isomorphisms of Hopf rings.*

*Proof.* First, because $MU_{(p)}$ splits multiplicatively as a product of $BP$s, we deduce from Theorem 48 the case of $E = H\mathbb{F}_p$. Since $H\mathbb{F}_{p*}\underline{BP}_{2*}$ is even, it follows that $H\mathbb{Z}_{(p)*}\underline{BP}_{2*}$ is torsion–free on a lift of a basis, and similarly (working across primes) $H\mathbb{Z}_*\underline{MU}_{2*}$ is torsion–free on a simultane-

You changed from *BP* to *MU* – is this intentional?

ous lift of basis. Next, using torsion–freeness, we conclude from an Atiyah–Hirzebruch spectral sequence that $MU_* \underline{MU}_{2*}$ is even and torsion–free itself, and moreover that the comparison is an isomorphism. Lastly, using naturality of Atiyah–Hirzebruch spectral sequences, given a complex–orientation $MU \to E$ we deduce that the spectral sequence

$$E_* \otimes H_*(\underline{MU}_{2*}; \mathbb{Z}) \cong E_* \otimes_{MU_*} MU_* \underline{MU}_{2*} \Rightarrow E_* \underline{MU}_{2*}$$

collapses, and similarly for the case of $BP$. The theorem follows.

This is an impressively broad theorem: the loopspaces $\underline{MU}_{2*}$ are quite complicated, and that any general statement can be made about them is remarkable. That this fact follows from a calculation in $H\mathbb{F}_p$–homology and some niceness observations is meant to showcase the density of $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{F}_p} \cong \widehat{\mathbb{G}}_a$ inside of $\mathcal{M}_{\mathbf{fg}}$. However, Remark 45 indicates that this Corollary does not cover all possible cases that the comparison map in Definition 62 becomes an isomorphism. In the remainder of the Case Study, we will investigate two other classes of $E$ and $G$ where this holds.

Should you mention the odd–dimensional stuff? You passed through it in the course of the proof anyway: you can see that $H_* \underline{BP}_{2*+1}$ is exterior on homology suspensions of algebra generators on degree below. This comports with what you already suspected about these unstable algebras.

## 4.5 Apr 1: Dieudonné modules

Our goal today is strictly algebraic. Because the category of finite type commutative and cocommutative Hopf algebras over a ground field $k$ is an abelian category, it admits a presentation as the module category for some (possibly noncommutative) ring. The description of this ring and of the explicit assignment from a group scheme to linear algebraic data is the subject of *Dieudonné theory*. We will give a survey of some of the results of Dieudonné theory today, including three different presentations of the equivalence.[1]

Start with a formal line $V$ over a ground ring $A$, let $\widehat{\mathbb{G}}$ denote $V$ equipped with a group structure, and let $\Omega^1_{V/k}$ be the module of Kähler differentials on $V$. We have previously been interested in the *invariant differentials* $\omega_{\widehat{\mathbb{G}}} \subseteq \Omega^1_{V/A}$ on $V$, back when we first discussed logarithms in Theorem 11. Such a differential gave rise to a logarithm through integration, in the case that $A$ was a $\mathbb{Q}$–algebra. However, if $A$ had positive characteristic $p$ then there would be an obstruction to integrating terms with exponents of the form $-1 \pmod p$, which in turn led us to the notion of $p$–height explored in Lecture 3.3.

**Cite me:** Weinstein's geometry of Lubin–Tate spaces notes.

[1] Emphasis on "*some of the results*". Dieudonné theory is an enormous subject with many interesting results both internal and connected to arithmetic geometry, which we'll explore almost none of.

A slightly different twist on this set-up leads to a new story entirely. Recall that $\Omega^1_{V/A}$ forms the first level of the *algebraic de Rham complex* $\Omega^*_{V/A}$. The translation invariant differentials studied in the theory of the logarithm are those differentials so that the identity $\mu^* - \pi_1^* - \pi_2^* = 0$ holds *at the chain level*. We can weaken this to request only that that difference be *exact*, or zero at the level of cohomology of the algebraic de Rham complex. This condition begets a sub–$A$–module $D(\widehat{\mathbb{G}}/A)$ of $H^1_{dR}(\widehat{\mathbb{G}}/A)$ consisting of those 1–forms which are cohomologically translation invariant.

What are $\mu$, $\pi_1$ and $\pi_2$?

*Example 26.* Let $A$ be a $\mathbb{Z}$–flat ring, let $\widehat{\mathbb{G}}$ be a formal group over $A$, and let $x$ be a coordinate on $\widehat{\mathbb{G}}$. Set $K = A \otimes \mathbb{Q}$, so that $A \to K$ is an injection. There is then a diagram of exact rows

question: so here, the point is that you are restricting to integrals with no constant term, and that is the same as taking the quotient of the actual differentials, since the map $A[\![x]\!] -> \Omega^1_{A[\![x]\!]/A}$ is injective on that submodule? AY

$$\begin{tikzcd}
0 \arrow[r] & \left\{\begin{matrix}\text{integrals with}\\ A\text{ coefficients}\end{matrix}\right\} \arrow[r] \arrow[d, equal] & \{\text{all conceivable integrals}\} \arrow[r] \arrow[d, equal] & \{\text{missing integrals}\} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & xA[\![x]\!] \arrow[r] \arrow[d, equal] & \{f \in xK[\![x]\!] \mid f' \in A[\![x]\!]\} \arrow[r, "d"] & H^1_{dR}(\widehat{\mathbb{G}}/A) \arrow[r] & 0 \\
0 \arrow[r] & xA[\![x]\!] \arrow[r] & \left\{ f \in xK[\![x]\!] \middle| \begin{matrix} f' \in A[\![x]\!], \\ \delta f \in A[\![x,y]\!] \end{matrix} \right\} \arrow[r, "d"] \arrow[u] & D(\widehat{\mathbb{G}}/A) \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

This is just the definition of $H^1_{dR}$ in terms of cocycles and coboundaries, right?

What's $\delta$?

where $\delta$ is induced by $\delta[\omega] = (\mu^* - \pi_1^* - \pi_2^*)[\omega]$.

The flatness condition in the Example is important to getting the calculation to work out right, and of course it is not satisfied when working over a perfect field of positive characteristic $p$ — our favorite setting in Lecture 3.3 and Case Study 3 more generally. However, de Rham cohomology has the following remarkable lifting property (which we have specialized to $H^1_{dR}$):

**Theorem 50.** *Let $A$ be a $\mathbb{Z}_{(p)}$–flat ring, let $f_1(x), f_2(x) \in A[\![x]\!]$ be power series without constant term. If $f_1 \equiv f_2 \pmod p$, then for any differential $\omega \in A[\![x]\!]dx$ the difference $f_1^*(\omega) - f_2^*(\omega)$ is exact.*

*Proof.* Write $\omega = dg$ for $g \in K[\![x]\!]$, and write $f_2 = f_1 + p\Delta$. Then

$$\begin{aligned}\int (f_2^*\omega - f_1^*\omega) &= g(f_2) - g(f_1) = g(f_1 + p\Delta) - g(f_1) \\ &= \sum_{n=1}^{\infty} \frac{(p\Delta)^n}{n!} g^{(n)}(f_1).\end{aligned}$$

Since $g' = \omega$ has coefficients in $A$, so does $g^{(n)}$ for all $n$, and the fraction $p^n/n!$ lies in the $\mathbb{Z}_{(p)}$–algebra $A$.

**Corollary 38 ($H^1_{dR}$ is "crystalline").** *If $f_1, f_2 \colon V \to V'$ are maps of pointed formal varieties which agree mod $p$, then they induce the same map on $H^1_{dR}$.* $\square$

Several well–behavedness results of the functor $D$ follow directly from Corollary 38. For instance, any map $f \colon \widehat{\mathbb{G}}' \to \widehat{\mathbb{G}}$ of pointed varieties which is a group homomorphism mod $p$ restricts to give a map $f^* \colon D(\widehat{\mathbb{G}}/A) \to D(\widehat{\mathbb{G}}'/A)$. Additionally, if $f_1$, $f_2$, and $f_3$ are three such maps of pointed varieties with $f_3 \equiv f_1 + f_2 \pmod p$ in $\mathsf{FormalGroups}(\widehat{\mathbb{G}}'/p, \widehat{\mathbb{G}}/p)$, then $f_3^* = f_1^* + f_2^*$ as maps $D(\widehat{\mathbb{G}}/A) \to D(\widehat{\mathbb{G}}'/A)$.

In the case that $k$ is a *perfect* field, the ring $\mathbb{W}_p(k)$ of $p$–typical Witt vectors on $k$ is simultaneously torsion-free and universal among nilpotent thickenings of the residue field $k$. This emboldens us to make the following definition:[2]

Maybe cite a reference that does this?

**Definition 63.** Let $k$ be a perfect field of characteristic $p > 0$, and let $\widehat{\mathbb{G}}_0$ be a formal group over $k$. Then, choose a lift $\widehat{\mathbb{G}}$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, and define the *(contravariant) Dieudonné module* of $\widehat{\mathbb{G}}_0$ by $M(\widehat{\mathbb{G}}_0) := D(\widehat{\mathbb{G}}/W(k))$.

This functor is best adapted to $p$–divisible groups, so typically $\widehat{\mathbb{G}}_0 = \widehat{\mathbb{G}}_a$ is disallowed. The definition and the most basic properties seem to work OK though...

*Remark 48.* This is independent of choice of lift up to coherent isomorphism. Given any other lift $\widehat{\mathbb{G}}'$ of $\widehat{\mathbb{G}}_0$ to $\mathbb{W}_p(k)$, we can find *some* power series — not necessarily a group homomorphism — covering the identity on $\widehat{\mathbb{G}}_0$. Corollary 38 then shows that this map induces a canonical isomorphism between the two potential definitions of $M(\widehat{\mathbb{G}}_0)$.

Note that the module $M(\widehat{\mathbb{G}}_0)$ carries some natural operations:

- Arithmetic: $M(\widehat{\mathbb{G}}_0)$ is naturally a $\mathbb{W}_p(k)$–module, with the action by $\ell$ corresponding to multiplication–by–$\ell$ on $\widehat{\mathbb{G}}_0$.

  well, $\widehat{\mathbb{G}}_0$ is a formal group over $k$, so maybe it would be better to write $\ell \pmod p$ or action on $\widehat{\mathbb{G}}$? AY
- Frobenius: The map $x \mapsto x^p$ is a group homomorphism mod $p$, so it induces a $\varphi$–semilinear map $F \colon M(\widehat{\mathbb{G}}_0) \to M(\widehat{\mathbb{G}}_0)$. That is, $F(\alpha v) = \alpha^\varphi F(v)$, where $\varphi$ is a lift of the Frobenius on $k$ to $\mathbb{W}_p(k)$.
- Verschiebung: The Verschiebung map is given by the mysterious formula
$$V \colon \sum_{n=1}^{\infty} a_n x^n \mapsto p \sum_{n=1}^{\infty} a_{pn}^{\varphi^{-1}} x^n.$$
  It satisfies anti-semilinearity, $aV(v) = V(a^\varphi v)$, and also $FV = p$.

With this, we come to the main theorem of this Lecture:

**Theorem 51.** *The functor $M$ determines a contravariant equivalence of categories between smooth 1–dimensional formal groups over $k$ of finite $p$–height and finite free*

[2] There is a better definition one might hope for, which instead assigns to each potential thickening and lift a "Dieudonne module", and then work to show that they all arise as base-changes of this universal one. This is possible and technically superior to the approach we are taking here.

*$\mathbb{W}_p(k)$–modules equipped with appropriate operations $F$ and $V$, called Dieudonné modules.* □

Add in words about being uniform and reduced?

You never define the Dieudonné ring explicitly. I think the statement of this theorem and some later statements (e.g., $\mathbb{W}_p(k)\{x\}$) can be made more clear by giving notation to the Dieudonné ring.

*Remark 49.* Several invariants of the formal group associated to a Dieudonné module can be read off from the functor $M$. For example, the $\mathbb{W}_p(k)$–rank of $M$ is equal to the $p$–height of $\widehat{\mathbb{G}}_0$. Additionally, the quotient $M/FM$ is canonically isomorphic to the cotangent space $T_0^*\widehat{\mathbb{G}}_0 \cong \omega_{\widehat{\mathbb{G}}_0}$.

*Example 27.* The Dieudonné module associated to $\widehat{\mathbb{G}}_m$ is the easiest to compute. For $x$ the usual coordinate, we have $[p](x) = x^p$, and hence the Frobenius $F$ acts on $M(\widehat{\mathbb{G}}_m)$ by $Fx = px$. It follows that $Vx = x$ and $M(\widehat{\mathbb{G}}_m) \cong \mathbb{W}_p(k)\{x\}$ with this action.

*Example 28 (cf. Example 7).*

This example is not done.

Dieudonné theory admits an extension to finite (flat) group schemes as well, and the torsion quotient of the Dieudonné module of a formal group agrees with the Dieudonné module associated to its torsion subscheme:

$$M(\widehat{\mathbb{G}}_0[p^j]) = M(\widehat{\mathbb{G}}_0)/p^j.$$

Start by calculating $V$ for $\widehat{\mathbb{G}}_a$, use this to motivate the presence of torsion.

The Dieudonné module associated to $\widehat{\mathbb{G}}_a$ is the infinite–dimensional torsion $\mathbb{W}_p(k)$–module $M(\widehat{\mathbb{G}}_a) = k\{x, Fx, F^2x, \ldots\}$. Set $p = 2$, and consider the subgroup scheme $\alpha_2 \subseteq \widehat{\mathbb{G}}_a$ with Dieudonné module

$$M(\alpha_2) = M(\widehat{\mathbb{G}}_a)/F^2 = k\{x, Fx\}.$$

We can now verify the four claims from Example 7:

- *The group scheme $\alpha_2$ has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$ but is not isomorphic to it.* This follows from calculating the Dieudonné module of homomorphisms:
- *There is no commutative group scheme $G$ of rank four such that $\alpha_2 = G[2]$.* This follows from calculating the space of rank four objects and noticing that $VF = 2$ gets you into trouble.
- *If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence*

  $$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

  *However, this short exact sequence doesn't split (even after making a base change).* This follows from calculating the action of $F$ and $V$: the exact sequence is split as modules, of course, but not as Dieudonné modules.

- *The subgroups of $\alpha_2 \times \alpha_2$ of order two are parameterized by* $\mathbb{P}^1$. This follows from calculating the Dieudonné module of the product, as well as its space of projections of the appropriate rank.

We can also use Dieudonné theory to compute the automorphism group of a fixed Honda formal group, which is information we wanted back in Lecture 3.8:

**Corollary 39.** *For $\Gamma_d$ the Honda formal group law of height $d$ over $\mathbb{F}_{p^d}$, we compute*

$$\operatorname{Aut}\Gamma_d \cong \mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \begin{pmatrix} Fw = w^{\varphi}F, \\ F^d = p \end{pmatrix}^{\times}.$$

*Proof.* The Dieudonné module associated to $\Gamma_d$ satisfies $F^d = p$, and hence $M(\Gamma_d/k)$ is presented as a *quotient* of the ring of operators on Dieudonné modules. The endomorphism ring of such a module is canonically isomorphic to the module itself.

We now turn to alternative presentations of the Dieudonné module functor, which have their own advantages and disadvantages. Let $\widehat{\mathbb{G}}$ again be a formal Lie group over a field $k$ of positive characteristic $p$, and consider Cartier's *functor of curves*

$$C\widehat{\mathbb{G}} = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}).$$

This is, again, a kind of relaxing of familiar data from Lie theory: rather than studying exponential curves, $C\widehat{\mathbb{G}}$ tracks all possible curves. In Lecture 3.3, we considered three kinds of operations on a given curve $\gamma\colon \widehat{\mathbb{A}}^1 \to \widehat{\mathbb{G}}$:

- Homothety: given a scalar $a \in A$, we define $[a] \cdot \gamma(t) = \gamma(at)$.
- Verschiebung: given an integer $n \geq 1$, we define $V_n\gamma(t) = \gamma(t^n)$.
- Arithmetic: given two curves $\gamma_1$ and $\gamma_2$, we can use the group law on $\widehat{\mathbb{G}}$ to define $\gamma_1 +_{\widehat{\mathbb{G}}} \gamma_2$. Moreover, given $\ell \in \mathbb{Z}$, the $\ell$–fold sum in $\widehat{\mathbb{G}}$ gives an operator
$$\ell \cdot \gamma = \overbrace{\gamma +_{\widehat{\mathbb{G}}} \cdots +_{\widehat{\mathbb{G}}} \gamma}^{\ell \text{ times}}.$$
This extends to an action by $\ell \in \mathbb{W}_p(k)$.
- Frobenius: given an integer $n \geq 1$, we define
$$F_n\gamma(t) = \sum_{i=1}^{n}{}_{\widehat{\mathbb{G}}}\, \gamma(\zeta_n t^{1/n}),$$
where $\zeta_n$ is an $n^{\text{th}}$ root of unity. (This formula is invariant under permuting the root of unity chosen, so determines a curve defined over the original ground ring.)

**Definition 64.** A curve $\gamma$ on a formal group is $p$–typical when $F_n\gamma = 0$ for $n \neq p^j$. Write $D_p\widehat{\mathbb{G}} \subseteq C\widehat{\mathbb{G}}$ for the subset of $p$–typical curves. In the case that the base ring is $p$–local, $C\widehat{\mathbb{G}}$ splits as a sum of copies of $D_p\widehat{\mathbb{G}}$, and there is a natural section $C\widehat{\mathbb{G}} \to D_p\widehat{\mathbb{G}}$ called $p$–typification, given by the same formula as in Lemma 46.

*Remark 50.* Precomposing with a coordinate $\widehat{\mathbb{A}}^1 \cong \widehat{\mathbb{G}}$ allows us to think of a logarithm $\log\colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}_a$ as a curve on $\widehat{\mathbb{G}}_a$. The definition of $p$–typicality given in Definition 48 coincides with the one given here.

Surprisingly, this construction captures the same data as the previous one.

**Theorem 52.** *The functor $D_p$ determines a* covariant *equivalence of categories between smooth 1–dimensional formal groups over $k$ of finite $p$–height and finite free $\mathbb{W}_p(k)$–modules equipped with appropriate operations $F$ and $V$. In fact, $D_p(\widehat{\mathbb{G}}) \cong M(\widehat{\mathbb{G}}/k)^*$.* □

Add in words about being uniform and reduced?

Can you give more intuition about how these two presentations are related, for example from Lie theory? Somehow (integral) curves (through a point) should correspond to the tangent space, while left-invariant forms correspond to the cotangent space. Is "cohomologically-invariant" analogous to "$p$-typification", perhaps along the lines of the "crystalline"-ness of $H^1_{dR}$?

Finally, we turn to a third presentation of Dieudonné theory using more pedestrian methods, with the aim of developing a theory more directly adapted to algebraic topology. One can show that the category of *finite–type* graded connected Hopf algebras is an abelian category, and hence must admit a presentation as modules over some (perhaps noncommutative) ring. The first step to accessing this presentation is to find a collection of projective generators for this category.

**Theorem 53 ([37]).** *Let $S(n)$ denote the free graded-commutative Hopf algebra on a single generator in degree $n$. There is a projective cover $H(n) \twoheadrightarrow S(n)$, given by the formula*

- *If either $p = 2$ and $n = 2^m k$ for $2 \nmid k$ and $m > 0$* or *$p \neq 2$ and $n = 2p^m k$ for $p \nmid k$ and $m > 0$, then $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_k]$ with the Witt vector diagonal.*
- *Otherwise, $H(n) = S(n)$ is the identity.* □

I would've thought that the important word to emphasize here is "graded", not "finite-type", since we've been assuming finite-type throughout this lecture.

**Corollary 40.** *The category $\mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}$ of finite–type graded connected Hopf algebras is a full subcategory of modules over*

Put in a citation about what "the Witt vector diagonal" means: the elements $w_i = x_0^{p^i} + px_1^{p^{i-1}} + \cdots + x_i$ are primitive.

$$\bigoplus_{n,m} \mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}(H(n), H(m)).$$

**Definition 65.** Let $\mathsf{GradedDMods}$ denote the category of graded abelian groups $M$ satisfying

1. $M_{<1} = 0$.
2. If $n$ is odd, then $pM_n = 0$.

3. There are homomorphisms $V\colon M_{pn} \to M_n$ and $F\colon M_n \to M_{pn}$ (where $n$ is even if $p \neq 2$), together satisfying $FV = p = VF$. (These are induced by the inclusion $H(n) \subseteq H(pn)$ and by the map $H(pn) \to H(n)$ sending $x_n$ to $x_{n-1}^p$.)

*Remark 51.* Combining these, if $n$ is even, taking the form $n = 2p^m k$ with $p \nmid k$ at odd primes $p$ or $n = 2^m k$ with $2 \nmid k$ at $p = 2$, then $p^{m+1} M_n = F^{m+1} V^{m+1} M_n = 0$.

**Theorem 54.** *The functor* $D_*\colon \mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/} \to \mathsf{GradedDMods}$ *defined by*

$$D_*(H) = \bigoplus_n D_n(H) = \bigoplus_n \mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}(H(n), H)$$

*is an exact equivalence of categories. Moreover,* $D_* H(n)$ *is characterized by the equation*

$$\mathsf{GradedDMods}(D_* H(n), M) = M_n. \square$$

It would be nice to tie these presentations together, at least with unjustified claims. What curve does a cohomologically left-invariant form get sent to? What does the appearance of the Witt scheme in the third presentation tell you about the relationship to the second presentation?

This last presentation could use some examples too.

Dieudonné theory is also about taking primitives in some sort of cohomology. Can this be connected to the additivity condition on unstable operations?

Weinstein's Section 1 also ends with a discussion of the Dieudonné functor extended to the crystalline site. This is necessary to get access to the period map.

Definition of the inverse functor to the Dieudonné module functor? I think this appears in the formal groups notes.

We know of a connection between $H_* BU$ and the Witt scheme. Is there a connection between $E_* MU$ and curves, or $E_* BP$ and $p$–typical curves, which is visible from this perspectve? Almost definitely! Also, a connection between curves and divisors: the zero locus of a given curve...

You could also try to give the Devinatz–Hopkins formula for the stabilizer action. It's entirely a matter of different presentations of the Dieudonné module... although it may require you to understand Dieudonné crystals, which you are *not* up for.

Include this remark: However, the subspace of $M$ spanned by $\omega_{\widetilde{G}}$ is *sensitive to choice of lift*, unlike the rest of this construction. This observation is the wellspring of the Gross–Hopkins period map.

## 4.6 Apr 4: Ordinary cooperations for Landweber flat theories

**Convention: We will write $H$ for $H\mathbb{F}_p$ for the duration of the lecture.**

Today we will put Dieudonné modules to work for us in algebraic topology. Our goal is to prove the following Theorem:

**Theorem 55.** *For* $F = H$ *and* $E$ *a Landweber flat homology theory, the comparison map*

$$H^R_* \underline{E}_{2*} \to H_* \underline{E}_{2*}$$

*is an isomorphism of Hopf rings.*

The essential observation about this is that the associated Dieudonné module $D_* H_* \underline{E}_{2*}$ is a *stable object*, in the sense of the following result of Goerss–Lannes–Morel:

**Theorem 56 ([7, Lemma 2.8]).** *Let* $X \to Y \to Z$ *be a cofiber sequence of spectra. Then, provided* $n > 1$ *satisfies* $n \not\equiv \pm 1 \pmod{2p}$*, there is an exact sequence*

$$D_n H_* \Omega^\infty X \to D_n H_* \Omega^\infty Y \to D_n H_* \Omega^\infty Z. \square$$

**Corollary 41 ([7, Theorem 2.1]).** *For $n > 1$ an integer satisfying $n \not\equiv \pm 1 \pmod{2p}$, there is a spectrum $B(n)$ satisfying*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

*(As convention, when $n \equiv \pm 1 \pmod{2p}$ we set $B(n) := B(n-1)$, and $B(0) := S^0$.)* □

Remark 2.9 has a helpful discussion of how to extend to the case $B(0)$.

Before exploiting this result to compute something about unstable cooperations, we will prove a sequence of small results making these spectra somewhat more tangible.

**Lemma 61 ([7, Lemma 3.2]).** *The spectrum $B(n)$ is connective and $p$–complete.*

*Proof.* First, rearrange:

$$\pi_k B(n) = B(n)_n S^{n-k} = D_n H_* \Omega^\infty \Sigma^\infty S^{n-k}.$$

If $k < 0$, $n$ is below the connectivity of $\Omega^\infty \Sigma^\infty S^{n-k}$ and hence this vanishes. The second assertion follows from the observation that $H\mathbb{Z}_* B(n)$ is an $\mathbb{F}_p$–vector space. To see this, restrict to the case $n \not\equiv \pm 1 \pmod{2p}$ and calculate

$$H\mathbb{Z}_k B(n) = B(n)_n \Sigma^{n-k} H\mathbb{Z} = D_n H_* K(\mathbb{Z}, n-k) = [Q^* H_* K(\mathbb{Z}, n-k)]_n. \square$$

We can use a similar trick to calculate $H^* B(n)$:

**Definition 66 ([7, Example 3.6]).** Let $G(n)$ be the free unstable $\mathcal{A}$–module on one generator of degree $n$, so that

We should reconcile this notation with what's used in Lecture 4.1 and what's been said historically. These are specifically modules for the unstable *additive* cooperations.

$$\mathsf{UnstableModules}_{\mathcal{A}_*}(G(n), M) = M_n.$$

This module admits a presentation as

$$G(n) = \begin{cases} \Sigma^n \mathcal{A} / \{\beta^\varepsilon P^i \mid 2pi + 2\varepsilon > n\} \mathcal{A} & \text{if } p > 2, \\ \Sigma^n \mathcal{A} / \{\mathrm{Sq}^i \mid 2i > n\} \mathcal{A} & \text{if } p = 2. \end{cases}$$

The Spanier–Whitehead dual of this right-module, $DG(n)$, is characterized by the left-module

Is this parenthesization right?

$$\Sigma^n (DG(n))^* = \begin{cases} \mathcal{A} / \mathcal{A} \{\chi(\beta^\varepsilon P^i) \mid 2pi + 2\varepsilon > n\} & \text{if } p > 2, \\ \Sigma^n \mathcal{A} / \mathcal{A} \{\chi \mathrm{Sq}^i \mid 2i > n\} & \text{if } p = 2. \end{cases}$$

Be careful about $n \not\equiv \pm 1 \pmod{p}$?

**Theorem 57 ([7, Proof of Theorem 3.1]).** *There is an isomorphism*

$$H^* B(n) \cong \Sigma^n (DG(n))^*.$$

*Proof.* Start, as before, by computing:

$$H_k B(n) = B(n)_n \Sigma^{n-k} H = D_n H_* K(\mathbb{F}_p, n-k).$$

The unstable module $G(n)$ also enjoys a universal property in the category of stable $\mathcal{A}$–modules:

$$\mathsf{Modules}_{\mathcal{A}/}(G(n), M) \cong [\Omega^\infty M]_n.$$

Hence, we can continue our computation:

$$\begin{aligned} H_k B(n) &= D_n H_* K(\mathbb{F}_p, n-k) \\ &= \mathsf{Modules}_{\mathcal{A}/}(G(n), \Sigma^{n-k}\mathcal{A}) \\ &= \mathsf{Modules}_{\mathbb{F}_p/}(G(n)_{n-k}, \mathbb{F}_p). \end{aligned}$$

We learn immediately that $H_* B(n)$ is finite. We would like to show, furthermore, that $H_* B(n)$ is the Spanier–Whitehead dual $\Sigma^n DG(n)$. It suffices to show

$$\mathsf{Modules}_{\mathcal{A}/}(G(n), \Sigma^j \mathcal{A}) = \mathsf{Modules}_{\mathcal{A}/}(\mathbb{F}_p, \Sigma^j \mathcal{A} \otimes H_* B(n))$$

for all values of $j$. This follows from calculating $B(n)_n \Sigma^{n+j} H$ using the same method. Finally, linear-algebraic duality and Definition 66 give the Theorem.

Additionally, the following Lemma is almost a consequence of basic understanding of unstable modules over $\mathcal{A}_*$, with minor fuss at the bad indices $n \equiv \pm 1 \pmod p$:

**Lemma 62 ([7, Lemma 3.3]).** *There is a natural onto map $B(n)_n X \to H_n X$.* $\square$

Let's now work toward using the $B(n)$ spectra to analyze the Hopf rings arising from unstable cooperations. We have previously computed that the comparison map

$$H_*^R \underline{BP}_{2*} \to H_* \underline{BP}_{2*}$$

is an isomorphism. We will begin by reimagining this statement in terms of Dieudonné theory.

**Cite me:** You probably also want to cite Hunton–Turner and Buchstaber–Lazarev.

To begin with, Dieudonné theory as we have described it is concerned with *Hopf algebras* rather than Hopf rings. However, a Hopf ring is not much structure on top of a system of graded Hopf algebras $A_*$: it is a map

$$\circ \colon A_* \boxtimes A_* \to A_*,$$

where "$\boxtimes$" is the tensor product of Hopf algebras. Since $D_*$ gives an equivalence of categories between graded Hopf algebras and graded Dieudonné modules, we should be able to find an analogous formula for the tensor product of Dieudonné modules.

**Definition 67.** The naive tensor product $M \otimes N$ of Dieudonné modules $M$ and $N$ receives the structure of a $\mathbb{W}(k)[V]$–module, where $V(x \otimes y) = V(x) \otimes V(y)$. We define the *tensor product of Dieudonné modules* by

$$M \boxtimes N = \mathbb{W}(k)[F, V] \otimes_{\mathbb{W}(k)[V]} (M \otimes N) \Big/ \begin{pmatrix} 1 \otimes Fx \otimes y = F \otimes x \otimes Vy, \\ 1 \otimes x \otimes Fy = F \otimes Vx \otimes y \end{pmatrix}.$$

This definition is easier than it should be in generality, because not only are you working with a *field* but you're even working with $\mathbb{F}_p$, which has no Frobenius.

Since the category of Dieudonné modules is the category of modules over some ring, there is also a tensor product over that ring instead over the partial ring $\mathbb{W}(k)[V]$. This seems like to natural thing to consider, unless the equivalence provided by the Freyd-Mitchell embedding theorem is not monoidal?

There's a wrinkle here: if you want to say that $D_*$ is an equivalence of symmetric monoidal categories, then there needs to be a unit object. I don't think that's the case in the setting you're in.

**Lemma 63 ([8, Theorem 7.7]).** *The natural map*

$$D_*(M) \boxtimes D_*(N) \to D_*(M \boxtimes N)$$

*is an isomorphism.* □

**Definition 68.** For a ring $R$, a *Dieudonné R–algebra* $A_*$ is a graded Dieudonné module equipped with an $R$–action and an algebra product

$$\circ\colon A_* \boxtimes A_* \to A_*.$$

This citation at least says that the R–W relation holds over the topology $E_*$–Dieudonné algebra.

*Example 29 ([8, Proposition 10.2]).* For a complex-oriented homology theory $E$, we define its Dieudonné $E_*$–algebra of algebraic unstable cooperations by

$$R_E = E_*[b_1, b_2, \ldots] / \left( b(s+t) = b(s) +_E b(t) \right),$$

How?

where $V$ is multiplicative, $V$ fixes $E_*$, and $V$ satisfies $Vb_{pj} = b_j$. (This determines the behavior of $F$.) We also write $D_E = \{D_{2m} H_* \underline{E}_{2n}\}$ for the even part of the topological Dieudonné algebra, and these come with natural comparison maps

$$R_E \to D_E \leftarrow D_* H_* \underline{E}_{2*}.$$

**Theorem 58 ([8, Theorem 11.7]).** *Restricting attention to the even parts, the maps*

$$R_E \to D_E \leftarrow D_* H_* \underline{E}_{2*}$$

*are isomorphisms for E Landweber flat.*

*Proof.* In Corollary 37, we showed that these maps are isomorphisms for $E = BP$. However, the right–hand object can be identified via Brown–Gitler juggling:

Is this really legal? We won't get intro trouble with odd indices and the semi-parity condition on $n$?

$$D_n H_* \underline{E}_{2j} = B(n)_n \Sigma^{2j} E = E_{2j+n} B(n).$$

If $E$ is Landweber flat, then the middle– and right–terms are determined by change-of-base from the respective $BP$ terms by definition of flatness. Finally, the left term commutes with change-of-base by its algebraic definition, and the theorem follows.

You made this sound more complicated in class than what's written here (with bar sequence arguments, etc.) Can you provide some of those details here?

*Remark 52.* The proof of Theorem 58 originally given by Goerss [8] involved a lot more work, essentially because he didn't want to assume Theorem 48 or

Corollary 37. Instead, he used the fact that $\Sigma^\infty_+ \Omega^2 S^3$ is a regrading of the ring spectrum $\bigvee_n B(n)$, together with knowledge of $BP_* \Omega^2 S^3$. Since we already spent time with Theorem 48, we're not obligated to pursue this other line of thought.

*Remark 53 ([8, Proposition 11.6]).* The Dieudonné algebra framework also makes it easy to add in the odd part after the fact. Namely, suppose that $E$ is a torsion–free ring spectrum and suppose that $E_* B(n)$ is even for all $n$. In this setting, we can verify the purely topological version of this statement: the map

$$D_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*$$

is an isomorphism.

To see this, note that because $E_{2n-2k-1} B(2n) \to D_{2n} H_* \underline{E}_{2k+1}$ is onto and $E_{2n-2k-1} B(2n)$ is assumed zero, the group $D_{2n} H_* \underline{E}_{2k+1}$ vanishes as well. A bar spectral sequence argument shows that $D_{2n+1} H_* \underline{E}_{2k+2}$ is also empty [8, Lemma 11.5.1]. Hence, the map on even parts

$$(D_E[e]/(e^2 - b_1))_{*,2n} \to (D_* H_* \underline{E}_*)_{*,2n}$$

is an isomorphism, and we need only show that

$$D_* H_* \underline{E}_{2n} \xrightarrow{e \cdot -} D_* H_* \underline{E}_{2n+1}$$

is an isomorphism as well. Since $e(Fx) = F(Ve \circ x) = 0$ and $D_* A / F D_* A \cong Q^* A$ for a Hopf algebra $A$, we see that $e$ kills decomposables and suspends indecomposables:

$$e D_* H_* \underline{E}_{2n} = \Sigma Q H_* \underline{E}_{2n}.$$

This is also what happens in the bar spectral sequence, and the claim follows. In light of Theorem 58, this means that for Landweber flat $E$, the comparison isomorphism can be augmented to a further isomorphism

$$R_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*.$$

Jeremy asked whether there was a connection between Goerss's original proof and the free $E_2$–algebra with $p$ killed which we keep dancing around this semester. I don't know, and it's a good question.

