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Formal Geometry and Bordism Operations

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Class information

Course ID: MATH 278 (159627).

Meeting times: Spring 2016, MWF 12pm–1pm.

Goals: The primary goal of this class is to teach students to view results in algebraic topology through the lens of (formal) algebraic geometry.

Grading: This class won't have any official assignments. I'll give references as readings for those who would like a deeper understanding, though I'll do my best to ensure that no extra reading is required to follow the arc of the class.

I do want to assemble course notes from this class, but it's unlikely that I will have time to type *all* of them up. Instead, I would like to "crowdsource" this somewhat: I'll type up skeletal notes for each lecture, and then we as a class will try to flesh them out as the semester progresses. As incentive to help, those who contribute to the document will have their name included in the acknowledgements, and those who contribute *substantially* will have their name added as a coauthor. Everyone could use more CV items. (Publication may take a while. I suspect the course won't run perfectly smoothly the first time, so this may takes a second semester pass to become fully workable. But, since topics courses only come around once in a while, this will necessarily mean a delay.)

The source for this document can be found at

```
https://github.com/ecpeterson/FormalGeomNotes.
```

If you're taking the class or otherwise want to contribute, you can write me at

```
                    ecp@math.harvard.edu
```

to request write access.

Acknowledgements

Contents

I would like as few section titles as possible to involve people's names.

A bunch of broken displaymode tombstones can have their positions fixed by using http://tex.stackexchange.com/a/66221/2671.

Compile an index by replacing all the `textit` commands in definition environments with some more fancy macro that tags it for inclusion.

Remember to use $f\colon A \to B$ everywhere.

Should sections have subsections? Does that help organize the TOC?

Hood wrote a macro called sumfgl (see also sumF and sumG) that will make a bunch of formal group law expressions typeset better. Propagate that change through. Consider also the adjunct macro in the preamble. (Jay says that left-adjoints should be on top in general.)

Make sure you use either id or 1 everywhere to denote the identity morphism.

You're not very consistent about using $\widehat{G}$ or Γ to denote an arbitrary formal group. It seems like you use one or the other based on your preference of whether it has finite height or not.

Eliminate contractions.

Eliminate filler words like "things".

Make sure that "Case Study" and "Lecture" are OK names by Springer's standards. If they aren't, do a careful search-and-replace for them.

Double check that you're careful about choosing consistent names for your objects: $S = \operatorname{Spec} R$ is the base scheme, that sort of thing.

Be consistent about $\mathcal{O}_X$ vs $\mathcal{O}(X)$, and similarly with $\mathcal{I}_D$ versus $\mathcal{I}(D)$.

Be consistent about "$\mathbb{S}$" versus "$\mathbb{S}^0$" for the standard sphere spectrum.

Chapter 0
Introduction

The goal of this book is to communicate a certain *weltanschauung* uncovered in pieces by many different people working in bordism theory, and the goal just for this introduction is to tell a story about one theorem where it is especially apparent.

To begin, we will define a homology theory called "bordism homology". Recall that the singular homology of a space X is comes about by probing X with simplices: beginning with the collection of continuous maps $\sigma\colon \Delta^n \to X$, we take the free $\mathbb{Z}$–module on each of these sets and construct a chain complex

$$\cdots \xrightarrow{\partial} \mathbb{Z}\{\Delta^n \to X\} \xrightarrow{\partial} \mathbb{Z}\{\Delta^{n-1} \to X\} \xrightarrow{\partial} \cdots .$$

Bordism homology is constructed analogously, but using manifolds Z as the probes instead of simplices:[1]

$$\cdots \xrightarrow{\partial} \{Z^n \to X \mid Z^n \text{ a compact } n\text{–manifold}\}$$
$$\xrightarrow{\partial} \{Z^{n-1} \to X \mid Z^{n-1} \text{ a compact } (n-1)\text{–manifold}\}$$
$$\xrightarrow{\partial} \cdots .$$

Lemma 1 ([Koc78, Section 4]). *This forms a chain complex of monoids under disjoint union of manifolds, and its homology is written $MO_*(X)$. These are naturally abelian groups, and moreover they satisfy the axioms of a generalized homology theory.* □

In fact, we can define a bordism theory MG for any suitable family of structure groups $G(n) \to O(n)$. The coefficient ring of MG, or its value $MG_*(*)$ on a point, gives the ring of G–bordism classes, and generally

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[1] One doesn't need to take the free abelian group on anything, since the disjoint union of two manifolds is already a (disconnected) manifold, whereas the disjoint union of two simplices is not a simplex.

$MG_*(Y)$ gives a kind of "bordism in families over the space Y". There are comparison morphisms for the most ordinary kinds of bordism, given by replacing a chain of manifolds with an equivalent simplicial chain:

$$MO \to H\mathbb{Z}/2, \quad MSO \to H\mathbb{Z}.$$

In both cases, we can evaluate on a point to get ring maps, called "genera":

$$MO_*(*) \to \mathbb{Z}/2 \quad \text{and} \quad MSO_*(*) \to \mathbb{Z},$$

neither of which is very interesting, since they're both zero in positive degrees.

However, having maps of homology theories (rather than just maps of coefficient rings) is considerably more data then just the genus. For instance, we can use it to extract a theory of integration as follows. Consider the following special case of oriented bordism, where we evaluate MSO_* on an infinite loopspace:

$$MSO_n K(\mathbb{Z}, n) = \{\text{oriented } n\text{–manifolds mapping to } K(\mathbb{Z}, n)\} / \sim$$
$$= \left\{ \begin{array}{c} \text{oriented } n\text{–manifolds } Z \\ \text{with a specified class } \omega \in H^n(Z; \mathbb{Z}) \end{array} \right\} \Big/ \sim .$$

Associated to such a representative (Z, ω), the yoga of stable homotopy theory then allows us to build a composite

$$\mathsf{S} \xrightarrow{(Z,\omega)} MSO \wedge (\mathsf{S}^{-n} \wedge \Sigma_+^\infty K(\mathbb{Z}, n))$$
$$\xrightarrow{\text{colim}} MSO \wedge H\mathbb{Z}$$
$$\xrightarrow{\varphi \wedge 1} H\mathbb{Z} \wedge H\mathbb{Z}$$
$$\xrightarrow{\mu} H\mathbb{Z},$$

where φ is the orientation map. Altogether, this composite gives us an element of $\pi_0 H\mathbb{Z}$, i.e., an integer.

Lemma 2. *The integer obtained by the above process is $\int_Z \omega$.* $\qquad\qquad\square$

This definition of $\int_Z \omega$ via stable homotopy theory is pretty nice, in the sense that many theorems accompany it for free. It is also very general: given a ring map off of any bordism spectrum, a similar sequence of steps will furnish us with an integral tailored to that situation.

Now take $G = e$ to be the trivial structure group, which is the bordism theory of manifolds with trivialized tangent bundle. In this case, the Pontryagin–Thom construction gives an equivalence $\mathsf{S} \xrightarrow{\simeq} Me$. It is thus possible (and some people have indeed taken up this viewpoint) that stable

homotopy theory can be done solely through the lens of "framed bordism". We will prefer to view this the other way: the sphere spectrum $\mathbb{S}$ often appears to us as a natural object, and we will occasionally replace it by Me, the framed bordism spectrum. For example, given a ring spectrum E with unit map $\mathbb{S} \to E$, we can reconsider this as a ring map

$$Me \xrightarrow{\simeq} \mathbb{S} \to E.$$

Following along the lines of the previous paragraph, we learn that any ring spectrum E is automatically equipped with a theory of integration for framed manifolds.

Sometimes, as in the examples above, this unit map factors:

$$\mathbb{S} \simeq Me \to MO \to H\mathbb{Z}/2.$$

This is a witness to the overdeterminacy of $H\mathbb{Z}/2$'s integral for framed bordism: if the framed manifold is pushed all the way down to an unoriented manifold, there is still enough residual data to define the integral.[2] Given any ring spectrum E, we can ask the analogous question: If we filter O by a system of structure groups, through what stage does the unit map $Me \to E$ factor? For instance, the map

$$\mathbb{S} = Me \to MSO \to H\mathbb{Z}$$

considered above does *not* factor further through MO — an orientation is *required* to define the integral of an integer–valued cohomology class. Recognizing $SO \to O$ as the 0th Postnikov–Whitehead truncation of O, we are inspired to use the rest of the Postnikov filtration as our filtration of structure groups. Here is a diagram of this filtration and some interesting minimally-factored integration theories related to it:

This is the situation homotopy theorists found themselves in some decades ago, when Ochanine and Witten proved the following mysterious theorem using analytical and physical methods:

Theorem 1 (Ochanine [Och87], Witten [Wit87, Wit88]). *There is a map of rings*

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[2] It's literally more information than this: even unframeable unoriented manifolds acquire a compatible integral.

$$\sigma : MSpin_* \to \mathbb{C}((q)).$$

Moreover, if Z is a Spin manifold such that twice its first Pontryagin class vanishes — that is, if Z lifts to a String–manifold — then $\sigma(Z)$ lands in the subring $MF \subseteq \mathbb{Z}[\![q]\!]$ of modular forms with integral coefficients.   □

However, neither party gave indication that their result should be valid "in families" (in our sense), and no theory of integration was formally produced (in our sense). From the perspective of the homotopy theorist, it wasn't even totally clear what such a claim would mean: to give a topological enrichment of these theorems would mean finding a ring spectrum E such that $E_*(*)$ had something to do with modular forms.

Around the same time, Landweber, Ravenel, and Stong began studying "elliptic cohomology" for independent reasons; sometime much earlier, Morava had constructed an object "K^{Tate}" associated to the Tate elliptic curve; and a decade later Ando, Hopkins, and Strickland put all these things together in the following theorem:

Cite me: Landweber–Ravenel–Stong, Morava's *Forms of K–theory*, and Ando–Hopkins–Strickland.

Theorem 2 (Ando–Hopkins–Strickland). *If E is an "elliptic cohomology theory", then there is a canonical map of homotopy ring spectra $MString \to E$ called the σ–orientation (for E). Additionally, there is an elliptic spectrum K^{Tate} whose σ–orientation gives Witten's genus $MString_* \to K^{\text{Tate}}_*$.*   □

We now come to the motivation for this class: the homotopical σ–orientation was actually first constructed using formal geometry. The original proof of Ando–Hopkins–Strickland begins with a reduction to maps of the form

$$MU[6,\infty) \to E.$$

They then work to show that in especially good cases they can complete the missing arrow in the diagram

$$
\begin{array}{ccc}
MU[6,\infty) & \longrightarrow & MString \\
 & \searrow & \vdots \\
 & & E.
\end{array}
$$

Leaving aside the extension problem for the moment, their main theorem is the following description of the cohomology ring $E^*MU[6,\infty)$:

Theorem 3 (Ando–Hopkins–Strickland [AHS01], cf. Singer [Sin68] and Stong [Sto63]). *For E an even–periodic cohomology theory,*

$$\operatorname{Spec} E_* MU[6,\infty) \cong C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0)),$$

where "$C^3(\widehat{\mathbb{G}}_E; \mathcal{I}(0))$" is a certain scheme. When E is taken to be elliptic, so that there is a specified isomorphism $\widehat{\mathbb{G}}_E \cong C_0^{\wedge}$ for C an elliptic curve, the theory of el-

liptic curves furnishes the scheme with a canonical point. Hence, there is a preferred
class MU[6, ∞) → *E, natural in the choice of elliptic E.*   □

Our real goal is to understand theorems like this last one, where algebraic geometry asserts some real control over something squarely in the domain of homotopy theory.

The structure of the class will be to work through a sequence of case studies where this perspective shines through most brightly. We'll start by working through Thom's calculation of the homotopy of MO, which holds the simultaneous attractive features of being approachable while revealing essentially all of the structural complexity of the general situation, so that we know what to expect later on. Having seen that through to the end, we'll then venture on to other examples: the complex bordism ring, structure theorems for finite spectra, unstable cooperations, and, finally, the σ–orientation and its extensions. The overriding theme of the class will be that algebraic geometry is a good organizing principle that gives us one avenue of insight into how homotopy theory functions. In particular, it allows us to organize "operations" of various sorts between spectra derived from bordism theories.

We should also mention that we will specifically *not* discuss the following aspects of this story:

- Analytic techniques will be completely omitted. Much of modern research stemming from the above problem is an attempt to extend index theory across Witten's genus, or to find a "geometric cochains" model of certain elliptic cohomology theories. These often mean heavy analytic work, and we will strictly confine ourselves to the domain of homotopy theory.
- As sort of a sub-point (and despite the motivation provided in this Introduction), we will also mostly avoid manifold geometry. (We do give a proof of Quillen's theorem on the structure of MU_* which invokes some mild amount of manifold geometry.)Again, much of the contemporary research about *tmf* is an attempt to find a geometric model, so that geometric techniques can be imported — including equivariance and the geometry of quantum field theories, to name two.

- In a different direction, our focus will not linger on actually computing bordism rings MX_*, nor will we consider geometric constructions on manifolds and their behavior after imaging into the bordism ring. This is also the source of active research: the structure of the symplectic bordism ring remains, to large extent, mysterious, and what we do understand of it comes through a mix of formal geometry and raw manifold geometry. This could be a topic that fits logically into this document, were it not for time limitations and the author's inexpertise.

- The geometry of E_∞ rings will also be avoided, at least to the extent possible. Such objects become inescapable by the conclusion of our story, but there are better resources from which to learn about E_∞ rings, and the pre–E_∞ story is not told so often these days. So, we will focus on the unstructured part and leave E_∞ rings to other authors.
- There will be plenty of places where we will avoid stating things in maximum generality or with maximum thoroughness. The story we are interested in telling draws from a blend of many others from different subfields of mathematics, many of which have their own topics courses. Sometimes this will mean avoiding stating the most beautiful theorem in a subfield in favor of a theorem we will find more useful. Other times this will mean abbreviating someone else's general definition to one more specialized to the task at hand. In any case, we will give references to other sources where you can find these things cast in starring roles.

Finally, we must mention that there are several good companions to these notes. Essentially none of the material here is original — it's almost all cribbed either from published or unpublished sources — but the source documents are quite scattered and individually dense. We will make a point to cite useful references as we go. One document stands out above all others, though: Neil Strickland's *Functorial Philosophy for Formal Phenomena* [Strb]. These lecture notes can basically be viewed as an attempt to make it through this paper in the span of a semester.

Case Study 1
Unoriented bordism

This Case Study culminates in the calculation of $MO_*(*)$, the bordism ring of unoriented manifolds, but we mainly take this as an opportunity to introduce several key concepts that will serve us throughout the book. First and foremost, we will require a definition of bordism spectrum that we can manipulate computationally, using just the tools of abstract homotopy theory. Once that is established, we immediately begin to bring algebraic geometry into the mix: the main idea is that the cohomology ring of a space is better viewed as a scheme (with plenty of extra structure), and the homology groups of a spectrum are better viewed as representation for a certain elaborate algebraic group. This data actually finds familiar expression in homotopy theory: we show that a form of group cohomology for this representation forms the input to the classical Adams spectral sequence. Finally, we calculate this representation structure for $H\mathbb{F}_{2*}MO$, find that it is suitably free, and thereby gain control of the Adams spectral sequence computing $MO_*(*)$.

1.1 Thom spectra and the Thom isomorphism

Our goal is a sequence of theorems about the unoriented bordism spectrum MO. We will begin by recalling a definition of the spectrum MO using just abstract homotopy theory, because it involves ideas that will be useful to us throughout the semester and because we cannot compute effectively with the chain-level definition given in the Introduction.

Definition 1. For a spherical bundle $S^{n-1} \to \xi \to X$, its Thom space is given by the cofiber

$$\xi \to X \xrightarrow{\text{cofiber}} T_n(\xi).$$

Proof ("Proof" of definition). There is a more classical construction of the Thom space: take the associated disk bundle by gluing an n–disk fiberwise, and add a point at infinity by collapsing ξ:

$$T_n(\xi) = (\xi \sqcup'_{S^{n-1}} D^n)^+.$$

To compare this with the cofiber definition, recall that the thickening of ξ to an n–disk bundle is the same thing as taking the mapping cylinder on $\xi \to X$. Since the inclusion into the mapping cylinder is now a cofibration, the quotient by this subspace agrees with both the cofiber of the map and the introduction of a point at infinity.

Before proceeding, here are two important examples:

Example 1. If $\xi = S^{n-1} \times X$ is the trivial bundle, then $T_n(\xi) = S^n \wedge (X_+)$. This is supposed to indicate what Thom spaces are "doing": if you feed in the trivial bundle then you get the suspension out, so if you feed in a twisted bundle you should think of it as a *twisted suspension*.

Example 2. Let ξ be the tautological S^0–bundle over $\mathbb{RP}^\infty = BO(1)$. Because ξ has contractible total space, $EO(1)$, the cofiber degenerates and it follows that $T_1(\xi) = \mathbb{RP}^\infty$. More generally, arguing by cells shows that the Thom space for the tautological bundle over $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$.

Now we catalog a bunch of useful properties of the Thom space functor. Firstly, recall that a spherical bundle over X is the same data as a map $X \to BGL_1 S^{n-1}$, where $GL_1 S^{n-1}$ is the subspace of $F(S^{n-1}, S^{n-1})$ expressed by the pullback of spaces

$$
\begin{array}{ccc}
GL_1 S^{n-1} & \longrightarrow & F(S^{n-1}, S^{n-1}) \\
\downarrow & & \downarrow \\
\mathrm{Aut}_{h\mathsf{Spaces}}\, S^{n-1} & \longrightarrow & \mathrm{End}_{h\mathsf{Spaces}}\, S^{n-1} = \pi_0 F(S^{n-1}, S^{n-1}).
\end{array}
$$

Lemma 3. *The construction T_n can be viewed as a functor from the slice category over $BGL_1 S^{n-1}$ to* Spaces. *Maps of slices*

induce maps $T_n(f^*\xi) \to T_n(\xi)$, and T_n is suitably homotopy-invariant. □

Next, the spherical subbundle of a vector bundle gives a common source of spherical bundles. The action of $O(n)$ on $\mathbb{R}^n$ preserves the unit sphere,

and hence gives a map $O(n) \to GL_1 S^{n-1}$. These are maps of topological groups, and the block–inclusion maps $i^n \colon O(n) \to O(n+1)$ commute with the suspension map $GL_1 S^{n-1} \to GL_1 S^n$. In fact, much more can be said:

Lemma 4. *The block–sum maps $O(n) \times O(m) \to O(n+m)$ are compatible with the join maps $GL_1 S^{n-1} \times GL_1 S^{m-1} \to GL_1 S^{n+m-1}$.* $\square$

Again taking a cue from K–theory, we take the colimit as n grows large, using the maps

$$
\begin{array}{ccccccc}
BGL_1 S^{n-1} & = & BGL_1 S^{n-1} \times * & \xrightarrow{\mathrm{id} \times \mathrm{triv}} & BGL_1 S^{n-1} \times BGL_1 S^0 & \xrightarrow{\quad * \quad} & BGL_1 S^n, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
BO(n) & = & BO(n) \times * & \xrightarrow{\mathrm{id} \times \mathrm{triv}} & BO(n) \times BO(1) & \xrightarrow{\quad \oplus \quad} & BO(n+1).
\end{array}
$$

Corollary 1. *The operations of block–sum and topological join imbue the colimiting spaces BO and $BGL_1 S$ with the structure of H–groups. Moreover, the colimiting map*

$$
J_{\mathbb{R}} \colon BO \to BGL_1 S,
$$

called the stable J–homomorphism, is a morphism of H–groups. $\square$

> Should you justify "group" rather than "space"?

Finally, we can ask about the compatibility of Thom constructions with all of this. In order to properly phrase the question, we need a version of the construction which operates on stable spherical bundles, i.e., whose source is the slice category over $BGL_1 S$. By calculating

> Does this calculation need justification?

$$
T_{n+1}(\xi * \mathrm{triv}) \simeq \Sigma T_n(\xi),
$$

we are inspired to make the following definition:

Definition 2. For ξ an S^{n-1}–bundle, we define the *Thom spectrum* of ξ to be

$$
T(\xi) := \Sigma^{-n} \Sigma^\infty T_n(\xi).
$$

By filtering the base space by compact subspaces, this begets a functor

$$
T \colon \mathsf{Spaces}_{/BGL_1 S} \to \mathsf{Spectra}.
$$

Lemma 5. *T is monoidal: it carries external fiberwise joins to smash products of Thom spectra. Correspondingly, $T \circ J_{\mathbb{R}}$ carries external direct sums of stable vector bundles to smash products of Thom spectra.* $\square$

Definition 3. The spectrum *MO* arises as the universal example of all these constructions, strung together:

$$
MO := T(J_{\mathbb{R}}) = \operatorname*{colim}_n T(J_{\mathbb{R}}^n) = \operatorname*{colim}_n \Sigma^{-n} T_n J_{\mathbb{R}}^n.
$$

The spectrum MO has several remarkable properties. The most basic such property is that it is a ring spectrum, and this follows immediately from $J_{\mathbb{R}}$ being a homomorphism of H–spaces. Much more excitingly, we can also deduce the presence of Thom isomorphisms just from the properties stated thus far. That $J_{\mathbb{R}}$ is a homomorphism means that the following square commutes:

$$
\begin{array}{ccccc}
BO \times BO & \xrightarrow{\;\sigma\;} & BO \times BO & \xrightarrow{\;\mu\;} & BO \\
 & & \downarrow{\scriptstyle J_{\mathbb{R}} \times J_{\mathbb{R}}} & & \downarrow{\scriptstyle J_{\mathbb{R}}} \\
 & & BGL_1 S \times BGL_1 S & \xrightarrow{\;\mu\;} & BGL_1 S.
\end{array}
$$

We have extended this square very slightly by a certain shearing map σ defined by $\sigma(x,y) = (xy^{-1}, y)$. It is evident that σ is a homotopy equivalence, since just as we can de-scale the first coordinate by y we can re-scale by it. We can calculate directly the behavior of the long composite:

$$
J_{\mathbb{R}} \circ \mu \circ \sigma(x,y) = J_{\mathbb{R}} \circ \mu(xy^{-1}, y) = J_{\mathbb{R}}(xy^{-1}y) = J_{\mathbb{R}}(x).
$$

It follows that the second coordinate plays no role, and that the bundle classified by the long composite can be written as $J_{\mathbb{R}} \times 0$.[1] We are now in a position to see the Thom isomorphism:

Lemma 6 (Thom isomorphism, universal example). *As MO–modules,*

$$
MO \wedge MO \simeq MO \wedge \Sigma^{\infty}_{+} BO.
$$

Proof. Stringing together the naturality properties of the Thom functor outlined above, we can thus make the following calculation:

$$
\begin{aligned}
T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}})) &\simeq T(\mu \circ (J_{\mathbb{R}} \times J_{\mathbb{R}}) \circ \sigma) && \text{(homotopy invariance)} \\
&\simeq T(J_{\mathbb{R}} \times 0) && \text{(constructed lift)} \\
&\simeq T(J_{\mathbb{R}}) \wedge T(0) && \text{(monoidality)} \\
&\simeq T(J_{\mathbb{R}}) \wedge \Sigma^{\infty}_{+} BO && \text{(Example 1)} \\
T(J_{\mathbb{R}}) \wedge T(J_{\mathbb{R}}) &\simeq T(J_{\mathbb{R}}) \wedge \Sigma^{\infty}_{+} BO && \text{(monoidality)} \\
MO \wedge MO &\simeq MO \wedge \Sigma^{\infty}_{+} BO. && \text{(definition of } MO) \;\square
\end{aligned}
$$

The equivalence is one of MO–modules because the MO–module structure of both sides comes from smashing with MO on the left.

From here, the general version of Thom's theorem follows quickly:

---

[1] This factorization does *not* commute with the rest of the diagram, just with the little lifting triangle it forms.

Definition 4. A map $\varphi\colon MO \to E$ of homotopy ring spectra is called an *orientation* of E (by MO).[2]

Theorem 4 (Thom isomorphism). *Let* $\xi\colon X \to BO$ *classify a vector bundle and let* $\varphi\colon MO \to E$ *be a map of ring spectra. Then there is an equivalence of* *E–modules*

$$E \wedge T(\xi) \simeq E \wedge \Sigma_+^\infty X.$$

Proof (Modifications to above proof). To accommodate X rather than BO as the base, we redefine $\sigma\colon BO \times X \to BO \times X$ by

$$\sigma(x, y) = \sigma(x\xi(y)^{-1}, y).$$

Follow the same proof as before with the diagram

This gives an equivalence $\theta_{MO}\colon MO \wedge T(\xi) \to MO \wedge \Sigma_+^\infty X$. To introduce E, note that there is a diagram

The bottom arrow θ_E exists by applying the action map to both sides and pushing the map $\theta_{MO} \wedge E$ down. Since θ_{MO} is an equivalence, it has an inverse α_{MO}. Therefore, the middle map has inverse $\alpha_{MO} \wedge E$, and we can similarly push this down to a map α_E, which we now want to show is the inverse to θ_E. From here it is a simple diagram chase: we have renamed three of the maps in the diagram to f, g, and h for brevity. Noting that $g \circ f$ is the identity map because of the unit axiom, we conclude

---

[2] Later, we will refer to analogous ring spectrum maps $MU \to E$ off of the complex bordism spectrum as *complex-orientations* of E. However, calling ring maps $MO \to E$ "unoriented-orientations" is rightfully considered distasteful.

$$g \circ f \simeq g \circ (\alpha_{MO} \wedge E) \circ (\theta_{MO} \wedge E) \circ f$$
$$\simeq \alpha_E \circ h \circ (\theta_{MO} \wedge E) \circ f \qquad \text{(action map)}$$
$$\simeq \alpha_E \circ \theta_E \circ g \circ f \qquad \text{(action map)}$$
$$\simeq \alpha_E \circ \theta_E. \qquad \qquad \square$$

It follows that α_E gives an inverse to θ_E.

Remark 1. One of the tentpoles of the theory of Thom spectra is that Theorem 4 has a kind of converse: if a ring spectrum E has suitably natural and multiplicative Thom isomorphisms for Thom spectra formed from real vector bundles, then one can define an essentially unique ring map $MO \to E$ realizing these isomorphisms via the machinery of Theorem 4.

Remark 2. There is also a cohomological version of the Thom isomorphism. Suppose that E is a ring spectrum under MO and let ξ be the spherical bundle of a vector bundle on a space X. The spectrum $F(\Sigma_+^\infty X, E)$ is a ring spectrum under E (hence under MO), so there is a Thom isomorphism as well as an evaluation map

$$F(\Sigma_+^\infty X, E) \wedge T(\xi) \xrightarrow{\simeq} F(\Sigma_+^\infty X, E) \wedge \Sigma_+^\infty X \xrightarrow{\text{eval}} E.$$

Passing through the exponential adjunction, the map

$$F(\Sigma_+^\infty X, E) \xrightarrow{\simeq} F(T(\xi), E)$$

gives the cohomological Thom isomorphism

$$E^* X \cong E^* T(\xi).$$

Example 3. We will close out this section by using this to actually make a calculation. Recall from Example 2 that $T(\mathcal{L} \downarrow \mathbb{RP}^n) = \mathbb{RP}^{n+1}$. Because MO is a connective spectrum, the diagram

$$MO \wedge MO \longrightarrow (MO \wedge MO)(-\infty, 0] \longrightarrow MO(-\infty, 0] \wedge MO(-\infty, 0] = H\pi_0 MO \wedge H\pi_0 MO$$

$$MO \longrightarrow MO(-\infty, 0] = H\pi_0 MO$$

shows that

$$MO \to MO(-\infty, 0] = H\pi_0 MO = H\mathbb{F}_2$$

is a map of ring spectra. Hence, we can apply the Thom isomorphism theorem to the mod–2 homology of Thom complexes coming from real vector bundles:

Have you really proved this? It should be easy, but I don't think "this just works" alone suffices.

This is *still* not a proof of this. Ugh.

$$\pi_*(H\mathbb{F}_2 \wedge T(\mathcal{L} - 1)) \cong \pi_*(H\mathbb{F}_2 \wedge T(0)) \qquad \text{(Thom isomorphism)}$$

$$\pi_*(H\mathbb{F}_2 \wedge \Sigma^{-1}\Sigma^\infty \mathbb{R}\mathrm{P}^{n+1}) \cong \pi_*(H\mathbb{F}_2 \wedge \Sigma_+^\infty \mathbb{R}\mathrm{P}^n) \qquad \text{(Example 2)}$$

$$\widetilde{H\mathbb{F}}_{2*+1}\mathbb{R}\mathrm{P}^{n+1} \cong H\mathbb{F}_{2*}\mathbb{R}\mathrm{P}^n. \qquad \text{(generalized homology)}$$

This powers an induction that shows $H\mathbb{F}_{2*}\mathbb{R}\mathrm{P}^\infty$ has a single class in every degree. The cohomological version of the Thom isomorphism in Remark 2, together with the $H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^n$–module structure of $H\mathbb{F}_2^*T(\mathcal{L} - 1)$, also gives the ring structure:

$$H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^n = \mathbb{F}_2[x]/x^{n+1}.$$

1.2 Cohomology rings and affine schemes

An abbreviated summary of this book is that we are going to put "Spec" in front of rings appearing in algebraic topology and see what happens. Before doing any algebraic topology, let me remind you what this means on the level of algebra. The core idea is to replace a ring R by the functor it corepresents, $\mathrm{Spec}\, R$. For any "test $\mathbb{F}_2$–algebra" T, we set

$$(\mathrm{Spec}\, R)(T) := \mathsf{Algebras}_{\mathbb{F}_2/}(R, T) \cong \mathsf{Schemes}_{/\mathbb{F}_2}(\mathrm{Spec}\, T, \mathrm{Spec}\, R).$$

More generally, we have the following definition:

Definition 5. An *affine $\mathbb{F}_2$–scheme* is a functor $X : \mathsf{Algebras}_{\mathbb{F}_2/} \to \mathsf{Sets}$ which is (noncanonically) isomorphic to $\mathrm{Spec}\, R$ for some $\mathbb{F}_2$–algebra R. Given such an isomorphism, we will refer to $\mathrm{Spec}\, R \to X$ as a *parameter* for X and its inverse $X \to \mathrm{Spec}\, R$ as a *coordinate* for X.

Lemma 7. *There is an equivalence of categories*

$$\mathrm{Spec} : \mathsf{Algebras}_{\mathbb{F}_2/}^{\mathrm{op}} \to \mathsf{AffineSchemes}_{/\mathbb{F}_2}. \quad \Box$$

The centerpiece of thinking about rings in this way, for us and for now, is to translate between a presentation of R as a quotient of a free algebra and a presentation of $(\mathrm{Spec}\, R)(T)$ as selecting tuples of elements in T subject to certain conditions. Consider the following example:

Example 4. Set $R_1 = \mathbb{F}_2[x]$. Then

$$(\mathrm{Spec}\, R_1)(T) = \mathsf{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x], T)$$

is determined by where x is sent — i.e., this Hom–set is naturally isomorphic to T itself. Consider also what happens when we impose a relation by passing to $R_2 = \mathbb{F}_2[x]/(x^{n+1})$. The value

$$(\operatorname{Spec} R_2)(T) = \operatorname{Algebras}_{\mathbb{F}_2/}(\mathbb{F}_2[x]/(x^{n+1}), T)$$

of the associated affine scheme is again determined by where x is sent, but now x can only be sent to elements which are nilpotent of order $n+1$. These schemes are both important enough that we give them special names:

$$\mathbb{A}^1 := \operatorname{Spec} \mathbb{F}_2[x], \qquad \mathbb{A}^{1,(n)} := \operatorname{Spec} \mathbb{F}_2[x]/(x^{n+1}).$$

The symbol "$\mathbb{A}^1$" is pronounced "the affine line" — reasonable, since the value $\mathbb{A}^1(T)$ is, indeed, a single T's worth of points. Note that the quotient map $R_1 \to R_2$ induces an inclusion $\mathbb{A}^{1,(n)} \to \mathbb{A}^1$ and that $\mathbb{A}^{1,(0)}$ is a constant functor:

$$\mathbb{A}^{1,(0)}(T) = \{f : \mathbb{F}_2[x] \to T \mid f(x) = 0\}.$$

Accordingly, we pronounce "$\mathbb{A}^{1,(0)}$" as "the origin on the affine line" and "$\mathbb{A}^{1,(n)}$" as "the $(n+1)^{\text{st}}$ order (nilpotent) neighborhood of the origin in the affine line".

We can also use this language to re-express another common object arising in algebraic topology: the Hopf algebra, which appears when taking the mod–2 cohomology of an H–group. In addition to the usual ring structure on cohomology groups, the H–group multiplication, unit, and inversion maps induce an additional diagonal map Δ, an augmentation map ε, and an antipode χ respectively. Running through the axioms, one quickly checks the following:

Lemma 8. *For a Hopf $\mathbb{F}_2$–algebra R, the functor $\operatorname{Spec} R$ is naturally valued in groups. Such functors are called group schemes. Conversely, a choice of group structure on $\operatorname{Spec} R$ endows R with the structure of a Hopf algebra.*

Proof (Proof sketch). This is a matter of recognizing the product in $\operatorname{Algebras}_{\mathbb{F}_2/}^{\text{op}}$ as the tensor product, then using the Yoneda lemma to transfer structure around.

Example 5. The functor $\mathbb{A}^1$ introduced above is naturally valued in groups: since $\mathbb{A}^1(T) \cong T$, we can use the addition on T to make it into an abelian group. When considering $\mathbb{A}^1$ with this group scheme structure, we notate it as $\mathbb{G}_a$. Applying the Yoneda lemma, one deduces the following formulas for the Hopf algebra structure maps:

$$\mathbb{G}_a \times \mathbb{G}_a \xrightarrow{\mu} \mathbb{G}_a \qquad\qquad x_1 + x_2 \leftarrow\!\!\!| \ x,$$
$$\mathbb{G}_a \xrightarrow{\chi} \mathbb{G}_a \qquad\qquad -x \leftarrow\!\!\!| \ x,$$
$$\operatorname{Spec} \mathbb{F}_2 \xrightarrow{\eta} \mathbb{G}_a \qquad\qquad 0 \leftarrow\!\!\!| \ x.$$

As an example of how to reason this out, consider the following diagram:

$$\begin{array}{ccc} \mathbb{G}_a \times \mathbb{G}_a & \xrightarrow{\mu} & \mathbb{G}_a \\ x_1 \uparrow \simeq \; x_2 \uparrow \simeq & {}^{x_1+x_2} & \\ \operatorname{Spec} \mathbb{F}_2[x_1] \times \operatorname{Spec} \mathbb{F}_2[x_2] & & x \uparrow \simeq \\ \| & & \\ \operatorname{Spec} \mathbb{F}_2[x_1,x_2] & \xrightarrow{\Delta^*} & \operatorname{Spec} \mathbb{F}_2[x]. \end{array}$$

It follows that the bottom map of affine schemes is induced by the algebra map

$$\mathbb{F}_2[x] \xrightarrow{\Delta} \mathbb{F}_2[x_1,x_2], \qquad\qquad x \mapsto x_1 + x_2.$$

Remark 3. In fact, $\mathbb{A}^1$ is naturally valued in *rings*. It models the inverse functor to Spec in the equivalence of categories above, i.e., the elements of a ring R always form a complete collection of $\mathbb{A}^1$–valued functions on some affine scheme $\operatorname{Spec} R$.

Example 6. We define the *multiplicative group scheme* by

$$\mathbb{G}_m = \operatorname{Spec} \mathbb{F}_2[x,y]/(xy-1).$$

Its value $\mathbb{G}_m(T)$ on a test algebra T is the set of pairs (x,y) such that y is a multiplicative inverse to x, and hence $\mathbb{G}_m$ is valued in groups. Applying the Yoneda lemma, we deduce the following formulas for the Hopf algebra structure maps:

$$\begin{array}{cc} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m & \begin{aligned} x_1 \otimes x_2 &\leftarrowtail x \\ y_1 \otimes y_2 &\leftarrowtail y, \end{aligned} \\[1em] \mathbb{G}_m \xrightarrow{\chi} \mathbb{G}_m & (y,x) \leftarrowtail (x,y), \\[0.5em] \operatorname{Spec} R \xrightarrow{\eta} \mathbb{G}_m & 1 \leftarrowtail x,y. \end{array}$$

Remark 4. As presented above, the multiplicative group comes with a natural inclusion $\mathbb{G}_m \to \mathbb{A}^2$. Specifically, the subset $\mathbb{G}_m \subseteq \mathbb{A}^2$ consists of pairs (x,y) in the graph of the hyperbola $y = 1/x$. However, the element x also gives an $\mathbb{A}^1$–valued function $x \colon \mathbb{G}_m \to \mathbb{A}^1$, and because multiplicative inverses in a ring are unique, we see that this map too is an inclusion. These two inclusions have rather different properties relative to their ambient spaces, and we will think harder about these essential differences later on.

Example 7 (cf. Example 32). This example showcases the complications that algebraic geometry introduces to this situation, and is meant as discouragement from thinking of the theory of affine group schemes as a strong ana-

logue of the theory of linear complex Lie groups. We set $\alpha_2 = \operatorname{Spec} \mathbb{F}_2[x]/(x^2)$, with group scheme structure given by

$$\alpha_2 \times \alpha_2 \xrightarrow{\mu} \alpha_2 \qquad\qquad x_1 + x_2 \hookleftarrow x,$$
$$\alpha_2 \xrightarrow{\chi} \alpha_2 \qquad\qquad -x \hookleftarrow x,$$
$$\operatorname{Spec}\mathbb{F}_2 \xrightarrow{\eta} \alpha_2 \qquad\qquad 0 \hookleftarrow x.$$

This group scheme has several interesting properties, which we will merely state for now, reserving their proofs for Example 32.

1. α_2 has the same underlying structure ring as $\mu_2 := \mathbb{G}_m[2]$, the 2–torsion points of $\mathbb{G}_m$, but is not isomorphic to it. (For instance, $\underline{\operatorname{GroupSchemes}}(\mu_2, \mu_2)$ gives the constant group scheme $\underline{\mathbb{Z}/2}$, but $\underline{\operatorname{GroupSchemes}}(\alpha_2, \mu_2) = \alpha_2$.)
2. There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.
3. If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence
$$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$
However, this short exact sequence does not split (even after base change).
4. The subgroups of $\alpha_2 \times \alpha_2$ of order 2 are parameterized by the scheme $\mathbb{P}^1$, i.e., for R an $\mathbb{F}_2$-algebra the subgroup schemes of $\alpha_2 \times \alpha_2$ of order two *which are defined over R* are parameterized by the set $\mathbb{P}^1(R)$.

We now turn to a different class of examples, which will wind up being the key players in our upcoming topological story. To begin, consider the colimit of the sets $\operatorname{colim}_{n\to\infty} \mathbb{A}^{1,(n)}(T)$, which is of use in algebra: it is the collection of nilpotent elements in T. These kinds of conditions which are "unbounded in n" appear frequently enough that we are moved to give these functors a name too:

Definition 6. An *affine formal scheme* is an ind-system of finite affine schemes.[3] The morphisms between two formal schemes are computed by

$$\operatorname{FormalSchemes}(\{X_\alpha\}, \{Y_\beta\}) = \lim_\alpha \operatorname{colim}_\beta \operatorname{Schemes}(X_\alpha, Y_\beta).$$

Given affine charts $X_\alpha = \operatorname{Spec} R_\alpha$, we will glibly suppress the system from the notation and write
$$\operatorname{Spf} R := \{\operatorname{Spec} R_\alpha\}.$$

Example 8. The individual schemes $\mathbb{A}^{1,(n)}$ do not support group structures. After all, the sum of two elements which are nilpotent of order $n+1$ can

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[3] This has the effect of formally adjoining colimits of filtered diagrams to the category of finite affine schemes.

only be guaranteed to be nilpotent of order $2n + 1$. It follows that the entire ind-system $\{\mathbb{A}^{1,(n)}\} =: \widehat{\mathbb{A}}^1$ supports a group structure, even though none of its constituent pieces do. We call such an object a *formal group scheme*, and this particular formal group scheme we denote by $\widehat{\mathbb{G}}_a$.

Example 9. Similarly, one can define the scheme $\mathbb{G}_m[n]$ of elements of unipotent order n:

$$\mathbb{G}_m[n] = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy - 1, x^n - 1)} \subseteq \mathbb{G}_m.$$

These *are* all group schemes, and they nest together in a complicated way: there is an inclusion of $\mathbb{G}_m[n]$ into $\mathbb{G}_m[nm]$. There is also a second filtration along the lines of the one considered in Example 8:

$$\mathbb{G}_m^{(n)} = \operatorname{Spec} \frac{\mathbb{F}_2[x,y]}{(xy - 1, (x-1)^n)}.$$

Additionally, the only $\mathbb{G}_m[n]$ which are infinitesimal thickenings of $\mathbb{G}_m[1]$ are those with $n = 2^j$.

These schemes form a sequential system, but they are only occasionally group schemes. Specifically, $\mathbb{G}_m^{(2^j)}$ is a group scheme, in which case $\mathbb{G}_m^{(2^j)} \cong \mathbb{G}_m[2^j]$. We define $\widehat{\mathbb{G}}_m$ using this common subsystem:

... of infinitesimal thickenings

$$\widehat{\mathbb{G}}_m := \{\mathbb{G}_m^{(2^j)}\}_{j=0}^{\infty}.$$

Let us now consider the example that we closed with last time, where we calculated $H\mathbb{F}_2^*(\mathbb{RP}^n) = \mathbb{F}_2[x]/(x^{n+1})$. Putting "Spec" in front of this, we could reinterpret this calculation as

$$\operatorname{Spec} H\mathbb{F}_2^*(\mathbb{RP}^n) \cong \mathbb{A}^{1,(n)}.$$

This is such a useful thing to do that we will give it a notation all of its own:

Definition 7. Let X be a finite cell complex, so that $H\mathbb{F}_2^*(X)$ is a ring which is finite–dimensional as an $\mathbb{F}_2$–vector space. We will write

$$X_{H\mathbb{F}_2} = \operatorname{Spec} H\mathbb{F}_2^* X$$

for the corresponding finite affine scheme.

Example 10. Putting together the discussions from this time and last time, in the new notation we have calculated

$$\mathbb{RP}^n_{H\mathbb{F}_2} \cong \mathbb{A}^{1,(n)}.$$

So far, this example just restates things we knew in a mildly different language. Our driving goal for the next section is to incorporate as much information as we have about these cohomology rings $H\mathbb{F}_2^*(\mathbb{RP}^n)$ into this description, which will result in us giving a more "precise" name for this

object. Along the way, we will discover why *X* had to be a *finite* complex and how to think about more general *X*. For now, though, we will content ourselves with investigating the Hopf algebra structure on $H\mathbb{F}_2^*\mathbb{RP}^\infty$, the cohomology of an infinite complex.

Example 11. Recall that $\mathbb{RP}^\infty$ is an *H*–space in two equivalent ways:

1. There is an identification $\mathbb{RP}^\infty \simeq K(\mathbb{F}_2, 1)$, and the *H*–space structure is induced by the sum on cohomology.
2. There is an identification $\mathbb{RP}^\infty \simeq BO(1)$, and the *H*–space structure is induced by the tensor product of real line bundles.

> I thought we came up with an instructive third example of where to find the *H*–space structure.

In either case, this induces a Hopf algebra diagonal

$$H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{RP}^\infty$$

which we would like to analyze. This map is determined by where it sends the class *x*, and because it must respect gradings it must be of the form $\Delta x = ax_1 + bx_2$ for some constants $a, b \in \mathbb{F}_2$. Furthermore, because it belongs to a Hopf algebra structure, it must satisfy the unitality axiom

$$H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\binom{\varepsilon\otimes\mathrm{id}}{\mathrm{id}\otimes\varepsilon}} H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{RP}^\infty.$$

$$\underbrace{\phantom{H\mathbb{F}_2^*\mathbb{RP}^\infty \otimes H\mathbb{F}_2^*\mathbb{RP}^\infty \xleftarrow{\Delta} H\mathbb{F}_2^*\mathbb{RP}^\infty}}_{\mathrm{id}}$$

and hence it takes the form

$$\Delta(x) = x_1 + x_2.$$

Noticing that this is exactly the diagonal map in Example 5, we tentatively identify "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" with the additive group. This is extremely suggestive but does not take into account the fact that $\mathbb{RP}^\infty$ is an infinite complex, so we have not yet allowed ourselves to write "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$". In light of the rest of the material discussed in this section, we have left open a very particular point: it is not clear if we should use the name "$\mathbb{G}_a$" or "$\widehat{\mathbb{G}}_a$". We will straighten this out tomorrow.

1.3 The Steenrod algebra

We left off in the previous section with an ominous finiteness condition in Definition 7, and we produced a pair of reasonable guesses as to what "$\mathbb{RP}^\infty_{H\mathbb{F}_2}$" could mean in Example 11. We will decide which of the two

guesses is reasonable by rigidifying the target category so as to incorporate the following extra structures:

1. Cohomology rings are *graded*, and maps of spaces respect this grading.
2. Cohomology rings receive an action of the Steenrod algebra, and maps of spaces respect this action.
3. Both of these are made somewhat more complicated when taking the cohomology of an infinite complex.
4. (Cohomology rings for more elaborate cohomology theories are only skew-commutative, but "Spec" requires a commutative input.)

Today we will fix all these deficiencies of $X_{H\mathbb{F}_2}$ except for #4, which does not matter with mod–2 coefficients but which will be something of a bugbear throughout the rest of the book.

We will begin by considering the grading on $H\mathbb{F}_2^* X$, where X is a finite complex. In algebraic geometry, the following standard construction is used to track gradings:

Definition 8 ([Str99b, Definition 2.95]). A $\mathbb{Z}$–*grading* on a ring R is a system of additive subgroups R_k of R satisfying $R = \bigoplus_k R_k$, $1 \in R_0$, and $R_j R_k \subseteq R_{j+k}$. Additionally, a map $f \colon R \to S$ of graded rings is said to *respect the grading* if $f(R_k) \subseteq S_k$.[4]

Lemma 9 ([Str99b, Proposition 2.96]). *A graded ring R is equivalent data to an affine scheme* Spec R *with an action by* $\mathbb{G}_m$. *Additionally, a map $R \to S$ is homogeneous exactly when the induced map* Spec $S \to$ Spec R *is* $\mathbb{G}_m$–*equivariant.*

Proof. A $\mathbb{G}_m$–action on Spec R is equivalent data to a coaction map

$$\alpha^* : R \to R \otimes \mathbb{F}_2[x^{\pm}].$$

Define R_k to be those points in r satisfying $\alpha^*(r) = r \otimes x^k$. It is clear that we have $1 \in R_0$ and that $R_j R_k \subseteq R_{j+k}$. To see that $R = \bigoplus_k R_k$, note that every tensor can be written as a sum of pure tensors. Conversely, given a graded ring R, define the coaction map on R_k by

$$(r_k \in R_k) \mapsto x^k r_k$$

and extend linearly.

This notion from algebraic geometry is somewhat different from what we are used to in algebraic topology, essentially because the algebraic topologist's "cohomology ring" is not *really* a ring at all — one is only allowed

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[4] The terminology "$\mathbb{Z}$–filtering" might be more appropriate, but this is the language commonly used.

to consider sums of homogeneous degree elements. This restriction stems directly from the provenance of cohomology rings: recall that

$$H\mathbb{F}_2^n X := \pi_{-n} F(\Sigma_+^\infty X, H\mathbb{F}_2).$$

One can only form sums internal to a *particular* homotopy group, using the cogroup structure on $\mathbb{S}^{-n}$. On the other hand, the most basic ring of algebraic geometry is the polynomial ring, and hence their notion is adapted to handle, for instance, the potential degree drop when taking the difference of two (nonhomogeneous) polynomials of the same degree.

We can modify our perspective very slightly to arrive at the algebraic geometers', by replacing $H\mathbb{F}_2$ with the periodified spectrum

$$H\mathbb{F}_2 P = \bigvee_{j=-\infty}^{\infty} \Sigma^j H\mathbb{F}_2.$$

This spectrum becomes a ring in the homotopy category by using the factorwise-defined multiplication maps

$$\Sigma^j H\mathbb{F}_2 \wedge \Sigma^k H\mathbb{F}_2 \simeq \Sigma^{j+k}(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \xrightarrow{\Sigma^{j+k}\mu} \Sigma^{j+k} H\mathbb{F}_2.$$

This spectrum has the property that $H\mathbb{F}_2 P^0(X)$ is isomorphic to $\bigoplus_n H\mathbb{F}_2^n(X)$ as ungraded rings, but now we can make topological sense of the sum of two classes which used to live in different $H\mathbb{F}_2$–degrees. At this point we can manually craft the desired coaction map α^* from Lemma 9, but we will shortly find that algebraic topology gifts us with it on its own.

Our route to finding this internally occurring α^* is by turning to the next supplementary structure: the action of the Steenrod algebra. Naively approached, this does not fit into the framework we have been sketching so far: the Steenrod algebra arises as the homotopy endomorphisms of $H\mathbb{F}_2$ and so is a *noncommutative* algebra. In turn, the action map

$$\begin{array}{ccc}
\mathcal{A}^* \otimes H\mathbb{F}_2^* X & \longrightarrow & H\mathbb{F}_2^* X \\
\| & & \| \\
[H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [X, H\mathbb{F}_2]_* & \xrightarrow{\circ} & [X, H\mathbb{F}_2]
\end{array}$$

will be difficult to squeeze into any kind of algebro-geometric framework. Milnor was the first person to see a way around this, with two crucial observations. First, the Steenrod algebra is a Hopf algebra[5], using the map

$$[H\mathbb{F}_2, H\mathbb{F}_2]_* \xrightarrow{\mu^*} [H\mathbb{F}_2 \wedge H\mathbb{F}_2, H\mathbb{F}_2]_* \cong [H\mathbb{F}_2, H\mathbb{F}_2]_* \otimes [H\mathbb{F}_2, H\mathbb{F}_2]_*$$

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[5] The construction of both the Hopf algebra diagonal here and the coaction map below is somewhat ad hoc. We will give a more robust presentation in Remark 29.

as the diagonal. This Hopf algebra structure is actually cocommutative — this is a rephrasing of the symmetry of the Cartan formula:

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(y).$$

It follows that the linear-algebraic dual of the Steenrod algebra $\mathcal{A}_*$ is a commutative ring, and hence $\mathrm{Spec}\,\mathcal{A}_*$ would make a reasonable algebro-geometric object.

Second, we want to identify the role of $\mathcal{A}_*$ in acting on $H\mathbb{F}_2^*X$. By assuming that X is a finite complex, we can write it as the Spanier–Whitehead dual $X = DY$ of some other finite complex Y. Starting with the action map on $H\mathbb{F}_2^*Y$:

$$\mathcal{A}^* \otimes H\mathbb{F}_2^*Y \to H\mathbb{F}_2^*Y$$

we take the $\mathbb{F}_2$–linear dual to get a coaction map

$$\mathcal{A}_* \otimes H\mathbb{F}_{2*}Y \leftarrow H\mathbb{F}_{2*}Y,$$

then use $X = DY$ to return to cohomology

$$\mathcal{A}_* \otimes H\mathbb{F}_2^*X \xleftarrow{\lambda^*} H\mathbb{F}_2^*X.$$

Finally, we re-interpret this as an action map

$$\mathrm{Spec}\,\mathcal{A}_* \times X_{H\mathbb{F}_2} \xrightarrow{\alpha} X_{H\mathbb{F}_2}.$$

Having produced the action map α, we are now moved to study α as well as the structure group $\mathrm{Spec}\,\mathcal{A}_*$ itself. Milnor works out the Hopf algebra structure of $\mathcal{A}_*$ by defining elements $\xi_j \in \mathcal{A}_*$ dual to $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} \in \mathcal{A}^*$. Taking $X = \mathbb{R}\mathrm{P}^n$ and $x \in H\mathbb{F}_2^1(\mathbb{R}\mathrm{P}^n)$ the generator, then since $\mathrm{Sq}^{2^{j-1}} \cdots \mathrm{Sq}^{2^0} x = x^{2^j}$ he deduces the formula

$$\lambda^*(x) = \sum_{j=0}^{\lfloor \log_2 n \rfloor} x^{2^j} \otimes \xi_j \quad (\text{in } H\mathbb{F}_2^*\mathbb{R}\mathrm{P}^n).$$

Noticing that we can take the limit $n \to \infty$ to get a well-defined infinite sum, he then makes the following calculation, stable in n:

$$(\lambda^* \otimes \mathrm{id}) \circ \lambda^*(x) = (\mathrm{id} \otimes \Delta) \circ \lambda^*(x) \qquad \text{(coassociativity)}$$

$$(\lambda^* \otimes \mathrm{id})\left(\sum_{j=0}^{\infty} x^{2^j} \otimes \xi_j\right) =$$

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^i} \otimes \xi_i\right)^{2^j} \otimes \xi_j = \qquad \text{(ring homomorphism)}$$

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j}\right) \otimes \xi_j = \qquad \text{(characteristic 2)}.$$

Then, turning to the right-hand side:

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j}\right) \otimes \xi_j = (\mathrm{id} \otimes \Delta)\left(\sum_{m=0}^{\infty} x^{2^m} \otimes \xi_m\right)$$

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} x^{2^{i+j}} \otimes \xi_i^{2^j}\right) \otimes \xi_j = \sum_{m=0}^{\infty} x^{2^m} \otimes \Delta(\xi_m),$$

from which it follows that

$$\Delta \xi_m = \sum_{i+j=m} \xi_i^{2^j} \otimes \xi_j.$$

Finally, Milnor shows that this is the complete story:

Theorem 5 (Milnor [Mil58, Theorem 2], [MT68, Chapter 6]). $\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_j, \ldots]$.

Proof (Flippant proof). There is at least a map $\mathbb{F}_2[\xi_1, \xi_2, \ldots] \to \mathcal{A}_*$ given by the definition of the elements ξ_j above. This map is injective, since these elements are distinguished by how they coact on $H\mathbb{F}_2^*\mathbb{RP}^\infty$. Then, since these rings are of graded finite type, Milnor can conclude his argument by counting how many elements he has produced, comparing against how many Adem and Cartan found (which we will do ourselves in Lecture 4.2), and noting that he has exactly enough.

We are now in a position to uncover the desired map α^* from earlier. In order to retell Milnor's story with $H\mathbb{F}_2P$ in place of $H\mathbb{F}_2$, note that there is a topological construction involving $H\mathbb{F}_2$ from which $\mathcal{A}_*$ emerges:

$$\mathcal{A}_* := \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2).$$

Performing substitution on this formula gives the periodified dual Steenrod algebra:

$$\mathcal{A}P_0 := \pi_0(H\mathbb{F}_2P \wedge H\mathbb{F}_2P) = H\mathbb{F}_2P_0(H\mathbb{F}_2P) = \mathcal{A}_*[\xi_0^{\pm}].$$

Lemma 10 ([Goe08, Formula 3.4, Remark 3.14]). *Projecting to the quotient Hopf algebra $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ gives exactly the coaction map α^*.*

Proof (Calculation). Starting with an auxiliary cohomology class $x \in H\mathbb{F}_2^n(X)$, we produce a homogenized cohomology class $x \cdot u^n \in H\mathbb{F}_2 P^0(X)$. Under the coaction map, this is sent to

$$H\mathbb{F}_2 P^0(X) \xrightarrow{\alpha^*} H\mathbb{F}_2 P^0(X) \otimes \mathcal{A}P_0 \longrightarrow H\mathbb{F}_2 P^0(X) \otimes \mathbb{F}_2[\xi_0^{\pm}]$$

$$x \cdot u^n \longmapsto \qquad\qquad\qquad\qquad\qquad\qquad x \cdot u^n \otimes \xi_0^n.$$

Applying Lemma 9 to this coaction thus selects the original degree n classes.

