QUILLEN'S PROOF WITHOUT MANIFOLDS: THE CASE $G = C_2$

Contents

Section 7 of Chapter VII of Rudyak's book 'On Thom Spectra, Orientability, and Cobordism' contains a very helpful exposition of Quillen's paper. I was able to piece together this argument with his hints.

1. Construction of the MU Power Operations

Remark 1.1. If you are very scared of highly commutative ring spectra, there is a different, classical definition of the power operations in Rudyak's book (definition VII.7.4). It is a bit complicated, but he doesn't use manifolds. Instead he uses vector bundles and the Thom isomorphism.

We suppose MU is known to be a commutative ring spectrum in a sufficiently strong sense that the multiplication map admits a factorization

$$MU \wedge MU \to (MU \wedge MU)_{hC_2} \to MU.$$

If X is any spectrum, we construct a natural map

$$P^r : MU^r(X) \to MU^{2r}\left(X \wedge \Sigma^\infty_+ \mathbb{RP}^\infty\right)$$

which we call a power operation. If X is a space, applying P^r to $\Sigma^\infty_+ X$ yields a map which we also denote

$$P^r : MU^r(X) \to MU^{2r}\left(X \times \mathbb{RP}^\infty\right).$$

To construct P^r we describe its effect on arbitrary $a : X \to \Sigma^r MU$. Consider the diagram of spectra

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{(a,a)} \Sigma^{2r} MU \wedge MU.$$

If we endow X with the trivial C_2-action but $X \wedge X$ and $MU \wedge MU$ with the swap action, we can apply homotopy orbits to the above diagram and obtain

$$X \wedge \Sigma^\infty_+ BC_2 \to (X \wedge X) \wedge_{C_2} (\Sigma^\infty_+ EC_2) \to \Sigma^{2r}(MU \wedge MU)_{hC_2}$$

Composing with the multiplication map $(MU \wedge MU)_{hC_2} \to MU$ we obtain the desired element

$$P^r(a) \in MU^{2r}\left(X \wedge \Sigma^\infty_+ \mathbb{RP}^\infty\right).$$

2. The core calculation

Let $t \in MU^2(\mathbb{CP}^\infty)$ denote the universal complex orientation. This section is devoted to the computation of $P^2(t) \in MU^4(\mathbb{CP}^\infty \times \mathbb{RP}^\infty)$.

Remark 2.1. There is a natural sequence

$$S^1 \to \mathbb{RP}^\infty \xrightarrow{\gamma} \mathbb{CP}^\infty \xrightarrow{\cdot 2} \mathbb{CP}^\infty,$$

involving the multiplication by 2 map. We let z denote $\gamma^*(t) \in MU^2(\mathbb{RP}^\infty)$, and we let L denote the line bundle corresponding to γ. If one knows a bit about MU_*, including for example the fact that $[2](t)$ is not a 0 divisor in $MU^*(\mathbb{CP}^\infty)$, it follows from the above sequence that

$$MU^*(\mathbb{RP}^\infty) \cong MU_*[[z]]/[2](z)$$

and

$$MU^*(\mathbb{CP}^\infty \times \mathbb{RP}^\infty) \cong MU_*[[t, z]]/[2](z)$$

These facts are presumably much easier to prove than the entire calculation of MU_*. I am not sure to what extent I implicitly assumed them below. We will eventually make reference to the bundle L to express $P^2(t)$ in terms of the elements t and z, but I don't believe I assumed anywhere that z is non-zero.

The relation of $P^2(t)$ to MU chern classes.

Remark 2.2. For the remainder of this document we adopt the following notation: If A is a space and B a spectrum an arrow $A \to B$ denotes a map $\Sigma^\infty_+ A \to B$.

We need to understand the sequence of maps

$$(\mathbb{CP}^\infty)_{hC_2} = \mathbb{CP}^\infty \times \mathbb{RP}^\infty \to (\mathbb{CP}^\infty \times \mathbb{CP}^\infty)_{hC_2} \xrightarrow{(t,t)} \Sigma^4(MU \wedge MU)_{hC_2} \to \Sigma^4 MU.$$

The direct sum map $(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)_{hC_2} \to BU(2)$ naturally endows $\mathbb{CP}^\infty \times \mathbb{RP}^\infty$ with a rank 2 complex vector bundle. Call this rank 2 vector bundle η. The long composite above is the 2nd MU Chern class of η, so it will be relatively easy to finish once we understand the isomorphism class of η.