## 4.7 Apr 6: Cooperations and geometric points on $\mathcal{M}_{\mathbf{fg}}$

Compare also with the main result of [12].

Throughout today, we will write $K$ for a Morava $K$–theory $K_\Gamma$ (which, if you like, you can take to be $K(d)$) and $A$ for a finitely generated abelian group, and $H$ for the associated Eilenberg–Mac Lane spectrum. Our goal is to study the unstable mixed cooperations $K_* \underline{H}_*$, which we expect to be connected to formal group homomorphisms $\Gamma \to \widehat{\mathbb{G}}_a$ but which isn't covered by any

Remark 11.4 in the Hopf Ring paper says that the failure of the odd primary case to be an isomorphism is measured by the suspension homomorphism operator $e$, and the kernel of the natural surjective map is exactly the kernel of multiplication by $e$. Have a look.

Ask Mike (and Jacob?) if there are analogues of these results for $kO$ which explain Mahowald's generalized $K$–theoretic Brown–Gitler spectra. 3/29: I did ask Mike, he said he didn't know. I also asked Paul, and he said this seemed unreasonable, since $kO$ isn't valued in co/commutative Hopf algebras. This is a fair of the cases studied thus far. This calculation is interesting to us for two reasons:

1. These cooperations appear naturally when pursuing a "fiberwise analysis" of cooperations, or a chromatic analysis of unstable homotopy theory, along the lines of Case Study 3.
2. The Eilenberg–Mac Lane spaces $\underline{H}_*$ appear as the layers of Postnikov towers. If we were to want to analyze the $K$–homology of a Postnikov tower (as we will in Case Study 5), we will naturally encounter pieces of $K_*\underline{H}_*$, and we would be wise to have a firm handle on these objects. It is a tribute to the perspective offered here that the successful way to approach this computation is not one-at-a-time, handicrafted for each possible Eilenberg–Mac Lane space, but rather all-at-once, as suggested by the unstable cooperations picture.

Unsurprisingly, our analysis will rest on the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{K_*\underline{H}_q}(K_*, K_*) \Rightarrow K_*\underline{H}_{q+1}.$$

However, because $K$–theory is naturally a 2–periodic theory, our method in Lecture 4.4 of inducting on homological degree and working with a triangular corner of the spectral sequence will fail because it is not a first-quadrant spectral sequence. Instead, we will induct on the Eilenberg–Mac Lane index $q$ as in Lecture 4.2, and as such we will begin with analyzing the base case of $q = 0$ where we are interested in manually computing $K_*BA$ for a reasonable abelian group $A$. Since $K$–theory has Künneth isomorphisms and $B(A_1 \times A_2) \simeq BA_1 \times BA_2$, it suffices to do the computation just for $A = C_{p^j}$.

This only takes care of finite abelian groups $A$. Is this what reasonable means?

**Theorem 59 ([35, Theorem 5.7], [14, Proposition 2.4.4]).** *There is an isomorphism*

$$BS^1[p^j]_K \cong BS^1_K[p^j].$$

I'm a little confused about the objects in this definition. Does the RHS mean take the scheme and then take p-torsion in the ring, then go back to the scheme? And the LHS is K theory of the cofiber of multiplication by $p^j$? Maybe would help if this was slightly explained

Put in a pullback corner here.

*Proof.* Consider the diagram of spherical fibrations:

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d, equal] & B(S^1[p^j]) \arrow[r] \arrow[d] & BS^1 \arrow[d, "p^j"] \\
S^1 \arrow[r] & ES^1 \arrow[r] & BS^1.
\end{tikzcd}$$

The induced long exact sequence (known as the Gysin sequence, or as the couple in the Serre spectral sequence for the first fibration) takes the form

$$\begin{tikzcd}
 & K^*BS^1 \arrow[dl] & \\
K^*(BS^1[p^j]) \arrow[rr, "\partial"'] & & K^*BS^1 \arrow[ul, "-\smile[p^j](x)"']
\end{tikzcd}$$

One of the stars on the right half of the diagram needs a dimension shift. The bottom $\partial$ arrow should be dotted.

where $x$ is a coordinate on $BS^1_K$. Because $BS^1_K$ is of finite height, the right diagonal map is injective. It follows that $\partial = 0$, and so this gives a short exact sequence of Hopf algebras, which we can reinterpret as a short exact sequence of group schemes

You should explain how you're using the map of spherical fibrations to give this.

I'm confused: does it make sense to say that $BS^1_K$ is of finite height? It seems like what you're really using is that the formal group $\Gamma$ associated to $K$ has finite height.

$$B(S^1[p^j])_K \to BS^1_K \xrightarrow{p^j} BS^1_K. \square$$

*Remark 54.* Dually, there is also an exact sequence of Hopf algebras

$$\begin{tikzcd}
 & K_*BS^1 \arrow[dr, "-\frown[p^j](x)"] & \\
K_*(BS^1[p^j]) \arrow[ur] & & K_*BS^1, \arrow[ll, "\partial"]
\end{tikzcd}$$

where again $\partial = 0$ and hence $K_*(BS^1[p^j])$ is presented as the kernel of the map "cap with $[p^j](x)$". We will revisit the duality next time.

There are a couple of approaches to the rest of this calculation, i.e., $K_*\underline{H}_q$ for $q > 1$. The original, due to Ravenel and Wilson [35], is to complete the calculation for the smallest abelian group $C_p$ and then induct upward toward more complicated groups like $C_{p^j}$ and $C_{p^\infty}$. More recently, there is also a preprint of Hopkins and Lurie [14] that begins with $A = C_{p^\infty}$ and then works downward. We will do the *easy* parts of both calculations, to give a feel for their relative strengths and deficiencies.

The Ravenel–Wilson version of the calculation proceeds much along the same lines as Lecture 4.2. Setting $H = H\mathbb{Z}/p$, we will study the bar spectral sequences

$$\operatorname{Tor}^{K_*\underline{H}_q}_{*,*}(K_*, K_*) \Rightarrow K_*\underline{H}_{q+1}$$

for different indices $q$ and use the $\circ$–product to push differentials around among them. Our first move, as in Lecture 4.2, is to study the bar spectral sequence

$$\operatorname{Tor}^{K_*\mathbb{Z}/p}_{*,*}(K_*, K_*) \Rightarrow K_*B\mathbb{Z}/p$$

and analyze what *must* happen in order to reach the conclusion of Theorem 59. In the input to this spectral sequence, the ground algebra is given by

$$K_*\underline{H\mathbb{Z}/p}_0 = K_*[[1]]/\langle [1]^p - 1\rangle = K_*[[1]-[0]]/\langle [1]-[0]\rangle^p.$$

Did you also use angle brackets to denote ideals in the rest of the document? Do you care to? Probably not.

The Tor–algebra for this truncated polynomial algebra $K_*[a_\emptyset]/a_\emptyset^p$ is then given by the formula

$$\mathrm{Tor}_{*,*}^{K_*[a_\varnothing]/a_\varnothing^p}(K_*, K_*) = \Lambda[\sigma a_\varnothing] \otimes \Gamma[\varphi a_\varnothing],$$

What is the reason for the $\sigma$ and $\varphi$ notation?

the combination of an exterior algebra and a divided power algebra. We know which classes are supposed to survive this spectral sequence, and hence we know where the differentials must be:

$$d(\varphi a_\varnothing)^{[p^d]} = \sigma a_\varnothing,$$
$$\Rightarrow d(\varphi a_\varnothing)^{[i+p^d]} = \sigma a_\varnothing \cdot (\varphi a_\varnothing)^{[i]}.$$

The spectral sequence collapses after this differential.[3]

With the base case analysis completed, we turn to the induction on $q$:

**Theorem 60 ([35, Theorem 9.2 and Theorem 11.1]).** *Using the $\circ$–product,*

$$K_* \underline{H\mathbb{Z}/p}_q = \mathrm{Alt}^q \ \underline{H\mathbb{Z}/p}_1.$$

*Proof (Proof sketch).* The inductive step turns out to be extremely index-rich, so I won't be so explicit or complete, but I'll point out the major landmarks. It will be useful to use the shorthand $a_{(i)} = a_\varnothing^{[p^i]}$, where $(i)$ is thought of as a multi-index with one entry.

We proceed by induction, assuming that $K_* \underline{H\mathbb{Z}/p}_q = \mathrm{Alt}^q \ \underline{H\mathbb{Z}/p}_1$ for a fixed $q$. Computing the Tor–algebra of $K_* \underline{H\mathbb{Z}/p^j}_q$ again yields a tensor of divided power and exterior classes, a pair for each algebra generator of $K_* \underline{H\mathbb{Z}/p^j}_q$. In analogy to the rewriting formula used in Theorem 46, there is also a rewriting formula in this context [35, Lemmas 9.5-6]:

Did you get this citation right? It doesn't look like I remember.

$$(\varphi a_{(i_1,\ldots,i_q)})^{[p^n]} \equiv (\varphi a_{(i_1,\ldots,i_{q-1})})^{[p^n]} \circ a_{(i_q+n)} \quad \text{mod } *\text{–decomposables.}$$

Since every class can be so decomposed, all the differentials are determined by the previous spectral sequence. In particular, classes are hit by differentials exactly when $i_q + n$ is large enough. Chasing this through shows that the inductive assumption that $K_* \underline{H\mathbb{Z}/p}_{q+1}$ is an exterior power holds, and the class $(\varphi a_{(i_1,\ldots,i_q)})^{[p^n]}$ represents $a_{(n,i_1+n,\ldots,i_q+n)}$.

*Remark 55.* When reworking this computation for the case

$$\mathrm{Tor}_{*,*}^{K_*\underline{H\mathbb{Z}/p^j}_q}(K_*, K_*) \Rightarrow K_* \underline{H\mathbb{Z}/p^j}_{q+1},$$

the main difference is that there are various algebra extensions to keep track of. These are controlled using the group maps

These are Erick's suggestions of how to denote these group maps, so that it's clearer which is the projection and which is the inclusion. I should go back through the rest of the notes and enforce this notation elsewhere too.

[3] In the $j > 1$ version of this analysis, there are some multiplicative extensions to sort out. Of course, these are all determined by already knowing the multiplicative structure on $K_* \underline{H\mathbb{Z}/p^j}_1$.

$$\mathbb{Z}/p^{j+1} \to \mathbb{Z}/p^j, \qquad \left(\frac{1}{p^j}\mathbb{Z}\right)/\mathbb{Z} \to \left(\frac{1}{p^{j+1}}\mathbb{Z}\right)/\mathbb{Z},$$

together with knowledge of how the extensions strung together at the previous $j$–stage. Then, these tools are revisited [35, Theorem 12.4] to give a computation in the limiting case $A = C_{p^\infty}$, where there's a $p$–adic equivalence $HC_{p^\infty} \simeq_p^\wedge \Sigma H\mathbb{Z}$.The calculation in this setting is the most interesting one of all — after all, it contains the case $BS^1_K$, which is of special interest to us.

Mike has said something about the pairing $C_{p^j} \times C^*_{p^j} \to \mathbb{Q}/\mathbb{Z}$ not being functorial in $j$ (so as to pass to the direct limit) which gave me pause. I should make sure I'm not messing something up here.

Remarkably, this maximally interesting case is easier to access directly than passing through all of this intermediate work, and this is the perspective of Hopkins and Lurie. We will pursue an inductive calculation of the formal group schemes $(\underline{HC_{p^\infty}}_q)_K$ by iterating the cohomological bar spectral sequence, culminating in the following Theorem:

Expand out the "limiting case as $j \to \infty$" of the differentials in the earlier spectral sequences that you talked about in class.

**Theorem 61.** *There is an isomorphism of formal group schemes*

$$(\underline{HC_{p^\infty}}_q)_K \cong (\mathbb{C}\mathrm{P}^\infty_K)^{\wedge q}.$$

*In particular, $(\underline{HC_{p^\infty}}_q)_K$ is a "$p$–divisible formal group" of dimension $\binom{d-1}{q-1}$ and height $\binom{d}{q}$.*

Assume that this Theorem is true for a fixed value of $q$. First, the cohomological bar spectral sequence lets us calculate just the *formal scheme* structure of $(\underline{HC_{p^\infty}}_{q+1})_K$, using the *formal group* structure of $(\underline{HC_{p^\infty}}_q)_K$. It has signature ([21], [14, Example 2.3.5])

$$H^*((\underline{HC_{p^\infty}}_q)_K; \widehat{\mathbb{G}}_a) \otimes_{K_0} K_* \Rightarrow K^* \underline{HC_{p^\infty}}_{q+1},$$

and hence we are moved to calculate the formal group cohomology of $(\underline{HC_{p^\infty}}_q)_K$. The following Lemma furthers the calculations of formal group cohomology in Lemma 43 and Lemma 51 to the situation of connected $p$–divisible groups of higher dimension:

**Lemma 64 ([14, Theorem 2.2.10 and Example 2.2.12]).** *If $\widehat{\mathbb{G}}$ is a connected $p$–divisible group over a field $k$, then $H^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is isomorphic to the symmetric algebra on $\Sigma H^1(\widehat{\mathbb{G}}[p^j]; \widehat{\mathbb{G}}_a)$, with generators concentrated in degree* 2. ☐

$j$ is a positive integer

**Corollary 42 ([14, Proposition 2.4.11]).** *As a formal scheme, $(\underline{HC_{p^\infty}}_{q+1})_K$ is a formal variety of dimension $\binom{d-1}{q}$.*

*Proof.* By setting $\widehat{\mathbb{G}} = (\underline{HC_{p^\infty}}_q)_K$, the Lemma gives us access to the $E^2$–page of our cohomological bar spectral sequence. We can calculate the dimension of $H^1$ to be

I have tried several times to figure out where this comes from, using tools from the beginning of H–L Section 2.2, but I haven't really convinced myself. It must have something to do with the dimension of the Dieudonné module of $\widehat{\mathbb{G}}$ minus the dimension of the image of $F$ — or something?

$$\dim_k H^1((\underline{H\mathbb{C}_{p^q}})_K; \widehat{\mathbb{G}}_a) = \operatorname{ht}(\underline{HC_{p^\infty}}_q)_K - \dim(\underline{HC_{p^\infty}}_q)_K = \binom{d}{q-1} - \binom{d-1}{q-1} = \binom{d-1}{q}.$$

It follows that the $E_2$–page of this spectral sequence is a polynomial $k$–algebra on $\binom{d-1}{q}$ generators, concentrated in even degrees, so that the spectral sequence collapses and $K^0 \underline{HC_{p^\infty}}_{q+1}$ is a power series algebra on as many generators.

In order to continue the induction, we now have to identify the group structure on $(\underline{HC_{p^\infty}}_{q+1})_K$. This is done using the theory of Dieudonné modules:

**Theorem 62 ([14, Proposition 2.4.12]).**

*Let $q \geq 1$ be an integer. Suppose, in addition to the inductive hypotheses above, that the sequence of group schemes*

Figure out this formatting. I should be able to just set the fraction to be "inline" rather than displaymode, but I don't know how to do that.

$$(\underline{H(\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p)}_q)_K \to (\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)}_q)_K \to (\underline{H(\mathbb{Q}_p/\frac{1}{p}\mathbb{Z}_p)}_q)_K \to K_0$$

*is exact, and that the map*

$$\theta^q \colon \mathbb{Q}_p/\mathbb{Z}_p \otimes M(\mathbb{CP}^\infty_K)^{\wedge q} \to M((\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)}_q)_K)$$

*is an isomorphism. Then $\theta^{q+1}$ is an isomorphism and the formal group $\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)}_{q+1}$ is a connected $p$–divisible group with height $\binom{d}{q+1}$ and dimension $\binom{d-1}{q}$.*

At one point this statement contained the phrase "$K(n)$-good" whose meaning I do not know.

*Proof (Proof sketch).* By applying the snake lemma to the diagram

Jay asked about the edge case of $\theta^1$, where the only interesting thing happening is the $p$-adic equivalence. That might be worth expanding in a Remark.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} & \longrightarrow & 0 \\
 & & \downarrow V & & \downarrow V & & \downarrow V & & \\
0 & \longrightarrow & M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} & \longrightarrow & 0
\end{array}$$

The Qs in the middle should be $\mathbb{Q}_p$, right?

and knowing that the middle map is an isomorphism, we learn that $V$ is a surjective endomorphism of $M^{\wedge(q+1)} M \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and that there is an isomorphism

$$\ker(V \colon \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} \to \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)}) \cong \operatorname{coker}(V \colon M^{\wedge(q+1)} \to M^{\wedge(q+1)}).$$

Why is the first component singled out? Also, what's $m$? What's $Y$?

The right-hand side is spanned by elements $V^I x$ with $I_1 = 0$, and hence the left-hand side has $k$–vector–space dimension $\binom{d-1}{q}$. By very carefully studying the bar spectral sequence, one can learn that $\theta^m$ induces a surjection

$$\ker V|_{\mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge m}} \to \ker V|_{D(Y)}.$$

In fact, since these two have the same rank, $\theta^m$ is an isomorphism on these subspaces. Since the action of $V$ is locally nilpotent, this is enough to show that $\theta^m$ is an isomorphism, without restriction to subspaces: if it failed to be an injection, we could apply $V$ enough times to get an example of a nontrivial element in $\ker V|_{\mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge m}}$ mapping to zero, and we can manually construct preimages through successive approximation.

*Remark 56 ([14, Proposition 2.4.13]).* With this in hand, you now have to pull apart the full $p$–divisible group to get a calculation of $(\underline{H\mathbb{Z}/p^j}_q)_K$. From this perspective, this is the hardest part with the longest, most convoluted proof.

*Remark 57 ([14, Section 3]).* Because $K_\Gamma^* \underline{H\mathbb{Z}/p^j}_q$ is even, you can hope to augment this to a calculation of $E_\Gamma^* \underline{H\mathbb{Z}/p^j}_q$. This is indeed possible, and the analogous formula is true at the level of Hopf algebras:

**Cite me:** You could also cite the alternating powers guy [10, 11] and a source for Dieudonné crystals.

$$(E_\Gamma)_* \underline{HC_{p^j}}_q \cong \mathrm{Alt}^q (E_\Gamma)_* \underline{HC_{p^j}}_1.$$

However, the attendant algebraic geometry is quite complicated: you either need a form of Dieudonné theory that functions over $\mathcal{M}_{E_\Gamma}$ (and then attempt to drag the proof above through that setting), or you need to directly confront what "alternating power of a $p$–divisible group" means at the level of $p$–divisible groups (and forego all of the time-saving help afforded to you by Dieudonné theory).

Actually say "Dieudonné crystal".

*Remark 58.* You'll notice that in $K_* \underline{H}_{q+1}$ if we let the $q$–index tend to $\infty$, we get the $K$–homology of a point. This is another way to see that the stable cooperations $K_* H$ vanish, meaning that the *only* information present comes from unstable cooperations.

We could even provide a quick proof of the stable calculation? Cf. http://mathoverflow.net/questions/220952/localization-at-the-johnson-wilson-spectrum-and-rationalization, http://mathoverflow.net/a/99211/1094 .

## *Things that belong in this chapter*

Maybe talk about some consequences: the Hopkins–Ravenel–Wilson results on finite Postnikov towers and so on?

Theorem 6.1 of R–W *The Hopf ring for complex bordism* sounds like something related to Quillen's elementary proof.

There's also a document by Boardman, Johnson, and Wilson (Chapter 2 of the *Handbook of Algebraic Topology*) that discusses an equivalence between Steve's approach and "unstable comodules". Please read this.

I was thinking that this would give a counterexample to the idea that the additive unstable cooperations always present the functions on the scheme of homomorphisms, but now I see that this example works too. As lazy evidence, I think counting the ranks of $Q^* K(d)_* \underline{H\mathbb{Z}/p}_*$ and $\underline{\mathsf{FormalGps}}(\Gamma_d, \widehat{\mathbb{G}}_a)$ (using Dieudonné theory, or using Callan's tangent space trick) gives the same number. More seriously, I think if you write out the scheme of homomorphisms, you'll see enough things degenerate (because $[-1]_{\widehat{\mathbb{G}}_a}(x) = -x$) that you do get this alternating algebra.

# Case Study 5
# The $\sigma$-orientation

Write an introduction for me. Use unstable cooperations from Morava's theories to classical complex and real $K$–theory.

Part of the theme of this chapter should be to use the homomorphism from topological vector bundles to algebraic line bundles — Neil's $\mathbb{L}$ construction — as inspiration for what to do, given suitable algebraic background.

An artifact of these being lecture notes for an actual class is that this last section is getting compressed due to end-of-semester time constraints. I think a published version of these notes would contain proofs of a bunch of these facts which, at present, are getting omitted either because the proofs take too long or because simply understanding the theorem statements takes too long. Really, the class in general is becoming quite strained at this point: it's hard to keep everything straight both for the students and for the teacher.

## 5.1 Apr 8: Coalgebraic formal schemes

Today we will discuss an elephant that has been lingering in our room: we began the class talking about the formal scheme associated to the *cohomology* of a space, but we have since become primarily interested in a construction converting the *homology* of a spectrum to a sheaf over a context. Our goal for today is to, when possible, put these on even footing. Our motivation for finally addressing this lingering discrepancy is more technical than aesthetic: we have previously wanted access to certain colimits of formal schemes (e.g., in Theorem 13). While such colimits are generally forbidding, similarly to colimits of manifolds, we will produce certain conditions under which they are accessible.

Suppose that $E$ is a ring spectrum, and recall the usual way to produce the structure of an $E^*$–algebra on $E^*X$ for $X$ a space. The space $X$ has a diagonal map $\Delta\colon X \to X \times X$, which on $E$–cohomology induces a multiplication map

$$E^*X \otimes_{E^*} E^*X \xrightarrow{\mu_E} E^*(X \times X) \xrightarrow{E^*\Delta} E^*X.$$

Dually, applying $E$–homology, we have a pair of maps

$$E_*X \xrightarrow{E_*\Delta} E_*(X \times X) \xleftarrow{\text{Künneth}} E_*X \otimes_{E_*} E_*X.$$

In the case that the Künneth map is an isomorphism, the right–hand map is invertible, and the long composite induces the structure of an $E_*$–coalgebra on $E_* X$. In the most generous case that $E$ is a field spectrum (in the sense of Corollary 31), $E^* X$ is functorially the linear dual of $E_* X$, which motivates us to consider the following purely algebraic construction:

**Definition 69.** Let $C$ be a coalgebra over a field $k$. The scheme $\operatorname{Sch} C$ *associated to* $C$ is defined by

$$(\operatorname{Sch} C)(T) = \left\{ f \in C \otimes T \,\middle|\, \begin{array}{c} \Delta f = f \otimes f \in (C \otimes T) \otimes_T (C \otimes T), \\ \varepsilon f = 1 \end{array} \right\}.$$

**Lemma 65.** *For $A$ a $k$–algebra, finite–dimensional as a $k$–module, one has* $\operatorname{Spec} A \cong \operatorname{Sch} A^*$.

*Proof (Proof sketch).* A point $f \in (\operatorname{Sch} A^*)(T) \subseteq A^* \otimes T$ gives rise to a map $f_* \colon A \to T$ by the duality pairing, which is a ring homomorphism by the condition. The finiteness assumption is present exactly so that $A$ is its own double–dual, giving an inverse assignment.

If we drop the finiteness assumption, then a lot can go wrong. For instance, the multiplication on our $k$–algebra $A$ gives rise only to maps

$$A^* \to (A \otimes_k A)^* \leftarrow A^* \otimes_k A^*,$$

which is not enough to make $A^*$ into a $k$–coalgebra. However, if we start instead with a $k$–coalgebra $C$ of infinite dimension, the following result is very telling:

**Cite me:** Demazure's book has this somewhere in it, also see your thesis for a more modern reference.

**Lemma 66.** *For $C$ a coalgebra over a field $k$, any finite–dimensional $k$–linear subspace of $C$ can be finitely enlarged to a subcoalgebra of $C$. Accordingly, taking the colimit gives a canonical equivalence*

$$\operatorname{Ind}(\mathsf{Coalgebras}_k^{\mathrm{fin}}) \xrightarrow{\simeq} \mathsf{Coalgebras}_k. \square$$

This result allows us to leverage our duality Lemma pointwise: for an arbitrary $k$–coalgebra, we break it up into a lattice of finite $k$–coalgebras, and take their linear duals to get a reversed lattice of finite $k$–algebras. Altogether, this indicates that $k$–coalgebras generally want to model *formal schemes*.

**Corollary 43.** *For $C$ a coalgebra over a field $k$ expressed as a colimit $C = \operatorname{colim}_k C_k$ of finite subcoalgebras, there is an equivalence*

$$\operatorname{Sch} C \cong \{\operatorname{Spec} C_k^*\}_k.$$

*This induces a* covariant *equivalence of categories*

$$\mathsf{Coalgebras}_k \cong \mathsf{FormalSchemes}_{/k}. \square$$

This covariant algebraic model for formal schemes is very useful. For instance, this equivalence makes the following calculation trivial:

Do you also need to compare the (Cartesian) monoidal structures?

**Lemma 67 ((cf. Theorem 13 and )).** *Select a coalgebra $C$ over a field $k$ together with a pointing $k \to C$. Write $M$ for the coideal $M = C/k$. The free formal monoid on the pointed formal scheme $\operatorname{Sch} k \to \operatorname{Sch} C$ is given by*

Backreference the relevance of the Div construction to homotopy theory.

Describe the diagonal map on this guy.

$$F(\operatorname{Sch} k \to \operatorname{Sch} C) = \operatorname{Sch} \operatorname{Sym}^* M. \square$$

It is unfortunate, then, that when working over an object more general than a field Lemma 66 fails. Nonetheless, it is possible to bake into the definitions the machinery needed to get a good-enough analogue of Corollary 43.

Include (a reference to) an example.

**Definition 70.** Let $C$ be an $R$–coalgebra which is free as an $R$–module. A basis $\{x_j\}$ of $C$ is said to be a *good basis* when any finite subcollection of $\{x_j\}$ can be finitely enlarged to a subcollection that spans a subcoalgebra. The coalgebra $C$ is itself said to be *good* when it admits a good basis. A formal scheme $X$ is said to be *coalgebraic* when it is isomorphic to $\operatorname{Sch} C$ for a good coalgebra $C$.

**Theorem 63 ([42, Proposition 4.64]).** *Suppose that $F\colon \mathsf{I} \to \mathsf{Coalgebras}_R$ is a colimit diagram of coalgebras such that each object in the diagram, including the colimit point, is a good coalgebra. Then*

$$\operatorname{Sch} \circ F\colon \mathsf{I} \to \mathsf{FormalSchemes}$$

*is a colimit diagram of formal schemes.* $\square$

This Theorem gives us access to many constructions on formal schemes, provided we assume that the input is coalgebraic. This covers many of the cases of interest to us, as every formal variety is coalgebraic. For an example of the sort of constructions that become available, one can prove the following Corollary by analyzing the symmetric power of coalgebras:

I guess you would prove this by first dévissaging to the case of $\widehat{\mathbb{A}}^n$. How does this work? Then you'd just need to prove it for $\widehat{\mathbb{A}}^n$ – Jay asked you how to do this for $\widehat{\mathbb{A}}^1$, and I think you should include the argument here.

**Corollary 44 ([42, Proposition 6.4]).** *For $X$ a coalgebraic formal scheme, $X^{\times n}_{\Sigma_n}$ exists. In fact, $\coprod_{n \ge 0} X^{\times n}_{\Sigma_n}$ models the free formal monoid on $X$. If $\operatorname{Spec} k \to X$ is a pointing, then $\operatorname{colim}_n \{X^{\times n}_{\Sigma_n}\}_n$ models the free formal monoid on the pointed formal scheme.* $\square$

In the specific case that $\operatorname{Spec} k \to X$ is a formal *curve*, we can prove something more:

**Corollary 45 ([42, Proposition 6.13]).** *For $\operatorname{Spec} k \to X$ a pointed formal curve, the free formal monoid is automatically an abelian group.* $\square$

*Proof (Proof sketch).* The idea is that the symmetric algebra on the coalgebra associated to a formal curve admits a sufficiently nice filtration that one can iteratively solve for a Hopf algebra antipode.

We now reconnect this algebraic discussion with the algebraic topology that spurred it.

**Lemma 68.** *If $E$ and $X$ are such that $E_*X$ is an $E_*$–coalgebra and*

$$E^*X = \mathsf{Modules}_{E_*}(E_*X, E_*),$$

*then there is an equivalence*

$$\operatorname{Sch} E_*X \cong X_E. \square$$

*Proof (Proof sketch).* The main point is that the formal topology on $X_E$ is induced by the compactly generated topology of $X$, and this same topology can also be used to write $\operatorname{Sch} E_*X$ as the colimit of finite $E_*$–coalgebras.

*Example 30 (Theorem 59 and Remark 54).* For a Morava $K$–theory $K_\Gamma$ associated to a formal group $\Gamma$ of finite height, we have seen that there is an exact sequence of Hopf algebras

$$K^0_\Gamma(BS^1) \xrightarrow{[p^j]} K^0_\Gamma(BS^1) \to K^0_\Gamma(BS^1[p^j]),$$

presenting $(BS^1[p^j])_K$ as the $p^j$–torsion formal subscheme $BS^1_K[p^j]$. The Hopf algebra calculation also holds in $K$–homology, where there is instead the following exact sequence

$$(K_\Gamma)_0 B(S^1[p^j]) \to (K_\Gamma)_0 BS^1 \xrightarrow{(-)^{*p^j}} (K_\Gamma)_0 BS^1,$$

presenting $(K_\Gamma)_0 B(S^1[p^j])$ as the $p^j$–order $*$–nilpotence in the middle Hopf algebra. Applying Sch to this last line covariantly converts this second statement about Hopf algebras to the corresponding statement above about the associated formal schemes — i.e., the behavior of the homology coalgebra is a direct reflection of the behavior of the formal schemes.

The example above also spurs us to consider an intermediate operation. We have seen that the algebra structure of the $K$–cohomology of a space and the coalgebra structure of the $K$–homology of the same space contain equivalent data: they both give rise to the same formal scheme. However, in the case at hand, $BS^1$ and $BS^1[p^j]$ are commutative $H$–spaces and hence give rise to *commutative and cocommutative Hopf algebras* on both $K$–cohomology and $K$–homology. Hence, in addition to considering the coalgebraic formal scheme $\operatorname{Sch}(K_\Gamma)_0 B(S^1[p^j])$, we can also consider the affine scheme $\operatorname{Spec}(K_\Gamma)_0 B(S^1[p^j])$. This, too, should contain identical information, and this is the subject of Cartier duality.

**Definition 71 ([42, Section 6.4]).** The *Cartier dual* of a finite group scheme $G$ is defined by

$$DG = \underline{\mathsf{GroupSchemes}}(G, \mathbb{G}_m).$$

**Lemma 69 ([42, Proposition 6.19]).** *On the level of Hopf algebras $A = \mathcal{O}_G$, this has the effect*

$$DG = \operatorname{Sch} A = \operatorname{Spec} A^*. \square$$

You could stand to include a proof of this. It's been a while since you actually proved something serious with formal schemes, and this is pretty nice.

You could also include mention of the motivation: $C_k\widehat{\mathbb{G}}$ is hard to exhibit, but $(C_k\widehat{\mathbb{G}})^\vee$ is easy. It's not clear how to do this without leaping too far ahead, though.

Your definition is for finite group schemes, but then you use it for formal groups. One way to extend the definition is by $\underline{\mathsf{FormalGroups}}(G, \widehat{\mathbb{G}}_m)$. Another is to take the limit of the objectwise duals in the direct system defining the formal group. Are they the same? Perhaps you should assure the reader that nothing goes wrong?

*Remark 59.* The effect of Cartier duality on the Dieudonné module of a formal group is linear duality. Hence, the covariant and contravariant Dieudonné modules described in Lecture 4.5 are related by Cartier duality.

*Remark 60.* The topological summary of Cartier duality is that, when $X$ is a free even commutative $H$–space,

$$DX_E = \underline{\mathsf{GroupSchemes}}(X_E, \mathbb{G}_m) = \operatorname{Spec} E_0 X.$$

## 5.2 Apr 11: Special divisors and the special splitting principle

You should be a little careful here: are the Triumvirate theorems a little stronger than the culmination of (1), (2), and (3)? I think the proof of the equivalence of (2) and (3) on $T$–points rather than on $E_0$–points is something slightly harder than the most basic equivalence between (2) and (3) (which concerns only $E_0$–points).

Starting today, after our extended interludes on chromatic homotopy theory and cooperations, we are going to return to thinking about bordism orientations directly. To begin, we will recall the various perspectives adopted in Case Study 2 when we were studying complex–orientations of ring spectra.

1. A complex–orientation of $E$ is, definitionally, a map $MUP \to E$ of ring spectra in the homotopy category.
2. A complex–orientation of $E$ is also equivalent to a multiplicative system of Thom isomorphisms for complex vector bundles. Such a system is determined by its value on the universal line bundle $\mathcal{L}$ over $\mathbb{C}\mathrm{P}^\infty$. We can also phrase this algebro-geometrically: such a Thom isomorphism is the data of a trivialization of the Thom sheaf $\mathbb{L}(\mathcal{L})$ over $\mathbb{C}\mathrm{P}^\infty_E$.
3. Ring spectrum maps $MUP \to E$ induce on $E$–homology maps $E_0 MUP \to E_0$ of $E_0$–algebras. This, too, can be phrased algebro-geometrically: these are elements of $(\operatorname{Spec} E_0 MUP)(E_0)$.

We can summarize our main result about these perspectives as follows:

**Theorem 64 ([2, Example 2.53]).** *Take $E$ to be* complex–orientable. *The functor*

**Cite me:** Put in cross references.

$$\begin{aligned}&\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets},\\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^*\mathbb{L}(\mathcal{L}) \textit{ over } u^*\mathbb{C}\mathrm{P}^\infty_E\}\end{aligned}$$

*is isomorphic to the affine scheme* $\operatorname{Spec} E_0 MUP$. *Moreover, the* $E_0$*–points of this scheme biject with ring spectrum maps* $MUP \to E$.

*Proof (Proof summary).* The equivalence between (1) and (2) is given by the splitting principle for complex line bundles. The equivalence between (1) and (3) follows from calculating that $E_0 MUP$ is a free $E_0$–module.

An analogous result holds for ring spectrum maps $MU \to E$ and the line bundle $\mathcal{L} - 1$, and it is proven in analogous way. In particular, we will want a version of the splitting principle for virtual vector bundles of virtual rank 0. Given a finite complex $X$ and such a rank 0 virtual vector bundle, write

$$\tilde{V}\colon\ X \to BU$$

for the classifying map. Because $X$ is a finite complex, there exists an integer $n$ so that $\tilde{V} = V - n \cdot 1$ for an honest rank $n$ vector bundle $V$ over $X$. Using Corollary 15, the splitting $f^*V \cong \bigoplus_{i=1}^n \mathcal{L}_i$ over $Y$ gives a trivialization of $\tilde{V}$ *internally to* $BU$ as

$$\tilde{V} = V - n \cdot 1 = \bigoplus_{i=1}^{n} (\mathcal{L}_i - 1),$$

as each bundle $\mathcal{L}_i - 1$ itself has the natural structure of a rank 0 virtual vector bundle. This begets the following analogue of the previous result:

Cite me: Put in cross references.