Early on in this discussion, trading the language "graded map" for "$\mathbb{G}_m$–equivariant map" did not seem to have much of an effect on our mathematics. The thrust of this Lemma is that "Steenrod–equivariant map" already includes "$\mathbb{G}_m$–equivariant map", which is a visible gain in brevity. To study the rest of the content of Steenrod equivariance algebro–geometrically, we need only identify what the series $\lambda^*(x)$ embodies. Note that this necessarily involves some creativity, and the only justification we can supply will be moral, borne out over time, as our narrative encompasses more and more phenomena. With that caveat in mind, here is one such description. Recall the map induced by the H–space multiplication

$$H\mathbb{F}_2^* \mathbb{R}P^\infty \otimes H\mathbb{F}_2^* \mathbb{R}P^\infty \leftarrow H\mathbb{F}_2^* \mathbb{R}P^\infty.$$

Taking a colimit over finite complexes, we produce an coaction of $\mathcal{A}_*$, and since the map above comes from a map of spaces, it is equivariant for the coaction. Since the action on the left is diagonal, we deduce the formula

$$\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2).$$

Lemma 11. *The series $\lambda^*(x) = \sum_{j=0}^\infty x^{2^j} \otimes \xi_j$ is the universal example of a series satisfying $\lambda^*(x_1 + x_2) = \lambda^*(x_1) + \lambda^*(x_2)$. The set $(\operatorname{Spec} \mathcal{A}P_0)(T)$ is identified with the set of power series f with coefficients in the $\mathbb{F}_2$–algebra T satisfying*

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Proof. Given a point $f \in (\operatorname{Spec} \mathcal{A}P_0)(T)$, we extract such a series by setting

$$\lambda_f^*(x) = \sum_{j=0}^\infty f(\xi_j) x^{2^j} \in T[\![x]\!].$$

Conversely, any series $\lambda(x)$ satisfying this homomorphism property must have nonzero terms appearing only in integer powers of 2, and hence we

can construct a point f by declaring that f sends ξ_j to the $(2^j)^{\text{th}}$ coefficient of λ.

We close our discussion by codifying what Milnor did when he stabilized against n. Each $\mathbb{RP}^n_{H\mathbb{F}_2}$ is a finite affine scheme, and to make sense of the object $\mathbb{RP}^\infty_{H\mathbb{F}_2}$ Milnor's technique was to consider the ind-system $\{\mathbb{RP}^n_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite affine schemes. We will record this as our technique to handle general infinite complexes:

Definition 9 (cf. Definition 20). When X is an infinite complex, filter it by its subskeleta $X^{(n)}$ and define $X_{H\mathbb{F}_2}$ to be the ind-system $\{X^{(n)}_{H\mathbb{F}_2}\}_{n=0}^\infty$ of finite schemes.

This choice to follow Milnor resolves our uncertainty about the topological example from last time:

Example 12 (cf. Examples 8 and 11). Write $\widehat{\mathbb{G}}_a$ for the ind-system $\mathbb{A}^{1,(n)}$ with the group scheme structure given in Example 11. That this group scheme structure filters in this way is a simultaneous reflection of two facts:

1. Algebraic: The set $\widehat{\mathbb{G}}_a(T)$ consists of all nilpotent elements in T. The sum of two nilpotent elements of orders n and m is guaranteed to itself be nilpotent with order at most $n + m$.
2. Topological: There is a factorization of the multiplication map on $\mathbb{RP}^\infty$ as $\mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{n+m}$ purely for dimensional reasons.

As group schemes, we have thus calculated

$$\mathbb{RP}^\infty_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a.$$

Example 13. Given the appearance of a homomorphism condition in Lemma 11, we would like to connect $\operatorname{Spec} \mathcal{A}P_0$ with $\widehat{\mathbb{G}}_a$ more directly. Toward this, we define a "hom functor" for two formal schemes:

$$\underline{\mathsf{FormalSchemes}}(X, Y)(T) = \left\{ (u, f) \,\middle|\, \begin{array}{l} u : \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2, \\ f : u^* X \to u^* Y \end{array} \right\}.$$

Restricting attention to homomorphisms, we see that a proper name for $\operatorname{Spec} \mathcal{A}P_0$ is

$$\operatorname{Spec} \mathcal{A}P_0 \cong \underline{\operatorname{Aut}} \widehat{\mathbb{G}}_a.$$

To check this, consider a point $g \in (\operatorname{Spec} \mathcal{A}P_0)(T)$ for an $\mathbb{F}_2$–algebra T. The $\mathbb{F}_2$–algebra structure of T (which is uniquely determined by a property of T) gives rise to a map $u \colon \operatorname{Spec} T \to \operatorname{Spec} \mathbb{F}_2$. The rest of the data of g gives rise to a power series in $T[\![x]\!]$ as in the proof of Lemma 11, which

can be re-interpreted as an automorphism $g: u^*\widehat{\mathbb{G}}_a \to u^*\widehat{\mathbb{G}}_a$ of formal group schemes.[6]

Remark 5. The projection $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^{\pm}]$ is split as Hopf algebras, and hence there is a decomposition

$$\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a \cong \mathbb{G}_m \times \underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a,$$

where $\underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a$ consists of those automorphisms with leading coefficient ξ_0 exactly equal to 1. This can be read to mean that the "interesting" part of the Steenrod algebra, $\underline{\mathrm{Aut}}_1\,\widehat{\mathbb{G}}_a$, consists of stable operations, in the sense that their action is independent of the degree–tracking mechanism.

Example 14. Remembering the slogan

$$\mathrm{Spec}\,\mathcal{A}P_0 \cong \underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$$

also makes it easy to recall the structure formulas for the dual Steenrod algebra. For instance, consider the antipode map, which has the effect on $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$ of sending a power series to its compositional inverse. That is:

$$\sum_{j=0}^{\infty} \chi(\xi_j)\left(\sum_{k=0}^{\infty} \xi_k x^{2^k}\right)^{2^j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \chi(\xi_j)\xi_k^{2^j} x^{2^{j+k}} = \sum_{n=0}^{\infty}\left(\sum_{j+k=n} \chi(\xi_j)\xi_k^{2^j}\right) x^{2^n} = 1,$$

from which we can extract formulas like

$$\chi(\xi_0) = \xi_0^{-1}, \qquad \chi(\xi_1) = \xi_0^{-3}\xi_1, \qquad \chi(\xi_2) = \xi_0^{-7}\xi_1^3 + \xi_0^{-5}\xi_2, \qquad \ldots.$$

In summary, the formula $\mathbb{RP}^{\infty}_{H\mathbb{F}_2} \cong \widehat{\mathbb{G}}_a$ is meant to point out that this language of formal schemes has an extremely good compression ratio — you can fit a lot of information into a very tiny space. This formula simultaneously encodes the cohomology ring of $\mathbb{RP}^{\infty}$ as the formal scheme, its diagonal as the group scheme structure, and the coaction of the dual Steenrod algebra by the identification with $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$. As a separate wonder, it is also remarkable that there is a single cohomological calculation — that of $\mathbb{RP}^{\infty}_{H\mathbb{F}_2}$ — which exerts such enormous control over mod–2 cohomology itself (e.g., the entire structure of the dual Steenrod algebra). This will turn out to be a surprisingly common occurrence as we progress.

---

[6] This description, too, is sensitive to the difference between $\widehat{\mathbb{G}}_a$ and $\mathbb{G}_a$. The scheme $\underline{\mathrm{End}}\,\mathbb{G}_a$ is populated by *polynomials* satisfying a homomorphism condition, and essentially none of them have inverses.

1.4 Hopf algebra cohomology

In this section, we will focus on an important classical tool: the Adams spectral sequence. We are going to study this in greater earnest later on, so I will avoid giving a satisfying construction today. But, even without a construction, it is instructive to see how such a thing comes about from a moral perspective.

Remark 6. Throughout today, we will work with graded homology groups, rather than with periodified cohomology as was the case in Lecture 1.3. This choice will remain mysterious for now, but we can at least reassure ourselves that it carries the same data as we were studying previously. Referring to our discussion of the construction of the coaction map, we see that without taking Spanier–Whitehead duals we already have an analogous coaction map on homology:

$$H\mathbb{F}_{2*}X \to H\mathbb{F}_{2*}X \otimes \mathcal{A}_*.$$

Additionally, building on the discussion in Remark 5, the splitting of the Hopf algebra shows that we are free to work gradedly or work with the periodified version of mod–2 homology, while still retaining the rest of the framework.

With this caveat out of the way, begin by considering the following three self-maps of the stable sphere:

$$\mathbb{S}^0 \xrightarrow{0} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{1} \mathbb{S}^0, \qquad \mathbb{S}^0 \xrightarrow{2} \mathbb{S}^0.$$

If we apply mod–2 homology to each line, the induced maps are

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{\mathrm{id}} \mathbb{F}_2, \qquad \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2.$$

We see that mod–2 homology can immediately distinguish between the null map and the identity map just by its behavior on morphisms, but it cannot distinguish between the null map and the multiplication-by-2 map. To try to distinguish between these two, we use the only other tool available to us: homology theories send cofiber sequences to long exact sequences, and moreover the data of a map f and the data of the inclusion map $\mathbb{S}^0 \to C(f)$ into its cone are equivalent in the stable category. So, we trade our maps 0 and 2 for the following cofiber sequences:

$$\mathbb{S}^0 \longrightarrow C(0) \longrightarrow \mathbb{S}^1, \qquad \mathbb{S}^0 \longrightarrow C(2) \longrightarrow \mathbb{S}^1.$$

Applying homology, these again appear to be the same:

[1]   • ————————→ •   • ————————→ •

[0]   • ————————→ •   • ————————→ •

$$HF_{2*}S^0 \to HF_{2*}C(0) \to HF_{2*}S^1, \qquad HF_{2*}S^0 \to HF_{2*}C(2) \to HF_{2*}S^1,$$

where we have drawn a "•" for a generator of an F_2–vector space, graded vertically, and arrows indicating the behavior of each map. However, if we enrich our picture with the data we discussed last time, we can finally see the difference. Recall the topological equivalences

$$C(0) \simeq S^0 \vee S^1, \quad C(2) \simeq \Sigma^{-1} RP^2.$$

> It would be nice if the dots aligned directly beneath the spaces in the cofiber sequences above. This can't be accomplished by a "column sep" attribute, since this doesn't control the width of a column but rather its literal separation from its neighbor.

In the two cases, the coaction map λ_* is given by

$$\lambda_* : HF_{2*}C(0) \to HF_{2*}C(0) \otimes \mathcal{A}_* \quad \lambda_* : HF_{2*}C(2) \to HF_{2*}C(2) \otimes \mathcal{A}_*$$
$$\lambda^* : e_0 \mapsto e_0 \otimes 1 \qquad\qquad\qquad \lambda^* : e_0 \mapsto e_0 \otimes 1 + e_1 \otimes \xi_1$$
$$\lambda^* : e_1 \mapsto e_1 \otimes 1, \qquad\qquad\qquad \lambda^* : e_1 \mapsto e_1 \otimes 1.$$

We draw this into the diagram as

[1]   • ————————→ •   • ————————→ •
                                                                             |
                                                                            ξ_1
                                                                             |
[0]   • ————————→ •   • ————————→ •

$$HF_{2*}S^0 \to HF_{2*}C(0) \to HF_{2*}S^1, \qquad HF_{2*}S^0 \to HF_{2*}C(2) \to HF_{2*}S^1,$$

where the vertical line indicates the nontrivial coaction involving ξ_1. We can now see what trading maps for cofiber sequences has bought us: mod–2 homology can distinguish the defining sequences for $C(0)$ and $C(2)$ by considering their induced extensions of comodules over $\mathcal{A}_*$. The Adams spectral sequence bundles this thought process into a single machine:

> Can this be phrased so as to indicate how this works for longer extensions? I've never tried to think about even what happens for $C(4)$.

Theorem 6 ([Rav86, Definition 2.1.8, Lemma 2.1.16], [MT68, Chapter 18]). *There is a convergent spectral sequence of signature*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(F_2, F_2) \Rightarrow (\pi_* S^0)^\wedge_2. \quad \square$$

In effect, this asserts that the above process is *exhaustive*: every element of $(\pi_* S^0)^\wedge_2$ can be detected and distinguished by some representative class of

extensions of comodules for the dual Steenrod algebra. Mildly more generally, if X is a bounded-below spectrum, then there is even a spectral sequence of signature

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X) \Rightarrow \pi_* X^{\wedge}_2.$$

We could now work through the construction of the Adams spectral sequence, but it will fit more nicely into a story later on in Lecture 3.1. Before moving on to other pursuits, however, we will record the following utility Lemma. It is believable based on the above discussion, and we will need to use it before we get around to examining the guts of the construction.

Lemma 12. *The* 0*–line of the Adams spectral sequence consists of exactly those elements visible to the Hurewicz homomorphism.* $\square$

For the rest of the section, we will focus on the algebraic input "$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$", which will require us to grapple with the homological algebra of comodules for a Hopf algebra. To start that discussion, it's both reassuring and instructive to see that homological algebra can, in fact, even be done with comodules. In the usual development of homological algebra for *modules*, the key observations are the existence of projective and injective modules, and there is something at work similar here.

Remark 7 ([Rav86, Appendix A1]). Much of the results below do not rely on working with a Hopf algebra over the field $k = \mathbb{F}_2$. In fact, k can usually be taken to be a ring rather than a field. More generally, the theory goes through in the context of comodules over flat Hopf algebroids.

Lemma 13 ([Rav86, Definition A1.2.1]). *Let A be a Hopf k–algebra, let M be an A–comodule, and let N be a k–module. There is a cofree adjunction:*

$$\mathsf{Comodules}_A(M, N \otimes_k A) \cong \mathsf{Modules}_k(M, N),$$

where $N \otimes_k A$ is given the structure of an A–comodule by the coaction map

$$N \otimes_k A \xrightarrow{\mathrm{id}\otimes\Delta} N \otimes_k (A \otimes_k A) = (N \otimes_k A) \otimes_k A.$$

Proof. Given a map $f\colon M \to N$ of k–modules, we can build the composite

$$M \xrightarrow{\psi_M} M \otimes_k A \xrightarrow{f\otimes\mathrm{id}_A} N \otimes_k A.$$

Alternatively, given a map $g\colon M \to N \otimes_k A$ of A–comodules, we build the composite

$$M \xrightarrow{g} N \otimes_k A \xrightarrow{\mathrm{id}_N\otimes\varepsilon} N \otimes_k k = N. \square$$

Corollary 2 ([Rav86, Lemma A1.2.2]). *The category* $\mathsf{Comodules}_A$ *has enough injectives. Namely, if M is an A–comodule and $M \to I$ is an inclusion of k–modules*

into an injective k–module I, then $M \to I \otimes_k A$ is an injective A–comodule under M. □

Remark 8. In our case, M itself is always k–injective, so there's already an injective map $\psi_M : M \to M \otimes A$: the coaction map. The assertion that this map is coassociative is identical to saying that it is a map of comodules.

Satisfied that "Ext" at least makes sense, we're free to pursue more conceptual ends. Recall from algebraic geometry that a module M over a ring R is equivalent data to quasicoherent sheaf $\widetilde{M}$ over $\operatorname{Spec} R$. We now give a definition of "quasicoherent sheaf" that fits with our functorial perspective:

Definition 10 ([Hov02, Definition 1.1], [Str99b, Definition 2.42]). A presheaf (of modules) over a scheme X is an assignment $\mathcal{F}: X(T) \to \operatorname{Modules}_T$, satisfying a kind of functoriality in T: for each map $f: T \to T'$, there is a compatible choice of natural transformation

$$
\begin{array}{ccc}
X(T) & \xrightarrow{\mathcal{F}(T)} & \operatorname{Modules}_T \\
{\scriptstyle X(f)}\Big\downarrow {\scriptstyle \tau(f)} & & \Big\downarrow {\scriptstyle -\otimes_T T'} \\
X(T') & \xrightarrow{\mathcal{F}(T')} & \operatorname{Modules}_{T'}.
\end{array}
$$

(We think of the image of a particular point $t: \operatorname{Spec} T \to X$ in $\operatorname{Modules}_T$ as the module of "sections over t".) Such a presheaf is said to be a *quasicoherent sheaf* when these natural transformations are all natural isomorphisms.

Lemma 14 ([Str99b, Proposition 2.47]). *An R–module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ by the rule*

$$(\operatorname{Spec} T \to \operatorname{Spec} R) \mapsto M \otimes_R T.$$

Conversely, every quasicoherent sheaf over an affine scheme arises in this way. □

The tensoring operation appearing in the definition of a presheaf appears more generally as an operation on the category of sheaves.

Definition 11. A map $f: \operatorname{Spec} S \to \operatorname{Spec} R$ induces maps $f^* \dashv f_*$ of categories of quasicoherent sheaves. At the level of modules, these are given by

$$
\begin{array}{ccc}
\operatorname{QCoh}_{\operatorname{Spec} R} & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \operatorname{QCoh}_{\operatorname{Spec} S} \\
\Big\| & & \Big\| \\
\operatorname{Modules}_R & \underset{N \leftmapsto N}{\overset{M \mapsto M \otimes_R S}{\rightleftarrows}} & \operatorname{Modules}_S.
\end{array}
$$

Jay was frustrated with which adjoint I put on top (and perhaps which went on which side). Apparently there's some convention, which I should look up and obey.

One of the main uses of these operations is to define the cohomology of a sheaf. Let $\pi\colon X \to \operatorname{Spec} k$ be a scheme over $\operatorname{Spec} k$, k a field, and let $\mathcal{F}$ be a sheaf over X. The adjunction above induces a derived adjunction

$$\operatorname{Ext}_X(\pi^* k, \mathcal{F}) \cong \operatorname{Ext}_{\operatorname{Spec} k}(k, R\pi_* \mathcal{F}),$$

which is used to translate the *definition* of sheaf cohomology to that of the cohomology of the derived pushforward $R\pi_* \mathcal{F}$, itself interpretable as a mere complex of k–modules. This pattern is very general: the sense of "cohomology" relevant to a situation is often accessed by taking the derived pushforward to a suitably terminal object.[7] To invent a notion of cohomology for comodules over a Hopf algebra, we are thus moved to produce push and pull functors for a map of Hopf algebras, and this is best motivated by another example.

> I would like to explain the sense in which a comodule for k^G gives rise to a G–representation when evaluating $\operatorname{Spec}(k^G)$ on, say, k. This might also belong in Lecture 3.1.

Example 15. A common source of Hopf algebras is through group–rings: given a group G, we can define the Hopf k–algebra $k[G]$ consisting of formal k–linear combinations of elements of G. This Hopf algebra is commutative exactly when G is abelian, and $k[G]$–modules are naturally equivalent to k–linear G–representations. Dually, the ring k^G of k–valued functions on G is always commutative, using pointwise multiplication of functions, and it is *cocommutative* exactly when G is abelian. If G is finite, then k^G and $k[G]$ are k–linear dual Hopf algebras, and hence finite–dimensional k^G–comodules are naturally equivalent to finite–dimensional k–linear G–representations.

A map of groups $f\colon G \to H$ induces a map $k^f\colon k^H \to k^G$ of Hopf algebras, and it is reasonable to expect that the induced push and pull maps of comodules mimic those of G– and H–representations. Namely, given an H–representation M, we can produce a corresponding G–representation by precomposition with f. However, given a G–representation N, two things may have to be corrected to extract an H–representation:

1. If f is not surjective, we must decide what to do with the extra elements in G.
2. If f is not injective — say, $f(g_1) = f(g_2)$ — then we must force the behavior of the extracted H–representation to agree on $f(g_1)$ and $f(g_2)$, even if g_1 and g_2 act differently on N. In the extreme case of $f\colon G \to 1$, we expect to recover the fixed points of N, since this computes $H^0_{\mathrm{gp}}(G; N)$.

These concerns, together with the definition of a tensor product as a coequalizer, motivate the following:

Definition 12. Given A–comodules M and N, their cotensor product is the k-module defined by the equalizer

---

[7] This perspective often falls under the heading of "six–functor formalism".

$$M\square_A N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k A \otimes_k N.$$

Lemma 15. *Given a map $f\colon A \to B$ of Hopf k–algebras, the induced adjunction $f^* \dashv f_*$ is given at the level of comodules by*

$$
\begin{array}{ccc}
\mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} A} & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathsf{QCoh}_{\operatorname{Spec} k /\!/ \operatorname{Spec} B} \\[4pt]
\| & & \| \\[4pt]
\mathsf{Comodules}_A & \underset{N\square_B A \leftarrow\!\mapsto N}{\overset{M \mapsto M}{\rightleftarrows}} & \mathsf{Comodules}_B. \square
\end{array}
$$

Remark 9. In Lecture 3.1 (and Remark 29 specifically), we will explain the notation "$\operatorname{Spec} k /\!/ \operatorname{Spec} A$" used above. For now, suffice it to say that there again exists a functor-of-points notion of "quasicoherent sheaf" associated to a Hopf k–algebra A, and such sheaves are equivalent to A–comodules.

As an example application, cotensoring gives rise to a concise description of what it means to be a comodule map:

Lemma 16 ([Rav86, Lemma A1.1.6b]). *Let M and N be A–comodules with M projective as a k–module. Then there is an equivalence*

$$\mathsf{Comodules}_A(M, N) = \mathsf{Modules}_k(M, k)\square_A N. \square$$

From this, we can deduce a connection between the push–pull flavor of comodule cohomology described above and the input to the Adams spectral sequence.

Corollary 3. *Let $N = N' \otimes_k A$ be a cofree comodule. Then $N\square_A k = N'$.*

Proof. Picking $M = k$, we have

$$
\begin{aligned}
\mathsf{Modules}_k(k, N') &= \mathsf{Comodules}_A(k, N) \\
&= \mathsf{Modules}_k(k, k)\square_A N \\
&= k\square_A N. \square
\end{aligned}
$$

Corollary 4. *There is an isomorphism*

$$\mathsf{Comodules}_A(k, N) = \mathsf{Modules}_k(k, k)\square_A N = k\square_A N$$

and hence

$$\operatorname{Ext}_A(k, N) \cong \operatorname{Cotor}_A(k, N) \left(= H^* R\pi_* N\right).$$

Proof. Resolve N using the cofree modules described above, then apply either functor $\mathsf{Comodules}_A(k, -)$ or $k\square_A -$. In both cases, you get the same complex.

Example 16. In Lecture 1.3, we identified $\mathcal{A}_*$ with the ring of functions on the group scheme $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ of strict automorphisms of $\widehat{\mathbb{G}}_a$, which is defined by the kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Today's punchline is that this is analogous to Example 15 above: $\mathrm{Cotor}_{\mathcal{A}_*}(\mathbb{F}_2, H\mathbb{F}_{2*}X)$ is thought of as "the derived fixed points" of "$G = \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$" on the "$G$–module" $H\mathbb{F}_{2*}X$.

We now give several examples to get a sense of how the Adams spectral sequence behaves.

Example 17. Consider the degenerate case $X = H\mathbb{F}_2$. Then $H\mathbb{F}_{2*}(H\mathbb{F}_2) = \mathcal{A}_*$ is a cofree comodule, and hence Cotor is concentrated on the 0–line:

$$\mathrm{Cotor}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_2, H\mathbb{F}_{2*}(H\mathbb{F}_2)) = \mathbb{F}_2.$$

The Adams spectral sequence collapses to show the wholly unsurprising equality $\pi_* H\mathbb{F}_2 = \mathbb{F}_2$, and indeed this is the element in the image of the Hurewicz map $\pi_* H\mathbb{F}_2 \to H\mathbb{F}_{2*}H\mathbb{F}_2$.

> I can't tell if this example belongs.

Example 18. Next, we consider $X = kO$, the connective real K–theory spectrum. The main input we need is the structure of $H\mathbb{F}_{2*}kO$ as an $\mathcal{A}_*$–comodule, so that we can compute

$$\mathrm{Cotor}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_2, H\mathbb{F}_{2*}kO) \Rightarrow \pi_* kO_2^\wedge.$$

There is a slick trick for doing this: by working in the category of kO–modules rather than in all spectra, we can construct a relative Adams spectral sequence

$$\mathrm{Cotor}_{\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2}^{*,*}(\mathbb{F}_2, \pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2)) \Rightarrow \pi_*(kO \wedge H\mathbb{F}_2).$$

The second argument is easy to identify:

$$\pi_* H\mathbb{F}_2 \wedge_{kO} (kO \wedge H\mathbb{F}_2) = \pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 = \mathcal{A}_*.$$

The Hopf algebra requires further input. Consider the following trio of cofiber sequences[8]:

$$\Sigma kO \xrightarrow{\cdot\eta} kO \to kU, \quad \Sigma^2 kU \xrightarrow{\cdot\beta} kU \to H\mathbb{Z}, \quad H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \to H\mathbb{F}_2.$$

---

[8] This first sequence, known as the Wood cofiber sequence, is a consequence of a very simple form of Bott periodicity [Har80, Section 5]: there is a fiber sequence of infinite-loopspaces $O/U \to BO \to BU$, and $\underline{kO}_1 = O/U$.

These combine to give a resolution of $H\mathbb{F}_2$ via an iterated cofiber of free kO–modules, with Poincaré series

$$((1+t^2)+t^3(1+t^2))+t((1+t^2)+t^3(1+t^2)) = 1+t+t^2+2t^3+t^4+t^5+t^6.$$

Repeatedly using the identity $kO \wedge_{kO} H\mathbb{F}_2 \simeq H\mathbb{F}_2$ gives a small presentation of the Hopf algebra $\pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$: it is a commutative Hopf algebra over $\mathbb{F}_2$ with the above Poincaré series. The Borel–Milnor–Moore [MM65, Theorem 7.11] classification of commutative Hopf algebras over $\mathbb{F}_p$ shows that the algebra structure is either

$$\frac{\mathbb{F}_2[a,b,c]}{(a^2=0,b^2=0,c^2=0)} \quad \text{or} \quad \frac{\mathbb{F}_2[a,b,c]}{(a^2=b,b^2=0,c^2=0)}$$

for $|a|=1$, $|b|=2$, and $|c|=3$. By knowing that the natural map $\mathcal{A} \to \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2$ winds up inducing an isomorphism $\pi_{*\leq 2}\mathbb{S} \to \pi_{*\leq 2}kO$, we conclude that we are in the latter case, which gives a presentation of the Hopf algebra as a whole:

$$
\begin{array}{ccc}
\pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 & \longrightarrow & \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \\
\| & & \| \\
\mathcal{A}_* & \longrightarrow & \dfrac{\mathbb{F}_2[\xi_1,\xi_2,\xi_3,\xi_4,\ldots]}{(\xi_1^4,\xi_2^2),(\xi_n \mid n \geq 3)}.
\end{array}
$$

This Hopf algebra is commonly denoted $\mathcal{A}(1)_*$, and its corresponding subgroup scheme $\operatorname{Spec} \pi_* H\mathbb{F}_2 \wedge_{kO} H\mathbb{F}_2 \subseteq \underline{\operatorname{Aut}}_1 \widehat{\mathbb{G}}_a$ admits easy memorization: it is the subscheme of automorphisms of the form $x + \xi_1 x^2 + \xi_2 x^4$, with exactly the additional relations imposed on ξ_1 and ξ_2 so that this set is stable under composition and inversion.[9,10] Its cohomology is periodic with period 8, and it is pictured through a range in Figure 1.1.

Example 19. At the other extreme, we can pick the extremely nondegenerate case $X = \mathbb{S}$, where $\underline{\operatorname{Aut}}_1 \widehat{\mathbb{G}}_a$ acts maximally nonfreely on $\operatorname{Spec} \mathbb{F}_2$. The resulting spectral sequence is pictured through a range in Figure 1.2.

> Jon asked: spectral sequences coming from π_* of a Tot tower increase Tot degree. ANSS differentials decrease degree: they run against the multiplicative structure in pictures. What's going on with this? I think this is a duality effect: working with the Steenrod algebra versus its dual.

---

[9] A similar analysis shows that $H\mathbb{F}_{2*}H\mathbb{Z}$ corepresents the subscheme of automorphisms of the form $x + \xi_1 x^2$ which are stable under composition and inversion.

[10] There is also an accidental isomorphism of this Hopf algebra with $\mathbb{F}_2^{D_4}$, where D_4 is the dihedral group with 8 elements.

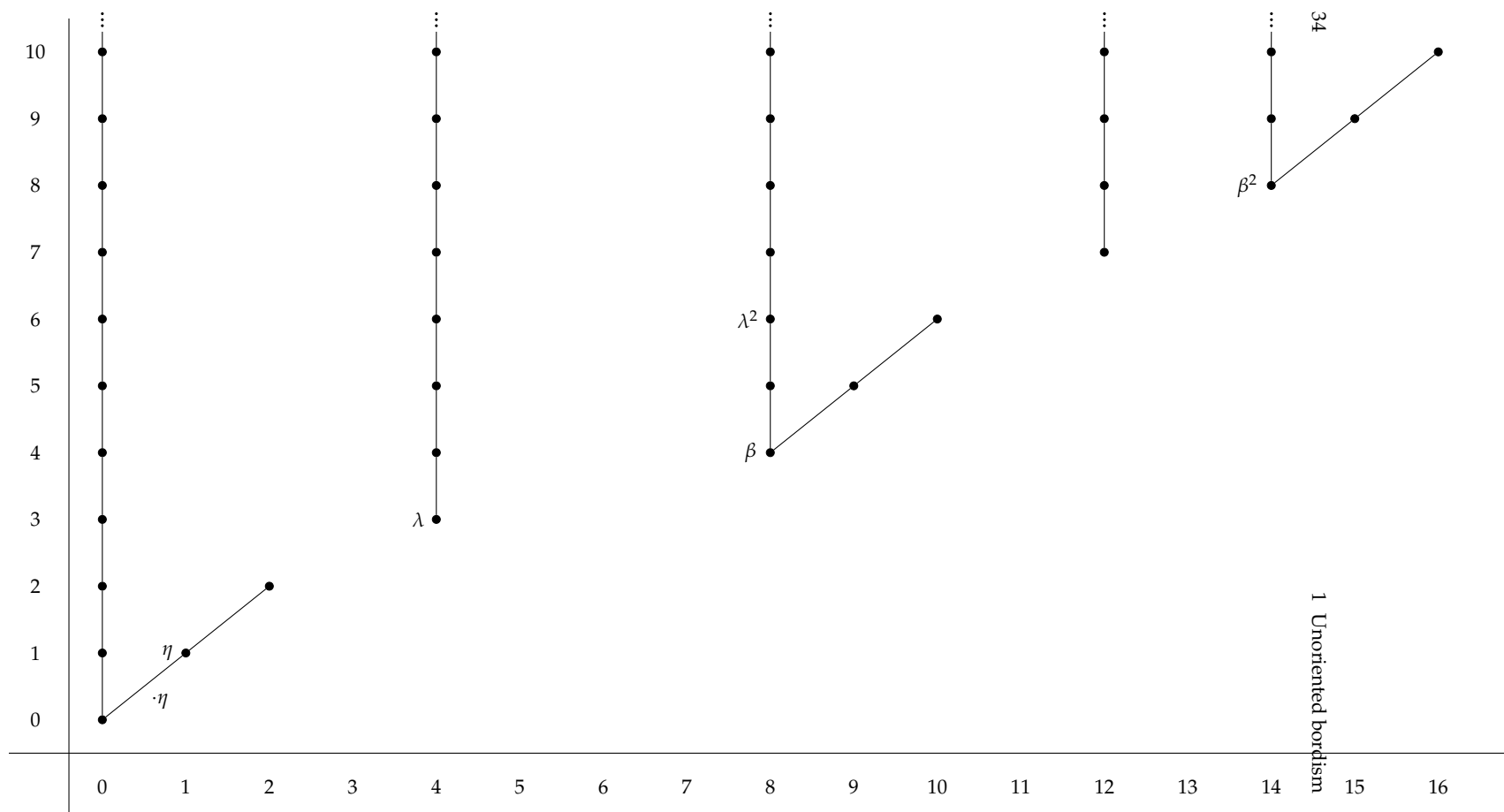**Fig. 1.1** The $H\mathbb{F}_2$–Adams spectral sequence for kO, which collapses at the second page. North and north-east lines denote multiplication by 2 and by η.
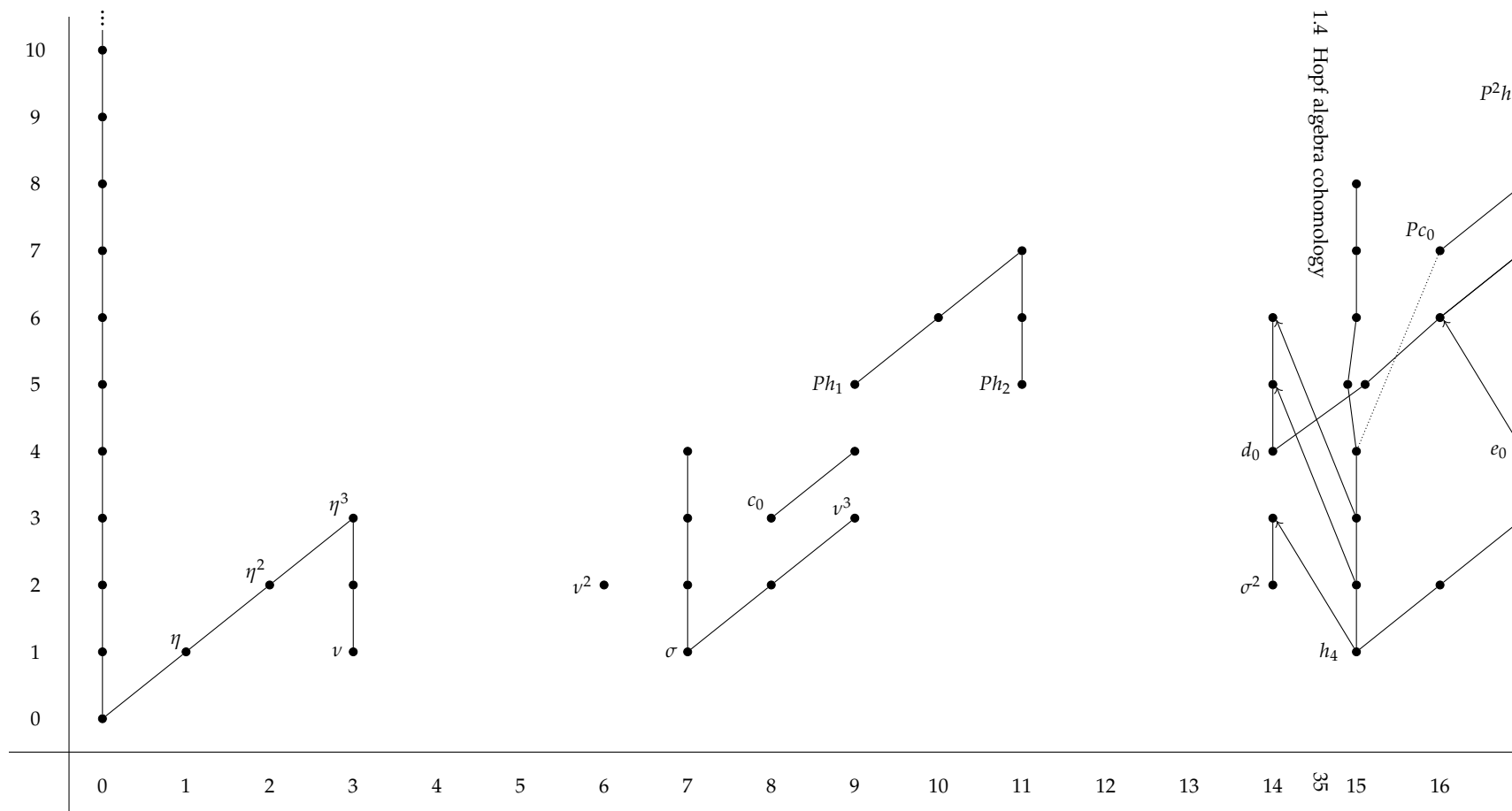
Fig. 1.2 A small piece of the $H\mathbb{F}_2$–Adams spectral sequence for the sphere, beginning at the second page. North and north-east lines denote multiplication by 2 and by η, north-west lines denote d_2– and d_3–differentials.

1.5 The unoriented bordism ring

Our goal in this section is to use our results so far to make a calculation of $\pi_* MO$, the unoriented bordism ring. Our approach is the same as in the examples at the end of the previous section: we will want to use the Adams spectral sequence of signature

$$H^*_{\text{gp}}(\underline{\text{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)\widehat{\ }) \Rightarrow \pi_* MO,$$

which requires understanding $H\mathbb{F}_2 P_0(MO)$ as a comodule for the dual Steenrod algebra.

Our first step toward this is the following calculation:

You could describe how the coaction map is induced on the symmetric algebra: I think it's applied simultaneously to each factor in the tensor element, then all the Steenrod algebra runoff is collected together by a multiplication.

Lemma 17 ([Swi02, Theorem 16.17]). *The natural map*

$$\text{Sym}\,\widetilde{H\mathbb{F}_2 P_0}(BO(1)) \to H\mathbb{F}_2 P_0(BO).$$

induces an isomorphism of Hopf algebras and of comodules for the dual Steenrod algebra

$$\text{Sym}\,\widetilde{H\mathbb{F}_2 P_0}(BO(1)) = \frac{\text{Sym}\,H\mathbb{F}_2 P_0(BO(1))}{\beta_0 = 0} \xrightarrow{\simeq} H\mathbb{F}_2 P_0(BO).$$

You could cite these standard facts.

Proof. This follows from a combination of standard facts about Stiefel–Whitney classes. First, these classes generate the cohomology ring $H\mathbb{F}_2^* BO(n)$:

$$H\mathbb{F}_2^* BO(n) \cong \mathbb{F}_2[\![w_1, \dots, w_n]\!].$$

... which you haven't defined.

Second, the total Stiefel–Whitney classis exponential, in the sense of

$$w(V \oplus W) = w(V) \cdot w(W).$$

From this, it follows that the natural map

$$H\mathbb{F}_2^* BO(n) \xrightarrow{\oplus_{j=1}^n \mathcal{L}_j} H\mathbb{F}_2^* BO(1)^{\times n} \cong (H\mathbb{F}_2^* BO(1))^{\otimes n}$$

is the inclusion of the symmetric polynomials, by calculating the total Stiefel–Whitney class

$$w\left(\bigoplus_{j=1}^n \mathcal{L}_j\right) = \prod_{j=1}^n (1 + w_1(\mathcal{L}_j)) = \sum_{j=0}^n \sigma_j(w_1(\mathcal{L}_1), \dots, w_1(\mathcal{L}_n))t^j.$$

Dually, the homological map

$$(H\mathbb{F}_{2*} BO(1))^{\otimes n} \to H\mathbb{F}_{2*} BO(n)$$

is surjective, modeling the quotient from the tensor product to the symmetric tensor product. Stabilizing as $n \to \infty$, we recover the statement of the Lemma.

With this in hand, we now turn to the homotopy ring $H\mathbb{F}_2 P_0 MO$. There are two equivalences that we might consider employing. We have the Thom isomorphism:

$$H\mathbb{F}_2 P_0(BO(1)) =\!=\!=\!= H\mathbb{F}_2 P_0(MO(1))$$
$$\beta_j, j \geq 0 \longmapsto \beta'_j, j \geq 0,$$

and we also have the equivalence induced by the topological map in Example 2:

$$\widetilde{H\mathbb{F}_2 P_0}(BO(1)) =\!=\!=\!= H\mathbb{F}_2 P_0(\Sigma MO(1))$$
$$\beta_j, j \geq 1 \longmapsto \beta'_{j-1}, j \geq 1.$$

We will use them both in turn.

Corollary 5 ([Ada95, Section I.3], [Hop, Proposition 6.2]). *There is an isomorphism*

$$H\mathbb{F}_2 P_0(MO) \cong \frac{\mathrm{Sym}\, H\mathbb{F}_2 P_0 MO(1)}{b'_0 = 1}.$$

Proof. The block sum maps

$$BO(n) \times BO(m) \to BO(n + m)$$

Thomify to give compatible maps

$$MO(n) \wedge MO(m) \to MO(n + m).$$

Taking the colimit, this gives a ring structure on MO compatible with that on $\Sigma^\infty_+ BO$ and compatible with the Thom isomorphism. $\square$

We now seek to understand the utility of the scheme $\mathrm{Spec}\, H\mathbb{F}_2 P_0(MO)$, as well as its action of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$. The first of these tasks comes from untangling some of the topological dualities we've been using thus far.

Lemma 18. *The following square commutes:*

$$\begin{array}{ccc}
\mathrm{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2 P_0(MO), \mathbb{F}_2) & =\!=\!=\!= & \mathrm{Spectra}(MO, H\mathbb{F}_2 P) \\
\uparrow & & \uparrow \\
\mathrm{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2 P_0(MO), \mathbb{F}_2) & =\!=\!=\!= & \mathrm{RingSpectra}(MO, H\mathbb{F}_2 P).
\end{array}$$

Proof. The top isomorphism asserts only that $\mathbb{F}_2$–cohomology and $\mathbb{F}_2$–homology are linearly dual to one another. The second follows immediately from investigating the effect of the ring homomorphism diagrams in the bottom-right corner in terms of the subset they select in the top-left.

Corollary 6. *There is a bijection between homotopy classes of ring maps $MO \to H\mathbb{F}_2P$ and homotopy classes of factorizations*

$$
\begin{array}{ccc}
S^0 & \longrightarrow & MO(1) \\
& \searrow & \downarrow \\
& & H\mathbb{F}_2P.
\end{array}
$$

Proof. We extend the square in the Lemma 18 using the following diagram:

$$
\begin{array}{ccc}
\mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO(1)), \mathbb{F}_2) & \longleftarrow & \mathsf{Modules}_{\mathbb{F}_2}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2) \\
\uparrow & & \uparrow \\
\{f\colon H\mathbb{F}_2P_0(MO(1)) \to \mathbb{F}_2 \,|\, f(\beta_0') = 1\} & = & \mathsf{Algebras}_{\mathbb{F}_2/}(H\mathbb{F}_2P_0(MO), \mathbb{F}_2),
\end{array}
$$

where the equality at bottom follows from the universal property of $H\mathbb{F}_2P_0(MO)$ in $\mathbb{F}_2$–algebras expressed in Corollary 5. Noting that β_0' is induced by the topological map $S^0 \to MO(1)$, the condition $f(\beta_0') = 1$ is exactly the condition expressed in the statement of the Corollary.

Corollary 7. *There is an $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$–equivariant isomorphism of schemes*

$$
\mathrm{Spec}\, H\mathbb{F}_2P_0(MO) \cong \mathrm{Coord}_1(\mathbb{RP}^\infty_{H\mathbb{F}_2P}),
$$

where the latter is the subscheme of functions $\mathbb{RP}^\infty_{H\mathbb{F}_2P} \to \widehat{\mathbb{A}}^1$ which are coordinates (i.e., which are isomorphisms of formal schemes — or, equivalently, which restrict to the canonical identification of tangent spaces $\mathbb{RP}^1_{H\mathbb{F}_2P} = \widehat{\mathbb{A}}^{1,(1)}$).

Proof. The conclusion of the previous Corollary is that the $\mathbb{F}_2$–points of $\mathrm{Spec}\, H\mathbb{F}_2P_0(MO)$ biject with classes $H\mathbb{F}_2P^0MO(1) \cong \widetilde{H\mathbb{F}_2P}^0 \mathbb{RP}^\infty$ satisfying the condition that they give an isomorphism $\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. Because $H\mathbb{F}_2P_0(MO)$ is a polynomial algebra, this holds in general: for $u\colon \mathbb{F}_2 \to T$ an $\mathbb{F}_2$–algebra, the T–points of $\mathrm{Spec}\, H\mathbb{F}_2P_0(MO)$ will biject with coordinates on $u^*\mathbb{RP}^\infty_{H\mathbb{F}_2P}$. The isomorphism of schemes follows, though we have not yet discussed equivarience.

To compute the action of $\underline{\mathrm{Aut}}\,\widehat{\mathbb{G}}_a$, we turn to the map in Example 2:

$$
\Sigma^\infty BO(1) \xrightarrow{c,\simeq} \Sigma MO(1).
$$

Writing $\beta(t) = \sum_{j=0}^{\infty} \beta_j t^j$ and $\xi(t) = \sum_{k=0}^{\infty} \xi_k t^{2^k}$, the dual Steenrod coaction on $H\mathbb{F}_2 P_0 BO(1)$ is encoded by the formula

$$\sum_{j=0}^{\infty} \psi(\beta_j) t^j = \psi(\beta(t)) = \beta(\xi(t)) = \sum_{j=0}^{\infty} \beta_j \left(\sum_{k=0}^{\infty} \xi_k t^{2^k} \right)^j.$$

Because $c_*(\beta_j) = \beta'_{j-1}$, this translates to the formula $\psi(\beta'(t)) = \beta'(\xi(t))$, where

$$\beta'(t) = \sum_{j=0}^{\infty} \beta'_j t^{j+1}.$$

Passing from $H\mathbb{F}_2 P_0(MO(1))$ to $H\mathbb{F}_2 P_0(MO) \cong \mathrm{Sym}\, H\mathbb{F}_2 P_0(MO(1))/(\beta'_0 = 1)$, this is precisely the formula for precomposing a coordinate with a strict automorphism — i.e., a point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ acts on a point in $\mathrm{Coord}(\mathbb{RP}^{\infty}_{H\mathbb{F}_2 P})$ in the way claimed.

We are now ready to analyze the group cohomology of $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ with coefficients in the comodule $H\mathbb{F}_2 P_0(MO)$. This is the last piece of input we need to assess the Adams spectral sequence computing $\pi_* MO$.

Theorem 7 ([Str06, Theorem 12.2], [Mit83, Proposition 2.1]). *The action of* $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ *on* $\mathrm{Coord}_1(\widehat{\mathbb{G}}_a)$ *is free:*

$$\mathrm{Coord}_1(\widehat{\mathbb{G}}_a) \cong \mathrm{Spec}\, \mathbb{F}_2[b_j \mid j \neq 2^k - 1] \times \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a).$$

Proof. Recall, again, that $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$ is defined by the (split) kernel sequence

$$0 \to \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a) \to \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a) \to \mathbb{G}_m \to 0.$$

Consider a point $f \in \mathrm{Coord}_1(\widehat{\mathbb{G}}_a)(R)$, which in terms of the standard coordinate can be expressed as

$$f(x) = \sum_{j=1}^{\infty} b_{j-1} x^j,$$

where $b_0 = 1$. Decompose this series as $f(x) = f_2(x) + f_{\mathrm{rest}}(x)$, with

$$f_2(x) = \sum_{k=0}^{\infty} b_{2^k-1} x^{2^k}, \qquad f_{\mathrm{rest}}(x) = \sum_{j \neq 2^k} b_{j-1} x^j.$$

Because we assumed $b_0 = 1$ and f_2 is concentrated in power–of–2 degrees, it follows that f_2 gives a point $f_2 \in \underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$. We can use it to de-scale and get a new coordinate $g(x) = f_2^{-1}(f(x))$, which has an analogous decomposition into series $g_2(x)$ and $g_{\mathrm{rest}}(x)$. Finally, note that $g_2(x) = x$ and that f_2 is the unique point in $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)(R)$ that has this property.

Corollary 8 ([Str06, Remark 12.3]). $\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1, j \geq 1]$ *with* $|b_j| = j$.

Proof. Set $M = \mathbb{F}_2[b_j \mid j \neq 2^k - 1]$. It follows from Corollary 3 applied to Theorem 7 that the $\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a)$–cohomology of $H\mathbb{F}_2 P_0(MO)$ has Cotor–amplitude 0:

$$\mathrm{Cotor}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_2, H\mathbb{F}_2 P_0(MO)) = \mathrm{Cotor}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_2, \mathcal{A}_* \otimes_{\mathbb{F}_2} M)$$
$$= \mathbb{F}_2 \square_{\mathcal{A}_*}(\mathcal{A}_* \otimes_{\mathbb{F}_2} M)$$
$$= \mathbb{F}_2 \otimes_{\mathbb{F}_2} M = M.$$

Since the Adams spectral sequence

$$H_{\mathrm{gp}}^*(\underline{\mathrm{Aut}}_1(\widehat{\mathbb{G}}_a); H\mathbb{F}_2 P_0(MO)) \Rightarrow \pi_* MO$$

is concentrated on the 0–line, it collapses. Using the residual $\mathbb{G}_m$–action to infer the grading, we deduce

$$\pi_* MO = \mathbb{F}_2[b_j \mid j \neq 2^k - 1].\square$$

The notation here is a little sloppy. I've written $\mathcal{A}_*$ for the kernel of the map $\mathcal{A}P_0 \to \mathbb{F}_2[\xi_0^\pm]$.

This is pretty remarkable: some statement about manifold geometry came down to understanding how we could reparametrize a certain formal group, itself a (fairly simple) purely algebraic problem. The connection between these two problems seems fairly miraculous: we needed a small object, $\mathbb{RP}^\infty$, which controlled the whole story; we needed to be able to compute everything about it; and we needed various other "generation" or "freeness" results to work out in our favor. It is not obvious that we will get this lucky twice, should we try to reapply these ideas to other cases. Nevertheless, trying to push our luck as far as possible is the main thrust of the rest of the book. We could close this section with this accomplishment, but there are two easy consequences of this calculation that are worth recording before we leave.

Lemma 19. *MO splits as a wedge of shifts of* $H\mathbb{F}_2$.

Proof. Referring to Lemma 12, we find that the Hurewicz map induces a π_*–injection $MO \to H\mathbb{F}_2 \wedge MO$. Pick an $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha$ for $\pi_* MO$ and extend it to a $\mathbb{F}_2$–basis $\{v_\alpha\}_\alpha \cup \{w_\beta\}_\beta$ for $\pi_* H\mathbb{F}_2 \wedge MO$. Altogether, this larger basis can be represented as a single map

$$\bigvee_\alpha \Sigma^{|v_\alpha|}\mathbb{S} \vee \bigvee_\beta \Sigma^{|w_\beta|}\mathbb{S} \xrightarrow{\vee_\alpha v_\alpha \vee \vee_\beta w_\beta} H\mathbb{F}_2 \wedge MO.$$

Smashing through with $H\mathbb{F}_2$ gives an equivalence

$$\bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|} H\mathbb{F}_2 \xrightarrow{\simeq} H\mathbb{F}_2 \wedge MO.$$

The composite map

$$MO \to H\mathbb{F}_2 \wedge MO \xleftarrow{\simeq} \bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2 \vee \bigvee_\beta \Sigma^{|w_\beta|} H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{|v_\alpha|} H\mathbb{F}_2$$

is a weak equivalence.

Remark 10. Just using that $\pi_* MO$ is connective and $\pi_0 MO = \mathbb{F}_2$, we can produce a ring spectrum map $MO \to H\mathbb{F}_2$. What we've learned is that this map has a splitting: MO is also an $H\mathbb{F}_2$–algebra.

remove me.

Remark 11. You should mention the stable cooperations Spec $MO_* MO$. Rather than coming with a specified logarithm, it's an isomorphism between any pair of additive formal groups — or, I suppose, a pair of logarithms.

Case Study 2
Complex bordism

Having totally dissected unoriented bordism, we can now turn our attention to other sorts of bordism theories, and there are many available: oriented, *Spin*, *String*, complex, . . . — the list continues. We would like to replicate the results from Case Study 1 for these other cases, but upon even a brief inspection we quickly see that only one of the bordism theories mentioned supports this program. Specifically, the space $\mathbb{RP}^\infty = BO(1)$ was a key player in the unoriented bordism story, and the only other similar ground object is $\mathbb{CP}^\infty = BU(1)$ in complex bordism. This informs our choice to spend this Case Study focused on it. To begin, the contents of Lecture 1.1 can be replicated essentially *mutatis mutandis*, resulting in the following theorems:

Theorem 8 (cf. Lemma 4 and surrounding discussion). *There is a map of infinite–loopspaces*

$$J_\mathbb{C} \colon BU \to BGL_1 \mathbb{S}$$

called the complex J–homomorphism.

Definition 13 (cf. Definition 3). The associated Thom spectrum is written "*MU*" and called *complex bordism*. A map $MU \to E$ of ring spectra is said to be a *complex orientation of E.*

Theorem 9 (cf. Theorem 4). *For a complex vector bundle ξ on a space X and a complex-oriented ring spectrum E, there is a natural equivalence*

$$E \wedge T(\xi) \simeq E \wedge \Sigma_+^\infty X. \square$$

Corollary 9 (cf. Example 3). *In particular, for a complex-oriented ring spectrum E it follows that $E^* \mathbb{CP}^\infty$ is isomorphic to a one–dimensional power series ring.* $\square$

We would like to then review the results of Lecture 1.3 and conclude (by reinterpreting Corollary 9) that $\mathbb{CP}_E^\infty$ gives a 1–dimensional formal group

Something I've seen more than once is an equivalence $MU(k) \simeq BU(k)/BU(k-1)$. It's not immediately obvious to me where this comes from. Where does it come from? Is it helpful to think about?

Maybe I'm confused about grading issues, but I thought $E^* \mathbb{CP}^\infty$ was a polynomial ring and $EP^0 \mathbb{CP}^\infty$ is the power series ring?

Also, this is a nice argument. Usually this computation proceeds through the AHSS. Can this method be adapted to spaces other than $\mathbb{CP}^\infty$?

over Spec E_*. In order to make this statement honestly, however, we are first required to describe more responsibly the algebraic geometry we outlined in Lecture 1.2. Specifically, the characteristic 2 nature of the unoriented bordism ring was a major simplifying feature which made it wholly amenable to study by $H\mathbb{F}_2$. In turn, $H\mathbb{F}_2$ has many nice properties — for example, it has a duality between homology and cohomology, and it supports a Künneth isomorphism — and these are reflected in the extremely simple algebraic geometry of Spec $\mathbb{F}_2$. By contrast, the complex bordism ring is considerably more complicated, not least because it is a characteristic 0 ring, and more generally we have essentially no control over the behavior of the coefficient ring E_* of some other complex–oriented theory. Nonetheless, once the background theory and construction of "X_E" are taken care of in Lecture 2.1, we indeed find that $\mathbb{CP}^\infty_E$ is a 1–dimensional formal group over Spec E_*.

However, where we could explicitly calculate $\mathbb{RP}^\infty_{H\mathbb{F}_2}$ to be $\widehat{\mathbb{G}}_a$, we again have little control over what formal group $\mathbb{CP}^\infty_E$ could possibly be. In the universal case, $\mathbb{CP}^\infty_{MU}$ comes equipped with a natural coordinate, and this induces a map

$$\operatorname{Spec} MU_* \to \mathcal{M}_{\mathbf{fgl}}$$

from the spectrum associated to the coefficient ring of complex bordism to the moduli of formal group laws. The conclusion of this Case Study in Corollary 25 (modulo an algebraic result, shown in the next Case Study as Theorem 25) states that this map is an isomorphism, so that $\mathbb{CP}^\infty_{MU}$ is the universal — i.e., maximally complicated — formal group. Our route for proving this passes through the foothills of the theory of "p^{th} power operations", which simultaneously encode many possible natural transformations from MU–cohomology to itself glommed together in a large sum, one term of which is the literal p^{th} power. Remarkably, the identity operation also appears in this family of operations, and the rest of the operations are in some sense controlled by this naturally occuring formal group law. A careful analysis of this sum begets the inductive proof in Corollary 23 that $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ is surjective.

The execution of this proof requires some understanding of cohomology operations for complex-oriented cohomology theories generally. Stable such operations correspond to homotopy classes $MU \to E$, i.e., elements of $E^0 MU$, which correspond via the Thom isomorphism to elements of $E^0 BU$. This object is the repository of E–characteristic classes for complex vector bundles, which we describe in terms of divisors on formal curves. This amounts to a description of the formal schemes $BU(n)_E$, which underpins our understanding of the whole story and which significantly informs our study of connective orientations in Case Study 5.