Determining the Bundle η over $\mathbb{CP}^\infty \times \mathbb{RP}^\infty$.

Remark 2.3. Let **Vect** denote the topological category of complex vector spaces and isomorphisms. A complex vector space with C_2 action is just a functor in $\mathrm{Hom}(BC_2, \mathbf{Vect}) \cong \mathrm{Hom}(\mathbb{RP}^\infty, \mathbf{Vect})$. If X is a space (considered only up to homotopy equivalence), a vector bundle over X is a functor $E \in \mathrm{Hom}(X, \mathbf{Vect})$. By a vector bundle over X with C_2-action we just mean a functor $E \in \mathrm{Hom}(X, \mathrm{Hom}(\mathbb{RP}^\infty, \mathbf{Vect})) \cong \mathrm{Hom}(X \times \mathbb{RP}^\infty, \mathbf{Vect})$, or in other words an ordinary vector bundle over $X \times \mathbb{RP}^\infty$.

Remark 2.4. In more classical terminology, if $E \to X$ is a vector bundle with C_2 acting on the fibers of E but acting trivially on X, then $E_{hC_2} \to X_{hC_2} \simeq X \times \mathbb{RP}^\infty$ is also a vector bundle.

Example 2.5. Let us enumerate some important examples of complex vector spaces with C_2-action (corresponding to ordinary vector bundles over $\mathbb{RP}^\infty$):

- There is $\mathbb{C}$ with the trivial action, which we denote simply by $\mathbb{C}$. This is the trivial line bundle over $\mathbb{RP}^\infty$.
- There is $\mathbb{C}$ with the multiplication by -1 action, which we denote by $\bar{\rho}$. Over $\mathbb{RP}^\infty$, this is the line bundle coming from the sequence $BC_2 \to BS^1 \xrightarrow{\cdot 2} BS^1$. Recall that we use L to denote this line bundle over $\mathbb{RP}^\infty$.
- There is ρ, the 2-dimensional regular representation of C_2, which splits as a direct sum $\rho \cong \mathbb{C} \oplus \bar{\rho}$. The corresponding ordinary rank 2 vector bundle over $\mathbb{RP}^\infty$ similarly splits.

The composite $\mathbb{CP}^\infty \xrightarrow{\Delta} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\oplus} BU(2)$ endows $\mathbb{CP}^\infty$ with the vector bundle $\xi \oplus \xi$, where ξ is the canonical bundle. This vector bundle naturally has a C_2 action which swaps the two factors, and it corresponds to the desired vector bundle η over $\mathbb{CP}^\infty \times \mathbb{RP}^\infty$. Instead of thinking about η, it will be more convenient to consider the C_2-vector bundle over $\mathbb{CP}^\infty$, which we denote by ζ. If π denotes the projection from $\mathbb{CP}^\infty$ to a point, then

$$\zeta \cong \xi \otimes \pi^*(\rho),$$

with ξ given the trivial C_2-action. By the discussion above,

$$\xi \otimes \pi^*(\rho) \cong (\xi \otimes \pi^*(\mathbb{C})) \oplus (\xi \otimes \pi^*(\bar{\rho})) \cong \xi \oplus (\xi \otimes \pi^*(\bar{\rho})).$$

Suppose that π_1 and π_2 are the two natural projections with domain $\mathbb{CP}^\infty \times \mathbb{RP}^\infty$. The above splitting shows

Lemma 2.6. *The bundle η over $\mathbb{CP}^\infty \times \mathbb{RP}^\infty$ is isomorphic to $\pi_1^*(\xi) \oplus (\pi_1^*(\xi) \otimes \pi_2^*(L))$*

Finishing the computation. Let c_1 and c_2 denote the first two MU chern classes. Recall that we are interested in computing $P^2(t)$, where $t \in MU^2(\mathbb{CP}^\infty)$ is the universal complex orientation. Our plan is to use the isomorphism

$$P^2(t) \cong c_2(\eta) \in MU^4(\mathbb{CP}^\infty \times \mathbb{RP}^\infty).$$