**Theorem 65 ([2, Example 2.54]).** *Take E to be complex–orientable. The functor*

$$\begin{aligned}&\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets},\\ &(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \{\textit{trivializations of } u^*\mathbb{L}(\mathcal{L}-1) \textit{ over } u^*\mathbb{C}\mathrm{P}^\infty_E\}\end{aligned}$$

*is isomorphic to the affine scheme* $\operatorname{Spec} E_0 MU$. *Moreover, the* $E_0$*–points of this scheme biject with ring spectrum maps* $MU \to E$. □

Identify this bundle $\mathbb{L}(\mathcal{L}-1)$ as $\omega \otimes \mathcal{I}(0)^{-1}$, we can think of sections as elements of $E^0 T(\mathcal{L} - 1 \to \mathbb{C}\mathrm{P}^\infty)$ which restrict to the identity under the inclusion

$$S^0 \to P^{\mathcal{L}-1}.$$

Justify this.

*Remark 61.* The map $BU \to BU \times \mathbb{Z}$ induces a map $MU \to MUP$. The induced map on schemes is normalization:

$$\begin{aligned}\operatorname{Spec} E_0 MUP &\to \operatorname{Spec} E_0 MU,\\ f &\mapsto \frac{f'(0)}{f}.\end{aligned}$$

These two Thom spectra are the beginning of a larger pattern. Their base spaces $BU \times \mathbb{Z}$ and $BU$ are both infinite loopspaces: they are $\underline{kU}_0$ and $\underline{kU}_2$ respectively, where $kU$ is the connective complex $K$–theory spectrum. In general, the space $\underline{kU}_{2k}$ is given as a connective cover:

Justify this as a Lemma.

$$\underline{kU}_{2k} = BU[2k, \infty),$$

and so the next Thom spectrum in the sequence is $MSU$, the bordism theory of $SU$–structured manifolds. The special unitary group $SU$ is explicit enough that these orientations can be fully understood along similar lines to what we have done so far. Our jumping off point for that story will be, again, an extension of the splitting principle.

**Lemma 70.** *Let $X$ be a finite complex, and let $\tilde{V}\colon X \to BU$ classify a virtual vector bundle of rank 0 over $X$. Select a factorization $\tilde{\tilde{V}}\colon X \to BSU$ of $\tilde{V}$ through $BSU$. Then, there is a space $f\colon Y \to X$, where $f_E\colon Y_E \to X_E$ is finite and flat, as well as a collection of line bundles $\mathcal{H}_j$, $\mathcal{H}'_j$ expressing a $BSU$–internal decomposition*

$$\tilde{\tilde{V}} = -\bigoplus_{j=1}^{n} (\mathcal{H}_j - 1)(\mathcal{H}'_j - 1).$$

Hood had a question here: what symmetry group acts on this splitting? I think it's possible to show that the full symmetric group action on the $BU \times \mathbb{Z}$ splitting induces a full symmetric group action on the $BSU$ splitting. I don't know if there's any more or any less.

The minus sign: I understand mechanically where it comes from, but I still find it odd. Can we recover the $BU$-splitting principle from this splitting principle – and what happens to the signs?

*Proof.* Begin by using Corollary 15 on $V$ to get an equality of $BU$–bundles

$$\tilde{\tilde{V}} = V' + \mathcal{L}_1 + \mathcal{L}_2 - n \cdot 1.$$

Adding $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)$ to both sides, this gives

$$\begin{aligned} \tilde{\tilde{V}} + (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) &= V' + \mathcal{L}_1 + \mathcal{L}_2 + (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) - n \cdot 1 \\ &= V' + \mathcal{L}_1\mathcal{L}_2 - (n-1) \cdot 1. \end{aligned}$$

By thinking of $(\mathcal{L}_j - 1)$ as an element of $kU^2(Y) = \mathsf{Spaces}(Y, BU)$, we see that the product element $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) \in kU^4(Y) = \mathsf{Spaces}(Y, BSU)$ has the natural structure of a $BSU$–bundle and hence so does the sum on the left-hand side[1]. The right-hand side is the rank 0 virtualization of a rank $(n-1)$ vector bundle, hence succumbs to induction. Finally, because $SU(1)$ is the trivial group, there are no nontrivial complex line bundles with structure group $SU(1)$, grounding the induction.

**Corollary 46.** *Ring spectrum maps $MSU \to E$ biject with trivializations of*

$$\mathbb{L}((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)) \downarrow (\mathbb{CP}^\infty)^{\times 2}_E. \square$$

*Remark 62.* Since we used the product map

[1] In the language of last section, we are making use of the Hopf ring $\circ$–product.

$$kU^2(Y) \otimes kU^2(Y) \to kU^4(Y)$$

in the course of the proof, it is also interesting to consider the product map

$$kU^4(Y) \otimes kU^0(Y) \to kU^4(Y).$$

Taking one of our splitting summands $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)$ and acting by some line bundle $\mathcal{H}$ gives

$$\begin{gathered}(\mathcal{L}_1 - 1)\mathcal{H}(\mathcal{L}_2 - 1) = \\ (\mathcal{H}\mathcal{L}_1 - \mathcal{H})(\mathcal{L}_2 - 1) = (\mathcal{L}_1 - 1)(\mathcal{H}\mathcal{L}_2 - \mathcal{H}) \\ (\mathcal{H}\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) - (\mathcal{H} - 1)(\mathcal{L}_2 - 1) = (\mathcal{L}_1 - 1)(\mathcal{H}\mathcal{L}_2 - 1) - (\mathcal{L}_1 - 1)(\mathcal{H} - 1).\end{gathered}$$

I think it's more symmetric if you write $\mathcal{L}_1\mathcal{H}$ instead of $\mathcal{H}\mathcal{L}_1$.

This "$kU^0$–linearity" is sometimes called a "2–cocycle condition", in reference to the similarity with the formula in Definition 46.

If we can show that $E_*BSU$ is even–concentrated and free as an $E_0$–module, then this will complete the $BSU$–analogue of Theorems 64 and 65. This is quite easy, following directly from the Serre spectral sequence:

**Lemma 71 ([3, Lemma 6.1]).** *The Postnikov fibration*

$$BSU \to BU \xrightarrow{B\det} BU(1)$$

*induces a short exact sequence of Hopf algebras*

$$E^0BSU \leftarrow E^0BU \xleftarrow{c_1 \leftarrow c_1} E^0BU(1). \square$$

**Corollary 47.** *The functor*

$$\{\mathrm{Spec}\, T \xrightarrow{u} \mathrm{Spec}\, E_0\} \to \{\textit{trivializations of } u^*\mathbb{L}((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)) \textit{ over } u^*\mathbb{C}\mathrm{P}^\infty_E\}$$

*is isomorphic to the affine scheme* $\mathrm{Spec}\, E_0MSU$*. Moreover, the* $E_0$*–points of this scheme biject with ring spectrum maps* $MSU \to E$. $\square$

However, the use of Lemma 71 inspires us to spend a moment longer with the associated formal schemes. An equivalent statement is that there is a short exact sequence of formal group schemes

$$\begin{tikzcd} BSU_E \arrow[r] \arrow[d, equal] & BU_E \arrow[r, "B\det"] \arrow[d, equal] & BU(1) \arrow[d, equal] \\ \mathrm{SDiv}_0\, \mathbb{C}\mathrm{P}^\infty_E \arrow[r] & \mathrm{Div}_0\, \mathbb{C}\mathrm{P}^\infty_E \arrow[r, "\mathrm{sum}"] & \mathbb{C}\mathrm{P}^\infty_E, \end{tikzcd}$$

where the scheme "$\mathrm{SDiv}_0\, \mathbb{C}\mathrm{P}^\infty_E$" of "special divisors" consists of those divisors which vanish under the summation map. However, where the com-

Is the sum map induced by the diagonal on $\mathbb{C}P^\infty$ or the multiplication on $E$ – does it matter? Also, is it clear why the right square in the diagram commutes? Does summation mean anything geometrically? What does having zero sum mean? The group of divisors is free abelian (before inverting the pointing), so wouldn't each point in the divisor have to be the basepoint in order for the sum to be zero?

parison map $BU(1)_E \to BU_E$ has an identifiable universal property — it presents $BU_E$ as the universal formal group on the pointed curve $BU(1)_E$ — the description of $BSU_E$ as a scheme of special divisors does not bear much immediate resemblance to a free object on the special divisor $([a]-[0])([b]-[0])$ classified by

$[0] = 1?$

$$(\mathbb{C}P^\infty)_E^{\times 2} \xrightarrow{(\mathcal{L}_1-1)(\mathcal{L}_2-1)_E} BSU_E \to BU_E = \operatorname{Div}_0 \mathbb{C}P^\infty_E.$$

It would be wise of us to straighten this out before moving on.

**Definition 72.** If it exists, let $C_2\widehat{\mathbb{G}}$ denote the symmetric square of $\operatorname{Div}_0 \widehat{\mathbb{G}}$, thought of as a module over $\operatorname{Div} \widehat{\mathbb{G}}$. This scheme has the property that a formal group homomorphism $\varphi\colon C_2\widehat{\mathbb{G}} \to H$ is equivalent data to a symmetric function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying a rigidity condition $(\psi(x,0) = 0)$and a 2–cocycle condition as in Remark 62.

Allen wanted to know why this deserved to be called "rigid".

**Theorem 66 (Ando–Hopkins–Strickland, unpublished).** *$\operatorname{SDiv}_0 \widehat{\mathbb{G}}$ is a model for $C_2\widehat{\mathbb{G}}$.*

**Cite me:** This is Prop 3.2 of the AHS preprint or Prop 2.13 of Strickland's FSKS preprint.

*Proof (Proof sketch).* Consider the map

$$\begin{aligned} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} &\to \operatorname{Div}_0 \widehat{\mathbb{G}}, \\ (a,b) &\mapsto ([a]-[0])([b]-[0]) \end{aligned}$$

for which there is a factorization of formal schemes

$$\begin{tikzcd} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \arrow[d, dotted] \arrow[dr] & & \\ F \arrow[r, "\ker"] & \operatorname{Div}_0 \widehat{\mathbb{G}} \arrow[r, "\sigma"] & \widehat{\mathbb{G}} \end{tikzcd}$$

because

$$\sigma(([a]-[0])([b]-[0])) = (a+b) - a - b + 0 = 0.$$

One can check that a homomorphism $F \to H$ pulls back to a function $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying the properties of Definition 72. To go the other way[2], we select a function $\psi\colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ and mimic the construction in Lemma 70. Expanding the definition of $\operatorname{Div}_0 \widehat{\mathbb{G}}$, we are moved to consider the object $\widehat{\mathbb{G}}^{\times k}$ parametrizing weight $k$ divisors with a full set of sections, where we define a map

You can say this better, right? Do you need the thing to have a full set of sections to define this map? Probably not...

[2] To get insight into how this part of the proof works, actually write out the expressions for $\tilde{\tilde{V}} = \bigoplus_{i=1}^{6} \mathcal{L}_i - 6 \cdot 1 = \bigoplus_{i=1}^{3}(\mathcal{L}_i - 1) \oplus \bigoplus_{j=4}^{6}(\mathcal{L}_j - 1)$ and see what happens.

$$\widehat{\mathbb{G}}^{\times k} \to H,$$
$$(a_1, \ldots, a_k) \mapsto -\sum_{j=2}^{k} \psi\left(\sum_{i=1}^{j-1} a_i, a_j\right).$$

This gives a compatible system of symmetric maps and hence bundles together to give a map $\tilde{\varphi}\colon \operatorname{Div}_0 \widehat{\mathbb{G}} \to H$ off of the colimit. In general, this map is not a homomorphism, but it is a homomorphism when restricted to $\varphi\colon F \to \operatorname{Div}_0 \widehat{\mathbb{G}} \to H$. Finally, one checks that any homomorphism $F \to H$ of formal groups restricting to the zero map $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ was already the zero map, and this gives the desired identification of $F$ with the universal property of $C_2\widehat{\mathbb{G}}$.

In the next interesting case of $\underline{kU}_6 = BU[6, \infty)$, there is not an accessible splitting principle. This not only makes the topology harder, but it also makes proving the existence of the symmetric cube $C_3\widehat{\mathbb{G}}$ harder, as there is no model to work from. Nonetheless, there is a $BU[6, \infty)$–analogue of Theorem 64, Theorem 65, and Corollary 47. In order to prove it, since we don't have access to an equivalence between viewpoints (1) and (2), we will have to instead prove an equivalence between (2) and (3) directly.

There's some stuff buried here about moving to the Thom spectrum after doing the analysis of the classifying space. I'm not sure where it lives just yet.

## 5.3 Apr 13: Elliptic curves and $\theta$–functions

The goal of this lecture should be to set up all the algebraic geometry we'll need, in a coherent-enough way that the students will be able to think back and at least mumble "yeah, ok, reasonable".

Today will constitute something of a résumé on elliptic curves. We'll hardly prove anything, and we also won't cover many topics that a sane introduction to elliptic curves would make a point to cover. Instead, we'll try to restrict attention to those concepts which will be of immediate use to us in the coming couple of lectures — in particular, we will discover a place where "$C_3\widehat{\mathbb{G}}$" appears internally to the theory of elliptic curves.

To begin with, recall that an elliptic curve in the complex setting is a torus, and it admits a presentation by selecting a lattice $\Lambda$ of full rank in $\mathbb{C}$ and forming the quotient
$$\mathbb{C} \xrightarrow{\pi_\Lambda} E_\Lambda = \mathbb{C}/\Lambda.$$
The meromorphic functions $f$ on $E_\Lambda$ pull back to give meromorphic functions $\pi_\Lambda^* f$ on $\mathbb{C}$ satisfying a periodicity constraint in the form of the functional equation
$$\pi_\Lambda^* f(z + \Lambda) = \pi_\Lambda^* f(z).$$
From this, it follows that there are no holomorphic such functions, save the constants — such a function would be bounded, and Liouville's theorem

would apply. It is, however, possible to build the following meromorphic special function, which has poles of order 2 at the lattice points:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

Its derivative is also a meromorphic function satisfying the periodicity constraint:

$$\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}.$$

In fact, these two functions generate all other meromorphic functions on $E_\Lambda$, in the sense that the subsheaf spanned by the algebra generators $\wp_\Lambda$ and $\wp'_\Lambda$ is exactly $\pi^*_\Lambda \mathcal{M}_{E_\Lambda}$. This algebra is subject to the following relation, in the form of a differential equation:

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda),$$

for some special values $g_2(\Lambda)$ and $g_3(\Lambda)$. Accordingly, writing $C \subseteq \mathbb{C}\mathrm{P}^2$ for the projective curve $wy^2 = 4x^3 - g_2(\Lambda)w^2x - g_3(\Lambda)w^3$, there is an analytic group isomorphism

$$\begin{aligned} E_\Lambda &\to C, \\ z \pmod{\Lambda} &\mapsto [1 : \wp_\Lambda(z) : \wp'_\Lambda(z)]. \end{aligned}$$

This is sometimes referred to as the Weierstrass presentation of $E_\Lambda$.

There is a second standard embedding of a complex elliptic curve into projective space, using *$\theta$–functions*, which are most naturally expressed *multiplicatively*. To begin, select a lattice $\Lambda$ and a basis for it, and rescale the lattice so that the basis takes the form $\{1, \tau\}$ with $\tau$ in the upper half-plane. Then, the normalized exponential function $z \mapsto \exp(2\pi i z)$ has $1 \cdot \mathbb{Z}$ as its kernel, and setting $q = \exp(2\pi i \tau)$ we get a second presentation of $E_\Lambda$ as $\mathbb{C}^\times / q^{\mathbb{Z}}$. I think it's helpful to draw a picture here of an annulus with some identification made.

The associated $\theta$–function is defined by

$$\theta_q(u) = \prod_{m \geq 1} (1 - q^m)(1 + q^{m - \frac{1}{2}} u)(1 + q^{m - \frac{1}{2}} u^{-1}) = \sum_{n \in \mathbb{Z}} u^n q^{\frac{1}{2} n^2}.$$

It vanishes on the set $\{\exp(2\pi i (\frac{1}{2} m + \frac{\tau}{2} n))\}$, i.e., at the center of the fundamental annulus. However, since it has no poles it cannot descend to give a function on $\mathbb{C}^\times / q^{\mathbb{Z}}$. A different obstruction to this descent is its imperfect periodicity relation:

$$\theta_q(qu) = u^{-1} q^{\frac{-1}{2}} \theta_q(u).$$

We can also shift the zero-set of $\theta_q$ by rational rescalings $a$ of $q$ and $b$ of 1: This isn't stated well.

$$\theta_q^{a,b}(u) = q^{\frac{a^2}{2}} \cdot u^a \cdot \exp(2\pi iab)\theta_q(uq^a \exp(2\pi ib)).$$

*Remark 63 ([17, Proposition 10.2.6]).* For any $N > 0$, define $V_\tau[N]$ to be the space of entire functions $f$ with $f(z+N) = f(z)$ and $f(z+\tau) = e^{-2\pi iNz - \pi iN^2\tau} f(z)$. Then, $V_\tau[N]$ has $\mathbb{C}$–dimension $N^2$, and the functions $\theta_\tau^{a,b}$ give a basis by picking representatives $(a_i, b_i)$ of the classes in $((1/N)\mathbb{Z}/\mathbb{Z})^2$.

At this point you swapped notation from $\theta_q$ to $\theta_\tau$.

Even though these functions do not themselves descend to $\mathbb{C}^\times/q^\mathbb{Z}$, we can collectively use them to construct a map to complex projective space, where the quasi-periodicity relations will mutually cancel in homogeneous coordinates.

**Theorem 67 ([17, Proposition 10.3.2]).** *Consider the map*

$$\begin{aligned} \mathbb{C}/N(\Lambda) &\xrightarrow{f_{(N)}} \mathbb{P}^{N^2-1}(\mathbb{C}), \\ z &\mapsto [\cdots : \theta_\tau^{i/N, j/N}(z) : \cdots]. \end{aligned}$$

What's $N(\Lambda)$?

*For $N > 1$, this map is an embedding.* □

*Example 31.* One can work out how it goes for $N = 2$, which will cause some of our pesky $\frac{1}{2}$'s to cancel. The four functions there are $\theta_q^{0,0}$ with zeroes on $\Lambda + \frac{\tau+1}{2}$, $\theta_q^{0,1/2}$ with zeroes on $\Lambda + \frac{\tau}{2}$, $\theta_q^{1/2,0}$ with zeroes on $\Lambda + \frac{1}{2}$, and $\theta_q^{1/2,1/2}$ with zeroes on $\Lambda$ exactly. The image of $f_{(2)}$ in $\mathbb{P}^{2^2-1}(\mathbb{C})$ is cut out by the equations

$$A^2x_0^2 = B^2x_1^2 + C^2x_2^2, \qquad A^2x_3^2 = C^2x_1^2 - B^2x_2^2,$$

where

$$x_0 = \theta_\tau^{0,0}(2z), \quad x_1 = \theta_\tau^{0,1/2}(2z), \quad x_2 = \theta_\tau^{1/2,0}(2z), \quad x_3 = \theta_\tau^{1/2,1/2}(2z)$$

and

$$A = \theta_\tau^{0,0}(0) = \sum_n q^{n^2}, \quad B = \theta_\tau^{0,1/2}(0) = \sum_n (-1)^n q^{n^2}, \quad C = \theta_\tau^{1/2,0}(0) = \sum_n q^{(n+1/2)^2}$$

upon which there is the additional "Jacobi" relation

$$A^4 = B^4 + C^4.$$

*Remark 64.* This embedding of $E_\Lambda$ as an intersection of quadric surfaces in $\mathbb{C}\mathrm{P}^3$ is quite different from the Weierstrass embedding. Nonetheless, the embeddings are analytically related. Namely, there is an equality

$$\frac{d^2}{dz^2} \log \theta_{\exp 2\pi i\tau}(\exp 2\pi iz) = \wp_\Lambda(z).$$

Separately, Weierstrass considered a function $\sigma_\Lambda$, defined by

$$\sigma_\Lambda(z) = z \prod_{\omega \in \Lambda \setminus 0} \left(1 - \frac{z}{\omega}\right) \cdot \exp\left[\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right],$$

which also has the property that its second logarithmic derivative is $\wp$ and so is "basically $\theta_q^{1/2,1/2}$". In fact, any elliptic function can be written in the form

$$c \cdot \prod_{i=1}^n \frac{\sigma_\Lambda(z-a_i)}{\sigma_\Lambda(z-b_i)}.$$

The $\theta$–functions version of the story has two main successes. One is that there is a version of this story for an arbitrary abelian variety. It turns out that all abelian varieties are projective, and the theorem sitting at the heart of this claim is

What's the other success?

**Cite me:** Milne's abelian varieties, Theorem 7.1.

**Corollary 48 (Theorem of the cube).** *Let $A$ be an abelian variety, let $p_i : A \times A \times A \to A$ be the projection onto the $i^{th}$ factor, and let $p_{ij} = p_i +_A p_j$, $p_{ijk} = p_i +_A p_j +_A p_k$. Then for any invertible sheaf $\mathcal{L}$ on $A$, the sheaf*

Several people wanted some sketch of how to use this theorem to prove the projectivity thing. I don't have any good sketch, and neither did Erick (offhand).

**Cite me:** Milne's Abelian Varieties chapter, Corollary 6.4 and Theorem 7.1.

$$\Theta^3(\mathcal{L}) := \frac{p_{123}^*\mathcal{L} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}}{p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_{31}^*\mathcal{L} \otimes p_\varnothing^*\mathcal{L}} = \bigotimes_{I \subseteq \{1,2,3\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}$$

*on $A \times A \times A$ is trivial. If $\mathcal{L}$ is rigid, then $\Theta^3(\mathcal{L})$ is canonically trivialized by a section $s(A;\mathcal{L})$.* □

*Remark 65.* The section $s(A;\mathcal{L})$ satisfies three familiar properties:

- It is symmetric: pulling back $\Theta^3\mathcal{L}$ along a shuffle automorphism of $A^3$ yields $\Theta^3\mathcal{L}$ again, and the pullback of the section $s(A;\mathcal{L})$ along this shuffle agrees with the original $s(A;\mathcal{L})$ across this identification.
- It is rigid: by restricting to $* \times A \times A$, the tensor factors in $\Theta^3\mathcal{L}$ cancel out to give the trivial bundle over $A \times A$. The restriction of the section $s(A;\mathcal{L})$ to this pullback bundle agrees with the extension of the rigidifying section.

What is the rigidifying section?

- It satisfies a 2–cocycle condition: in general, we define

$$\Theta^k\mathcal{L} := \bigotimes_{I \subseteq \{1,\ldots,k\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}.$$

  In fact, $\Theta^{k+1}\mathcal{L}$ can be written as a pullback of $\Theta^k\mathcal{L}$:

$$\Theta^{k+1}\mathcal{L} = \frac{(p_{12} \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}}{(p_1 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L} \otimes (p_2 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}},$$

  and pulling back a section $s$ along this map gives a new section

$$(\delta s)(x_0, x_1, \ldots, x_k) := \frac{s(x_0 +_A x_1, x_2, \ldots, x_k)}{s(x_0, x_2, \ldots, x_k) \cdot s(x_1, x_2, \ldots, x_k)}.$$

Performing this operation on the first and second factors yields the defining equation of a 2–cocycle.

You should write this equation out – it never hurts to see it again.

*Remark 66.* The proof of projectivity arising from this method rests on choosing an ample line bundle on $A$ and constructing some generating global sections to get an embedding into $\mathbb{P}(\mathcal{L}^{\oplus n})$. Mumford showed that a choice of "$\theta$–structure" on $A$, which is only slightly more data given in terms of Heisenberg representations, gives a canonical identification of $\mathbb{P}(\mathcal{L}^{\oplus n})$ with a *fixed* projective space. This is suitable for studying how these equations change as one considers different points in the *moduli* of abelian varieties. Separately, Breenshowed that if $\mathcal{L}$ is a line bundle on $A$ with a chosen trivialization of $\Theta^3\mathcal{L}$ and $\pi\colon A' \to A$ is an epimorphism that trivializes $\mathcal{L}$, then one can also associate to this a theory of $\theta$–functions.

This is a little strange to say: the existence of an ample line bundle is by definition equivalent to projectivity, so we're not so much "choosing" as we are "constructing" such a line bundle using the theorem of the cube.

I think.

**Cite me:** Breen.

You could expand on what this is supposed to mean.

Akhil's class notes mention a "very easy" proof for the theorem of the cube for complex varieties at the end of lecture 5. Consider fleshing out that argument and see whether it does what you need.

## 5.4 Apr 15: Unstable chromatic cooperations for $kU$

Let $\Gamma$ be a formal group of finite $p$–height of a field $k$ of positive characteristic $p$, and let $E = E_\Gamma$ denote the associated Morava $E$–theory. Our goal in this section is to get a partial description of the Hopf ring of unstable cooperations $(E_\Gamma)_* \underline{kU}_{2*}$. Our results in previous sections give a foothold into this analysis by computing

$$E_0(BU \times \mathbb{Z}),\ E_0 BU,\ E_0 BSU$$

in terms of the affine schemes they represent. We also saw that these results were the cornerstone for accessing descriptions of the schemes

$$\operatorname{Spec} E_0 MUP,\ \operatorname{Spec} E_0 MU,\ \operatorname{Spec} E_0 MSU.$$

In particular, the next step is to understand $E_0 BU[6, \infty)$, and our main tool for doing this will be the Postnikov fibration

$$\underline{H\mathbb{Z}}_3 \to BU[6, \infty) \to BSU.$$

Our main goals are to construct a model sequence of formal schemes, then show that $E$–theory is well-behaved enough that the formal schemes it constructs exactly match the model.

The main tool used to build the model is the following construction:

**Definition 73.** A map $f\colon X \to Y$ of spaces induces a map $f_E\colon X_E \to Y_E$ of formal schemes. In the case that $Y$ is a commutative $H$–space and $Y_E$ is connected, we can construct a map according to the composite

$$\begin{tikzcd}
X_E \times \underline{\mathsf{GroupSchemes}}(Y_E, \mathbb{G}_m) \arrow[rrr, dotted] \arrow[d, equal] & & & \widehat{\mathbb{A}}^1 \\
X_E \times \underline{\mathsf{FormalGroups}}(Y_E, \widehat{\mathbb{G}}_m) \arrow[r, "f \times 1"] & Y_E \times \underline{\mathsf{FormalGroups}}(Y_E, \widehat{\mathbb{G}}_m) \arrow[rr, "\mathrm{ev}"] & & \widehat{\mathbb{G}}_m. \arrow[u]
\end{tikzcd}$$

This is called *the adjoint map*, and we write $\hat{f}$ for the version of this map valued variously in $\widehat{\mathbb{G}}_m$, $\mathbb{G}_m$, and $\widehat{\mathbb{A}}^1$. It encodes equivalent information to the map of $E_*$–modules

How do you recover the adjoint map and vice versa?

$$E_* \to E_* Y \widehat{\otimes}_{E_*} E^* X$$

by applying the map to $1 \in E_*$.

What's with the completed tensor product? Also, this is just the dual of the map $E_* X \to E_* Y$?

**Lemma 72.** *This construction converts many properties of $f$ into corresponding properties of this adjoint element. For instance:*

I think this could be connected more strongly to the cooperations stuff in the previous Chapter.

- *It is natural in the source: for $w \in F^n(X)$ and $\gamma\colon \underline{E}_n \to \underline{D}_n$, there is*

  $$(1 \times \operatorname{Spec} E_0\gamma) \circ \hat{w} = \widehat{\gamma_* w}.$$

Expand on these, perhaps.

- *It converts sums of classes to products of maps to $\mathbb{G}_m$.* $\square$

This is confusing. You write $\hat{w}$, but this is not the adjoint map $X_E \times \underline{\text{GroupSchemes}}((\underline{E}_n)_E, \mathbb{G}_m) \to \mathbb{G}_m$, but rather the "equivalent" map $E_* \to E_* \underline{E}_n \otimes E^* X$. In that case, moreover, the formula should be $(\operatorname{Spec} E_*\gamma \otimes 1) \circ \hat{w} = \widehat{\gamma_* w}$.

What about naturality in the target?

In Lecture 5.2, we became interested in the class $\Pi_2$, defined by

$$\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{\Pi_2 := (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)} \underline{kU}_4 = BSU.$$

The adjoint to this cohomology class is a map of formal schemes

$$\hat{\Pi}_2\colon (\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \times_{\mathrm{S}_E} \operatorname{Spec} E_0 BSU \to \mathbb{G}_m,$$

which using the exponential adjunction can be interpreted as a map

$\mathrm{S}_E$ is just $\operatorname{Spec} E_*$?

$$\operatorname{Spec} E_0 BSU \to \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m).$$

Because the adjoint construction preserves properties of the class $\Pi_2$, we learn that this map factors through a particular closed subscheme

$$C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \subseteq \underline{\mathsf{FormalSchemes}}((\mathbb{C}\mathrm{P}^\infty_E)^{\times 2}, \mathbb{G}_m)$$

of symmetric, rigid functions satisfying the 2–cocycle condition. By careful manipulation of divisors in Theorem 66, we showed that $BSU_E \cong \mathrm{SDiv}_0\, \mathbb{C}\mathrm{P}^\infty_E$, which on applying Cartier duality shows that our induced map

Cartier duality seems quite important to these examples. Could you say more about this construction, either here or on the day about Dieudonné modules?

$$\operatorname{Spec} E_0 BSU \to C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m)$$

is an isomorphism.

> This is not really a definition…

**Definition 74.** Similarly, we define a cohomology class

$$\Pi_3 = (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)(\mathcal{L}_3 - 1) \in kU^6(\mathbb{CP}^\infty)^{\times 3}.$$

> Is this bad notation?

It induces an adjoint map

$$\hat{\Pi}_3 \colon \operatorname{Spec} E_0 BU[6, \infty) \to C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m),$$

where $C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m)$ is the scheme of $\mathbb{G}_m$–valued trivariate functions on $\mathbb{CP}^\infty_E$ satisfying symmetry, rigidity, and a 2–cocycle condition. (If $C_3\mathbb{CP}^\infty_E := \operatorname{Sym}^3_{\operatorname{Div}\mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E$ were to exist, this would be its Cartier dual.)

> Where did $\delta$ come from? A previous day? Talk about this some, probably.

> Jun Hou asked if this map fits into a larger chain complex — presumably because of the name "$\delta$". The problem is that $\delta$ is always bound to moving from 1–cocycles to 2–cocycles. But maybe "$\tau$" is worth mentioning further on...

**Lemma 73 ([3, Lemma 7.1]).** *There is a commutative square*

$$\begin{tikzcd}
\operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\Pi_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[d, "\Pi_3"] \\
C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m),
\end{tikzcd}$$

*where*

$$\delta(f)(x_1, x_2, x_3) := \frac{f(x_1 +_E x_2, x_3)}{f(x_1, x_3) f(x_2, x_3)}.$$

> You wrote $\Pi_2$ and $\Pi_3$, but these are not exactly the maps $\Pi_2$ and $\Pi_3$. They're not even the adjoint maps $\hat{\Pi}_2$ and $\hat{\Pi}_3$. I don't think you have any other options since introducing even more notation would be cumbersome, but perhaps you could make a small remark when you write down the induced maps that you're also calling them $\Pi_2$ and $\Pi_3$.

*Proof.* This is checked by calculating $\Pi_3 = (\mu_{12}^* - \pi_1^* - \pi_2^*)\Pi_2$.

With this now in hand, we have constructed the solid maps in the following diagram:

$$\begin{tikzcd}
\operatorname{Spec} E_0 BSU \arrow[r] \arrow[d, "\cong"', "\Pi_2"] & \operatorname{Spec} E_0 BU[6,\infty) \arrow[r] \arrow[d, "\Pi_3"] & \operatorname{Spec} E_0 \underline{H\mathbb{Z}_3} \arrow[d, "\cong"'] \\
C^2(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\mathbb{CP}^\infty_E; \mathbb{G}_m) \arrow[r, dotted, "e_*"] & \underline{\mathsf{FormalGroups}}((\mathbb{CP}^\infty_E)^{\wedge 2}, \widehat{\mathbb{G}}_m)
\end{tikzcd}$$

> The notation $\mu$ and $\pi$ does not comport with that from the previous lecture.

We would like to prove enough about this diagram to show that is an isomorphism of short exact sequences.

> You haven't said anything about why the rightmost vertical map is an isomorphism.

Before we begin testing exactness, we first need a pair of sequences — i.e., we must construct the map $e$. There is a candidate construction, coming from the theory of $\theta$–functions:

> Was there any motivation or reason why one might look to elliptic function theory to supply this map?

**Definition 75.** Let $A$ be an abelian variety equipped with a line bundle $\mathcal{L}$. Suppose that $s$ is a symmetric, rigid section of $\Theta^3\mathcal{L}$, i.e., a *cubical structure on* $\mathcal{L}$. This induces the structure of a *symmetric biextension* on $\Theta^2\mathcal{L}$ by furnishing compatible multiplication maps

> Hood wanted me to write out what the relevant pullbacks were, and I pretty well refused. We should here, at least. The point is that $(p_{12} - p_1 - p_2)^*\Theta^2\mathcal{L} = \Theta^3\mathcal{L}$.