2.1 Calculus on formal varieties

In light of the introduction, we see that it would be prudent to develop some of the theory of formal schemes and formal varieties outside of the context of $\mathbb{F}_2$–algebras. However, writing down a list of definitions and checking that they have good enough properties is not especially enlightening or fun. Instead, it will be informative to understand where these objects come from in algebraic geometry, so that we can carry the accompanying geometric intuition along with us as we maneuver our way back toward homotopy theory and bordism. Our overarching goal in this Lecture is to develop a notion of calculus (and analytic expansions in particular) in the context of affine schemes. The place to begin is with a definition of cotangent and tangent spaces, as well as some supporting vocabulary.

Definition 14 (cf. Definition 5). For an R–algebra A, the functor $\operatorname{Spec} A \colon \mathsf{Algebras}_R \to \mathsf{Sets}$ defined by

$$(\operatorname{Spec} A)(T) := \mathsf{Algebras}_R(A, T)$$

is called the *spectrum of A*. A functor X which is naturally isomorphic to to $\operatorname{Spec} A$ for some A is called an *affine (R–)scheme*, and $A = \mathcal{O}_{\operatorname{Spec} A}$ is called its *ring of functions*. A subfunctor $Y \subseteq X$ is said to be a *closed*[1] *subscheme* when an identification[2] $X \cong \operatorname{Spec} A$ induces a further identification

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} \\
\operatorname{Spec}(R/I) & \longrightarrow & \operatorname{Spec} R.
\end{array}
$$

Definition 15. Take $S = \operatorname{Spec} R$ to be our base scheme, let $X = \operatorname{Spec} A$ be an affine scheme over S, and consider an S–point $x \colon S \to X$ of X. The point x is automatically closed, so that x is presented as $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for some ideal I. The *cotangent space $T_x^* X$* is defined by the quotient R–module

$$T_x^* X := I/I^2,$$

consisting of functions vanishing at x as considered up to first order. Examples of these include the linear parts of curves passing through x, so we additionally define the *tangent space $T_x X$* by

$$T_x X = \mathsf{Schemes}_{\operatorname{Spec} R/}(\operatorname{Spec} R[\varepsilon]/\varepsilon^2, X),$$

i.e., maps $\operatorname{Spec} R[\varepsilon]/\varepsilon^2 \to X$ which restrict to $x \colon S \to X$ upon setting $\varepsilon = 0$.

---

[1] The word "closed" is meant to suggest properties of these inclusions: in suitable senses, they are closed under finite unions and arbitrary intersections.

[2] This property is independent of choice of chart.

Remark 12. In the situation above, there is a natural map $T_x X \to \mathrm{Modules}_R(T_x^* X, R)$. A map $\mathcal{O}_X \to R[\varepsilon]/\varepsilon^2$ induces a map $I \to (\varepsilon)$ and hence a map

$$I/I^2 \to (\varepsilon)/(\varepsilon^2) \cong R,$$

i.e., a point in $T_x^* X$.

Harkening back to Example 13, the definition of the R–module tangent space begs promotion to an S–scheme.

Lemma 20. *There is an affine scheme $T_x X$ defined by*

$$(T_x X)(T) := \left\{ (u, f) \,\middle|\, \begin{matrix} u \colon \ \mathrm{Spec}\, T \to S, \\ f \in T_{u^* x} u^* X \end{matrix} \right\}.$$

Proof (Proof sketch). We specialize an argument of Strickland [Str99b, Proposition 2.94] to the case at hand.[3] We start by seeking an R–algebra B such that R–algebra maps $B \to T$ biject with pairs of maps $u \colon R \to T$ and T–algebra maps

$$f \colon A \otimes_R T \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

Such maps f biject with R–algebra maps

$$A \to R[\varepsilon]/\varepsilon^2 \otimes_R T.$$

Noting that $R[\varepsilon]/(\varepsilon)^2$ is free and finite–dimensional as an R–module, we forget from R–algebra maps down to just R–module maps, use R–linear duality to move it to the domain, promote it back to an R–algebra by forming the symmetric algebra, then finally try to pick out the maps of interest by imposing a quotient. By expanding Strickland's formulas, we arrive at the equation

$$\underline{\mathrm{Schemes}}_S(\mathrm{Spec}\, R[\varepsilon]/\varepsilon^2, X) = \mathrm{Spec}\, A\{1, da \mid a \in A\} \Big/ \left(\begin{matrix} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{matrix} \right).$$

To extract the scheme $T_x X$ from this, we construct the pullback

$$T_x X := \underline{\mathrm{Schemes}}_S(\mathrm{Spec}\, R[\varepsilon]/\varepsilon^2, X) \times_X S,$$

where the structure maps are given on the left by setting $\varepsilon = 0$ and on the right using the point x. Expanding the formulas again shows that the coordinate ring of this affine scheme is given by

---

[3] Strickland also shows that mapping schemes between formal schemes exist considerably more generally [Str99b, Theorem 4.69]. The source either has to be "finite" in some sense, in which case the proof proceeds along the lines presented here, or it has to be *coalgebraic*, which is an important technical tool that we discuss much later in Definition 74.

$$\mathcal{O}_{T_x X} = A/I^2 \cong R \oplus T_x^* X. \square$$

Definition 16. The ring of functions appearing in the proof above fits into an exact sequence

$$0 \to \Omega_{A/R} \to A\{1, da \mid a \in A\} \Big/ \left(\begin{array}{c} dr = 0 \text{ for } r \in R, \\ d(a_1 a_2) = da_1 \cdot a_2 + a_1 \cdot da_2 \end{array} \right) \xrightarrow{da=0} A\{1\} \to 0.$$

The kernel $\Omega_{A/R}$ is called the module of *Kähler differentials* (of A, relative to R). The map $d \colon R \to \Omega^1_{A/R}$ is the universal R–linear derivation into an A–module, i.e.,

$$\mathsf{Derivations}_R(A, M) = \mathsf{Modules}_A(\Omega^1_{A/R}, M).$$

The upshot of this calculation is that $\operatorname{Spec} A/I^2$ is a natural place to study the linear behavior of functions on X near x. We have also set the definitions up so that we can easily generalize to higher-order approximations:

Definition 17. More generally, the nth *jet space* of X at x, or the nth *order neighborhood* of x in X, is defined by

$$\underline{\mathsf{Schemes}}_S(\operatorname{Spec} R[\varepsilon]/\varepsilon^{n+1}, X) \times_X S \cong \operatorname{Spec} A/I^{n+1}.$$

Each jet space has an inclusion from the one before, modeled by the closed subscheme $\operatorname{Spec} A/I^n \to \operatorname{Spec} A/I^{n+1}$.

In order to study analytic expansions of functions, we bundle these jet spaces together into a single object embodying formal expansions in X at x:

Definition 18. Fix a scheme S. A *formal S–scheme* $X = \{X_\alpha\}_\alpha$ is an ind-system of S–schemes X_α.[4] Given a closed subscheme Y of an affine S–scheme X, we define the nth *order neighborhood of Y in X* to be the scheme $\operatorname{Spec} R/I^{n+1}$. The *formal neighborhood of Y in X* is then defined to be the formal scheme

$$X_Y^\wedge := \operatorname{Spf} R_I^\wedge := \left\{ \operatorname{Spec} R/I \to \operatorname{Spec} R/I^2 \to \operatorname{Spec} R/I^3 \to \cdots \right\}.$$

In the case that $Y = S$, this specializes to the system of jet spaces as in Definition 17.

---

[4] This definition, owing to Strickland [Str99b, Definition 4.1], is somewhat idiosyncratic. Its generality gives it good categorical properties, but it is somewhat disconnected from the formal schemes familiar to algebraic geometers, which primarily arise through linearly topologized rings **Find a citation for this style of definition**. For functor-of-points definitions that hang more tightly with the classical definition, the reader is directed toward Strickland's solid formal schemes [Str99b, Section 4.2] or to Beilinson and Drinfel'd [BD, Section 7.11.1].

Although we will make use of these definitions generally, the following ur-example captures the most geometrically-intuitive situation.

Example 20. Picking the affine scheme $X = \operatorname{Spec} R[x_1, \ldots, x_n] = \mathbb{A}^n$ and the point $x = (x_1 = 0, \ldots, x_n = 0)$ gives a formal scheme known as *formal affine n–space*, given explicitly by

$$\widehat{\mathbb{A}}^n = \operatorname{Spf} R[\![x_1, \ldots, x_n]\!].$$

Evaluated on a test algebra T, $\widehat{\mathbb{A}}^1(T)$ yields the ideal of nilpotent elements in T and $\widehat{\mathbb{A}}^n(T)$ its n–fold Cartesian power.

Lemma 21. *Pointed maps $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ naturally biject with m–tuples of n–variate power series with no constant term.* $\qquad\square$

The preceding Lemma shows how formal varieties are especially nice, because maps between them can be boiled down to statements about power series.[5] In particular, this allows local theorems from analytic differential geometry to be imported, including a version of the inverse function theorem.

> Cite me: This is 3.1.8 in the Crystals notes..

Theorem 10. *A pointed map $f \colon V \to W$ of finite–dimensional formal varieties is an isomorphism if and only if the induced map $T_0 f \colon T_0 V \to T_0 W$ is an isomorphism of R–modules.*

> Consider deleting this proof? Who cares?

Proof. First, reduce to the case where $V \cong \widehat{\mathbb{A}}^n$ and $W \cong \widehat{\mathbb{A}}^n$ have the same dimension, and select charts for both. Then, $T_0 f$ is a matrix of dimension $n \times n$. If $T_0 f$ fails to be invertible, we are done, and if it is invertible, we replace f by $f \circ (T_0 f)^{-1}$ so that $T_0 f$ is the identity matrix.

We now construct the inverse function by induction on degree. Set $g^{(1)}$ to be the identity function, so that f and $g^{(1)}$ are mutual inverses when restricted to the first-order neighborhood. So, suppose that $g^{(r-1)}$ has been constructed, and consider its interaction with f on the rth order neighborhood:

$$g_i^{(r-1)}(f(x)) = x_i + \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n} + o(r+1).$$

By adding in the correction term

$$g_i^{(r)} = g_i^{(r-1)} - \sum_{|J|=r} c_J x_1^{J_1} \cdots x_n^{J_n},$$

we have $g_i^{(r)}(f(x)) = x_i + o(r)$.

---

[5] In some sense, this Lemma is a full explanation for why anyone would even think to involve formal geometry in algebraic topology (nevermind how useful the program has been in the long run). Calculations in algebraic topology have long been expressed in terms of power series rings, and with this Lemma we are provided geometric interpretations for such statements.

Part of the point of the geometric language is to divorce abstract rings from concrete presentations, so we additionally reserve some vocabulary for the property of being isomorphic to $\widehat{\mathbb{A}}^n$:

Definition 19. A *formal affine variety* (of dimension n) is a formal scheme V which is (noncanonically) isomorphic to $\widehat{\mathbb{A}}^n$. The two maps in an isomorphism pair

$$V \xrightarrow{\cong} \widehat{\mathbb{A}}^n, \quad V \xleftarrow{\cong} \widehat{\mathbb{A}}^n$$

are called a *coordinate (system)* and a *parameter (system)* respectively. Finally, an S–point $x\colon S \to X$ is called *formally smooth* when $X_x^{\wedge}$ gives a formal variety.

With all this algebraic geometry in hand, we now return to our original motivation: extracting formal schemes from the rings appearing in algebraic topology.

Definition 20 (cf. Definition 9). Let E be an even-periodic ring spectrum, and let X be a CW–space. Because X is compactly generated, it can be written as the colimit of its compact subspaces $X^{(\alpha)}$, and we set[6]

$$X_E := \operatorname{Spf} E^0 X := \{\operatorname{Spec} E^0 X^{(\alpha)}\}_\alpha.$$

Consider the example of $\mathbb{C}P_E^\infty$ for E a complex-oriented cohomology theory. We saw in Corollary 9 that the complex-orientation determines an isomorphism $\mathbb{C}P_E^\infty \cong \widehat{\mathbb{A}}^1$ (i.e., an isomorphism $E^0\mathbb{C}P^\infty \cong E^0[\![x]\!]$). However, the object "$E^0\mathbb{C}P^\infty$" is something that exists independent of the orientation map $MU \to E$, and the language of Definition 19 allows us to make the distinction between the property and the data:

Lemma 22. *A cohomology theory E is complex orientable (i.e., it is able to receive a ring map from MU) precisely when $\mathbb{C}P_E^\infty$ is a formal curve (i.e., it is a formal variety of dimension 1). A choice of orientation $MU \to E$ determines a coordinate $\mathbb{C}P_E^\infty \cong \widehat{\mathbb{A}}^1$.* □

As in Example 12, the formal scheme $\mathbb{C}P_E^\infty$ has additional structure: it is a group. We close today with some remarks about such objects.

---

[6] The careful reader will immediately notice that the rings in the pro-system underlying Definition 20 run the risk of not being even-concentrated. We are thus required to make the following technical compromise: for any pro-isomorphic system of even E^*–algebras $\{R_\beta \otimes_{E^0} E^*\} \cong \{E^0 X^{(\alpha)}\}_\alpha$ we set

$$X_E := \{\operatorname{Spec} R_\beta\}_\beta,$$

and otherwise we leave X_E undefined. For example, the technical condition of Definition 20 is satisfied if there exists a cofinal subsystem of $\{X^{(\alpha)}\}_\alpha$ with $E^* X^{(\alpha)}$ even–concentrated. This follows, for instance, from $H\mathbb{Z}_* X$ being free and even [Str99b, Definition 8.15, Proposition 8.17].

Definition 21. A formal group is a formal variety endowed with an abelian group structure.[7] If E is a complex-orientable cohomology theory, then $\mathbb{CP}^\infty_E$ naturally forms a (1–dimensional) formal group using the map classifying the tensor product of line bundles.

Remark 13. As with formal schemes, formal groups can arise as formal completions of an algebraic group at its identity point. It turns out that there are many more formal groups than come from this procedure, a phenomenon that is of keen interest to stable homotopy theorists — see Appendix A.4.

We give the following Corollary as an example of how nice the structure theory of formal varieties is.

Corollary 10. *As with physical groups, the formal group addition map on $\widehat{\mathbb{G}}$ determines the inverse law.*

Proof. Consider the shearing map

$$\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \xrightarrow{\sigma} \widehat{\mathbb{G}} \times \widehat{\mathbb{G}},$$
$$(x, y) \mapsto (x, x + y).$$

The induced map $T_0\sigma$ on tangent spaces is evidently invertible, so by Theorem 10 there is an inverse map $(x, y) \mapsto (x, y - x)$. Setting $y = 0$ and projecting to the second factor gives the inversion map. ∎

Definition 22. Let $\widehat{\mathbb{G}}$ be a formal group. In the presence of a coordinate $\varphi \colon \widehat{\mathbb{G}} \cong \widehat{\mathbb{A}}^n$, the addition law on $\widehat{\mathbb{G}}$ begets a map

$$
\begin{array}{ccc}
\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} & \longrightarrow & \widehat{\mathbb{G}} \\
\| & & \| \\
\widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n & \longrightarrow & \widehat{\mathbb{A}}^n,
\end{array}
$$

> Again, the vertical arrows should be arrows, not equal signs?

and hence a n-tuple of $(2n)$-variate power series "$+_\varphi$", satisfying

$$
\begin{aligned}
\underline{x} +_\varphi \underline{y} &= \underline{y} +_\varphi \underline{x}, && \text{(commutativity)} \\
\underline{x} +_\varphi \underline{0} &= \underline{x}, && \text{(unitality)} \\
\underline{x} +_\varphi (\underline{y} +_\varphi \underline{z}) &= (\underline{x} +_\varphi \underline{y}) +_\varphi \underline{z}. && \text{(associativity)}
\end{aligned}
$$

Such a tuple $+_\varphi$ is called a *formal group law*, and it is the concrete data associated for a formal group.

---

[7] Formal groups in dimension 1 are automatically commutative if and only if the ground ring has no elements which are simultaneously nilpotent and torsion [Haz12, Theorem I.6.1].

Let's now consider two examples of E which are complex-orientable and describe these invariants for them.

Example 21. There is an isomorphism $\mathbb{CP}^\infty_{H\mathbb{Z}P} \cong \widehat{\mathbb{G}}_a$. This follows from reasoning identical to that given in Example 12.

Example 22. There is also an isomorphism $\mathbb{CP}^\infty_{KU} \cong \widehat{\mathbb{G}}_m$. A reasonablechoice of first Chern class is given by the natural topological map

$$c_1 \colon \Sigma^{-2}\Sigma^\infty\mathbb{CP}^\infty \xrightarrow{1-\beta\mathcal{L}} KU,$$

and a formula for the first Chern class of the tensor product is thus

$$
\begin{aligned}
c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) &= 1 - \beta(\mathcal{L}_1 \otimes \mathcal{L}_2) \\
&= -\beta^{-1}\left((1-\beta\mathcal{L}_1)\cdot(1-\beta\mathcal{L}_2)\right) + (1-\beta\mathcal{L}_1) + (1-\beta\mathcal{L}_2) \\
&= c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - \beta^{-1}c_1(\mathcal{L}_1)c_1(\mathcal{L}_2).
\end{aligned}
$$

In this coordinate on $\mathbb{CP}^\infty_{KU}$, the group law is then $F(x_1, x_2) = x_1 + x_2 - \beta^{-1}x_1x_2$. Using the coordinate function $1 - t$, this is also the coordinate that arises on the formal completion of $\mathbb{G}_m$ at $t = 1$:

$$
\begin{aligned}
x_1(t_1) +_{\mathbb{G}_m} x_2(t_2) &= 1 - (1-t_1)(1-t_2) \\
&= t_1 + t_2 - t_1t_2.
\end{aligned}
$$

As an application of all these tools, we will show that the rational theory of formal groups is highly degenerate: every rational formal group is isomorphic to $\widehat{\mathbb{G}}_a$. Suppose now that R is a $\mathbb{Q}$–algebra and that $A = R[\![x]\!]$ is the coordinatized ring of functions on a formal line over R. What's special about this rational curve case is that differentiation gives an isomorphism between the Kähler differentials $\Omega^1_{A/R}$ and the ideal (x) of functions vanishing at the origin (i.e., the ideal sheaf selecting the closed subscheme $0 \colon \operatorname{Spec} R \to \operatorname{Spf} A$). Its inverse is formal integration:

$$\int \colon \left(\sum_{j=0}^\infty c_j x^j\right) dx \mapsto \sum_{j=0}^\infty \frac{c_j}{j+1}x^{j+1}.$$

Theorem 11. *For R a $\mathbb{Q}$–algebra, there is a canonical isomorphism of formal groups*

$$\log \colon \widehat{\mathbb{G}} \to T_0\widehat{\mathbb{G}} \otimes \widehat{\mathbb{G}}_a.$$

Proof. Taking a cue from classical Lie theory, we attempt to use integration to define exponential and logarithm functions for a given formal group law F. This is typically accomplished by studying invariant differentials: a 1–form

$\omega \in \Omega^1_{A/R}$ is said to be *invariant (under F)* when $\omega = T^*_y \omega$ for all translations $T_y(x) = x +_F y$. In terms of a coordinate $\omega = f(x)dx$, this condition becomes

$$f(x)dx = f(y +_F x)d(y +_F x) = f(y +_F x)\frac{\partial(y +_F x)}{\partial x}dx.$$

Restricting to the origin by setting $x = 0$, we deduce the condition

$$f(0) = f(y) \cdot \left.\frac{\partial(y +_F x)}{\partial x}\right|_{x=0}.$$

Since R is a Q–algebra, integrating against y yields

$$\log_F(y) = \int f(y)\,dy = f(0)\int \left(\left.\frac{\partial(y +_F x)}{\partial x}\right|_{x=0}\right)^{-1} dy.$$

To see that the series $\log_F$ has the claimed homomorphism property, note that

$$\frac{\partial \log_F(y +_F x)}{\partial x}dx = f(y +_F x)d(y +_F x) = f(x)dx = \frac{\partial \log_F(x)}{\partial x}dx,$$

so $\log_F(y +_F x)$ and $\log_F(x)$ differ by a constant. Checking at $y = 0$ shows that the constant is $\log_F(x)$, hence

$$\log_F(x +_F y) = \log_F(x) + \log_F(y).$$

The choice of boundary value $f(0)$ corresponds to the choice of vector in $T_0\widehat{G}$.

Example 23. Consider the formal group law $x_1(t_1) +_{\widehat{G}_m} x_2(t_2) = t_1 + t_2 - t_1 t_2$ studied in Example 22. Its associated rational logarithm is computed as

$$\log_{\widehat{G}_m}(t_2) = f(0) \cdot \int \frac{1}{1 - t_2}dt_2 = -f(0)\log(1 - t_2) = -f(0)\log(x_2),$$

where "$\log(x_2)$" refers to the classical natural logarithm of x_2.

2.2 Divisors on formal curves

We continue to develop vocabulary and accompanying machinery used to give algebro-geometric reinterpretations of the results in the introduction to this Case Study. In the previous section we deployed the language of formal schemes to recast Corollary 9 in geometric terms, and we now turn towards reencoding Theorem 9. In Definition 10 and Lemma 14 we discussed a gen-

eral correspondence between R–modules and quasicoherent sheaves over $\operatorname{Spec} R$, and the isomorphism of 1–dimensional $E_* X$–modules appearing in Theorem 9 moves us to study sheaves over X_E which are 1–dimensional — i.e., line bundles. In fact, for the purposes of Theorem 9, we will find that it suffices to understand the basics of the geometric theory of line bundles *just over formal curves*. This is our goal in this Lecture, and we leave the applications to algebraic topology aside for later. For the rest of this section we fix the following three pieces of data: a base formal scheme S, a formal curve C over S, and a distinguished point $\zeta \colon S \to C$ on C.

To begin, we will be interested in a very particular sort of line bundle over C: for any function f on C which is not a zero-divisor, the subsheaf $\mathcal{I}_f = f \cdot \mathcal{O}_C$ of functions on C which are divisible by f form a 1–dimensional $\mathcal{O}_C$–submodule of the ring of functions $\mathcal{O}_C$ itself — i.e., a line bundle on C. By interpreting $\mathcal{I}_f$ as an ideal sheaf, this gives rise to a second interpretation of this data in terms of a closed subscheme

$$\operatorname{Spec} \mathcal{O}_C / f \subseteq C,$$

which we will refer to as the *divisor* associated to $\mathcal{I}_f$. In general these can be somewhat pathological, so we specialize further to an extremely nice situation:

Definition 23 ([Str99b, Section 5.1]). An *effective Weil divisor D* on a formal curve C is a closed subscheme of C whose structure map $D \to S$ presents D as finite and free. We say that the *rank* of D is n when its ring of functions $\mathcal{O}_D$ is free of rank n over $\mathcal{O}_S$.

Lemma 23 ([Str99b, Proposition 5.2], see also [Str99b, Example 2.10]). *There is a scheme $\operatorname{Div}_n^+ C$ of effective Weil divisors of rank n. It is a formal variety of dimension n. In fact, a coordinate x on C determines an isomorphism $\operatorname{Div}_n^+ C \cong \widehat{\mathbb{A}}^n$ where a divisor D is associated to a monic polynomial $f_D(x)$ with nilpotent lower-order coefficients.*

Proof (Proof sketch). To pin down the functor we wish to analyze, we make the definition

$$\operatorname{Div}_n^+(C)(R) = \left\{ (a, D) \,\middle|\, \begin{array}{c} a : \operatorname{Spec} R \to S, \\ D \text{ is an effective divisor on } C \times_S \operatorname{Spec} R \end{array} \right\}.$$

To show that this is a formal variety, we pursue the final claim and select a coordinate x on C, as well as a point $(a, D) \in \operatorname{Div}_n^+(C)(T)$. The coordinate presents $C \times_S \operatorname{Spec} T$ as

$$C \times_X \operatorname{Spec} T = \operatorname{Spf} T[\![x]\!],$$

and the characteristic polynomial $f_D(x)$ of x in $\mathcal{O}_D$ presents D as the closed subscheme

$$D = \mathrm{Spf}\, R[\![x]\!]/(f_D(x))$$

for $f_D(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ monic. Additionally, for any prime ideal $\mathfrak{p} \in R$ we can form the field $R_{\mathfrak{p}}/\mathfrak{p}$, over which the module $\mathcal{O}_D \otimes_R R_{\mathfrak{p}}/\mathfrak{p}$ must still be of rank n. It follows that

$$f_D(x) \otimes_R R_{\mathfrak{p}}/\mathfrak{p} \equiv x^n,$$

hence that each a_j lies in the intersection of all prime ideals of R, hence that each a_j is nilpotent.

In turn, this means that the polynomial f_D is selected by a map $\mathrm{Spec}\, R \to \widehat{\mathbb{A}}^n$. Conversely, given such a map, we can form the polynomial $f_D(x)$ and the divisor D.

Remark 14. This Lemma effectively connects several simple dots: especially nice polynomials $f_D(x) \in \mathcal{O}_C$, their vanishing loci $D \subseteq C$, and the ideal sheaves $\mathcal{I}_D$ of functions divisible by f — i.e., functions with a partially prescribed vanishing set. Basic operations on polynomials affect their vanishing loci in predictable ways, and these operations are also reflected on the level of divisor schemes. For instance, there is a unioning map

$$\mathrm{Div}_n^+ C \times \mathrm{Div}_m^+ C \to \mathrm{Div}_{n+m}^+ C,$$
$$(D_1, D_2) \mapsto D_1 \sqcup D_2.$$

At the level of ideal sheaves, we use their 1–dimensionality to produce the formula

$$\mathcal{I}_{D_1 \sqcup D_2} = \mathcal{I}_{D_1} \otimes_{\mathcal{O}_C} \mathcal{I}_{D_2}.$$

Under a choice of coordinate x, the map at the level of polynomials is given by

$$(f_{D_1}, f_{D_2}) \mapsto f_{D_1} \cdot f_{D_2}.$$

Next, note that there is a canonical isomorphism $C \to \mathrm{Div}_1^+ C$. Iterating the above addition map gives the vertical map in the following triangle:

$$
\begin{array}{ccc}
 & & C^{\times n} \\
 & \swarrow & \downarrow{\scriptstyle \sqcup} \\
C_{\Sigma_n}^{\times n} & \xdashrightarrow{\ \cong\ } & \mathrm{Div}_n^+ C.
\end{array}
$$

Lemma 24. *The object $C_{\Sigma_n}^{\times n}$ exists as a formal variety, it factors the iterated addition map, and the dotted arrow is an isomorphism.*

Proof. The first assertion is a consequence of Newton's theorem on symmetric polynomials: the subring of symmetric polynomials in $R[x_1, \ldots, x_n]$ is

itself polynomial on generators

$$\sigma_j(x_1, \ldots, x_n) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=j}} x_{S_1} \cdots x_{S_j},$$

and hence

$$R[\sigma_1, \ldots, \sigma_n] \subseteq R[x_1, \ldots, x_n]$$

gives an affine model of $C^{\times n} \to C^{\times n}_{\Sigma_n}$. Picking a coordinate on C allows us to import this fact into formal geometry to deduce the existence of $C^{\times n}_{\Sigma_n}$. The factorization then follows by noting that the iterated $\sqcup$ map is symmetric. Finally, Remark 14 shows that the horizontal map pulls the coordinate a_j back to σ_j, so the third assertion follows.

Remark 15. The map $C^{\times n} \to C^{\times n}_{\Sigma_n}$ is an example of a map of schemes which surjective *as a map of sheaves*. This is somewhat subtle: for any given test ring T, it is not necessarily the case that $C^{\times n}(T) \to C^{\times n}_{\Sigma_n}(T)$ is surjective on T–points — this amounts to the claim that not every polynomial can be written as a product of linear factors. However, for a fixed point $f \in C^{\times n}_{\Sigma_n}(T)$, we are guaranteed a flat covering $T \to \prod_j T_j$ such that there are individual lifts $\widetilde{f}_j$ of f over each T_j.

> Add an example here. Remember that the coefficients are supposed to be nilpotents.

Now we use the pointing $\zeta \colon S \to C$ to interrelate divisor schemes of varying ranks. Together with the $\sqcup$ operation, ζ gives a composite

$$\mathrm{Div}_n^+ C \longrightarrow C \times \mathrm{Div}_n^+ C \longrightarrow \mathrm{Div}_1^+ C \times \mathrm{Div}_n^+ C \longrightarrow \mathrm{Div}_{n+1}^+ C,$$

$$D \longmapsto (\zeta, D) \longmapsto ([\zeta], D) \longmapsto [\zeta] \sqcup D.$$

Definition 24. We define the following variants of "stable divisor schemes":

$$\mathrm{Div}^+ C = \coprod_{n \geq 0} \mathrm{Div}_n^+ C,$$

$$\mathrm{Div}_n C = \mathrm{colim}\left(\mathrm{Div}_n^+ C \xrightarrow{[\zeta]+-} \mathrm{Div}_{n+1}^+ C \xrightarrow{[\zeta]+-} \cdots \right),$$

$$\mathrm{Div}\, C = \mathrm{colim}\left(\mathrm{Div}^+ C \xrightarrow{[\zeta]+-} \mathrm{Div}^+ C \xrightarrow{[\zeta]+-} \cdots \right)$$

$$\cong \coprod_{n \in \mathbb{Z}} \mathrm{Div}_n C.$$

Theorem 12 (cf. Corollary 45). *The scheme* $\mathrm{Div}^+ C$ *models the free formal monoid on the unpointed formal curve C. The scheme* $\mathrm{Div}\, C$ *models the free formal group on the unpointed formal curve C. The scheme* $\mathrm{Div}_0 C$ *simultaneously models the free formal monoid and the free formal group on the* pointed *formal curve C.* □

We will postpone the proof of this Theorem until later, once we've developed a theory of coalgebraic formal schemes.

Remark 16. Given $q\colon C \to C'$ a map of formal curves over S and $D \subseteq C$ a divisor on C, the composite $D \to C \to C'$ is also a divisor, denoted q_*D. Theorem 12 gives a second construction of q_*D in the stable case, using the composite

$$C \xrightarrow{q} C' \cong \mathrm{Div}_1^+ C' \to \mathrm{Div}\, C'.$$

Since the target of this map is a formal group scheme, universality induces a map

$$q_*\colon \ \mathrm{Div}\, C \to \mathrm{Div}\, C'.$$

On the other hand, for a general q the pullback $D \times_{C'} C$ of a divisor $D \subseteq C'$ will not be a divisor on C. It is possible to impose conditions on q so that this is so, and in this case q is called an *isogeny*. We will return to this in the future.

Our final goal for the section is to broaden this discussion to line bundles on formal curves generally, using this nice case as a model. To begin, we need some vocabulary that connects the general case to the one studied above.

Definition 25 (cf. [Vak15, Section 14.2]). Suppose that $\mathcal{L}$ is a line bundle on C and select a section u of $\mathcal{L}$. There is a largest closed subscheme $D \subseteq C$ where the condition $u|_D = 0$ is satisfied. If D is a divisor, u is said to be *divisorial* and $D = \mathrm{div}\, u$.

Lemma 25 (cf. [Vak15, Exercise 14.2.E]). *A divisorial section u of a line bundle $\mathcal{L}$ induces an isomorphism $\mathcal{L} \cong \mathcal{I}_D$.*   □

Line bundles which admit divisorial sections are thus those that arise through our construction above. However, in the classical situation, such line bundles account for roughly "half" of the available line bundles: line bundles are also used to house meromorphic functions with prescribed zeroes *and poles*, and we have not encountered such sections yet.

Definition 26 ([Str99b, Definition 5.20 and Proposition 5.26]). The ring of meromorphic functions on C, $\mathcal{M}_C$, is obtained by inverting all coordinates in $\mathcal{O}_C$.[8] Additionally, this can be augmented to a scheme $\mathrm{Mer}(C, \mathbb{G}_m)$ of meromorphic functions on C by

$$\mathrm{Mer}(C, \mathbb{G}_m)(R) := \left\{ (u, f) \middle| \begin{array}{l} u : \mathrm{Spec}\, R \to S, \\ f \in \mathcal{M}_{C \times_S \mathrm{Spec}\, R}^{\times} \end{array} \right\}.$$

---

[8] In fact, it suffices to invert any single one [Str99b, Lemma 5.21].

Thinking of a meromorphic function as the formal expansion of a rational function, we are moved to study the monoidality of divisoriality.

Lemma 26. *If u_1 and u_2 are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then $u_1 \otimes u_2$ is a divisorial section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ and $\operatorname{div}(u_1 \otimes u_2) = \operatorname{div} u_1 + \operatorname{div} u_2$.*

$\square$

Definition 27. A *meromorphic divisorial section* of a line bundle $\mathcal{L}$ is a decompositon $\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ together with an expression of the form u_+/u_-, where u_+ and u_- are divisorial sections of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. We set $\operatorname{div}(u_+/u_-) = \operatorname{div} u_+ - \operatorname{div} u_-$.

In the case of a formal curve, the fundamental theorem is that meromorphic functions (or "Cartier divisors"), line bundles, and stable Weil divisors all essentially agree. A particular meromorphic function spans a 1–dimensional $\mathcal{O}_C$–submodule sheaf of $\mathcal{M}_C$, and hence it determines a line bundle. Conversely, a line bundle is determined by local gluing data, which is exactly the data of a meromorphic function. However, it is clear that there is some overdeterminacy in this first operation: scaling a meromorphic function by a nowhere vanishing entire function will not modify the submodule sheaf. Additionally, the function div gives an assignment from meromorphic functions to stable Weil divisors which is also insensitive to rescaling by a nowhere vanishing function.

Theorem 13 ([Str99b, Proposition 5.26]). *In the case of a formal curve C, there is a short exact sequence of formal groups*

$$0 \to \underline{\mathsf{FormalSchemes}}(C, \mathbb{G}_m) \to \operatorname{Mer}(C, \mathbb{G}_m) \to \operatorname{Div}(C) \to 0. \square$$

2.3 Line bundles associated to Thom spectra

Today we will exploit all of the algebraic geometry we set up yesterday to deduce a load of topological results.

Definition 28. Let E be a complex-orientable theory and let $V \to X$ be a complex vector bundle over a space X. According to Theorem 9, the cohomology of the Thom spectrum $E^*T(V)$ forms a 1–dimensional E^*X–module. Using Lemma 14, we construct a line bundle over X_E

$$\mathbb{L}(V) := \widetilde{E^*T(V)},$$

called the *Thom sheaf* of V.

Remark 17. One of the main utilities of this definition is that it only uses the *property* that E is complex-orientable, and it begets only the *property* that $\mathbb{L}(V)$ is a line bundle.

This construction enjoys many properties already established.

Corollary 11. *A vector bundle V over Y and a map $f\colon X \to Y$ induce an isomorphism*

$$\mathbb{L}(f^*V) \cong (f_E)^*\mathbb{L}(V).$$

There is also is a canonical isomorphism

$$\mathbb{L}(V \oplus W) = \mathbb{L}(V) \otimes \mathbb{L}(W).$$

Finally, this property can then be used to extend the definition of $\mathbb{L}(V)$ to virtual bundles:

$$\mathbb{L}(V - W) = \mathbb{L}(V) \otimes \mathbb{L}(W)^{-1}.$$

Proof. The first claim is justified by Lemma 3, the second by Lemma 5, and the last is a direct consequence of the first two.

We use these properties to work the following Example, which connects Thom sheaves with the major players from Lecture 2.1.

Various people have been uncomfortable about whether the grading matters here, whether E is period-ified, …. **Actually, something is almost definitely wrong:** the functor T is defined to give the reduced Thom spectrum. Shit.

Example 24 ([AHS04, Section 8]). Take $\mathcal{L}$ to be the canonical line bundle over $\mathbb{CP}^\infty$. Using the same mode of argument as in Example 2, the zero-section

$$\Sigma^\infty \mathbb{CP}^\infty \xrightarrow{\simeq} T(\mathcal{L})$$

gives an identification

$$E^0\mathbb{CP}^\infty \supseteq \widetilde{E}^0\mathbb{CP}^\infty \xleftarrow{\simeq} E^0 T(\mathcal{L})$$

of $E^0 T(\mathcal{L})$ with the augmentation ideal in $E^0\mathbb{CP}^\infty$. At the level of Thom sheaves, this gives an isomorphism

$$\mathcal{I}(0) \xleftarrow{\simeq} \mathbb{L}(\mathcal{L})$$

of $\mathbb{L}(\mathcal{L})$ with the sheaf of functions vanishing at the origin of $\mathbb{CP}^\infty_E$. Pulling $\mathcal{L}$ back along

$$0\colon * \to \mathbb{CP}^\infty$$

gives a line bundle over the one-point space, which on Thom spectra gives the inclusion

$$\Sigma^\infty \mathbb{CP}^1 \to \Sigma^\infty \mathbb{CP}^\infty.$$

Stringing many results together, we can now calculate: [9]

---

[9] The identification with $\mathcal{I}(0)/\mathcal{I}(0) \cdot \mathcal{I}(0)$ below deserves further explanation. The functor 0^* is right-exact, so sends the short exact sequence

$$\widetilde{\pi_2 E} \cong \widetilde{E^0 \mathbb{CP}^1} \qquad\qquad\qquad (S^2 \simeq \mathbb{CP}^1)$$

$$\cong \mathbb{L}(0^* \mathcal{L}) \qquad\qquad\qquad\qquad (\text{Definition 28})$$

$$\cong 0^* \mathbb{L}(\mathcal{L}) \qquad\qquad\qquad\qquad (\text{Corollary 11})$$

$$\cong 0^* \mathcal{I}(0) \qquad\qquad\qquad (\text{preceding calculation})$$

$$\cong \mathcal{I}(0)/(\mathcal{I}(0) \cdot \mathcal{I}(0)) \quad (\text{definition of } 0^* \text{ from Definition 11})$$

$$\cong T_0^* \mathbb{CP}_E^\infty \qquad\qquad\qquad\qquad (\text{Definition 15})$$

$$\cong \omega_{\mathbb{CP}_E^\infty}, \qquad\qquad\qquad\quad (\text{proof of Theorem 11})$$

where $\omega_{\mathbb{CP}_E^\infty}$ denotes the sheaf of invariant differentials on $\mathbb{CP}_E^\infty$. Consequently, if $k \cdot \varepsilon$ is the trivial bundle of dimension k over a point, then

$$\widetilde{\pi_{2k} E} \cong \mathbb{L}(k \cdot \varepsilon) \cong \mathbb{L}(k \cdot 0^* \mathcal{L}) \cong \mathbb{L}(0^* \mathcal{L})^{\otimes k} \cong \omega_{\mathbb{CP}_E^\infty}^{\otimes k}.$$

Finally, given an E–algebra $f \colon E \to F$ (e.g., $F = E^{X_+}$), then we have

$$\widetilde{\pi_{2k} F} \cong f_E^* \omega_{\mathbb{CP}_E^\infty}^{\otimes k}.$$

Outside of this Example, it is difficult to find line bundles $\mathbb{L}(V)$ which we can analyze so directly. In order to get a handle on on $\mathbb{L}(V)$ in general, we now seek to strengthen this bond between line bundles and vector bundles by finding inside of algebraic topology the alternative presentations of line bundles given in Lecture 2.2. In particular, we would like a topological construction on vector bundles which produces divisors — i.e., finite schemes over X_E. This has the scent of a certain familiar topological construction called projectivization, and we now work to justify the relationship.

Definition 29. Let V be a complex vector bundle of rank n over a base X. Define $\mathbb{P}(V)$, the *projectivization of V*, to be the $\mathbb{CP}^{n-1}$–bundle over X whose fiber of $x \in X$ is the space of complex lines in the original fiber $V|_x$.

Theorem 14. *Take E to be* complex-oriented. *The E–cohomology of $\mathbb{P}(V)$ is given by the formula*
$$E^* \mathbb{P}(V) \cong E^*(X)[\![t]\!]/c(V)$$
for a certain monic polynomial

$$c(V) = t^n - c_1(V) t^{n-1} + c_2(V) t^{n-2} - \cdots + (-1)^n c_n(V).$$

Proof. We fit all of the fibrations we have into a single diagram:

$$0 \to \mathcal{I}(0)^2 \to \mathcal{I}(0) \to \mathcal{I}(0)/\mathcal{I}(0)^2 \to 0$$

to a right-exact sequence, and we need only check that the map $0^* \mathcal{I}(0)^2 \to 0^* \mathcal{I}(0)$ is zero. This is the statement that a function vanishing to second order also has vanishing first derivative.

$$
\begin{array}{ccccccc}
& & \mathbb{C}^{\times} & & & & \\
& & \parallel \quad \searrow & & & & \\
\mathbb{C}^n & \longleftarrow & \mathbb{C}^n \setminus \{0\} & \longrightarrow & \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}^{\infty} \\
\downarrow & & \parallel & & \downarrow & & \parallel \\
& & \mathbb{C}^{\times} & & & & \\
& & \quad \searrow & & & & \\
V & \longleftarrow & V \setminus \zeta & \longrightarrow & \mathbb{P}(V) & \longrightarrow & \mathbb{C}\mathrm{P}^{\infty} \\
\zeta \uparrow \downarrow & & \downarrow & & \downarrow \pi & & \downarrow \\
X & == & X & == & X & \longrightarrow & *.
\end{array}
$$

We read this diagram as follows: on the far left, there's the vector bundle we began with, as well as its zero-section ζ. Deleting the zero-section gives the second bundle, a $\mathbb{C}^n \setminus \{0\}$–bundle over X. Its quotient by the scaling $\mathbb{C}^{\times}$–action gives the third bundle, a $\mathbb{C}\mathrm{P}^{n-1}$–bundle over X. Additionally, the quotient map $\mathbb{C}^n \setminus \{0\} \to \mathbb{C}\mathrm{P}^{n-1}$ is itself a $\mathbb{C}^{\times}$–bundle, and this induces the structure of a $\mathbb{C}^{\times}$–bundle on the quotient map $V \setminus \zeta \to \mathbb{P}(V)$. Thinking of these as complex line bundles, they are classified by a map to $\mathbb{C}\mathrm{P}^{\infty}$, which can itself be thought of as the last vertical fibration, fibering over a point.

Note that the map between these two last fibers is surjective on E–cohomology. It follows that the Serre spectral sequence for the third vertical fibration is degenerate, since all the classes in the fiber must survive.[10] We thus conclude that $E^*\mathbb{P}(V)$ is a free $E^*(X)$–module on the classes $\{1, t, t^2, \ldots, t^{n-1}\}$ spanning $E^*\mathbb{C}\mathrm{P}^{n-1}$, where t encodes the chosen complex-orientation of E. To understand the ring structure, we need only compute $t^{n-1} \cdot t$, which must be able to be written in terms of the classes which are lower in t–degree:

$$
t^n = c_1(V)t^{n-1} - c_2(V)t^{n-2} + \cdots + (-1)^{n-1}c_n(V)
$$

for some classes $c_j(V) \in E^*X$. The main claim follows.

In coordinate-free language, we have the following Corollary:

Corollary 12 (Theorem 14 redux). *Take E to be* complex-orientable. *The map*

$$
\mathbb{P}(V)_E \to X_E \times \mathbb{C}\mathrm{P}^{\infty}_E
$$

is a closed inclusion of X_E–schemes, and the structure map $\mathbb{P}(V)_E \to X_E$ is free and finite of rank n. It follows that $\mathbb{P}(V)_E$ is a divisor on $\mathbb{C}\mathrm{P}^{\infty}_E$ considered over X_E, i.e.,

$$
\mathbb{P}(V)_E \in \left(\mathrm{Div}_n^+(\mathbb{C}\mathrm{P}^{\infty}_E)\right)(X_E). \square
$$

---

[10] This is called the Leray–Hirsch theorem.

Definition 30. The classes $c_j(V)$ of Theorem 14 are called the *Chern classes* of V (with respect to the complex-orientation t of E). They are visibly natural with respect to pullback of bundles, and the Chern polynomial $c(-)$ is multiplicative:

$$c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2).$$

The next major theorems concerning projectivization are the following:

Corollary 13. *The sub-bundle of $\pi^*(V)$ consisting of vectors $(v, (\ell, x))$ such that v lies along the line ℓ splits off a canonical line bundle.* □

Corollary 14 ("Splitting principle" / "Complex–oriented descent"). *Associated to any n–dimensional complex vector bundle V over a base X, there is a canonical map $i_V \colon Y_V \to X$ such that $(i_V)_E \colon (Y_V)_E \to X_E$ is finite and faithfully flat, and there is a canonical splitting into complex line bundles:*

$$i_V^*(V) \cong \bigoplus_{i=1}^{n} \mathcal{L}_i. \square$$

This last Corollary is extremely important. Its essential contents is to say that any question about characteristic classes can be checked for sums of line bundles. Specifically, because of the injectivity of i_V^*, any relationship among the characteristic classes deduced in $E^* Y_V$ must already be true in the ring $E^* X$. The following theorem is a consequence of this principle:

Theorem 15. *Again take E to be complex-oriented. The coset fibration*

$$U(n-1) \to U(n) \to S^{2n-1}$$

deloops to a spherical fibration

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

The associated Serre spectral sequence

$$E_2^{*,*} = H^*(BU(n); E^* S^{2n-1}) \Rightarrow E^* BU(n-1)$$

degenerates at E_{2n} and induces an isomorphism

$$E^* BU(n) \cong E^*[\![\sigma_1, \dots, \sigma_n]\!].$$

Now, let $V \colon X \to BU(n)$ classify a vector bundle V. Then the coefficient c_j in the polynomial $c(V)$ is selected by σ_j:

$$c_j(V) = V^*(\sigma_j).$$

Proof (Proof sketch). The first part is a standard calculation. To prove the relation between the Chern classes and the σ_j, the splitting principle states that we can factor complete the map $V \colon X \to BU(n)$ to a square

$$
\begin{array}{ccc}
Y_V & \xrightarrow{\ \oplus_{i=1}^n \mathcal{L}_i\ } & BU(1)^{\times n} \\
\downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle \oplus} \\
X & \xrightarrow{\ \ V\ \ } & BU(n).
\end{array}
$$

The equation $c_j(f_V^* V) = V^*(\sigma_j)$ can be checked in $E^* Y_V$.

We now see that not only does $\mathbb{P}(V)_E$ produce a point of $\mathrm{Div}_n^+(\mathbb{CP}_E^\infty)$, but actually the scheme $\mathrm{Div}_n^+(\mathbb{CP}_E^\infty)$ itself appears internally to topology:

Corollary 15. [11] *For a complex orientable cohomology theory E, there is an isomorphism*

$$
BU(n)_E \cong \mathrm{Div}_n^+ \mathbb{CP}_E^\infty,
$$

so that maps $V \colon X \to BU(n)$ are transported to divisors $\mathbb{P}(V)_E \subseteq \mathbb{CP}_E^\infty \times X_E$. Selecting a particular complex orientation of E begets two isomorphisms

$$
BU(n)_E \cong \widehat{\mathbb{A}}^n, \qquad\qquad \mathrm{Div}_n^+ \mathbb{CP}_E^\infty \cong \widehat{\mathbb{A}}^n,
$$

and these are compatible with the centered isomorphism above. $\square$

This description has two remarkable features. One is its "faithfulness": this isomorphism of formal schemes means that the entire theory of characteristic classes is captured by the behavior of the divisor scheme. The other aspect is its coherence with topological operations we find on $BU(n)$. For instance, the Whitney sum map translates as follows:

Lemma 27. *The sum map*

$$
BU(n) \times BU(m) \xrightarrow{\ \oplus\ } BU(n+m)
$$

induces on Chern polynomials the identity

$$
c(V_1 \oplus V_2) = c(V_1) \cdot c(V_2).
$$

In terms of divisors, this means

$$
\mathbb{P}(V_1 \oplus V_2)_E = \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E,
$$

and hence there is an induced square

---

[11] See [Str99b, Proposition 8.31] for a proof that recasts Theorem 15 itself in coordinate-free terms.

$$
\begin{array}{ccc}
BU(n)_E \times BU(m)_E & \xrightarrow{\ \oplus\ } & BU(n+m) \\
\| & & \| \\
\mathrm{Div}_n^+ \mathbb{CP}_E^\infty \times \mathrm{Div}_m^+ \mathbb{CP}_E^\infty & \xrightarrow{\ \sqcup\ } & \mathrm{Div}_{n+m}^+ \mathbb{CP}_E^\infty .\square
\end{array}
$$

The following is a consequence of combining this Lemma with the splitting principle:

Corollary 16. *The map* $Y_E \xrightarrow{f_V} X_E$ *pulls* $\mathbb{P}(V)_E$ *back to give*

$$
Y_E \times_{X_E} \mathbb{P}(V)_E \cong \bigsqcup_{i=1}^{n} \mathbb{P}(\mathcal{L}_i)_E.
$$

Proof (Interpretation). This says that the splitting principle is a topological enhancement of the claim that a divisor can be base-changed along a finite flat map where it splits as a sum of points.

The other constructions from Lecture 2.2 are also easily matched up with topological counterparts:

Corollary 17. *There are natural isomorphisms* $BU_E \cong \mathrm{Div}_0 \mathbb{CP}_E^\infty$ *and* $(BU \times \mathbb{Z})_E \cong \mathrm{Div}\,\mathbb{CP}_E^\infty$. *Additionally,* $(BU \times \mathbb{Z})_E$ *is the free formal group on the curve* $\mathbb{CP}_E^\infty$. $\square$

Space	classifies	Scheme	classifies
$BU(n)$	vector bundles of rank n	$\mathrm{Div}_n^+ \mathbb{CP}_E^\infty$	effective Weil divisors of rank n
$\coprod_n BU(n)$	unstable vector bundles	$\mathrm{Div}^+ \mathbb{CP}_E^\infty$	semiring of effective divisors
$BU \times \mathbb{Z}$	stable virtual bundles	$\mathrm{Div}\,\mathbb{CP}_E^\infty$	ring of stable Weil divisors
$BU \times \{0\}$	stable virtual bundles of rank 0	$\mathrm{Div}_0 \mathbb{CP}_E^\infty$	ideal of stable divisors of rank 0

Fig. 2.1 Different notions of vector bundles and their associated divisors

Corollary 18. *There is a commutative diagram*

$$
\begin{array}{ccc}
BU(n)_E \times BU(m)_E & \xrightarrow{\ \otimes\ } & BU(nm)_E \\
\| & & \| \\
\mathrm{Div}_n^+ \mathbb{CP}_E^\infty \times \mathrm{Div}_m^+ \mathbb{CP}_E^\infty & \xrightarrow{\ \cdot\ } & \mathrm{Div}_{nm}^+ \mathbb{CP}_E^\infty ,
\end{array}
$$

where the bottom map acts by

$$
(D_1, D_2 \subseteq \mathbb{CP}_E^\infty \times X_E) \mapsto (D_1 \times D_2 \subseteq \mathbb{CP}_E^\infty \times \mathbb{CP}_E^\infty \xrightarrow{\mu} \mathbb{CP}_E^\infty),
$$

and μ *is the map induced by the tensor product map* $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$.

Proof. By the splitting principle, it is enough to check this on sums of line bundles. A sum of line bundles corresponds to a totally decomposed divisor, so we consider the case of a pair of such divisors $\bigsqcup_{i=1}^{n}\{a_i\}$ and $\bigsqcup_{j=1}^{m}\{b_j\}$. Referring to Definition 21, the map acts by

$$\left(\bigsqcup_{i=1}^{n}\{a_i\}\right)\left(\bigsqcup_{j=1}^{m}\{b_j\}\right) = \bigsqcup_{i,j}\{\mu_{\mathbb{CP}_E^\infty}(a_i,b_j)\}. \quad\square$$

Finally, we can connect our analysis of the divisors coming from topological vector bundles with the line bundles studied at the start of the section.

Lemma 28. *Let $\zeta : X_E \to X_E \times \mathbb{CP}_E^\infty$ denote the pointing of the formal curve $\mathbb{CP}_E^\infty$, and let $\mathcal{I}(\mathbb{P}(V)_E)$ denote the ideal sheaf on $X_E \times \mathbb{CP}_E^\infty$ associated to the divisor subscheme $\mathbb{P}(V)_E$. There is a natural isomorphism of sheaves over X_E:*

> In FPFP, Neil has a $-D$ rather than a D.

$$\zeta^*\mathcal{I}(\mathbb{P}(V)_E) \cong \mathbb{L}(V).$$

Proof (Proof sketch). In terms of a complex-oriented E and Theorem 14, the effect of pulling back along the zero section is to set $t = 0$, which collapses the Chern polynomial to just the top class $c_n(V)$. This element, called *the Euler class of V*, provides the E^*X–module generator of $E^*T(V)$ — or, equivalently, the trivializing section of $\mathbb{L}(V)$.

> Do you have the right spectrum here: *MU* versus *MUP*?

Theorem 16 (cf. Theorem 65). *A trivialization $t\colon \mathbb{L}(\mathcal{L}-1) \cong \mathcal{O}_{\mathbb{CP}_E^\infty}$ of the Thom sheaf associated to the canonical bundle induces a ring map $MU \to E$.*

Proof. Suppose that V is a rank n vector bundle over X, and let $f\colon Y \to X$ be the space guaranteed by the splitting principle to provide an isomorphism $f^*V \cong \bigoplus_{j=1}^{n}\mathcal{L}_j$. The chosen trivialization t then pulls back to give a trivialization of $\mathcal{I}(\mathbb{P}(f^*V)_E)$, and by finite flatness this descends to also give a trivialization of $\mathcal{I}(\mathbb{P}(V)_E)$. Pulling back along the zero section gives a trivialization of $\mathbb{L}(V)$. Then note that the system of trivializations produced this way is multiplicative, as a consequence of $\mathbb{P}(V_1 \oplus V_2)_E \cong \mathbb{P}(V_1)_E \sqcup \mathbb{P}(V_2)_E$. In the universal examples, this gives a sequence of compatible maps $MU(n) \to E$ which assemble on the colimit $n \to \infty$ to give the desired map of ring spectra.

2.4 Cohomology operations for complex bordism

Our eventual goal, like in Case Study 1, is to give an algebro-geometric description of $MU_*(*)$ and of the cooperations MU_*MU. It is possible to approach this the same way as last time, using the Adams spectral sequence.

However, $MU_*(*)$ is an integral algebra and so we cannot make do with working out the mod–2 Adams spectral sequence alone — we would at least have to work out the mod–p Adams spectral sequence for every p. At odd primes p, there is the following unfortunate theorem:

Theorem 17. *There is an isomorphism*

$$H\mathbb{F}_p P_0 H\mathbb{F}_p P \cong \mathbb{F}_p[\xi_0^{\pm}, \xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots]$$

with $|\xi_j| = 2p^j - 2$ and $|\tau_j| = 2p^j - 1$. □

There are odd–dimensional classes in this algebra, and the *graded-commutativity* of the dual mod–p Steenrod algebra means that these classes anti-commute. This prohibits us from writing "$\text{Spec}(H\mathbb{F}_p P_0 H\mathbb{F}_p)$", and this is the first time we have encountered Hindrance #4 from Lecture 1.3 in the wild. For now, we will simply avoid these methods and find another approach.

There is such an alternative proof, due to Quillen, that bypasses the Adams spectral sequence. This approach has some deficiencies of its own: it requires studying the algebra of stable operations MU^*MU, which we do not expect to be at all commutative, and it requires studying *power operations*, which are in general very technical creatures. However, we will eventually want to talk about power operations anyway (cf. Case Study 6), and because this is the road less traveled we will elect to take it. Our goal in this Lecture is to define these two kinds of cohomology operations, specialized to the particular setting we will need for Quillen's proof.

The description of the first class of operations, the stable operations, follows immediately from our discussion of complex cobordism up to this point, so we will begin there. We learned in Corollary 15 that for any complex-oriented cohomology theory E we have the calculation

$$E^*BU \cong E^*[\![\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots]\!],$$

and we gave a rich interpretation of this in terms of divisor schemes:

$$BU_E \cong \text{Div}_0 \, \mathbb{CP}_E^{\infty}.$$

We would like to leverage the Thom isomorphism to gain a description of E^*MU generally and MU^*MU specifically. However, the former is *not* a ring, and although the latter is a ring its multiplication is exceedingly complicated[12], which means that our extremely compact algebraic description of E^*BU in Corollary 15 will be of limited use. Instead, we will have to content ourselves with an E_*–module basis of E^*MU.