We calculate

$$c_2(\eta) = c_2(\pi_1^*(\xi) \oplus \pi_1^*(\xi) \otimes \pi_2^*(L)) = c_1(\pi_1^*(\xi))c_1(\pi_1^*(\xi) \otimes \pi_2^*(L)) = (t)(t +_{MU} z),$$

so

$$P^2(t) = (t)(t +_{MU} z) = t(z + t + \sum_{i,j=1}^{\infty} a_{ij} z^i t^j),$$

where the $a_{ij} \in MU_*$ are coefficients in MU's formal group law. Being somewhat perverse, we could further rewrite this as

$$P^2(t) = \left(\sum_{l(\alpha) \leq 1} z^{1-l(\alpha)} a_\alpha(z) S_\alpha(t) \right),$$

where α ranges over sequences of positive integers of length ≤ 1, $a_\alpha(z)$ is a polynomial with formal group law coefficients, and $S_\alpha(t)$ is the Landweber-Novikov operation.

3. Power operations on $MU^{2n}(MU(n))$

For each positive integer n, let $MU(n)$ denote the Thom spectrum of the tautological vector bundle over $BU(n)$. The defining inclusion $\Sigma^{-2n}MU(n) \to MU$ gives a canonical element $U_n \in MU^{2n}(MU(n))$. Our goal in this section will be to compute $P^{2n}(U_n) \in MU^{4n}(MU(n) \wedge \Sigma^\infty_+ \mathbb{RP}^\infty)$.

Consider the natural map $\mathbb{CP}^\infty \times ... \times \mathbb{CP}^\infty \xrightarrow{\oplus} BU(n)$. If we take Thom spectra and apply MU cohomology, we obtain a natural map

$$MU_*[[t_1, t_2, ..., t_n]] \cong \bigotimes_{i=1}^n \widetilde{MU}^*(\mathbb{CP}^\infty) \cong \widetilde{MU}^*\left(\bigwedge_{i=1}^n \mathbb{CP}^\infty\right) \leftarrow MU^*(MU(n))$$

This map takes U_n to the product $t_1t_2...t_n$, and the splitting principle says that the map is a monomorphism. By naturality, $P^{2n}(U_n)$ maps to $P^{2n}(t_1t_2...t_n) \in \widetilde{MU}^{4n}\left((\bigwedge_{i=1}^n \mathbb{CP}^\infty) \wedge \mathbb{RP}^\infty_+\right)$. Power operations are multiplicative, so this later object is just

$$P^2(t_1)P^2(t_2)...P^2(t_n) = \prod_{i=1}^n \left(\sum_{l(\alpha)\leq 1} z^{1-l(\alpha)}a_\alpha(z)S_\alpha(t_i)\right) = \sum_{l(\alpha)\leq n} z^{n-l(\alpha)}a_\alpha(z)S_\alpha(t_1t_2...t_n)$$

for some polynomials $a_\alpha(z)$ with formal group law coefficients. By the splitting principle and naturality, it follows that

$$P^{2n}(U_n) = \sum_{l(\alpha)\leq n} z^{n-l(\alpha)}a_\alpha(z)S_\alpha(U_n).$$

4. Proving Quillen's Formula

Our goal will be to prove the following theorem from class:

Theorem 4.1. *(Result from class) Let X be a finite pointed space and suppose $x \in \widetilde{MU}^{2q}(X)$. Choose m large enough that x may be represented by a map $f : \Sigma^{2m}X \to MU(m+q)$. Then*

$$z^m P^{2q}(x) = \sum_\alpha z^{q+m-l(\alpha)}a_\alpha(z)s_\alpha(x).$$

Remark 4.2. Contrary to what we discussed in class, I do not believe this shows the power operations to be additive, precisely because of the z^m business on the left.

Proof. Note that $f^*U_{m+q} = \sigma^{2m}x$, where multiplication by σ^{2m} executes the suspension isomorphism

$$\widetilde{MU}^{2q}(X) \xrightarrow{\cong} \widetilde{MU}^{2m+2q}(\Sigma^{2m}X)$$

The theorem follows immediately from the multiplicativity of the power operations and the following fact:

$$P^{2m}(\sigma^{2m}) = z^m\sigma^{2m}.$$

Thinking of σ^{2m} as induced from a map $S^{2m} \to MU(m)$, the fact also follows from our universal calculation in the previous section. $\square$