> You might also want to give a second definition to the alternating-fraction one, using something like the tensor product of reduced line bundles, so that the link to the topology is more directly understood.

$$(\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x',y} \to (\Theta^2\mathcal{L})_{x+x',y}, \quad (\Theta^2\mathcal{L})_{x,y} \otimes (\Theta^2\mathcal{L})_{x,y'} \to (\Theta^2\mathcal{L})_{x,y+y'}.$$

There is a canonical piece of gluing data on this biextension, in the form of

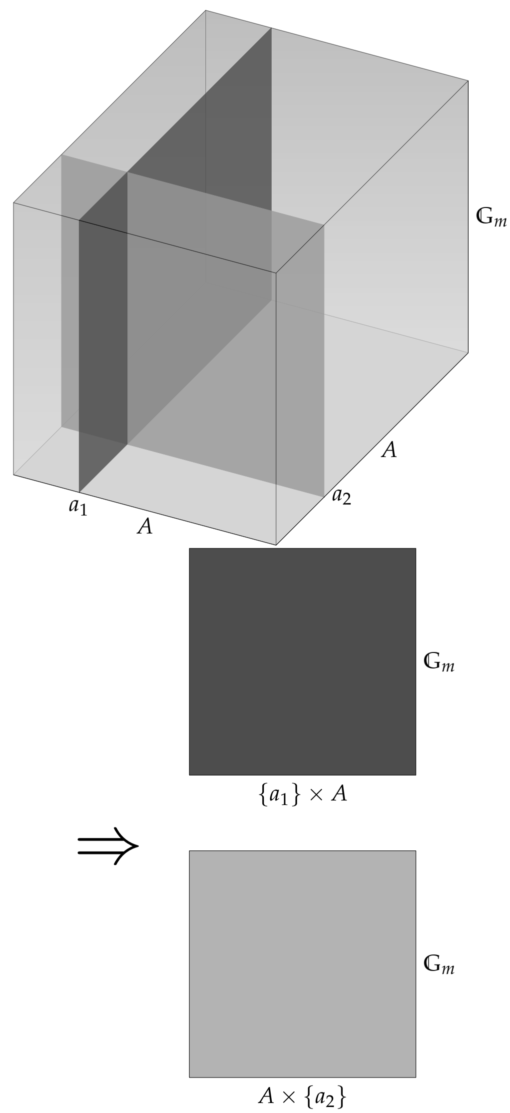



**Fig. 5.1** Extensions contained in a biextension.

an isomorphism of pullback bundles

$$\begin{aligned} e_{p^j}\colon\ (p^j \times 1)^*\mathcal{L}|_{A[p^j]\times A[p^j]} &\cong (1\times p^j)^*\mathcal{L}|_{A[p^j]\times A[p^j]}, \\ (\ell, x, y) &\mapsto \left(\ell \cdot \prod_{k=1}^{p^j-1} \frac{s(x,[k]x,y)}{s(x,[k]y,y)}\right). \end{aligned}$$

This function $e_{p^j}$ is called the *($p^{jth}$) Weil pairing*.

Can you prove some parts of this more carefully?

*Remark 67.* In the case that $A$ is an elliptic curve, this agrees with the usual definition of its "Weil pairing".

Cite me: AS, I guess.

**Lemma 74.** *The Weil pairings assemble into a total Weil pairing on the p–divisible group associated to A. Together, the total Weil pairing is alternating and biexponential.* □

What does biexponential mean?

We can use the same formula in the setting of a cubical structure on a line bundle over a finite height formal group $\widehat{\mathbb{G}}$ to produce the desired map

$$e_*\colon\ C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2}, \mathbb{G}_m).$$

How? What is $\ell$?

In fact, this is the right map:

**Lemma 75 ([3, Theorem 4.2, Corollary 4.4]).** *The square commutes (up to sign):*

$$\begin{array}{ccc} \operatorname{Spec} E_0 BU[6,\infty) & \longrightarrow & \operatorname{Spec} E_0 K(\mathbb{Z},3) \\ \big\downarrow{\scriptstyle \Pi_3} & & \big\downarrow{\scriptstyle b_*} \\ C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m) & \xrightarrow{\quad e \quad} & \mathsf{Weil}(\mathbb{CP}^\infty_E). \end{array}$$

You have not introduced the notation Weil.

*Proof (Proof sketch).* This is reasonably difficult. The main points are to show that the restriction

$$d_{p^j}\colon\ BC_{p^j}^{\times 2} \xrightarrow{\beta\circ\mu} \underline{H\mathbb{Z}}_3 \to BU[6,\infty)$$

can be expressed by the Weil pairing formula:

$$d_{p^j} = \bigoplus_{k=1}^{p^j-1} \Big((\mathcal{L}_1 - 1)(\mathcal{L}_1^{\otimes k} - 1)(\mathcal{L}_2 - 1) - (\mathcal{L}_1 - 1)(\mathcal{L}_2^{\otimes k} - 1)(\mathcal{L}_2 - 1)\Big).$$

Where does $d_{p^j}$ appear in the diagram above?

After this is accomplished, what's left is to use naturality properties of the adjoint construction to compute the clockwise and counterclockwise composites.

This fills out the diagram we are considering. We now assemble just enough exactness results:

**Lemma 76 ([3, Lemma 7.2]).** *The map $\delta\colon C^2 \to C^3$ is injective for $\mathbb{C}\mathrm{P}^\infty_E$ a finite height formal group.*

*Proof.* Being finite height means that the multiplication-by-$p$ map of $\mathbb{C}\mathrm{P}^\infty_E$ is fppf–surjective. The kernel of $\delta$ consists of alternating, biexponential maps $(\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \to \mathbb{G}_m$. By restricting such a map $f$ to

$$f\colon \mathbb{C}\mathrm{P}^\infty_E[p^j] \times \mathbb{C}\mathrm{P}^\infty_E \to \mathbb{G}_m,$$

we can calculate

$$f(x, p^j y) = f(p^j x, y) = f(0, y) = 1.$$

But since $p^j$ is surjective on $\mathbb{C}\mathrm{P}^\infty_E$, every point on the right-hand side can be so written, so at every left-hand stage the map is trivial. Finally, $\mathbb{C}\mathrm{P}^\infty_E = \operatorname{colim}_j \mathbb{C}\mathrm{P}^\infty_E[p^j]$, so this filtration is exhaustive and we conclude that the kernel is trivial.

This doesn't sound like the proof that you and Hood gave in class? Is that a different proof?

**Lemma 77 ([3, Lemma 7.3]).** *In fact, the following sequence is exact*

$$0 \to C^2(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \xrightarrow{\delta} C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m) \to \mathrm{Weil}(\mathbb{C}\mathrm{P}^\infty_E).$$

*Proof.* This is hard work. Breen's idea is to show that picking a preimage under $\delta$ is the same as picking a trivialization of the underlying symmetric biextension of the cubical structure. Then (following Mumford), one shows that the underlying symmetric biextension is trivial exactly if the Weil pairing is trivial.

You could put in references here. AS gives them.

Finally, the top row falls quickly:

**Lemma 78 ([3, Lemma 7.5]).** *The top row of the main diagram is a short exact sequence of group schemes.*

Remark: This proof also shows that the $E$–theory of $\underline{kU}_8$ fits into a sexseq.

*Proof.* This is easiest proved by considering the sequence of homology Hopf algebras instead. Since the integral homology of $BSU$ and the $E$–homology of $\underline{H\mathbb{Z}}_3$ are both free and even, the Atiyah–Hirzebruch spectral sequence for $E_* BU[6, \infty)$ collapses.

**Corollary 49.** *The map*

$$\Pi_3\colon \operatorname{Spec} E_0 BU[6, \infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E; \mathbb{G}_m)$$

*is an isomorphism, and the map*

$$e_* \colon C^3(\mathbb{C}P^\infty_E; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}((\mathbb{C}P^\infty_E)^{\wedge 2}, \mathbb{G}_m)$$

*is a surjection.*

*Proof.* This is a direct consequence of the 5–lemma.

## 5.5 Apr 18: Unstable additive cooperations for $kU$

Write $H = H\mathbb{F}_p$. Today we will study the effect of the map $\hat{\Gamma}_3$ in ordinary homology. Many parts of the proof we explored for $E$–theory break. Topologically, the Serre spectral sequence for $H^*BU[6,\infty)$ is not even-concentrated and so is not forced to collapse. Algebraically, because $\widehat{\mathbb{G}}_a$ is not $p$–divisible the behavior of the model exact sequence is also suspect. Because the situation has fewer insulating good properties, we are forced to actually consider it carefully. The upside, however, is that the standard group law on $\widehat{\mathbb{G}}_a$ is simple enough that we can compute the problem to death.

We begin with the topological half of our tasks. The Serre spectral sequence

$$E_2^{*,*} = H\mathbb{F}_p^* BSU \otimes H\mathbb{F}_p^* \underline{H\mathbb{Z}}_3 \Rightarrow H\mathbb{F}_p^* BU[6,\infty)$$

is quite accessible, and we will recount the case of $p = 2$. In this case, the spectral sequence has $E_2$–page

$$E_2^{*,*} = H\mathbb{F}_2^* BSU \otimes H\mathbb{F}_2^* \underline{H\mathbb{Z}}_3 \cong \mathbb{F}_2[c_2, c_3, \ldots] \otimes \mathbb{F}_2 \left[ \mathrm{Sq}^I \iota_3 \,\middle|\, \begin{array}{c} I_j \geq 2I_{j+1}, \\ 2I_1 - I_+ \end{array} \right].$$

$2I_1 - I_+$ doesn't look like a condition. Also, what is $I_+$? Somehow this should mean that you start with $\mathrm{Sq}^2$ not $\mathrm{Sq}^1$.

Because the target is 6–connective, we must have the transgressive differential $d_4 \iota_3 = c_2$, which via the Kudo transgression theorem spurs the much larger family of differentials

$$d_{4+I_+} \mathrm{Sq}^I \iota_3 = \mathrm{Sq}^I c_2.$$

**Cite me:** Wu formulas, maybe May's concise book.

Can these formulas be read off from the divisorial calculation? Maybe not, since it's easy to read off the Milnor primitives but harder to see the Steenrod squares.

This necessitates understanding the action of the Steenrod operations on the cohomology of $BSU$, which is due to Wu:

$$\mathrm{Sq}^{2^j} \cdots \mathrm{Sq}^4 \mathrm{Sq}^2 c_2 = c_{1+2^j}.$$

This spectral sequence can be drawn in using Hood's package.

Accounting for the squares of classes left behind, this culminates in the following calculation:

**Theorem 68.** *There is an isomorphism*

$$H\mathbb{F}_2^* BU[6,\infty) \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid j \neq 2^k+1, j \geq 3)} \otimes F_2[\iota_3^2, (\mathrm{Sq}^2 \iota_3)^2, \ldots]. \square$$

**Theorem 69.** *More generally, there is an isomorphism*

Cite me: Stong, Singer.

$$H\mathbb{F}_2^* \underline{kU}_{2k} \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid \sigma_2(j-1) < k-1)} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-1}],$$

*where $\sigma_2$ is the dyadic digital sum and "Op" denotes the subalgebra of $H\mathbb{F}_2^* \underline{H\mathbb{Z}}_{2k-1}$ generated by the indicated class.*

*Proof (Remarks on proof).* Stong specialized to $p = 2$ and carefully applied the Serre spectral sequence to the fibrations

$$\underline{kU}_{2(k+1)} \to \underline{kU}_{2k} \to \underline{H\mathbb{Z}}_{2k}.$$

Singer worked at an arbitrary prime and used the Eilenberg–Moore spectral sequence for the fibrations

$$\underline{H\mathbb{Z}}_{2k-1} \to \underline{kU}_{2(k+1)} \to \underline{kU}_{2k}.$$

Both used considerable knowledge of the interaction of these spectral sequences with the Steenrod algebra.

*Remark 68.* These methods and results generalize directly to odd primes. The necessary modifications come from understanding the unstable mod–$p$ Steenrod algebra, using the analogues of Wu's formulas due to Shay, and employing Singer's Eilenberg–Moore calculation. Again, $H\mathbb{F}_p^* BU[6,\infty)$ is presented as a quotient by $H\mathbb{F}_p^* BU$ by certain Chern classes satisfying a $p$–adic sum condition, tensored up with the subalgebra of $H\mathbb{F}_p^* \underline{H\mathbb{Z}}_3$ generated by a certain element.

**Cite me:** Wu formulas, Shay's extension in *mod–p Wu formulas for the Steenrod algebra and the Dyer–Lashof algebra.*

*Remark 69.* We can already see from Theorem 68 that our map of short exact sequences in Lecture 5.4 does not have a full analogue in the setting of additive homology. By considering the edge homomorphism in the Serre spectral sequence, we see that

$$\operatorname{Spec} HP_0 BSU \to \operatorname{Spec} HP_0 BU[6,\infty)$$

is not a monomorphism.

Now, we turn to the algebra. The main idea, as already used in Lemma 51, is to first perform a tangent space calculation

Is this really the best example to reference?

$$T_0 C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \cong C^k(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a),$$

then study the behavior of the different tangent directions to determine the full object $C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$. As a warm-up, we will first consider the case $k = 2$:

**Corollary 50 (cf. Lemma 43).** *The unique symmetric* 2*–cocycle of homogeneous degree n has the form*

$$c_n(x,y) = \begin{cases} (x+y)^n - x^n - y^n & \text{if } n \neq p^j, \\ \frac{1}{p}\left((x+y)^n - x^n - y^n\right) & \text{if } n = p^j. \end{cases} \quad \square$$

Our goal, then, is to select such a 2–cocycle $f$ and study the minimal conditions needed on a symbol $a$ to produce a multiplicative 2–cocycle of the form $1 + af + \cdots$. Since $c_n = \frac{1}{d_n}\delta(x^n)$ is itself produced by an additive formula, life would be easiest if we had access to an exponential, so that we could build

The notation is a little odd, since you're conflating $x^n$ with the function $x \mapsto x^n$.

$$\text{“}\delta \exp(a_n x^n)^{1/d_n} = \exp(\delta a_n x^n / d_n) = \exp(a_n c_n).\text{”}$$

However, the existence of an exponential series is equivalent to requiring that $a_n$ have all fractions, which turns out not to be minimal. In fact, *no* conditions on $a_n$ are required *at all*, if we tweak the definition of an exponential series:

**Definition 76.** The *Artin–Hasse exponential* is the power series

Where does this come from? I've never learned a universal property for it. That bothers me. It must have something to do with $p$–typification.

$$E_p(t) = \exp\left(\sum_{j=0}^{\infty} \frac{t^{p^j}}{p^j}\right) \in \mathbb{Z}_{(p)}[\![t]\!].$$

This series has excellent properties, mimicking those of $\exp(t)$ as closely as possible while keeping coefficients in $\mathbb{Z}_{(p)}$ rather than in $\mathbb{Q}$. Writing $\delta\colon C^1 \to C^2$ and

$$d_n = \begin{cases} 1 & \text{if } n = p^j, \\ 0 & \text{otherwise,} \end{cases}$$

we set

$$g_n(x,y) := \delta E_p(a_n x^n)^{1/p^{d_n}} = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j}\, \delta x^{np^j}}{p^{j+d_n}}\right) = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} c_{np^j}(x,y)}{p^j}\right).$$

This gives a point in $C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)(\mathbb{Z}_{(p)}[a_n])$, and exhaustion of the tangent spaceproves the following Lemma:

Granting that this exhausts the tangent space, how do we recover the global scheme? Even if we had an inverse function theorem (like Thm. 2.1.4), I would only expect that to be at most a local isomorphism – unless these are all really formal schemes masquerading as ordinary schemes?

**Lemma 79 ([2, Proposition 3.9]).** *The map*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_n \mid n \geq 2] \xrightarrow{\prod_{n\geq 2} g_2} C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}$$

*is an isomorphism.* $\square$

The case $k = 3$ is similar, with one important new wrinkle. Over an $\mathbb{F}_2$–algebra there is an equality $c_n^2 = c_{2n}$. However, this relation does not generalize to trivariate 2–cocycles:

$$\frac{1}{2}\delta(c_6) = x^2y^2z^2 + x^4yz + xy^4z + xyz^4, \qquad \left(\frac{1}{2}\delta c_3\right)^2 = x^2y^2z^2.$$

This pattern is generic and exhaustive for $\mathbb{F}_p$–algebras:

**Lemma 80 ([2, Proposition 3.20, Proposition A.12]).** *The $p$–primary residue of the scheme of trivariate symmetric* 2*–cocycles is presented by*

$$\operatorname{Spec}\mathbb{F}_p[a_d \mid d \geq 3] \times \operatorname{Spec}\mathbb{F}_p[b_d \mid d = p^j(1+p^k)] \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a;\widehat{\mathbb{G}}_a) \times \operatorname{Spec}\mathbb{F}_p. \square$$

Similar juggling of the Artin–Hasse exponential yields the following multiplicative classification:

**Theorem 70 ([2, Proposition 3.28]).** *There is an isomorphism*

$$\operatorname{Spec}\mathbb{Z}_{(p)}[a_d \mid d \geq 3, d \neq 1+p^t] \times \operatorname{Spec}\Gamma[b_{1+p^t}] \to C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m) \times \operatorname{Spec}\mathbb{Z}_{(p)}.$$

*Proof (Proof sketch).* The main claim is that the Artin–Hasse exponential trick used in the case $k = 2$ works here as well, provided $d \neq 1 + p^t$ so that taking an appropriate $p^{\text{th}}$ root works out. They then show that the remaining exceptional cases extend to multiplicative cocycles only when the $p^{\text{th}}$ power of the leading coefficient vanishes. Finally, a rational calculation shows how to bind these truncated generators together into a divided power algebra.

In our pursuit of the map of exact sequences of Lecture 5.4, we are missing one piece: a link from topology to the scheme of Weil pairings, $\operatorname{Weil}(\widehat{\mathbb{G}}_a)$. The object "Spec $HP_0\underline{H\mathbb{Z}}_3$" is insuitable because it doesn't exist — the homology algebra $HP_0\underline{H\mathbb{Z}}_3$ is not even–concentrated. However, analyzing the edge homomorphism from our governing Serre spectral sequence shows that the map

$$HP^0BU[6,\infty) \to HP^0\underline{H\mathbb{Z}}_3$$

factors through the subalgebra $A$ generated by the *squares* of the polynomial generators. Accordingly, we aim to place Spec $A^*$ in the top-right corner of our map of

**Lemma 81 ([2, Lemma 3.36, Proposition 4.13, Lemma 4.11]).** *The scheme* Spec $A^*$ *models* $\operatorname{Weil}(\widehat{\mathbb{G}}_a)$ *by an isomorphism $\lambda$ commuting with $e \circ \hat{\Pi}_3$.*

*Proof (Proof sketch).* The $\mathbb{F}_p$–scheme $\operatorname{Weil}(\widehat{\mathbb{G}}_a)$ is simple to describe:

$$(a_{mn})_{m,n} \longmapsto \prod_{m<n} \operatorname{texp}\left(a_{mn}(x^{p^m}y^{p^n} - x^{p^n}y^{p^m})\right)$$

$$\operatorname{Spec} \mathbb{F}_p[a_{mn} \mid m < n]/(a_{mn}^p) \xrightarrow{\cong} \operatorname{Weil}(\widehat{\mathbb{G}}_a),$$

where $\operatorname{texp}(t) = \sum_{j=0}^{p-1} t^j/j!$ is the truncated exponential series. It is easy to check that this ring of functions agrees with $A^*$, and it requires hard work (although not much creativity) to check the remainder of the statement: that $e \circ \hat{\Pi}_3$ factors through $\operatorname{Spec} A^*$ and that the factorization is an isomorphism.

It's maybe not obvious to a reader why this sequence is exact in the middle, although you have secretly proven this in the mess above.

We have now finally assembled our map of right-exact sequences:

$$\begin{tikzcd}
\operatorname{Spec} HP_0 BSU \arrow[r] \arrow[d, "\hat{\Pi}_2", "\cong"'] & \operatorname{Spec} HP_0 BU[6,\infty) \arrow[r] \arrow[d, "\hat{\Pi}_3"] & \operatorname{Spec} A^* \arrow[r] \arrow[d, "\lambda", "\cong"'] & 0 \\
C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "\delta"] & C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \arrow[r, "e"] & \operatorname{Weil}(\widehat{\mathbb{G}}_a) \arrow[r] & 0.
\end{tikzcd}$$

Our calculations now pay off:

**Corollary 51.** *The map $\hat{\Pi}_3$ is an isomorphism:*

$$\hat{\Pi}_3 \colon \operatorname{Spec} HP_0 BU[6,\infty) \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m).$$

*Proof (Proof sketch).* The main point is that we don't actually have to compute much about the middle map. Because the squares commute and the sequences are exact as indicated, we at least learn that $\hat{\Pi}_3$ is an epimorphism after base-change to $\operatorname{Spec} \mathbb{Q}$ and $\operatorname{Spec} \mathbb{F}_p$ for each prime $p$. But, since both source and target are affine schemes of graded finite type with equal Poincaré series in each case, our epimorphism is an isomorphism.

**Corollary 52 ([2, Theorem 2.31]).** *The map $\hat{\Pi}_3$ is an isomorphism for any complex-orientable $E$.*

Jay asked in class for a summary of exactly what exactness statements are true for a general $E$ (especially relative to the extreme case of $E$–theory, where everything is exact and pleasant).

*Proof (Proof sketch).* This follows much along the lines of Corollary 37. We check that the statement holds for $E = MUP$ using a tangent space argument, and then an Atiyah–Hirzebruch argument gives the statement for any complex-oriented $E$.

*Remark 70.* Our analysis in Lecture 4.6 gives us full access to the Hopf ring structure of the *nonconnective* cooperations $H\mathbb{F}_{p*}\underline{KU}_{2*}$. Using a variety of techniques, Morton and Strickland calculated the Hopf ring structure of $H\mathbb{F}_{2*}\underline{K}_{2*}$ where $K$ ranges among the nonconnective objects $KO$, $KU$, $KSp$, and the less common "$KT$", which is self–conjugated $K$–theory [27, 28, 40].

This could be expanded some, as not all these references are relevant to what's written.

## 5.6 Apr 20: Covariance, Θ–structures on Thom sheaves

Today we will (despite appearances) mostly leave the algebraic geometry alone, instead attending to two lingering topological concerns. First, over the past few lectures we have been concerned with the homology schemes $\operatorname{Spec} E_0 \underline{kU}_{2*}$. We were originally motivated by a sequence of cohomological isomorphisms

$$\begin{aligned}(BU \times \mathbb{Z})_E &\cong \operatorname{Sym}^0_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_0 \mathbb{CP}^\infty_E, \\ BU_E &\cong \operatorname{Sym}^1_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_1 \mathbb{CP}^\infty_E, \\ BSU_E &\cong \operatorname{Sym}^2_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_2 \mathbb{CP}^\infty_E,\end{aligned}$$

along with identifications

$$BU \times \mathbb{Z} \simeq \underline{kU}_{2\cdot 0}, \qquad BU \simeq \underline{kU}_{2\cdot 1}, \qquad BSU \simeq \underline{kU}_{2\cdot 2}.$$

Our analysis of Cartier duality in Remark 60 gave us isomorphisms like

$$\operatorname{Spec} E_0 BSU \cong \underline{\mathsf{GroupSchemes}}(BSU_E, \mathbb{G}_m) \cong \underline{\mathsf{FormalGroups}}(C_2 \mathbb{CP}^\infty_E, \widehat{\mathbb{G}}_m).$$

Following the universal property of this particular symmetric square, we were led to consider the scheme of symmetric bivariate functions on $\mathbb{CP}^\infty_E$ satisfying a 2–cocycle condition. Our next move was to show that $\operatorname{Spec} E_0 BU[6, \infty)$ was modeled by a similar scheme of *trivariate* functions — but we proved this directly, while avoiding the "predual" cohomological statement

$$BU[6, \infty)_E \cong C_3 \mathbb{CP}^\infty_E := \operatorname{Sym}^3_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E.$$

This is because the homological statement is the low-hanging fruit: it is easy to demonstrate that the scheme of such functions exists as a closed subscheme of all functions. It is considerably harder to show that a symmetric cube exists at all.

**Lemma 82.** *$DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all formal varieties for $k \le 3$.* **Cite me:** 4.41 in the AHS preprint.

*Proof.* We know that $\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all free $\mathbb{Z}$–modules of graded finite rank in the range $k \le 3$, so we may write Cite this from a couple lectures ago.

$$\mathcal{O}(DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)) \cong (\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*.$$

We will show that this later Hopf algebra $\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*$ is a power series ring, specializing for the moment to the case $k = 2$. It will suffice to show that it is a power series ring modulo $p$ for every prime $p$. Such graded connected finite-type Hopf algebras over $\mathbb{F}_p$ were classified by Borel (and exposited by Milnor–Moore [25, Theorem 7.11]) as either polynomial or truncated polynomial. These two cases are distinguished by the Frobenius operation: the Frobenius on a polynomial ring is injective, whereas the Frobenius on a truncated polynomial ring is not. It is therefore equivalent to show that the *Verschiebung* on the original ring $\mathcal{O}(C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m)) \otimes \mathbb{F}_p$ is *surjective*. Recalling that $c_n^p = c_{np}$ at the level of bivariate 2–cocycles, we compute

$$p^* a_n = a_{np}^p,$$

and since $Fa_{np} = a_{np}^p$ and $FV = p^*$, we learn

$$V(a_{np}) = a_n.$$

You should remind us that $c_n$ , $a_n$ were the things defined in the last lecture.

Essentially the same proof handles the cases $k = 1$ and $k = 0$. The case $k = 3$ requires a small modification, to cope with the two classes of trivariate 2–cocycles. On the polynomial tensor factor of $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m))$ we can reuse the same Verschiebung argument to see that its dual Hopf algebra is polynomial. The dual of the divided power tensor factor is immediately a primitively generated polynomial algebra, without any further argument .

I'm confused about this statement. Why doesn't something similar hold for the polynomial factor, given that it is true?

**Theorem 71.** *The scheme $C_3\mathbb{CP}^\infty_E$ exists, and it is modeled by $BU[6,\infty)_E$.*

This regurgitated proof is pretty broken.

**Cite me:** Prop 3.3 in AHS preprint.

*Proof (Proof sketch).* Let $\widehat{\mathbb{G}}$ be an arbitrary formal group. Note first that if $C^3(\widehat{\mathbb{G}};\mathbb{G}_m)$ is coalgebraic, then $C_3\widehat{\mathbb{G}}$ exists and is its Cartier dual: the diagram presenting $\mathcal{O}(C^3(\widehat{\mathbb{G}};\mathbb{G}_m))$ as a reflexive coequalizer of free Hopf algebras is also the diagram meant to present $C_3\widehat{\mathbb{G}}$ as a coalgebraic formal scheme. So, if the coequalizing Hopf algebra has a good basis, it will follow from Theorem 63 that the resulting diagram is a colimit diagram in formal schemes, with $C_3\widehat{\mathbb{G}}$ sitting at the cone point. It will additionally follow from there that the isomorphism from Corollary 52

$$\operatorname{Spec} E_0 BU[6,\infty) \xrightarrow{\cong} C^3(\mathbb{CP}^\infty_E;\mathbb{G}_m)$$

will re-dualize to an isomorphism

$$BU[6,\infty)_E \xleftarrow{\cong} C_3\mathbb{CP}^\infty_E.$$

**Cite me:** Prop 3.4 in the AHS preprint.

Here we are again, making grading arguments. We've been bad about this earlier in the paper too.

So, we reduce to checking that $\mathcal{O}(C^3(\widehat{\mathbb{G}};\mathbb{G}_m))$ admits a good basis. By a base change argument, it suffices to take $\widehat{\mathbb{G}}$ to be the universal formal group over the Lazard ring. Noting that $\mathcal{O}(C^3(\widehat{\mathbb{G}};\mathbb{G}_m))$ must be of graded finite type, we will work to show that it is free on a basis we have good control over.

Specializing from $\widehat{\mathbb{G}}$ over $\mathcal{M}_{\mathbf{fgl}}$ to $\widehat{\mathbb{G}}_a$ over $\operatorname{Spec}\mathbb{Z}$, we know from Lemma 82 that $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m))$ is a free abelian group, and we know from Theorem 25 that $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$ is as well. By picking a $\mathbb{Z}$–basis $\mathbb{Z}\{\beta_j\}_j$ of $\mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m)$, we can choose a map of $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$–modules lifting it

$$\begin{array}{ccc} \mathcal{O}(\mathcal{M}_{\mathbf{fgl}})\{\tilde{\beta}_j\}_j & \xrightarrow{\alpha} & \mathcal{O}C^3(\widehat{\mathbb{G}};\mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathbb{Z}\{\beta_j\}_j & \xrightarrow{\cong} & \mathcal{O}C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m). \end{array}$$

By induction on degree, one sees that $\alpha$ is surjective, and since the source is a free abelian group we need only check that the source and target have the same Poincaré series to conclude that $\alpha$ is an isomorphism.

We proceed to test this rationally: over $\operatorname{Spec}\mathbb{Q}$ we can use the logarithm to construct an isomorphism

$$\operatorname{Spec}\mathbb{Q}\times(\mathcal{M}_{\mathbf{fgl}}\times C^k(\widehat{\mathbb{G}};\mathbb{G}_m))\to\operatorname{Spec}\mathbb{Q}\times(\mathcal{M}_{\mathbf{fgl}}\times C^k(\widehat{\mathbb{G}}_a;\mathbb{G}_m)),$$

hence the Poincaré series agree, so $\alpha\otimes\mathbb{Q}$ and $\alpha$ are both isomorphisms. Having established freenss, our other goal was to show that $M$ has a sequence of good subcoalgebras. These come by considering the subcoalgebras, indexed on an integer $d$, spanned by the basis vectors of degree at most $d$.

Our second task today is to address the gap between $\underline{kU}_{2k}$ and its Thom spectrum $T(\underline{kU}_{2k})$. After all, our motivation for all of this algebraic geometry is to give a description to the set

$$(\operatorname{Spec}E_0MU[6,\infty))(E_0),$$

but what we have done so far is describe the scheme $\operatorname{Spec}E_0BU[6,\infty)$. Since these two spectra are related by a Thom construction, we should be able to deduce the description that we want by thinking about Thom sheaves. We now straighten this out. The place to start is with a construction:

**Lemma 83.** *For $\xi\colon G\to BGL_1\mathbb{S}$ a group map, the Thom spectrum $T\xi$ is a $(\Sigma^\infty_+G)$–cotorsor.*

*Proof (Proof sketch).* The Thom isomorphism $T\xi\wedge T\xi\simeq T\xi\wedge\Sigma^\infty_+G$ composes with the unit map $\mathbb{S}\to T\xi$ to give the *Thom diagonal*

Which factor are we composing into? Does it matter?

$$T\xi\to T\xi\wedge\Sigma^\infty_+G.\square$$

**Corollary 53.** *In addition to our interpretation of $\mathbb{L}(\xi)$ as a $\mathbb{G}_m$–torsor over $G_E$, $\mathbb{L}(\xi)^{-1}$ is furthermore a $(G_E\times\mathbb{G}_m)$–torsor over $S_E$.* □

$S_E$ vs. $\mathbb{S}_E$, and how does this follow?

We expand this idea out in our situation. A morphism $MU[6,\infty)\to E$ produces a trivialization of $\mathbb{L}(\bigotimes_{j=1}^3(\mathcal{L}_j-1))$ over $(\mathbb{C}\mathrm{P}^\infty_E)^{\times 3}$, and the associated trivializing section is symmetric and rigid. This prompts us to make the following definition:

**Cite me:** Lemma 2.49 of AHS.

**Definition 77.** We write $C^k(\widehat{\mathbb{G}};\mathcal{L})$ for the functor of trivializing sections of $\Theta^k\mathcal{L}$ over $\widehat{\mathbb{G}}^{\times k}$ which are symmetric and rigid. This construction has some nice properties:

- Note that taking the trivial sheaf $\mathcal{L} = \mathcal{O}_{\widehat{\mathbb{G}}}$ recovers the scheme $C^k(\widehat{\mathbb{G}};\mathbb{G}_m)$ of $\mathbb{G}_m$–valued such functions from before.
- By consequence, if $\mathcal{L}$ is *trivializable* then this functor is an affine scheme.
- Affine or not, there is a pairing map

$$C^k(\widehat{\mathbb{G}};\mathcal{L}) \times C^k(\widehat{\mathbb{G}};\mathcal{H}) \to C^k(\widehat{\mathbb{G}};\mathcal{L}\otimes\mathcal{H}).$$

  In particular, this recovers the group structure on $C^k(\widehat{\mathbb{G}};\mathbb{G}_m)$ and it makes $C^k(\widehat{\mathbb{G}};\mathcal{L})$ into a $C^k(\widehat{\mathbb{G}};\mathbb{G}_m)$–torsor.

Thus, we have constructed a map

$$\varphi\colon\ \operatorname{Spec} E_0MU[6,\infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathcal{I}(0)).$$

The following Lemma is a matter of fully expanding definitions:

**Lemma 84 ([2, Theorem 2.50]).** *The map*

$$\operatorname{Spec} E_0MU[6,\infty) \to C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathcal{I}(0))$$

*is an isomorphism.*

You shouldn't call this anything more than a "proof sketch" unless you expand out the reasoning for why this square commutes.

*Proof (Proof sketch).* This map is equivariant, in the sense that the following square commutes:

$$\begin{tikzcd}
\operatorname{Spec} E_0MU[6,\infty) \times \operatorname{Spec} E_0BU[6,\infty) \arrow[r] \arrow[d, "\varphi\times 52"] & \operatorname{Spec} E_0MU[6,\infty) \arrow[d, "\varphi"] \\
C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathcal{I}(0)) \times C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathbb{G}_m) \arrow[r] & C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathcal{I}(0)).
\end{tikzcd}$$

Any equivariant map of torsors is automatically an isomorphism.

This, finally, gives us access to the analogue of Theorem 64, Theorem 65, and Corollary 47:

**Corollary 54.** *Take E to be* complex–orientable. *The functor* $C^3(\mathbb{C}\mathrm{P}^\infty_E;\mathcal{I}(0))$ *defined by*

$$\begin{aligned}
\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} &\to \mathsf{Sets},\\
(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) &\mapsto \left\{ \begin{array}{l} \textit{symmetric, rigid trivializations} \\ \textit{of } u^*\Theta^3\mathbb{L}(\mathcal{L}) \textit{ over } u^*(\mathbb{C}\mathrm{P}^\infty_E)^{\times 3} \end{array} \right\}
\end{aligned}$$

*is isomorphic to the affine scheme* $\operatorname{Spec} E_0MU[6,\infty)$*. Moreover, the $E_0$–points of this scheme biject with ring spectrum maps $MU[6,\infty) \to E$.* ☐

This is a leftover todo from the following section. We accomplish all of this here, but I haven't looked at the references tacked on at the end of this to see if they say anything I've missed. — — — — — I want to sketch the reduction for even-periodic elliptic cohomology theories to the case of *MUP*, then from there to *HkP* for the prime fields *K*, then from there to questions about additive cocycles. We certainly don't need to recall any of these calculations, but I think it's a nice example of the philosophy that the additive formal group is such a knotted point

## 5.7 Apr 22: Modular forms from $MU[6,\infty)$–manifolds

Our goal today is to actually leverage the arithmetic geometry in Corollary 48, rather than just using the body of results about $\theta$–functions as inspiration. In order to do this, we need to place ourselves in a situation where algebraic topology is directly linked to abelian varieties.

**Definition 78.** An *elliptic spectrum* consists of a even–periodic ring spectrum $E$, a (generalized) elliptic curve $C$ over $\operatorname{Spec} E_0$, and a fixed isomorphism

$$\varphi\colon\ C_0^\wedge \xrightarrow{\cong} \mathbb{CP}_E^\infty.$$

A map among such spectra consists of a map of ring spectra $f\colon\ E \to E'$ together with a specified isomorphism of elliptic curves $\psi\colon\ f^*C \to C'$.