---

[12] For a space X, E^*X has a ring structure because X has a diagonal, and MU does not have a diagonal. In the special case of $E = MU$, there is a ring product coming from endomorphism composition.

Definition 31. Take $MU \to E$ to be a complex-oriented ring spectrum, which presents $E^* BU$ as the subalgebra of symmetric functions inside of an infinite–dimensional polynomial algebra:

$$E^* BU \subseteq E^* BU(1)^{\times \infty} \cong E^*[\![x_1, x_2, \ldots]\!].$$

For any nonnegative multi-index $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finitely many entries nonzero, there is an associated *monomial symmetric function* b_α, which is the sum of those monomials whose exponent lists contain exactly α_j many instances of j.[13] We then set $s_\alpha \in E^* MU$ to be the image of b_α under the Thom isomorphism of E_*–modules

$$E^* MU \cong E^* BU.$$

It is called the αth *Landweber–Novikov operation* with respect to the orientation $MU \to E$.

Definition 32. In the case of the identity orientation $MU \xrightarrow{\mathrm{id}} MU$, the resulting system of Chern classes is called the *Conner–Floyd–Chern classes* and the associated system of cohomology operations is called the *Landweber–Novikov operations* without further qualification.

Remark 18. For a vector bundle V and a complex-oriented cohomology theory E, we define the *total symmetric Chern class* of V by the sum

$$c_{\mathbf{t}}(V) = \sum_\alpha c_\alpha(V) \mathbf{t}^\alpha.$$

In the case of a line bundle $\mathcal{L}$ with first Chern class $c_1(\mathcal{L}) = x$, this degenerates to the sum

$$c_{\mathbf{t}}(\mathcal{L}) = \sum_{j=0}^\infty x^j t_j.$$

For a direct sum $U = V \oplus W$, the total symmetric Chern class satisfies a Cartan formula:

$$c_{\mathbf{t}}(U) = c_{\mathbf{t}}(V \oplus W) = c_{\mathbf{t}}(V) \cdot c_{\mathbf{t}}(W).$$

Again specializing to line bundles $\mathcal{L}$ and $\mathcal{H}$ with first Chern classes $c_1(\mathcal{L}) = x$ and $c_1(\mathcal{H}) = y$, this gives

---

[13] For example, $\alpha = (1, 2, 0, 0, \ldots)$ corresponds to the sum

$$b_\alpha = \sum_i \sum_{j \neq i} \sum_{\substack{k \neq i \\ k > j}} x_i x_j^2 x_k^2 = x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3 + \cdots.$$

$$c_{\mathbf{t}}(U) = c_{\mathbf{t}}(\mathcal{L} \oplus \mathcal{H}) = \left(\sum_{j=0}^{\infty} x^j t_j\right) \cdot \left(\sum_{k=0}^{\infty} y^k t_k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^j y^k t_j t_k$$

$$= 1 + (x+y)t_1 + (xy)t_1^2 + (x^2+y^2)t_2 + (xy^2 + x^2 y)t_1 t_2 + \cdots$$

$$= 1 + c_1(U)t_1 + c_2(U)t_1^2 + (c_1^2(U) - 2c_2(U))t_2 + (c_1(U)c_2(U))t_1 t_2 + \cdots,$$

where we have expanded out some of the pieces of the total symmetric Chern class in polynomials in the usual Chern classes.

Definition 33 ([Ada95, Theorem I.5.1]). Take the orientation to be $MU \xrightarrow{\mathrm{id}} MU$, so that we are considering MU^*MU and the Landweber–Novikov operations arising from the Conner–Floyd–Chern classes. These account for the *stable* operations in MU–cohomology, analogous to the Steenrod operations for $H\mathbb{F}_2$. They satisfy the following properties:

- s_0 is the identity.
- s_α is natural: $s_\alpha(f^*x) = f^*(s_\alpha x)$.
- s_α is stable: $s_\alpha(\sigma x) = \sigma(s_\alpha x)$.
- s_α is additive: $s_\alpha(x+y) = s_\alpha(x) + s_\alpha(y)$.
- s_α satisfies a Cartan formula. Define

$$s_{\mathbf{t}}(x) := \sum_\alpha s_\alpha(x)\mathbf{t}^\alpha := \sum_\alpha s_\alpha(x) \cdot t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n} \cdots \in MU^*(X)[\![t_1, t_2, \ldots]\!]$$

  for an infinite sequence of indeterminates $t_1, t_2, \ldots$. Then

$$s_{\mathbf{t}}(xy) = s_{\mathbf{t}}(x) \cdot s_{\mathbf{t}}(y).$$

- Let $\xi\colon X \to BU(n)$ classify a vector bundle and let φ denote the Thom isomorphism

$$\varphi\colon MU^*X \to MU^*T(\xi).$$

Then the Chern classes of ξ are related to the Landweber–Novikov operations on the Thom spectrum by the formula

$$\varphi(c_\alpha(\xi)) = s_\alpha(\varphi(1)).$$

I feel like there's a punchline missing here. Maybe a connection to the Steenrod algebra is due? Or some kind of computation?

Should you include Quillen's "norm" perspective on these operations (cf. between equations 2.2 and 2.3)? You were just almost-but-not-quite talking about isogenies, and you're going to want to talk about norm constructions eventually...

We now turn to the construction of the other cohomology operations we will be interested in: the power operations. Power operations get their name from their *multiplicative* properties, and correspondingly we do not expect them to be additive operations — the Landweber–Novikov operations account for all of the additive operations, and they satisfy a Cartan formula rather than splitting outright over products. In the abstract, power operations arise from "E_∞" or "H_∞" structures on ring spectra, but in practice

most such structures arise from geometric models of cohomology theories. To produce power operations for complex bordism and to avoid speaking in suffocating generality, we will return to the geometry of complex vector bundles.

Definition 34 ([Rud98, Definition VII.7.4]). Suppose that $\xi\colon X \to BU(k)$ presents a complex vector bundle of rank k on X. The n–fold direct sum of this bundle gives a new bundle

$$X^{\times n} \xrightarrow{\xi^{\oplus n}} BU(k)^{\times n} \xrightarrow{\oplus} BU(n \cdot k)$$

of rank nk on which the cyclic group C_n acts. By taking the (homotopy) C_n–quotient, we produce a vector bundle $\xi(n)$ on $X_{hC_n}^{\times n}$ participating in the diagram

$$
\begin{array}{ccccc}
X^{\times n} & \xrightarrow{\xi^{\oplus n}} & BU(k)^{\times n} & \longrightarrow & BU(nk) \\
\downarrow & & \downarrow & {\scriptstyle \mu} \nearrow & \\
X_{hC_n}^{\times n} & \xrightarrow{\xi(n)} & BU(k)_{hC_n}^{\times n}. &
\end{array}
$$

The universal case gives the map μ.

Lemma 29 ([Rud98, Equation VII.7.3]). *There is an isomorphism of Thom spectra*

$$T(\xi(n)) \simeq (T\xi)_{hC_n}^{\wedge n}.$$

Proof. This proof is mostly a matter of having had the idea to write down the Lemma to begin with. From here, we string basic properties together:

$$
\begin{aligned}
T(\xi(n)) &= T(\xi_{hC_n}^{\oplus n}) & \text{(definition)} \\
&= T(\xi^{\oplus n})_{hC_n} & \text{(colimits commute with colimits)} \\
&= T(\xi)_{hC_n}^{\wedge n}. & \text{(T is monoidal: Lemma 5)} \square
\end{aligned}
$$

Applying the Lemma to the universal case produces a factorization

$$MU(k)^{\wedge n} \to MU(k)_{hC_n}^{\wedge n} \to MU(nk)$$

of the unstable multiplication map, and hence a stable factorization

$$MU^{\wedge n} \to MU_{hC_n}^{\wedge n} \xrightarrow{\mu} MU.$$

Compatible such systems of factorizations are what beget *power operations*, which we will now extract in the case at hand.

Definition 35. Starting with a class

$$f \colon \Sigma^{2r}\Sigma_+^\infty X \to MU,$$

we apply $(-)_{hC_n}^{\wedge n}$ to produce the composite

$$(\Sigma^{2r}\Sigma_+^\infty X)_{hC_n}^{\wedge n} \xrightarrow{f_{hC_n}^{\wedge n}} MU_{hC_n}^{\wedge n} \xrightarrow{\mu} MU$$
$$\Big\| \qquad\qquad \searrow^{P_{\mathrm{ext}}^n(f)}$$
$$\Sigma^{2nr}(\Sigma_+^\infty X)_{hC_n}^{\wedge n}.$$

This defines the *external n^{th} Steenrod power of f*. Employing the diagonal map on X, we can also pull back to get a map

$$P^n(f) \colon \ \Sigma^{2nr}\Sigma_+^\infty X \wedge \Sigma_+^\infty BC_n \simeq \Sigma^{2nr}X_{hC_n} \xrightarrow{\Delta_{hC_n}} \Sigma^{2nr}X_{hC_n}^{\wedge n} \xrightarrow{P_{\mathrm{ext}}^n(f)} MU.$$

This defines the *internal n^{th} Steenrod power of f*.

Remark 19. This assignment

$$\{\Sigma_+^\infty X \to MU\} \xrightarrow{P^n} \{\Sigma_+^\infty X \wedge \Sigma_+^\infty BC_n \to MU\}$$

extends to a map of function spaces

$$\mathsf{Spaces}(X, MU) \xrightarrow{P^n} \mathsf{Spaces}(X \times BC_n, MU).$$

However, this is *not* an infinite loop map, i.e., it does *not* descend from a map of function spectra. The basic obstruction is that power operations are not additive, which is what the infinite loopspace structure governs, but rather satisfy a kind of binomial theorem [BMMS86, pg. 25].

Remark 20. These operations P^n are visibly multiplicative:

$$P^n(f \cdot g) = P^n(f) \cdot P^n(g).$$

Additionally, $P^n(f)$ reduces to the n–fold internal cup product f^n upon restriction to the basepoint in BC_n. In general, classes in $MU^* BC_n$ can be used to give rise to other "restricted power operations" that encode unstable operations other than the literal n^{th} cup power.

Remark 21. Power operations can also be constructed for other groups than C_n acting on the index set $\{1, \dots, n\}$ of smash factors in $X^{\wedge n}$. We do not need these for now, but they show up naturally when trying to understand compositions like $P_{\mathrm{ext}}^n \circ P_{\mathrm{ext}}^m$, which involve $X^{\wedge(nm)}$ and the action of the wreath product $C_n \wr C_m$.

This is pretty much the definition of an H_∞ ring spectrum, but this is not enough to extract power operations on positive-degree classes. There, you want compatible systems of maps like $(\Sigma^j E)_{h\Sigma_n}^{\wedge n} \to \Sigma^{nj} E$. In general, you get such systems only for all j which are divisible by some fixed d, and the resulting mess is called an H_∞^d–structure. For example, KU has an H_∞^2–structure and KO has an H_∞^8–structure. My suspicion with Nat is that power operation maps come from applying the power operation functor to the identity class in $[E, E]$, and that the H_∞^d–structures on these two spectra come from applying the power operation functor to the autoequivalence supplied by the relevant Bott element.

Is this right? You can't interpret the image of an element of π_{2r} on the left as being the power operation applied to another cohomology class, since you'd expect the image to lie in degree π_{2nr}.

Remark 22 ([Qui71, Equations 3.10-11]). The construction Quillen gives in the course of his proof is considerably different from the one recounted here. He begins by giving a different presentation of the complex cobordism groups of a manifold M: a complex orientation of a smooth map $Z \to M$ is a factorization

$$Z \xrightarrow{i} E \xrightarrow{\pi} M$$

through a complex vector bundle $\pi \colon E \to M$ by an embedding i, as well as a complex structure on the normal bundle ν_i. Up to suitable notions of stability (in the dimension of E) and homotopy equivalence (involving, in particular, isotopies of different embeddings i), these quotient to give cobordism classes of maps complex-oriented maps $Z \to M$. The collection of cobordism classes over M of codimension q over is isomorphic to $MU^q(M)$ [Qui71, Proposition 1.2]. Quillen's definition of the power operations is then given in terms of this geometric model: a representative $f \colon Z \to M$ of a cobordism class gives rise to another complex-oriented map $f^{\times n} \colon Z^{\times n} \to M^{\times n}$, and he defines $P^n_{\text{ext}}(f)$ to be the postcomposition with $M^{\times n} \to M^{\times n}_{hC_n}$.

Remark 23. The chain model for ordinary homology is actually rigid enough to define power operations there, too. Curiously, they are all generated by the quadratic power operations (i.e., the "squares"), and all the quadratic power operations turn out to be *additive* — that is, you just get the Steenrod squares again! Specifically, the C_2–construction

$$(\Sigma^n \Sigma^\infty_+ X \xrightarrow{f} H\mathbb{F}_2) \xrightarrow{P^2} (\Sigma^{2n} \Sigma^\infty_+ X \wedge \Sigma^\infty_+ BC_2 \xrightarrow{P^2(f)} H\mathbb{F}_2)$$

gives a class in $H\mathbb{F}_2^{2n-*}(X) \otimes H\mathbb{F}_2^*(\mathbb{RP}^\infty)$, which decomposes as

$$P^2(f) = \sum_{j=0}^{2n} \mathrm{Sq}^{2n-j}(X) \otimes x^j.$$

This appears to be a lucky degeneracy, but in the next Lecture we will exploit something similar with a particular power operation in complex cobordism.

2.5 Stabilizing the MU–Steenrod operations

We have left untouched a major component from the previous Lecture: any kind of example calculation of the Steenrod power operations. Power operations turn out to be prohibitively complicated, preventing us from ever really doing this — but in trying, we *will* find a natural approximation formula in terms of the stable Landweber–Novikov operations. Giving a precise formula for this approximation is the goal of this Lecture.

For X a finite complex and $f \in MU^{2q}(X)$ some *MU*–cohomology class, we set our sights on the too-lofty goal of computing $P^n(f)$. Because of the definition $MU = \mathrm{colim}_k MU(k)$ and because P^n is natural under pullback, it will suffice for us to study the effect of P^n on the universal classes

$$u_m \colon MU(m) \to MU,$$

beginning with the canonical orientation

$$u_1 = x \in h\mathsf{Spectra}(MU(1), MU) \cong MU^2\mathbb{CP}^\infty.$$

In order to understand the effect $P^n(x)$ of the power operation on x, we recall a different interpretation of x: it is also the 1$^{\text{st}}$ Conner–Floyd–Chern class of the tautological bundle $\mathcal{L}$ on $\mathbb{CP}^\infty$, i.e.,

$$x \colon MU(1) \to MU$$

is the Thomification of the block inclusion

$$\mathcal{L} \colon BU(1) \to BU.$$

The construction defining $P^n_{\text{ext}}(x)$ thus fits into the following diagram:

$$
\begin{array}{ccccc}
\Sigma^\infty_+ BU(1) & \longrightarrow & \Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n & & \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta_{hC_n}} & & \\
(\Sigma^\infty_+ BU(1))^{\wedge n} & \longrightarrow & \Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} & \xrightarrow{\ \mathcal{L}(n)\ } & \Sigma^\infty_+ BU(n) \\
\downarrow{\scriptstyle c_1^{\wedge n}} & {\scriptstyle \mathcal{L}^{\oplus n}} & \downarrow & {\scriptstyle P^n_{\text{ext}}} & \downarrow{\scriptstyle c_n} \\
(\Sigma^2 MU)^{\wedge n} & \longrightarrow & (\Sigma^2 MU)^{\wedge n}_{hC_n} & \longrightarrow & \Sigma^{2n} MU.
\end{array}
$$

The commutativity of the widest rectangle (i.e., the justification for the name "c_n" on the right-most vertical arrow) comes from the Cartan formula for Chern classes: because $\mathcal{L}^{\oplus n}$ splits as the sum of n line bundles, $c_n(\mathcal{L}^{\oplus n})$ is computed as the product of the 1$^{\text{st}}$ Chern classes of those line bundles. Second, the commutativity of the right-most square is not trivial: it is a specific consequence of how the multiplicative structure on *MU* arises from the direct sum of vector bundles.[14] The commutativities of the other two squares comes from the natural transformation from a C_n–space to its homotopy orbit space.

---

[14] In general, any notion of first Chern class $\Sigma^\infty_+ BU(1) \to \Sigma^2 E$ gives rise to a *noncommuting* diagram of this same shape. The two composites $\Sigma^\infty_+ BU(1)^{\wedge n}_{hC_n} \to \Sigma^{2n} E$ need not agree, since $\mathcal{L}(n)$ has no *a priori* reason to be compatible with the factorization appearing in the H^2_∞–structure. They turn out to be related nonetheless, and their exact relation (as well as a procedure for making them agree) is the subject of **PUT REFERENCE TO CHAPTER 6**.

Hence, the the internal Steenrod power operation $P^n(x)$ is defined by the composite

$$\Sigma^\infty_+ BU(1) \wedge \Sigma^\infty_+ BC_n \simeq \Sigma^\infty_+ BU(1)_{hC_n} \xrightarrow{\Delta_{hC_n}} (\Sigma^\infty_+ BU(1))^{\wedge n}_{hC_n} \to \Sigma^{2n} MU^{\wedge n}_{hC_n} \to \Sigma^{2n} MU,$$

which is to say

$$P^n(x) = c_n(\Delta^*_{hC_n} \mathcal{L}(n)).$$

We have thus reduced to computing a particular Conner–Floyd–Chern class of a particular bundle.

Our next move is to realize that we have not lost information by passing from the bundle $\Delta^* \mathcal{L}^{\oplus n}$ to the bundle $\Delta^*_{hC_n} \mathcal{L}(n)$.

Theorem 18. *There is a natural bijection between G–equivariant vector bundles over a base X on which G acts trivially and vector bundles on $X \times BG$.*

Proof. This is the exponential adjunction

$$\mathsf{Spaces}(X \times BG, BU) \xrightleftharpoons{\longrightarrow} \mathsf{Spaces}(*/\!/G, \mathsf{Spaces}(X, BU)),$$

$$V_{hG} \xmapsto{\hspace{3cm}} V.$$

The right-hand side consists of G–equivariant vector bundles over the G–trivial base X, and the left-hand side consists of vector bundles over $X \times BG$.

> This isn't much of a proof.

Corollary 19. *Under this bijection, the vector bundle $\Delta^*_{hC_n} \mathcal{L}(n)$ on $BU(1) \times BC_n$ corresponds to the C_n–equivariant vector bundle $\Delta^* \mathcal{L}^{\oplus n}$ on $BU(1)$.* $\square$

We thus proceed to analyze $\Delta^*_{hC_n} \mathcal{L}(n)$ by studying the C_n–equivariant bundle $\Delta^* \mathcal{L}^{\oplus n}$ instead. The C_n–action is given by permutation of the factors, and hence we have an identification

$$\Delta^* \mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^* \rho,$$

where ρ is the permutation representation of C_n (considered as a vector bundle over a point) and $\pi \colon BU(1) \to *$ is the constant map. The permutation representation for the abelian group C_n, also known as its regular representation, is accessible by character theory. The generating character $\chi \colon U(1)[n] \to U(1)$ gives a decomposition

$$\rho \cong \bigoplus_{j=0}^{n-1} \chi^{\otimes j}.$$

Applying this to our situation, we get a sequence of isomorphisms of C_n–equivariant vector bundles

$$\Delta^* \mathcal{L}^{\oplus n} \cong \mathcal{L} \otimes \pi^* \rho \cong \mathcal{L} \otimes \bigoplus_{j=0}^{n-1} \pi^* \chi^{\otimes j} \cong \bigoplus_{j=0}^{n-1} \mathcal{L} \otimes \pi^* \chi^{\otimes j}.$$

Applying Theorem 18, we recast this as a calculation of the bundle $\Delta^*_{hC_n} \mathcal{L}(n)$:

$$\Delta^*_{hC_n} \mathcal{L}(n) = \bigoplus_{j=0}^{n-1} \pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j},$$

where η is the bundle classified by $\eta \colon BU(1)[n] \to BU(1)$ and π_1, π_2 are the two projections off of $BU(1) \times BC_n$.

We now use this to access $c_n(\Delta^*_{hC_n} \mathcal{L}(n))$. As the top Chern class of this n–dimensional vector bundle, we think of this as a calculation of its Euler class, which lets us lean on multiplicativity:

$$P^n(x) = c_n(\Delta^*_{hC_n} \mathcal{L}(n)) = e\left(\bigoplus_{j=0}^{n-1} \pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j} \right) = \prod_{j=0}^{n-1} e\left(\pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j} \right)$$

$$= \prod_{j=0}^{n-1} c_1\left(\pi_1^* \mathcal{L} \otimes \pi_2^* \eta^{\otimes j} \right) = \prod_{j=0}^{n-1} (x +_{MU} [j]_{MU}(t)).$$

Here x is still the 1st Conner–Floyd–Chern class of $\mathcal{L}$ and t is the Euler class of η. We now try to make sense of this product expression for $c_n(\Delta^*_{hC_n} \mathcal{L}(n))$ by expanding it in powers of x and identifying its component pieces.

Lemma 30. *There is a series expansion*

$$P^n(x) = \prod_{j=0}^{n-1} (x +_{MU} [j]_{MU}(t)) = w + \sum_{j=1}^{\infty} a_j(t) x^j,$$

where $a_j(t)$ is a series with coefficients in the subring $C \subseteq MU_*$ spanned by the coefficients of the natural MU–formal group law. The leading term

$$w = e(\rho) = \prod_{j=0}^{n-1} e(\eta^{\otimes j}) = \prod_{j=0}^{n-1} [j]_{MU}(e(\eta)) = (n-1)! t^{n-1} + \sum_{j \geq n} b_j t^j$$

is the Euler class of the reduced permutation representation, and, again, the elements b_j lie in the subring C. $\qquad\qquad\square$

This is about as much information as we can hope to extract in the 1st universal case. We thus return to our original goal: understanding the action of P^n on each of the canonical classes

$$u_m \colon MU(m) \to MU.$$

Our best hope to approach this is to use the splitting principle, so we rewrite the formula in Lemma 30 in a form amenable to direct sums:

$$P^n(x) = \sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x), \qquad a_\alpha(t) = \prod_{j=0}^{\infty} a_j(t)^{\alpha_j}.$$

Corollary 20. *There is the universal formula*

$$P^n(u_m) = \sum_{|\alpha| \leq m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m).$$

Proof. This follows directly from the splitting principle and the Cartan formula:

$$P^n(u_m) = \overbrace{P^n(x_1) \cdot \cdots \cdot P^n(x_m)}^{\text{each of the } m \text{ factors}}$$

$$= \left(\sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_1) \right) \cdot \cdots \cdot \left(\sum_{|\alpha| \leq 1} w^{1-|\alpha|} a_\alpha(t) s_\alpha(x_m) \right)$$

$$= \sum_{|\alpha| \leq m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m). \square$$

We will use this to power the following conclusion about cohomology classes on in general, starting with an observation about the fundamental class of a sphere:

Lemma 31 (cf. [Rud98, Corollary VII.7.14]). *For $f \in MU^{2q}(X)$ a cohomology class in a finite complex X, there is the suspension relation*

$$P^n(\sigma^{2m} f) = w^m \sigma^{2m} P^n(f).$$

Proof. We calculate P^n applied to the fundamental class

$$S^{2m} \xrightarrow{\iota_{2m}} T_m BU(m) \simeq \Sigma^{2m} MU(m) \xrightarrow{\Sigma^{2m} u_m} \Sigma^{2m} MU$$

by restricting the universal formula:

$$P^n(\iota_{2m}^* u_m) = \iota_{2m}^* P^n(u_m) = \iota_{2m}^* \left(\sum_{|\alpha| \leq m} w^{m-|\alpha|} a_\alpha(t) s_\alpha(u_m) \right) = w^m \iota_{2m},$$

since $s_\alpha(\iota_{2m}) = 0$ for any nonzero α, as the cohomology of S^{2m} is too sparse. Because $\sigma^{2m} f = \iota_{2m} \wedge f$, we conclude the proof by multiplicativity of P^n.

Theorem 19 (cf. [Qui71, Proposition 3.17], [Rud98, Corollary VII.7.14]). *Let X be a finite pointed space and let f be a cohomology class*

$$f \in \widetilde{MU}^{2q}(X).$$

For $m \gg 0$, there is a formula

$$w^m P^n(f) = \sum_{|\alpha| \leq m+q} w^{q+m-|\alpha|} a_\alpha(t) s_\alpha(f),$$

with t, w, and $a_\alpha(t)$ as defined above.[15]

Proof. We take m large enough so that f is represented by an unstable map

$$g \colon \Sigma^{2m} X \to T_{m+q} BU(m+q),$$

in the sense that g intertwines f with the universal class u_{m+q} by the formula

$$g^* u_{m+q} = \sigma^{2m} f.$$

We use Lemma 31 and naturality to conclude

$$w^m \sigma^{2m} P^n(f) = P^n(\sigma^{2m} f) = P^n(g^* u_{m+q}) = g^* P^n(u_{m+q})$$

$$= g^* \left(\sum_{|\alpha| \leq m+q} w^{m+q-|\alpha|} a_\alpha(t) s_\alpha(u_{m+q}) \right)$$

$$= \sum_{|\alpha| \leq m+q} w^{m+q-|\alpha|} a_\alpha(t) \sigma^{2m} s_\alpha(f). \square$$

Our conclusion, then, is that P^n is *almost* naturally expressible in terms of the Landweber–Novikov operations, where the "*almost*" is controlled by some w–torsion. Additionally, the various factors in this statement — including w itself — are controlled by the formal group law "$+_{MU}$" associated to the tautological complex orientation of *MU* and the subring C.

Remark 24. You can put a remark here about how power operations become additive after passing to the Tate construction, now that you sort of understand this. Or, you can put a forward reference in to where you're going to talk about this, way out in the Power Operations chapter. Actually, probably do both: mention it in passing and put in "cf.".

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[15] The reader comparing with Quillen's paper will notice various apparent discrepancies between the statements of our Theorem and of his. These are notational: he grades his cohomology functor homologically, which occasionally causes our q to match his $-q$, so that his n is comparable to our $m - q$. **GET THIS WARNING RIGHT**

2.6 The complex bordism ring

With Theorem 19 in hand, we will deduce Quillen's major structural theorem about MU_*. We will preserve the notation used in Lemma 30 and Theorem 19:

- ρ is the reduced regular representation of C_n, which coincides with its reduced permutation representation, and $w = e(\rho)$ is its Euler class.
- $\eta \colon BU(1)[n] \to BU(1)$ is the line bundle associated to a generating character for C_n, and $t = e(\eta)$ its Euler class.
- C is the subring of MU_* generated by the coefficients of the formal group law associated to the identity complex–orientation.

In the course of working out the main Theorem, we will want to make use of some properties of the class t.

Lemma 32 (cf. [RW80, Theorem 5.7], [HL, Proposition 2.4.4], [Rud98, Theorem VII.7.9]). *There is an isomorphism of formal groups*

$$BU(1)[n]_{MU} \cong \mathbb{CP}^\infty_{MU}[n].$$

Proof. Consider the pullback diagram of spherical fibrations:

$$
\begin{array}{ccccc}
U(1) & \longrightarrow & BU(1)[n] & \longrightarrow & BU(1) \\
\| & & \downarrow & & \downarrow{\scriptstyle n} \\
U(1) & \longrightarrow & EU(1) & \longrightarrow & BU(1).
\end{array}
$$

The Euler class of the first bundle is the class x, and it pulls back along the map $n \colon BU(1) \to BU(1)$ to give the Euler class $[n](x)$ of the second bundle. The induced long exact sequence[16] takes the form

$$
\begin{array}{ccc}
 & MU^*BU(1) & \\
\nearrow & & \nwarrow{\scriptstyle -\smile[n](x)} \\
MU^*(BU(1)[n]) & \xrightarrow[\;\;\partial\;\;]{} & MU^{*+2}BU(1)
\end{array}
$$

where x is the coordinate on $BU(1)_{MU}$. Because $[n]_{MU}(x) = nx + \cdots$ and because $\widetilde{MU}^2\mathbb{CP}^1$ has no n–torsion, the right diagonal map is injective and hence $\partial = 0$. This therefore gives a short exact sequence of Hopf algebras, which we can reinterpret as a short exact sequence of group schemes

$$0 \to BU(1)[n]_{MU} \to BU(1)_{MU} \xrightarrow{n} BU(1)_{MU} \to 0. \qquad \square$$

---

[16] This sequence is known as the Gysin sequence. It arises as the exact couple for the Serre spectral sequence for the spherical fibration.

Corollary 21. *The Künneth map*

$$MU^*(X) \otimes_{MU^*} MU^*(BU(1)[n]) \to MU^*(X \times BU(1)[n])$$

is an isomorphism. In terms of coordinate rings, this gives isomorphisms

$$(X \times BU(1)[n])_{MU} \cong X_{MU} \times BU(1)[n]_{MU} \cong X_{MU} \times BU(1)_{MU}[n].$$

Proof. This follows from the evenness of $MU^*(BU(1)[n])$.

Corollary 22 ([Qui71, Proposition 4.4]). *Write*

$$\langle n \rangle_{MU}(x) = \frac{[n]_{MU}(x)}{x}.$$

If $\omega \in MU^*BU(1)[n]$ satisfies $t \cdot \omega = 0$, then there exists a class y with $\omega = y \cdot \langle n \rangle_{MU}(t)$.

Proof. By Lemma 32 we know $MU^*BU(1)[n] \cong MU^*[\![t]\!]/[n]_{MU}(t)$, so the kernel of multiplication by t is exactly $\langle n \rangle_{MU}(t)$.[17]

In all, we learn that the Euler class $t = e(\eta)$ corresponds to the restriction of the coordinate x along the closed inclusion

$$BU(1)[n]_{MU} \cong BU(1)_{MU}[n] \to BU(1)_{MU}.$$

We now turn to the main Theorem.

Theorem 20 ([Qui71, Theorem 5.1]). *If X has the homotopy type of a finite complex, then*

$$MU^*(X) = C \cdot \sum_{q \geq 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \sum_{q > 0} MU^q(X).$$

Remark 25. In what follows, the reader should carefully remember the degree conventions stemming from the formula

$$MU^*X = \pi_{-*}F(\Sigma^\infty_+ X, MU).$$

The homotopy ring MU_* appears in the *negative* degrees of $MU^*(*)$, but the fundamental class of S^m appears in the *positive* degree $MU^m(S^m)$.

Proof (Proof of Theorem 20). We can immediately reduce the claim in two ways. First, it is true if and only if it is also true for reduced cohomology.

---

[17] Quillen considers this as coming from the Gysin sequence for $S^1 \to S(\eta) \to BU(1)[n]$, which has Euler class t.

Second, because $MU^{2*+1}(*) = 0$, we can restrict attention just to $MU^{2*}(X)$, since we can then handle the odd-degree parts of $MU^*(X)$ by suspending X once. Defining

$$R^{2*} := C \cdot \sum_{q>0} MU^{2q}(X),$$

we can thus focus on the claim

$$\widetilde{MU}^{2*}(X) \overset{?}{=} C \cdot \sum_{q>0} MU^{2q}(X).$$

Noting that the claim is trivially true for all positive values of $*$, we will show this by working p–locally and inducting on the value of "$-*$".

Suppose that

$$R_{(p)}^{-2j} = \widetilde{MU}^{-2j}(X)_{(p)}$$

for $j < q$ and consider $x \in \widetilde{MU}^{-2q}(X)$. Then, for $m \gg 0$, we have

$$w^m P^p(x) = \sum_{|\alpha| \leq m-q} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x = w^{m-q} x + \sum_{\substack{|\alpha| \leq m-q \\ \alpha \neq 0}} w^{m-q-|\alpha|} a(t)^\alpha s_\alpha x.$$

Recall that w is a power series in t with coefficients in C and leading term $(p-1)! \cdot t^{p-1}$, so that $t^{p-1} = w \cdot \theta(t)$ for some multiplicatively invertible series $\theta(t)$ with coefficients in C. Since s_α raises degree, we have $s_\alpha x \in R$ by the inductive hypothesis, and we may thus collect all those terms into a series $\psi_x(t) \in R_{(p)}[\![T]\!]$ to write

$$t^{m-q}(w^q P^p(x) - x) = \psi_x(t).$$

Suppose $m > q$ is the least integer for which we can write such an equation — we will show $m = q + 1$ in a moment. Using Remark 20, we find that restricting this equation along the inclusion $i \colon X \to X \times BU(1)[p]$ sets $t = 0$ and yields $\psi_x(0) = 0$. It follows that $\psi_x(t) = t\varphi_x(t)$ is at least once t–divisible, and thus

$$t(t^{m-q-1}(w^q Px - x) - \varphi_x(t)) = 0.$$

Appealing to Corollary 22, we produce a class $y \in \widetilde{MU}^{-2q+2(m-1)}(X)$ with

$$t^{m-q-1}(w^q P^p(x) - x) = \varphi_x(x) + y\langle p \rangle(t).$$

If $m > q + 1$, then $y \in R_{(p)}$ for degree reasons and hence the right-hand side gives a series expansion contradicting our minimality hypothesis. So, $m =$

$q + 1$, and the outer factor of t^{m-q-1} is not present in the last expression.[18] Restricting along i again to set $w = t = 0$ and $P^p(x) = x^p$, we obtain the equation

$$\left.\begin{array}{l} -x \text{ if } q > 0 \\ x^p - x \text{ if } q = 0 \end{array}\right\} = \varphi_x(0) + py.$$

In the first case, where $q > 0$, it follows that $MU^{-2q}(X) \subseteq R^{-2q} + pMU^{-2q}(X)$, and since $MU^{-2q}(X)$ has finite order torsion[19] it follows that $MU^{-2q}(X) = R^{-2q}$. In the other case, x can be rewritten as a sum of elements in R^0, elements in $p\widetilde{MU}^0(X)$, and elements in $(\widetilde{MU}^0(X))^p$. Since the ideal $\widetilde{MU}^0(X)$ is nilpotent, it again follows that $\widetilde{MU}^0(X) = R^0$, concluding the induction.

Corollary 23. *The coefficients of the formal group law generate MU_*.*

Proof. This is the case $X = *$.

Remark 26. This proof actually also goes through for MO as well. In that case, it's even easier, since the equation $2 = 0$ in $\pi_0 MO$ causes much of the algebra to collapse. The proof does not extend further to cases like MSO or MSp, as explained in the introduction to this Case Study: these bordism theories do not have associated formal group laws, and so we lose the control we had in Lecture 2.5.

Take $\mathcal{M}_{\mathbf{fgl}}$ to be the moduli of formal group laws. Since a formal group law is a power series satisfying some algebraic identities, this moduli object is an affine scheme with coordinate ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$. A rephrasing of Corollary 23 is that the natural map

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$$

is *surjective*. This is reason enough to start studying $\mathcal{M}_{\mathbf{fgl}}$ in earnest, which we take up in the next Case Study — but while we're here, if we anachronistically assume one algebraic fact about $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ we can prove that the natural map is actually an *isomorphism*. The place to start is with the following topological observation about mixing complex–orientations:

Lemma 33 ([Ada95, Lemma 6.3 and Corollary 6.5]). *Let $\varphi: MU \to E$ be a complex–oriented ring spectrum and consider the two orientations on $E \wedge MU$ given by*

---

[18] One can interpret the proof thus far as giving a bound on the amount of w–torsion needed to get the stability relation described in Theorem 19. Our answer is quite surprising: we have found that we need just a single w, which isn't much stability at all!

[19] This is a consequence of X having finitely many cells, MU having finitely many cells in each degree, and each homotopy group of the stable sphere being finitely generated.

$$\mathsf{S} \wedge MU \xrightarrow{\eta_E \wedge 1} E \wedge MU, \qquad MU \wedge \mathsf{S} \xrightarrow{\varphi \wedge \eta_{MU}} E \wedge MU.$$

The two induced coordinates x^E and x^{MU} on $\mathbb{C}\mathrm{P}^\infty_{E \wedge MU}$ are related by the formulas

$$x^{MU} = \sum_{j=0}^{\infty} b_j^E (x^E)^{j+1} =: g(x^E),$$

$$g^{-1}(x^{MU} +_{MU} y^{MU}) = g^{-1}(x^E) +_E g^{-1}(y^E).$$

where $E_* MU \cong \dfrac{\mathrm{Sym}_{E_*} E_*\{\beta_1, \beta_2, \beta_3, \ldots\}}{\beta_1 = 1} \cong E_*[b_1, b_2, \ldots]$, as in Lemma 17, Corollary 5, and Corollary 9.

Proof. The second formula is a direct consequence of the first. The first formula comes from taking the module generators $\beta_{j+1} \in E_{2(j+1)}\mathbb{C}\mathrm{P}^\infty = E_{2j}MU(1)$ and pushing them forward to get the algebra generators $b_j \in E_{2j}MU$. Then, the triangle

$$[\mathbb{C}\mathrm{P}^\infty, MU] \xrightarrow{\hspace{9cm}} [\mathbb{C}\mathrm{P}^\infty, E \wedge MU]$$
$$\mathrm{Modules}_{E_*}(E_* \mathbb{C}\mathrm{P}^\infty, E_* MU)$$

allows us to pair x^{MU} with $(x^E)^{j+1}$ to determine the coefficients of the series.

Corollary 24 ([Ada95, Corollary 6.6]). *In particular, for the orientation $MU \to H\mathbb{Z}$ we have*

$$x_1 +_{MU} x_2 = \exp^H(\log^H(x_1) + \log^H(x_2)),$$

where $\exp^H(x) = \sum_{j=0}^{\infty} b_j x^{j+1}$. $\qquad\qquad\square$

However, one also notes that $H\mathbb{Z}_* MU = \mathbb{Z}[b_1, b_2, \ldots]$ carries the universal example of a formal group law with a logarithm — this observation is independent of any knowledge about the coefficient ring MU_*. It turns out that this brings us one step away from understanding MU_*:

Theorem 21 (To be proven as Theorem 25). *There is a ring $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$ carrying the universal formal group law, and it is free: it is a polynomial ring over $\mathbb{Z}$ in countably many generators.* $\qquad\qquad\square$

Corollary 25. *The natural map $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \to MU_*$ classifying the formal group law on MU_* is an isomorphism.*

Proof. We proved in Corollary 23 that this map is surjective. We also proved in Theorem 11 that every rational formal group law has a logarithm, i.e., the long composite on the second row

$$\begin{array}{ccccc}
\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} & \longrightarrow & MU_* & \longrightarrow & (H\mathbb{Z}_* MU) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \otimes \mathbb{Q} & \longrightarrow & MU_* \otimes \mathbb{Q} & \xrightarrow{\simeq} & (H\mathbb{Z}_* MU) \otimes \mathbb{Q}
\end{array}$$

is an isomorphism. It follows from Theorem 21 that the left-most vertical map is injective, hence the top-left horizontal map is injective, hence it is an isomorphism.

Corollary 26. *The ring $\pi_*(MU \wedge MU)$ carries the universal example of two strictly isomorphic formal group laws. Additionally, the ring $\pi_0(MUP \wedge MUP)$ carries the universal example of two isomorphic formal group laws.*

Proof. Combine Lemma 33 and Corollary 25.

This todo belongs in the Remark at the end of 2.5 / in the footnote of 2.6.1 / in an appendix. Not here. There's buzz about a "Frobenius map" for structured rings going around these days. I guess the point is that an E_2–algebra structure is enough to get a multiplicative map $E^0 X \to E^0 X \otimes E^0 BC_p$. This isn't additive, so it can't come from an infinite loop map, but it becomes additive when passing to the Tate construction: $E^X \to (E^X)^{tC_p}$, using the fact that the genuine C_p fixed points of $E^{X \times p}$ is E^X, and the square relating genuine, homotopy, and geometric fixed points. Mike has been claiming that these results of Quillen's can be interpreted in this way, but I'm not sure what the interpretation is. He says it has something to do with inverting the Euler class and the part of Quillen's argument that involves walking down the multiples of Euler classes on both sides of the equation.

Case Study 3
Finite spectra

Our goal in this Case Study is to thoroughly examine one of the techniques from Case Study 1 that has not yet resurfaced: the idea that $H\mathbb{F}_2$–homology takes values in quasicoherent sheaves over some algebro-geometric object encoding the coaction of the dual Steenrod Hopf algebra. We will find that this situation is quite generic: associated to mildly nice ring spectra E, we will construct a very rich algebro-geometric object $\mathcal{M}_E$, called its context, such that E–homology sends spaces X to sheaves $\mathcal{M}_E(X)$ over $\mathcal{M}_E$. In still nicer situations, the difference between the E_*–module $E_*(X)$ and the sheaf $\mathcal{M}_E(X)$ tracks exactly the analogue of the action of the dual Steenrod algebra, called the *Hopf algebroid of stable E–homology cooperations*. From this perspective, we will reinterpret Quillen's Corollary 25 as giving a presentation

$$\mathcal{M}_{MUP} \xrightarrow{\cong} \mathcal{M}_{\mathbf{fg}},$$

where $\mathcal{M}_{\mathbf{fg}}$ is the *moduli of formal groups*. This indicates a program for studying periodic complex bordism, which we will spend the rest of this introduction outlining.

Abstractly, one can hope to study any sheaf, including $\mathcal{M}_E(X)$, by analyzing its stalks. The main utility of Quillen's theorem is that it gives us access to a concrete model of the context $\mathcal{M}_{MUP}$, so that we can determine where to even look for those stalks. However, even this is not really enough to get off the ground: the stalks of some sheaf can exhibit nearly arbitrary behavior. In particular, there is little reason to expect the stalks of $\mathcal{M}_E(X)$ to vary nicely with X. Accordingly, given a map f in the diagram

$$\begin{array}{ccccc}
\operatorname{Spec} R & \xrightarrow{\ f\ } & \mathcal{M}_{\mathbf{fgl}} & = & \mathcal{M}_{MUP}[0] & = & \operatorname{Spec} MUP_0 \\
& \searrow & \downarrow & & \downarrow & \\
& & \mathcal{M}_{\mathbf{fg}} & = & \mathcal{M}_{MUP}, &
\end{array}$$

life would be easiest if the R–module determined by $f^* \mathcal{M}_{MUP}(X)$ were itself the value of a homology theory $R_0(X) = MUP_0 X \otimes_{MUP_0} R$ — this is exactly what it would mean for $R_0(X)$ to "vary nicely with X". Of course, this is unreasonable to expect in general: homology theories are functors which convert cofiber sequences of spectra to long exact sequences of groups, but base–change from $\mathcal{M}_{\mathbf{fg}}$ to $\operatorname{Spec} R$ preserves exact sequences exactly when the diagonal arrow is *flat*. However, if flatness is satisfied, this gives the following theorem:

Theorem 22 (Landweber). *Given such a diagram where the diagonal arrow is flat, the functor*

$$R_0(X) := MUP_0(X) \otimes_{MUP_0} R$$

is a homology theory.

In the course of proving this theorem, Landweber additionally devised a method to recognize flat maps. Recall that a map f is flat exactly when for any closed substack $i \colon A \to \mathcal{M}_{\mathbf{fg}}$ with ideal sheaf $\mathcal{I}$ there is an exact sequence

$$0 \to f^* \mathcal{I} \to f^* \mathcal{O}_{\mathcal{M}_{\mathbf{fg}}} \to f^* i_* \mathcal{O}_A \to 0.$$

Landweber classified the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, thereby giving a precise list of conditions needed to check maps for flatness.

This appears to be a moot point, however, as it is unreasonable to expect this idea to apply to computing stalks: the inclusion of a geometric point is flat only in highly degenerate cases. We will see that this can be repaired: the inclusion of the formal completion of a substack of a Noetherian[1] stack is flat, and so we naturally become interested in the infinitesimal deformation spaces of the geometric points Γ on $\mathcal{M}_{\mathbf{fg}}$. If we can analyze those, then Landweber's theorem will produce homology theories called *Morava E_Γ–theories*. Moreover, if we find that these deformation spaces are *smooth*, it will follow that their deformation rings support regular sequences. In this excellent case, by taking the regular quotient we will be able to recover *Morava K_Γ–theory*, a *homology theory*, which plays the role[2] of computing the stalk of

---

[1] $\mathcal{M}_{\mathbf{fg}}$ is not Noetherian, but we will find that each geometric point except $\widehat{\mathbb{G}}_a$ lives in an open substack that happens to be Noetherian.

[2] To be clear: $K_\Gamma(X)$ may not actually compute the literal stalk of $\mathcal{M}_{MUP}(X)$ at Γ, since the homotopical operation of quotienting out the regular sequence is potentially sensitive to torsion sections of $\mathcal{M}_{MUP}(X)$.

$\mathcal{M}_{MUP}(X)$ at Γ.[3]

We have thus assembled a task list:

- Describe the open and closed substacks of $\mathcal{M}_{\mathbf{fg}}$.
- Describe the geometric points of $\mathcal{M}_{\mathbf{fg}}$.
- Analyze their infinitesimal deformation spaces.

This list will occupy us for the first half of this Case Study. In the second half, we will exploit these homology theories E_Γ and K_Γ, as well as their connection to $\mathcal{M}_{\mathbf{fg}}$ and to MU, to make various structural statements about the category Spectra. These homology theories are especially well-suited to understanding the subcategory Spectra$^{\text{fin}}$ of finite spectra, and we will recount several important statements in that setting. Together with these homology theories, these celebrated results, collectively called the nilpotence and periodicity theorems, form the basis of *chromatic homotopy theory*. In fact, our *real* goal in this Case Study is to give an introduction to the chromatic perspective that remains in line with our algebro-geometrically heavy narrative.

> still need to talk about closed and open substacks, their basic properties

> A blanket todo from a conversation with Eva: (1) we should expand the comparison of a Cartesian q.c. sheaf on an affinely presented stack with a module plus structural data; (2) it's not clear why the complex $B\check{G}$ computes the claimed Cotor groups; (3) we aren't very clear about where flat maps come from, which should be stated early on and which should be reiterated down in the big list of homology theories; (4) the Baer-criterion-style argument is fuzzy, and could be stated up in the big program and also in the proof of the LEFT; (5) the purpose of the Witt ring in deformation theory is not spelled out; (6) the difference between 0–truncation (Grothendieck's faithfully flat descent) and 1–truncation (having a nice ring spectrum) is blurred, which is perhaps unhealthy; (7) the business with descent data is still unclear, why $S \otimes_R N \cong N \otimes_R S$ is worth anything; (8) the coarse moduli uses the smoothness property, which should be referenced; (9) in the statement of $\mathcal{M}_{MU}$, the analysis of $MU_* MU$ could be referenced (or, really, the chain of thought leading up to this could be sketched); (10) "invariant prime ideal" is not defined, nor is it compared to the geometric objects you'd call "closed substacks".

> Figure out where the label StructureOfMfgl is referenced and make sure it points appropriately to either this introduction or to MfgI:Height.

> Danny has been nervous lately about the LEFT yielding module spectra and ring spectra. It would be good to write out how LEFT applied to a sheaf of rings gives a ring spectrum.

3.1 The context of a spectrum

> I don't think I mention Hopf algebroids during this lecture! This is a miserable oversight that *must* be corrected. Also, I should mention the cotensor product for Hopf algebroids. Update by d.s.: fixed. My reference was appendix I of Ravenel. Feel free to delete anything you think is unnecessary for our purposes. I think some of the homological algebra near the end can be left out.

> Be sure to explain what the groupoid quotient of a Hopf algebroid is in this lecture. You promised an explanation back in Lecture 1.4.

> **Cite me:** Pridham's article *Presenting higher stacks as simplicial schemes* seems like a good reference? Maybe some Toen things are appropriate? I don't really know where this simplicial scheme stuff is written down..

Definition 36. A *Hopf Algebroid* over a commutative ring K is a pair (A, Γ) of commutative K-algebras with structure maps such that for any other commutative K-algebra B, the sets $\text{Hom}(A, B)$ and $\text{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid. The structure maps are

1. $\eta_L : A \to \Gamma$ (source)
2. $\eta_R : A \to \Gamma$ (target)

---

[3] Incidentally, this program has no content when applied to $\mathcal{M}_{H\mathbb{F}_2}$, as $\text{Spec}\,\mathbb{F}_2$ is simply too small.

3. $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ (composition)
4. $\varepsilon : \Gamma \to A$ (identity)
5. $c : \Gamma \to \Gamma$ (inverse)

There are some relations among the structure maps that mimics the defining properties of a groupoid. I won't mention them here but they can be found in Ravenel's green book, appendix I. A graded Hopf algebroid is *connected* if the left and right sub A-modules generated by Γ_0 are both isomorphic to A. If $\eta_R = \eta_L$, then Γ is a commutative Hopf algebra over A.

Definition 37. A left Γ-*comodule* M is a left A-module M together with a left A-linear map $\psi : M \to \Gamma \otimes_A M$ that is both counitary and coassociative.

From now on, we assume that Γ is flat over A.

Definition 38. Let M be a right Γ-comodule, and N a left Γ-comodule. Their *cotensor product* over Γ is the K-module defined by the exact sequence $0 \to M\square_\Gamma N \to M \otimes_A N \xrightarrow{\psi \otimes N - M \otimes \psi} M \otimes_A \Gamma \otimes_A N$, where ψ are the comodule structure maps for M and N.

Notice that if M is a left comodule, then it can be given the structure of a right comodule by the composition

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M \otimes c} M \otimes \Gamma,$$

where T swaps the two factors and c is the conjugation map. From this, it is easy to deduce that $M\square_\Gamma N = N\square_\Gamma M$. The following lemma relates cotensor products to Hom.

Lemma 34. *Let M and N be left Γ-comodules with M projective over A. Then*

1. *$\mathrm{Hom}_A(M, A)$ is a right Γ-module.*
2. *$\mathrm{Hom}_\Gamma(M, N) = \mathrm{Hom}_A(M, A)\square_\Gamma N$. In particular, when $M = A$, we have $\mathrm{Hom}_\Gamma(A, N) = A\square_\Gamma N$.*

Proof. There exist maps ψ_M^*, $\psi_N^* : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, \Gamma \otimes_A N)$, defined by

$$M \xrightarrow{\psi_M} \Gamma \otimes M \xrightarrow{\Gamma \otimes f} \Gamma \otimes_A N,$$

$$M \xrightarrow{f} N \xrightarrow{\psi_N} \Gamma \otimes_A N.$$

Since M is projective over A, there is a canonical isomorphism

$$\mathrm{Hom}_A(M, A) \otimes_A N \simeq \mathrm{Hom}_A(M, N).$$

When $N = A$, we obtain the map

$$\psi_M^* : \mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_A(M, A) \otimes_A \Gamma.$$

It is easy to check that this map satisfies the coassociativity axiom. For the second part, note that by definition, we have

$$\mathrm{Hom}(M, N) = \ker(\psi_M^* - \psi_N^*) \subset \mathrm{Hom}_A(M, N),$$
$$\mathrm{Hom}_A(M, A) \square_\Gamma N = \ker(\psi_M^* \otimes N - \mathrm{Hom}_A(M, A) \otimes \psi_N).$$

The claim then follows from the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}(M, A) \otimes N & \xrightarrow{\;\simeq\;} & \mathrm{Hom}_A(M, N) \\
{\scriptstyle \psi_M^* \otimes N}\downarrow\downarrow{\scriptstyle \mathrm{Hom}(M,A)\otimes\psi_N} & & {\scriptstyle \psi_M^*}\downarrow\downarrow{\scriptstyle \psi_N^*} \\
\mathrm{Hom}(M, A) \otimes \Gamma \otimes N & \xrightarrow{\;\simeq\;} & \mathrm{Hom}_A(M, \Gamma \otimes_A N)
\end{array}
$$

Definition 39. A *map of Hopf algebroids* $f : (A, \Gamma) \to (B, \Sigma)$ is a pair of K-algebra maps $f_1 : A \to B, f_2 : \Gamma \to \Sigma$ such that $f_1\varepsilon = \varepsilon f_2$, $f_2\eta_R = \eta_R f_1$, $f_2\eta_L = \eta_L f_1$, $f_2 c = c f_2$, and $\Delta f_2 = (f_2 \otimes f_2)\Delta$.

Now we will discuss some homological algebra of Hopf algebroids. It turns out that the category of Γ-comodules has enough injectives, and so we can make the following definition:

Definition 40. For left Γ-comodules M and N, $\mathrm{Ext}_\Gamma^i(M, N)$ is the ith right derived functor of $\mathrm{Hom}_\Gamma(M, N)$, regarded as a functor of N. For M a right Γ-module, $\mathrm{Cotor}_\Gamma^i(M, N)$ is the ith right derived functor of $M\square_\Gamma N$, also regarded as a functor of N. The corresponding graded groups are denoted $\mathrm{Ext}_\Gamma(M, N)$ and $\mathrm{Cotor}_\Gamma(M, N)$, respectively.

The next lemma shows that the resolution can satisfy a weaker condition than being injective.

Lemma 35. *Let*

$$0 \to N \to R^0 \to R^1 \to \cdots$$

be a long exact sequence of left Γ-comodules such that $\mathrm{Cotor}_\Gamma^n(M, R^i) = 0$ for all $n > 0$. Then $\mathrm{Cotor}_\Gamma(M, N)$ is the cohomology of the complex

$$\mathrm{Cotor}_\Gamma^0(M, R^0) \xrightarrow{\delta_0} \mathrm{Cotor}_\Gamma^0(M, R^1) \xrightarrow{\delta_1} \cdots.$$

There are many candidates that satisfy the condition of 35. If M is a projective A-module and N is an A-module, then $\mathrm{Cotor}_\Gamma^i(M, \Gamma \otimes_A N) = 0$ for $i > 0$ and $\mathrm{Cotor}_\Gamma^0(M, \Gamma \otimes_A N) = M \otimes_A N$. A relative injective Γ-comodule is a direct summand of comodules of the form $\Gamma \otimes_A N$.

Definition 41. (Cobar Resolution) Let M be a left Γ-comodule. For a right Γ-comodule L that is projective over A, the cobar complex $C_\Gamma^*(L, M)$ is $C_\Gamma^s(L, M) = L \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M$ (with the obvious differential). When $L = \Gamma$, $D_\Gamma(M) = C_\Gamma(\Gamma, M)$ is called the cobar resolution of M.

It turns out that $D_\Gamma(M)$ is a resolution of M by relative injectives, and we have the following proposition:

Proposition 1. *If L is projective over A, then $H(C_\Gamma^*(L, M)) = Cotor_\Gamma(L, M)$. In particular, if $L = A$, then $H(C_\Gamma^*(A, M)) = Ext_\Gamma(A, M)$.*

end of Hopf algebroid. Actual class starts

Think about what sorts of simplicial sheaves you really want. They seem like they should be valued in something like quasicategories: the stable operations are valued in space-like simplicial sets, the isogenies pile is a sheaf of *categories*, then unstable operations generically have some other weird structure...

Today we will make good on our promise, made during our investigation of the unoriented bordism ring, to explain where the Adams spectral sequence comes from. This story neatly divides into two parts, and the first half is just an investigation of how rich of an algebraic category C we can find that supports a factorization

$$\text{Spectra} \xrightarrow{\quad E_* \quad} \text{Modules}_{E_*}$$
$$\text{C.}$$

Our answer to this question will come out of considering Grothendieck's framework of descent. Classically, descent concerns itself with a map $f\colon R \to S$ of a rings and an S–module N, and it asks questions like:

- When is there an R–module M such that $N \cong M \otimes_R S = f^*M$?
- What extra data can be placed on N, called *descent data*, so that the category of descent data for N is equivalent to the category of R–modules under the map f^*?
- What conditions can be placed on f so that the category of descent data for any given module is always contractible, called *effectivity*?