*Remark 71.* We have chosen to consider *isomorphisms* of elliptic curve rather than general homomorphisms because this is what algebraic topology suggests that we do. After all, the mixed cooperations of complex-oriented ring spectra are modeled by the isomorphisms of the associated formal groups. In the next Case Study, we will develop a theory (with an attendant notion of a "context") which incorporates isogenies of elliptic curves in addition to isomorphisms.

Coupling Definition 78 to Corollary 54 and Corollary 48, we learn the following result:

**Corollary 55.** *An elliptic spectrum $(E, C, \varphi)$ receives a canonical map of ring spectra*

$$MU[6,\infty) \to E.$$

*This map is natural in choice of elliptic spectrum: if $(E, C, \varphi) \to (E', C', \varphi')$ is a map of elliptic spectra, then the triangle*

$$\begin{tikzcd} & MU[6,\infty) \arrow[dl] \arrow[dr] & \\ E \arrow[rr] & & E' \end{tikzcd}$$

*commutes.* □

*Example 32.* Our basic example of an elliptic curve was $E_\Lambda = \mathbb{C}/\Lambda$, with $\Lambda$ a complex lattice. The projection $\mathbb{C} \to \mathbb{C}/\Lambda$ has a local inverse which defines an isomorphism of formal groups

$$\varphi\colon\ (E_\Lambda)_0^\wedge \xrightarrow{\cong} \widehat{\mathbb{G}}_a \otimes \mathbb{C}.$$

Accordingly, we define an elliptic spectrum $HE_\Lambda P$ whose underlying ring spectrum is $H\mathbb{C}$ and whose associated elliptic curve and isomorphism are $E_\Lambda$ and $\varphi$. This spectrum receives a natural map

$$MU[6,\infty) \to HE_\Lambda P,$$

which to a bordism class $M \in MU[6,\infty)_{2n}$ assigns an element $\Phi_\Lambda(M) \cdot u_\Lambda^n \in HE_\Lambda P_{2n}$ for some $\Phi_\Lambda(M) \in \mathbb{C}$.

What is $u_\Lambda^n$? How is it normalized?

*Example 33.* The naturality of the $MU[6,\infty)$–orientation moves us to consider more than one elliptic spectrum at a time. If $\Lambda'$ is another lattice with $\Lambda' = \lambda \cdot \Lambda$, then the multiplication map $\lambda\colon \mathbb{C} \to \mathbb{C}$ descends to an isomorphism $E_\Lambda \to E_{\Lambda'}$ and hence a map of elliptic spectra $HE_{\Lambda'}P \to HE_\Lambda P$ acting by $u_{\Lambda'} \mapsto \lambda u_\Lambda$. The commuting triangle in Corollary 55 then begets the *modularity relation*

$$\Phi(M; \lambda \cdot \Lambda) = \lambda^{-n}\Phi(M; \Lambda).$$

*Example 34.* This equation leads us to consider all curves $E_\Lambda$ simultaneously — or, equivalently, to consider modular forms. The lattice $\Lambda$ can be put into a standard form, by picking a basis and scaling it so that one vector lies at 1 and the other vector lies in the upper half-plane. This gives a cover

$$\mathfrak{h} \to \mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{C}$$

Is the notation $\mathfrak{h}$ for the upper half plane standard here? Usually it's denoted $\mathbb{H}$?

which is well-behaved away from the special points $i$ and $e^{2\pi i/6}$. A *complex modular form of weight $k$* is an analytic function $\mathfrak{h} \to \mathbb{C}$ which satisfies a certain decay condition and which is quasi-periodic for the action of $SL_2(\mathbb{Z})$, i.e.,[3]

$$f\left(M; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(M; \tau).$$

Using these ideas, we construct a cohomology theory $H\mathcal{O}_\mathfrak{h}P$, where $\mathcal{O}_\mathfrak{h}$ is the ring of complex-analytic functions on the upper half-plane. The $\mathfrak{h}$–parametrized family of elliptic curves

$$\mathfrak{h} \times \mathbb{C}/(1, \tau) \to \mathfrak{h},$$

together with the logarithm, present $H\mathcal{O}_\mathfrak{h}P$ as an elliptic spectrum $H\mathfrak{h}P$. The canonical map $\Phi\colon MU[6,\infty) \to H\mathfrak{h}P$ specializes at a point to give the functions $\Phi(-;\Lambda)$ considered above, and hence $\Phi(M)$ is itself a complex modular form of weight $k$.

In fact, this is a ghost of Ochanine and Witten's modular genus from Theorem 1, as a bordism class in $MU[6,\infty)_{2n}$ is, in particular, a bordism class

[3] That is, for the action of change of basis vectors.

in $MString_{2n}$. However, they know more about this function than we can presently see: they claim that it has an integral $q$–expansion. In terms of the modular form, its $q$–expansion is given by building the Taylor expansion "at $\infty$" (using that unspoken decay condition). In order to use our topological methods, it would be nice to have an elliptic spectrum embodying these $q$–expansions in the same way that $H\mathfrak{h}P$ embodied holomorphic functions, together with a comparison map that trades a modular form for its $q$–expansion. The main ideas leading to such a spectrum come from considering the behavior of $E_\Lambda$ as $\tau$ tends to $i \cdot \infty$.

**Definition 79.** Note that as $\tau \to i \cdot \infty$, the parameter $q = \exp(2\pi i \tau)$ tends to 0. In the multiplicative model of Lecture 5.3, we considered $D'$ the punctured complex disk with associated family of elliptic curves

$$C'_{\mathrm{an}} = \mathbb{C}^\times \times D'/(u,q) \sim (qu,q).$$

The fiber of $C'$ over a particular point $q \in D'$ is the curve $\mathbb{C}^\times/q^{\mathbb{Z}}$. The Weierstrass equations give an embedding of $C'_{\mathrm{an}}$ into $D' \times \mathbb{C}\mathrm{P}^2$ described by

$$wy^2 + wxy = x^3 - 5\alpha_3 w^2 x + -\frac{5\alpha_3 + 7\alpha_5}{12} w^3$$

for certain functions$\alpha_3$ and $\alpha_5$ of $q$. At $q = 0$, these curve collapses to the twisted cubic

Do these have weights?

$$wy^2 + wxy = x^3,$$

and over the whole open unit disc $D$ we call this extended family $C_{\mathrm{an}}$.

Now let $A \subseteq \mathbb{Z}[\![q]\!]$ by the subring of power series which converge absolutely on the open unit disk. It turns out that the coefficients of the Weierstrass cubic (i.e., $5\alpha_3$ and $\frac{1}{12}(5\alpha_3 + 7\alpha_5)$) lie in $A$, so it determines a generalized elliptic curve $C$ over Spec $A$, and $C_{\mathrm{an}}$ is the curve given by base-change from $A$ to the ring of holomorphic functions on $D$.The *Tate curve* $C_{\mathrm{Tate}}$ is defined to be the family over the intermediate object $D_{\mathrm{Tate}} = \operatorname{Spec} \mathbb{Z}[\![q]\!]$ base-changed from $A$.

**Cite me:** Find a reference for this. You may be able to look in Morava's Section 5.

The singular fiber at $q = 0$ prompts us to enlarge our notion of elliptic curve slightly.

**Definition 80 ([2, Definitions B.1-2]).** A *Weierstrass curve* is *any* curve of the form

**Cite me:** Is this right? Check the published AHS to make sure.

$$C(a_1, a_2, a_3, a_4, a_6) := \left\{ ([x:y:z], s) \in \mathbb{P}^2 \times S \;\middle|\; \begin{array}{c} y^2z + a_1(s)xyz + a_3(s)yz^2 = \\ x^3 + a_2(s)x^2z + a_4(s)xz^2 + a_6(s)z^3 \end{array} \right\}$$

Erick didn't like the name "generalized elliptic curve", which already means something else in arithmetic (disallowing cuspidal singularities and also allowing other funny behavior). He didn't have a second suggestion, though.

You use $w$s above but $z$s here.

A *generalized elliptic curve* over $S$ is a scheme $C$ equipped with maps

$$S \xrightarrow{0} C \xrightarrow{\pi} S$$

such that $C$ is Zariski–locally isomorphic to a system of Weierstrass curves (in a way preserving 0 and $\pi$).[4]

**Cite me:** Some kind of reference would be appreciated.

*Remark 72.* The singularities of a degenerate Weierstrass equation always occur outside of a formal neighborhood of the marked identity point, which in fact still carries the structure of a formal group. The formal group associated to the twisted cubic is the multiplicative group, and the isomorphism making the identification extends a family of such isomorphisms $\varphi$ over the nonsingular part of the Tate curve.

The smooth locus of the twisted cubic is actually also the *informal* multiplicative group. The $\theta$–function we're going to describe below also has the property that it specializes at $q = 0$ to the usual coordinate $(1-u)$ on $\mathbb{G}_m$, which is nice. This degeneracy to honest–$\mathbb{G}_m$ is probably also related to the use of the Todd orientation further down.

**Definition 81 ([26, Section 5], [2, Section 2.7]).** The elliptic spectrum $K_{\mathrm{Tate}}$, called *Tate K–theory*, has as its underlying spectrum $KU[\![q]\!]$. The associated elliptic curve is $C_{\mathrm{Tate}}$, and the isomorphism $\mathbb{C}\mathrm{P}^{\infty}_{KU[\![q]\!]} \cong (C_{\mathrm{Tate}})^{\wedge}_{0}$ is $\varphi$ from Remark 72.

The trade for the breadth of this definition is that theorems pulled from the study of abelian varities have to be shown to extend uniquely to those generalized elliptic curves which are not smooth curves.

**Theorem 72 ([2, Propositions 2.57 and B.25]).** *For a generalized elliptic curve $C$, there is a canonicaltrivialization $s$ of $\Theta^3\mathcal{I}(0)$ which is compatible with change of base and with isomorphisms. If $C$ is a smooth elliptic curve, then $s$ agrees with that of Corollary 48.* □

You should compare this with the unicity statement in the previous iteration of this theorem, plus how it's *not* unique at these singular points.

**Corollary 56.** *The trivializing section $s$ associated to $C_{\mathrm{Tate}}$ is given by $\delta^{\circ 3}\theta$, where*

$$\theta_q(u) = (1-u)\prod_{n>0}(1-q^n u)(1-q^n u^{-1}).$$

This is a little funny, because this is *not* one of the usual $\theta$–functions from Lecture 5.3. I think we should state its quasiperiodicity relation again, since it doesn't agree with the old one.

*Proof.* Even though $\theta$ is not a function on $C_{\mathrm{Tate}}$ because of quasiperiodicity, it does trivialize both $\pi^*\mathcal{I}(0)$ for $\pi\colon \mathbb{C}^\times \times D \to C_{\mathrm{Tate}}$ and $\mathcal{I}(0)$ for $(C_{\mathrm{Tate}})^{\wedge}_{0}$. Moreover, the quasiperiodicities in the factors in the formula defining $\delta^3\theta|_{(C_{\mathrm{Tate}})^{\wedge}_{0}}$ cancel each other out, and the function does descend to give a trivialization of $\Theta^3\mathcal{I}(0)$. By the unicity clause in Theorem 72, it must give a formula expressing $s$.

This use of unicity is a little opaque to me. I guess we're using that $\delta^3\theta$ is "obviously" the natural trivializing section for nonsingular values of $q$? Or maybe we just mean that the Tate curve covers much of $\mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec}\mathbb{C}$.

**Definition 82.** The induced map

$$\sigma_{\mathrm{Tate}}\colon MU[6,\infty) \to K_{\mathrm{Tate}}$$

is called the *complex $\sigma$–orientation*.

**Corollary 57.** *Let $M \in \pi_{2n}MU[6,\infty)$ be a bordism class. The $q$–expansion of Witten's modular form $\Phi(M)$ has integral coefficients.*

[4] An elliptic curve in the usual sense turns out to be a generalized elliptic curve which is smooth, i.e., the discriminant of the Weierstrass equations is a unit.

*Proof.* The span of elliptic spectra equipped with $MU[6,\infty)$–orientations

$$\begin{tikzcd} & MU[6,\infty) \arrow[dl, "\sigma_{\mathrm{Tate}}"'] \arrow[d] \arrow[dr, "\Phi"] & \\ K_{\mathrm{Tate}} \arrow[r] & K_{\mathrm{Tate}} \otimes \mathbb{C} & H\mathfrak{h}P \arrow[l] \end{tikzcd}$$

models $q$–expansion. The arrow $K_{\mathrm{Tate}} \to K_{\mathrm{Tate}} \otimes \mathbb{C}$ is injective on homotopy, which shows that the $q$–expansion of $\Phi(M)$ lands in the subring of integral power series.

In fact, Corollary 56 gives us access to a formula for $\sigma_{\mathrm{Tate}} = \delta^3\theta$, where $\theta$ here is interpreted as a coordinate on $(C_{\mathrm{Tate}})^\wedge_0$. This means that $\sigma_{\mathrm{Tate}}$ belongs to the commutative triangle

$$\begin{tikzcd} MU[6,\infty) \arrow[d, "\delta"'] \arrow[rrr, "\sigma_{\mathrm{Tate}}"] & & & KU[\![q]\!]. \\ MSU \arrow[r, "\delta"'] & MU \arrow[r, "\delta"'] & MUP \arrow[ur, "\theta"'] & \end{tikzcd}$$

To begin, the usual map $MUP \to KU$ selects the coordinate $f(u) = 1 - u$ on the formal completion of $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[u^\pm]$. The induced map

$$MU \xrightarrow{\delta} MUP \to KU$$

sends $f$ to the rigid section $\delta f$ of

$$\Theta^1\mathcal{I}(0) = \mathcal{I}(0)_0 \otimes \mathcal{I}(0)^{-1} \cong \omega \otimes \mathcal{I}(0),$$

and in terms of the right-hand side,

$$\delta f = \frac{f'(0)}{f(u)} Du = \frac{1}{1-u}\left(-\frac{du}{u}\right),$$

where $Du$ is the invariant differential. We can augment this to a calculation of $\delta\theta$ by considering the composite

This first map is one of the units. Which is it?

$$\delta\theta\colon MU \xrightarrow{\eta_R} MU \wedge MU \simeq MU \wedge BU_+ \xrightarrow{\delta(1-u)\wedge\theta'} K_{\mathrm{Tate}}$$

where $\theta'$ is the element of $BU^{K_{\mathrm{Tate}}} \cong C^1(\widehat{C}_{\mathrm{Tate}}; \mathbb{G}_m)$ given by the formula

I'm confused. Where did this class come from? Is it the derivative of $\theta$? The $\delta$–derivative of $\theta$? Something else? Actually, it sort of looks like $(1-u) \cdot \theta(1)/\theta(u)$...?

$$\theta' = \prod_{n \ge 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}.$$

This means that its effect on a line bundle is determined by

$$\theta'(1-\mathcal{L}) = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n\mathcal{L})(1-q^n\mathcal{L}^{-1})},$$

and its effect on vector bundles in general is determined by the splitting principle and an exponential law. In fact, one can work this exponential law out to mean

$$\theta'(\dim V \cdot 1 - V) = \bigotimes_{n\geq 1} \bigoplus_{j\geq 0} \mathrm{Sym}^j(\dim V \cdot 1 - V \otimes_{\mathbb{R}} \mathbb{C}) q^{jn} =: \bigotimes_{n\geq 1} \mathrm{Sym}_{q^n}(-\bar{V}_{\mathbb{C}}).$$

Knowing that $(\eta_R)_*\colon MU_* \to \pi_*(MU \wedge \Sigma^\infty_+ BU)$ sends a manifold $M$ with stable normal bundle $\nu$ to the pair $(M, \nu)$, we compute the composite on homotopy to be

$$\begin{aligned}\sigma_{\mathrm{Tate}}(M \in \pi_{2n} MU[6,\infty)) &= (\delta(1-u) \wedge \theta')(M,\nu) \\ &=: \mathrm{Td}\left(M; \bigotimes_{n\geq 1} \mathrm{Sym}_{q^n}(\bar{\tau}_{\mathbb{C}})\right).\end{aligned}$$

This is exactly Witten's formula for his genus, as applied to complex manifolds with first two Chern classes trivialized.

This is missing something. We have written $\mathrm{Td}(M)(-du/u)^n$ for $MU_{2n} \to KU_{2n}$, the effect in homotopy of the usual coordinate on $KU$. You can also think of this as $p_! 1$ for $1 \in K^{2n}M$ and $p\colon M \to *$. More generally, then, $\mathrm{Td}(M;V) := p_! V$. But, also, we did describe pushforwards earlier... maybe we can make use of this?

## 5.8 Apr 25: *Spin* and *String* orientations

A different approach to this is to work through the $p \neq 2$ case, since you can totally, correctly work it out and see the $\Sigma$–structure relation get imposed. That's kind of attractive. Too bad you had this through after the class was complete.

In the previous Lecture, we proved that elliptic spectra receive canonical $MU[6,\infty)$–orientations, that complex elliptic spectra collectively give rise to a genus valued in modular forms, and that the $q$–expansions of these modular forms are integral. However, the original Theorem 1 of Ochanine and Witten claimed to describe a genus on *Spin*– and *String*–manifolds, which we have only managed to approximate with our study of $MU[6,\infty)$–orientations. Our last goal for this Case Study is to show that the chromatic formal schemes associated to spaces like *BString* are somewhat accessible, and so chromatically-amenable elliptic spectra receive canonical *MString*–orientations.

Fix a formal group $\Gamma$ of finite height $d$, and write $K = K_\Gamma$ for the associated Morava $K$–theory. We will start with the more modest goal of understanding the bottom few layers of the Postnikov tower for $\underline{kO}_0 \simeq BO \times \mathbb{Z}$, which have the names

$$BO[2,\infty) := BSO, \qquad BO[4,\infty) := BSpin, \qquad BO[8,\infty) := BString.$$

The := should be =: perhaps?

*Remark 73.* Unlike $kU$, there is *not* an equivalence

$$\underline{kO}_n \not\simeq (BO \times \mathbb{Z})[n,\infty),$$

unless $n$ happens to take the form $n = 8k$ for a nonnegative integer $k$. The reasoning for this stray equivalence is similar to that for $kU$: the homotopy ring of $kO$ has a polynomial factor of degree 8, and the other elements lie in a band of dimensions smaller than 8. Otherwise, other things happen — for instance,

$$\underline{kO}_1 \simeq O/U, \qquad\qquad \underline{kO}_6 \simeq Spin/SU.$$

*Remark 74 ([19, Section 5.2]).* We may as well take the ground field of our Morava $K$–theory to have characteristic $p = 2$, since at odd characteristics there is little distinction between $kO$ and $kU$, owing to the fiber sequence

$$\Sigma kO \xrightarrow{\cdot\eta=0} kO \to kU.$$

However, this reveals a disadvantage of Morava $K$–theory that will finally cause us real consternation: Morava $K$–theories at the prime 2 are not commutative ring spectra. Accordingly, $(K_\Gamma)_*G$ for a commutative $H$–group $G$ may fail to give a commutative algebra. Luckily, Remark 35 tells us that if $(K_\Gamma)_*G$ happens to be even-concentrated, then the obstructions to commutativity identically vanish. So, we can be somewhat indelicate about this noncommutativity issue, provided that we continually check that the algebras we are forming are even-concentrated.

The way you wrote this fiber sequence doesn't show why $p = 2$ is special.

In order to get off the ground, we will need the following Lemma about the behavior of the Atiyah–Hirzebruch spectral sequence for a Morava $K$–theory:

**Lemma 85 ([45, Lemma 2.1]).** *Let $k_\Gamma$ be the connective cover of the Morava $K$–theory $K_\Gamma$. In the Atiyah–Hirzebruch spectral sequence*

Does this Lemma admit a coordinate-free statement? They probably aren't all identically controlled by $Q_d$, but rather by $Q_d$ plus decomposables.

$$E_2^{*,*} = Hk^*X \otimes_k k_\Gamma^* \Rightarrow k_\Gamma^* X,$$

*the differentials are given by*

$$d_r(x) = \begin{cases} 0 & \text{if } r \le 2(p^d - 1), \\ \lambda Q_d x \otimes v_d & \text{if } r = 2(p^d-1)+1 \end{cases}$$

*where $\lambda \neq 0$ and $Q_d$ is the $d^{th}$ Milnor primitive.* □

**Corollary 58 ([36, Section 2.5], [19, Equation 3.1]).** *There is a bi-Cartesian square of coalgebraic formal schemes*

$$\begin{tikzcd}
& \mathrm{Div}_0\, \overline{\mathbb{G}}[2] \arrow[rr] \arrow[dl] & & \mathrm{Div}_0\, \widehat{\mathbb{G}}[2] \arrow[dl] \\
\mathrm{Div}_0\, \overline{\mathbb{G}} \arrow[rr] & & BO_K. &
\end{tikzcd}$$

*Proof.* We apply Lemma 85 to the analysis of the spectral sequence of Hopf algebras

$$H\mathbb{F}_{2*}BO \otimes (K_\Gamma)_* \Rightarrow (K_\Gamma)_*BO.$$

**Cite me:** I don't know where these Milnor primitives are calculated. I guess we could have done them using formal geometry.

We have $H\mathbb{F}_{2*}BO \cong \mathbb{F}_2[b_1, b_2, \ldots]$ and

$$Q_d b_{2^{d+1}+2j} = b_{2j+1},$$

Only the squares of even generators survive.

from which it follows that all the odd generators are killed, all their squares survive, and only the even generators of low degree are permanent cycles. This results in a decomposition

$$(K_\Gamma)_*BO \cong (K_\Gamma)_*[b_2, b_4, b_{2^{d+1}-2}] \underset{(K_\Gamma)[b_{2j}^2 | j < 2^d]}{\otimes} (K_\Gamma)_*[b_{2j}^2],$$

and so we are tasked with assigning names to the coalgebraic formal schemes appearing in this formula.

You write $\Gamma$ but also $\widehat{\mathbb{G}}$.

The left-hand factor is the free Hopf algebra on the coalgebra determined by the 2–torsion in the formal group $\Gamma$. The right-hand factor is the free Hopf algebra on the formal curve $\overline{\mathbb{G}} := \mathbb{HP}_K^\infty$, using the isogeny

$$\begin{aligned} \mathbb{HP}_K^\infty &\to \mathbb{CP}_K^\infty \\ y &\mapsto x \cdot [-1](x) \end{aligned}$$

induced by desymplectification. Because $\mathbb{H}^\times$ is not commutative, $\overline{\mathbb{G}}$ is not a formal group, but we pull back the multiplication-by-2 isogeny from $\widehat{\mathbb{G}}$ to $\overline{\mathbb{G}}$ and define the subscheme $\overline{\mathbb{G}}[2]$ of points mapping to zero.

It would be nice if you could put in a bit more detail about why these formal schemes correspond to the coalgebras from above.

Suppose that the Postnikov sections

$$X(n, \infty) \to X[n, \infty) \to X[n, n]$$

induce short exact sequences of formal groups. A presentation of $X_K$ as a bi-Cartesian square then acquires value via the following algebraic Lemma:

**Lemma 86.** *Consider the cube of formal group schemes constructed by taking pointwise fibers of the composite to G:*

$$
\begin{tikzcd}[row sep=small, column sep=small]
& A' \arrow[rr] \arrow[dl] \arrow[dd] & & B' \arrow[dl] \arrow[dd] & \\
C' \arrow[rr, crossing over] \arrow[dd] & & D' & & \\
& A \arrow[rr] \arrow[dl] & & B \arrow[dl] & \\
C \arrow[rr] & & D \arrow[from=uu, crossing over] \arrow[rr] & & G.
\end{tikzcd}
$$

*If the bottom face is bi-Cartesian, then so is the top.* ☐

**Corollary 59.** *There is a bi-Cartesian square*

$$
\begin{tikzcd}
& \operatorname{Div}_0 \overline{\mathbb{G}}[2] \arrow[rr] \arrow[dl] & & \operatorname{SDiv}_0 \widehat{\mathbb{G}}[2] \arrow[dl] \\
\operatorname{Div}_0 \overline{\mathbb{G}} \arrow[rr] & & BSO_K. &
\end{tikzcd}
$$

*Proof (Proof sketch).* The fibration $BSO \to BO \to BO(1)$ gives a short exact sequence of Hopf algebras , so using Corollary 58 we are in the situation of Lemma 86. To compute the pointwise kernels, begin by considering the commuting square of Postnikov sections

after applying $K$-homology – what is the argument for this? Do I need to look at some spectral sequence?

$$
\begin{tikzcd}
BO_K \arrow[r] \arrow[d] & BU_K \arrow[d] \\
\widehat{\mathbb{G}}[2] \arrow[r] & \widehat{\mathbb{G}}.
\end{tikzcd}
$$

Both horizontal maps are injections. Since $\operatorname{Div} \overline{\mathbb{G}} \to BU_K \cong \operatorname{Div}_0 \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ is null, the composite $\operatorname{Div}_0 \overline{\mathbb{G}} \to \widehat{\mathbb{G}}[2]$ is null. Similarly, the composite $\operatorname{Div} \widehat{\mathbb{G}}[2] \to \widehat{\mathbb{G}}[2]$ acts by summation, and its kernel is $\operatorname{SDiv}_0 \widehat{\mathbb{G}}[2]$.

This diagram seems to be missing $\operatorname{Div}_0 \overline{\mathbb{G}}$ and $\operatorname{Div} \overline{\mathbb{G}}$. Also, you are identifying $K(\mathbb{Z}/2.1)_K$ with $\widehat{\mathbb{G}}[2]$. You argue using $BU$, but it also seems that whatever argument you are using to show that the right vertical arrows have null composite should also just work for $BO$.

From here, the computation gets harder.

**Corollary 60 ([19, Section 5.3]).** *There is a bi-Cartesian square*

$$
\begin{tikzcd}
& \operatorname{Div}_0 \overline{\mathbb{G}}[2] \arrow[rr] \arrow[dl] & & \ker \omega \arrow[dl] \\
\operatorname{Div}_0 \overline{\mathbb{G}} \arrow[rr] & & BSpin_K, &
\end{tikzcd}
$$

*where* $\omega\colon C_2\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}[2]^{\wedge 2}$ *is the map* $([a]-[0])([b]-[0]) \mapsto a \wedge b$.

*Proof.* This goes similarly to Corollary 59, once you know that the Postnikov section induces a short exact sequence of formal groups. The composite $\operatorname{Div}\overline{\mathbb{G}} \to \widehat{\mathbb{G}}[2]^{\wedge 2}$ is shown to be zero using an identical technique. To identify the behavior on the other factor, we need the following diagram of exact sequences of Hopf algebras from Kitchloo, Laures, and Wilson [19, Theorem 6.4]:

**Cite me:** Cite the relevant part of KLW.

Really? This is what I have written down in my notes, but I can't expand it out.

$$\begin{tikzcd}
 & & & K_* \arrow[d] \\
 & K_* \arrow[d] & & K_*K(\mathbb{Z},3) \arrow[d] \\
K_* \arrow[r] & K_*BSpin \arrow[r] \arrow[d] & K_*BSU \arrow[r, "\tau"] \arrow[d, equal] & K_*BU[6,\infty) \arrow[d, "\delta"] \\
K_* \arrow[r] & K_*BSO \arrow[r, "i"] \arrow[d] & K_*BSU \arrow[r, "1-\xi"] & K_*BSU \arrow[d] \\
 & K_*K(C_2,2) \arrow[d] \arrow[uurr, dotted, bend right] & & K_* \\
 & K_*, & &
\end{tikzcd}$$

where $\tau\colon C_2\widehat{\mathbb{G}} \to C_3\widehat{\mathbb{G}}$ is specified at the level of formal schemes by

$$\tau\colon ([a]-[0])([b]-[0]) \mapsto ([a]-[0])([b]-[0])([-a-b]-[0]).$$

Since $(1-\xi)\circ i = 0$, we have that $\delta \circ \tau \circ i = 0$ and hence that $\tau \circ i$ lifts to $K_*K(\mathbb{Z},3)$. Identifying $\operatorname{SDiv}_0 \widehat{\mathbb{G}}[2]$ with $C_2\widehat{\mathbb{G}}[2]$, we check that the composites

$$C_2\widehat{\mathbb{G}}[2] \xrightarrow{\omega} \widehat{\mathbb{G}}[2]^{\wedge 2} \xrightarrow{\varepsilon} C_3\widehat{\mathbb{G}}$$

and

$$C_2\widehat{\mathbb{G}}[2] \to C_2\widehat{\mathbb{G}} \xrightarrow{\tau} C_3\widehat{\mathbb{G}}$$

agree. For a point $[a,b] \in C_2\widehat{\mathbb{G}}$, this is the claim

$$\begin{aligned}
0 &= \varepsilon(a \wedge b) - \tau[a,b] \\
&= [a,a,b] - [b,a,b] - [-a-b,a,b] \\
&= [a,a,b] - [b+a,a,b] + [b,a+a,b] - [b,a,b],
\end{aligned}$$

and this is forced null in $C_3\widehat{\mathbb{G}}$, as it looks like a 2–cocycle shuffle.

Where is $\widehat{\mathbb{G}}[2]^{\wedge 2}$ in the diagram? What is $\xi$? What is $\varepsilon$? Is the dotted arrow the Bockstein? In the diagram you wrote $K_*K(C_2,2)$, but I think we already have too many $C_2$'s. I'm also confused about the appearances of all the symmetric powers $C_2$ and $C_3$. Where do they appear in the diagram?

I'm a little fuzzy on the coherence of this with the Bockstein: this computes the lift of $\tau \circ f$ into $K(\mathbb{Z}, 3)_K$, and it does happen to factor through the subscheme $K(\mathbb{Z}/2, 2)_K$ determined by the Bockstein. However, I don't immediately see why this agrees with the bottom Postnikov section of $BSO$: that's a map off of $BSO$ and this is a rotated map into $BU[6, \infty)$, so it's not an immediate consequence of naturality. It has to do with rotating the Wood cofiber sequence just right, and in particular where the horizontal sequences come from: they're stitched-together from two consecutive Wood cofiber sequences.

**Ideally, we would use this presentation of $BSpin_K$ to say something about $MSpin$–orientations. I just realized I don't know how, though!**

*Remark 75.* This computation becomes almost unfeasible for $BString$, but we will sketch two approaches. One is that the sequence

$$\underline{HC_{2}}_{2} \to \underline{HC_{2^\infty}}_{2} \to BString \to BSpin \to \underline{HC_{2^\infty}}_{4} \xrightarrow{2} \underline{HC_{2^\infty}}_{4}$$

induces an exact sequence of group schemes. The other avenue of access is the pair of fiber sequences

$$\underline{H\mathbb{Z}}_3 \to \widetilde{BSpin} \to BSpin, \qquad BString \to \widetilde{BSpin} \to \underline{HC_{2}}_{3},$$

formed by considering the pullback of the corner

$$BSpin \to \underline{H\mathbb{Z}}_4 \leftarrow \underline{HC_{2}}_{3}.$$

Both of these fibrations induce short exact sequences of Hopf algebras.

However, since we are specifically interested in $MString$–orientations, there is an alternative approach that avoids describing the formal scheme $BString_K$. Again appealing to results of Kitchloo, Laures, and Wilson, we find that the sequence

$$K_* Spin/SU \to K_* BU[6, \infty) \to K_* BString$$

is exact and right-exact. The kernel of the map $K_* Spin/SU \to K_* BU[6, \infty)$ is Cite me: KLW.
a Hopf algebra they call "$CK_3$", where Cite me: Theorem 2.3.5.vi of KLW.

$$CK_j = \bigoplus_{k=j}^{\infty} K_* K(\mathbb{Z}/2, k).$$

More than that, KLW even say where the polynomial and nonpolynomial parts of $K_* Spin/SU$ land inside of $K_* BU[6, \infty)$. I thinkthat this means that But I have not checked!
$K_* BU[6, \infty)$ is a flat $K_* Spin/SU$–module at heights $d \leq 2$.

Applying the Thom spectrum functor to the fiber square gives the pushout diagram

$$\begin{tikzcd}
\Sigma^\infty_+ Spin/SU \arrow[r] \arrow[d] & MU[6, \infty) \arrow[d] \\
\mathbb{S} \arrow[r] & MString,
\end{tikzcd}$$

or, equivalently, an equivalence

$$MString \simeq MU[6,\infty) \wedge_{\Sigma^\infty_+ Spin/SU} \mathbb{S}.$$

It seems that this approach, which I presume is the one that ultimately works, is independent of all the earlier work involving *BSpin*.

This in turn gives a Tor spectral sequence of signature

$$\mathrm{Tor}_{*,*}^{K_* Spin/SU}(K_* MU[6,\infty), K_*) \Rightarrow K_* MString.$$

So, under the flatness hypothesis above, there are no higher Tor terms so the spectral sequence collapses to give

$$K_* MString \cong K_* MU[6,\infty) /\!/ K_* Spin/SU.$$

You have not introduced $\Sigma$–structures yet.

So, what remains to be shown is that $K_* Spin/SU$ picks out the correct extra relation for $\Sigma$–structures . Then, we need a density argument to show that this handles all of the at-a-point cases of elliptic cohomology.

*Remark 76.* However, the spectra $HE_\Lambda P$ do not qualify as "chromatically amenable" from the perspective of this last argument, and so we lose access to our genus valued in modular forms. Additionally, $K_{\mathrm{Tate}}$ does not qualify, essentially because it is integral rather than $p$–adic. The methods described here give rise to putative $p$–adic $q$–expansions of modular forms, but we have not been able to check that they satisfy the modularity condition, nor that they assemble into a single, integral object. Amazingly, such theorems are achievable, and they are the impetus for the study of topological modular forms and algebraic geometry done with $E_\infty$–rings.

Expand this last sentence. Point to the Appendix, or cite the *TMF* book, or idk.

The Atiyah–Bott–Shapiro orientation and the fibration $BSU \to BSpin$. This is possibly somewhere around Theorem 2.3.5.iv in KLW, the last fibration in 2.3.2 at $k = -2$, and sections 5.3 and 5.13

Cite Kitchloo–Laures's "Real structures and Morava $K$–theories"

What follows is the analysis for *MString*. Is the one for *MSpin* analogous and doable? Does it involve $CK_2$ and maybe a clever choice of *MSU*–orientation?

If you managed to get all this to work, you could try to understand the *MSpin*–orientation of $K_{\mathrm{Tate}}$, as in page 628 of the published AHS.

### *Some other things that might belong in this chapter*

The cubical structure on a singular (generalized) elliptic curve is not unique, but (published) AHS has an argument showing that the unicity of the choice on the nonsingular "bulk" extends to a unique choice on the "boundary" of the compactified moduli too.

There's also the work of Ando–French–Ganter on factorized / iterated $\Theta$ structures and how they give rise to the "two–variable Jacobi genus".

# Case Study 6
# Power operations

I wish this had a better title.

Write an introduction for me.