The essential structure of these answers is easy to guess if we proceed by example, using the few tools available to us. Suppose that we begin instead with an R–module M and we set $N = M \otimes_R S$. By tensoring up, we have two R–algebra maps $S \to S \otimes_R S$, given by including along either factor, and we can further tensor N up to $N \otimes_R S$ or $S \otimes_R N$. Since N came from the R–module M, these are canonically isomorphic:

$$\varphi\colon ((f \otimes 1) \circ f)^*M \cong ((1 \otimes f) \circ f)^*M.$$

The notation f^* confused me briefly: you push-forward modules but pull-back quasicoherent sheaves.

Repeating this process produces more isomorphisms which compose according to the triangle

$$N \otimes_R S \otimes_R S \xrightarrow[\simeq]{\varphi_{13}} S \otimes_R S \otimes_R N$$

with φ_{12} (with $\simeq$) down to $S \otimes_R N \otimes_R S$, and $\simeq$, φ_{23} up to $S \otimes_R S \otimes_R N$.

$$S \otimes_R N \otimes_R S,$$

where φ_{ij} denotes applying φ to the ith and j^{th} coordinates.

> **Cite me:** Allen said he knew a good reference for this descent picture.

> I think the isomorphism here is supposed to be an isomorphism $N \otimes_R S \to N \otimes_R S$ that takes the obvious $S \otimes_R S$-module structure to the swapped-factor $S \otimes_R S$-module structure. So for example, if $N = S$, the only (obvious?) descent isomorphism $S \otimes_R S \to S \otimes_R S$ would be the swap map. This is spelled out nicely in 4.5.1 of Mona Merling's thesis. -EB

Definition 42. Let $f \colon R \to S$ be a map of rings as above. An S–module N equipped with an isomorphism $S \otimes_R N \cong N \otimes_R S$ of $S \otimes_R S$–modules which causes the above triangle to commute is called a *descent datum* for the map f.

Remark 27. There are other ways to view this data. For example, later on we will revisit it from the categorical perspective of *comonads*. However, there is another perspective which we have already encountered earlier on: that of the *canonical coalgebra* or *Amitsur complex*. Associated to the map $f \colon R \to S$, we can form the ring $S \otimes_R S$, which supports a map

$$S \otimes_R S \simeq S \otimes_R R \otimes_R S \to S \otimes_R S \otimes_R S \simeq (S \otimes_R S) \otimes_S (S \otimes_R S).$$

One can check that descent data on a S–module is the same as the data of a coaction against $S \otimes_R S$. As a first step, notice the similarity of function signatures:

$$N \xrightarrow{\psi} N \otimes_S (S \otimes_R S) \simeq N \otimes_R S.$$

The following theorem is the usual culmination of an initial investigation into descent:

Theorem 23 (Grothendieck). *If $f \colon R \to S$ is faithfully flat, then there is an equivalence of R–modules and S–modules equipped with descent data.*

Proof (Jumping off point). The basic observation is that $0 \to R \to S \to S \otimes_R S$ is an exact sequence of R–modules. This makes much of the homological algebra involved work out.

For details and additional context, see Section 4.2.1 of [Vis05]; the story in the context of Hopf algebroids is also spelled out in detail in [Mil].

> I added the citation requested above, but make a pass over this since I may have put it in an awkward place - AY.

In our situation, this hypothesis will essentially never be satisfied, so we will pursue a less dramatic statement of the properties of descent. To see what kind of theorem one might expect, consider the example of $f \colon \mathbb{Z} \to \mathbb{F}_p$, which is neither faithful nor flat. Then, consider the following list of problems (and their partial solutions):

- The tensor functor f^* cannot distinguish even between the $\mathbb{Z}$–modules $\mathbb{Z}$ and $\mathbb{Z}/p$. However, if we use Lf^* and resolve $\mathbb{Z}/p$ as $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$, the complexes $Lf^*(\mathbb{Z})$ and $Lf^*(\mathbb{Z}/p)$ do look distinct.
- Once we pass to the derived category, then we are no longer in a situation where we can expect the single cocycle condition from the descent data above to suffice. Instead, we can form a simplicial scheme, called the *descent object*, by the formula

$$
\mathcal{D}_{\mathbb{Z}\to\mathbb{F}_p} := \left\{ \mathrm{Spec}\,\mathbb{F}_p \;\substack{\longleftarrow\\[-2pt]\longrightarrow} \; \times_{\mathrm{Spec}\,\mathbb{Z}} \substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow} \mathrm{Spec}\,\mathbb{F}_p \substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow} \times_{\mathrm{Spec}\,\mathbb{Z}} \substack{\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow} \mathrm{Spec}\,\mathbb{F}_p \substack{\longleftarrow\\[-2pt]\longrightarrow} \cdots \right\}.
$$

  This is meant to look like the Čech nerve for the "cover" $\mathrm{Spec}\,\mathbb{F}_p \to \mathrm{Spec}\,\mathbb{Z}$.
- Accordingly, we need to update our notion of quasicoherent sheaf to live over a simplicial scheme [Sta14, Tag 09VK]. Given a simplicial scheme X, a sheaf $\mathcal{F}$ on X will be a sequence of sheaves $\mathcal{F}[n]$ on $X[n]$ as well as, for each map $\varphi : [m] \to [n]$ in the simplicial indexing category inducing a map $X(\varphi) : X[n] \to X[m]$, a choice of map of sheaves

$$
\mathcal{F}(\varphi)_* \colon \mathcal{F}[m] \to X(\varphi)_* \mathcal{F}[n].
$$

  Such a sheaf will be called *quasicoherent* when it is levelwise quasicoherent.
- Finally, we can characterize the structure a quasicoherent sheaf over $\mathcal{D}_{\mathbb{Z}\to\mathbb{F}_p}$ receives when it is tensored down from $\mathbb{Z}$. Such a sheaf enjoys that the adjoint map

$$
\mathcal{F}(\varphi)^* \colon X(\varphi)^* \mathcal{F}[m] \to \mathcal{F}[n]
$$

  is an isomorphism, and in this case we say that $\mathcal{F}$ is *Cartesian*.

Lemma 36. *Without passing to the derived category, there is an equivalence of categories between Cartesian quasicoherent sheaves on the descent object and quasicoherent sheaves equipped with descent data.*   □

The real utility of this framework is that it pulls apart the question of descent into two distinct pieces, summarized in the following theorem:

Theorem 24. *Let $i \colon A \to X$ be a closed subscheme, and consider the formal completion*

$$A \xrightarrow{\quad i \quad} X$$

with j and k completing the diagram to $X_A^\wedge$.

If X is Noetherian, then k^* is flat as a functor of sheaves, j^* is conservative as a functor in the derived category of sheaves, and there is an equivalence of derived categories of sheaves over $X_A^\wedge$ and sheaves over the descent object $\mathcal{D}_{A \to X}$. ☐

Remark 28. The usual theorem about faithfully flat descent then follows by using the hypotheses on i to deduce that, e.g., if i^* and j^* are both conservative, then so must k^* be.

We now transfer what we've learned to the situation of homotopical algebra. Recalling that spectra are equivalent to $\mathbb{S}$–modules, $\mathbb{S}$ the usual sphere spectrum, then any other ring spectrum comes equipped with a unit map $\eta \colon \mathbb{S} \to E$ and hence push and pull functors

$$\eta_* \colon M \mapsto M, \qquad\qquad \eta^* \colon X \mapsto E \wedge X.$$

Correspondingly, to any spectrum X we can define the following cosimplicial spectrum:

Definition 43. Let $\mathcal{D}_E(X)$ be the cosimplicial spectrum determined by the formula

$$\mathcal{D}_E(X) := \left\{ \begin{array}{c} E \longrightarrow \\ E \longrightarrow \wedge \longleftarrow \\ E \xrightarrow{\eta_L} \wedge \longleftarrow E \longrightarrow \\ \wedge \xleftarrow{\mu} E \xrightarrow{\Delta} \wedge \longleftarrow \cdots \\ X \xrightarrow{\eta_R} \wedge \longleftarrow E \longrightarrow \\ X \longrightarrow \wedge \longleftarrow \\ X \longrightarrow \end{array} \right\}.$$

It is called *the descent object for X from E to $\mathbb{S}$*.

Lemma 37. *When E is an A_∞–ring spectrum, the descent object $\mathcal{D}_E(X)$ can be naturally considered as a cosimplicial object in the ∞–category of spectra.* ☐

Definition 44. Let E be an A_∞–ring spectrum. Then $X_E^\wedge := \operatorname{Tot} \mathcal{D}_E(X)$ is called the *E–nilpotent completion* of X. The spectral sequence resulting from the coskeletal filtration is called the *E–Adams spectral sequence* (for X).

It is not always the case that $X_E^\wedge$ can be lifted from a cosimplicial object in the homotopy category to a sufficiently structured cosimplicial object that we could take its totalization or homotopy colimit.

In general, it's quite rare that the E–nilpotent completion of a spectrum X recovers X, but in the nice cases we typically work in, it has been known to happen. In particular, there is the following theorem:

Lemma 38. *Let E be a connective A_∞ ring spectrum and let X be any connective spectrum. Then $X_E^\wedge$ is equivalent to the "$\pi_0 E$–localization" of X, i.e., for a prime p the spectrum $X_E^\wedge$ is p–local if $\pi_0 E$ is p–local, it is p–complete if $\pi_0 E$ is p–torsion, and otherwise it is just X.* $\square$

You really can just look at the Adams tower...

Proof (Proof sketch).

Finally, we can compare the topological situation with the algebraic situation. To have any hope of applying algebra and algebraic geometry, we must impose some nicety properties. Here is the first:

Definition 45. E satisfies **CH**, the **C**ommutativity **H**ypothesis, when $\pi_* E^{\wedge j}$ is commutative for all $j \geq 1$.

Definition 46. Suppose that E is a ring spectrum satisfying **CH**. We define a simplicial scheme associated to E, called its *context*, to be

$$\mathcal{M}_E := \operatorname{Spec} \pi_* \mathcal{D}_E(\mathbf{S})$$

$$= \left\{ \operatorname{Spec} \pi_* E \longrightarrow \operatorname{Spec} \pi_* \begin{pmatrix} E \\ \wedge \\ E \end{pmatrix} \rightleftarrows \operatorname{Spec} \pi_* \begin{pmatrix} E \\ \wedge \\ E \\ \wedge \\ E \end{pmatrix} \rightrightarrows \cdots \right\}.$$

The context is the wellspring of the algebraic category C dreamed of in the introduction to this lecture.

Definition 47. For a ring spectrum E satisfying **CH** and input spectrum X, we define the following diagram of abelian groups:

$$\Gamma \mathcal{M}_E(X) := \left\{ \pi_* \begin{pmatrix} E \\ \wedge \\ X \end{pmatrix} \rightleftarrows \pi_* \begin{pmatrix} E \\ \wedge \\ E \\ \wedge \\ X \end{pmatrix} \rightrightarrows \pi_* \begin{pmatrix} E \\ \wedge \\ E \\ \wedge \\ E \\ \wedge \\ X \end{pmatrix} \rightrightarrows \cdots \right\},$$

The jth object is a module for $\mathcal{O}(\mathcal{M}_E[j])$, and hence determines a quasicoherent sheaf over the scheme $\mathcal{M}_E[j]$. Suitably interpreted, the maps of abelian groups determine maps of pushforwards so that $\mathcal{M}_E(X)$ is a quasicoherent sheaf over the simplicial scheme $\mathcal{M}_E$.

There is also a common hypothesis on E that brings us back into the world of coalgebra, down from simplicial schemes.

Definition 48. Take E_*E to be an E_*–module using the left-unit map. We will say that E satisfies **FH**, the **F**latness **H**ypothesis, when the right-unit map $E_* \to E_*E$ is a flat map of E_*–modules.[4]

Remark 29. The main utility of this is that it obviates us from working through the homological algebra of sheaves over simplicial schemes. Instead, since **FH** causes $\mathcal{M}_E$ to become 1–truncated, we can refer to Remark 27 and simply refer back to the homological algebra of comodules. In light of the discussion in Examples 15 and 16, we also see an interpretation of these groupoid–valued simplicial schemes: they are valued in sets equipped with an action by $\operatorname{Spec} E_*E$, which acts also on the base $\operatorname{Spec} E_*$. To denote this "homotopical quotient" or "action groupoid", we will write

$$\operatorname{Spec} E_* /\!\!/ \operatorname{Spec} E_*E.$$

Such affine groupoid–valued schemes are themselves quite tangible: their rings of functions form *Hopf algebroids*, and Cartesian quasicoherent sheaves on the groupoid scheme correspond to comodules for the Hopf algebroid.

Cite me: A lot of this could use citation. Most of it is in Ravenel's appendix or Hovey's paper.

Remark 30. This homotopical perspective is quite useful — for instance, a map of groupoid–schemes which induces on points a natural weak equivalence of groupoids also induces an equivalence of comodule categories. In fact, the *derived* comodule category depends only upon the stack associated to the groupoid–scheme, which allows still more contexts to be identified. We won't need this observation in what's to come, though, and it introduces substantial technical distractions. **However, we *may* got sloppy and say "stack" from time to time.**

I'm still hazy over these two remarks. I understand what they are trying to say but I'm feeling hazy on the details. You should explain it to me sometime - d.s.

The standard Johnson–Wilson chart for $\mathcal{M}_{\mathbf{fg}}^{<d}$ is a standard example of a place where stackiness is actually relevant. Admit to this here and put a forward reference.

Example 25. Most of the homology theories we will discuss have this property. For an easy example, $H\mathbb{F}_2 P$ certainly has this property: there is only

Could also explain the difference: levelwise sheaves of 0–types vs sheaves of ∞–types.

This feigned sloppiness could also be ameliorated by referencing the equivalence between comodules over a Hopf algebroid and q.c. sheaves over the associated stack. This is in COCTALOS, for instance.

I think there is a notion of quasicoherent sheaf directly over $\mathcal{D}_E$ and an interpretation of Cartesian sheaves in that setting. I think that a different view on **FH** is that it causes the functor π_* to preserve Cartesianness.

---

[4] The essential point of this is that it causes $E_*E \otimes_{E_*} E_*X$ to become a homology theory and $E_*E \otimes_{E_*} E_*X \to (E \wedge E)_*X$ to become an isomorphism on a point. Alternatively, this can be viewed as a degeneration condition on the Künneth spectral sequence for $E_*(E \wedge E)$.

one possible algebraic map $\mathbb{F}_2 \to \mathcal{A}_*$, so **FH** is necessarily satisfied. This grants us access to a description of the context for $H\mathbb{F}_2$:

$$\mathcal{M}_{H\mathbb{F}_2 P} = \operatorname{Spec} F_2 /\!\!/ \underline{\operatorname{Aut}} \widehat{\mathbb{G}}_a.$$

Example 26. The context for *MUP* is considerably more complicated, but Quillen's theorem can be equivalently stated as giving a description of it. It is isomorphic to the moduli of formal groups:

$$\mathcal{M}_{MUP} \simeq \mathcal{M}_{\mathbf{fg}} := \mathcal{M}_{\mathbf{fgl}} /\!\!/ \mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}},$$

where $\mathcal{M}_{\mathbf{ps}} = \underline{\operatorname{End}}(\widehat{\mathbb{A}}^1)$ is the moduli of self-maps of the affine line (i.e., of power series) and $\mathcal{M}_{\mathbf{ps}}^{\mathrm{gpd}}$ is the multiplicative subgroup of invertible such maps.

Remark 31. If E is a complex-oriented ring spectrum, then the simplicial sheaf $\mathcal{M}_{MU}(E)$ has an extra degeneracy, which causes the MU–based Adams spectral sequence for E to degenerate. In this sense, the "stackiness" of $\mathcal{M}_{MU}(E)$ is a measure of the failure of E to be orientable.

3.2 The structure of $\mathcal{M}_{\mathbf{fg}}$ I: Distance from $\widehat{\mathbb{G}}_a$

Today, we will embark on this analysis by studying the scheme $\mathcal{M}_{\mathbf{fgl}}$ which naturally covers the stack $\mathcal{M}_{\mathbf{fg}}$.

Definition 49. There is an affine scheme $\mathcal{M}_{\mathbf{fgl}}$ classifying formal group laws. Begin with the scheme classifying all bivariate power series:

$$\operatorname{Spec} \mathbb{Z}[a_{ij} \mid i,j \geq 0] \leftrightarrow \{\text{bivariate power series}\},$$
$$f \in \operatorname{Spec} \mathbb{Z}[a_{ij} \mid i,j \geq 0](R) \leftrightarrow \sum_{i,j \geq 0} f(a_{ij}) x^i y^j.$$

Then, set $\mathcal{M}_{\mathbf{fgl}}$ to be the closed subscheme selected by the formal group law axioms in Definition 22.

This presentation of $\mathcal{M}_{\mathbf{fgl}}$ as a subscheme appears to be extremely complicated in that its ideal is generated by many hard-to-describe elements, but $\mathcal{M}_{\mathbf{fgl}}$ itself is actually not complicated at all. We will prove the following theorem:

Theorem 25 ([Laz55, Théorème II]). *There is a noncanonical isomorphism*

$$L_\infty := \mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}} \cong \mathbb{Z}[t_n \mid 1 \le n < \infty].\square$$

The most important consequence of this is *smoothness*:

Corollary 27. *Given a formal group law φ over a ring R and a surjective ring map $f\colon S \to R$, there exists a formal group law $\widetilde{\varphi}$ over S with*

$$\varphi = f^*\widetilde{\varphi}.\square$$

Remark 32. One might hope that the filtration above has an immediate geo- metric realization. After all, one can consider the nth order formal neighbor- hood $\widehat{\mathbb{A}}^{1,(n)}$ of Example 4. The appropriate analogue of Lemma 21 shows that a map

$$\widehat{\mathbb{A}}^{1,(n)} \times \widehat{\mathbb{A}}^{1,(n)} \to \widehat{\mathbb{A}}^{1,(n)}$$

is represented by a bivariate power series, *modulo the ideal* (x^{n+1}, y^{n+1}). This ideal is distinct from $(x, y)^{n+1}$, and so the source scheme of a formal n–bud is not the square of $\widehat{\mathbb{A}}^{1,(n)}$, and a formal n–bud does *not* determine a group object on some finite scheme. This is actually a good thing: there are struc- ture theorems preventing many of these intermediate group structures on finite schemes from existing.

> What filtration?

> Cite me: Akhil is who reminded me of this, back in Berkeley..

> There's some hidden text here about n– buds, but I don't think we ever care about it.

Proof (Proof of Theorem 25). Let $U = \mathbb{Z}[b_0, b_1, b_2, \ldots]/(b_0 - 1)$ be the universal ring supporting a "strict" exponential

$$\exp(x) := \sum_{j=0}^\infty b_j x^{j+1}$$

with compositional inverse

$$\log(x) := \sum_{j=0}^\infty m_j x^{j+1}.$$

They induce a formal group law on U by the formula

$$x +_U y = \exp(\log(x) + \log(y)),$$

classified by a map $u\colon L_\infty \to U$. Modulo decomposables, this map can be computed as

> Why are you writ- ing L_∞ instead of $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}}$?

$$u(a_{i(n-i)}) = \binom{n}{i} b_{n-1} \quad (\text{mod decomposables}).$$

Writing $d_n = \gcd\left(\binom{n}{i}\big|0 < k < n\right)$, the map Qu on degree $2n$ has image the subgroup generated by $d_{n+1}b_n$. We write T_{2n} for this subgroup. Using the splitting of Qu from Lemma 39.4 below, we use the freeness of U to *choose* an algebra splitting

$$U \xrightarrow{v} L_\infty \xrightarrow{u} U.$$

The map v is an isomorphism because uv is injective and because we have checked that v is surjective on indecomposables.

> This last sentence was a little quick for me. For example, I don't think you're "checking" that v is surjective but more observing that it is so by construction.

> Do the intermediate rings matter here?

Definition 50. In order to prove the missing Lemma 39, it will be useful to study the series $+_\varphi$ "up to degree n", i.e., modulo $(x, y)^{n+1}$. Such a truncated series satisfying the analogues of the formal group law axioms is called a *formal n–bud*. Additionally, a *symmetric 2–cocycle* is a symmetric polynomial $f(x, y)$ satisfying the equation

$$f(x, y) - f(t + x, y) + f(t, x + y) - f(t, x) = 0.$$

Lemma 39 (Symmetric 2–cocycle lemma (Part 1)). *The following are equivalent:*

1. *Symmetric 2–cocycles that are homogeneous polynomials of degree n are spanned by*
$$c_n = \frac{1}{d_n} \cdot ((x + y)^n - x^n - y^n).$$

2. *For F is an r–bud, the set of $(r + 1)$–buds extending F form a torsor under addition for $R_{2n-2} \otimes c_r$.*

> Things suddenly become graded here — and you really make use of this. Explain yourself.

3. *Any homomorphism $(QL)_{2n} \to A$ factors through the map $(QL)_{2n} \to T_{2n}$.*
4. *There is a canonical splitting $T_{2n} \to (QL)_{2n}$.*

Proof (Equivalences). Verifying that Claims 1 and 2 are equivalent is a matter of writing out the purported $(r + 1)$–buds and taking their difference. To see that Claim 2 is equivalent to Claim 3, follow the chain

> Do you mean write out a $(r + 1)$-bud and applying the associativity axiom?

$$\mathsf{Groups}((QL)_{2n}, A) \cong \mathsf{Rings}(\mathbb{Z} \oplus (QL)_{2n}, \mathbb{Z} \oplus \Sigma^{2n} A) \cong \mathsf{Rings}(L, \mathbb{Z} \oplus \Sigma^{2n} A).$$

This shows that such a homomorphism of groups determines an extension of the n–bud $\widehat{\mathbb{G}}_a$ to an $(n + 1)$–bud, which takes the form of a 2–cocycle with coefficients in A, and hence factors through T_{2n}. Finally, Claim 4 is the universal case of Claim 3.

> Stitch these together better.

Lemma 40 (Claim 1 of Lemma 39). *Symmetric 2–cocycles that are homogeneous polynomials of degree n are spanned by*

$$c_n = \frac{1}{d_n} \cdot ((x + y)^n - x^n - y^n).$$

Proof. It suffices to show the Lemma over a finitely generated ring. In fact, the Lemma is true for $A \oplus B$ if and only if it's true for A and for B, so the structure theorem for finitely generated abelian groups reduces to the cases of $\mathbb{Z}$ and $\mathbb{Z}/p^r$. If $A \subseteq B$ and the Lemma is true for B, it's true for A, so we can further reduce the $\mathbb{Z}$ case to $\mathbb{Q}$. We can also reduce from $\mathbb{Z}/p^r$ to $\mathbb{Z}/p$ using an inductive, Bockstein-style argument. Hence, we can now freely assume that our ground object is a prime field.

> **Cite me:** This follows Chapter 3 of COCTA-LOS.

> Rephrase this in terms of localizations.

For a formal group scheme $\widehat{\mathbb{G}}$, we can form a simplicial scheme $B\widehat{\mathbb{G}}$ in the usual way:

$$B\widehat{\mathbb{G}} := \left\{ \begin{array}{c} * \longleftarrow \\ * \longleftarrow \times \longrightarrow \\ * \longleftarrow \times \longrightarrow \widehat{\mathbb{G}} \longleftarrow \\ \times \longrightarrow \widehat{\mathbb{G}} \longleftarrow \times \longrightarrow \cdots \\ * \longleftarrow \times \longrightarrow \widehat{\mathbb{G}} \longleftarrow \\ * \longleftarrow \times \longrightarrow \\ * \longleftarrow \end{array} \right\}.$$

By applying the functor $\underline{\mathsf{FormalGroups}}(-, \widehat{\mathbb{G}}_a)(k)$, we get a cosimplicial abelian group, hence a cochain complex, of which we can take the cohomology. In the case $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$, the 2–cocycles in this cochain complex are *precisely* the things we've been calling 2–cocycles[5], so we are interested in computing H^2. The first observation in this direction is that $d^1(x^k) = d_k c_k$. Secondly, one may check that this complex also computes

$$\mathrm{Cotor}_{\mathcal{O}_{\widehat{\mathbb{G}}}^*}(k, k) \cong \mathrm{Ext}_{\mathcal{O}_{\widehat{\mathbb{G}}}^*}(k, k),$$

which we're now going to compute using a more efficient complex.

Q: There is a resolution

$$0 \to \mathbb{Q}[t] \xrightarrow{\cdot t} \mathbb{Q}[t] \to \mathbb{Q} \to 0,$$

from which this follows:

$$H^* \underline{\mathsf{FormalGroups}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{when } * = 0, \\ \mathbb{Q} & \text{when } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This means that every 2–cocycle is a coboundary, symmetric or not. $\mathbb{F}_p$: Again, we switch to working with Ext over a free divided power algebra. Such an algebra splits as a tensor of truncated polynomial algebras, and

---

[5] They aren't obligated to be symmetric, though.

again computing a minimal free resolution results in the calculation

$$H^* \underline{\mathsf{FormalGroups}}(B\widehat{\mathbb{G}}_a, \widehat{\mathbb{G}}_a)(\mathbb{F}_p) = \Lambda[\alpha_k \mid k \geq 0] \otimes \mathbb{F}_p[\beta_k \mid k \geq 0],$$

with $\alpha_k \in \mathrm{Ext}^1$ and $\beta_k \in \mathrm{Ext}^2$. In fact, α_k is represented by x^{p^k} and β_k is represented by $c_{p^k}(x, y)$, and in the case $p = 2$ the exceptional class α_{k-1}^2 is represented by $C_{2^k}(x, y)$. Since we have representatives for the surviving homology classes and we know where the bounding class lives, it follows that $c_n(x, y)$ and $x^{p^a} y^{p^b}$ give a basis for *all* of the 2–cocycles. It's easy to select the symmetric ones, and it agrees with the prediction of the statement of the Lemma.

> I'm not sure how to do any of these calculations! Ha.

This finally concludes the proof of Theorem 25.

> Also, people seem to say things about the Mischenko logarithm rather than the invariant differential, but I wonder if we should phrase things in those terms.

> Section 12 of Neil's FG notes talk about the infinite height subscheme of $\mathcal{M}_{\mathbf{fgl}}$. He compares it to $H_* MO$ and to the Hurewicz image $\pi_* MU \to H_*(MU; \mathbb{F}_p)$.

Having described the structure of $\mathcal{M}_{\mathbf{fgl}}$, we turn to understanding the quotient stack $\mathcal{M}_{\mathbf{fg}}$. Earlier, we proved the following theorem:

> Be careful about $\star$-isomorphisms versus isomorphisms.

Theorem 26. *Let k be any field of characteristic 0. Then there is a unique map*

$$\mathrm{Spec}\, k \to \mathcal{M}_{\mathbf{fg}}. \square$$

Proof. This is a rephrasing of Theorem 11 in the language of stacks.

We would like to have a similar classification of the closed points in positive characteristic. We proved the theorem above by solving a certain differential equation, which necessitated integrating a power series. Integration is what we expect to fail in positive characteristic. The following definition tracks *where* it fails:

Definition 51. Let $+_\varphi$ be a formal group law. Let n be the largest degree such that there exists a formal power series ℓ with

$$\ell(x +_\varphi y) = \ell(x) + \ell(y) \pmod{(x, y)^n},$$

> Should that be a $n + 1$?

i.e., ℓ is a logarithm for the n–bud determined by $+_\varphi$. The *p–height of* $+_\varphi$ is defined to be $\log_p(n)$.

We will show that this definition is well-behaved, in the following sense:

Lemma 41. *Over a field of positive characteristic p, the p–height of a formal group law is always an integer. (That is, $n = p^d$ for some natural number d.)*

We will have to develop some machinery to get there. First, notice that this definition of height really depends on the formal group rather than the formal group law.

Lemma 42. *The height of a formal group law is an isomorphism invariant, i.e., it descends to a function on $\mathcal{M}_{\mathbf{fg}}$.*

Proof. The series ℓ is a partial logarithm for the formal group law φ, i.e., an isomorphism between the formal group defined by φ and the additive group. Since isomorphisms compose, this statement follows.

With this in mind, we look for a more standard form for formal group laws, where Lemma 41 will hopefully be obvious. In light of our goal, the most obvious standard form is as follows:

Definition 52. Suppose that a formal group law $+_\varphi$ does have a logarithm. We say that $+_\varphi$ has a *p–typical logarithm* in the case that its logarithm has the form

$$\log_\varphi(x) = \sum_{j=0}^{\infty} \ell_j x^{p^j}.$$

Lemma 43. *Every formal group law $+_\varphi$ with a logarithm $\log_\varphi$ is naturally iso-morphic to one whose logarithm is p–typical, called the p–typification of $+_\varphi$.*

> **Cite me:** This is due to Hazewinkel.

Proof. Let $\widehat{\mathbb{G}}$ denote the formal group associated to $+_\varphi$, and consider its inherited coordinate

$$g_0 \colon \widehat{\mathbb{A}}^1 \xrightarrow{\cong} \widehat{\mathbb{G}},$$

so that

$$\log_\varphi = \log \circ g_0 = \sum_{n=1}^{\infty} a_n x^n.$$

Our goal is to perturb this coordinate to a new coordinate g which has the property that it couples with the logarithm

$$\widehat{\mathbb{A}}^1 \xrightarrow{g} \widehat{\mathbb{G}} \xrightarrow{\log} \widehat{\mathbb{G}}_a$$

to give a power series expansion

$$\log(g(x)) = \sum_{n=0}^{\infty} a_{p^n} x^{p^n}.$$

To do this, we introduce four operators on *curves*:

- Given $r \in R$, we can define a *homothety* by rescaling the coordinate by r:

$$\log(g(rx)) = \sum_{n=1}^{\infty} (a_n r^n) x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a shift operator (or *Verschiebung*) by

$$\log(V_\ell g(x)) = \log(g(x^\ell)) = \sum_{n=1}^{\infty} a_n x^{n\ell}.$$

- Given an $\ell \in \mathbb{Z}_{(p)}$, we define the ℓ–series by

$$\log([\ell](g(x))) = \ell \log(g(x)) = \sum_{n=1}^{\infty} \ell a_n x^n.$$

- For $\ell \in \mathbb{Z}$, we can define a *Frobenius operator* by

$$\log(F_\ell g(x)) = \log\left(\sum_{j=1}^{\ell}{}_{\widehat{G}} g(\zeta_\ell^j x^{1/\ell})\right) = \sum_{n=1}^{\infty} \ell a_{n\ell} x^n,$$

where ζ_ℓ is a primitive ℓ^{th} root of unity. Because this formula is Galois–invariant in choice of primitive root, it actually expands to a series which lies over the ground ring (without requiring an extension by ζ_ℓ). But, by pulling the logarithm through and noting

$$\sum_{j=1}^{\ell} \zeta_\ell^{jn} = \begin{cases} \ell & \text{if } \ell \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

we can explicitly compute the behavior of F_ℓ.[6]

Stringing these together, for $p \nmid \ell$ we have

$$\log([1/\ell]V_\ell F_\ell g(x)) = \sum_{n=1}^{\infty} a_{n\ell} x^{n\ell}.$$

Hence, we can consider the curve $g -_{\widehat{G}} \sum_{p \nmid \ell}[1/\ell]V_\ell F_\ell g$, which is another coordinate on the same formal group $\widehat{G}$, but with a p–typical logarithm.

Of course, not every formal group law supports a logarithm — after all, this is the point of "height". There are two ways around this: one is to pick a surjection $S \to R$ from a torsion-free ring S, *choose* a lift of the formal group law to S, then pass to $S \otimes \mathbb{Q}$ and study how much of the resulting logarithm descends to R. However, it is not clear that this procedure is independent of choice. We therefore pursue an alternative approach: an intermediate definition that applies to all formal group laws and which specializes to the one above in the presence of a logarithm. To do this, we consider what computations are made easier with this sort of formula for a logarithm, and we arrive at the following:

---

[6] The definition of Frobenius comes from applying the Verschiebung to the character group (or "Cartier dual") of $\widehat{G}$.

The sum is taken over all $\ell \in \mathbb{Z}$ not a multiple of p, using the formal group addition, right? In that case, aren't you subtracting too much? If n is not a power of p, then the logarithm of this expression is a_n times $1-$ the number of factors of n coprime to p. Also, why is this still an isomorphism?

Some of these gs should be g_0s.

Definition 53. The *p–series* of a formal group law $+_\varphi$ is given by the formula

$$[p]_\varphi(x) := \overbrace{x +_\varphi \cdots +_\varphi x}^{p \text{ times}}.$$

Lemma 44. *If $+_\varphi$ is a formal group law with p–typical logarithm, then there are elements v_n with*

$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots .$$

Proof (Proof sketch). This comes from comparing the two series

$$\log_\varphi(px) = px + \cdots ,$$
$$\log_\varphi([p]_\varphi(x)) = p \log_\varphi(x) = px + \cdots .$$

The difference is concentrated in degrees of the form p^d, beginning in degree p, so one can find an element v_1 so that

$$p \log_\varphi(x) - (\log_\varphi(px) + \log_\varphi(v_1 x^p))$$

starts in degree p^2, and so on. In all, this gives the equation

$$p \log_\varphi(x) = \log_\varphi(px) + \log_\varphi(v_1 x^p) + \log_\varphi(v_2 x^{p^2}) + \cdots$$

at which point we can use formal properties of the logarithm to deduce

$$\log_\varphi[p]_\varphi(x) = \log_\varphi \left(px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots \right)$$
$$[p]_\varphi(x) = px +_\varphi v_1 x^p +_\varphi v_2 x^{p^2} +_\varphi \cdots +_\varphi v_n x^{p^n} +_\varphi \cdots \square$$

Definition 54. A formal group law is itself said to be *p–typical* when its *p*–series has the above form. (In particular, the logarithm of a *p*–typical formal group law is a *p*–typical logarithm.)

Corollary 28 (Lemma 43). *Every formal group law is naturally isomorphic to a p–typical one.*

Proof. The procedure applied to the formal group law $+_\varphi$ in the proof of Lemma 43 applies equally well to an arbitrary formal group law, even without a logarithm — it just wasn't clear what was being gained. Now, it is clear: we are gaining the conclusion of this Corollary.

Remark 33. There is an inclusion of groupoid–valued sheaves from *p*–typical formal group laws with isomorphisms to all formal group laws with isomorphisms. Corollary 28 can be viewed as presenting this inclusion as a deformation retraction, and in particular the inclusion is a natural *equivalence* of

I'm still not sure about this. Lemma 3.3.7 doesn't say anything about *p*-series, and the previous two lemmas use the existence of a logarithm rather substantially.

Cite me: This is Lemma 6.6.3 in the crystals notes. Definitely copy this proof! It avoids invoking a logarithm, which you haven't managed to do otherwise. Additionally: this gives a proof that the *p*–series of a *p*–typical coordinate has the usual *p*–typical expression. To go the other way, a *p*–typical coordinate on a finite height formal group can't be killed by *p* (in the curves formulation of the Dieudonné module), and hence because its *p*–series is nonvanishing (and lies in $DG \subseteq C_pG \cong (DG)^{\times\infty}$) its *p*–division must as

groupoids. It follows that they both present the same stack: $\mathcal{M}_{\mathbf{fg}}$. In fact, the moduli of p–typical formal group laws is isomorphic to $\operatorname{Spec}\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ — every possible p–series is realized by a unique p–typical formal group law.

Remark 34. In fact, the rational logarithm coefficients can be recursively recovered from the coefficients v_d, using a similar manipulation:

$$p\log_\varphi(x) = \log_\varphi\left([p]_\varphi(x)\right)$$

$$p\sum_{n=0}^\infty m_n x^{p^n} = \log_\varphi\left(\sum_{d=0}^\infty {}_\varphi v_d x^{p^d}\right) = \sum_{d=0}^\infty \log_\varphi\left(v_d x^{p^d}\right)$$

$$\sum_{n=0}^\infty p m_n x^{p^n} = \sum_{d=0}^\infty \sum_{j=0}^\infty m_j v_d^{p^j} x^{p^{d+j}} = \sum_{n=0}^\infty \left(\sum_{k=0}^n m_k v_{n-k}^{p^k}\right) x^{p^n},$$

implicitly taking $m_0 = 1$ and $v_0 = p$.

Proof (Proof of Lemma 41). Replace the formal group law by its p–typification. Based on Remark 34, we see that the height of a p–typical formal group law over a field of characteristic p is coincides with the appearance of the first nonzero coefficient in its p–series.

3.3 The structure of $\mathcal{M}_{\mathbf{fg}}$ II: Large scales

With the notion of "height" firmly in hand, we are now in a position to classify the geometric points of $\mathcal{M}_{\mathbf{fg}}$.

Theorem 27 ([Laz55, Théorème IV]). *Let $\bar{k}$ be an algebraically closed field of positive characteristic p. There is a bijection between maps*

$$\Gamma : \operatorname{Spec}\bar{k} \to \mathcal{M}_{\mathbf{fg}}$$

and numbers $1 \le d \le \infty$ given by $\Gamma \mapsto \operatorname{ht}(\Gamma)$.

Proof. The easy part of the proof is surjectivity: recalling Remark 33, take the p–typical formal group law over $\mathbb{F}_p$ determined by the p–series $[p]_{\varphi_d}(x) = x^{p^d}$, sometimes called the *Honda formal group law*.

To show injectivity, we must show that every p–typical formal group law φ over $\bar{k}$ is isomorphic to the appropriate Honda group law. Suppose that the p–series for φ begins

$$[p]_\varphi(x) = x^{p^d} + ax^{p^{d+k}} + \cdots.$$

Then, we will construct a coordinate transformation $g(x) = \sum_{j=1}^\infty b_j x^j$ satisfying

$$g(x^{p^d}) \equiv [p]_\varphi(g(x)) \qquad\qquad (\text{mod } x^{p^{d+k}+1})$$

$$\sum_{j=1}^\infty b_j x^{jp^d} \equiv \sum_{j=1}^\infty b_j^{p^d} x^{jp^d} + \sum_{j=1}^\infty ab_j^{p^{d+k}} x^{jp^{d+k}} \qquad (\text{mod } x^{p^{d+k}+1}).$$

For g to be a coordinate transformation, we must have $b_1 = 1$, which in the critical degree $x^{p^{d+k}}$ forces the relation

$$b_{p^k} = b_{p^k}^{p^d} + a.$$

Since $\bar{k}$ is algebraically closed, this relation is solvable, and the coordinate can be perturbed so that the term $x^{p^{d+k}}$ does not appear in the p–series. If we set the earlier terms in the series to be 0, then we can induct on d.

Remark 35. From this, it follows that the "coarse moduli of formal groups" — i.e., the functor from rings to isomorphism classes of formal groups over that ring — is not representable by a scheme. The infinitely many isomorphism classes over $\mathrm{Spec}\,\mathbb{F}_p$ produce infinitely many over $\mathrm{Spec}\,\mathbb{Z}$ as well. On the other hand, there is a single $\mathbb{Q}$–valued point of the coarse moduli, whereas the $\mathbb{Z}$–points of a representable functor would inject into its $\mathbb{Q}$–points.

We now turn to the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, which also admit a reasonable presentation in terms of height.

Lemma 45 ([Wil82, Theorem 4.6]). *Recall that the moduli scheme of p–typical formal group laws is presented as*

$$\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} = \mathrm{Spec}\,\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots].$$

Suppose $g(x) = \sum_{j=0}^\infty {}_L t_j x^{p^j}$ is the universal p-typical coordinate transformation, which we can use to conjugate the universal group law "$+_L$" to a second p–typical group law "$+_R$", whose p–series has the form

$$[p]_R(x) = \sum_{d=0}^\infty {}_R \eta_R(v_d) x^{p^d}.$$

Modulo p, there is the relation:

$$\sum_{\substack{i \geq 0 \\ j > 0}} {}_L t_i \eta_R(v_j)^{p^i} \equiv \sum_{\substack{i > 0 \\ j \geq 0}} {}_L v_i t_j^{p^i} \qquad (\text{mod } p).$$

Proof (Proof sketch). Work modulo p, one can Freshman's Dream the identity $[p]_L(g(x)) = g([p]_R(x))$ to death.

Corollary 29 ([Wil82, Lemmas 4.7-8]). *Write I_d for the ideal $I_d = (p, v_1, \ldots, v_{d-1})$. Then*

$$\eta_R(v_d) \equiv v_d \pmod{I_d}.$$

It follows that the ideals I_d are invariant for all d. □

What is *much* harder to prove is the following:

Theorem 28 ([Wil82, Theorem 4.9]). *If I is an invariant prime ideal, then $I = I_d$ for some d.*

Proof (Proof sketch). Inductively assume that $I_d \subseteq I$. If this is not an equality, we want to show that $I_{d+1} \subseteq I$ is forced. Take $y \in I \setminus I_d$; if we could show

$$\eta_R(y) = a v_d^j t^K + \cdots,$$

we could proceed by primality to show that $v_d \in I$ and hence $I_{d+1} \subseteq I$. This is possible (and, indeed, this is how the proof goes), but it requires serious bookkeeping.

The equivalent statement in terms of stacks is:

Theorem 29 (Landweber). *The unique closed substack of $\mathcal{M}_{\mathbf{fg},(p)} := \mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ of codimension d is selected by $\mathcal{O}_{\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}}} / (p, v_1, \ldots, v_{d-1})$.* □

Remark 36. The complementary open substack of dimension d is harder to describe. From first principles, we can say only that it is the locus where the coordinate functions $p, v_1, \ldots, v_d$ do not *all simultaneously vanish*. It turns out that:

1. On a cover, at least one of these coordinates can be taken to be invertible.
2. Once one of them is invertible, a coordinate change on the formal group law can be used to make v_d (and perhaps others in the list) invertible. Hence, we can use $v_d^{-1} \mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}}$ as a coordinate chart.
3. Over a further base extension and a further coordinate change, the higher coefficients v_{d+k} can be taken to be zero. Hence, we can also use $v_d^{-1} \mathbb{Z}_{(p)}[v_1, \ldots, v_d]$ as a coordinate chart.

Margin notes:

To understand where the η_R comes from in the formal below it's the best to mention that the two formal group laws coming from $\eta_L, \eta_R : BP_* \to BP_* BP$ are isomorphic. And the formula for this isomorphism is the one in the lemma below. I don't think we've mentioned BP yet, though. But I guess $\mathcal{M}_{BPP}$ is the moduli of p-typical formal group laws mentioned above, so we are very close to introducing BP. I have part of this written up about the Hopf algebroid $(BP_*, BP_* BP)$. I am wary of including it and risk botching up the big picture. -d.s.

Under η_R?

I'm gonna try tidying this proof up a bit. -d.s.

How does the base case go?

I can't extrapolate this series just from one term, but I'm guessing the rest obviously lives in I_d? Also, what is a and t and why can't they be in I instead of v_d?

Cite me: Landweber must have a paper?.

This actually uses the Zariski topology on the affine site, and hence may really use stackiness instead of levelwise schemei-ness. This is a problem, since you just told your students that stackiness won't come up...

What is the (global sections of the) sheaf of rings on a simplicial scheme (stack)? If you take sheaves of rings, and then global sections, levelwise, you end up with a simplicial ring, not a ring. -EB

Hood wanted to know: What, exactly, is required here?

I'm not sure where this goes — maybe it goes right here — but Allen told me how to give a nice proof of this fact. He didn't give me a citation for this, but the main point was that flatness can be checked locally, so the general formula for the pullback $\mathcal{M}_{(A_1, \Gamma_1)} \to \mathcal{M}_{(L,W)} \leftarrow \mathcal{M}_{(A_2, \Gamma_2)}$ being $\mathcal{M}_{(A_1 \otimes W \otimes A_2, \Gamma_1 \otimes W \otimes \Gamma_2)}$ specializes to compute the pullback of $\operatorname{Spec} A \to \mathcal{M}_{(L,W)} \leftarrow \operatorname{Spec} E(d)_*$ to be $\mathcal{M}_{(A \otimes W \otimes E(d)_*, A \otimes W \otimes E(d)_*)} = \operatorname{Spec}(A \otimes W \otimes E(d)_*)$. But the map $\operatorname{Spec} A \to \mathcal{M}_{(L,W)}$ factors through $\operatorname{Spec} L$ (*hold on*), so it suffices just to check flatness for $\operatorname{Spec} L \otimes_L W \otimes_L E(d)_* \cong \operatorname{Spec} W \otimes_L E(d)_* \to \operatorname{Spec} E(d)_*$, which you finally do by hand. For (*hold on*), Lurie gives a nice proof at the end of Lecture 11 (internal to an actual discussion of representability and stacks) of this factorization Lemma, where he uses an iterated decomposition of $\mathcal{M}_{\mathbf{ps}}^{\mathbf{gpd}}$ into extensions of $\mathbb{G}_m$ by $\mathbb{G}_a$s, and noting that $H^*(\operatorname{Spec} R; \mathbb{G}_a)$ vanishes for $* > 0$.

We can now rephrase Theorem 22 in terms of algebraic conditions.

Theorem 30 (Landweber, cf. Theorem 22, see also [Hop, Theorem 21.4 and Proposition 21.5]). *Let M be a module over*

$$\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}} \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_d, \ldots].$$

If $(p, v_1, \ldots, v_d, \ldots)$ forms an infinite regular sequence on M, then

$$X \mapsto M \otimes_{\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}^{p\text{-typ}}}} MU_0(X)$$

determines a homology theory on finite spectra X. Moreover, if $M/I_d = 0$ for some $d \gg 0$, then the same formula determines a homology theory on all spectra X.

Proof. This is a direct consequence of the classification of closed substacks of $\mathcal{M}_{\mathbf{fg},(p)}$ in Theorem 29. Specifically, M determines a flat quasicoherent sheaf on $\mathcal{M}_{\mathbf{fg},(p)}$ when $\mathrm{Tor}_1(M, N) = 0$ for any other comodule N. Using the classification of closed substacks and the regularity condition, one can iteratively use the short exact sequences

$$0 \to M/I_d \xrightarrow{\cdot v_d} M/I_d \to M/I_{d+1} \to 0$$

to trade this condition for the list of conditions

- $\mathrm{Tor}_1(p^{-1}M, N) = 0$.
- $\mathrm{Tor}_2(v_1^{-1}M/p, N) = 0$.
- ...
- $\mathrm{Tor}_d(v_{d-1}^{-1}M/I_{d-1}, N) = 0$.
- $\mathrm{Tor}_{d+1}(M/I_d, N) = 0$.

for any d. If N is coherent, as in the case $N = MU_*(X)$ for a finite spectrum X, then this final condition is satisfied automatically for $d \gg 0$. (Alternatively, we can assume that M eventually satisfies this condition on its own.) By observing the length of the Koszul resolution associated to the cover $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$, one finally sees that

$$\mathrm{Tor}_d(v_{d-1}^{-1}M/I_{d-1}, N) = 0$$

is satisfied for *any* quasicoherent sheaf.

Remark 37. It's worth pointing out how strange all of this is. In Euclidean geometry, open subspaces are always top-dimensional, and closed subspaces can drop dimension. Here, proper open substacks of every dimension appear, and every nonempty closed substack is ∞–dimensional (albeit of positive codimension).

3.4 The structure of $\mathcal{M}_{\mathbf{fg}}$ III: Small scales

We now turn to the deformation theory of formal groups, which is about the appearance of formal groups in families. Specifically, following **??** we will be interested in infinitesimal deformations of formal groups over fields of positive characteristic.

Definition 55. Given a formal group Γ classified by a map $\operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$, then a *deformation of Γ to a scheme X* is a factorization

$$\operatorname{Spec} k \to X \to \mathcal{M}_{\mathbf{fg}}.$$

If X is a nilpotent thickening of $\operatorname{Spec} k$ (or an ind-system of such), then the deformation is said to be *infinitesimal*.

The study of all possible infinitesimal deformations of a particular map $\operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ has a geometric interpretation, embodied by the following Lemma:

Lemma 46. *Let $\operatorname{Spec} k \to Y$ be any map, and let $\operatorname{Spec} k \to X \to Y$ be a factorization through a nilpotent thickening X of $\operatorname{Spec} k$. Then there is a natural further factorization*

$$\operatorname{Spec} k \to X \dashrightarrow Y_X^\wedge \to Y. \square$$

The spirit of the Lemma, then, is that the study of infinitesimal deformations of $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ is equivalent to the study of $(\mathcal{M}_{\mathbf{fg}})_\Gamma^\wedge$ itself. So, this fits into our program of analyzing the (local) structure of $\mathcal{M}_{\mathbf{fg}}$.

Example 27. It's also helpful to expand what an infinitesimal deformation is in our case of interest. Set $Y = \mathcal{M}_{\mathbf{fg}}$, and fix a map $\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathbf{fg}}$ classifying a formal group Γ over $\operatorname{Spec} k$. Let S be a local ring with maximal ideal $\mathfrak{m}$ so that S is a nilpotent thickening of $S/\mathfrak{m}$. A deformation of Γ to S is the data of a formal group $\widehat{\mathbb{G}}$ over $\operatorname{Spec} S$, an identification $i\colon \operatorname{Spec} k \to \operatorname{Spec} S/\mathfrak{m}$, and a choice of an isomorphism f fitting together into the following diagram:

$$
\begin{array}{ccc}
\Gamma \xrightarrow[\simeq]{f} i^*j^*\widehat{\mathbb{G}} & \longrightarrow & \widehat{\mathbb{G}} \\
\downarrow & & \downarrow \\
\operatorname{Spec} k \xrightarrow{i} \operatorname{Spec} S/\mathfrak{m} & \xrightarrow{j} & \operatorname{Spec} S.
\end{array}
$$

Example 28. Consider the case of an infinitesimal parameter space $X = \widehat{\mathbb{A}}^1$. A map $\widehat{\mathbb{A}}^1 \to \mathcal{M}_{\mathbf{fg}}$ can be presented by a map $\widehat{\mathbb{A}}^1 \to \mathcal{M}_{\mathbf{fgl}}$, which corresponds to a "family" of formal group laws $+_{\varphi_h}$ of the form

$$x +_{\varphi_h} y = (x +_\varphi y) + h(x +_{\varphi(1)} y) + h^2(x +_{\varphi(2)} y) + \cdots$$

for some series $+_{\varphi(n)}$. In particular, $+_{\varphi(0)}$ is a formal group law over k.

The analysis of $(\mathcal{M}_{\mathbf{fg}})^\wedge_\Gamma$ is due to Lubin and Tate, but we first follow a more structured approach written down by Lazarev.

Cite me: Cite both of these.

Definition 56. Let $+_\varphi$ be a formal group law over R, and let M be an R–module. The deformation complex $\widehat{C}^*(\varphi; M)$ is defined by

Can this be phrased geometrically?

$$M \to M[\![x_1]\!] \to M[\![x_1, x_2]\!] \to M[\![x_1, x_2, x_3]\!] \to \cdots$$

with differential

$$
\begin{aligned}
(df)(x_1, \ldots, x_{n+1}) = &\; \varphi_1\left(\sum_{i=1}^n {}_\varphi x_i, x_{n+1}\right) \cdot f(x_1, \ldots, x_n) \\
&+ \sum(-1)^i f(x_1, \ldots, x_i +_\varphi x_{i+1}, \ldots, x_{n+1}) \\
&+ (-1)^{n+1}\left(\varphi_2\left(x_1, \sum_{i=2}^{n+1} {}_\varphi x_i\right) \cdot f(x_2, \ldots, x_{n+1})\right),
\end{aligned}
$$

where we have written

$$\varphi_1(x, y) = \frac{\partial(x +_\varphi y)}{\partial x}, \qquad\qquad \varphi_2(x, y) = \frac{\partial(x +_\varphi y)}{\partial y}.$$

This complex tracks the data of infinitesimal deformations. For instance, consider a deformed automorphism f of $+_\varphi$, expressed as

$$f(x) = f_0(x) + h f_1(x) + h^2 f_2(x) + \cdots,$$

and satisfying

$$f(x +_\varphi y) = f(x) +_\varphi f(y).$$

Applying $\left.\frac{\partial}{\partial h}\right|_{h=0}$ to this equality gives

$$f_1(x +_\varphi y) = \varphi_1(x, y) f_1(x) + \varphi_2(x, y) f_1(y)$$

and thus f_1 is a 1–cocycle in the deformation complex. A similar sequence of observations culminates in the following theorem:

Why is this $\varphi_1(x, y)$ and not $\varphi_1(f_0(x), f_0(y))$? Is it the point that $f(x) \equiv x$ mod h?

Theorem 31 ([Laz97, p. 1320]). *Let $+_\varphi$ be a formal group law over a ring R and let $S \to R$ be a square–zero extension with kernel M.*

1. *Automorphisms of $+_\varphi$ over S covering the identity on R correspond to elements in $\widehat{Z}^1(\varphi; M)$.*
2. *Extensions of $+_\varphi$ to S correspond to elements in $\widehat{Z}^2(\varphi; M)$.*
3. *Two such extensions are isomorphic as formal group laws over S if their cocycles differ by an element in $\widehat{B}^2(\varphi; M)$.* □

So, this complex contains all the information we're interested in. Miraculously, we actually already studied the main input to computing this complex yesterday:

Lemma 47 ([Laz97, p. 1320]). *This is quasi-isomorphic to the usual bar complex:*

$$\widehat{C}^*(\varphi; M) \to \mathsf{FormalSchemes}(B\widehat{\mathbb{G}}_\varphi, M \otimes \widehat{\mathbb{G}}_a)$$

$$f \mapsto \varphi_1 \left(0, \sum_{i=1}^n {}_\varphi x_i \right)^{-1} f(x_1, \dots, x_n). \square$$

[margin note: What is $\widehat{\mathbb{G}}_\varphi$? Also, you stop using $\widehat{\mathbb{G}}_\varphi$ below.]

Of course, yesterday we computed the specific case of $\widehat{\mathbb{G}} = \widehat{\mathbb{G}}_a$. However, by filtering the multiplication on $\widehat{\mathbb{G}}$ by degree, we can use this specific calculation to get up to the general one.

Lemma 48. *Let $\widehat{\mathbb{G}}$ be a formal group of finite height d over a field k. Then the group $H^2(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$ classifying isomorphism classes of deformations is a free k–vector space of dimension $(d-1)$.*

[margin note: d or $(d-1)$? There's β_0 through $\beta_{d-1}\cdots$]

Proof (Proof (after Hopkins)). Using p–typification, we select a coordinate on $\widehat{\mathbb{G}}$ of the form

$$x +_\varphi y = x + y + (\mathsf{unit})c_{p^d} + \cdots.$$

Then, filter $\widehat{\mathbb{G}}$ by degree and consider the resulting spectral sequence of signature

$$H^*(\widehat{\mathbb{G}}_a; M \otimes \widehat{\mathbb{G}}_a) \cong M \otimes (\Lambda_k[\alpha_j \mid j \geq 0] \otimes k[\beta_j \mid j \geq 0]) \Rightarrow H^*(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a).$$

To compute the differentials in this spectral sequence, one computes by hand the formula for the differential in the bar complex, working up to lowest visible degree. In order to compute

$$(x +_\varphi y)^{p^r} - (x^{p^r} + y^{p^r}) = (\mathsf{unit}) \cdot c_{p^{d+r}}(x, y) + \cdots,$$

[margin note: This last "sentence" is missing a few words. Also, maybe you should remind us what α_j and β_j are in terms of x and y.]

where we used $c_{p^d}^{p^r} = c_{p^{r+d}}$. So, we see that there are $d-1$ things at the bottom of the spectral sequence which are not coboundaries, and we need to check that they are indeed permanent cocycles. To do this, we need only show that they are realized by deformations, which Lubin and Tate accomplish in Lemma 49.

Lemma 49 ([LT66, Proposition 1.1]). *Let W be a local ring with residue field k, and let φ be a group law of height d on k. There is a group law $\tilde{\varphi}$ over $W[\![u_1, \ldots, u_{d-1}]\!]$ restricting to φ on k such that for some $j \geq 1$,*

$$x +_{\tilde{\varphi}} y \equiv x + y + u_j c_{p^j}(x, y) \pmod{u_1, \ldots, u_{j-1}, (x,y)^{p^j+1}}. \;\square$$

Picking $W = \mathbb{W}_p(k)$ to be the ring of Witt vectors, Lemma 49 produces the universal example of a deformation of a group law φ to $\tilde{\varphi}$.