There should be a context-based presentation of this chapter's material too. What do contexts for structured ring spectra look like? Why would you consider them — what object are you trying to approximate? How do you guess that the algebraic model is reasonable until you're aware of something like Strickland's theorem?

Since you spend so much time talking about descent in other parts of these notes, maybe you should also read the end of the AHS $H_\infty$ paper where they claim to recast their results in the usual language of descent.

Conversation with Nat on 2/9 suggests taking the following route in this chapter: contexts for $E_\infty$ mapping spaces in general; Subgroups and level structures; the Drinfel'd ring and the universal level structure; the isogenies pile; power operations and Adams operations, after Ando (naturally indexed vs indexed on subgroups; have a look at the Screenshot you took on this day); comparison of comodules $M$ for the isogenies pile with the action of $M_n(\mathbb{Z}_p)$ on $M \otimes_{E_n^*} D_\infty$ (this is a modern result due to Tomer, Tobi, Lukas, and Nat); $H_\infty$ $MU$–orientations and Matt's thesis; the analogous results for $\Theta^k$–structures. In particular, leave character theory, $p$–divisible groups, and rational phenomena for spillover at the end of the year. They aren't strictly necessary to telling the story; you just need to know a little about the Drinfel'd ring to construct Matt's maps. (If you have time, though, the point is that the rationalized Drinfel'd ring carries the universal level structure which is also an isomorphism.)

The stuff around 4.3.1-2 of Matt's published thesis talks about $H_\infty$–maps being determined by their values on $*$ and $\mathbb{C}P^\infty$, which is an interesting result. You might also compare with Butowiez–Turner.

Work in height 1 (and height 2??) examples through this? $K$–theory is pretty accessible, and the height 2 examples are somewhat understood (Charles, Yifei), and they're both relevant for the elliptic *MString* story. (There's also the pile of elliptic curves with isogenies...)

Nat warns that the very end of Matt's thesis uses character theory for $S^1$, which you have to be very careful about to pull off correctly. ($S^1$ is not a finite group, but in certain contexts it can be approximated by its torsion subgroups...)

Yifei warned me that Matt's "there exists a unique coordinate..." Lemma is specifically about lifting the *Honda* formal group law over $\mathbb{F}_q$. If you want to do this with elliptic cohomology or something, then you need a stronger statement (and it's clear what this statement should be, but no one has proven it).

—— Here are various notes from conversations with Nat, recorded and garbled well after they happened. ——

We could try to understand Matt's thesis's Section 4.2. It identifies the action of the internal power operation on $E_n$ using the internal theory of quotient isogenies to the Lubin–Tate deformation problem (2.5.4). Conditions 1 and 3 of 4.2.1 are easy to verify: they are 4.2.3 (evaluate on a point) and 4.2.4 (the power operation does raise things to a power) respectively. Condition 2 takes more work, and it's about identifying the divisor associated to the isogeny granted by Condition 1. It's worked out in 4.2.5, which is not very hard, and 4.2.6, which shows that *the* Thom class associated to a vector bundle is sent under a power operation to *some* Thom class. 4.2.5 then uses that the quotient of *some* Thom classes has to be a unit in the underlying ring.

(Q: Can 4.2.5 be phrased about two coordinates on the same formal group, rather than two presentations of the same divisor? There's a comparison between functions on the quotient with invariant functions on the original group — and perhaps with functions invariant by pulling back along the isogeny?)

Prop 8.3 in "Character of the Total Power Operation" provides an algebro-geometric proof of something in AHS04, using the fact that for $R$ a nice complete local ring and $G$, $G'$ $p$–divisible groups over $R$, there is an injection

$$\mathrm{Isog}_R(G, G') \hookrightarrow \mathrm{Isog}_{R/\mathfrak{m}}(G, G').$$

Nat thinks that using the power operation internal to $MU$ is what gets you Lubin's product formula for the quotient (cf. the calculation in Quillen's theorem), and using the power operation internal to $E$–theory gives you *something* called $\psi^H$, which you separately calculate on Euler classes. The point (cf. Ando's thesis's Theorem 1) is to pick a coordinate so that these coincide. (Lubin–Tate theory and Lubin's theory of isogenies says that they always coincide up to unique $\star$–isomorphism — after all, automorphisms (and isogenies) don't deform — and the point is that for particular coordinates on particular formal groups, you can take the $\star$–isomorphisms to all be the identity.)

Ando's thesis only deals with power operations internal to $E$–theory starting in Section 4. Before then, he shows that the pushforward of the power operations internal to $MU$ can be lifted through maps on $E$–theory (although these maps may not be topologically induced). It's not clear to me what the value of this is — if you're constructing the operations on the $E$–theory side, then surely you're going to construct them so that they're on-the-nose equal to the $MU$–operations?

The meaty part of AHS04 is Theorem 6.1, that the necessary condition is sufficient. It falls into steps: first, we can restrict attention to $\Sigma_p$, and even inside of there we can restrict attention to $C_p$. Then, the two directions around the $H_\infty$ square give two trivializations (cf. 4.2.6 of Ando's thesis) $g_{cl(ockwise)}$ and $g_{c(ounter)c(lockwise)}$ of $\Theta^k \mathcal{I}(0)$. The fact that they're both trivializations means there's an equation $g_{cl} = r g_{cc}$ for $r \in E^0 D_{C_p} BU[2k, \infty)^\times$. Then, he wants to study the map

$$E^0 D_{C_p} BU[2k, \infty)_+ \xrightarrow{\Delta^* \times i^*} E^0 BC_p^* \times BU[2k, \infty) \times E^0 BU[2k, \infty)^{\times p},$$

which they know to be an injection by work of McClure, but for some reason they can restrict attention to just the left-hand factor. The left-hand factor is the ring of functions on $\underline{\mathsf{FormalGps}}(A, \widehat{\mathbb{G}}_F) \times BU[2k, \infty)_E$, and they can further restrict attention to level structures, where there are only two: the injective one and the null map. They then check these two cases by hand, and it follows that $r = 0$, so the two ways of navigating the diagram agree at the level of topology.

(Section 8 of Hopkins–Lawson has an injectivity proof that smells similar to the above injectivity trick with McClure's map.)

Just working in the case $k = 1$ (or $k = 0$), which is supposed to recover the "classical" results of Ando's thesis, we can try to recursively expand the various arguments and definitions. The counterclockwise map appears to be the easy one, and it's discussed around 4.11. The clockwise map appears to be the hard one, and it's discussed in 3.21. For $\chi_\ell = \chi_\ell \times \widehat{\mathbb{G}}$ given by

$$T \times \widehat{\mathbb{G}} \xrightarrow{\chi_\ell} \underline{\mathrm{Hom}}(A, \widehat{\mathbb{G}}) \times \widehat{\mathbb{G}},$$

the main content of 3.21 is an equality

$$\chi_\ell^* s_{cl} = \psi_\ell^{\mathcal{L}}(s_g) = (\psi_\ell^{\widehat{\mathbb{G}}/E})^* (\psi_\ell^E)^* s_g,$$

where $\psi_\ell^E$ is defined in 3.9, $\psi_\ell^{\widehat{\mathbb{G}}/E}$ is defined in 3.14 and the preceding remarks, $s_g$ is the section describing the source coordinate (cf. part 2 of 3.21), and $\psi_\ell^{\mathcal{L}}$ is described between the paragraph before 3.16 and Definition 3.20. Trying to rewrite $\psi_\ell^{\mathcal{L}}$ into the form required for 3.21 requires pushing through 8.11 and 10.15.

We spent a lot of time just writing out the definitions of things, trying to get them straight in the universal case (which AHS04 wants to avoid for some reason — maybe they didn't yet have a good form of Strickland's theorem?). It was helpful in the moment, but hard to read now.

All of this rests, most importantly, on how a quotient of the Lubin–Tate universal deformation by a subgroup still gives a Lubin–Tate universal deformation. This is Section 12.3 of AHS04, and it's Section 9 of Neil's Finite Subgroups paper. (Nat says there's something to look out for in here. Watch where they say they have $E_0$–algebra maps versus ring maps.)

## 6.1 $E_\infty$ ring spectra and their contexts

Mike has suggested looking at the paper *The K–theory localization of an unstable sphere*, by Mahowald and Thompson. In it, they manually construct a resolution of $S^{2n+1}$ suitable for computing the unstable Adams spectral sequence for $K$–theory, but the resolution that they build is also exactly what you would use to compute the mapping spectral sequence for $E_\infty(K^{S^{2n-1}}, K)$. Additionally, because the unstable $K$–theoretic operations are exhausted by the power operations, these two spectral sequences converge to the same target.
Purely in terms of the $E_\infty$ version, one can consider the composition of spectral sequences

$$\mathrm{Ext}_{\mathbb{Z}[\theta]}(\mathbb{Z}, \mathrm{Der}_{K_*-alg}(K^*X, K^*)) \Rightarrow \mathrm{Der}_{K_*-Dyer-Lashof-alg}(K^*X, K^*) \Rightarrow E_\infty(\widehat{\mathbb{S}^0}^X, K_p^\wedge)$$

and

$$E_\infty(\widehat{\mathbb{S}^0}^X, K_p^\wedge)^{h\mathbb{Z}_p^*} = E_\infty(\widehat{\mathbb{S}^0}^X, \widehat{\mathbb{S}^0})$$

where the first spectral sequence is a composition spectral sequence for derivations in $K_*$–algebras and then derivations respecting the Mandell's $\theta$–operation. If $X$ is an odd sphere, then $K^*X$ has no derivations and this composite spectral sequence collapses, making the composition possible.
This is also related to recent work of Behrens–Rezk on the Bousfield–Kuhn functor...

Another unpublished theorem of Hopkins and Lurie is that the natural map $Y = F(*, Y) \to E_\infty(E_n^Y, E_n)$ is an equivalence when $Y$ is a finite Postnikov tower in the range of degrees that $E_n$ can see.

## 6.2 Subgroups and level structures

Something that these notes routinely fail to do is to lead into the algebraic geometry in a believable way. "Today we're going to talk about isogenies" — and then, lo' and behold, isogenies appear the next day in algebraic topology. This book would read much better if it showed how these structures were guessed to exist to begin with.

Here's a definition of an isogeny. Weierstrass preparation can be phrased as saying that a Weierstrass map is a coordinate change and a standard isogeny.

**Cite me:** Definition 5.17 of FSFG.

**Definition 83.** Take $C$ and $D$ to be formal curves over $X$. A map $f : C \to D$ is an *isogeny* when the induced map $C \to C \times_X D$ exhibits $C$ as a divisor on $C \times_X D$ as $D$–schemes.

Be sure to compare this definition with the usual one for formal groups: surjections with finite cokernel. These are easy to come up with lots of examples for! (Don't ditch this definition, though, since it's the one that lets you prove something about Weierstrass preparation geometrically.)

In fact, every map in positive characteristic can be factored as a coordinate change and an isogeny, which is a weak form of preparation.

Lubin's finite quotients of formal groups. (Interaction with the Lubin–Tate moduli problem? Or does this belong in the next day?)

Write out isogenies of the additive formal group, note that you just get the unstable Steenrod algebra again. This is a remarkable accident.

Push and pull maps for divisor schemes

Moduli of subgroup divisors

The Drinfel'd moduli ring, level structures

—-

**Cite me:** Prop 6.2 of HKR.

**Lemma 87.** *The following conditions on a homomorphism*

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

*are equivalent:*

1. *For all $\alpha \neq 0$ in $\Lambda_r^*$, $\varphi(\alpha)$ is a unit (resp., not a zero-divisor).*
2. *The Hopf algebra homomorphism*

$$R[\![x]\!]/[p^r](x) \to R^{\Lambda_r^*}$$

*is an isomorphism (resp., a monomorphism).* □

**Lemma 88.** *Let $\mathcal{L}_r(R)$ be the set of all group homomorphism*

Cite me: Shortly after Prop 6.2 of HKR. Section 7?.

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

*satisfying either of the conditions 1 or 2 above. This functor is representable by a ring*

$$L_r(E^*) := S^{-1}E^*(B\Lambda_r)$$

*that is finite and faithfully flat over $p^{-1}E^*$. (Here $S$ is generated by the $\varphi(\alpha)$ with $\alpha \neq 0$, $\varphi : \Lambda_r^* \to F[p^r](E^*B\Lambda_r)$ the canonical map.)*

—

Section 2: complete local rings

"Galois" means $R \to S$ a finite extension of integral domains has $R$ as the fixed subring for $\mathrm{Aut}_R(S)$ and $S$ is free over $R$. Galois extension of rings implies the extension of fraction fields is Galois. The converse holds for finite (finitely generated as a module) dominant (kernel of $f$ is nilpotent) maps of smooth (regular local ring) schemes.

Section 3: basic facts about formal groups

definition of height

Section 4: basic facts about divisors

Since $x -_F a \doteq x - a$, you can treat the divisor $[a]$ (defined in a coordinate by the ideal sheaf generated by $x - x(a)$) as generated just by $x - a$.

**Lemma 89.** *Let $D$ and $D'$ be two divisors on $\widehat{\mathbb{G}}$ over $X$. There is then a closed subscheme $Y \leq X$ such that for any map $a : Z \to X$ we have $a^*D \leq a^*D'$ if and only if $a$ factors through $Y$.* □

Cite me: Prop 4.6 of Finite Subgroups.

Section 5: quotient by a finite sbgp is again a fml gp

**Definition 84.** A *finite subgroup* of $\widehat{\mathbb{G}}$ will mean a divisor $K$ on $\widehat{\mathbb{G}}$ which is also a subgroup scheme. Let $\mathcal{O}_{\widehat{\mathbb{G}}/K}$ be the equalizer

$$\mathcal{O}_{\widehat{\mathbb{G}}/K} \longrightarrow \mathcal{O}_{\widehat{\mathbb{G}}} \overset{\mu^*}{\underset{\pi^*}{\rightrightarrows}} \mathcal{O}_K \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{\mathbb{G}}}.$$

**Lemma 90.** *Write $y = N_\pi \mu^* x \in \mathcal{O}_{\widehat{\mathbb{G}}}$.*[1] *Then $y \equiv x^{p^m} \pmod{\mathfrak{m}_X}$ and $\mathcal{O}_{\widehat{\mathbb{G}}/K} =$*

Cite me: Theorem 5.3 of Finite Subgroups.

$\mathcal{O}_X[\![y]\!]$. *Moreover, the projection* $\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/K$ *is the categorical cokernel of* $K \to \widehat{\mathbb{G}}$. *This all commutes with base change: given* $f : Y \to X$ *we have* $f^*\widehat{\mathbb{G}}/f^*K = f^*(\widehat{\mathbb{G}}/K)$. □

Expand this out in the case of a subgroup scheme given by a sum of point divisors.

cf. also Prop 2.2.2 of Matt's thesis

Section 6: coordinate-free lubin-tate theory

nothing you haven't already seen. in fact, most of it is done in coordinates, with only passing reference to the decoordinatization.

Section 7: level–$A$ structures: smooth, finite, flat

Be careful to distinguish the physical group $A$ from the associated *constant group scheme*.

As discussed long ago, for finite abelian $p$–groups there's a scheme

$$\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}})(Y) = \underline{\mathsf{Groups}}(A, \widehat{\mathbb{G}}(Y)).$$

If $\widehat{\mathbb{G}}$ were a discrete group, we could decompose this as

$$\text{“}\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}}) = \coprod_{B \leq A} \mathrm{Mono}(A/B, \widehat{\mathbb{G}})\text{”}$$

along the different kernel types of homomorphisms, but Mono does not exist as a scheme.Level structures approximate this as best one can be approximating $\widehat{\mathbb{G}}$ by something essentially discrete: an étale group scheme.

Come up with a really compelling example. You had one when you were talking to Danny and Jeremy. Probably you got it *from* Jeremy.

For a map $\varphi : A \to \widehat{\mathbb{G}}(Y)$, we write $[\varphi A] = \sum_{a \in A} [\varphi(a)]$. We also write $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, so that $\Lambda[p^m] = (\mathbb{Z}/p^m)^{\times n}$. Note

$$|\mathsf{AbelianGroups}(A, \Lambda)| = |A|^n = \operatorname{rank}\left(\underline{\mathsf{FormalGroups}}(A, \widehat{\mathbb{G}}) \to X\right).$$

**Definition 85.** A *level–$A$ structure* on $\widehat{\mathbb{G}}$ over an $X$–scheme $Y$ is a map $\varphi : A \to \widehat{\mathbb{G}}(Y)$ such that $[\varphi A[p]] \leq G[p]$ as divisors. A *level–$m$ structure* means a level–$\Lambda[p^m]$ structure.

**Cite me:** Prop 7.2-4 of Finite Subgroups.

**Lemma 91.** *The functor from schemes over* $X$ *to sets given by*

$$Y \mapsto \{\textit{level–}A \textit{ structures on } \widehat{\mathbb{G}} \textit{ over } Y\}$$

*is represented by a finite flat scheme* $\mathrm{Level}(A, \widehat{\mathbb{G}})$ *over* $X$. *It is contravariantly functorial for monomorphisms of abelian groups. Also, if* $\varphi : A \to \widehat{\mathbb{G}}$ *is a level structure then* $[\varphi A]$ *is a subgroup divisor and* $[\varphi A[p^k]] \leq \widehat{\mathbb{G}}[p^k]$ *for all k. In fact, if* $A = \Lambda[p^m]$ *then* $[\varphi A] = \widehat{\mathbb{G}}[p^m]$. □

I can't imagine proving this. It's worth noting that it's proven by considering just the universal case, which we know to be smooth.

In Section 26 of FPFP Neil says there's a decomposition into irreducible components

[1] Remember that if $f : X \to Y$ is a finite flat map, then $N_f : \mathcal{O}_X \to \mathcal{O}_Y$ is the nonadditive map sending $u$ to the determinant of multiplication by $u$, considered as an $\mathcal{O}_Y$–linear endomorphism of $\mathcal{O}_X$.

$$\mathrm{Hom}(A, \widehat{\mathbb{G}}) = \mathrm{Hom}(A, \widehat{\mathbb{G}}_{\mathrm{red}}) = \bigcup_B \mathrm{Level}(A/B, \widehat{\mathbb{G}})$$

and this $\bigcup$ turns into a $\coprod$ after inverting $p$. He also mentions this as motivation in Finite Subgroups, but he doesn't appear to prove it?

Section 8: maps among level–$A$ schemes, their Galois behavior

**Theorem 73.** *Let $A$, $B$ be finite abelian $p$–groups of rank at most $n$, and let $u : A \to B$ be a monomorphism. Then:*

**Cite me:** Theorem 8.1 of Finite Subgroups.

1. $$\mathsf{FormalSchemes}_X(\mathrm{Level}(B, \widehat{\mathbb{G}}), \mathrm{Level}(A, \widehat{\mathbb{G}})) = \mathrm{Mono}(A, B).$$
2. *Such homomorphisms are detected by the behavior at the generic point.*
3. *The map $u^! : \mathrm{Level}(B, \widehat{\mathbb{G}}) \to \mathrm{Level}(A, \widehat{\mathbb{G}})$ is finite and flat.*
4. *If $B \simeq \Lambda[p^m]$, then $u^!$ is a Galois covering.*
5. *The torsion subgroup of $\widehat{\mathbb{G}}(\mathrm{Level}(A, \widehat{\mathbb{G}}))$ is $A$.* $\square$

Section 9: epimorphisms of groups become maps of level schemes, quotients by level structures

Let $\widehat{\mathbb{G}}_0$ be a formal group of height $n$ over $X_0 = \operatorname{Spec} \kappa$. For every $m$, the divisor $p^m[0]$ is a subgroup of $\widehat{\mathbb{G}}_0$. We write $\widehat{\mathbb{G}}_0\langle p^m \rangle$ for the quotient group $\widehat{\mathbb{G}}_0/p^m[0]$ and $\widehat{\mathbb{G}}\langle m \rangle \to X\langle m \rangle$ for the universal deformation of $\widehat{\mathbb{G}}_0\langle m \rangle \to X_0$. Note that $\widehat{\mathbb{G}}_0[p] = p^n[0]$, which induces isomorphisms $\widehat{\mathbb{G}}_0\langle m+n \rangle \to \widehat{\mathbb{G}}_0\langle m \rangle$, and we use this to make as many identifications as we can.

**Lemma 92.** *Let $u : A \to B$ be an epimorphism of abelian $p$–groups wit kernel $|\ker(u)| = p^\ell$. Then $u$ induces a map*

**Cite me:** 9.1 of Finite Subgroups.

$$u_! : \mathrm{Level}(A, \widehat{\mathbb{G}}\langle m \rangle) \to \mathrm{Level}(B, \widehat{\mathbb{G}}\langle m+\ell \rangle).$$

*Also, if $A = \Lambda[p^m]$, then $u_!$ is a Galois covering with Galois group*

$$\Gamma = \{\alpha \in \mathrm{Aut}(A) \mid u\alpha = u\}.\square$$

**Corollary 61.** *In particular, the map $A \to 0$ induces a map*

**Cite me:** Interstitial text between 9.1 and 9.2 of Finite Subgroups.

$$0_! : \mathrm{Level}(A, \widehat{\mathbb{G}}\langle m \rangle) \to \mathrm{Level}(0, \widehat{\mathbb{G}}\langle m+\ell \rangle) = X\langle m+\ell \rangle$$

*which extracts quotient formal groups from level structures. In the case $A = \Lambda[p^\ell]$, $0_!$ is just the projection $0^!$.* $\square$

Section 10: moduli of subgroup schemes

**Theorem 74.** *The functor*

**Cite me:** Theorem 10.1 of Finite Subgroups.

$$Y \mapsto \{\textit{subgroups of } \widehat{\mathbb{G}} \times_X Y \textit{ of degree } p^m\}$$

*is represented by a finite flat scheme* $\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})$ *over* $X$ *of degree* $|\mathrm{Sub}_{p^m}(\Lambda)|$. *The formation commutes with base change.* $\square$

We can at least give the construction: let $D$ be the universal divisor defined over $Y = \mathrm{Div}_{p^m}(\widehat{\mathbb{G}})$ with equation $f_D(x) = \sum_{k=0}^{p^m} c_k x^k$. There are unique elements $a_{ij} \in \mathcal{O}_Y$ such that

$$f(x +_F y) = \sum_{i,j=0}^{p^m-1} a_{ij} x^i y^j \pmod{f(x), f(y)}.$$

Define

$$\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}}) = \mathrm{Spf}\,\mathcal{O}_Y/(c_0, a_{ij} \mid 0 \le i,j < p^m).$$

Finiteness, flatness, and rank counting are what take real work, starting with an arithmetic fracture square.

Section 13: deformation theory of isogenies

**Definition 86.** Suppose we have a morphism of formal groups

$$\begin{tikzcd}
\widehat{\mathbb{G}}_0 \arrow[r, "q_0"] \arrow[d] & \widehat{\mathbb{G}}'_0 \arrow[d] \\
X_0 \arrow[r, "f_0"] & X'_0
\end{tikzcd}$$

such that the induced map $\widehat{\mathbb{G}}_0 \to f_0^* \widehat{\mathbb{G}}'_0$ is an isogeny of degree $p^m$. By a deformation of $q_0$ we mean a prism

$$\begin{tikzcd}
\mathbb{H} \arrow[dr, "q"] \arrow[dd] & & \mathbb{H}_0 \arrow[ll] \arrow[rr] \arrow[dr] \arrow[dd] & & \widehat{\mathbb{G}}_0 \arrow[dr, "q_0"] \arrow[dd] & \\
& \mathbb{H}' \arrow[dd] & & \mathbb{H}'_0 \arrow[ll] \arrow[rr] \arrow[dd] & & \widehat{\mathbb{G}}'_0 \arrow[dd] \\
Y \arrow[dr, "1"] & & Y_0 \arrow[ll] \arrow[rr] \arrow[dr, "1"] & & X_0 \arrow[dr, "f_0"] & \\
& Y & & Y_0 \arrow[ll] \arrow[rr] & & X'_0,
\end{tikzcd}$$

where the middle face is the pullback of the left face, the back-right and front-right faces are pullbacks, so that $q$ is also an isogeny of degree $p^m$.

Let $\widehat{\mathbb{G}}/X$ be the universal deformation of $\widehat{\mathbb{G}}_0$, let $a : \mathrm{Sub}_{p^m}(\widehat{\mathbb{G}}) \to X$ be the usual projection, and let $K < a^*\widehat{\mathbb{G}}$ be the universal example of a subgroup of degree $p^m$. As $\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})$ is a closed subscheme of $\mathrm{Div}_{p^m}(\widehat{\mathbb{G}})$ and $\mathrm{Div}_{p^m}(\widehat{\mathbb{G}})_0 = X_0$, we see that $\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})_0 = X_0$. There is a unique subgroup of order $p^m$ of $\widehat{\mathbb{G}}_0$ defined over $X_0$, viz. the divisor $p^m[0] = \operatorname{Spf} \mathcal{O}_{\widehat{\mathbb{G}}_0}/x^{p^m}$. In particular, $K_0 = p^m[0] = \ker(q_0)$. It follows that there is a pullback diagram as shown below:

$$\begin{array}{ccccc} (a^*\widehat{\mathbb{G}}/K)_0 & \xrightarrow{\simeq} & \widehat{\mathbb{G}}_0/p^m[0] & \xrightarrow{\overline{q}_0,\simeq} & \widehat{\mathbb{G}}'_0 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})_0 & \xrightarrow{a_0,\simeq} & X_0 & \xrightarrow{f_0,\simeq} & X'_0. \end{array}$$

We see that $a^*\widehat{\mathbb{G}} \to a^*\widehat{\mathbb{G}}/K$ is a deformation of $q_0$, and it is terminal in the category of such.

Now let $\widehat{\mathbb{G}}'/X'$ be the universal deformation of $\widehat{\mathbb{G}}'_0/X'_0$. The above construction also exhibits $a^*\widehat{\mathbb{G}}/K$ as a deformation of $\widehat{\mathbb{G}}'_0$, so it is classified by a map $b : \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}}) \to X'$ extending the map $b_0 = f_0 \circ a_0 : \operatorname{Sub}_{p^m}(\widehat{\mathbb{G}})_0 \to X'_0$.

**Theorem 75.** *b is finite and flat of degree* $|\operatorname{Sub}_{p^m}(\Lambda)|$. $\square$

**Cite me:** Prop 13.1 of Finite Subgroups, *hard*.

Cf. Matt's thesis's Prop 2.5.1: $\Phi$ is a formal group over $\mathbb{F}_p$, $F$ a lift of $\Phi$ to $E_n$, $H$ a finite subgroup of $F(D_k)$, then $F/H$ is a lift of $\Phi$ to $D_k$. (This is because the quotient map to $F/H$ reduces to $t \mapsto t^{p^r}$ for some $r$ over $\mathbb{F}_p$, which is an endomorphism of $\Phi$, so the quotient map over the residue field doesn't do anything!) See also Prop 2.5.4, where he characterizes all isogenies of this sort as arising from this construction.

Section 14: connections to AT

Neil's *Finite Subgroups of Formal Groups* has (in addition to lots of results) a section 14 where he talks about the action of a generalized Hecke algebra on the $E$–theory of a space. Let $a$ and $b$ be two points of $X$, with fibers $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_b$, and let $q : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_b$ be an isogeny. Then there's an induced map $(Z_E)_a \to (Z_E)_b$, functorial in $q$ and natural in $Z$. "Certain Ext groups over this Hecke algebra form the input to spectral sequences that compute homotopy groups of spaces of maps of strictly commutative ring spectra, for example." **This sounds like the beginning of an answer to my context question.**

Section 11: flags of controlled rank ascending to $\widehat{\mathbb{G}}[p]$ and a map $\operatorname{Level}(1, \widehat{\mathbb{G}}) \to \operatorname{Flag}(\lambda, \widehat{\mathbb{G}})$. Section 12: the orbit scheme $\operatorname{Type}(A, \widehat{\mathbb{G}}) = \operatorname{Level}(A, \widehat{\mathbb{G}}) / \operatorname{Aut}(A)$: smooth, finite, flat Section 15: formulas for computation Section 16: examples

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**Theorem 76.** *Let R be a complete local domain with positive residue characteristic p, and let F be a formal group of finite height d over R. If $\mathcal{O}$ is the ring of integers in the algebraic closure of the fraction field of R, then $F(\mathcal{O})[p^k] \cong (\mathbb{Z}/p^k)^d$ and $F(\mathcal{O})_{\mathrm{tors}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^d$.* $\square$

**Cite me:** See Theorem 2.4.1 of Ando's thesis, though he just cites other people.

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Section 20 of FPFP is about "full sets of points" and the comparison with the cohomology of the flag variety of a vector bundle.

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Talk with Nat:

- Definitions in terms of divisors.
- Equalizer diagram for quotients by finite subgroups.
- The image of a level structure $\ell$ is a subgroup divisor.
- The schemes classifying subgroups and level structures (which are hard and easy respectively, and which have hard and easy connections to topology respectively).
- It's easy to give explicit examples of the behavior of level structures based on cyclic groups.
- Galois actions on the rings of level structures.

## 6.3 The Drinfel'd ring and the universal level structure

Talk with Nat:

- Recall the Lubin–Tate moduli problem.
- Show that quotients of deformations by finite subgroups give deformations again.
- Define the Drinfel'd ring.
- As an $E^0$–algebra, it carries the universal level structure.
- As an ind–(complete local ring), it corepresents deformations (by precomposition with the map $E^0 \to D_n$) *equipped with level structures*.
- Describe the action by $GL_n(\mathbb{Z}_p)$. (Hint at the action by $M_{n\times n}(\mathbb{Z}_p)$ with $\det \neq 0$.)
- Describe the isogenies pile and its relation to all this? (This doesn't really fit precisely, but it may be good to put here, on an algebraic day.)

## 6.4 Descending coordinates along level structures

It's not clear to me what theorems about level structures and so forth are best included on this day and which belong back in the lecture above. We should be able to split things apart into stuff desired for character theory and stuff desired for descent.

Ando's Theorem 3.4.4: Let $D_j$ be the ring extension of $E_n$ which trivializes the $p^j$–torsion subgroup of $\widehat{\mathbb{G}}_{E_n}$. Let $H$ be a finite subgroup of $\widehat{\mathbb{G}}_{E_n}(D_k)$. There is an unstable transformation of ring-valued functors

$$E_n X \xrightarrow{\Psi^H} D_j \otimes E_n X,$$

and if $F$ is an Ando coordinate then for any line bundle $\mathcal{L} \to X$ there is a formula

$$\psi^H(e\mathcal{L}) = \prod_{h \in H} (h +_F e\mathcal{L}) \in D_j \otimes E_n(X).$$

$D_j$ is Galois over $E_n$ with Galois group $GL_n(\mathbb{Z}/p^j)$. If $\rho$ is a collection of finite subgroups weighted by elements of $E_n$ which is stable under the action of the Galois group, then $\Psi^\rho$ descends to take values in just $E_n$. (For example, the entire subgroup has this property.)

This is built by a character map. Take $H \subseteq F(D_j)[p^j]$ to be a finite subgroup again; then there is a map

$$\chi^H : E_n(D_{H^*}X) \to D_j \otimes E_n(X),$$

where $D_{H^*}$ denotes the extended power construction on $X$ using the Pontryagin dual of $H$. This composes to give an operation

$$Q^H : MU^{2*}(X) \xrightarrow{P_{H^*}} MU^{2|H|*}(D_{H^*}X) \to E_n^{2|H|*}(D_{H^*}X) \xrightarrow{\chi^H} D_j \otimes E_n^{2|H|*}(X).$$

Then $Q^H$ is a ring homomorphism with effects

$$Q^H F^{MU} = F/H, \qquad Q^H(e_{MU}\mathcal{L}) = \prod_{h\in H} h +_F e\mathcal{L}.$$

Then we need to factor $Q^H : MU(X) \to D_j \otimes E_n(X)$ across the orienting map $MU \to E_n$. Since $E_n$ is Landweber flat and $Q^H$ is a ring map, it suffices to do this for the one–point space, i.e., to construct a ring homomorphism

$$\Psi^H : E_n \to D_j$$

so that $\Psi^H = \Psi^H(*) \otimes Q^H$. The first condition above then translates to $\Psi^H F^{MU} = F/H$.

**Theorem 77.** *For each $\star$–isomorphism class of lift $F$ of $\Phi$ to $E_n$, there is a unique choice of coordinate $x$ on $F$, lifting the preferred coordinate on $\Phi$, such that $\alpha_*^H F_x = F_x/H$, or equivalently that $l_H^x = f_H^x$, for all finite subgroups $H$. (These morphisms are arranged in the following diagram:)*

Cite me: Theorem 2.5.7 of Matt's thesis.

$$\begin{tikzcd}
F_x \arrow[r, "f_H^x"] \arrow[rr, bend left, "l_H^x"] & F_x/H \arrow[r, "g_H^x"] & \alpha_*^H F_x \\
F \arrow[u, "x"] \arrow[r] & F/H \arrow[u, "x_H"] \arrow[r, "g_H"] & \alpha_*^H F, \arrow[u, "\alpha_* x"']
\end{tikzcd}$$

*where $\alpha_H : E_n \to D_k$ is the unique ring homomorphism such that there is a $\star$–isomorphism $g_H : F/H \to \alpha_*^H F$.* ∎

This is some serious work, and I don't think we'll prove it. The main point is that $\alpha_*^p F_x = F_x/p$ can be reimagined as $f_p^x(t) = [p]_{F_x}(t)$, and this already is enough to determine what $x$ is by descending along the power of the maximal ideal in $E_n$, the length of a full level structure, and pieces of a smaller level structure inside of the full one. It really is a long argument.

Section 2.7 of Matt's thesis works the example of a normalized coordinate for $\hat{\mathbb{G}}_m$. It's *not* the $p$–typical coordinate. It *is* the standard one! Cool.

**Lemma 93.** $P_r(x+y) = \sum_{j=0}^{r} \mathrm{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$.

*This expresses the non-additivity of the power operations on MU. It's apparently needed in the proof that $Q^H$ acts as it should on Euler classes. It involves transfer formulas, which may mean we need to work that section of HKR into that day.*

**Cite me:** Lemma 3.2.7 of Matt's thesis, BMMS86 page 25, AHS $H_\infty$ appendix.

*Proof.* Represent $x$ and $y$ by maps

$$U \xrightarrow{f} X, \qquad V \xrightarrow{g} Y.$$

Then $P_r(x+y)$ is represented by

$$D_r(U \sqcup V) \xrightarrow{D_r(f \sqcup g)} D_r X.$$

There is a decomposition

$$D_r(U \sqcup V) = \coprod_{j=0}^{r} E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j}),$$

and on the $j$ factor the map $D_r(f \sqcup g)$ restricts to

$$\begin{tikzcd}
E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j}) \arrow[r, "E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (f^j \times g^{r-j})"] \arrow[d] & E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} X^r \arrow[d] \\
D_r(U \sqcup V) \arrow[r, "D_r(f \sqcup g)"] & D_r X,
\end{tikzcd}$$

where the vertical maps are projections. The counterclockwise composite represents the $j$ summand of $P_r(x+y)$ coming from the decomposition above; the clockwise composite represents the class $\mathrm{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$. □

There's also this useful naturality Lemma for power operations and Euler classes: $P_\pi(eV) = e(D_\pi V \to D_\pi X)$. Does that come up in the Quillen chapter? Maybe it should.

**Cite me:** Prop 3.2.10 of Matt's thesis, see also p. 42 of Quillen.

**Lemma 94.** *Write $\Delta : B\pi \times X \to D_\pi X$ and let $\mathcal{L}$ be a complex line bundle on $X$.*

$$\Delta^* P_\pi(e\mathcal{L}) = \prod_{u \in \pi^*} \left( e \begin{pmatrix} E\pi \times_u \mathbb{C} \\ \downarrow \\ B\pi \end{pmatrix} +_{MU} e(\mathcal{L}) \right).$$

**I need to already know:** Matt claims that 3.2.10, the above Lemma, is the beating heart of the paper. Look how similar it looks to the formal group law quotient formula! That's why an expanded formula must be included in the previous days, not just Neil's geometric scribblings.

Matt in and before Theorem 3.3.2 describes the ring $D_k$ as the *image* of the localization map $E_n(B\Lambda_k) \to S^{-1}E_n(B\Lambda_k)$ rather than as the whole target. Why?? He cites HKR for this, but the citation is meaningless because the theorem numbering scheme is so old. Ah, comparing with Lemma 3.3.3 yields a clue: $D_k$ has a universal property as it sits under $E_n$, rather than under $E_n(B\Lambda_k)$...

Now, suppose that we pass down to the $k^{\text{th}}$ Drinfel'd ring, so that the $p^k$–torsion in the formal group is presented as a discrete group $\Lambda^*[p^k]$. Pick such a subgroup $H \subseteq \Lambda^*[p^k]$ with $|H| = r$, and consider also the dual map $\pi : \Lambda[p^k] \to H^*$. We define the character map associated to $H$ to be the composite

$$\chi^H\colon E_n(D_{H^*}X) \xrightarrow{\Delta^*} E_n(BH^*) \otimes_{E_n} E_n(X) \xrightarrow{\chi\pi\otimes 1} D_k \otimes_{E_n} E_n(X) =: D_k(X).$$

This definition is set up so that

$$\chi^H \left( e \begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} \right).$$

In the presence of a coordinate $x$, this sews together to give a cohomology operation:

$$\begin{aligned} Q^H\colon\ MU^{2*}(X) &\xrightarrow{P_G^{MU}} MU^{2r*}(D_{H^*}X) \\ &\xrightarrow{\Delta^*} MU^{2r*}(BH^* \times X) \\ &\xrightarrow{t_x} E_n(BH^* \times X) \\ &\xrightarrow{\simeq} E_nBH^* \otimes_{E_n} E_nX \\ &\xrightarrow{\chi^H\otimes 1} D_kX. \end{aligned}$$

It turns out that $Q^H$ is a ring homomorphism (cf. careful manipulation of HKR's Theorem C, which may not be worth it to write out, but it seems like the main manipulation is the last line of Proof of Theorem 3.3.8 on pg. 466), so each choice of $H$ (and $x$) determines a new coordinate on $D_k$.

**Theorem 78.** *The effect of $Q^H$ on Euler classes is*

$$Q^H e_{MU}\mathcal{L} = f_H^x e_x \mathcal{L} \in D_k(X),$$

*and its effect on coefficients is*

$$Q_*^H F_{MU} = F_x/H.$$

*Proof.* We chase through results established so far:

$$\begin{aligned} Q^H(e_{MU}\mathcal{L}) &= (\chi^H \otimes 1) \circ t_x \circ \Delta^* \circ P_G^{MU}(e_{MU}\mathcal{L}) \\ &= (\chi^H \otimes 1) \circ t_x \left( \prod_{u \in H^* * = H} e_{MU} \begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{MU} e_{MU}\mathcal{L} \right) \\ &= (\chi^H \otimes 1) \left( \prod_{u \in H} e_{E_n} \begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{F_x} e_{E_n}\mathcal{L} \right) \\ &= \prod_{u \in H} (\varphi_{univ}(u) +_{F_x} e_{E_n}\mathcal{L}) = f_H^x(e_{E_n}\mathcal{L}). \end{aligned}$$

Then, "since $D_k$ is a domain, $F_x/H$ is completely determined by the functional equation"

$$f_H^x(F_x(t_1,t_2)) = F_x/H(f_H^x(t_1, f_H^x(t_2))).$$

Take $t_1$ and $t_2$ to be the Euler classes of the two tautological bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$, so that

$$\begin{aligned} Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) &= Q^H\left(e_{MU}\left(\begin{array}{c}\mathcal{L}_1\otimes\mathcal{L}_2\\ \downarrow\\ \mathbb{C}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty\end{array}\right)\right) \\ &= f_H^x\left(e_{E_n}\left(\begin{array}{c}\mathcal{L}_1\otimes\mathcal{L}_2\\ \downarrow\\ \mathbb{C}\mathrm{P}^\infty\times\mathbb{C}\mathrm{P}^\infty\end{array}\right)\right) = f_H^x(t_1 +_{F_x} t_2). \end{aligned}$$

On the other hand, $Q^H$ is a ring homomorphism, so we can also split it over the sum first:

$$\begin{aligned} Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) &= Q^H(e_{MU}\mathcal{L}_1) +_{Q_*^H F^{MU}} Q^H(e_{MU}\mathcal{L}_2) \\ &= f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2), \end{aligned}$$

hence $f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2) = f_H^x(t_1 +_{F_x} t_2)$ and $Q_*^H F^{MU} = F_x/H$.

Finally, we would like to produce a factorization

$$MU \xrightarrow{\Psi^H} E_n \to D_k$$

of the long natural transformation $Q^H$. Since $E_n$ was built by Landweber flatness, it suffices to do this on coefficient rings, i.e., when applying the functors in the diagram to the one-point space. On a point, our calculations above show that $\Psi^H$ exists exactly when $\alpha_*^H F_x = F_x/H$. We did this algebraic calculation earlier: given any coordinate, there is a unique coordinate $P$ that is $\star$–isomorphic to it and through which the operations $Q^H$ factor to give ring operations $\Psi^H$ for all subgroups $H \subseteq \Lambda_k^* = F_P(D_k)[p^k]$. This solves the problem of giving the operations the right *source*.

Leave a remark in here about this: McClure in BMMS works along similar lines to show that the Quillen idempotent is not $H_\infty$, but he doesn't get any positive results (and, in particular, he can't complete his analysis as we do because he doesn't have access to the $BP$–homology of finite groups and to HKR character theory). One wonders whether the stuff here does say something about $BP$ as the height tends toward $\infty$. So far as I know, no one has written much about this. Surely it remains a bee in Matt's bonnet.

Now we focus on giving the operations the right *target*. This is considerably easier. The group $\mathrm{Aut}(\Lambda_k^*)$ acts on the set of subgroups of $\Lambda_k^*$, and we define a ring $Op^k$ by the fixed points of $\mathrm{Aut}(\Lambda_k^*)$ acting on the polynomial ring $E_n$[subgroups of $\Lambda_k^*$]. Note that $Op^k \subseteq Op^{k+1}$, and define $Op = \mathrm{colim}_k\, Op^k$, which consists of elements $\rho = \sum_{i\in I} a_i \prod_{H\in\alpha_i} H$, $I$ a finite set, $a_i \in E_n$, and $\alpha_i$ are certain $\mathrm{Aut}(\Lambda_k^*)$–stable lists of subgroups of

$\Lambda_k^*$, $k \gg 0$, with possible repetitions. For such a $\rho$, we define the associated operations

$$Q^\rho \colon MU^{2*}(X) \xrightarrow{\sum_{i \in I} a_i \prod_{H \in \alpha_i} Q^H} D_k(X),$$
$$\Psi^\rho \colon E_n(X) \xrightarrow{\sum_{i \in I} a_i \prod_{H \in \alpha_i} \Psi^H} D_k(X).$$

The theorem is that these actually land in $E_n(X)$, as they definitely land in $D_k^{\mathrm{Aut}(\Lambda_k^*)} \otimes_{E_n} E_n(X)$, and Galois descent for level structures says that left–hand factor is just $E_n$.

Matt runs the example of the subgroups $\widehat{\mathbb{G}}_m[p^j]$ in $p$–adic $K$–theory and he compares it with some Hopf ring analysis of $E_n \underline{E_n}_*$ due to Wilson

## 6.5 The moduli of subgroup divisors

Following... the original? Following Nat?

Continuing on from the above, if we expected $E_n$ to be $E_\infty$ (or even $H_\infty$) so that it had power operations, then we would want to understand $E_n B\Sigma_{p^j}$ and match that with the operations we see.

—

There are union maps

$$B\Sigma_j \times B\Sigma_k \to B\Sigma_{j+k},$$

stable transfer maps

$$B\Sigma_{j+k} \to B\Sigma_j \times B\Sigma_k,$$

and diagonal maps

$$B\Sigma_j \to B\Sigma_j \times B\Sigma_j.$$

These induce a coproduct $\psi$ as well as products $\times$ and $\bullet$ on $E^0\mathbb{PS}^0$, where $\mathbb{PS}^0 = \coprod_{j=0}^\infty B\Sigma_j$ is the free $E_\infty$–ring on $\mathbb{S}^0$. This is a Hopf ring, and under $\times$ alone it is a formal power series ring. The $\times$–indecomposables (which, I guess, are analogues of considering additive unstable cooperations) are

$$Q^\times E^0\mathbb{PS}^0 = \prod_{k \geq 0} \left( E^0 B\Sigma_{p^k} / \operatorname{tr} E^0 B\Sigma_{p^{k-1}}^p \right),$$

where the $k^{\text{th}}$ factor in the product is naturally isomorphic to $\mathcal{O}_{\mathrm{Sub}_{p^k}(\widehat{\mathbb{G}})}$. The primitives are also accessible as the kernel of the dual restriction map.

Theorem 3.2 shows that $E^0 B\Sigma_k$ is free over $E^0$, Noetherian, and of rank controlled by generalized binomial coefficients. Prop 3.4 is the only place where work gets done, and it's all in terms of $K$–theory and HKR characters.

There's actually an extra coproduct, coming from applying $D$ to the fold map $S^0 \vee S^0 \to S^0$.

The main content of Prop 5.1 (due to Kashiwabara) is that $K_0\mathbb{P}\mathbb{S}^0$ injects into $K_0\underline{BP}_0$. Grading $K_0\mathbb{P}\mathbb{S}^0$ using the $k$–index in $B\Sigma_k$, you can see that it's of graded finite type, so we need only know it has no nilpotent elements to see that $K_0\mathbb{P}\mathbb{S}^0$ is $*$–polynomial. This follows from our computation that $K_0\underline{BP}_0$ is a tensor of power series and Laurent series rings. Corollary 5.2 is about $K_0QS^0$, which is the group completion of $K_0\mathbb{P}\mathbb{S}^0$, so it's the tensor of $K_0\mathbb{P}\mathbb{S}^0$ with a graded field.

Prop 5.6, using a double bar spectral sequence method, shows that $K^0QS^2$ is a formal power series algebra. Tracking the spectral sequences through, you'll find that $Q^\times K^0QS^0$ agrees with $PK^0QS^2$. (You'll also notice that $K^0QS^2$ only has one product on it, cf. Remark 5.4.)

Snaith's theorem says $\Sigma^\infty QX = \Sigma^\infty \mathbb{P}X$ for connected spaces $X$. You can also see (just after Theorem 6.2) the nice equivalences

$$\mathbb{P}_k S^2 \simeq B\Sigma_k^{V_k} \simeq \mathbb{P}_k(S^0)^{V_k},$$

where superscript denotes Thom complex. So, for a complex-orientable cohomology theory, you can learn about $\mathbb{P}_k S^0$ from $\mathbb{P}_k S^2$. In particular, we finally learn that $E^0\mathbb{P}S^0$ is a formal power series $\times$–algebra (once checking that the Thom isomorphism is a ring map). (We already knew the homological version of this claim.)

Section 8 has a nice discussion about indecomposables and primitives, to help move back and forth between homology and cohomology. It probably helps most with the dimension count argument below that we aren't going to get into.

Start again with $D_{p^k}S^2 \simeq B\Sigma_{p^k}^{V_{p^k}}$. We can associate to this a divisor $\mathbb{D}(V_{p^k})$ on $(B\Sigma_{p^k})_E$, which we know little about, but it is classified by a map to $\operatorname{Div}_{p^k} \mathbb{CP}_E^\infty$. This receives a closed inclusion from $\operatorname{Sub}_{p^k} \mathbb{CP}_E^\infty$, so their pullback $Z_k$ is the largest subscheme of $(B\Sigma_{p^k})_E$ over which $\mathbb{D}(V_{p^k})$ is a subgroup divisor.

$$\begin{tikzcd}
H_k \arrow[rr] \arrow[dd] & & \mathbb{D}(V_{p^k}) \arrow[dd] & & \\
& Z_k \arrow[rr] \arrow[dr] & & \operatorname{Sub}_{p^k} \mathbb{CP}_E^\infty \arrow[dr] & \\
\operatorname{Spf} E^0 B\Sigma_{p^k}/\mathrm{tr} \arrow[rr] \arrow[ur, dotted] & & (B\Sigma_{p^k})_E \arrow[rr] & & \operatorname{Div}_{p^k} \mathbb{CP}_E^\infty
\end{tikzcd}$$

We will show the existence of the dashed map, implying that the restricted divisor $H_k$ is a subgroup divisor on $Y_k = \operatorname{Spf} E^0 B\Sigma_{p^k}/\mathrm{tr}$.

(Prop 9.1:) This proof falls into two parts: first we construct a family of maps to $(B\Sigma_{p^k})_E$ on whose image $\mathbb{D}(V_{p^k})$ restricts to a subgroup divisor, and then we show that the union of their images is exactly $Y_k$. Let $A$ be an abelian $p$–subgroup of $\Sigma_{p^k}$ that acts transitively on $\{1, \ldots, p^k\}$ (i.e., it is not boosted from some transfer). The restriction of $V_{p^k}$ to $A$ is the regular representation, which splits as a sum of characters $V_{p^k}|_A = \bigoplus_{\mathcal{L} \in A^*} \mathcal{L}$. Identifying $BA_E = \underline{\mathsf{FormalGroups}}(A^*, \mathbb{CP}^\infty_E)$, $\mathbb{D}(V_{p^k})$ restricts all the way to $\sum_{\mathcal{L} \in A^*} [\varphi(\mathcal{L})]$, with $\varphi : A^* \to$"$\Gamma(\mathrm{Hom}(A^*, \widehat{\mathbb{G}}), \widehat{\mathbb{G}})$". In Finite Subgroups of Formal Groups (see Props 22 and 32), we learned that the restriction of $\mathbb{D}(V_{p^k})$ further to $\mathrm{Level}(A^*, \mathbb{CP}^\infty_E)$ is a subgroup divisor. So, our collection of maps are those of the form

$$\mathrm{Level}(A^*, \mathbb{CP}^\infty_E) \to \underline{\mathsf{FormalGroups}}(A^*, \mathbb{CP}^\infty_E) = BA_E \to (B\Sigma_{p^k})_E.$$

Here, finally, is where we have to do some real work involving Chern classes and commutative algebra, so I'm inclined to skip it in the lectures. Finally, you do a dimension count to see that $Z_k$ and $\operatorname{Spf} E^0 B\Sigma_{p^k}/\mathrm{tr}$ have the same dimension (which requires checking enough commutative algebra to see that "dimension" even makes sense), and so you show the map is injective and you're done.

—

Here's Neil's proof of the joint images claim. It seems like a clear enough use of character theory that we should include it, if we can make character theory itself clear.

Recall from [18, Theorem 23] that $\mathrm{Level}(A^*, \widehat{\mathbb{G}})$ is a smooth scheme, and thus that $D(A) = \mathcal{O}_{\mathrm{Level}(A^*, \widehat{\mathbb{G}})}$ is an integral domain. Using [18, Proposition 26], we see that when $\mathcal{L} \in A^*$ is nontrivial, we have $\varphi(\mathcal{L}) \neq 0$ as sections of $\widehat{\mathbb{G}}$ over $\mathrm{Level}(A^*, \widehat{\mathbb{G}})$, and thus $e(\mathcal{L}) = x(\varphi(\mathcal{L})) \neq 0$ in $D(A)$. It follows that that $c_{p^k} = \prod_{\mathcal{L} \neq 1} e(\mathcal{L})$ is not a zero-divisor in $D(A)$. On the other hand, if $A'$ is an Abelian $p$-subgroup of $\Sigma_{p^k}$ which does not act transitively on $\{1, \ldots, p^k\}$, then the restriction of $V_{p^k}1$ to $A'$ has a trivial summand, and thus $c_{p^k}$ maps to zero in $D(A')$. Next, we recall the version of generalised character theory described in [8, Appendix A].

$$p^{-1}E^0 BG = \left( \prod_A p^{-1} D(A) \right)^G$$

where $A$ runs over all Abelian $p$-subgroups of $G$. As $\overline{R}_k = E^0(B\Sigma_{p^k})/ann(c_{p^k})$ and everything in sight is torsion-free, we see that $p^1\overline{R}_k$ is the quotient of $p^1 E^0 B\Sigma_{p^k}$ by the annihilator of the image of $c_{p^k}$ . Using our analysis of the images of $c_{p^k}$ in the rings $D(A)$, we conclude that

$$p^{-1}\overline{R}_k = \left(\prod_A p^1 D(A)\right)^{\Sigma_{p^k}},$$

where the product is now over all transitive Abelian $p$-subgroups. This implies that for such $A$, the map $E^0 B\Sigma_{p^k} \to D(A)$ factors through $\overline{R}_k$, and that the resulting maps $\overline{R}_k \to D(A)$ are jointly injective. This means that $Y_k = \operatorname{Spf} \overline{R}_k$ is the union of the images of the corresponding schemes $\operatorname{Level}(A^*, \widehat{\mathbb{G}})$, as required.

## 6.6 Interaction with Θ–structures

The Ando–Hopkins–Strickland result that the $\sigma$–orientation is an $H_\infty$–map

The main classical point is that an $MU\langle 0\rangle$–orientation is $H_\infty$ when the following diagram commutes for every choice of $A$:

$$\begin{tikzcd}
(BA^* \times \mathbb{C}P^\infty)^{V_{reg} \otimes \mathcal{L}} \arrow[r] & D_n MU\langle 0\rangle \arrow[r] \arrow[d] & D_n E \arrow[d] \\
 & MU\langle 0\rangle \arrow[r] & E
\end{tikzcd}$$

(This is equivalent to the condition given in the section on Matt's thesis. In fact, maybe I should try writing this so that Matt's thesis uses the same language?) If you write out what this means, you'll see that a given coordinate on $E$ pulls back to give two elements in the $E$–cohomology of that Thom spectrum (or: sections of the Thom sheaf), and the orientation is $H_\infty$ when they coincide.

Similarly, an $MU\langle 6\rangle$–orientation corresponds to a section of the sheaf of cubical structures on a certain Thom sheaf. Using the $H_\infty$ structures on $MU\langle 6\rangle$ and on $E$ give two sections of the pulled back sheaf of cubical structures, and the $H_\infty$ condition is that they agree for all choices of group $A$.

Then you also need to check that the $\sigma$–orientation actually satisfies this.

The AHS document really restrictions attention to $E_2$. Is there a version of this story that gives non-supersingular orientations too, or even the $K_{\mathrm{Tate}}$ orientation? I can't tell if the restriction in AHS's exposition comes from not knowing that $K_{\mathrm{Tate}}$ has an $E_\infty$ structure or if it comes from a restriction on the formal group. (At one point it looks like they only need to know that $p$ is regular on $\pi_0 E$, cf. 16.5...)

Section 3.1: Intrinsic description of the isogenies story for an $H_\infty$ *complex orientable* ring spectrum, without mention of a specific orientation / coordinate. This is nice: it means that a complex orientation has to be a coordinate which is compatible with the descent picture already extant on the level of formal groups, which is indeed the conclusion of Matt's thesis.

Section 3.2: They define an abelian group indexed extended power construction
$$D_A(X) = \mathcal{L}(U^{A^*}, U) \wedge_{A^*} X^{(A^*)},$$
where $\mathcal{L}(U^{A^*}, U)$ is the space of linear isometries from the $A^{*\text{th}}$ power of a universe $U$ down to itself. Yuck. Then, given a level structure $(i\colon \operatorname{Spf} R \to S_E, \ell\colon A_{\operatorname{Spf} R} \to i^*\widehat{\mathbb{G}})$, they construct a map

$$\psi_\ell^E\colon \pi_0 E \xrightarrow{D_A} \pi_0 \mathsf{Spectra}(D_A S^0, E) = \pi_0 E^{BA^*_+} \to \mathcal{O}((BA^*)_E) \xrightarrow{\chi_\ell} R,$$

where $\chi_\ell$ is the map classifying the homomorphism $\ell$. This is a continuous map of rings: it's clearly multiplicative, it's additive up to transfers (but those vanish for an abelian group), and it's continuous by an argument in Lemma 3.10. (You don't actually need an abelian group here; you can work in the scheme of subgroups — i.e., in the cohomology of $B\Sigma_k$ modulo transfers — and this will still work.) This construction is natural in $H_\infty$ maps $f\colon E \to F$:

$$\begin{tikzcd}
i^*S_F \arrow[drr, bend left, "\psi_\ell^F"] \arrow[dr, dotted, "\psi_\ell^{F/E}"] \arrow[ddr, bend right] & & \\
& \operatorname{Spf} R \times_{i,S_E,S_f} S_F \arrow[r, "\psi_\ell^F"] \arrow[d] & S_F \arrow[d, "S_f"] \\
& \operatorname{Spf} R \arrow[r, "\psi_\ell^E"] & S_E,
\end{tikzcd}$$

begetting the relative map $\psi_\ell^{F/E}\colon i^*S_F \to (\psi_\ell^E)^* S_F$ as indicated. For example, take $F = E^{\mathbb{C}\mathrm{P}^\infty_+}$, so that $\widehat{\mathbb{G}} = S_F$, giving the (group) map

$$\psi_\ell^{\widehat{\mathbb{G}}/E}\colon i^*\widehat{\mathbb{G}} \to (\psi_\ell^E)^*\widehat{\mathbb{G}}.$$

One of the immediate goals is to show that this is an isogeny. A different construction we can do is take $V$ to be a virtual bundle over $X$ and set $F = E^{X_+}$. Given $m \in \pi_0\mathsf{Spectra}(X^V, E)$ applying the construction of $D_A$ above gives an element

$$\psi_\ell^V(m) \in R \mathop{\widehat{\otimes}}_{\chi_\ell, \hat{\pi}_0 E^{BA^*_+}} \hat{\pi}_0 \mathsf{Spectra}((BA^* \times X)^{V_{reg} \otimes V}, E).$$

This map is additive and also $\psi_\ell^V(xm) = \psi_\ell^F(x)\psi_\ell^V(m)$, so we can interpret this as a map
$$\psi_\ell^V\colon (\psi_\ell^F)^*\mathbb{L}(V) \to \chi_\ell^* \mathbb{L}(V_{reg} \otimes V)$$
of line bundles over $i^*S_F = i^*X_E$.

Cite me: Lemma 3.19 of AHS $H_\infty$.

**Lemma 95.** *The map $\psi_\ell^V$ has the following properties:*

1. *If $m$ trivializes $\mathbb{L}(V)$ then $\psi_\ell^V(m)$ trivializes $\chi_\ell^*\mathbb{L}(V_{reg}\otimes V)$.*
2. $\psi_\ell^{V_1\oplus V_2} = \psi_\ell^{V_1}\otimes\psi_\ell^{V_2}$.
3. *For $f\colon Y\to X$ a map, $\psi_\ell^{f^*V} = f^*\psi_\ell^V$.* □

In particular, we can apply this to $X = \mathbb{C}P^\infty$ and $\mathbb{L}(\mathcal{L}-1) = \mathcal{I}(0)$. Then 8.11 gives

$$\psi_\ell^{\mathcal{L}-1}\colon (\psi_\ell^F)^*\mathcal{I}_{\widehat{\mathbb{G}}}(0)\to\chi_\ell^*\mathbb{L}(V_{reg}\otimes(\mathcal{L}-1)) = \mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell).$$

Cite me: Prop 3.21.

**Theorem 79.** *The map $\psi_\ell^{\widehat{\mathbb{G}}/E}\colon i^*\widehat{\mathbb{G}}\to(\psi_\ell^E)^*\widehat{\mathbb{G}}$ of 3.15 is an isogeny with kernel $[\ell(A)]$. Using $\psi_\ell^{\widehat{\mathbb{G}}/E}$ to make the identification*

$$(\psi_\ell^{\widehat{\mathbb{G}}/E})^*\mathcal{I}_{(\psi_\ell^E)^*\widehat{\mathbb{G}}}(0)\cong\mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell),$$

*the map $\psi_\ell^{\mathcal{L}-1}$ sends a coordinate $x$ on $\widehat{\mathbb{G}}$ to the trivialization $(\psi_\ell^{\widehat{\mathbb{G}}/E})^*(\psi_\ell^E)^*x$ of $\mathcal{I}_{i^*\widehat{\mathbb{G}}}(\ell)$.* □

3.24 might be interesting.

So far, it seems like the point is that the identity map on $MU\langle 0\rangle$ classifies a section of the ideal sheaf at zero of the universal formal group which is compatible with descent for level structures, so any $H_\infty$ map out of $MU\langle 0\rangle$ classifies not just a section of the ideal sheaf at zero of whatever other formal group but does so in a way that is, again, compatible with descent for level structures.

Cite me: Prop 4.13.

The discussion leading up to this theorem seems interesting, especially equations 4.10,12.

**Theorem 80.** *Let $g : MU\langle 0\rangle \to E$ be a homotopy multiplicative map, and let $s = s_g$ be the corresponding trivialization of $\mathcal{I}_{\widehat{\mathbb{G}}}(0)$. If the map $g$ is $H_\infty$, then for any level structure $\ell : A\to i^*\widehat{\mathbb{G}}$ the section $s$ satisfies the identity*

$$N_{\psi_\ell^{\widehat{\mathbb{G}}/E}}i^*s = (\psi_\ell^E)^*s,$$

*in which the isogeny $\psi_\ell^{\widehat{\mathbb{G}}/E}$ has been used to make the identification*

$$N_{\psi_\ell^{\widehat{\mathbb{G}}/E}}i^*\mathcal{I}_{\widehat{\mathbb{G}}}(0)\cong\mathcal{I}_{(\psi_\ell^E)^*\widehat{\mathbb{G}}}(0).\square$$

Cite me: Eqn 5.3, generalizes Quillen's splitting formula.

**Lemma 96.** *For $V$ a vector bundle on a space $X$ and $V_{reg}$ the (vector bundle over $BA^*$ induced from) the regular representation on $A$, there is an isomorphism of sheaves over $(BA^*\times X)_E$*

$$\mathbb{L}(V_{reg}\otimes V)\cong\bigotimes_{a\in A}\widetilde{T}_a\mathbb{L}(V).$$

Eqn 5.4 claims to use 5.3 but seems to be using something about the behavior of the norm map on line bundles vs the translated sum of divisors appearing in 5.3.

The beginning of the proof of 6.1 appears to be a simplification of some of the descent arguments appearing in the algebraic parts of Matt's thesis's main calculations. On the other hand, I can't even read what the McClure reference in 6.1 is doing. What's $\Delta^*$??

**Lemma 97.** *Take $\pi_0 E$ to be a complete local ring and $\widehat{\mathbb{G}}_E$ to be of finite height. If $B^* \subset A^*$ is a proper subgroup, then the following composite map of $\pi_0 E$–modules is zero:*

Cite me: Prop 7.5.

$$\pi_0 E^{BB^*_+} \xrightarrow{transfer} \pi_0 E^{BA^*_+} \xrightarrow{\chi_\ell} \mathcal{O}(T).$$

*Proof.* It suffices to consider the tautological level structure over $\mathrm{Level}(A, \widehat{\mathbb{G}})$. We may take $A$ to be a $p$–group, and indeed for now we set $A = \mathbb{Z}/p$, $B = 0$. For $t \in \pi_0 E^{\mathbb{CP}^\infty_+}$ a coordinate with formal group law $F$, we have

$$\pi_0 E^{BA^*_+} \cong \pi_0 E[\![t]\!]/[p]_F(t)$$

and $\tau : \pi_0 E^{BB^*_+} = \pi_0 E \to \pi_0 E^{BA^*_+}$ is given by $\tau(1) = \langle p \rangle_F(t)$, where $\langle p \rangle_F(t) = [p]_F(t)/t$ is the "reduced $p$–series". The result then follows from the isomorphism $\mathcal{O}(\mathrm{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}_E)) \cong \pi_0 E[\![t]\!]/\langle p \rangle_F(t)$. The result then follows in general by induction: $B^*$ can be taken to be a *maximal* proper subgroup of $A^*$, with cokernel $\mathbb{Z}/p$.

*Example 35.* Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group with coordinate $x$ so that the group law is

$$x +_{\widehat{\mathbb{G}}_m} y = x + y - xy, \quad [p](x) = 1 - (1-x)^p.$$

The monomorphism $\mathbb{Z}/p \to \widehat{\mathbb{G}}_m(\mathbb{Z}[\![y]\!]/[p](y))$ given by $j \mapsto [j](y)$ becomes the zero map under the base change

$$\begin{aligned} \mathbb{Z}[\![y]\!]/[p](y) &\to \mathbb{Z}/p, \\ y &\mapsto 0. \end{aligned}$$

*Remark 77.* If $R$ is a domain of characteristic 0, then a level structure over $R$ actually induces a monomorphism on points.

Cite me: Corollary 9.21, Prop 9.17.

**Lemma 98.** *The natural map*

Cite me: Prop 9.24.

$$\mathcal{O}(\underline{\mathsf{FormalGroups}}(\mathbb{Z}/p, \widehat{\mathbb{G}})) \to R \times \mathcal{O}(\mathrm{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}))$$

One of the reduction steps in Prop 6.1 is handled by 9.24, which is in turn equivalent to a basic case of an HKR theorem, so should be stated on that day (or in the algebraic day).

*is injective.*

*Proof.*

Fill this.

I left off at Section 10.

—— Descent along level structures, simplicially (Section 11) ——

Actually, this section appears *not* to be about FGps, and instead it's about the *coarse moduli quotient* to the functor of formal groups, which is not locally representable. I'm a little confused about this — I intend to ask Mike what's going on.

Write $\mathrm{Level}(A) \to \mathrm{FGps}$ for the parameter space of a formal group equipped with a level–$A$ structure, together with its structure map (to the *coarse moduli of formal groups!!!*). We define a sequence of schemes by: $\mathrm{Level}_0 = \mathrm{FGps}$, $\mathrm{Level}_1 = \coprod_{A_0} \mathrm{Level}(A_0)$ for finite abelian groups $A_0$, and most generally

$$\mathrm{Level}_n = \coprod_{0 = A_n \subseteq \cdots \subseteq A_0} \mathrm{Level}(A_0).$$

There are two maps $\mathrm{Level}_1 \to \mathrm{Level}_0$. One is the structural one, where we simply peel off the formal group and forget the level structure. The other comes from the quotient map: $\ell \colon A \to \widehat{\mathbb{G}}$ yields a quotient isogeny $q \colon \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}/\ell$, and we take the second map $\mathrm{Level}_1 \to \mathrm{Level}_0$ to send $\ell$ to $\widehat{\mathbb{G}}/\ell$. Then, consider the following Lemma:

Cite me: AHS Lemma 11.3.

**Lemma 99.** *For $\ell \colon A \to \widehat{\mathbb{G}}$ a level structure and $B \subseteq A$ a subgroup, the induced map $\ell|_B \colon B \to \widehat{\mathbb{G}}$ is a level structure and the quotient $\widehat{\mathbb{G}}/\ell|_B$ receives a level structure $\ell' \colon A/B \to \widehat{\mathbb{G}}/\ell|_B$.* □

This gives us enough compatibility among quotients to use the two maps above to assemble the $\mathrm{Level}_*$ schemes into a simplicial object. Most face maps just omit a subgroup, except for the last face map, since the zero subgroup is not permitted to be omitted. Instead, the last face map sends the string of subgroups $0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_0$ and level structure $\ell \colon A_0 \to \widehat{\mathbb{G}}$ to the quotient string $0 = A_{n-1}/A_{n-1} \subseteq \cdots \subseteq A_0/A_{n-1}$ and quotient level structure $\ell \colon A_0/A_{n-1} \to \widehat{\mathbb{G}}/\ell|_{A_{n-1}}$. The degeneracy maps come from lengthening one of these strings by an identity inclusion.

Cite me: Definition 11.10, Remark 11.11.