Theorem 32. *Let $\mathrm{Spf}\, R$ be an infinitesimal deformation of its residue field $\mathrm{Spec}\, k$. For each lift of φ to ψ over $\mathrm{Spf}\, R$, there is a unique homomorphism*

$$\alpha \in \mathsf{FormalSchemes}(\mathrm{Spf}\, R, \mathrm{Spf}\, \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!])$$

with $\alpha^* \tilde{\varphi}$ uniquely strictly isomorphic to ψ.

Proof. We will prove this inductively on the neighborhoods of $\mathrm{Spec}\, k = \mathrm{Spec}\, R/I$ in $\mathrm{Spf}\, R$. Suppose that we have demonstrated the Theorem for $\psi_{r-1} = R/I^{r-1} \otimes \psi$, so that there is a map $\alpha_{r-1}\colon \mathbb{W}_p(k)[\![u_1, \ldots, u_{d-1}]\!] \to R/I^{r-1}$ and a strict isomorphism $g_{r-1}\colon \psi_{r-1} \to \alpha_{r-1}^* \tilde{\varphi}$. The exact sequence

$$0 \to I^{r-1}/I^r \to R/I^r \to R/I^{r-1} \to 0$$

exhibits R/I^r as a square–zero extension of R/I^{r-1} by $M = I^{r-1}/I^r$.

Let β be *any* lift of α_{r-1} and h be *any* lift of g_{r-1} to R/I^r, and let A and B be the induced group laws

$$x +_A y = \beta^* \tilde{\varphi}, \qquad x +_B y = h\left(h^{-1}(x) +_{\psi_r} h^{-1}(y)\right).$$

Since these both deform the group law ψ_{r-1}, by Lemma 48 there exist $m_j \in M$ and $f(x) \in M[\![x]\!]$ satisfying

$$(x +_B y) - (x +_A y) = (df)(x,y) + \sum_{j=1}^{d-1} m_j v_j(x,y),$$

where $v_j(x,y)$ is the 2–cocycle associated to the cohomology 2–class β_j. The following definitions complete the induction:

$$g_r(x) = h(x) - f(x), \qquad \alpha_r(u_j) = \beta(u_j) + m_j. \;\square$$

Remark 38. Our calculation $H^1(\widehat{\mathbb{G}}_\varphi; M \otimes \widehat{\mathbb{G}}_a)$ also shows that there are *no* automorphisms of the formal group Γ over the special fiber which induce automorphisms of the universal deformation. Specifically, *any* deformation of a nontrivial automorphism of Γ acts nontrivially on Lubin–Tate space by permuting the deformations living over the various fibers. A consequence

of this observation is that the deformation space produced in Theorem 32 is a *formal scheme*, carrying only the previously-known inertial group of $\operatorname{Aut}\Gamma$ at the special fiber, rather than a full-on stack.

Remark 39. We also see that our analysis fails wildly for the case $\Gamma = \widehat{\mathbb{G}}_a$. The differential calculation in Lemma 48 are meant to give us an upper bound on the dimensions of $H^1(\Gamma; \widehat{\mathbb{G}}_a)$ and $H^2(\Gamma; \widehat{\mathbb{G}}_a)$, but this family of differentials is zero in the additive case. Accordingly, both of these vector spaces are infinite dimensional — the infinitesimal

Cite me: Neil's FG notes in the first half of section 18 talk about additive extensions and their relation to infinitesimal deformations. In the second half, he (more or less) talks about the de Rham crystal and shows that $\operatorname{Ext}_{\mathrm{rigid}}(G, \widehat{\mathbb{G}}_a) \cong \operatorname{Prim}(H^1_{dR}(G/X))$ in 18.37..

I still have some confusion about the formal similarity between deforming formal group laws over square-zero extensions of the base and deforming formal n–buds over the finite order nilpotent neighborhoods of a point. This would be a good place to sort that out.

Having accomplished all our major goals, we close our algebraic analysis of $\mathcal{M}_{\mathbf{fg}}$ with a diagram summarizing our results.

PICTURE GOES HERE.

3.5 Spectra detecting nilpotence

We have now arrived at the conclusion of our program from **??** for manufacturing interesting homology theories from Quillen's theorem: we have an ample supply of open and closed substacks of $\mathcal{M}_{\mathbf{fg}}$, and we have analyzed its geometric points as well as their deformation neighborhoods.

Definition 57. We define the following "chromatic" homology theories:

- Recall that the moduli of p–typical group laws is affine, presented by the scheme $\operatorname{Spec} BPP_0$, $BPP_0 := \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$. Since the inclusion of p–typical group laws into all group laws induces an equivalence of stacks, it follows that this formula determines a homology theory on finite spectra, called *Brown–Peterson homology*:

$$BPP_0(X) := MUP_0(X) \otimes_{MUP_0} BPP_0.$$

- A chart for the open substack $\mathcal{M}_{\mathbf{fg}}^{\leq d}$ in terms of $\mathcal{M}_{\mathbf{fgl}}^{p\text{-typ}} \cong \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ is $\operatorname{Spec} E(d)P_0 := \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d^{\pm}]$. It follows that there is a homology theory $E(d)P$, called *the dth Johnson–Wilson homology*, defined on all spectra by

$$E(d)P_0(X) := MUP_0(X) \otimes_{MUP_0} E(d)P_0.$$

- Similarly, for a formal group Γ of height $d < \infty$, there is a chart $\operatorname{Spf} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!]$ for its deformation neighborhood. Correspondingly, there is a homology theory E_Γ, called *the (discontinuous) Morava E–theory for Γ*, determined by

$$E_{\Gamma 0}(X) := MUP_0(X) \otimes_{MUP_0} \mathbb{Z}_p[\![u_1, \ldots, u_{d-1}]\!].$$

> Why is this not written as $E_\Gamma P_0$?

- Since $(p, u_1, \ldots, u_{d-1})$ forms a regular sequence on $E_{\Gamma *}$, we can form the regular quotient K_Γ in the homotopy category. This determines a spectrum, and hence determines a homology theory called *the Morava K–theory for Γ*. In the case where Γ comes from the Honda p–typical formal group law (of height d), this spectrum is often written as $K(d)$. As an edge case, we also set $K(\infty) = H\mathbb{F}_p$.[7]
- More delicately, there is a version of Morava E–theory which takes into account the formal topology on $(\mathcal{M}_{\mathbf{fg}})_\Gamma^\wedge$, called *continuous Morava E–theory*. It is defined by the pro-system $\{E_\Gamma(X)/u^I\}$, where I ranges over multi-indices and the quotient is taken in the "homotopical sense", i.e.,

> Can you be more precise about this?

---

[7] By Theorem 27, it often suffices to consider just these spectra to make statements about all K_Γ. With more care, it even often suffices to consider formal groups Γ of finite height.

u_j–torsion elements contribute to the odd-degree homotopy of the quotient.

- There is also a homology theory associated to the closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d}$. Since $I_d = (p, v_1, \ldots, v_{d-1})$ is generated by a regular sequence on BPP_0, we can directly define the spectrum $P(d)P$ by a regular quotient:

$$P(d)P = BP/(p, v_1, \ldots, v_{d-1}).$$

This spectrum does have the property $P(d)P_0 = BPP_0/I_d$, but it is *only* the case that $P(d)P_0 = BPP_0(X)/I_d$ when I_d forms a regular sequence on $BPP_0(X)$ — which is reasonably rare among the cases of interest.

Remark 40 ([KLW04, Section 5.2], [Str99a, Theorem 2.13]). Morava K–theory at the even prime is not commutative. Instead, there is a derivation $Q_d : K(d) \to \Sigma K(d)$ satisfying

$$ab - ba = uQ_d(a)Q_d(b).$$

In particular, $K(d)^*X$ is a commutative ring whenever $K(d)^1X = 0$.

Having constructed these "stalk" homology theories, I want to show that you can actually perform stalkwise analyses of the sheaves coming from bordism theory. Our example case is a famous theorem: the solution of Ravenel's nilpotence conjectures by Devinatz, Hopkins, and Smith. Their theorem concerns spectra which "detect nilpotence" in the following sense:

All the stuff after this point is written graded-ly. I guess we still haven't decided whether this is the right presentation.

Definition 58. A ring spectrum *E detects nilpotence* if, for any ring spectrum R, the kernel of the Hurewicz homomorphism $E_* : \pi_*R \to E_*R$ consists of nilpotent elements.

Cite me: Hopkins–Smith, or maybe the intro to D–H–S.

First, a word about why one would care about such a condition. The following theorem is classical:

Theorem 33 (Nishida). *Every homotopy class $\alpha \in \pi_{\geq 1}S$ is nilpotent.* $\square$

Cite me: Nishida.

However, people studying K–theory in the '70s discovered the following phenomenon:

Theorem 34 (Adams). *Let $M_{2n}(p)$ denote the mod–p Moore spectrum with bottom cell in degree $2n$. Then there is an index n and a map $v : M_{2n}(p) \to M_0(p)$ such that KU_*v acts by multiplication by the nth power of the Bott class. The minimal such n is given by the formula*

$$n = \begin{cases} p - 1 & \text{when } p \geq 3, \\ 4 & \text{when } p = 2. \end{cases} \square$$

Cite me: Adams.

In particular, this means that v cannot be nilpotent, since a null-homotopic map induces the zero map in any homology theory. Just as we took the non-nilpotent endomorphism p in $\pi_0 \operatorname{End} \mathbb{S}$ and coned it off, we can take the endomorphism v in $\pi_{2p-2} \operatorname{End} M_0(p)$ and cone it off to form a new spectrum called $V(1)$.[8]

One can ask, then, whether the pattern continues: does $V(1)$ have a non-nilpotent self-map, and can we cone it off to form a new such spectrum with a new such map? Can we then do that again, indefinitely? In order to study this question, we are motivated to find spectra E as above, since an E that detects nilpotence cannot send such a nontrivial self-map to zero. In fact, we found one such E already:

Theorem 35 ([DHS88, Theorem 1.i]). *Complex cobordism MU detects nilpotence.* $\square$

They also show that the MU is the universal object which detects nilpotence, in the sense that any other ring spectrum can have this property checked stalkwise on $\mathcal{M}_{MU}$:

Corollary 30 ([HS98, Theorem 3]). *A ring spectrum E detects nilpotence if and only if for all $0 \leq d \leq \infty$ and for all primes p, $K(d)_* E \neq 0$.*

Proof. If $K(d)_* E = 0$ for some d, then the non-nilpotent unit map $\mathbb{S} \to K(d)$ lies in the kernel of the Hurewicz homomorphism for E, so E fails to detect nilpotence.

Hence, for any d we must have $K(d)_* E \neq 0$. Because $K(d)_*$ is a field, it follows by picking a basis of $K(d)_* E$ that $K(d) \wedge E$ is a nonempty wedge of suspensions of $K(d)$. So, for $\alpha \in \pi_* R$, if $E_* \alpha = 0$ then $(K(d) \wedge E)_* \alpha = 0$ and hence $K(d)_* \alpha = 0$. So, we need to show that if $K(d)_* \alpha = 0$ for all n and all p then α is nilpotent. Taking Theorem 35 as given, it would suffice to show merely that $MU_* \alpha$ is nilpotent. This is equivalent to showing that the ring spectrum $MU \wedge R[\alpha^{-1}]$ is contractible or that the unit map is null:

$$\mathbb{S} \to MU \wedge R[\alpha^{-1}].$$

A nontrivial result of Johnson and Wilson shows that if $MU_* X = 0$ for any X, then for any d we have $K([0, d])_* X = 0$ and $P(d+1)_* X = 0$. (Specifically, it is immediate that $MU_* X = 0$ forces $P(d+1)_* X = 0$ and $v_{d'}^{-1} P(d')_*(X) = 0$ for all $d' < d$. What's nontrivial is showing that $v_{d'}^{-1} P(d')_*(X) = 0$ if and only if $K(d')_*(X) = 0$ [Rav84, Theorem 2.1.a], [JW75, Section 3].) Taking $X = R[\alpha^{-1}]$, we have assumed all of these

---

[8] The spectrum $V(1)$ is actually defined to be a finite spectrum with $BP_* V(1) \cong BP_*/(p, v_1)$. At $p = 2$ this spectrum doesn't exist and this is a misnomer. More generally, at odd primes p Nave shows that $V((p+1)/2)$ doesn't exist [Nav10, Theorem 1.3].

are zero except for $P(d+1)$. But $\text{colim}_d\, P(d+1) \simeq H\mathbb{F}_p \simeq K(\infty)$, and $\mathbb{S} \to K(\infty) \wedge R[\alpha^{-1}]$ is assumed to be null as well. By compactness of $\mathbb{S}$, that null-homotopy factors through some finite stage $P(d+1) \wedge R[\alpha^{-1}]$ with $d \gg 0$.

As another example of the primacy of these methods, we can show the following interesting result. Say that R is a field spectrum when every R–module (in the homotopy category) splits as a wedge of suspensions of R. It is easy to check (as mentioned in the proof above) that $K(d)$ is an example of such a spectrum.

Corollary 31. *Every field spectrum R splits as a wedge of Morava's $K(d)$ theories.*

> This might require some care. You've been talking about $K(d)P$ mostly. Why is this the right thing to switch to? If you restrict attention to 2–periodic field spectra, can you use $K(d)P$ instead?

Proof. Set $E = \bigvee_{\text{primes } p} \bigvee_{d \in [0,\infty]} K(d)$, so that E detects nilpotence. The class 1 in the field spectrum R is non-nilpotent, so it survives when paired with some K–theory $K(d)$, and hence $R \wedge K(d)$ is not contractible. Because both R and $K(d)$ are field spectra, the smash product of the two simultaneously decomposes into a wedge of $K(d)$s and a wedge of Rs. So, R is a retract of a wedge of $K(d)$s, and picking a basis for its image on homotopy shows that it is a sub-wedge of $K(d)$s.

Remark 41. This is interesting in its own right, because field spectra are exactly those spectra which have Künneth isomorphisms. So, even if you weren't neck-deep in algebraic geometry, you might still have struck across these homology theories just if you like to compute things, since Künneth formulas make things computable.

> Jake asked if there was a geometric interpretation of these cohomology theories K_Γ. At present, there isn't one. Maybe remark on this.

3.6 Periodicity in finite spectra

We're now well-situated to address Ravenel's question about finite spectra and periodic self-maps. The solution to this problem passes through some now-standard machinery for triangulated $\otimes$–categories.

> might suggest introducing a notation for (finite) p-local spectra? And say that $\mathbb{S}$ means the p-local sphere; AY

Definition 59. A full subcategory of a triangulated category (e.g., the homotopy category of p–local finite spectra) is *thick* if…

> there's an issue here because triangulated categories don't have weak equivalences - in any case, these are isomorphisms because we're talking about the homotopy category. However, 2 of 3 should imply that it's closed under isomorphism. You could either say the full subcategory is triangulated (i.e. has 0 and 2 of 3) and closed under retracts, or just state this for full subcategories of the category/homotopy category of p-local finite spectra since that's where all the relevant examples are; AY

- …it is closed under weak equivalences.
- …it is closed under retracts.
- …it has a 2-out-of-3 property for cofiber sequences.

Examples of thick subcategories include:

- The category C_d of p–local finite spectra which are $K(d-1)$–acyclic. (For instance, if $d = 1$, the condition of $K(0)$–acyclicity is that the spectrum have purely torsion homotopy groups.) These are called "finite spectra of type at least d".
- The category D_d of p–local finite spectra F which have a self-map $v : \Sigma^N F \to F$, $N \gg 0$ inducing multiplication by a unit in $K(d)$–homology. These are called "v_d–self–maps".

> Doesn't v have to be nilpotent in the other $K(m)$'s? AY

Ravenel shows the following useful result interrelating the C_d:

Lemma 50 ([Rav84, Theorem 2.11]). *For X a finite complex, there is a bound*

$$\dim K(d-2)_* X \leq \dim K(d-1)_* X.$$

In particular, there is an inclusion $C_{d-1} \subseteq C_d$. $\qquad\qquad\square$

> as an extension of what we talked briefly about today, I suppose this is the fact that the stalk dimension of a coherent sheaf is upper semi-continuous; maybe this is worth adding in. AY

Hopkins and Smith show the following classification theorem:

Theorem 36 ([HS98, Theorem 7]). *Any thick subcategory C of the category of p–local finite spectra must be C_d for some finite d.*

> Why do you not need C_∞ for this?

Proof. Since C_d are nested by Lemma 50 and they form an exhaustive filtration, it is thus sufficient to show that any object $X \in C$ with $X \in C_d$ induces an inclusion $C_d \subseteq C$. Write R for the endomorphism ring spectrum $R = F(X, X)$, and write F for the fiber of its unit map:

$$F \xrightarrow{f} S \xrightarrow{\eta_R} R.$$

Finally, let Y be *any* finite spectrum of type at least d. It suffices to show $Y \in C$.

Now consider applying $K(n)$–homology (for *arbitrary n*) to the map

$$1 \wedge f \colon Y \wedge F \to Y \wedge S.$$

The induced map is always zero:

- In the case that $K(n)_* X$ is nonzero, then $K(n)_* \eta_R$ is injective because $K(n)_*$ is a graded field, and so $K(n)_* f$ is zero.
- In the case that $K(n)_* X$ is zero, then $n \leq d$ and, because of the bound on type, $K(n)_* Y$ is zero as well.

By a small variant of local nilpotence detection (Corollary 30, [HS98, Corollary 2.5]), it follows for $j \gg 0$ that

$$Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge S^{\wedge j}$$

is null-homotopic. Hence, one can calculate the cofiber to be

$$\text{cofib}\left(Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge S^{\wedge j}\right) \simeq Y \wedge \text{cofib}\, f^{\wedge j} \simeq Y \vee (Y \wedge \Sigma F^{\wedge j}),$$

so that Y is a retract of this cofiber.

We now work to show that this smash product lies in the thick subcategory C of interest. First, note that $X \wedge Z$ lies in C for any finite complex Z, since Z can be expressed as a finite gluing diagram of spheres and smashing this through with X expresses $X \wedge Z$ as the iterated cofiber of maps with source and target in C. Next, consider the following smash version of the octahedral axiom: the factorization

$$F \wedge F^{\wedge(j-1)} \xrightarrow{f \wedge 1} S \wedge F^{\wedge(j-1)} \xrightarrow{1 \wedge f^{\wedge(j-1)}} S \wedge S^{\wedge(j-1)}$$

begets a cofiber sequence

$$F \wedge \text{cofib}\, f^{\wedge(j-1)} \to \text{cofib}\, f^{\wedge j} \to \text{cofib}\, f \wedge S^{\wedge(j-1)}.$$

Is this sequence backwards? (Note that it doesn't matter: you can just write the factorization in the other order...)

Now turn an eye toward induction. Noting that $\text{cofib}(f) = R = X \wedge DX$ lies in C, we can use the 2-out-of-3 property on the octahedral sequence to see that $\text{cofib}(f^{\wedge j})$ lies in C. It follows that $Y \wedge \text{cofib}(f^{\wedge j})$ also lies in C, and using the retraction Y belongs to C as well.

As an application of this classification, they also show the following considerably harder theorem:

Theorem 37 ([HS98, Theorem 9]). *A p–local finite spectrum is $K(d-1)$–acyclic exactly when it admits a v_d–self–map.*

It's also a corollary of these same methods that the inclusion $C_d \subsetneq C_{d-1}$ is proper.

Proof (Executive summary of proof). Given the classification of thick subcategories, if a property is closed under thickness then one need only exhibit a single spectrum with the property to know that all the spectra in the thick subcategory it generates also all have that property. Inductively, they manually construct finite spectra $M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}})$ for sufficiently large indices i_* which admit a self-map v governed by a commuting square

$$\begin{array}{ccc}
BP_* M_{|v_d|i_d}(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{\ v\ } & BP_* M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) \\
\| & & \| \\
\Sigma^{|v_d|i_d} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{\ -\cdot v_d^{i_d}\ } & BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}).
\end{array}$$

These maps are guaranteed by very careful study of Adams spectral sequences.

You wrote BPP_0 yesterday.

Remark 42. We ran into the asymptotic condition $I \gg 0$ yesterday, when we asserted that there is no root of the 2–local v_1–self–map $v \colon M_8(2) \to M_0(2)$.

There is a second interesting application of these ideas, investigated by Paul Balmer as part of a broad attempt to analyze a geometric object through its modules.

Definition 60. Given a triangulated $\otimes$–category C, define a thick subcategory $C' \subseteq C$ to be...

- ...a $\otimes$–*ideal* when it has the additional property that $x \in C'$ forces $x \otimes y \in C'$ for any $y \in C$.
- ...a *prime* $\otimes$–*ideal* when $x \otimes y \in C'$ also forces at least one of $x \in C'$ or $y \in C'$.

Finally, define the *spectrum* of C to be its collection of prime $\otimes$–ideals, topologized so that $U(x) = \{C' \mid x \in C'\}$ form a basis of opens.

Theorem 38 (Balmer). *The spectrum of $D^{\mathrm{perf}}(\mathrm{Mod}_R)$ is naturally homeomorphic to the Zariski spectrum of R.* $\square$

Balmer's construction applies much more generally. The category Spectra can be identified with $\mathrm{Modules}_S$, and so one is moved to attempt to compute the Balmer spectrum of $\mathrm{Modules}_S^{\mathrm{perf}} = \mathrm{Spectra}^{\mathrm{fin}}$. In fact, we just finished this.

Theorem 39. *The Balmer spectrum of $\mathrm{Spectra}_{(p)}^{\mathrm{fin}}$ consists of the thick subcategories C_d, and $\{C_n\}_{n=0}^d$ are its open sets.*

Proof. Using the characterization of C_d as the kernel of $K(d-1)_*$, we see that it is a prime $\otimes$–ideal:

$$K(d-1)_*(X \wedge Y) \cong K(d-1)_*X \otimes_{K(d-1)_*} K(d-1)_*Y$$

is zero exactly when at least one of X and Y is $K(d-1)$–acyclic.

Remark 43. In fact, our favorite functor $\mathcal{M}_{MU}(-) \colon \mathrm{Spectra} \to \mathrm{QCoh}(\mathcal{M}_{MU})$ induces a homeomorphism of the Balmer spectrum of $\mathrm{Spectra}^{\mathrm{fin}}$ to that of $\mathcal{M}_{\mathbf{fg}}$. However, this functor does *not* exist on the level triangulated categories, so this remark has to be interpreted somewhat lightly.

3.7 Chromatic localization

Balmer's construction is remarkably successful at describing the most salient features of the stable category, but it falls a ways short of the rich "spectrum"

object we've come to know from algebraic geometry. In particular, we have only a topological space, and not anything like a locally ringed space (or a space otherwise equipped locally with algebraic data). It's also totally unclear why *MU* plays such an important mediating role between geometry (i.e., the stable category) and algebra (i.e., the moduli of formal groups). Nonetheless, taking that as granted, we can use Bousfield's theory of homological localization to access "local" categories of spectra of the sort that a sheaf of local rings would supply us with.

Theorem 40 ([Bou79], [Mar83, Theorem 7.7]). *Let R_* denote the homology theory associated by Landweber's Theorem 22 to a flat map $j\colon \operatorname{Spec} R \to \mathcal{M}_{\mathbf{fg}}$. There is then a diagram*

$$
\begin{array}{ccc}
\operatorname{Spectra}_R & \xrightarrow[\text{conservative}]{R_*} & \operatorname{QCoh}(\operatorname{Spec} R) \\
i \dashv L_R & \quad R_* \quad & j_* \dashv j^* \\
\operatorname{Spectra} & \xrightarrow{MU_*} & \operatorname{QCoh}(\mathcal{M}_{MU}),
\end{array}
$$

such that i is fully faithful, i is left-adjoint to L_R, j^* is left-adjoint to j_*, i and j_* are inclusions of full subcategories, the red composites are all equal, and R_* is conservative on $\operatorname{Spectra}_R$.[9]   □

In the case when R models the inclusion of the deformation space around the point Γ_d, we will denote the localizer by

$$
\operatorname{Spectra} \xrightarrow{\widehat{L}_d} \operatorname{Spectra}_{\Gamma_d}.
$$

In the case when R models the inclusion of the open complement of the unique closed substack of codimension d, we will denote the localizer by

$$
\operatorname{Spectra} \xrightarrow{L_d} \operatorname{Spectra}_d = \operatorname{Spectra}_{\mathcal{M}_{\mathbf{fg}}^{\leq d}}.
$$

We have set up our situation so that the following properties of these localizations either have easy proofs or are intuitive from the algebraic analogue of $j^* \vdash j_*$:

1. There is an equivalence

$$
L_d X \simeq (L_d \mathbb{S}) \wedge X,
$$

---

[9] The meat of this theorem is in overcoming set-theoretic difficulties in the construction of $\operatorname{Spectra}_R$. Bousfield accomplished this by describing a model structure on $\operatorname{Spectra}$ for which R–equivalences create the weak–equivalences.

analogous to $j^*M \simeq R \otimes M$ in the algebraic setting [Rav92, Theorem 7.5.6]. Because $\widehat{L}_d$ is associated to the inclusion of a formal scheme (i.e., an ind-finite scheme), it has the formula

$$\widehat{L}_d X \simeq \lim_I \left(M_0(v^I) \wedge L_d X \right)$$

analogous to $j^*M \simeq \lim_j (R/I^j \otimes M)$ in the complete algebraic setting [Hov95, Proof of Lemma 2.3].

2. Because the open substack of dimension d properly contains both the open substack of dimension $(d-1)$ and the infinitesimal deformation neighborhood of the closed point of height d, there are natural factorizations

$$\mathrm{id} \to L_d \to L_{d-1}, \qquad\qquad \mathrm{id} \to L_d \to \widehat{L}_d.$$

In particular, $L_d X = 0$ implies both $L_{d-1} X = 0$ and $\widehat{L}_d X = 0$.

3. The inclusion of the open substack of dimension $d-1$ into the one of dimension d has relatively closed complement the point of height d. Algebraically, this gives a gluing square (or Mayer-Vietoris square), and this is reflected in homotopy theory by a homotopy pullback square (or chromatic fracture square):

$$\begin{array}{ccc} L_d & \longrightarrow & \widehat{L}_d \\ \downarrow & \lrcorner & \downarrow \\ L_{d-1} & \longrightarrow & L_{d-1}\widehat{L}_d. \end{array}$$

Remark 44. More generally, whenever $L_B L_A = 0$, there is a fracture square

$$\begin{array}{ccc} L_{A \vee B} & \longrightarrow & L_B \\ \downarrow & \lrcorner & \downarrow \\ L_A & \longrightarrow & L_A L_B. \end{array}$$

So, this last fact follows from $L_d \simeq L_{K(0) \vee \cdots \vee K(d)}$ and $L_{K(d)} L_{K(d-1)} = 0$. Similarly, there is an "arithmetic fracture square"

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^{\wedge} \\ \downarrow & \lrcorner & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_p^{\wedge} \right)_{\mathbb{Q}}, \end{array}$$

which is a topological instantiation of the adèlic decomposition of a $\mathbb{Z}$–module - see [Bou79] for details in the arithmetic case.

There are also considerably more complicated facts known about these functors:

Theorem 41 ([Rav92, Theorem 7.5.7]). *The homotopy limit of the tower*

$$\cdots \to L_d F \to L_{d-1} F \to \cdots \to L_1 F \to L_0 F$$

recovers the p–local homotopy type of any finite spectrum F.[10]

This suggests a productive method for analyzing the homotopy groups of spheres: study the homotopy groups of each $L_d S$ and perform the reassembly process encoded by this inverse limit. Writing M_d for the fiber in the sequence

$$M_d \to L_d \to L_{d-1},$$

the "geometric chromatic spectral sequence" associated to this tower takes the form

$$\pi_* M_* S \Rightarrow \pi_* S_{(p)}.$$

So, M_d means the difference between the assembled layers L_d and L_{d-1} — but this was also the heuristic job of $\widehat{L}_d$ above. It turns out that these are interrelated by the following two theorems:

Theorem 42. *There is a pair of natural equivalences*

$$\widehat{L}_d M_d \simeq \widehat{L}_d, \qquad\qquad M_d \widehat{L}_d \simeq M_d. \;\square$$

Cite me: Gross–Hopkins?.

Theorem 43. *Analogous to "1." above, there is a natural equivalence*

$$M_d X \simeq \operatorname*{colim}_I \left(M^0(v^I) \wedge L_d X \right),$$

where $M^0(v^I)$ denotes a generalized Moore spectrum with top *cell in dimension 0.*

Cite me: I forget who this is due to.

Referring to "1." is clumsy. Make the previous things into Lemmas or something, rather than a bulleted list.

Remark 45. It is possible to draw the chromatic fracture square and the definition of M_d in the same diagram:

---

[10] Spectra satisfying this limit property are said to be *chromatically complete*, which is closely related to being *harmonic*, i.e., being local with respect to $\bigvee_{d=0}^{\infty} K(d)$. (I believe this a joke about "music of the spheres".) It is known that nice Thom spectra are harmonic (so, in particular, every suspension and finite spectrum), that every finite spectrum is chromatically complete, and that there exist some harmonic spectra which are not chromatically complete.

$$
\begin{array}{ccc}
M_d X & \!\!=\!\!=\!\! & M_d X \\
\downarrow & & \downarrow \\
L_d X & \longrightarrow & \widehat{L}_d X \\
\downarrow & \llcorner & \downarrow \\
L_{d-1} X & \longrightarrow & L_{d-1} \widehat{L}_d X.
\end{array}
$$

From this, we see that there is a fiber sequence

$$
M_d X \to \widehat{L}_d X \to L_{d-1} \widehat{L}_d X.
$$

The case $d = 1$ gives the prototypical example of the difference between these two presentations of the "exact height d data", where the sequence becomes:

$$
\operatorname*{colim}_{j}(M^0(p^j) \wedge L_1 X) \to \lim_{j}(M_0(p^j) \wedge L_1 X) \to \left(\lim_{j}(M_0(p^j) \wedge L_1 X) \right)_{\mathbb{Q}}.
$$

If, for instance, $\pi_0 L_1 X = \mathbb{Z}_{(p)}$, then the long exact sequence of homotopy groups associated to this fiber sequence gives

$$
\begin{array}{ccccc}
\pi_0 \widehat{L}_1 X & \longrightarrow & \pi_0 L_0 \widehat{L}_1 X & \longrightarrow & \pi_{-1} M_1 X \\
\| & & \| & & \| \\
\mathbb{Z}_p^{\wedge} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & \mathbb{Z}/p^\infty.
\end{array}
$$

This is a model for what happens generally: the v_j–torsion–free groups get converted to infinitely v_j–divisible groups, with some dimension shifts. (*Exactly* what happens is often hard to work out, and I'm not aware of a totally general statement.)

In any case, one sees that it is also profitable to consider the homotopy groups of $\widehat{L}_d S$. The spectral version $\mathcal{D}_{S \to E}(F)$ of $\mathcal{M}_E(F)$ considered on the first day furnishes us with a tool by which we can approach this:

Theorem 44 (Example 24, Definition 44, and Remark 29). *The E_Γ–based Adams spectral sequence for the sphere converges strongly to $\pi_* \widehat{L}_d S$. Writing ω for the line bundle on $\mathcal{M}_{E_\Gamma}$ of invariant differentials, we have*

$$
E_2^{*,*} = H^*_{\mathrm{stack}}(\mathcal{M}_{E_\Gamma}; \omega^{\otimes *}) \Rightarrow \pi_* \widehat{L}_d S. \qquad\Box
$$

Show that the action of the stabilizer group lifts to an action on Lubin–Tate space. This is relevant for what you're about to write.

The utility of this theorem is in the identification with stack cohomology. Recalling the discussion in Examples 15 and 16, as well as the identification

$$\mathcal{M}_{E_{\Gamma_d}} = \left(\mathcal{M}_{\mathbf{fg}}\right)^{\wedge}_{\Gamma_d} \simeq \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)} /\!/ \underline{\mathrm{Aut}}(\Gamma_d)$$

in Remark 38, we become interested in the action of $\mathrm{Aut}\,\Gamma_d$ on LT_d. We will deduce the following description of $\mathrm{Aut}\,\Gamma_d$ later on:

Theorem 45 (Corollary 39). *For Γ_d the Honda formal group law of height d over a perfect field k of positive characteristic p, we compute*

$$\mathrm{Aut}\,\Gamma_d \cong \mathbb{W}_p(k)\langle S\rangle \Big/ \left(\begin{array}{c} Sw = w^\varphi S, \\ S^d = p \end{array}\right),$$

where φ denotes a lift of the Frobenius from k to $\mathbb{W}_p(k)$. $\qquad\square$

As a matter of emphasis, this Theorem does not give a description of the *representation* of $\mathrm{Aut}\,\Gamma_d$, which is very complicated. Nonetheless, we have reduced the computation of all of the stable homotopy groups of spheres to an arithmetically–founded problem in profinite group cohomology, so that arithmetic geometry might lend a hand.

Example 29 (Adams). In the case $d = 1$, $\mathrm{Aut}(\Gamma_1) = \mathbb{Z}_p^\times$ and it acts on $\pi_* E_1 = \mathbb{Z}_p[u^\pm]$ by $\gamma \cdot u^n \mapsto \gamma^n u^n$. At odd primes p (so that p is coprime to the torsion part of $\mathbb{Z}_p^\times$), one computes

$$H^s(\mathrm{Aut}(\Gamma_1); \pi_* E_1) = \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \bigoplus_{j=2(p-1)k} \mathbb{Z}_p\{u^j\}/(pku^j) & \text{when } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This, in turn, gives the calculation

$$\pi_t \widehat{L}_1 S^0 = \begin{cases} \mathbb{Z}_p & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = t|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

These groups are familiar to homotopy theorists: the J–homomorphism $J : BU \to BF$ described on the first day selects exactly these elements (for nonnegative t).

Case Study 4
Unstable cooperations

4.1 Unstable contexts

Today we will take the framework of contexts discussed in Lecture 3.1 and augment it in two important (and very distinct) ways. First, we will assume that X is a *space* rather than a spectrum, and try to encode the extra structure appearing on E_*X from this assumption. Toward that end, recall that the levels of $\mathcal{M}_E(X)$ are defined by repeatedly smashing X with E, and that we had arrived at this by considering descent for the adjunction

$$\mathsf{Spectra} = \mathsf{Modules}_\mathbb{S} \xrightarrow{\;-\wedge E\;} \mathsf{Modules}_E$$

induced by the algebra map $\mathbb{S} \to E$. Given a spectrum X, our framework was set up to give its best possible approximation $X_E^\wedge$ within E–module spectra.

We will extend this to spaces by sewing this adjunction together with another:

$$\mathsf{Spaces} \xrightarrow[\Omega^\infty]{\Sigma^\infty} \mathsf{Modules}_\mathbb{S} \xrightarrow{\;-\wedge E\;} \mathsf{Modules}_E.$$

We will write $E(-)$ for the induced monad on Spaces, given by the formula

$$E(X) = \operatorname*{colim}_{j \to \infty} \Omega^j(\underline{E}_j \wedge X) = \Omega^\infty(E \wedge \Sigma^\infty X),$$

where $\underline{E}_*$ are the constituent spaces in the Ω–spectrum of E. This space has the property that $\pi_* E(X) = \widetilde{E}_* X$ (in nonnegative dimensions). The monadic structure comes from the two evident natural transformations:

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$$\eta\colon X \simeq S^0 \wedge X$$
$$\to \underline{E}_0 \wedge X$$
$$\to \operatorname*{colim}_{j\to\infty} \Omega^j(\underline{E}_j \wedge X) = E(X),$$

$$\mu\colon E(E(X)) = \operatorname*{colim}_{j\to\infty} \Omega^j \left(\underline{E}_j \wedge \operatorname*{colim}_{k\to\infty} \Omega^k(\underline{E}_k \wedge X) \right)$$
$$\to \operatorname*{colim}_{\substack{j\to\infty \\ k\to\infty}} \Omega^{j+k}(\underline{E}_j \wedge \underline{E}_k \wedge X)$$
$$\xrightarrow{\mu} \operatorname*{colim}_{\substack{j\to\infty \\ k\to\infty}} \Omega^{j+k}(\underline{E}_{j+k} \wedge X) \xleftarrow{\simeq} E(X).$$

Just as in the stable situation, we can extract from this a cosimplicial space:

Definition 61. Consider the descent cosimplicial object

$$\mathcal{UD}_E(X) := \left\{ \begin{array}{c} \begin{array}{ccccccc} & & & & \longrightarrow & E & \longleftarrow \\ & & \xrightarrow{\eta_L} & E & \longleftarrow & \circ & \longrightarrow \\ E & \xleftarrow{\mu} & \circ & \xrightarrow{\Delta} & E & \longleftarrow \\ \circ & \xrightarrow{\eta_R} & E & \longleftarrow & \circ & \longrightarrow & \cdots \\ X & & \circ & & \longrightarrow & E & \longleftarrow \\ & & X & & \circ & \longrightarrow \\ & & & & X \end{array} \end{array} \right\}.$$

Its totalization gives the *unstable E–completion of X.*

Under suitable hypotheses, we can extract from this an unstable analog of $\mathcal{M}_E$. Recall that our goal in Lecture 3.1 was to associate to $E_* X$ a quasi-coherent sheaf over $\mathcal{M}_E$, a fixed object, dependent on E but independent of X. In the presence of further hypotheses called "**FH**", we saw in Remark 29 that this same data could be expressed as an E_*E–comodule structure on $E_* X$. In particular, **FH** caused the marked map in

$$E_* X \xrightarrow{\eta_R} E_*(E \wedge X) \xleftarrow{\star} E_* E \otimes_{E_*} E_* X$$

to become invertible.

In the present setting, consider the analogous composite

$$\pi_m E(X) \xrightarrow{\eta_R} \pi_m E(E(X))$$
$$\xleftarrow{\mu \circ 1} \pi_m E(E(E(X)))$$
$$\xleftarrow{\text{compose}} \pi_m E(E(S^n)) \times \pi_n E(X).$$

Definition 62. The *unstable context of E* is the collection of cosimplicial abelian groups $\pi_* \mathcal{UD}_E(S^n)$. In the case $n = 0$, this is a cosimplicial ring, and in the case $n \neq 0$ the 0–simplices merely form a module over $\pi_* \mathcal{UD}_E(S^0)[0]$.

> This is just E_*, right?

Remark 46. In the case that E has Künneth isomorphisms, the "backwards" maps above become invertible, which is a kind of unstable analogue of the condition **FH**. This is the situation in which most of the classical work on this topic was done.

> Sort out exactly what structure lives here.

> **Cite me:** BCM, BJW, ...

> I *don't* really understand what sort of algebraic structure this gives us. It would be nice to have an unstable scheme-theoretic analogue of the stable context, so that the homology of spaces gave us "quasi-coherent sheaves" over this unstable object (and, in good cases, the unstable Adams spectral sequence had its E_2–page computed by some homological algebra over this object; see BCM Section 6).

> Talk about how this motivates us to consider algebraically the 0– and 1–simplices along, hoping that an eventual analogue of **FH** will keep us from having to consider anything further.

Ignoring for the moment what the correct scheme-theoretic analogue of this might be, we will press onward and record the algebraic objects appearing in the presence of the unstable analogue of **FH**.

Definition 63. A Hopf ring $A_{*,*}$ over a graded ring R_* is itself a graded ring object in the category $\mathsf{Coalgebras}_{R_*}$, sometimes called an R_*–coalgebraic graded ring object. It has the following structure maps:

> Can this definition be made without specifying the grading as such and instead using a $\mathbb{G}_m$–action?

$$+\colon A_{s,t} \times A_{s,t} \to A_{s,t} \qquad\qquad (A_{s,t} \text{ is an abelian group})$$
$$\cdot\colon R_{s'} \otimes_{R_*} A_{s,t} \to A_{s+s',t} \qquad\qquad (A_{*,t} \text{ is a } R_*\text{–module})$$
$$\Delta\colon A_{s,t} \to \bigoplus_{s'+s''=s} A_{s',t} \otimes_{R_*} A_{s'',t} \qquad\qquad (A_{*,t} \text{ is a } R_*\text{–coalgebra})$$
$$*\colon A_{s,t} \otimes_{R_*} A_{s',t} \to A_{s+s',t} \qquad (\text{addition for the ring in } R_*\text{–coalgebras})$$
$$\eta_*\colon R_* \to A_{*,0} \qquad (\text{null element for ring addition})$$
$$\chi\colon A_{s,t} \to A_{s,t} \qquad (\text{negation for the ring in } R_*\text{–coalgebras})$$
$$\circ\colon A_{s,t} \otimes_{R_*} A_{s',t'} \to A_{s+s',t+t'} \qquad (\text{multiplication map for the ring in } R_*\text{–coalgebras})$$
$$\eta_\circ\colon R_* \to A_{*,0} \qquad (\text{null element for ring multiplication}).$$

> So this is sort of an algebra/algebroid except with an extra grading t, which corresponds to the fact that elements of $E_* \underline{E}_m$ correspond to maps where the source has no grading/ has any grading (just like the elements in stable $E_* E$), but the target is a specific level in the Ω-spectrum. Maybe mention something to this effect? -EB

These are required to satisfy various commutative diagrams. The least obvious is displayed in Figure 4.1, encoding the distributivity of $\circ$–"multiplication" over $*$–"addition".

> I think that the "skew–commutativity" of $\circ$–multiplication is also worth mentioning. This confused me for a good while, being most familiar with the material in Lecture 4.7.

Remark 47. A ring spectrum E with Künneth isomorphisms

$$E_*(\underline{E}_m \times \underline{E}_n) \cong E_*(\underline{E}_m) \otimes_{E_*} E_*(\underline{E}_n)$$

gives rise to a Hopf ring $E_* \underline{E}_n = \pi_* \mathcal{UD}_E(S^n)[1]$. For a space X, the homology groups $E_* X$ form a comodule for this Hopf ring.

One can modify this story in a number of minor ways.

Remark 48. One can restrict to the *additive* unstable cooperations by passing to the quotient $Q^* E_* \underline{E}_*$. These corepresent the morphisms in a cocategory

$$\begin{array}{ccc}
A_{s,t} \otimes_{R_*} \left(A_{s',t'} \otimes_{R_*} A_{s'',t'} \right) & \xrightarrow{\;1\otimes *\;} & A_{s,t} \otimes_{R_*} A_{s'+s'',t'} \\
\downarrow{\scriptstyle \Delta \otimes (1\otimes 1)} & & \\
\left(\bigoplus_{s_1+s_2=s} A_{s_1,t} \otimes_{R_*} A_{s_2,t} \right) \otimes_{R_*} \left(A_{s',t'} \otimes_{R_*} A_{s'',t'} \right) & & \\
\downarrow{\scriptstyle \simeq} & & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1,t} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s'',t'} \right) & & \circ \\
\downarrow{\scriptstyle 1\otimes\tau\otimes 1} & & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s'',t'} \right) & & \\
\downarrow{\scriptstyle \circ\otimes\circ} & & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1+s',t+t'} \otimes_{R_*} A_{s_2+s'',t+t'} \right) & \xrightarrow{\quad * \quad} & A_{s+s'+s'',t+t'},
\end{array}$$

Fig. 4.1 The distributivity axiom for $*$ over $\circ$ in a Hopf algebra.

object in Rings (using the $\circ$–product for multiplication, which descends to $*$–indecomposables). The ring E_* corepresents the objects in this cocategory object.

Remark 49. The procedure in Remark 47 can be generalized to the case of *two* ring spectra, E and F, equipped with Künneth isomorphisms

$$E_*(\underline{F}_m \times \underline{F}_n) \cong E_*(\underline{F}_m) \otimes_{E_*} E_*(\underline{F}_n).$$

Again, the bigraded object $E_*\underline{F}_*$ forms a Hopf ring. These "mixed cooperations" appear as part of the cooperations for the ring spectrum $E \vee F$ — or, from the perspective of spectral shemes, for the joint cover $\{\mathbb{S} \to E, \mathbb{S} \to F\}$. The role of the mixed cooperations in this setting is to prevent the $(E \vee F)$–based unstable Adams spectral sequence from double-counting homotopy elements visible to both the unstable E– and F–completions.

4.2 Unstable cooperations in ordinary homology

The objects discussed in the previous Lecture appear to be almost bottomlessly complicated: there are so many groups and so many structure maps. At first glance, it might seem like it's a hopeless enterprise to actually try to compute $\mathcal{UM}_E^*$ for any spectrum E, but in fact the plenty of structure maps give enough footholds that this is often feasible, provided we have

A lot of the homological algebra of unstable comodules exists only after passing to this quotient. Try to explain why.

Explain this Remark, really. (1) Why does passing to the indecomposables project onto the additive cooperations? This should be thought of as an operation dual to restricting to the primitive unstable *operations*. (2) Why a cocategory object?

Maybe put in the unstable $\subseteq$ unstable additive $\subseteq$ stable diagram? Talk about examples of these three things in, say, mod–2 homology? Talk about homology suspension on this day, and make clear the comparison to the stable situation.

I feel that this can be used to take an unstable comodule for E–theory and produce from it an unstable comodule for F–theory (up to a wrong-way map). Martin Bendersky thought this was strange, but I don't think it's so odd, and I would like to understand how to straighten it out.

Does "Cartesian" mean anything in this setting?

Section III.11 of Wilson's *Primer* has a synopsis of how additive unstable operations should be treated.

Cite me: Bendersky Curtis Miller's [BCM78] *The unstable Adams spectral sequence for generalized homology.*

Cite me: Boardman Johnson Wilson's [BJW95] *Unstable*

sufficiently strong stomachs. Today we will treat the case $E = H\mathbb{F}_2$, which requires us to introduce all of the relevant tools but whose computations turn out to be very straightforward.

The place to start is with a very old lemma:

Lemma 51. *If E is a spectrum with $\pi_{-1}E = 0$, then $\underline{E}_1 \simeq B\underline{E}_0$.* □

> clarification: should we specify E to be an Ω-spectrum so the condition gives us that it's connective?

The essential point is that B gives the connective delooping of $\underline{E}_0$, so if E is connective then this will yield the spaces in the Ω–spectrum of E. This is useful to us because $B\underline{E}_0$ comes with a natural skeletal filtration, and this gives rise to a spectral sequence:

Corollary 32 ([RW80, Theorem 2.1]). *There is a convergent spectral sequence of Hopf algebras of signature*

$$E^1_{*,j} = F_*(\Sigma\underline{E}_0)^{\wedge j} \Rightarrow F_*\underline{E}_1.$$

In the case that F has Künneth isomorphisms of the form

> In class you didn't write any Σ.

$$F((\Sigma\underline{E}_0)^{\wedge j}) \cong F(\Sigma\underline{E}_0)^{\otimes j},$$

the E^2–page is identifiable as

$$E^2_{*,*} \cong \mathrm{Tor}^{F_*\underline{E}_0}_{*,*}(F_*, F_*).\square$$

In general, if E is a connective spectrum, we get a family of spectral sequences of signature

$$E^2_{*,*} \cong \mathrm{Tor}^{F_*\underline{E}_j}_{*,*}(F_*, F_*) \Rightarrow F_*\underline{E}_{j+1}.$$

> For this you'll want an analogue of the lemma, something like $\underline{E}_j \simeq B\underline{E}_{j-1}$. What is the connective hypothesis for this? It can't be something like $\pi_{j-2}E = 0$, because that won't be satisfied?

That this spectral sequence is multiplicative for the $*$–product is useful enough, but the situation is actually much, much better than this:

Lemma 52 ([RW80, Theorem 2.2]). *Denote by $E^r_{*,*}(F_*\underline{E}_j)$ the spectral sequence considered above whose E^2–term is constructed from Tor over $F_*\underline{E}_j$. There are maps*

> **Cite me:** This isn't the right citation. They blame this generality on a Thomason–Wilson article..

$$E^r_{*,*}(F_*\underline{E}_j) \otimes_{F_*} F_*\underline{E}_m \to E^r_{*,*}(F_*\underline{E}_{j+m})$$

which agree with the map

$$F_*\underline{E}_{j+1} \otimes_{F_*} F_*\underline{E}_m \xrightarrow{\circ} F_*\underline{E}_{j+m+1}$$

on the E^∞–page and which satisfy

$$d^r(x \circ y) = (d^r x) \circ y.\square$$

This Lemma is obscenely useful: it means that differentials can be transported *between spectral sequences* for classes which can be decomposed as $\circ$–products. This means that the bottom spectral sequence (i.e., the case $j = 0$) exerts a large amount of control over the others — and this spectral sequence often turns out to be very computable.

We now turn to our example of $E = H\mathbb{F}_2$ and $F = H\mathbb{F}_2$. To ground our induction, we will consider the first spectral sequence

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}B\mathbb{F}_2.$$

Using that $\mathbb{R}P^\infty$ gives a model for $B\mathbb{F}_2$, we use Example 3 to analyze this spectral sequence: that Example states that as an $\mathbb{F}_2$–module, there is an isomorphism

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \mathbb{F}_2\{a_j \mid j \geq 0\}.$$

Using our further computation in Example 11, we can also give a presentation of the Hopf algebra structure on $H\mathbb{F}_{2*}B\mathbb{F}_2$: it is dual to the primitively-generated polynomial algebra on a single class, so forms a divided power algebra on a single class $a_\varnothing$. In characteristic 2, this decomposes as

$$H\mathbb{F}_{2*}B\mathbb{F}_2 \cong \Gamma[a_\varnothing] \cong \bigotimes_{j=0}^{\infty} \mathbb{F}_2[a_{(j)}]/a_{(j)}^2,$$

where we have written $a_{(j)}$ for $a_\varnothing^{[2^j]}$ in the divided power structure.

Corollary 33. *This* Tor *spectral sequence collapses at the E^2–page.*

Proof. As an algebra, the homology $H\mathbb{F}_{2*}(\mathbb{F}_2)$ of the discrete space $\mathbb{F}_2$ is presented by the truncated polynomial algebra

$$H\mathbb{F}_{2*}(\mathbb{F}_2) \cong \mathbb{F}_2[\mathbb{F}_2] = \mathbb{F}_2[[1] - [0]]/([1] - [0])^{*2}.$$

The Tor–algebra of this is then divided power on a single class:

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}(\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = \Gamma[a_\varnothing].$$

In order for the two computations to agree, there can therefore be no differentials in the spectral sequence.

Now we turn to the rest of the induction:

Theorem 46. *$H\mathbb{F}_{2*}\underline{H\mathbb{F}_2}_t$ is the exterior $*$–algebra on the t–fold $\circ$–products of the generators $a_{(j)} \in H\mathbb{F}_{2*}B\mathbb{F}_2$.*

Proof. Make the inductive assumption that this is true for some fixed value of t. It follows that the Tor groups of the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2t}}}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2t+1}}$$

form a divided power algebra generated by the same t–fold $\circ$–products. An analogue of another Ravenel–Wilson lemma [RW80, Lemma 9.5] gives a congruence

$$\left(a_{(j_1)} \circ \cdots \circ a_{(j_t)}\right)^{[2^{j_t+1}]} \equiv a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})} \quad (\text{mod decomposables}).$$

It follows from Lemma 52 that the differentials vanish:

$$d\left(\left(a_{(j_1)} \circ \cdots \circ a_{(j_t)}\right)^{[2^{j_t+1}]}\right) \equiv d\left(a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})}\right) \quad (\text{mod decomposables})$$
$$= a_{(j_1)} \circ d\left(a_{(j_2)} \circ \cdots \circ a_{(j_{t+1})}\right) = 0.$$

Hence, the spectral sequence collapses. To see that there are no multiplicative extensions, note that the only potentially undetermined multiplications occur as $*$–squares of exterior classes. However, the $*$–squaring map is induced by the topological map

$$\underline{H\mathbb{F}_{2t}} \xrightarrow{\cdot 2} \underline{H\mathbb{F}_{2t}},$$

which is already null on the level of spaces. It follows that there are no extensions and the induction holds.

Corollary 34. *It follows that*

$$H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}} \xLeftarrow{\cong} \bigoplus_{t=0}^{\infty}(H_*(\mathbb{R}P^\infty; \mathbb{F}_2))^{\wedge t},$$

where $(-)^{\wedge t}$ *denotes the* t^{th} *exterior power in the category of Hopf algebras.*

Proof. The leftward direction of this isomorphism is realized by the $\circ$–product.

Remark 50. Our computation of the full Hopf ring of unstable cooperations can be winnowed down to give information about particular classes of cooperations. For instance, the *additive* unstable cooperations are given by passing to the $*$–indecomposable quotient

$$Q_* H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}} \cong \mathbb{F}_2\left\{a_{(I_0)} \circ \cdots \circ a_{(I_t)}\right\}$$
$$\cong \mathbb{F}_2[\xi_0, \xi_1, \xi_2, \ldots].$$

In terms of Lemma 11, we have

$$\mathrm{Spec}\, Q_* H\mathbb{F}_{2*}\underline{H\mathbb{F}_{2*}} \cong \underline{\mathrm{End}}(\widehat{\mathbb{G}}_a).$$

One passes to the *stable* cooperations by taking the colimit along the homology suspension element $a_{(0)} = \xi_0$. This has the effect of adjoining a $\circ$–product inverse to $a_{(0)}$, i.e.,

$$(Q_* H\mathbb{F}_{2*} \underline{H\mathbb{F}_2}_*)[a_{(0)}^{\circ(-1)}] \cong \mathbb{F}_2[\xi_0^{\pm}, \xi_1, \xi_2, \ldots],$$

which is exactly the ring of functions on $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ considered in Lemma 11.

Remark 51 ([Wil82, Theorems 8.5 and 8.11]). The odd–primary analogue of this result appears in Wilson's book. In that situation, the bar spectral sequences do not degenerate but rather have a single family of differentials, and the result imposes a single relation on the free Hopf ring. The end result is

$$H\mathbb{F}_{p*} \underline{H\mathbb{F}_p}_* \cong \frac{\bigotimes_{I,J} \mathbb{F}_p[e_1 \circ \alpha_I \circ \beta^J, \alpha_I \circ \beta^J]}{(e_1 \circ \alpha_I \circ \beta^J)^{*2} = 0, (\alpha_I \circ \beta^J)^{*p} = 0, e_1 \circ e_1 = \beta_1},$$

where $e_1 \in (H\mathbb{F}_p)_1 \underline{H\mathbb{F}_p}_1$ is the homology suspension element, $\alpha_{(j)} \in (H\mathbb{F}_p)_{2p^j} \underline{H\mathbb{F}_p}_1$ are the analogues of the elements considered above, and $\beta_{(j)} \in (H\mathbb{F}_p)_{2p^j} \mathbb{CP}^\infty$ are the algebra generators of the Hopf algebra dual of the ring of functions on the formal group $\mathbb{CP}^\infty_{H\mathbb{F}_p}$ associated to $H\mathbb{F}_p$ by its natural complex orientation. (In particular, the Hopf ring is *free* on these Hopf algebras, subject to the single interesting relation $e_1 \circ e_1 = \beta_{(0)}$.)

4.3 Algebraic unstable cooperations

One of our goals for this Case Study is to study the mixed unstable cooperations $E_* \underline{G}_{2*}$ for complex-orientable cohomology theories E and G. These turn out to behave more regularly than one might expect, in the sense that there is a uniform algebraic model and a comparison map which is often an isomorphism. In order to formulate what will become our main result, we will need to begin with some algebraic definitions.

Definition 64. Let R and S be graded rings. We can form a Hopf ring over R by forming the "ring–ring" $R[S]$: as an R–module, this is free and generated by symbols $[s]$ for $s \in S$. The Hopf ring maps $*$, $\circ$, and Δ are determined by the formulas

$$R[S] \otimes_R R[S] \xrightarrow{*} R[S] \qquad\qquad [s] * [s'] = [s + s'],$$
$$R[S] \otimes_R R[S] \xrightarrow{\circ} R[S] \qquad\qquad [s] \circ [s'] = [s \cdot s'],$$
$$R[S] \xrightarrow{\Delta} R[S] \otimes_R R[S] \qquad\qquad \Delta[s] = [s] \otimes [s].$$

[margin notes]

Define what the homology suspension element e is. The point is that the equivalence $E_n \simeq \Omega E_{n+1}$ is adjoint to a map $\Sigma E_n \to E_{n+1}$, and the effect of this map on F-homology is $\circ$-ing with e.

Is this right? What happened to $\mathcal{A}_*$ versus AP_0?

Explain this. You messed it up in class.

I think this relation is supposed to be analogous to $S^1 \wedge S^1 \simeq S^2 = \mathbb{CP}^1$.

You wrote β^J and β_1 in the display. What is the relation between β^J and β_1 and $\beta_{(j)}$? Is $\beta_{(0)} = \beta_1$?

Neil's MO answer about $H_* K(\mathbb{Z}, 3)$: http://mathoverflow.net/a/216041/1094

For instance, the distributivity axiom is checked in the calculation

$$[s''] \circ ([s] * [s']) = ([s''] \circ [s]) * ([s''] \circ [s'])$$
$$[s''] \circ [s + s'] =$$
$$[s''(s + s')] =$$
$$= [s''s] * [s''s']$$
$$= [s''s + s''s'].$$

Definition 65. Let C be an R–coalgebra, and let S be an auxiliary ring. We can form a free Hopf ring $R[S][C]$ on C under $R[S]$, which has the property

$$\mathsf{HopfRings}_{R[S]/}(R[S][C], T) \cong \mathsf{Coalgebras}_{R/}(C, T).$$

In terms of elements, it is an R–module spanned by $R[S]$ and C, as well as free $*$– and $\circ$–products of elements of C, altogether subjected to the Hopf ring relations.

Remark 52. Given an R–coalgebra C, we can form the free commutative Hopf algebra on C by taking its associated symmetric algebra. This is a degenerate case of a free Hopf ring construction, where S is taken to be the zero ring.

Now we turn our eyes to topology. Let E and F be two complex-orientable cohomology theories where F has enough Künneth isomorphisms. Set $R = F_*$, $S = E_*$, and $C = F_*\mathbb{C}P^\infty$ to form the free Hopf ring $R[S][C] = F_*[E_*][F_*\mathbb{C}P^\infty]$.

Lemma 53. *Orientations of E induce maps $F_*[E^*][F_*\mathbb{C}P^\infty] \to F_*\underline{E}_*$.*

Proof. To construct this map using universal properties, we need to check that $F_*\underline{E}_*$ is a Hopf ring under $F_*[E^*]$, and then we need to produce a map $F_*\mathbb{C}P^\infty \to F_*[E^*]$. For the first task, $F_*\underline{E}_*$ is already an F_*–module. An element $v \in E^n$ corresponds to a path component $[v] \in \pi_0\underline{E}_n$, which pushes forward along

$$\pi_0\underline{E}_n \to F_0\underline{E}_n$$

to give an element $[v] \in F_0\underline{E}_n$. One can check that this determines a map of Hopf rings $F_*[E^*] \to F_*\underline{E}_*$.

Next, we will use our assumed data of orientations. The complex-orientation of E gives a preferred class $\mathbb{C}P^\infty \to \underline{E}_2$, representing the coordinate $x \in E^2\mathbb{C}P^\infty$. By applying F–homology to this representing map, we get a map of F_*–coalgebras

$$F_*\mathbb{C}P^\infty \to F_*\underline{E}_2 \subseteq F_*\underline{E}_*.$$

Universality gives the desired map of Hopf rings.

There is no reason to expect $F_*\underline{E}_*$ to be a free Hopf ring, and so it would be naive to expect this map to be an equivalence. Indeed, Ravenel and Wilson

show that orientations of E and F together beget an interesting relation. An orientation on E gives us a comparison map as above, and an orientation on F gives a collection of preferred elements $\beta_j \in F_{2j}\mathbb{C}P^\infty$. Their result is to show that these elements are subject to the formal group laws *both* of F and of E:

Theorem 47 ([RW77, Theorem 3.8], [Wil82, Theorem 9.7]). *Write $\beta(s)$ for the formal sum $\beta(s) = \sum_j \beta_j x^j$. Then, in $F_*\underline{E}_*[\![s, t]\!]$, there is an equation*

$$\beta(s +_F t) = \beta(s) +_{[E]} \beta(t),$$

where

$$\beta(s +_F t) = \sum_n \beta_n \left(\sum_{i,j} a_{ij}^F s^i t^j \right)^n,$$

$$\beta(s) +_{[E]} \beta(t) = \underset{i,j}{*} \left([a_{ij}^E] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right).$$

Proof (Proof sketch). This is a matter of calculating the behavior of

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty \xrightarrow{x} \underline{E}_2$$

in two different ways: using the effect of μ in F–homology and pushing forward in x, or using the effect of μ in E–cohomology and pushing forward along the Hurewicz map $\mathbb{S} \to F$.

Altogether, this motivates our algebraic model for the Hopf ring of unstable cooperations:

Definition 66. Define $F_*^R \underline{E}_*$ to be the quotient of $F_*[E^*][F_*\mathbb{C}P^\infty]$ by the relation above. There is a natural *comparison map*

$$F_*^R \underline{E}_* \to F_*\underline{E}_*.$$

We will show that for many such E and G this map is an isomorphism. Before embarking on this, however, we would like to explore the connection to formal groups suggested by the formula in Theorem 47. Note that the Hopf ring-ring $R[S]$ has a natural augmentation given by $[s] \mapsto 1$, so that $\langle s \rangle = [s] - [0]$ form a generating set of the augmentation ideal.

Lemma 54. *In the $*$–indecomposable quotient $Q^* R[S]$, there are the formulas*

$$\langle s \rangle + \langle s' \rangle = \langle s + s' \rangle, \qquad\qquad \langle s \rangle \circ \langle s' \rangle = \langle ss' \rangle.$$

Proof. Modulo $*$–decomposables, we can write

$$0 \equiv \langle s \rangle * \langle s' \rangle = [s] * [s'] - [s] - [s'] + [0] = \langle s + s' \rangle - \langle s \rangle - \langle s' \rangle.$$

We can also directly calculate

$$\langle s \rangle \circ \langle s' \rangle = [ss'] - [0] - [0] + [0] = \langle ss' \rangle. \square$$

Corollary 35. *Orientations of E and F induce isomorphisms*

$$\mathrm{Spec}\, Q^* F_*^R \underline{E}_* \cong \underline{\mathsf{FormalGroups}}(\mathbb{CP}_E^\infty, \mathbb{CP}_F^\infty).$$

Proof. This is a matter of calculating $Q^* F_*^R \underline{E}_*$. Using Lemma 54, we have

$$\underset{i,j}{*} \left([a_{ij}^E] \circ \left(\sum_k \beta_k s^k \right)^{\circ i} \circ \left(\sum_\ell \beta_\ell t^\ell \right)^{\circ j} \right) \equiv \sum_{i,j} a_{ij}^E \left(\sum_k \beta_k s^k \right)^i \left(\sum_\ell \beta_\ell t^\ell \right)^j \ (\text{in } Q^*).$$

It follows that

$$Q^* F_*^R \underline{E}_* = F_*[\beta_0, \beta_1, \beta_2, \ldots] / (\beta(s +_F t) = \beta(s) +_E \beta(t)) \ . \square$$

Next time, we will investigate $F_* \underline{E}_*$ in the more modest and concrete setting of $F = H\mathbb{F}_p$ and $E = BP$. One might think that this is merely a first guess at a topological computation that seems accomplishable after Lecture 4.2, but we will quickly show that it plays the role of a universal example of this sort of calculation.

4.4 Complex-orientable cooperations

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Today we are aiming for a proof of the following Theorem:

Theorem 48 ([RW77, Theorem 4.2]). *The natural homomorphism*

$$H_*^R \underline{BP}_{2*} \to H_* \underline{BP}_{2*}$$

is an isomorphism. (In particular, $H_* \underline{BP}_{2*}$ is even–concentrated.)

This is proved by a fairly elaborate counting argument, and as such our first move will be to produce an upper bound for the size of the source Hopf ring. To begin, consider the following consequence of Lemma 54:

Side notes:

This is a little sloppy. Where are the coefficients of E being sent? Is $Q^*R[S]$ really $R \otimes S$ like Hood calculated? Hm.

What happened to the E_* in $F_*[E_*]$ in this expression?

You could also include the odd part of the approximation, with $e \circ e = \beta_1$, and from that calculate the algebraic model of the stabilization.

Jeremy found a paper (Chan's *A simple proof that the unstable (co-)homology of the Brown–Peterson spectrum is torsion-free*, see also Wilson's Primer's Section 10) where $H_* \underline{BP}_{2*}$ is proven to be bipolynomial (and even!) without any Hopf ring rigamarole. It looks like the method of proof is not very different from the Hopf ring one, but it's much shorter... and maybe the result will fall out of the Dieudonne module calculations anyhow? Consider it as an option after you break this lecture in two.

Ravenel and Wilson also tell you how to get a basis of $*$– indecomposables, which they call "allowable" monomials. Might be worth mentioning.

Cite me: Pages 266–270 of Ravenel–Wilson, especially the bottom of 268..

Corollary 36. *As a ∘–algebra,*

$$Q^* H_0^R \underline{BP}_{2*} \cong \mathbb{F}_p[[v_n] - [0_{-|v_n|}] \mid n \geq 1],$$

where $0_{-|v_n|}$ denotes the null element of $BP^{|v_n|}(*)$.   □

Directly from the definition of $H_*^R \underline{BP}_{2*}$, we now know that $Q^* H_*^R \underline{BP}_{2*}$ is generated by $[v_n] - [0_{-|v_n|}]$ for $n \geq 1$ and b_j, $j \geq 0$. In fact, p–typicality shows [RW77, Lemma 4.14] that it suffices to consider $b_{p^d} = b_{(d)}$ for $i \geq 0$. Altogether, this gives a secondary comparison map

$$A := \mathbb{F}_p[[v_n], b_{(d)} \mid n > 0, d \geq 0] \twoheadrightarrow Q^* H_*^R \underline{BP}_{2*}.$$

This map is not an isomorphism, as these elements are subject to the following relation:

Lemma 55 ([RW77, Lemma 3.14], [Wil82, Theorem 9.13]). *Write $I = ([p], [v_1], [v_2], \ldots)$, and work in $Q^* H_* \underline{BP}_2 / I^{\circ 2} \circ Q^* H_* \underline{BP}_2$. For any n we have*

$$\sum_{i=1}^{n} [v_i] \circ b_{(n-i)}^{\circ p^i} \equiv 0.$$

Proof. Consider the series expansion of $\beta_0 = \beta(ps) = [p]_{[BP]}(\beta(s))$.

Let r_n, the nth relation, denote the same sum taken in A instead:

$$r_n := \sum_{i=1}^{n} [v_i] \circ b_{(n-i)}^{\circ p^i}.$$

The Lemma then shows that the pushforward of r_n into $Q^* H_* \underline{BP}_{2*}$ is in the ideal generated by $I^{\circ 2}$. Ravenel and Wilson show the following well-behavedness result about these relators, by a fairly tedious argument:

Lemma 56 ([RW77, Lemma 4.15.b]). *The sequence $(r_1, r_2, \ldots)$ is regular in A.*
                                                                                                  □

This is exactly what we need to get our size bound.

Lemma 57. *Set*

$$c_{i,j} = \dim_{\mathbb{F}_p} Q^* H_i^R \underline{BP}_{2j}, \qquad d_{i,j} = \dim_{\mathbb{F}_p} \mathbb{F}_p[[v_n], b_{(0)}]_{i,j}.$$

Then $c_{i,j} \leq d_{i,j}$ and $d_{i,j} = d_{i+2,j+1}$.

Proof. We have seen that $c_{i,j}$ is bounded by the $\mathbb{F}_p$–dimension of

$$\mathbb{F}_p[[v_n], b_{(d)} \mid d \geq 0]_{i,j} / (r_1, r_2, \ldots).$$

But, since this ideal is regular and $|r_j| = |b_{(j)}|$, this is the same count as $d_{i,j}$. The other relation among the $d_{i,j}$ follows from multiplication by $b_{(0)}$, with $|b_{(0)}| = (2, 1)$.

We also need that one of the bidegrees of $[v_n]$ is zero, right?

We now turn to showing that this estimate is *sharp* and that the secondary comparison map is *onto*, and hence an isomorphism, using the bar spectral sequence. Recalling that the bar spectral sequence converges to a the homology of the *connective* delooping, let $\underline{BP}'_{2*}$ denote the connected component of $\underline{BP}_{2*}$ containing $[0_{2*}]$. We will then demonstrate the following theorem inductively:

Theorem 49 ([RW77, Induction 4.18]). *The following hold for all values of the induction index k:*

1. $Q^* H_{\leq 2(k-1)} \underline{BP}'_{2*}$ *is generated by* $\circ$*–products of the* $[v_n]$ *and* $b_{(j)}$.
2. $H_{\leq 2(k-1)} \underline{BP}'_{2*}$ *is isomorphic to a polynomial algebra in this range.*
3. *For* $0 < i \leq 2(k-1)$, *we have* $d_{i,j} = \dim_{\mathbb{F}_p} Q^* H_i \underline{BP}_{2j}$.

Before addressing the theorem, we show that this finishes our calculation:

Proof (Proof of Theorem 48, assuming Theorem 49 for all k). Recall that we are considering the natural map

This could be typeset better, by numbering the parts of Theorem 48 and just referring to numbered claims.

$$H^R_* \underline{BP}_{2*} \to H_* \underline{BP}_{2*}.$$

The first part of Theorem 49 shows that this map is a surjection. The third part of Theorem 49 together with our counting estimate shows that the induced map

$$Q^* H^R_* \underline{BP}_{2*} \to Q^* H_* \underline{BP}_{2*}$$

is an isomorphism. Finally, the second part of Theorem 49 says that the original map, before passing to $*$–indecomposables, must be an isomorphism as well.

Proof (Proof of Theorem 49). The infinite loopspaces in $\underline{BP}_{2*}$ are related by $\Omega^2 \underline{BP}'_{2(*+1)} = \underline{BP}_{2*}$, so we will use two bar spectral sequences to extract information about $\underline{BP}'_{2(*+1)}$ from $\underline{BP}_{2*}$. Since we have assumed that $H_{\leq 2(k-1)} \underline{BP}_{2*}$ is polynomial in the indicated range, we know that in the first spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{H_* \underline{BP}_{2*}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_* \underline{BP}_{2*+1}$$

the E^2–page is, in the same range, exterior on generators in Tor–degree 1 and topological degree one higher than the generators in the polynomial algebra. Since differentials lower Tor–degree, the spectral sequence is multiplicative, and there are no classes on the 0–line, it collapses in the range $[0, 2k - 1]$. Additionally, since all the classes are in odd topological degree, there are

no algebra extension problems, and we conclude that $H_*\underline{BP}_{2*+1}$ is indeed exterior up through degree $(2k-1)$.

We now consider the second bar spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{H_*\underline{BP}_{2*+1}}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*\underline{BP}_{2(*+1)}.$$

The Tor algebra of an exterior algebra is divided power on a class of topological dimension one higher. Since these classes are now all in even degrees, the spectral sequence collapses in the range $[0, 2k]$. Additionally, these primitive classes are related to the original generating classes by double suspension, i.e., by circling with $b_{(0)}$. This shows the first inductive claim on the *primitive classes* through degree $2k$, and we must argue further to deduce our generation result for $x^{[p^j]}$ of degree $2k$ with $j > 0$. By inductive assumption, we can write

$$x = [y] \circ b^{\circ I_0}_{(0)} \circ b^{\circ I_1}_{(1)} \circ \cdots,$$

and one may as well consider the element

$$z := [y] \circ b^{\circ I_0}_{(j)} \circ b^{\circ I_1}_{(j+1)} \circ \cdots.$$

This element isn't $x^{[p^j]}$ on the nose, but the diagonal of $z - x^{[p^j]}$ lies in lower filtration degree — i.e., it is primitive as far as the filtration is concerned — and so we are again done.

The remaining thing to do is to use the size bounds: the only way that the map

$$H^R_* \underline{BP}_{2*} \to H_*\underline{BP}_{2*}$$

could be surjective is if there were multiplicative extensions in the spectral sequence joining $x^{[p]}$ to x^p. Granting this, we see that the module ranks of the algebra itself and of its indecomposables are exactly the right size to be a free (i.e., polynomial) algebra, and hence this must be the case.

Having accomplished Theorem 48, we reduce a general computation to it:

Corollary 37 ([RW77, Corollary 4.7]). *For a complex-orientable cohomology theory E, the natural maps*

$$E^R_* \underline{MU}_{2*} \to E_*\underline{MU}_{2*}, \qquad\qquad E^R_* \underline{BP}_{2*} \to E_*\underline{BP}_{2*}$$

are isomorphisms of Hopf rings.

Proof. First, because $MU_{(p)}$ splits multiplicatively as a product of BPs, we deduce from Theorem 48 the case of $E = H\mathbb{F}_p$. Since $H\mathbb{F}_{p*}\underline{BP}_{2*}$ is even, it follows that $H\mathbb{Z}_{(p)*}\underline{BP}_{2*}$ is torsion–free on a lift of a basis, and similarly (working across primes) $H\mathbb{Z}_*\underline{MU}_{2*}$ is torsion–free on a simultane-

You changed from *BP* to *MU* – is this intentional?

ous lift of basis. Next, using torsion–freeness, we conclude from an Atiyah–Hirzebruch spectral sequence that $MU_*\underline{MU}_{2*}$ is even and torsion–free itself, and moreover that the comparison is an isomorphism. Lastly, using naturality of Atiyah–Hirzebruch spectral sequences, given a complex–orientation $MU \to E$ we deduce that the spectral sequence

$$E_* \otimes H_*(\underline{MU}_{2*}; \mathbb{Z}) \cong E_* \otimes_{MU_*} MU_*\underline{MU}_{2*} \Rightarrow E_*\underline{MU}_{2*}$$

collapses, and similarly for the case of BP. The theorem follows.

This is an impressively broad theorem: the loopspaces $\underline{MU}_{2*}$ are quite complicated, and that any general statement can be made about them is remarkable. That this fact follows from a calculation in $H\mathbb{F}_p$–homology and some niceness observations is meant to showcase the density of $\mathbb{CP}^\infty_{H\mathbb{F}_p} \cong \widehat{\mathbb{G}}_a$ inside of $\mathcal{M}_{\mathbf{fg}}$. However, Remark 50 indicates that this Corollary does not cover all possible cases that the comparison map in Definition 66 becomes an isomorphism. In the remainder of the Case Study, we will investigate two other classes of E and G where this holds.

> Should you mention the odd–dimensional stuff? You passed through it in the course of the proof anyway: you can see that $H_*\underline{BP}_{2*+1}$ is exterior on homology suspensions of algebra generators on degree below. This comports with what you already suspected about these unstable algebras.

4.5 Dieudonné modules

Our goal today is strictly algebraic. Because the category of finite type commutative and cocommutative Hopf algebras over a ground field k is an abelian category, it admits a presentation as the module category for some (possibly noncommutative) ring. The description of this ring and of the explicit assignment from a group scheme to linear algebraic data is the subject of *Dieudonné theory*. We will give a survey of some of the results of Dieudonné theory today, including three different presentations of the equivalence.[1]

> Cite me: Weinstein's geometry of Lubin–Tate spaces notes.

Start with a formal line V over a ground ring A, let $\widehat{\mathbb{G}}$ denote V equipped with a group structure, and let $\Omega^1_{V/A}$ be the module of Kähler differentials on V. We have previously been interested in the *invariant differentials* $\omega_{\widehat{\mathbb{G}}} \subseteq \Omega^1_{V/A}$ on V, back when we first discussed logarithms in Theorem 11. Such a differential gave rise to a logarithm through integration, in the case that A was a $\mathbb{Q}$–algebra. However, if A had positive characteristic p then there would be an obstruction to integrating terms with exponents of the form $-1 \pmod p$, which in turn led us to the notion of p–height explored in Lecture 3.2.

---

[1] Emphasis on "*some of the results*". Dieudonné theory is an enormous subject with many interesting results both internal and connected to arithmetic geometry, which we'll explore almost none of.

A slightly different twist on this set-up leads to a new story entirely. Recall that $\Omega^1_{V/A}$ forms the first level of the *algebraic de Rham complex* $\Omega^*_{V/A}$. The translation invariant differentials studied in the theory of the logarithm are those differentials so that the identity $\mu^* - \pi_1^* - \pi_2^* = 0$ holds *at the chain level*. We can weaken this to request only that that difference be *exact*, or zero at the level of cohomology of the algebraic de Rham complex. This condition begets a sub–A–module $D(\widehat{G}/A)$ of $H^1_{dR}(\widehat{G}/A)$ consisting of those 1–forms which are cohomologically translation invariant.

Example 30. Let A be a $\mathbb{Z}$–flat ring, let $\widehat{G}$ be a formal group over A, and let x be a coordinate on $\widehat{G}$. Set $K = A \otimes \mathbb{Q}$, so that $A \to K$ is an injection. There is then a diagram of exact rows

$$0 \to \left\{ \begin{array}{c} \text{integrals with} \\ A \text{ coefficients} \end{array} \right\} \to \{\text{all conceivable integrals}\} \to \{\text{missing integrals}\} \to 0$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & xA[\![x]\!] & \longrightarrow & \{f \in xK[\![x]\!] \mid f' \in A[\![x]\!]\} & \xrightarrow{d} & H^1_{dR}(\widehat{G}/A) & \longrightarrow & 0 \\
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & xA[\![x]\!] & \longrightarrow & \left\{ f \in xK[\![x]\!] \, \middle| \, \begin{array}{c} f' \in A[\![x]\!], \\ \delta f \in A[\![x,y]\!] \end{array} \right\} & \xrightarrow{d} & D(\widehat{G}/A) & \longrightarrow & 0, \\
\end{array}$$

where δ is induced by $\delta[\omega] = (\mu^* - \pi_1^* - \pi_2^*)[\omega]$.

The flatness condition in the Example is important to getting the calculation to work out right, and of course it is not satisfied when working over a perfect field of positive characteristic p — our favorite setting in Lecture 3.2 and Case Study 3 more generally. However, de Rham cohomology has the following remarkable lifting property (which we have specialized to H^1_{dR}):

Theorem 50. *Let A be a $\mathbb{Z}_{(p)}$–flat ring, let $f_1(x), f_2(x) \in A[\![x]\!]$ be power series without constant term. If $f_1 \equiv f_2 \pmod{p}$, then for any differential $\omega \in A[\![x]\!]dx$ the difference $f_1^*(\omega) - f_2^*(\omega)$ is exact.*

Proof. Write $\omega = dg$ for $g \in K[\![x]\!]$, and write $f_2 = f_1 + p\Delta$. Then

$$\int (f_2^* \omega - f_1^* \omega) = g(f_2) - g(f_1) = g(f_1 + p\Delta) - g(f_1)$$

$$= \sum_{n=1}^{\infty} \frac{(p\Delta)^n}{n!} g^{(n)}(f_1).$$

Since $g' = \omega$ has coefficients in A, so does $g^{(n)}$ for all n, and the fraction $p^n/n!$ lies in the $\mathbb{Z}_{(p)}$–algebra A.

Corollary 38 (H^1_{dR} is "crystalline"). *If $f_1, f_2 \colon V \to V'$ are maps of pointed formal varieties which agree mod p, then they induce the same map on H^1_{dR}.* □

Several well–behavedness results of the functor D follow directly from Corollary 38. For instance, any map $f \colon \widehat{G}' \to \widehat{G}$ of pointed varieties which is a group homomorphism mod p restricts to give a map $f^* \colon D(\widehat{G}/A) \to D(\widehat{G}'/A)$. Additionally, if f_1, f_2, and f_3 are three such maps of pointed varieties with $f_3 \equiv f_1 + f_2 \pmod{p}$ in $\mathsf{FormalGroups}(\widehat{G}'/p, \widehat{G}/p)$, then $f_3^* = f_1^* + f_2^*$ as maps $D(\widehat{G}/A) \to D(\widehat{G}'/A)$.

In the case that k is a *perfect* field, the ring $\mathbb{W}_p(k)$ of p–typical Witt vectors on k is simultaneously torsion-free and universal among nilpotent thickenings of the residue field k. This emboldens us to make the following definition:[2]

> Maybe cite a reference that does this?

Definition 67. Let k be a perfect field of characteristic $p > 0$, and let $\widehat{G}_0$ be a formal group over k. Then, choose a lift $\widehat{G}$ of $\widehat{G}_0$ to $\mathbb{W}_p(k)$, and define the *(contravariant) Dieudonné module* of $\widehat{G}_0$ by $M(\widehat{G}_0) := D(\widehat{G}/W(k))$.

> This functor is best adapted to p–divisible groups, so typically $\widehat{G}_0 = \widehat{G}_a$ is disallowed. The definition and the most basic properties seem to work OK though...

Remark 53. This is independent of choice of lift up to coherent isomorphism. Given any other lift $\widehat{G}'$ of $\widehat{G}_0$ to $\mathbb{W}_p(k)$, we can find *some* power series — not necessarily a group homomorphism — covering the identity on $\widehat{G}_0$. Corollary 38 then shows that this map induces a canonical isomorphism between the two potential definitions of $M(\widehat{G}_0)$.

Note that the module $M(\widehat{G}_0)$ carries some natural operations:

- Arithmetic: $M(\widehat{G}_0)$ is naturally a $\mathbb{W}_p(k)$–module, with the action by ℓ corresponding to multiplication–by–ℓ on $\widehat{G}_0$.

> well, $\widehat{G}_0$ is a formal group over k, so maybe it would be better to write $\ell \pmod p$ or action on $\widehat{G}$? AY

- Frobenius: The map $x \mapsto x^p$ is a group homomorphism mod p, so it induces a φ–semilinear map $F \colon M(\widehat{G}_0) \to M(\widehat{G}_0)$. That is, $F(\alpha v) = \alpha^\varphi F(v)$, where φ is a lift of the Frobenius on k to $\mathbb{W}_p(k)$.
- Verschiebung: The Verschiebung map is given by the mysterious formula

$$V \colon \sum_{n=1}^{\infty} a_n x^n \mapsto p \sum_{n=1}^{\infty} a_{pn}^{\varphi^{-1}} x^n.$$

It satisfies anti-semilinearity, $aV(v) = V(a^\varphi v)$, and also $FV = p$.

With this, we come to the main theorem of this Lecture:

Theorem 51. *The functor M determines a contravariant equivalence of categories between smooth 1–dimensional formal groups over k of finite p–height and finite free*

---

[2] There is a better definition one might hope for, which instead assigns to each potential thickening and lift a "Dieudonne module", and then work to show that they all arise as base-changes of this universal one. This is possible and technically superior to the approach we are taking here.

$\mathbb{W}_p(k)$–*modules equipped with appropriate operations F and V, called Dieudonné modules.*   □

Remark 54. Several invariants of the formal group associated to a Dieudonné module can be read off from the functor M. For example, the $\mathbb{W}_p(k)$–rank of M is equal to the p–height of $\widehat{\mathbb{G}}_0$. Additionally, the quotient M/FM is canonically isomorphic to the cotangent space $T_0^*\widehat{\mathbb{G}}_0 \cong \omega_{\widehat{\mathbb{G}}_0}$.

Example 31. The Dieudonné module associated to $\widehat{\mathbb{G}}_m$ is the easiest to compute. For x the usual coordinate, we have $[p](x) = x^p$, and hence the Frobenius F acts on $M(\widehat{\mathbb{G}}_m)$ by $Fx = px$. It follows that $Vx = x$ and $M(\widehat{\mathbb{G}}_m) \cong \mathbb{W}_p(k)\{x\}$ with this action.

Example 32 (cf. Example 7).

> This example is not done.

Dieudonné theory admits an extension to finite (flat) group schemes as well, and the torsion quotient of the Dieudonné module of a formal group agrees with the Dieudonné module associated to its torsion subscheme:

$$M(\widehat{\mathbb{G}}_0[p^j]) = M(\widehat{\mathbb{G}}_0)/p^j.$$

The Dieudonné module associated to $\widehat{\mathbb{G}}_a$ is the infinite–dimensional torsion $\mathbb{W}_p(k)$–module $M(\widehat{\mathbb{G}}_a) = k\{x, Fx, F^2x, \ldots\}$. Set $p = 2$, and consider the subgroup scheme $\alpha_2 \subseteq \widehat{\mathbb{G}}_a$ with Dieudonné module

$$M(\alpha_2) = M(\widehat{\mathbb{G}}_a)/F^2 = k\{x, Fx\}.$$

We can now verify the four claims from Example 7:

- *The group scheme α_2 has the same underlying structure ring as $\mu_2 = \mathbb{G}_m[2]$ but is not isomorphic to it.* This follows from calculating the Dieudonné module of homomorphisms:
- *There is no commutative group scheme G of rank four such that $\alpha_2 = G[2]$.* This follows from calculating the space of rank four objects and noticing that $VF = 2$ gets you into trouble.
- *If $E/\mathbb{F}_2$ is the supersingular elliptic curve, then there is a short exact sequence*

$$0 \to \alpha_2 \to E[2] \to \alpha_2 \to 0.$$

However, this short exact sequence doesn't split (even after making a base change). This follows from calculating the action of F and V: the exact sequence is split as modules, of course, but not as Dieudonné modules.

Margin notes:

> Add in words about being uniform and reduced?

> You never define the Dieudonné ring explicitly. I think the statement of this theorem and some later statements (e.g., $\mathbb{W}_p(k)\{x\}$) can be made more clear by giving notation to the Dieudonné ring.

> Start by calculating V for $\widehat{\mathbb{G}}_a$, use this to motivate the presence of torsion.

- *The subgroups of $\alpha_2 \times \alpha_2$ of order two are parameterized by $\mathbb{P}^1$. This follows from calculating the Dieudonné module of the product, as well as its space of projections of the appropriate rank.*

We can also use Dieudonné theory to compute the automorphism group of a fixed Honda formal group, which is information we wanted back in Lecture 3.7:

Corollary 39. *For Γ_d the Honda formal group law of height d over $\mathbb{F}_{p^d}$, we compute*

$$
\mathrm{Aut}\,\Gamma_d \cong \mathbb{W}_p(\mathbb{F}_{p^d})\langle F\rangle \Big/ \begin{pmatrix} Fw = w^\varphi F, \\ F^d = p \end{pmatrix}^\times.
$$

Proof. The Dieudonné module associated to Γ_d satisfies $F^d = p$, and hence $M(\Gamma_d/k)$ is presented as a *quotient* of the ring of operators on Dieudonné modules. The endomorphism ring of such a module is canonically isomorphic to the module itself.

We now turn to alternative presentations of the Dieudonné module functor, which have their own advantages and disadvantages. Let $\widehat{G}$ again be a formal Lie group over a field k of positive characteristic p, and consider Cartier's *functor of curves*

$$
C\widehat{G} = \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{G}).
$$

This is, again, a kind of relaxing of familiar data from Lie theory: rather than studying exponential curves, $C\widehat{G}$ tracks all possible curves. In Lecture 3.2, we considered three kinds of operations on a given curve $\gamma\colon \widehat{\mathbb{A}}^1 \to \widehat{G}$:

- Homothety: given a scalar $a \in A$, we define $[a]\cdot\gamma(t) = \gamma(at)$.
- Verschiebung: given an integer $n \geq 1$, we define $V_n\gamma(t) = \gamma(t^n)$.
- Arithmetic: given two curves γ_1 and γ_2, we can use the group law on $\widehat{G}$ to define $\gamma_1 +_{\widehat{G}} \gamma_2$. Moreover, given $\ell \in \mathbb{Z}$, the ℓ–fold sum in $\widehat{G}$ gives an operator

$$
\ell\cdot\gamma = \overbrace{\gamma +_{\widehat{G}} \cdots +_{\widehat{G}} \gamma}^{\ell\ \text{times}}.
$$

  This extends to an action by $\ell \in \mathbb{W}_p(k)$.
- Frobenius: given an integer $n \geq 1$, we define

$$
F_n\gamma(t) = \sum_{i=1}^{n}{}_{\widehat{G}}\,\gamma(\zeta_n t^{1/n}),
$$

where ζ_n is an nth root of unity. (This formula is invariant under permuting the root of unity chosen, so determines a curve defined over the original ground ring.)

Definition 68. A curve γ on a formal group is p–typical when $F_n\gamma = 0$ for $n \neq p^j$. Write $D_p\widehat{\mathbb{G}} \subseteq C\widehat{\mathbb{G}}$ for the subset of p–typical curves. In the case that the base ring is p–local, $C\widehat{\mathbb{G}}$ splits as a sum of copies of $D_p\widehat{\mathbb{G}}$, and there is a natural section $C\widehat{\mathbb{G}} \to D_p\widehat{\mathbb{G}}$ called p–typification, given by the same formula as in Lemma 43.

Remark 55. Precomposing with a coordinate $\widehat{\mathbb{A}}^1 \cong \widehat{\mathbb{G}}$ allows us to think of a logarithm log: $\widehat{\mathbb{G}} \to \widehat{\mathbb{G}}_a$ as a curve on $\widehat{\mathbb{G}}_a$. The definition of p–typicality given in Definition 52 coincides with the one given here.

Surprisingly, this construction captures the same data as the previous one.

Theorem 52. *The functor D_p determines a* covariant *equivalence of categories between smooth 1–dimensional formal groups over k of finite p–height and finite free $\mathbb{W}_p(k)$–modules equipped with appropriate operations F and V. In fact,*
$$D_p(\widehat{\mathbb{G}}) \cong M(\widehat{\mathbb{G}}/k)^*. \qquad \square$$

Finally, we turn to a third presentation of Dieudonné theory using more pedestrian methods, with the aim of developing a theory more directly adapted to algebraic topology. One can show that the category of *finite–type graded* connected Hopf algebras is an abelian category, and hence must admit a presentation as modules over some (perhaps noncommutative) ring. The first step to accessing this presentation is to find a collection of projective generators for this category.

Theorem 53 ([Sch70]). *Let $S(n)$ denote the free graded-commutative Hopf algebra on a single generator in degree n. There is a projective cover $H(n) \twoheadrightarrow S(n)$, given by the formula*

- *If either $p = 2$ and $n = 2^m k$ for $2 \nmid k$ and $m > 0$ or $p \neq 2$ and $n = 2p^m k$ for $p \nmid k$ and $m > 0$, then $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_k]$ with the Witt vector diagonal.*
- *Otherwise, $H(n) = S(n)$ is the identity.* $\qquad \square$

Corollary 40. *The category $\mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}$ of finite–type graded connected Hopf algebras is a full subcategory of modules over*

$$\bigoplus_{n,m} \mathsf{HopfAlgebras}^{>0,\mathrm{fin}}_{\mathbb{F}_p/}(H(n), H(m)).$$

Definition 69. Let $\mathsf{GradedDMods}$ denote the category of graded abelian groups M satisfying

1. $M_{<1} = 0$.
2. If n is odd, then $pM_n = 0$.

3. There are homomorphisms $V\colon M_{pn} \to M_n$ and $F\colon M_n \to M_{pn}$ (where n is even if $p \neq 2$), together satisfying $FV = p = VF$. (These are induced by the inclusion $H(n) \subseteq H(pn)$ and by the map $H(pn) \to H(n)$ sending x_n to x_{n-1}^p.)

Remark 56. Combining these, if n is even, taking the form $n = 2p^m k$ with $p \nmid k$ at odd primes p or $n = 2^m k$ with $2 \nmid k$ at $p = 2$, then $p^{m+1} M_n = F^{m+1} V^{m+1} M_n = 0$.

Theorem 54. *The functor* $D_*\colon \mathsf{HopfAlgebras}_{\mathbb{F}_p/}^{>0,\mathrm{fin}} \to \mathsf{GradedDMods}$ *defined by*

$$D_*(H) = \bigoplus_n D_n(H) = \bigoplus_n \mathsf{HopfAlgebras}_{\mathbb{F}_p/}^{>0,\mathrm{fin}}(H(n), H)$$

is an exact equivalence of categories. Moreover, $D_*H(n)$ *is characterized by the equation*

$$\mathsf{GradedDMods}(D_*H(n), M) = M_n. \square$$

It would be nice to tie these presentations together, at least with unjustified claims. What curve does a cohomologically left-invariant form get sent to? What does the appearance of the Witt scheme in the third presentation tell you about the relationship to the second presentation?

This last presentation could use some examples too.

Dieudonné theory is also about taking primitives in some sort of cohomology. Can this be connected to the additivity condition on unstable operations?

Weinstein's Section 1 also ends with a discussion of the Dieudonné functor extended to the crystalline site. This is necessary to get access to the period map.

4.6 Ordinary cooperations for Landweber flat theories

Convention: We will write H for $H\mathbb{F}_p$ for the duration of the lecture.

Today we will put Dieudonné modules to work for us in algebraic topology. Our goal is to prove the following Theorem:

Definition of the inverse functor to the Dieudonné module functor? I think this appears in the formal groups notes.

Theorem 55. *For $F = H$ and E a Landweber flat homology theory, the comparison map*

$$H_*^R \underline{E}_{2*} \to H_* \underline{E}_{2*}$$

is an isomorphism of Hopf rings.

The essential observation about this is that the associated Dieudonné module $D_* H_* \underline{E}_{2*}$ is a *stable object*, in the sense of the following result of Goerss–Lannes–Morel:

Theorem 56 ([GLM93, Lemma 2.8]). *Let $X \to Y \to Z$ be a cofiber sequence of spectra. Then, provided $n > 1$ satisfies $n \not\equiv \pm 1 \pmod{2p}$, there is an exact sequence*

$$D_n H_* \Omega^\infty X \to D_n H_* \Omega^\infty Y \to D_n H_* \Omega^\infty Z. \square$$

We know of a connection between $H_* BU$ and the Witt scheme. Is there a connection between $E_* MU$ and curves, or $E_* BP$ and p–typical curves, which is visible from this perspective? Almost definitely! Also, a connection between curves and divisors: the zero locus of a given curve... — Actually, Remark 18.1 in FPFP says that this is an unhealthy way to think about $H_* BU$. Very interesting.

You could also try to give the Devinatz–Hopkins formula for the stabilizer action. It's entirely a matter of different presentations of the Dieudonné module... although it may require you to understand Dieudonné crystals, which you are *not* up for.

Include this remark: However, the subspace of M spanned by $\omega_{\hat{G}}$ is *sensitive to choice of lift,* unlike the rest of this construction. This

Corollary 41 ([GLM93, Theorem 2.1]). *For $n > 1$ an integer satisfying $n \not\equiv \pm 1$ (mod $2p$), there is a spectrum $B(n)$ satisfying*

$$B(n)_n X \cong D_n H_* \Omega^\infty X.$$

(As convention, when $n \equiv \pm 1 \pmod{2p}$ we set $B(n) := B(n-1)$, and $B(0) := S^0$.)   □

> Remark 2.9 has a helpful discussion of how to extend to the case $B(0)$.

Before exploiting this result to compute something about unstable cooperations, we will prove a sequence of small results making these spectra somewhat more tangible.

Lemma 58 ([GLM93, Lemma 3.2]). *The spectrum $B(n)$ is connective and p–complete.*

Proof. First, rearrange:

$$\pi_k B(n) = B(n)_n S^{n-k} = D_n H_* \Omega^\infty \Sigma^\infty S^{n-k}.$$

If $k < 0$, n is below the connectivity of $\Omega^\infty \Sigma^\infty S^{n-k}$ and hence this vanishes. The second assertion follows from the observation that $H\mathbb{Z}_* B(n)$ is an $\mathbb{F}_p$–vector space. To see this, restrict to the case $n \not\equiv \pm 1 \pmod{2p}$ and calculate

$$H\mathbb{Z}_k B(n) = B(n)_n \Sigma^{n-k} H\mathbb{Z} = D_n H_* K(\mathbb{Z}, n-k) = [Q^* H_* K(\mathbb{Z}, n-k)]_n. \square$$

We can use a similar trick to calculate $H^* B(n)$:

> We should reconcile this notation with what's used in Lecture 4.1 and what's been said historically. These are specifically modules for the unstable *additive* cooperations.

Definition 70 ([GLM93, Example 3.6]). Let $G(n)$ be the free unstable $\mathcal{A}$–module on one generator of degree n, so that

$$\mathsf{UnstableModules}_{\mathcal{A}_*}(G(n), M) = M_n.$$

This module admits a presentation as

$$G(n) = \begin{cases} \Sigma^n \mathcal{A} / \{\beta^\varepsilon P^i \mid 2pi + 2\varepsilon > n\} \mathcal{A} & \text{if } p > 2, \\ \Sigma^n \mathcal{A} / \{\mathrm{Sq}^i \mid 2i > n\} \mathcal{A} & \text{if } p = 2. \end{cases}$$

The Spanier–Whitehead dual of this right-module, $DG(n)$, is characterized by the left-module

> Is this parenthesization right?

$$\Sigma^n (DG(n))^* = \begin{cases} \mathcal{A} / \mathcal{A} \{\chi(\beta^\varepsilon P^i) \mid 2pi + 2\varepsilon > n\} & \text{if } p > 2, \\ \Sigma^n \mathcal{A} / \mathcal{A} \{\chi \mathrm{Sq}^i \mid 2i > n\} & \text{if } p = 2. \end{cases}$$

> Be careful about $n \not\equiv \pm 1 \pmod{p}$?

Theorem 57 ([GLM93, Proof of Theorem 3.1]). *There is an isomorphism*

$$H^* B(n) \cong \Sigma^n (DG(n))^*.$$

Proof. Start, as before, by computing:

$$H_k B(n) = B(n)_n \Sigma^{n-k} H = D_n H_* K(\mathbb{F}_p, n-k).$$

The unstable module $G(n)$ also enjoys a universal property in the category of stable $\mathcal{A}$–modules:

$$\mathsf{Modules}_{\mathcal{A}/}(G(n), M) \cong [\Omega^\infty M]_n.$$

Hence, we can continue our computation:

$$
\begin{aligned}
H_k B(n) &= D_n H_* K(\mathbb{F}_p, n-k) \\
&= \mathsf{Modules}_{\mathcal{A}/}(G(n), \Sigma^{n-k} \mathcal{A}) \\
&= \mathsf{Modules}_{\mathbb{F}_p/}(G(n)_{n-k}, \mathbb{F}_p).
\end{aligned}
$$

We learn immediately that $H_* B(n)$ is finite. We would like to show, furthermore, that $H_* B(n)$ is the Spanier–Whitehead dual $\Sigma^n D G(n)$. It suffices to show

$$\mathsf{Modules}_{\mathcal{A}/}(G(n), \Sigma^j \mathcal{A}) = \mathsf{Modules}_{\mathcal{A}/}(\mathbb{F}_p, \Sigma^j \mathcal{A} \otimes H_* B(n))$$

for all values of j. This follows from calculating $B(n)_n \Sigma^{n+j} H$ using the same method. Finally, linear-algebraic duality and Definition 70 give the Theorem.

Additionally, the following Lemma is almost a consequence of basic understanding of unstable modules over $\mathcal{A}_*$, with minor fuss at the bad indices $n \equiv \pm 1 \pmod{p}$:

Lemma 59 ([GLM93, Lemma 3.3]). *There is a natural onto map $B(n)_n X \to H_n X$.* $\qquad\square$

Let's now work toward using the $B(n)$ spectra to analyze the Hopf rings arising from unstable cooperations. We have previously computed that the comparison map

$$H_*^R \underline{BP}_{2*} \to H_* \underline{BP}_{2*}$$

is an isomorphism. We will begin by reimagining this statement in terms of Dieudonné theory.

> **Cite me:** You probably also want to cite Hunton–Turner and Buchstaber–Lazarev.

To begin with, Dieudonné theory as we have described it is concerned with *Hopf algebras* rather than Hopf rings. However, a Hopf ring is not much structure on top of a system of graded Hopf algebras A_*: it is a map

$$\circ\colon A_* \boxtimes A_* \to A_*,$$

where "$\boxtimes$" is the tensor product of Hopf algebras. Since D_* gives an equivalence of categories between graded Hopf algebras and graded Dieudonné

modules, we should be able to find an analogous formula for the tensor product of Dieudonné modules.

Definition 71. The naive tensor product $M \otimes N$ of Dieudonné modules M and N receives the structure of a $\mathbb{W}(k)[V]$–module, where $V(x \otimes y) = V(x) \otimes V(y)$. We define the *tensor product of Dieudonné modules* by

$$M \boxtimes N = \mathbb{W}(k)[F, V] \otimes_{\mathbb{W}(k)[V]} (M \otimes N) \Big/ \left(\begin{array}{l} 1 \otimes Fx \otimes y = F \otimes x \otimes Vy, \\ 1 \otimes x \otimes Fy = F \otimes Vx \otimes y \end{array} \right).$$

> This definition is easier than it should be in generality, because not only are you working with a *field* but you're even working with $\mathbb{F}_p$, which has no Frobenius.

Lemma 60 ([Goe99, Theorem 7.7]). *The natural map*

$$D_*(M) \boxtimes D_*(N) \to D_*(M \boxtimes N)$$

is an isomorphism.   □

> Since the category of Dieudonné modules is the category of modules over some ring, there is also a tensor product over that ring instead over the partial ring $\mathbb{W}(k)[V]$. This seems like to natural thing to consider, unless the equivalence provided by the Freyd-Mitchell embedding theorem is not monoidal?

Definition 72. For a ring R, a *Dieudonné R–algebra A_** is a graded Dieudonné module equipped with an R–action and an algebra product

$$\circ \colon A_* \boxtimes A_* \to A_*.$$

> There's a wrinkle here: if you want to say that D_* is an equivalence of symmetric monoidal categories, then there needs to be a unit object. I don't think that's the case in the setting you're in.

Example 33 ([Goe99, Proposition 10.2]). For a complex-oriented homology theory E, we define its Dieudonné E_*–algebra of algebraic unstable cooperations by

$$R_E = E_*[b_1, b_2, \ldots] \big/ \big(b(s + t) = b(s) +_E b(t) \big),$$

where V is multiplicative, V fixes E_*, and V satisfies $Vb_{pj} = b_j$. (This determines the behavior of F.) We also write $D_E = \{D_{2m}H_*\underline{E}_{2n}\}$ for the even part of the topological Dieudonné algebra, and these come with natural comparison maps

> This citation at least says that the R–W relation holds over the topology E_*–Dieudonné algebra.

> How?

$$R_E \to D_E \leftarrow D_*H_*\underline{E}_{2*}.$$

Theorem 58 ([Goe99, Theorem 11.7]). *Restricting attention to the even parts, the maps*

$$R_E \to D_E \leftarrow D_*H_*\underline{E}_{2*}$$

are isomorphisms for E Landweber flat.

Proof. In Corollary 37, we showed that these maps are isomorphisms for $E = BP$. However, the right–hand object can be identified via Brown–Gitler juggling:

> Is this really legal? We won't get intro trouble with odd indices and the semi-parity condition on n?

$$D_nH_*\underline{E}_{2j} = B(n)_n\Sigma^{2j}E = E_{2j+n}B(n).$$

If E is Landweber flat, then the middle– and right–terms are determined by change-of-base from the respective BP terms by definition of flatness. Finally, the left term commutes with change-of-base by its algebraic definition, and the theorem follows.

> You made this sound more complicated in class than what's written here (with bar sequence arguments, etc.) Can you provide some of those details here?

Remark 57. The proof of Theorem 58 originally given by Goerss [Goe99] involved a lot more work, essentially because he didn't want to assume Theorem 48 or Corollary 37. Instead, he used the fact that $\Sigma^{\infty}_{+}\Omega^2 S^3$ is a regrading of the ring spectrum $\bigvee_n B(n)$, together with knowledge of $BP_*\Omega^2 S^3$. Since we already spent time with Theorem 48, we're not obligated to pursue this other line of thought.

Remark 58 ([Goe99, Proposition 11.6]). The Dieudonné algebra framework also makes it easy to add in the odd part after the fact. Namely, suppose that E is a torsion–free ring spectrum and suppose that $E_* B(n)$ is even for all n. In this setting, we can verify the purely topological version of this statement: the map

$$D_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*$$

is an isomorphism.

To see this, note that because $E_{2n-2k-1} B(2n) \to D_{2n} H_* \underline{E}_{2k+1}$ is onto and $E_{2n-2k-1} B(2n)$ is assumed zero, the group $D_{2n} H_* \underline{E}_{2k+1}$ vanishes as well. A bar spectral sequence argument shows that $D_{2n+1} H_* \underline{E}_{2k+2}$ is also empty [Goe99, Lemma 11.5.1]. Hence, the map on even parts

$$(D_E[e]/(e^2 - b_1))_{*,2n} \to (D_* H_* \underline{E}_*)_{*,2n}$$

is an isomorphism, and we need only show that

$$D_* H_* \underline{E}_{2n} \xrightarrow{e\cdot -} D_* H_* \underline{E}_{2n+1}$$

is an isomorphism as well. Since $e(Fx) = F(Ve \circ x) = 0$ and $D_* A / F D_* A \cong Q^* A$ for a Hopf algebra A, we see that e kills decomposables and suspends indecomposables:

$$e D_* H_* \underline{E}_{2n} = \Sigma Q H_* \underline{E}_{2n}.$$

This is also what happens in the bar spectral sequence, and the claim follows. In light of Theorem 58, this means that for Landweber flat E, the comparison isomorphism can be augmented to a further isomorphism

$$R_E[e]/(e^2 - b_1) \to D_* H_* \underline{E}_*.$$

4.7 Cooperations and geometric points on $\mathcal{M}_{\mathbf{fg}}$

Throughout today, we will write K for a Morava K–theory K_Γ (which, if you like, you can take to be $K(d)$) and A for a finitely generated abelian group, and H for the associated Eilenberg–Mac Lane spectrum. Our goal is to study

Jeremy asked whether there was a connection between Goerss's original proof and the free E_2–algebra with p killed which we keep dancing around this semester. I don't know, and it's a good question.

Compare also with the main result of [HH95].

Remark 11.4 in the Hopf Ring paper says that the failure of the odd primary case to be an isomorphism is measured by the suspension homomorphism operator e, and the kernel of the natural surjective map is exactly the kernel of multiplication by e. Have a look.

Ask Mike (and Jacob?) if there are analogues of these results for kO which explain Mahowald's generalized K–theoretic Brown–Gitler spectra. 3/29: I did ask Mike, he said he didn't know. I also asked Paul, and he said this seemed

the unstable mixed cooperations $K_*\underline{H}_*$, which we expect to be connected to formal group homomorphisms $\Gamma \to \widehat{\mathbb{G}}_a$ but which isn't covered by any of the cases studied thus far. This calculation is interesting to us for two reasons:

1. These cooperations appear naturally when pursuing a "fiberwise analysis" of cooperations, or a chromatic analysis of unstable homotopy theory, along the lines of Case Study 3.

2. The Eilenberg–Mac Lane spaces $\underline{H}_*$ appear as the layers of Postnikov towers. If we were to want to analyze the K–homology of a Postnikov tower (as we will in Case Study 5), we will naturally encounter pieces of $K_*\underline{H}_*$, and we would be wise to have a firm handle on these objects. It is a tribute to the perspective offered here that the successful way to approach this computation is not one-at-a-time, handicrafted for each possible Eilenberg–Mac Lane space, but rather all-at-once, as suggested by the unstable cooperations picture.

Unsurprisingly, our analysis will rest on the bar spectral sequence

$$\mathrm{Tor}_{*,*}^{K_*\underline{H}_q}(K_*, K_*) \Rightarrow K_*\underline{H}_{q+1}.$$

However, because K–theory is naturally a 2–periodic theory, our method in Lecture 4.4 of inducting on homological degree and working with a triangular corner of the spectral sequence will fail because it is not a first-quadrant spectral sequence. Instead, we will induct on the Eilenberg–Mac Lane index q as in Lecture 4.2, and as such we will begin with analyzing the base case of $q = 0$ where we are interested in manually computing K_*BA for a reasonable abelian group A. Since K–theory has Künneth isomorphisms and $B(A_1 \times A_2) \simeq BA_1 \times BA_2$, it suffices to do the computation just for $A = C_{p^j}$.

> This only takes care of finite abelian groups A. Is this what reasonable means?

Theorem 59 ([RW80, Theorem 5.7], [HL, Proposition 2.4.4]). *There is an isomorphism*

$$BS^1[p^j]_K \cong BS^1_K[p^j].$$

> I'm a little confused about the objects in this definition. Does the RHS mean take the scheme and then take p-torsion in the ring, then go back to the scheme? And the LHS is K theory of the cofiber of multiplication by p^j? Maybe would help if this was slightly explained

Proof. Consider the diagram of spherical fibrations:

$$
\begin{array}{ccccc}
S^1 & \longrightarrow & B(S^1[p^j]) & \longrightarrow & BS^1 \\
\| & & \downarrow & & \downarrow{\scriptstyle p^j} \\
S^1 & \longrightarrow & ES^1 & \longrightarrow & BS^1.
\end{array}
$$

> Put in a pullback corner here.

The induced long exact sequence (known as the Gysin sequence, or as the couple in the Serre spectral sequence for the first fibration) takes the form

$$
\begin{array}{ccc}
& K^* BS^1 & \\
& \nearrow \quad \nwarrow {\scriptstyle -\smile [p^j](x)} & \\
K^*(BS^1[p^j]) & \xrightarrow{\quad \partial \quad} & K^* BS^1
\end{array}
$$

where x is a coordinate on BS^1_K. Because BS^1_K is of finite height, the right diagonal map is injective. It follows that $\partial = 0$, and so this gives a short exact sequence of Hopf algebras, which we can reinterpret as a short exact sequence of group schemes

$$
B(S^1[p^j])_K \to BS^1_K \xrightarrow{p^j} BS^1_K. \quad \square
$$

Remark 59. Dually, there is also an exact sequence of Hopf algebras

$$
\begin{array}{ccc}
& K_* BS^1 & \\
& \nearrow \quad \searrow {\scriptstyle -\frown [p^j](x)} & \\
K_*(BS^1[p^j]) & \xleftarrow{\quad \partial \quad} & K_* BS^1,
\end{array}
$$

where again $\partial = 0$ and hence $K_*(BS^1[p^j])$ is presented as the kernel of the map "cap with $[p^j](x)$". We will revisit the duality next time.

There are a couple of approaches to the rest of this calculation, i.e., $K_*\underline{H}_q$ for $q > 1$. The original, due to Ravenel and Wilson [RW80], is to complete the calculation for the smallest abelian group C_p and then induct upward toward more complicated groups like C_{p^j} and C_{p^∞}. More recently, there is also a preprint of Hopkins and Lurie [HL] that begins with $A = C_{p^\infty}$ and then works downward. We will do the *easy* parts of both calculations, to give a feel for their relative strengths and deficiencies.