**Definition 87.** Let $\widehat{\mathbb{G}} \colon F \to \mathrm{FGps}$ be a functor over formal groups, and define schemes $\mathrm{Level}(A, F) = \mathrm{Level}(A) \times_{\widehat{\mathbb{G}}} F$ and $\mathrm{Level}_n(F) = \mathrm{Level}_n \times_{\widehat{\mathbb{G}}} F$. Then, *descent data for level structures on $F$* is the structure of a simplicial scheme on $\mathrm{Level}_*(F)$, together with a morphism of simplicial schemes $\mathrm{Level}_*(F) \to \mathrm{Level}_*$. It is enough to specify a map $d_1 \colon \mathrm{Level}_1(F) \to F$, use that to build the simplicial scheme structure as in the above Lemma, and assert that the following square commutes:

$$\begin{tikzcd} \mathrm{Level}_1(F) \arrow[r] \arrow[d, "d_1"] & \mathrm{Level}_1 \arrow[d, "d_1"] \\ F \arrow[r] & \mathrm{FGps}. \end{tikzcd}$$

*Example 36.* Let $\widehat{\mathbb{G}} \colon S \to \mathrm{FGps}$ be a formal group of finite height over a $p$–local formal scheme $S$. The functor $\mathrm{Level}(A, \widehat{\mathbb{G}})$ is exactly the functor defined in Section 9 (see above), and in particular it is represented by an $S$–scheme. The maps $\psi_\ell$ and $f_\ell$ from Definition 3.1 amount to giving a map

$d_1\colon \operatorname{Level}_1(\widehat{\mathbb{G}}) \to S$ and an isogeny $q\colon d_0^*\widehat{\mathbb{G}} \to d_1^*\widehat{\mathbb{G}}$ whose kernel on $\operatorname{Level}(A,\widehat{\mathbb{G}})$ is $A$. The other conditions on Definition 3.1 exactly ensure that $(\operatorname{Level}_*(\widehat{\mathbb{G}}), d_*, s_*)$ is a simplicial functor and over $\operatorname{Level}_2(\widehat{\mathbb{G}})$ the relevant hexagonal diagram commutes:

$$\begin{tikzcd} & d_0^*d_0^*\widehat{\mathbb{G}} \arrow[dl, equal] \arrow[dr, "d_0^*q"] & \\ d_1^*d_0^*\widehat{\mathbb{G}} \arrow[d, "d_1^*q"] & & d_0^*d_1^*\widehat{\mathbb{G}} \arrow[d, equal] \\ d_1^*d_1^*\widehat{\mathbb{G}} \arrow[dr, equal] & & d_2^*d_0^*\widehat{\mathbb{G}} \arrow[dl, "d_2^*q"] \\ & d_2^*d_1^*\widehat{\mathbb{G}}. & \end{tikzcd}$$

*Example 37.* We now further package this into a single object. Let $\underline{\widehat{\mathbb{G}}}$ be the functor over FGps whose value on $R$ is the set of pullback diagrams

$$\begin{tikzcd} \widehat{\mathbb{G}}' \arrow[r, "f"] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\ \operatorname{Spf} R \arrow[r, "i"] & S \end{tikzcd}$$

such that the map $\widehat{\mathbb{G}}' \to i^*\widehat{\mathbb{G}}$ induced by $f$ is a homomorphism (hence isomorphism) of formal groups over $\operatorname{Spf} R$. For a finite abelian group $A$, write $\operatorname{Level}(A,\underline{\widehat{\mathbb{G}}})(R)$ for the set of diagrams

$$\begin{tikzcd} A_{\operatorname{Spf} R} \arrow[r, "\ell"] \arrow[dr] & \widehat{\mathbb{G}}' \arrow[r, "f"] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\ & \operatorname{Spf} R \arrow[r, "i"] & S \end{tikzcd}$$

where the square forms a point in $\underline{\widehat{\mathbb{G}}}(R)$ and $\ell$ is a level–$A$ structure. Giving a map of functors $d_1\colon \operatorname{Level}_1(\underline{\widehat{\mathbb{G}}}) \to \underline{\widehat{\mathbb{G}}}$ making the above square commute is to give a pullback diagram

$$\begin{tikzcd} \widehat{\mathbb{G}}/\ell \arrow[r] \arrow[d] & \widehat{\mathbb{G}} \arrow[d] \\ \operatorname{Level}_1(\widehat{\mathbb{G}}) \arrow[r] & S, \end{tikzcd}$$

or equivalently a map of formal schemes $\operatorname{Level}_1(\widehat{\mathbb{G}}) \to S$ and an isogeny $q\colon d_0^*\widehat{\mathbb{G}}d_1^*\widehat{\mathbb{G}}$ whose kernel on $\operatorname{Level}(A,\widehat{\mathbb{G}})$ is $A$. Therefore, descent data for level structures on the formal group $\widehat{\mathbb{G}}$ (in the sense of Section 3) are equivalent to descent data for level structures on the functor $\underline{\widehat{\mathbb{G}}}$.

—— Section 12: Descent for level structures on Lubin–Tate groups ——

Let $k$ be perfect of positive characteristic $p$, and let $\Gamma$ be a formal group of finite height over $k$. Recall that this induces a relative Frobenius

$$\begin{tikzcd}
\Gamma \arrow[r, "F"] \arrow[rr, bend left, "\varphi_\Gamma"] \arrow[dr] & \varphi_k^*\Gamma \arrow[r] \arrow[d] & \Gamma \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "\varphi_k"] & \operatorname{Spec} k.
\end{tikzcd}$$

The map $F$ is an isogeny of degree $p$, with kernel the divisor $p \cdot [0]$. Recall also that a deformation $H$ of $\Gamma$ to $T$ induces a map $\underline{H} \to \operatorname{Def}(\Gamma)$, and there is a universal such $\widehat{\mathbb{G}}$ over the ground scheme $S \cong \operatorname{Spf} \mathbb{W}(k)[\![u_1, \ldots, u_{d-1}]\!]$ such that $\underline{\widehat{\mathbb{G}}} \to \operatorname{Def}(\Gamma)$ is an isomorphism of functors over FGps.

Now consider a point in $\operatorname{Level}(A, \operatorname{Def}\Gamma)$:

$$\begin{tikzcd}
A_T \arrow[r, "\ell"] \arrow[dr] & H \arrow[d] & H_0 \arrow[l] \arrow[r, "f"] \arrow[d] & \Gamma \arrow[d] \\
& T & T_0 \arrow[l] \arrow[r, "j"] & \operatorname{Spec} k.
\end{tikzcd}$$

The level structure $\ell$ gives rise to a quotient isogeny $q\colon H \to H'$. Since $A$ is sent to 0 in $\mathcal{O}_{T_0}$, there is a canonical map $\bar{q}$ fitting into the diagram

$$\begin{tikzcd}
H \arrow[r, "q"] \arrow[dd] & H' \arrow[ddl] & & & & & \\
& & H_0 \arrow[ull] \arrow[r] \arrow[rr, bend left] \arrow[ddr] & H_0' \arrow[ull] \arrow[r, dotted, "\bar{q}"] \arrow[dd] & (\varphi^r)^* H_0 \arrow[r] \arrow[ddl] & H_0 \arrow[r, "f"] \arrow[dd] & \Gamma \arrow[dd] \\
T & & & & & & \\
& & & T_0 \arrow[ulll] \arrow[rr, "\varphi^r"] & & T_0 \arrow[r, "j"] & \operatorname{Spec} k.
\end{tikzcd}$$

The map $\bar{q}$ combines with the rest of the maps to exhibit $H'$ as a deformation of $\Gamma$, and hence we get a natural transformation

$$d_1\colon \operatorname{Level}_1(\operatorname{Def}(\Gamma)) \to \operatorname{Def}(\Gamma).$$

Since $\varphi^r \varphi^s = \varphi^{r+s}$, this gives descent data for level structures on $\operatorname{Def}(\Gamma)$. Identifying this functor with $\underline{\widehat{\mathbb{G}}}$ using Lubin–Tate theory, we equivalently have shown the existence of descent data for level structures on $\underline{\widehat{\mathbb{G}}}$.

Incidentally, the descent data constructed here is also the descent data that would come from the structure of an $E_\infty$–orientation on the Morava $E$–theory $E_d$, essentially because the divisor associated to the kernel of the relative Frobenius on the special fiber is forced to be $p[0]$, and everything is dictated by how the deformation theory *has* to go (and the fact that the topological operations we're studying induce deformation-theoretic-describable operations on algebra).

——Section 15: Level structures on elliptic curves, and the relation to the $\sigma$–orientation / the corresponding section of the $\Theta^3$–sheaf——

## *Tyler's argument*

There's an important injectivity result used by Ando and Ando–Hopkins–Strickland (though Matt blames it on Hopkins and Strickland both times) about the injectivity of a certain $p^{\text{th}}$ power map. They cite the McClure chapter of BMMS, but McClure's proof requires finite type hypotheses on the cohomology theory involved, which Morava $E$–theory does not satisfy. There is a similar proof in the recent paper of Hopkins–Lawson, and so Nat and I wrote to Tyler about whether there was a common generalization of the two theorems that would give a good replacement argument. Here is his reply:

—

Here are my current thought processes, which may be a bit messy at present. Fix a space $X$ and take $X^{(p)}$ for its smash power, as McClure does.

Let's write $M = F(\Sigma^\infty X^{(p)}, E)$ for the function spectrum which is now $C_p$–equivariant, and $N = F(\Sigma^\infty X, E)$. Let's assume that $E$ has an $E_\infty$ multiplication and that $X$ is nice in the following sense: $E^X$ is a wedge of copies of $E$ (unshifted). This is satisfied when $E$ is $E$–theory and $X$ is finite type with $\mathbb{Z}_{(p)}$-homology only in even degrees.

We get two maps:

$$M^{hC_p} \to M$$

This will realize our "forgetful" map $E^*(DX) \to E^*(X^{(p)})$.

$$M^{hC_p} \to N^{hC_p}$$

This will realize the "other" map $E^*(DX) \to E^*((BC_p)_+ \wedge X)$.

We want to prove that these are jointly monomorphisms.

The assumptions on $X$ actually imply that $E^{X^{(p)}} = (E^X)^{(p)}$ where the latter smash is taken over $E$. This decomposes, $C_p$–equivariantly, into a wedge of copies of $E$ with trivial action and a bunch of regular representations $E[C_p]$. Since $M$ is $E$–dual to this, we find that the map $M^{hC_p} \to M$ is a monomorphism on all the $E[C_p]$ components; on the $E$ parts with trivial action it decomposes as a product of projections $E_*[\![x]\!]/[p](x) \to E_*$. The kernel of this consists of the multiples of $x$. So if we want to prove a monomorphism, all we have to do now is show that these multiples of x map monomorphically into the homotopy of $N^{hC_p}$.

I now want to consider the composite to the Tate spectra

$$M^{hC_p} \to N^{hC_p} \to N^{tC_p}$$

or equivalently

$$M^{hC_p} \to M^{tC_p} \to N^{tC_p}.$$

The first composition shows that, if we can show that this composite is a monomorphism on the multiples of $x$, we will be done. The second composition has, as its first map, inverting $x$, and it's a monomorphism on the desired classes. So we just have to check that the second map preserves that.

This has the following benefit: instead of being born out of the unstable diagonal map $X \to X^{(p)}$, the constructions

$$M^{tC_p} = F(X^{(p)}, E)^{tC_p}$$

and

$$N^{tC_p} = F(X, E)^{tC_p}$$

take cofiber sequences in (finite) $X$ to fiber sequences of spectra. I think that this means that, instead of being functions of the unstable diagonal map on $X$, they are constructions that only require knowledge of the *stable* homotopy type of $X$. I believe in fact that, by checking the case $X = S^0$, we then find that the map $M^{tC_p} \to N^{tC_p}$ is an equivalence for any finite $X$, and that should hopefully be enough to buy us a monomorphism on any of the $X$'s that we're describing.

I don't understand this. I guess this has been a recurring theme in the Thursday seminar, and also in Chapters 2 and 6 of these notes (in some guise). I can ask Mike to explain it to me.

The stability of the Tate construction comes about for the following reason.

Say we have a cofiber sequence $X \to Y \to Y/X$. Then the $p$–fold smash power $Y^{(p)}$ has a natural, equivariant filtration: $Y$ is, for dumb reasons, the homotopy colimit of $(X \to Y)$, then $Y^{(p)}$ is the hocolim of a $p$–fold smash power of this diagram (now indexed on $\{0 \to 1\}^{\times p}$). We can filter this diagram, equivariantly, according to "distance from the initial vertex", and get an equivariant filtration of $Y^{(p)}$ whose associated graded in degree $k$ consists of all ways to smash $k$ copies of $Y/X$ with $(p-k)$ copies of $X$ in some order. In grading 0 this is $X^{(p)}$, and in grading $p$ this is $(Y/X)^{(p)}$; in all the gradings in between you get a wedge of terms where $C_p$ acts by permuting the wedge factors.

The Tate construction preserves $C_p$-equivariant cofiber sequences, and destroys anything where $C_p$ acts by permuting wedge factors. As a result, the only parts that survive are the bottom $(X^{(p)})^{tC_p}$ and the top $((Y/X)^{(p)})^{tC_p}$ in a cofiber sequence.

The standard reference should probably be Greenlees–May's *Generalized Tate cohomology theories* but I'm not as closely familiar with the contents there. Sometimes this is called the topological Singer construction and I originally learned about it from Lunoe–Nielsen–Rognes. This talk about the stability properties is present in section 2 of DAG XIII somewhere, and possibly also in Jacob's notes on the Sullivan conjecture. Charles pointed out to me recently that these properties of the Tate construction are really why the Steenrod operations and certain power operations are stable: it's because they don't come out of a homotopy orbit construction on a smash power, but instead out of a Tate construction.

## *Other stuff that goes in this chapter*

Dyer–Lashof operations, the Steenrod operations, and isogenies of the formal additive group

**Cite me:** See Neil's *Steenrod algebra* note, maybe? Talk to Mike?.

Another augmentation to the notion of a context: working not just with $E_*X$ but with $E_*(X \times BG)$ for finite $G$.

Charles's *The congruence criterion* paper codifies the Hecke algebra picture Neil is talking about, and in particular it talks about sheaves over the pile of isogenies.

If we're going to talk about that Hecke algebra, then maybe we can also talk about the period map, since one of the main points of it is that it's equivariant for that action.

Section 3.7 of Matt's thesis also seems to deal with the context question: he gives a character-theoretic description of the total power operation, which ties the behavior of the total power operation to a formula of type "decomposition into subgroups". Worth reading.

The rational story: start with a sheaf on the isogenies pile. Tensor everything with $\mathbb{Q}$. That turns this thing into a rational algebra under the Drinfel'd ring together with an equivariant action of $GL_n\mathbb{Q}_p$.

This is Nat's claim. Check back with him about how this is visible.

Matt's Section 4 talks about the $E_\infty$ structure on $E_n$ and compatibility with his power operations. It's not clear how this doesn't immediately follow from the stuff he proves in Section 3, but I think I'm just running out of stream in reading this thesis. One of the neat features of this later section is that it relies on calculations in $E_n D_\pi \underline{MU}_{2*}$, which is an interesting way to mix operations coming from instability and from an $H_\infty$–structure. This is yet another clue about what the relevant picture of a context should look like. He often cites VIII.7 of BMMS.

Mike says that Mahowald–Thompson analyzed $L_{K(n)}\Omega S^{2n+1}$ by writing down some clever finite resolution. The resolution that they produce by hand is actually exactly what you would get if you tried to understand the mapping spectral sequence for $E_\infty(E_n^{\Omega S^{2n+1}}, E_n)$.

Mike also says that a consequence of the unpublished Hopkins–Lurie ambidexterity follow-up is that the comparison map $\mathsf{Spaces}(*, Y) \to E_\infty(E_n^Y, E_n^*)$ is an equivalence if $Y$ is a finite Postnikov tower living in the range of degrees visible to Morava $E$–theory.

The final chapter of Matt's thesis has never really been published, where he investigates power operations on elliptic cohomology theories. That might belong in this chapter as an example of the techinques, since we've already defined elliptic cohomology theories.

# Appendix A
# Loose ends

I'd like to spend a couple of days talking about ways the picture in this class can be extended, finally, some actually unanswered questions that naturally arise. The following two section titles are totally made up and probably won't last.

## A.1 $E_\infty$ geometry

Example 34 is an inspiration for considering *tmf* as well.

### *The modularity of the MString orientation*

$E_\infty$ orientations by *MString*
*tmf*, *TMF*, and *Tmf* in terms of $\mathcal{M}_{\mathbf{ell}}$
Thom spectra and $\infty$–categories
The Bousfield–Kuhn functor and the Rezk logarithm

## A.2 Rational phenomena: character theory for Lubin–Tate spectra

There's a sufficient amount of reliance on character theory in Matt's thesis that we should talk about it. You should write that action and then backtrack here to see what you need for it.

See Morava's *Local fields* paper

*Remark 78.* Theorem 2.6 of Greenlees–Strickland for a nice transchromatic perspective. See also work of Stapleton and Schlank–Stapleton, of course. Flesh this out.

Cite me: Theorem A.

**Theorem 81.** *Let $E$ be any complex-oriented cohomology theory. Take $G$ to be a finite group and let $\mathsf{Ab}_G$ be the full subcategory of the orbit category of $G$ built out of abelian subgroups of $G$. Finally, let $X$ be a finite $G$–CW complex. Then, each of the natural maps*

$$E^*(EG \times_G X) \to \lim_{A \in \mathsf{Ab}_G} E^*(EG \times_A X) \to \int_{A \in \mathsf{Ab}_G} E^*(BA \times X^A)$$

*becomes an isomorphism after inverting the order of $G$. In particular, there is an isomorphism*

$$\frac{1}{|G|} E^* BG \to \lim_{A \in \mathsf{Ab}_G} \frac{1}{|G|} E^* BA. \square$$

This is an analogue of Artin's theorem:

**Theorem 82.** *There is an isomorphism*

$$\frac{1}{|G|} R(G) \to \lim_{C \in \mathsf{Cyclic}_G} \frac{1}{|G|} R(C). \square$$

HKR intro material connecting Theorem A to character theory:

Recall that classical characters for finite groups are defined in the following situation: take $L = \mathbb{Q}^{\mathrm{ab}}$ to be the smallest characteristic 0 field containing all roots of unity, and for a finite group $G$ let $Cl(G; L)$ be the ring of class functions on $G$ with values in $L$. The units in the profinite integers $\widehat{\mathbb{Z}}$ act on $L$ as the Galois group over $\mathbb{Q}$, and since $G = \mathsf{Groups}(\widehat{\mathbb{Z}}, G)$ they also act naturally on $G$. Together, this gives a conjugation action on $Cl(G; L)$: for $\varphi \in \widehat{\mathbb{Z}}$, $g \in G$, and $\chi \in Cl(G; L)$, one sets

$$(\varphi \cdot \chi)(g) = \varphi(\chi(\varphi^{-1}(g))).$$

The character map is a ring homomorphism

$$\chi : R(G) \to Cl(G; L)^{\widehat{\mathbb{Z}}},$$

and this induces isomorphisms

$$\chi : L \otimes R(G) \xrightarrow{\simeq} Cl(G; L)$$

and even

$$\chi : \mathbb{Q} \otimes R(G) \xrightarrow{\simeq} Cl(G; L)^{\widehat{\mathbb{Z}}}.$$

Now take $E = E_\Gamma$ to be a Morava $E$–theory of finite height $d = \mathrm{ht}(\Gamma)$. Take $E^*(B\mathbb{Z}_p^d)$ to be topologized by $B(\mathbb{Z}/p^j)^d$. A character $\alpha : \mathbb{Z}_p^d \to S^1$

will induce a map $\alpha^* : E^*\mathbb{C}P^\infty \to E^*B\mathbb{Z}_p^d$. We define $L(E^*) = S^{-1}E^*(B\mathbb{Z}_p^d)$, where $S$ is the set of images of a coordinate on $\mathbb{C}P_E^\infty$ under $\alpha^*$ for nonzero characters $\alpha$. Note that this ring inherits an $\mathrm{Aut}(\mathbb{Z}_p^d)$ action by $E^*$–algebra maps.

The analogue of $Cl(G; L)$ will be $Cl_{d,p}(G; L(E^*))$, defined to be the ring of functions $\chi : G_{d,p} \to L(E^*)$ stable under $G$–orbits. Noting that

$$G_{d,p} = \mathrm{Hom}(\mathbb{Z}_p^d, G),$$

one sees that $\mathrm{Aut}(\mathbb{Z}_p^d)$ acts on $G_{d,p}$ and thus on $Cl_{d,p}(G; L(E^*))$ as a ring of $E^*$–algebra maps: given $\varphi \in \mathrm{Aut}(\mathbb{Z}_p^d)$, $\alpha \in G_{d,p}$, and $\chi \in Cl_{d,p}(G; L(E^*))$ one lets

$$(\varphi \cdot \chi)(\alpha) = \varphi(\chi(\varphi^{-1}(\alpha))).$$

Now we introduce a finite $G$–CW complex $X$. Let

$$\mathrm{Fix}_{d,p}(G, X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)} X^{\mathrm{im}\,\alpha}.$$

This space has commuting actions of $G$ and $\mathrm{Aut}(\mathbb{Z}_p^d)$. We set

$$Cl_{d,p}(G, X; L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{d,p}(G, X))^G,$$

which is again an $E^*$–algebra acted on by $\mathrm{Aut}(Z_p^d)$. We define the character map "componentwise": a homomorphism $\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)$ induces

$$E^*(EG \times_G X) \to E^*(B\mathbb{Z}_p^d) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}) \to L(E^*) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}).$$

Taking the direct sum over $\alpha$, this assembles into a map

$$\chi_{d,p}^G : E^*(EG \times_G X) \to Cl_{d,p}(G, X; L(E^*))^{\mathrm{Aut}(Z_p^d)}.$$

Nat taught you how to say all these things with $p$–adic tori, which was *much* clearer.

**Cite me:** Theorem C.

Checking this invariant ring claim is easiest done by comparing the functors the two things corepresent.

**Theorem 83.** *The invariant ring is $L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^d)} = p^{-1}E^*$, and $L(E^*)$ is faithfully flat over $p^{-1}E^*$. The character map $\chi_{d,p}^G$ induces isomorphisms*

$$\chi_{d,p}^G \colon\ L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*)),$$
$$\chi_{d,p}^G \colon\ p^{-1}E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}.$$

*In particular, when $X = *$, there are isomorphisms*

$$\chi^G_{d,p}\colon\ L(E^*)\otimes_{E^*} E^*(BG) \xrightarrow{\simeq} Cl_{d,p}(G; L(E^*)),$$
$$\chi^G_{d,p}\colon\ p^{-1}E^*(BG) \xrightarrow{\simeq} Cl_{d,p}(G; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}.\square$$

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Jack gives an interpretation of this in terms of formal $\mathcal{O}_L$–modules.

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I also have this summary of Nat's of the classical case:

It's not easy to decipher if you weren't there for the conversation, but here's my take on it. First, the map we wrote down today was the non-equivariant chern character: it mapped non-equivariant $KU \otimes \mathbb{Q}$ to non-equivariant $H\mathbb{Q}$, periodified. The first line on Nat's board points out that if you use this map on Borel-equivariant cohomology, you get nothing interesting: $K^0(BG)$ is interesting, but $H\mathbb{Q}^*(BG) = H\mathbb{Q}^*(*)$ collapses for finite $G$. So, you have to do something more impressive than just directly marry these two constructions to get something interesting.

That bottom row is Nat's suggestion of what "more interesting" could mean. (Not really his, of course, but I don't know who did this first. Chern, I suppose.) For an integer $n$, there's an evaluation map of (forgive me) topological stacks

$$*//(\mathbb{Z}/n) \times \mathrm{Hom}(*//(\mathbb{Z}/n), *//G) \xrightarrow{\mathrm{ev}} *//G$$

which upon applying a global-equivariant theory like $K_G$ gives

$$K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod_{\text{conjugacy classes of } g \text{ in } G} *) \xleftarrow{ev^*} K_G(*).$$

Now, apply the genuine $G$-equivariant Chern character to the $K_G$ factor to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) \leftarrow K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod *),$$

where the coproduct is again taken over conjugacy classes in G. Now, compute $K_{\mathbb{Z}/n}(*) = R(\mathbb{Z}/n) = \mathbb{Z}[x]/(x^n - 1)$, and insert this calculation to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) = \mathbb{Q}(\zeta_n) \otimes (\bigoplus_{\text{conjugacy classes}} \mathbb{Q}),$$

where $\zeta_n$ is an $n^{\text{th}}$ root of unity. As $n$ grows large, this selects sort of the part of the complex numbers $\mathbb{C}$ that the character theory of finite groups cares about, and so following all the composites we've built a map

$$K_G(*) \to \mathbb{C} \otimes (\bigoplus_{\text{conjugacy classes}} \mathbb{C}).$$

The claim, finally, is that this map sends a $G$-representation (thought of as a point in $K_G(*)$) to its class function decomposition.

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## A.3 Apr 27: Knowns and unknowns

### *Higher orientations*

*TAF* and friends

The $\alpha_{1/1}$ argument: Prop 2.3.2 of Hovey's $v_n$–elements of ring spectra

### *Equivariance*

This is tied up with the theory of power operations in a way I've never really thought about. Seems complicated.

### *Index theorems*

Connections with analysis

The Stolz–Teichner program

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Contexts for structured ring spectra

Difficulty in computing $\mathbb{S}_d \circlearrowleft E_d^*$. (Gross–Hopkins and the period map.)

Barry's $p$–adic measures

Fixed point spectra and e.g. $L_{K(2)}tmf$.

Blueshift, A–M–S, and the relationship to A–F–G?

Does $E_n$ receive an $E_\infty$ orientation? Does $BP$?

Remark 12.13 of published $H_\infty$ AHS says their obstruction framework agrees with the $E_\infty$ obstruction framework (if you take everything in sight to have $E_\infty$ structures). This is almost certainly related to the discussion at the end of Matt's thesis about the $MU$–orientation of $E_d$.

Section 12.4 compares doing $H_\infty$ descent with doing $E_\infty$ descent and shows that they're the same (in the case of interest?).

Hovey's paper on $v_n$–periodic elements in ring spectra. He has a nice (and thorough!) exposition on why one should be interested in bordism spectra and their splittings: for instance, a careful analysis of $MSpin$ will inexorably lead one toward studying $KO$. It would be nice if studying $MString$ (and potentially higher analogues) would lead one toward non-completed, non-connective versions of $EO_n$. Talk about $BoP$, for instance.

Matt's short resolutions of chromatically localized $MU$.

# Appendix B
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# Material for lecture

Mike's 1995 announcement is a nice read. There are many snippets you could pull out of it for use here. "$H\mathbb{Q}$ serves as the target for the Todd genus, but actually the Todd genus of a manifold is an integer and it turns out that $KU$ refines the Todd genus." The end of section 3, with $\tau \mapsto 1/\tau$, is mysterious. In section 4, Mike claims that there's a $BU[6,\infty)$–structured splitting principle into sums of things of the form $(1-\mathcal{L}_1)(1-\mathcal{L}_2)(1-\mathcal{L}_3)$. He then says that one expects the characteristic series of a $BU[6,\infty)$–genus to be a series of 3 variables, which is nice intuition. Could mention that $\Theta^k$ is a kind of $k^{\text{th}}$ difference operator, so that things in the kernel of $\Theta^k$ are "$k^{\text{th}}$ order polynomials". (More than this, the theorem of the cube is reasonable from this perspective, since $\Theta^3$ kills "quadratic things" and the topological object $H^2(-;\mathbb{Z})$ classifying line bundles is indeed "quadratic".) If the bundle admits a symmetry operation, then the fiber over $(x, y, -x-y)$ is canonically trivialized, so a $\Sigma$–structure on a symmetric line bundle is a $\Theta^3$–structure that restricts to the identity on these canonical parts. Mike claims (Theorem 6.2) that if $1/2 \in E^0(*)$ or if $E$ is $K(n)$–local, $n \leq 2$, then $BString^E$ is the parameter space of $\Sigma$–structures on the sheaf of functions vanishing at the identity on $G_E$. The map $MString \to KO_{\text{Tate}}$ actually factors through $MSpin$, so even though this produces the right $q$–series, you really need to know that $MString$ factors through $tmf$ and $MSpin$ *doesn't* to deduce the modularity for *String*–manifolds. (You can prove modularity separately for $BU[6, infty)$–manifolds, though, by essentially the same technique: refer to the rest of the (complex!) moduli of elliptic curves, which exist as $MU[6, infty)$–spectra.)

Generally: if $X$ is a space, then $X_{H\mathbb{F}_2}$ is a scheme with an $\operatorname{Aut}\widehat{\widehat{\mathbb{G}}}_a$–action. If $X$ is a spectrum (so it fails to have a diagonal map) then $(H\mathbb{F}_2)_*X$ is just an $\mathbb{F}_2$–module, also with an $\operatorname{Aut}\widehat{\widehat{\mathbb{G}}}_a$–action.

The cohomology of a qc sheaf pushed forward from a scheme to a stack along a cover agrees with just the cohomology over the scheme. (In the case of $* \to *//G$, this probably uses the cospan $* \to *//G \leftarrow *$ with pullback $G$...)

Akhil Mathew has notes from an algebraic geometry class ( https://math.berkeley.edu/~amathew/232b.pc ) where lectures 3–5 address the theorem of the cube.

Equivalences of various sorts of cohomologies: Ext in modules and quasi-coherent cohomology (goodness. Hartshorne, I suppose); Ext in comodules and quasicoherent cohomology on stacks (COCTALOS Lemma 12.4); quasi-coherent cohomology on simplicial schemes (Stacks project 09VK).

Make clear the distinction between $E_n$ and $\widehat{E(n)}$. Maybe explain the Devinatz–Hopkins remark that $r : \widehat{E(n)} \to E_n$ is an inclusion of fixed points and as such does not classify the versal formal group law.

when describing Quillen's model, he makes a lot of use of Gysin maps and Thom / Euler classes. at this point, maybe you can introduce what a Thom sheaf / Thom class is for a pointed formal curve?

I think this theorem is motivated by Artin–Mazur formal groups, and the Crystals notes use it to extract a formal group from a Dieudonné module. Some motivation could go here.

**Theorem 84.** *Let $A$ be a Noetherian ring and $G : \mathsf{AdicAlgebras}_A \to \mathsf{AbelianGroups}$ be a functor such that*

1. *$G(A) = 0$.*
2. *$G$ takes surjective maps to surjective maps.*
3. *There is a finite, free $A$–module $M$ and a functorial isomorphism*

$$I \otimes_A M \to G(B) \to G(B')$$

*whenever $I$ belongs to a square-zero extension of adic $A$–algebras*

$$I \to B \to B'.$$

*Then, $G \cong \widehat{\mathbb{A}}^n$ as a functor to sets, where $n = \dim M$.*

*Proof.* This is 9.6.4 in the Crystals notes.

$MUP$ happens to be the Thom spectrum of $BU \times \mathbb{Z}$.

–Formal groups in algebraic topology

—Day 1

+ Warning: noncontinuous maps of high-dimensional formal affine spaces.

—Day 2

+ Three definitions of complex orientable / oriented cohomology theories. + Some proofs: the splitting principle, Chern roots, diagrammatic Adam's condition, ... .

—Day 3

+ Lemma and proof: homomorphisms $F \to G$ of $\mathbb{F}_p$-FGLs factor as $F \to G' \to G$, where $G' \to G$ is a Frobenius isogeny and $F \to G'$ is invertible. + Definition of height. Examples: $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$. + Redefinition of height as the log-$p$ rank of the $p$-torsion. + Logarithms for FGLs over torsion-free rings. The integral equation. Height as radius.

—Day 4

+ A picture of $\mathcal{M}_{\mathbf{fg}} \times \mathbb{Z}_{(p)}$ + Definition of "deformation" + Plausibility argument for square-zero deformations being classified by "$\mathrm{Ext}^1(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$" + Theorem statement: $\mathrm{Ext}^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is computed by $H^* \operatorname{Hom}(B\widehat{\mathbb{G}}, \widehat{\mathbb{G}}_a)(R)$. + Theorem statement: That cochain complex is quasi-isomorphic to Lazarev's infinitesimal complex. + Proofs: Infinitesimal homomorphisms gives 1-cocycles, infinitesimal deformations give 2-cocycles. + Theorem statement (LubinâĂŞ-Tate): $H^0$, $H^1$, $H^2$ calculations. + Implications for Bockstein spectral sequence computing infinitesimal deformations. + Clarification about relative deformations and what "arithmetic deformation" means

—Day 5

+ The rational complex bordism ring. + QuillenâĂŹs theorem as refining rational complex genera to integral ones. + HondaâĂŹs theorem about $\zeta$–functions as manufacturing integral genera. + A statement of Landweber's theorem about regularity, stacky interpretation, no proof. + Definition of forms of a module, map to Galois cohomology + Computation of the Galois cohomology for: $H\mathbb{F}_p$, $MU/p$, $KU/p$ + Computation of the Galois cohomology for $\widehat{\mathbb{G}}_m$, explicit description of the invariant via the $\zeta$–function + MoravaâĂŹs sheaf over $L_1(\mathbb{Z}_p^{nr})$, Gamma-equivariance and transitivity, ConnerâĂŞFloyd + Identification of $L_1/\Gamma$ with $\mathcal{M}_{\mathbf{fg}}^{\leq 1}$, connection to LEFT.

—Day 6

+ The invariant differential. + de Rham cohomology in positive characteristic. The de Rham cohomology of $\widehat{\mathbb{A}}^1/\mathbb{F}_p$. + Cohomologically invariant differentials and the functor $D$. + Crystalline properties of $D$. The map $F$. + The Dieudonne functor, the main theorem.

—Future topics

+ The main theorem of class field theory + Lubin and Tate's construction of abelian extensions of local number fields + Description of the Lubin–Tate tower and the local Langlands correspondence + Lazard's theorem + Uniqueness of $\mathcal{O}_K$-module structure in characteristic zero + $p$-typification + Construction of the spectra $MU$, $BP$, $E(n)$, $K(n)$, $E_n$ + Goerss–Hopkins–Miller and Devinatz–Hopkins + Gross–Hopkins period map and the calculation of the Verdier dualizing sheaf on $LT_n$ + The Ravenel–Wilson calculation, exterior powers of $p$-divisible groups + Kohlhaase's Iwasawa theory + Lubin's dynamical results on formal power series + Classification of field spectra

## *Ideas*

1. Singer–Stong calculation of $H^*BU[2k,\infty)$. [ASIDE: $H\mathbb{F}_2^* ko$ and the Hopf algebra quotient of $\widehat{\mathbb{A}}_*$.]
2. Ando, Hopkins, Strickland on $H_\infty$–orientations and the norm condition
3. The rigid, real $\sigma$–orientation: AHR. Its effect in homology.
4. The Rezk logarithm and the Bousfield–Kuhn functor
5. Statement of Lurie's characterization of *TMF*, using this to determine a map from *MString* by AHR
6. Dylan's paper on String orientations
7. Matt's calculation of $E_\infty$–orientations of $K(1)$–local spectra using the short free resolution of *MU* in the $K(1)$–local category

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8. Cartier duality
9. Subschemes and divisors
10. Coalgebraic formal schemes
11. *Forms of K–theory*, Elliptic spectra, Tate *K*–theory, *TMF*
12. What are Weil pairings for geometers?
13. The Atiyah–Bott–Shapiro orientation (Is there a complex version of this? I understand it as a splitting of *MSpin*...)
14. Sinkinson's calculation and $MBP\langle m\rangle$–orientations
15. Hovey–Ravenel on nonorientations of $E_n$ by $MO[k,\infty)$. Other things in H–R?
16. Wood's cofiber sequence and $KO_{(p\geq 3)}$
17. The Serre–Tate theorem
18. The fundamental domain of $\pi_{GH}$
19. Orientations and the functor $gl_1$.

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## *Resources*

Ando, Hopkins, Strickland (Theorem of the Cube)
Ando, Hopkins, Strickland ($H_\infty$ map)
Ando, Strickland
Ando, Hopkins, Rezk
Barry Walker's thesis
Bill Singer's thesis, Bob Stong's *Determination*
Morava's *Forms of K–theory*
Neil's Functorial Philosophy for Formal Phenomena

Ravenel, Wilson

Kitchloo, Laures, Wilson

Akhil wrote a couple of blog posts about Ochanine's theorem: `https://amathew.wordpress.com/201` and `https://amathew.wordpress.com/2012/05/31/the-other-direction-of-ochaines-theo` Mentioning a more precise result might lend to a more beefy introduction.

What follows are notes from other talks I've given about quasi-relevant material which can probably be cannibalized for this class.