The Ravenel–Wilson version of the calculation proceeds much along the same lines as Lecture 4.2. Setting $H = H\mathbb{Z}/p$, we will study the bar spectral sequences

$$
\mathrm{Tor}^{K_*\underline{H}_q}_{*,*}(K_*, K_*) \Rightarrow K_*\underline{H}_{q+1}
$$

for different indices q and use the $\circ$–product to push differentials around among them. Our first move, as in Lecture 4.2, is to study the bar spectral sequence

$$
\mathrm{Tor}^{K_*\mathbb{Z}/p}_{*,*}(K_*, K_*) \Rightarrow K_* B\mathbb{Z}/p
$$

and analyze what *must* happen in order to reach the conclusion of Theorem 59. In the input to this spectral sequence, the ground algebra is given by

$$
K_*\underline{H\mathbb{Z}/p}_0 = K_*[[1]]/\langle [1]^p - 1\rangle = K_*[[1]-[0]]/\langle [1]-[0]\rangle^p.
$$

The Tor–algebra for this truncated polynomial algebra $K_*[a_\varnothing]/a_\varnothing^p$ is then given by the formula

$$\mathrm{Tor}_{*,*}^{K_*[a_{\varnothing}]/a_{\varnothing}^p}(K_*, K_*) = \Lambda[\sigma a_{\varnothing}] \otimes \Gamma[\varphi a_{\varnothing}],$$

the combination of an exterior algebra and a divided power algebra. We know which classes are supposed to survive this spectral sequence, and hence we know where the differentials must be:

$$d(\varphi a_{\varnothing})^{[p^d]} = \sigma a_{\varnothing},$$

$$\Rightarrow d(\varphi a_{\varnothing})^{[i+p^d]} = \sigma a_{\varnothing} \cdot (\varphi a_{\varnothing})^{[i]}.$$

The spectral sequence collapses after this differential.[3]

With the base case analysis completed, we turn to the induction on q:

Theorem 60 ([RW80, Theorem 9.2 and Theorem 11.1]). *Using the $\circ$–product,*

$$K_* \underline{H\mathbb{Z}/p}_q = \mathrm{Alt}^q \, \underline{H\mathbb{Z}/p}_1.$$

Proof (Proof sketch). The inductive step turns out to be extremely index-rich, so I won't be so explicit or complete, but I'll point out the major landmarks. It will be useful to use the shorthand $a_{(i)} = a_{\varnothing}^{[p^i]}$, where (i) is thought of as a multi-index with one entry.

We proceed by induction, assuming that $K_* \underline{H\mathbb{Z}/p}_q = \mathrm{Alt}^q \, \underline{H\mathbb{Z}/p}_1$ for a fixed q. Computing the Tor–algebra of $K_* \underline{H\mathbb{Z}/p^j}_q$ again yields a tensor of divided power and exterior classes, a pair for each algebra generator of $K_* \underline{H\mathbb{Z}/p^j}_q$. In analogy to the rewriting formula used in Theorem 46, there is also a rewriting formula in this context [RW80, Lemmas 9.5-6]:

$$\left(\varphi a_{(i_1,\ldots,i_q)}\right)^{[p^n]} \equiv \left(\varphi a_{(i_1,\ldots,i_{q-1})}\right)^{[p^n]} \circ a_{(i_q+n)} \quad \mathrm{mod} \; *\text{–decomposables.}$$

Since every class can be so decomposed, all the differentials are determined by the previous spectral sequence. In particular, classes are hit by differentials exactly when $i_q + n$ is large enough. Chasing this through shows that the inductive assumption that $K_* \underline{H\mathbb{Z}/p}_{q+1}$ is an exterior power holds, and the class $\left(\varphi a_{(i_1,\ldots,i_q)}\right)^{[p^n]}$ represents $a_{(n,i_1+n,\ldots,i_q+n)}$.

Remark 60. When reworking this computation for the case

$$\mathrm{Tor}_{*,*}^{K_* \underline{H\mathbb{Z}/p^j}_q}(K_*, K_*) \Rightarrow K_* \underline{H\mathbb{Z}/p^j}_{q+1},$$

the main difference is that there are various algebra extensions to keep track of. These are controlled using the group maps

---

[3] In the $j > 1$ version of this analysis, there are some multiplicative extensions to sort out. Of course, these are all determined by already knowing the multiplicative structure on $K_* \underline{H\mathbb{Z}/p^j}_1$.

$$\mathbb{Z}/p^{j+1} \to \mathbb{Z}/p^j, \qquad \left(\frac{1}{p^j}\mathbb{Z}\right)/\mathbb{Z} \to \left(\frac{1}{p^{j+1}}\mathbb{Z}\right)/\mathbb{Z},$$

together with knowledge of how the extensions strung together at the previous j–stage. Then, these tools are revisited [RW80, Theorem 12.4] to give a computation in the limiting case $A = C_{p^\infty}$, where there's a p–adic equivalence $HC_{p^\infty} \simeq_p^\wedge \Sigma H\mathbb{Z}$. The calculation in this setting is the most interesting one of all — after all, it contains the case BS^1_K, which is of special interest to us.

> Mike has said something about the pairing $C_{p^j} \times C^*_{p^j} \to \mathbb{Q}/\mathbb{Z}$ not being functorial in j (so as to pass to the direct limit) which gave me pause. I should make sure I'm not messing something up here.

Remarkably, this maximally interesting case is easier to access directly than passing through all of this intermediate work, and this is the perspective of Hopkins and Lurie. We will pursue an inductive calculation of the formal group schemes $(\underline{HC_{p^\infty q}})_K$ by iterating the cohomological bar spectral sequence, culminating in the following Theorem:

> Expand out the "limiting case as $j \to \infty$" of the differentials in the earlier spectral sequences that you talked about in class.

Theorem 61. *There is an isomorphism of formal group schemes*

$$(\underline{HC_{p^\infty q}})_K \cong (\mathbb{C}P^\infty_K)^{\wedge q}.$$

In particular, $(\underline{HC_{p^\infty q}})_K$ is a "p–divisible formal group" of dimension $\binom{d-1}{q-1}$ and height $\binom{d}{q}$.

Assume that this Theorem is true for a fixed value of q. First, the cohomological bar spectral sequence lets us calculate just the *formal scheme* structure of $(\underline{HC_{p^\infty q+1}})_K$, using the *formal group* structure of $(\underline{HC_{p^\infty q}})_K$. It has signature ([Laz97], [HL, Example 2.3.5])

$$H^*((\underline{HC_{p^\infty q}})_K; \widehat{\mathbb{G}}_a) \otimes_{K_0} K_* \Rightarrow K^* \underline{HC_{p^\infty q+1}},$$

and hence we are moved to calculate the formal group cohomology of $(\underline{HC_{p^\infty q}})_K$. The following Lemma furthers the calculations of formal group cohomology in Lemma 40 and Lemma 48 to the situation of connected p–divisible groups of higher dimension:

Lemma 61 ([HL, Theorem 2.2.10 and Example 2.2.12]). *If $\widehat{\mathbb{G}}$ is a connected p–divisible group over a field k, then $H^*(\widehat{\mathbb{G}}; \widehat{\mathbb{G}}_a)$ is isomorphic to the symmetric algebra on $\Sigma H^1(\widehat{\mathbb{G}}[p^j]; \widehat{\mathbb{G}}_a)$, with generators concentrated in degree 2.* $\square$

> j is a positive integer

Corollary 42 ([HL, Proposition 2.4.11]). *As a formal scheme, $(\underline{HC_{p^\infty q+1}})_K$ is a formal variety of dimension $\binom{d-1}{q}$.*

Proof. By setting $\widehat{\mathbb{G}} = (\underline{HC_{p^\infty q}})_K$, the Lemma gives us access to the E^2–page of our cohomological bar spectral sequence. We can calculate the dimension of H^1 to be

> I have tried several times to figure out where this comes from, using tools from the beginning of H–L Section 2.2, but I haven't really convinced myself. It must have something to do with the dimension of the Dieudonné module of $\widehat{\mathbb{G}}$ minus the dimension of the image of F — or something?

> I think I see how this is supposed to work out, do you think the detail is worth adding? In any case maybe we can talk about it since I'm not sure I'm right -AY

$$\dim_k H^1((\underline{H\mathbb{C}_{pq}})_K; \widehat{\mathbb{G}}_a) = \mathrm{ht}(\underline{HC_{p^\infty q}})_K - \dim(\underline{HC_{p^\infty q}})_K = \binom{d}{q-1} - \binom{d-1}{q-1} = \binom{d-1}{q}.$$

It follows that the E_2–page of this spectral sequence is a polynomial k–algebra on $\binom{d-1}{q}$ generators, concentrated in even degrees, so that the spectral sequence collapses and $K^0 \underline{HC_{p^\infty q+1}}$ is a power series algebra on as many generators.

In order to continue the induction, we now have to identify the group structure on $(\underline{HC_{p^\infty q+1}})_K$. This is done using the theory of Dieudonné modules:

Theorem 62 ([HL, Proposition 2.4.12]).

Let $q \geq 1$ be an integer. Suppose, in addition to the inductive hypotheses above, that the sequence of group schemes

$$(\underline{H(\tfrac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p)_q})_K \to (\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)_q})_K \to (\underline{H(\mathbb{Q}_p/\tfrac{1}{p}\mathbb{Z}_p)_q})_K \to K_0$$

is exact, and that the map

$$\theta^q : \mathbb{Q}_p/\mathbb{Z}_p \otimes M(\mathbb{CP}_K^\infty)^{\wedge q} \to M((\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)_q})_K)$$

is an isomorphism. Then θ^{q+1} is an isomorphism and the formal group $\underline{H(\mathbb{Q}_p/\mathbb{Z}_p)_{q+1}}$ is a connected p–divisible group with height $\binom{d}{q+1}$ and dimension $\binom{d-1}{q}$.

Proof (Proof sketch). By applying the snake lemma to the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} & \longrightarrow & 0 \\
& & \downarrow V & & \downarrow V & & \downarrow V & & \\
0 & \longrightarrow & M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q} \otimes M^{\wedge(q+1)} & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} & \longrightarrow & 0
\end{array}
$$

and knowing that the middle map is an isomorphism, we learn that V is a surjective endomorphism of $M^{\wedge(q+1)} M \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and that there is an isomorphism

$$\ker(V : \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)} \to \mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge(q+1)}) \cong \mathrm{coker}(V : M^{\wedge(q+1)} \to M^{\wedge(q+1)}).$$

The right-hand side is spanned by elements $V^I x$ with $I_1 = 0$, and hence the left-hand side has k–vector–space dimension $\binom{d-1}{q}$. By very carefully studying the bar spectral sequence, one can learn that θ^m induces a surjection

$$\ker V|_{\mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge m}} \to \ker V|_{D(Y)}.$$

In fact, since these two have the same rank, θ^m is an isomorphism on these subspaces. Since the action of V is locally nilpotent, this is enough to show

[Margin notes:]

Figure out this formatting. I should be able to just set the fraction to be "inline" rather than displaymode, but I don't know how to do that.

At one point this statement contained the phrase "$K(n)$-good" whose meaning I do not know.

my understanding: in their induction, they need two properties of the EM spaces which together they call $K(n)$-good: first it means that the $K(n)$ homology is in even degrees, so that the $K(n)_0$ determines the $K(n)_*$. Second, it means that $K(n)_0(X)$ is the union of its p^t torsion over all t. It seems like this allows them to define the Dieudonne module on $K(n)_0(X)$, since DM is originally defined for connected Hopf algebras, and then it gets extended to p-nilpotent ones via a general construction (see section 1.4 of ambidexterity). I'm sort of curious why they use $K(n)$-homology throughout the paper though. -AY

Jay asked about the edge case of θ^1, where the only interesting thing happening is the p–adic equivalence. That might be worth expanding in a Remark.

The Qs in the middle should be $\mathbb{Q}_p$, right?

Why is the first component singled out? Also, what's m? What's Y?

that θ^m is an isomorphism, without restriction to subspaces: if it failed to be an injection, we could apply V enough times to get an example of a nontrivial element in $\ker V|_{\mathbb{Q}_p/\mathbb{Z}_p \otimes M^{\wedge m}}$ mapping to zero, and we can manually construct preimages through successive approximation.

Remark 61 ([HL, Proposition 2.4.13]). With this in hand, you now have to pull apart the full p–divisible group to get a calculation of $(\underline{H\mathbb{Z}/p^j}_q)_K$. From this perspective, this is the hardest part with the longest, most convoluted proof.

Remark 62 ([HL, Section 3]). Because $K^*_\Gamma \underline{H\mathbb{Z}/p^j}_q$ is even, you can hope to augment this to a calculation of $E^*_\Gamma \underline{H\mathbb{Z}/p^j}_q$. This is indeed possible, and the analogous formula is true at the level of Hopf algebras:

$$(E_\Gamma)_* \underline{HC}_{p^i q} \cong \mathrm{Alt}^q (E_\Gamma)_* \underline{HC}_{p^i 1}.$$

However, the attendant algebraic geometry is quite complicated: you either need a form of Dieudonné theory that functions over $\mathcal{M}_{E_\Gamma}$ (and then attempt to drag the proof above through that setting), or you need to directly confront what "alternating power of a p–divisible group" means at the level of p–divisible groups (and forego all of the time-saving help afforded to you by Dieudonné theory).

Remark 63. You'll notice that in $K_* \underline{H}_{q+1}$ if we let the q–index tend to ∞, we get the K–homology of a point. This is another way to see that the stable cooperations $K_* H$ vanish, meaning that the *only* information present comes from unstable cooperations.

Things that belong in this chapter

Theorem 6.1 of R–W *The Hopf ring for complex bordism* sounds like something related to Quillen's elementary proof.

There's also a document by Boardman, Johnson, and Wilson (Chapter 2 of the *Handbook of Algebraic Topology*) that discusses an equivalence between Steve's approach and "unstable comodules". Please read this.

Cite me: You could also cite the alternating powers guy [Hed, Hed14] and a source for Dieudonné crystals.

Actually say "Dieudonné crystal".

We could even provide a quick proof of the stable calculation? Cf. http://mathoverflow.net/questions/220952/localization-at-the-johnson-wilson-spectrum-and-rationalization, http://mathoverflow.net/a/99211/1094 .

Maybe talk about some consequences: the Hopkins–Ravenel–Wilson results on finite Postnikov towers and so on?

I was thinking that this would give a counterexample to the idea that the additive unstable cooperations always present the functions on the scheme of homomorphisms, but now I see that this example works too. As lazy evidence, I think counting the ranks of $Q^* K(d)_* \underline{H\mathbb{Z}/p}_*$ and $\underline{\mathrm{FormalGps}}(\Gamma_d, \hat{\mathbb{G}}_a)$ (using Dieudonné theory, or using Callan's tangent space trick) gives the same number. More seriously, I think if you write out the scheme of homomorphisms, you'll see enough things degenerate (because $[-1]_{\hat{\mathbb{G}}_a}(x) = -x$) that you do get this alternating algebra.

Case Study 5
The σ-orientation

5.1 Coalgebraic formal schemes

Today we will discuss an elephant that has been lingering in our room: we began the class talking about the formal scheme associated to the *cohomology* of a space, but we have since become primarily interested in a construction converting the *homology* of a spectrum to a sheaf over a context. Our goal for today is to, when possible, put these on even footing. Our motivation for finally addressing this lingering discrepancy is more technical than aesthetic: we have previously wanted access to certain colimits of formal schemes (e.g., in Theorem 12). While such colimits are generally forbidding, similarly to colimits of manifolds, we will produce certain conditions under which they are accessible.

Suppose that E is a ring spectrum, and recall the usual way to produce the structure of an E^*–algebra on $E^* X$ for X a space. The space X has a diagonal map $\Delta\colon X \to X \times X$, which on E–cohomology induces a multiplication map

$$E^* X \otimes_{E^*} E^* X \xrightarrow{\mu_E} E^* (X \times X) \xrightarrow{E^* \Delta} E^* X.$$

Dually, applying E–homology, we have a pair of maps

$$E_* X \xrightarrow{E_* \Delta} E_*(X \times X) \xleftarrow{\text{Künneth}} E_* X \otimes_{E_*} E_* X.$$

In the case that the Künneth map is an isomorphism, the right–hand map is invertible, and the long composite induces the structure of an E_*–coalgebra on $E_* X$. In the most generous case that E is a field spectrum (in the sense of Corollary 31), $E^* X$ is functorially the linear dual of $E_* X$, which motivates us to consider the following purely algebraic construction:

Definition 73. Let C be a coalgebra over a field k. The scheme $\text{Sch}\, C$ *associated to* C is defined by

$$(\text{Sch}\, C)(T) = \left\{ f \in C \otimes T \;\middle|\; \begin{matrix} \Delta f = f \otimes f \in (C \otimes T) \otimes_T (C \otimes T), \\ \varepsilon f = 1 \end{matrix} \right\}.$$

Lemma 62. *For A a k–algebra, finite–dimensional as a k–module, one has $\text{Spec}\, A \cong \text{Sch}\, A^*$.*

Proof (Proof sketch). A point $f \in (\text{Sch}\, A^*)(T) \subseteq A^* \otimes T$ gives rise to a map $f_* \colon A \to T$ by the duality pairing, which is a ring homomorphism by the condition. The finiteness assumption is present exactly so that A is its own double–dual, giving an inverse assignment.

If we drop the finiteness assumption, then a lot can go wrong. For instance, the multiplication on our k–algebra A gives rise only to maps

$$A^* \to (A \otimes_k A)^* \leftarrow A^* \otimes_k A^*,$$

which is not enough to make A^* into a k–coalgebra. However, if we start instead with a k–coalgebra C of infinite dimension, the following result is very telling:

<div style="border:1px solid green; background:green; color:white; padding:4px;">
Cite me: Demazure's book has this somewhere in it, also see your thesis for a more modern reference; also the ambidexterity paper section 1.1 treats this issue.
</div>

Lemma 63. *For C a coalgebra over a field k, any finite–dimensional k–linear subspace of C can be finitely enlarged to a subcoalgebra of C. Accordingly, taking the colimit gives a canonical equivalence*

$$\text{Ind}\big(\text{Coalgebras}_k^{\text{fin}}\big) \xrightarrow{\simeq} \text{Coalgebras}_k. \,\square$$

This result allows us to leverage our duality Lemma pointwise: for an arbitrary k–coalgebra, we break it up into a lattice of finite k–coalgebras, and take their linear duals to get a reversed lattice of finite k–algebras. Altogether, this indicates that k–coalgebras generally want to model *formal schemes*.

Corollary 43. *For C a coalgebra over a field k expressed as a colimit $C = \text{colim}_k C_k$ of finite subcoalgebras, there is an equivalence*

$$\text{Sch}\, C \cong \{\text{Spec}\, C_k^*\}_k.$$

This induces a covariant *equivalence of categories*

$$\mathsf{Coalgebras}_k \cong \mathsf{FormalSchemes}_{/k}. \qquad \square$$

This covariant algebraic model for formal schemes is very useful. For instance, this equivalence makes the following calculation trivial:

Lemma 64 ((cf. Theorem 12 and)). *Select a coalgebra C over a field k together with a pointing $k \to C$. Write M for the coideal $M = C/k$. The free formal monoid on the pointed formal scheme $\mathrm{Sch}\, k \to \mathrm{Sch}\, C$ is given by*

$$F(\mathrm{Sch}\, k \to \mathrm{Sch}\, C) = \mathrm{Sch}\, \mathrm{Sym}^* M. \qquad \square$$

It is unfortunate, then, that when working over an object more general than a field Lemma 63 fails. Nonetheless, it is possible to bake into the definitions the machinery needed to get a good-enough analogue of Corollary 43.

Definition 74. Let *C* be an *R*–coalgebra which is free as an *R*–module. A basis $\{x_j\}$ of *C* is said to be a *good basis* when any finite subcollection of $\{x_j\}$ can be finitely enlarged to a subcollection that spans a subcoalgebra. The coalgebra *C* is itself said to be *good* when it admits a good basis. A formal scheme *X* is said to be *coalgebraic* when it is isomorphic to $\mathrm{Sch}\, C$ for a good coalgebra *C*.

Theorem 63 ([Str99b, Proposition 4.64]). *Suppose that $F \colon \mathsf{I} \to \mathsf{Coalgebras}_R$ is a colimit diagram of coalgebras such that each object in the diagram, including the colimit point, is a good coalgebra. Then*

$$\mathrm{Sch} \circ F \colon \mathsf{I} \to \mathsf{FormalSchemes}$$

is a colimit diagram of formal schemes. $\qquad \square$

This Theorem gives us access to many constructions on formal schemes, provided we assume that the input is coalgebraic. This covers many of the cases of interest to us, as every formal variety is coalgebraic. For an example of the sort of constructions that become available, one can prove the following Corollary by analyzing the symmetric power of coalgebras:

Corollary 44 ([Str99b, Proposition 6.4]). *For X a coalgebraic formal scheme, $X_{\Sigma_n}^{\times n}$ exists. In fact, $\coprod_{n \geq 0} X_{\Sigma_n}^{\times n}$ models the free formal monoid on X. If $\mathrm{Spec}\, k \to X$ is a pointing, then $\mathrm{colim}_n \{X_{\Sigma_n}^{\times n}\}_n$ models the free formal monoid on the pointed formal scheme.* $\qquad \square$

In the specific case that $\mathrm{Spec}\, k \to X$ is a formal *curve*, we can prove something more:

Corollary 45 ([Str99b, Proposition 6.13]). *For $\mathrm{Spec}\, k \to X$ a pointed formal curve, the free formal monoid is automatically an abelian group.* $\qquad \square$

Margin notes:

- Do you also need to compare the (Cartesian) monoidal structures?
- Backreference the relevance of the Div construction to homotopy theory.
- Describe the diagonal map on this guy.
- Include (a reference to) an example.
- I guess you would prove this by first dévissaging to the case of $\hat{\mathbb{A}}^n$. How does this work? Then you'd just need to prove it for $\hat{\mathbb{A}}^n$ – Jay asked you how to do this for $\hat{\mathbb{A}}^1$, and I think you should include the argument here.

Proof (Proof sketch). The idea is that the symmetric algebra on the coalgebra associated to a formal curve admits a sufficiently nice filtration that one can iteratively solve for a Hopf algebra antipode.

We now reconnect this algebraic discussion with the algebraic topology that spurred it.

Lemma 65. *If E and X are such that E_*X is an E_*–coalgebra and*

$$E^*X = \mathsf{Modules}_{E_*}(E_*X, E_*),$$

then there is an equivalence

$$\mathrm{Sch}\, E_*X \cong X_E. \,\square$$

Proof (Proof sketch). The main point is that the formal topology on X_E is induced by the compactly generated topology of X, and this same topology can also be used to write $\mathrm{Sch}\, E_*X$ as the colimit of finite E_*–coalgebras.

Example 34 (Theorem 59 and Remark 59). For a Morava K–theory K_Γ associated to a formal group Γ of finite height, we have seen that there is an exact sequence of Hopf algebras

$$K_\Gamma^0(BS^1) \xrightarrow{[p^j]} K_\Gamma^0(BS^1) \to K_\Gamma^0(BS^1[p^j]),$$

presenting $(BS^1[p^j])_K$ as the p^j–torsion formal subscheme $BS^1_K[p^j]$. The Hopf algebra calculation also holds in K–homology, where there is instead the following exact sequence

$$(K_\Gamma)_0 B(S^1[p^j]) \to (K_\Gamma)_0 BS^1 \xrightarrow{(-)^{*p^j}} (K_\Gamma)_0 BS^1,$$

presenting $(K_\Gamma)_0 B(S^1[p^j])$ as the p^j–order $*$–nilpotence in the middle Hopf algebra. Applying Sch to this last line covariantly converts this second statement about Hopf algebras to the corresponding statement above about the associated formal schemes — i.e., the behavior of the homology coalgebra is a direct reflection of the behavior of the formal schemes.

The example above also spurs us to consider an intermediate operation. We have seen that the algebra structure of the K–cohomology of a space and the coalgebra structure of the K–homology of the same space contain equivalent data: they both give rise to the same formal scheme. However, in the case at hand, BS^1 and $BS^1[p^j]$ are commutative H–spaces and hence give rise to *commutative and cocommutative Hopf algebras* on both K–cohomology and K–homology. Hence, in addition to considering the coalgebraic formal scheme $\mathrm{Sch}(K_\Gamma)_0 B(S^1[p^j])$, we can also consider the affine

scheme $\mathrm{Spec}(K_\Gamma)_0 B(S^1[p^j])$. This, too, should contain identical information, and this is the subject of Cartier duality.

Definition 75 ([Str99b, Section 6.4]). The *Cartier dual* of a finite group scheme G is defined by
$$DG = \underline{\mathrm{GroupSchemes}}(G, \mathbb{G}_m).$$

Lemma 66 ([Str99b, Proposition 6.19]). *On the level of Hopf algebras $A = \mathcal{O}_G$, this has the effect*
$$DG = \mathrm{Sch}\, A = \mathrm{Spec}\, A^*. \square$$

> You could stand to include a proof of this. It's been a while since you actually proved something serious with formal schemes, and this is pretty nice.

Remark 64. The effect of Cartier duality on the Dieudonné module of a formal group is linear duality. Hence, the covariant and contravariant Dieudonné modules described in Lecture 4.5 are related by Cartier duality.

> You could also include mention of the motivation: $C_k\hat{G}$ is hard to exhibit, but $(C_k\hat{G})^\vee$ is easy. It's not clear how to do this without leaping too far ahead, though.

Remark 65. The topological summary of Cartier duality is that, when X is a free even commutative H–space,
$$DX_E = \underline{\mathrm{GroupSchemes}}(X_E, \mathbb{G}_m) = \mathrm{Spec}\, E_0 X.$$

> Your definition is for finite group schemes, but then you use it for formal groups. One way to extend the definition is by $\underline{\mathrm{FormalGroups}}(G, \hat{\mathbb{G}}_m)$. Another is to take the limit of the objectwise duals in the direct system defining the formal group. Are they the same? Perhaps you should assure the reader that nothing goes wrong?

5.2 Special divisors and the special splitting principle

Starting today, after our extended interludes on chromatic homotopy theory and cooperations, we are going to return to thinking about bordism orientations directly. To begin, we will recall the various perspectives adopted in Case Study 2 when we were studying complex–orientations of ring spectra.

> You should be a little careful here: are the Triumvirate theorems a little stronger than the culmination of (1), (2), and (3)? I think the proof of the equivalence of (2) and (3) on T–points rather than on E_0–points is something slightly harder than the most basic equivalence between (2) and (3) (which concerns only E_0–points).

1. A complex–orientation of E is, definitionally, a map $MUP \to E$ of ring spectra in the homotopy category.
2. A complex–orientation of E is also equivalent to a multiplicative system of Thom isomorphisms for complex vector bundles. Such a system is determined by its value on the universal line bundle $\mathcal{L}$ over $\mathbb{CP}^\infty$. We can also phrase this algebro-geometrically: such a Thom isomorphism is the data of a trivialization of the Thom sheaf $\mathbb{L}(\mathcal{L})$ over $\mathbb{CP}^\infty_E$.
3. Ring spectrum maps $MUP \to E$ induce on E–homology maps $E_0 MUP \to E_0$ of E_0–algebras. This, too, can be phrased algebro-geometrically: these are elements of $(\mathrm{Spec}\, E_0 MUP)(E_0)$.

We can summarize our main result about these perspectives as follows:

Theorem 64 ([AHS01, Example 2.53]). *Take E to be* complex–orientable. *The functor*

$$\mathsf{AffineSchemes}_{/\,\mathrm{Spec}\,E_0} \to \mathsf{Sets},$$

$$(\mathrm{Spec}\,T \xrightarrow{u} \mathrm{Spec}\,E_0) \mapsto \{\text{trivializations of } u^*\mathbb{L}(\mathcal{L}) \text{ over } u^*\mathbb{CP}^\infty_E\}$$

is isomorphic to the affine scheme $\mathrm{Spec}\,E_0 MUP$. *Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MUP \to E$.

Proof (Proof summary). The equivalence between (1) and (2) is given by the splitting principle for complex line bundles. The equivalence between (1) and (3) follows from calculating that $E_0 MUP$ is a free E_0–module.

An analogous result holds for ring spectrum maps $MU \to E$ and the line bundle $\mathcal{L} - 1$, and it is proven in analogous way. In particular, we will want a version of the splitting principle for virtual vector bundles of virtual rank 0. Given a finite complex X and such a rank 0 virtual vector bundle, write

$$\tilde{V} \colon X \to BU$$

for the classifying map. Because X is a finite complex, there exists an integer n so that $\tilde{V} = V - n \cdot 1$ for an honest rank n vector bundle V over X. Using Corollary 14, the splitting $f^*V \cong \bigoplus_{i=1}^n \mathcal{L}_i$ over Y gives a trivialization of $\tilde{V}$ internally to BU as

$$\tilde{V} = V - n \cdot 1 = \bigoplus_{i=1}^n (\mathcal{L}_i - 1),$$

as each bundle $\mathcal{L}_i - 1$ itself has the natural structure of a rank 0 virtual vector bundle. This begets the following analogue of the previous result:

Theorem 65 ([AHS01, Example 2.54]). *Take E to be complex–orientable. The functor*

$$\mathsf{AffineSchemes}_{/\,\mathrm{Spec}\,E_0} \to \mathsf{Sets},$$

$$(\mathrm{Spec}\,T \xrightarrow{u} \mathrm{Spec}\,E_0) \mapsto \{\text{trivializations of } u^*\mathbb{L}(\mathcal{L} - 1) \text{ over } u^*\mathbb{CP}^\infty_E\}$$

is isomorphic to the affine scheme $\mathrm{Spec}\,E_0 MU$. *Moreover, the* E_0*–points of this scheme biject with ring spectrum maps* $MU \to E$. $\qquad\square$

Remark 66. The map $BU \to BU \times \mathbb{Z}$ induces a map $MU \to MUP$. The induced map on schemes is normalization:

Margin notes:

Cite me: Put in cross references.

Cite me: Put in cross references.

Identify this bundle $\mathbb{L}(\mathcal{L} - 1)$ as $\omega \otimes \mathcal{I}(0)^{-1}$, we can think of sections as elements of $E^0 T(\mathcal{L} - 1 \to \mathbb{CP}^\infty)$ which restrict to the identity under the inclusion

$$S^0 \to P^{\mathcal{L}-1}.$$

Justify this.

$$\operatorname{Spec} E_0 MUP \to \operatorname{Spec} E_0 MU,$$

$$f \mapsto \frac{f'(0)}{f}.$$

These two Thom spectra are the beginning of a larger pattern. Their base spaces $BU \times \mathbb{Z}$ and BU are both infinite loopspaces: they are $\underline{kU}_0$ and $\underline{kU}_2$ respectively, where kU is the connective complex K–theory spectrum. In general, the space $\underline{kU}_{2k}$ is given as a connective cover:

> Justify this as a Lemma.

$$\underline{kU}_{2k} = BU[2k, \infty),$$

and so the next Thom spectrum in the sequence is MSU, the bordism theory of SU–structured manifolds. The special unitary group SU is explicit enough that these orientations can be fully understood along similar lines to what we have done so far. Our jumping off point for that story will be, again, an extension of the splitting principle.

Lemma 67. *Let X be a finite complex, and let $\tilde{V}\colon X \to BU$ classify a virtual vector bundle of rank 0 over X. Select a factorization $\tilde{V}\colon X \to BSU$ of $\tilde{V}$ through BSU. Then, there is a space $f\colon Y \to X$, where $f_E\colon Y_E \to X_E$ is finite and flat, as well as a collection of line bundles $\mathcal{H}_j, \mathcal{H}'_j$ expressing a BSU–internal decomposition*

> Hood had a question here: what symmetry group acts on this splitting? I think it's possible to show that the full symmetric group action on the $BU \times \mathbb{Z}$ splitting induces a full symmetric group action on the BSU splitting. I don't know if there's any more or any less.

$$\tilde{V} = -\bigoplus_{j=1}^{n}(\mathcal{H}_j - 1)(\mathcal{H}'_j - 1).$$

> The minus sign: I understand mechanically where it comes from, but I still find it odd. Can we recover the BU-splitting principle from this splitting principle – and what happens to the signs?

Proof. Begin by using Corollary 14 on V to get an equality of BU–bundles

$$\tilde{V} = V' + \mathcal{L}_1 + \mathcal{L}_2 - n \cdot 1.$$

Adding $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)$ to both sides, this gives

$$\begin{aligned}
\tilde{V} + (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) &= V' + \mathcal{L}_1 + \mathcal{L}_2 + (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) - n \cdot 1 \\
&= V' + \mathcal{L}_1 \mathcal{L}_2 - (n - 1) \cdot 1.
\end{aligned}$$

By thinking of $(\mathcal{L}_j - 1)$ as an element of $kU^2(Y) = \mathsf{Spaces}(Y, BU)$, we see that the product element $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) \in kU^4(Y) = \mathsf{Spaces}(Y, BSU)$ has the natural structure of a BSU–bundle and hence so does the sum on the left-hand side[1]. The right-hand side is the rank 0 virtualization of a rank $(n-1)$ vector bundle, hence succumbs to induction. Finally, because $SU(1)$ is the trivial group, there are no nontrivial complex line bundles with structure group $SU(1)$, grounding the induction.

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[1] In the language of last section, we are making use of the Hopf ring $\circ$–product.

Corollary 46. *Ring spectrum maps $MSU \to E$ biject with trivializations of*

$$\mathbb{L}((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)) \downarrow (\mathbb{CP}^\infty)_E^{\times 2}.\square$$

Remark 67. Since we used the product map

$$kU^2(Y) \otimes kU^2(Y) \to kU^4(Y)$$

in the course of the proof, it is also interesting to consider the product map

$$kU^4(Y) \otimes kU^0(Y) \to kU^4(Y).$$

Taking one of our splitting summands $(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)$ and acting by some line bundle $\mathcal{H}$ gives

$$(\mathcal{L}_1 - 1)\mathcal{H}(\mathcal{L}_2 - 1) =$$
$$(\mathcal{H}\mathcal{L}_1 - \mathcal{H})(\mathcal{L}_2 - 1) = (\mathcal{L}_1 - 1)(\mathcal{H}\mathcal{L}_2 - \mathcal{H})$$
$$(\mathcal{H}\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1) - (\mathcal{H} - 1)(\mathcal{L}_2 - 1) = (\mathcal{L}_1 - 1)(\mathcal{H}\mathcal{L}_2 - 1) - (\mathcal{L}_1 - 1)(\mathcal{H} - 1).$$

> I think it's more symmetric if you write $\mathcal{L}_1\mathcal{H}$ instead of $\mathcal{H}\mathcal{L}_1$.

This "kU^0–linearity" is sometimes called a "2–cocycle condition", in reference to the similarity with the formula in Definition 50.

If we can show that $E_* BSU$ is even–concentrated and free as an E_0–module, then this will complete the BSU–analogue of Theorems 64 and 65. This is quite easy, following directly from the Serre spectral sequence:

Lemma 68 ([AS01, Lemma 6.1]). *The Postnikov fibration*

$$BSU \to BU \xrightarrow{B\det} BU(1)$$

induces a short exact sequence of Hopf algebras

$$E^0 BSU \leftarrow E^0 BU \xleftarrow{c_1 \leftarrow c_1} E^0 BU(1).\square$$

Corollary 47. *The functor*

$$\{\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0\} \to \{\textit{trivializations of } u^*\mathbb{L}((\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)) \textit{ over } u^*\mathbb{CP}_E^\infty\}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MSU$. *Moreover, the E_0–points of this scheme biject with ring spectrum maps $MSU \to E$.* $\square$

However, the use of Lemma 68 inspires us to spend a moment longer with the associated formal schemes. An equivalent statement is that there is a short exact sequence of formal group schemes

$$BSU_E \longrightarrow BU_E \xrightarrow{B\det} BU(1)$$

$$\text{SDiv}_0 \, \mathbb{CP}_E^\infty \longrightarrow \text{Div}_0 \, \mathbb{CP}_E^\infty \xrightarrow{\text{sum}} \mathbb{CP}_E^\infty,$$

where the scheme "$\text{SDiv}_0 \, \mathbb{CP}_E^\infty$" of "special divisors" consists of those divisors which vanish under the summation map. However, where the comparison map $BU(1)_E \to BU_E$ has an identifiable universal property — it presents BU_E as the universal formal group on the pointed curve $BU(1)_E$ — the description of BSU_E as a scheme of special divisors does not bear much immediate resemblance to a free object on the special divisor $([a] - [0])([b] - [0])$ classified by

$$(\mathbb{CP}^\infty)_E^{\times 2} \xrightarrow{(\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)_E} BSU_E \to BU_E = \text{Div}_0 \, \mathbb{CP}_E^\infty.$$

It would be wise of us to straighten this out before moving on.

Definition 76. If it exists, let $C_2\widehat{\mathbb{G}}$ denote the symmetric square of $\text{Div}_0 \, \widehat{\mathbb{G}}$, thought of as a module over $\text{Div} \, \widehat{\mathbb{G}}$. This scheme has the property that a formal group homomorphism $\varphi \colon C_2\widehat{\mathbb{G}} \to H$ is equivalent data to a symmetric function $\psi \colon \widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying a rigidity condition ($\psi(x, 0) = 0$) and a 2–cocycle condition as in Remark 67.

Theorem 66 (Ando–Hopkins–Strickland, unpublished). $\text{SDiv}_0 \, \widehat{\mathbb{G}}$ is a model for $C_2\widehat{\mathbb{G}}$.

Proof (Proof sketch). Consider the map

$$\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to \text{Div}_0 \, \widehat{\mathbb{G}},$$
$$(a, b) \mapsto ([a] - [0])([b] - [0])$$

for which there is a factorization of formal schemes

$$\widehat{\mathbb{G}} \times \widehat{\mathbb{G}}$$

$$F \xrightarrow{\text{ker}} \text{Div}_0 \, \widehat{\mathbb{G}} \xrightarrow{\sigma} \widehat{\mathbb{G}}$$

because

$$\sigma(([a] - [0])([b] - [0])) = (a + b) - a - b + 0 = 0.$$

One can check that a homomorphism $F \to H$ pulls back to a function $\widehat{\mathbb{G}} \times \widehat{\mathbb{G}} \to H$ satisfying the properties of Definition 76. To go the other way[2], we

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[2] To get insight into how this part of the proof works, actually write out the expressions for $\tilde{V} = \bigoplus_{i=1}^6 \mathcal{L}_i - 6 \cdot 1 = \bigoplus_{i=1}^3 (\mathcal{L}_i - 1) \oplus \bigoplus_{j=4}^6 (\mathcal{L}_j - 1)$ and see what happens.

select a function $\psi\colon \widehat{G} \times \widehat{G} \to H$ and mimic the construction in Lemma 67. Expanding the definition of $\mathrm{Div}_0 \widehat{G}$, we are moved to consider the object $\widehat{G}^{\times k}$ parametrizing weight k divisors with a full set of sections, where we define a map

$$\widehat{G}^{\times k} \to H,$$

$$(a_1, \ldots, a_k) \mapsto -\sum_{j=2}^{k} \psi \left(\sum_{i=1}^{j-1} a_i, a_j \right).$$

This gives a compatible system of symmetric maps and hence bundles together to give a map $\tilde{\varphi}\colon \mathrm{Div}_0 \widehat{G} \to H$ off of the colimit. In general, this map is not a homomorphism, but it is a homomorphism when restricted to $\varphi\colon F \to \mathrm{Div}_0 \widehat{G} \to H$. Finally, one checks that any homomorphism $F \to H$ of formal groups restricting to the zero map $\widehat{G} \times \widehat{G} \to H$ was already the zero map, and this gives the desired identification of F with the universal property of $C_2\widehat{G}$.

In the next interesting case of $\underline{kU_6} = BU[6, \infty)$, there is not an accessible splitting principle. This not only makes the topology harder, but it also makes proving the existence of the symmetric cube $C_3\widehat{G}$ harder, as there is no model to work from. Nonetheless, there is a $BU[6, \infty)$–analogue of Theorem 64, Theorem 65, and Corollary 47. In order to prove it, since we don't have access to an equivalence between viewpoints (1) and (2), we will have to instead prove an equivalence between (2) and (3) directly.

5.3 Elliptic curves and θ–functions

Today will constitute something of a résumé on elliptic curves. We'll hardly prove anything, and we also won't cover many topics that a sane introduction to elliptic curves would make a point to cover. Instead, we'll try to restrict attention to those concepts which will be of immediate use to us in the coming couple of lectures — in particular, we will discover a place where "$C_3\widehat{G}$" appears internally to the theory of elliptic curves.

To begin with, recall that an elliptic curve in the complex setting is a torus, and it admits a presentation by selecting a lattice Λ of full rank in $\mathbb{C}$ and forming the quotient

$$\mathbb{C} \xrightarrow{\pi_\Lambda} E_\Lambda = \mathbb{C}/\Lambda.$$

The meromorphic functions f on E_Λ pull back to give meromorphic functions $\pi_\Lambda^* f$ on $\mathbb{C}$ satisfying a periodicity constraint in the form of the func-

tional equation

$$\pi_\Lambda^* f(z + \Lambda) = \pi_\Lambda^* f(z).$$

From this, it follows that there are no holomorphic such functions, save the constants — such a function would be bounded, and Liouville's theorem would apply. It is, however, possible to build the following meromorphic special function, which has poles of order 2 at the lattice points:

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}.$$

Its derivative is also a meromorphic function satisfying the periodicity constraint:

$$\wp_\Lambda'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3}.$$

In fact, these two functions generate all other meromorphic functions on E_Λ, in the sense that the subsheaf spanned by the algebra generators $\wp_\Lambda$ and $\wp_\Lambda'$ is exactly $\pi_\Lambda^* \mathcal{M}_{E_\Lambda}$. This algebra is subject to the following relation, in the form of a differential equation:

$$\wp_\Lambda'(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp_\Lambda(z) - g_3(\Lambda),$$

for some special values $g_2(\Lambda)$ and $g_3(\Lambda)$. Accordingly, writing $C \subseteq \mathbb{CP}^2$ for the projective curve $wy^2 = 4x^3 - g_2(\Lambda)w^2 x - g_3(\Lambda)w^3$, there is an analytic group isomorphism

$$E_\Lambda \to C,$$
$$z \pmod{\Lambda} \mapsto [1 : \wp_\Lambda(z) : \wp_\Lambda'(z)].$$

This is sometimes referred to as the Weierstrass presentation of E_Λ.

There is a second standard embedding of a complex elliptic curve into projective space, using θ–functions, which are most naturally expressed *multiplicatively*. To begin, select a lattice Λ and a basis for it, and rescale the lattice so that the basis takes the form $\{1, \tau\}$ with τ in the upper half-plane. Then, the normalized exponential function $z \mapsto \exp(2\pi i z)$ has $1 \cdot \mathbb{Z}$ as its kernel, and setting $q = \exp(2\pi i \tau)$ we get a second presentation of E_Λ as $\mathbb{C}^\times / q^{\mathbb{Z}}$.

I think it's helpful to draw a picture here of an annulus with some identification made.

The associated θ–function is defined by

$$\theta_q(u) = \prod_{m \geq 1} (1 - q^m)(1 + q^{m - \frac{1}{2}} u)(1 + q^{m - \frac{1}{2}} u^{-1}) = \sum_{n \in \mathbb{Z}} u^n q^{\frac{1}{2} n^2}.$$

It vanishes on the set $\{\exp(2\pi i(\frac{1}{2} m + \frac{\tau}{2} n))\}$, i.e., at the center of the fundamental annulus. However, since it has no poles it cannot descend to give a

function on $\mathbb{C}^\times/q^{\mathbb{Z}}$. A different obstruction to this descent is its imperfect periodicity relation:

$$\theta_q(qu) = u^{-1}q^{\frac{-1}{2}}\theta_q(u).$$

This isn't stated well.

We can also shift the zero-set of θ_q by rational rescalings a of q and b of 1:

$$\theta_q^{a,b}(u) = q^{\frac{a^2}{2}} \cdot u^a \cdot \exp(2\pi iab)\theta_q(uq^a \exp(2\pi ib)).$$

Remark 68 ([Hus04, Proposition 10.2.6]). For any $N > 0$, define $V_\tau[N]$ to be the space of entire functions f with $f(z + N) = f(z)$ and $f(z + \tau) = e^{-2\pi iNz - \pi iN^2\tau}f(z)$. Then, $V_\tau[N]$ has $\mathbb{C}$–dimension N^2, and the functions $\theta_\tau^{a,b}$

At this point you swapped notation from θ_q to θ_τ.

give a basis by picking representatives (a_i, b_i) of the classes in $((1/N)\mathbb{Z}/\mathbb{Z})^2$.

Even though these functions do not themselves descend to $\mathbb{C}^\times/q^{\mathbb{Z}}$, we can collectively use them to construct a map to complex projective space, where the quasi-periodicity relations will mutually cancel in homogeneous coordinates.

Theorem 67 ([Hus04, Proposition 10.3.2]). *Consider the map*

$$\mathbb{C}/N(\Lambda) \xrightarrow{f_{(N)}} \mathbb{P}^{N^2-1}(\mathbb{C}),$$

$$z \mapsto [\cdots : \theta_\tau^{i/N,j/N}(z) : \cdots].$$

What's $N(\Lambda)$?

For $N > 1$, this map is an embedding. $\qquad\qquad\square$

Example 35. One can work out how it goes for $N = 2$, which will cause some of our pesky $\frac{1}{2}$'s to cancel. The four functions there are $\theta_q^{0,0}$ with zeroes on $\Lambda + \frac{\tau+1}{2}$, $\theta_q^{0,1/2}$ with zeroes on $\Lambda + \frac{\tau}{2}$, $\theta_q^{1/2,0}$ with zeroes on $\Lambda + \frac{1}{2}$, and $\theta_q^{1/2,1/2}$ with zeroes on Λ exactly. The image of $f_{(2)}$ in $\mathbb{P}^{2^2-1}(\mathbb{C})$ is cut out by the equations

$$A^2x_0^2 = B^2x_1^2 + C^2x_2^2, \qquad\qquad A^2x_3^2 = C^2x_1^2 - B^2x_2^2,$$

where

$$x_0 = \theta_\tau^{0,0}(2z), \quad x_1 = \theta_\tau^{0,1/2}(2z), \quad x_2 = \theta_\tau^{1/2,0}(2z), \quad x_3 = \theta_\tau^{1/2,1/2}(2z)$$

and

$$A = \theta_\tau^{0,0}(0) = \sum_n q^{n^2}, \quad B = \theta_\tau^{0,1/2}(0) = \sum_n (-1)^n q^{n^2}, \quad C = \theta_\tau^{1/2,0}(0) = \sum_n q^{(n+1/2)^2}$$

upon which there is the additional "Jacobi" relation

$$A^4 = B^4 + C^4.$$

Remark 69. This embedding of E_Λ as an intersection of quadric surfaces in $\mathbb{C}P^3$ is quite different from the Weierstrass embedding. Nonetheless, the embeddings are analytically related. Namely, there is an equality

$$\frac{d^2}{dz^2} \log \theta_{\exp 2\pi i \tau}(\exp 2\pi i z) = \wp_\Lambda(z).$$

Separately, Weierstrass considered a function σ_Λ, defined by

$$\sigma_\Lambda(z) = z \prod_{\omega \in \Lambda \backslash 0} \left(1 - \frac{z}{\omega}\right) \cdot \exp\left[\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right],$$

which also has the property that its second logarithmic derivative is $\wp$ and so is "basically $\theta_q^{1/2,1/2}$". In fact, any elliptic function can be written in the form

$$c \cdot \prod_{i=1}^{n} \frac{\sigma_\Lambda(z - a_i)}{\sigma_\Lambda(z - b_i)}.$$

The θ–functions version of the story has two main successes. One is that there is a version of this story for an arbitrary abelian variety. It turns out that all abelian varieties are projective, and the theorem sitting at the heart of this claim is

Corollary 48 (Theorem of the cube). *Let A be an abelian variety, let $p_i : A \times A \times A \to A$ be the projection onto the ith factor, and let $p_{ij} = p_i +_A p_j$, $p_{ijk} = p_i +_A p_j +_A p_k$. Then for any invertible sheaf $\mathcal{L}$ on A, the sheaf*

$$\Theta^3(\mathcal{L}) := \frac{p_{123}^*\mathcal{L} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}}{p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_{31}^*\mathcal{L} \otimes p_\varnothing^*\mathcal{L}} = \bigotimes_{I \subseteq \{1,2,3\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}$$

on $A \times A \times A$ is trivial. If $\mathcal{L}$ is rigid, then $\Theta^3(\mathcal{L})$ is canonically trivialized by a section $s(A; \mathcal{L})$. $\qquad\square$

Remark 70. The section $s(A; \mathcal{L})$ satisfies three familiar properties:

- It is symmetric: pulling back $\Theta^3\mathcal{L}$ along a shuffle automorphism of A^3 yields $\Theta^3\mathcal{L}$ again, and the pullback of the section $s(A; \mathcal{L})$ along this shuffle agrees with the original $s(A; \mathcal{L})$ across this identification.
- It is rigid: by restricting to $* \times A \times A$, the tensor factors in $\Theta^3\mathcal{L}$ cancel out to give the trivial bundle over $A \times A$. The restriction of the section $s(A; \mathcal{L})$ to this pullback bundle agrees with the extension of the rigidifying section.
- It satisfies a 2–cocycle condition: in general, we define

$$\Theta^k\mathcal{L} := \bigotimes_{I \subseteq \{1,\dots,k\}} (p_I^*\mathcal{L})^{(-1)^{|I|-1}}.$$

What's the other success?

Cite me: Milne's abelian varieties, Theorem 7.1.

Several people wanted some sketch of how to use this theorem to prove the projectivity thing. I don't have any good sketch, and neither did Erick (offhand).

Cite me: Milne's Abelian Varieties chapter, Corollary 6.4 and Theorem 7.1.

What is the rigidifying section?

In fact, $\Theta^{k+1}\mathcal{L}$ can be written as a pullback of $\Theta^k\mathcal{L}$:

$$\Theta^{k+1}\mathcal{L} = \frac{(p_{12} \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}}{(p_1 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L} \otimes (p_2 \times \mathrm{id}_{A^{k-1}})^*\mathcal{L}},$$

and pulling back a section s along this map gives a new section

$$(\delta s)(x_0, x_1, \dots, x_k) := \frac{s(x_0 +_A x_1, x_2, \dots, x_k)}{s(x_0, x_2, \dots, x_k) \cdot s(x_1, x_2, \dots, x_k)}.$$

Performing this operation on the first and second factors yields the defining equation of a 2–cocycle.

Remark 71. The proof of projectivity arising from this method rests on choosing an ample line bundle on A and constructing some generating global sections to get an embedding into $\mathbb{P}(\mathcal{L}^{\oplus n})$. Mumford showed that a choice of "θ–structure" on A, which is only slightly more data given in terms of Heisenberg representations, gives a canonical identification of $\mathbb{P}(\mathcal{L}^{\oplus n})$ with a *fixed* projective space. This is suitable for studying how these equations change as one considers different points in the *moduli* of abelian varieties. Separately, Breen showed that if $\mathcal{L}$ is a line bundle on A with a chosen trivialization of $\Theta^3\mathcal{L}$ and $\pi\colon A' \to A$ is an epimorphism that trivializes $\mathcal{L}$, then one can also associate to this a theory of θ–functions.

5.4 Unstable chromatic cooperations for kU

Let Γ be a formal group of finite p–height of a field k of positive characteristic p, and let $E = E_\Gamma$ denote the associated Morava E–theory. Our goal in this section is to get a partial description of the Hopf ring of unstable cooperations $(E_\Gamma)_* \underline{kU}_{2*}$. Our results in previous sections give a foothold into this analysis by computing

$$E_0(BU \times \mathbb{Z}), \ E_0 BU, \ E_0 BSU$$

in terms of the affine schemes they represent. We also saw that these results were the cornerstone for accessing descriptions of the schemes

$$\mathrm{Spec}\, E_0 MUP, \ \mathrm{Spec}\, E_0 MU, \ \mathrm{Spec}\, E_0 MSU.$$

In particular, the next step is to understand $E_0 BU[6, \infty)$, and our main tool for doing this will be the Postnikov fibration

$$\underline{H\mathbb{Z}}_3 \to BU[6,\infty) \to BSU.$$

Our main goals are to construct a model sequence of formal schemes, then show that E–theory is well-behaved enough that the formal schemes it constructs exactly match the model.

The main tool used to build the model is the following construction:

Definition 77. A map $f\colon X \to Y$ of spaces induces a map $f_E\colon X_E \to Y_E$ of formal schemes. In the case that Y is a commutative H–space and Y_E is connected, we can construct a map according to the composite

$$
\begin{array}{ccc}
X_E \times \underline{\mathsf{GroupSchemes}}(Y_E, \mathbb{G}_m) & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \widehat{\mathbb{A}}^1 \\
\| & & \uparrow \\
X_E \times \underline{\mathsf{FormalGroups}}(Y_E, \widehat{\mathbb{G}}_m) & \xrightarrow{f \times 1} Y_E \times \underline{\mathsf{FormalGroups}}(Y_E, \widehat{\mathbb{G}}_m) \xrightarrow{\mathrm{ev}} & \widehat{\mathbb{G}}_m.
\end{array}
$$

This is called *the adjoint map*, and we write $\hat{f}$ for the version of this map valued variously in $\widehat{\mathbb{G}}_m$, $\mathbb{G}_m$, and $\widehat{\mathbb{A}}^1$. It encodes equivalent information to the map of E_*–modules

$$E_* \to E_* Y \widehat{\otimes}_{E_*} E^* X$$

by applying the map to $1 \in E_*$.

Lemma 69. *This construction converts many properties of f into corresponding properties of this adjoint element. For instance:*

- *It is natural in the source: for $w \in F^n(X)$ and $\gamma\colon \underline{E}_n \to \underline{D}_n$, there is*

$$(1 \times \mathrm{Spec}\, E_0 \gamma) \circ \hat{w} = \widehat{\gamma_* w}.$$

- *It converts sums of classes to products of maps to $\mathbb{G}_m$.* $\qquad\qquad\square$

  In Lecture 5.2, we became interested in the class Π_2, defined by

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\Pi_2 := (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)} \underline{kU}_4 = BSU.$$

The adjoint to this cohomology class is a map of formal schemes

$$\hat{\Pi}_2\colon (\mathbb{CP}^\infty_E)^{\times 2} \times_{\mathbb{S}_E} \mathrm{Spec}\, E_0 BSU \to \mathbb{G}_m,$$

which using the exponential adjunction can be interpreted as a map

$$\mathrm{Spec}\, E_0 BSU \to \underline{\mathsf{FormalSchemes}}((\mathbb{CP}^\infty_E)^{\times 2}, \mathbb{G}_m).$$

Because the adjoint construction preserves properties of the class Π_2, we learn that this map factors through a particular closed subscheme

$$C^2(\mathbb{CP}_E^\infty; \mathbb{G}_m) \subseteq \underline{\text{FormalSchemes}}((\mathbb{CP}_E^\infty)^{\times 2}, \mathbb{G}_m)$$

of symmetric, rigid functions satisfying the 2–cocycle condition. By careful manipulation of divisors in Theorem 66, we showed that $BSU_E \cong \text{SDiv}_0\, \mathbb{CP}_E^\infty$, which on applying Cartier duality shows that our induced map

$$\text{Spec}\, E_0 BSU \to C^2(\mathbb{CP}_E^\infty; \mathbb{G}_m)$$

is an isomorphism.

Definition 78. Similarly, we define a cohomology class

$$\Pi_3 = (\mathcal{L}_1 - 1)(\mathcal{L}_2 - 1)(\mathcal{L}_3 - 1) \in kU^6(\mathbb{CP}^\infty)^{\times 3}.$$

It induces an adjoint map

$$\hat{\Pi}_3 \colon\ \text{Spec}\, E_0 BU[6, \infty) \to C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m),$$

where $C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m)$ is the scheme of $\mathbb{G}_m$–valued trivariate functions on $\mathbb{CP}_E^\infty$ satisfying symmetry, rigidity, and a 2–cocycle condition. (If $C_3\mathbb{CP}_E^\infty := \text{Sym}^3_{\text{Div}\,\mathbb{CP}_E^\infty} \text{Div}_0\, \mathbb{CP}_E^\infty$ were to exist, this would be its Cartier dual.)

Lemma 70 ([AS01, Lemma 7.1]). *There is a commutative square*

$$
\begin{array}{ccc}
\text{Spec}\, E_0 BSU & \longrightarrow & \text{Spec}\, E_0 BU[6, \infty) \\
\downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_3} \\
C^2(\mathbb{CP}_E^\infty; \mathbb{G}_m) & \xrightarrow{\ \delta\ } & C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m),
\end{array}
$$

where

$$\delta(f)(x_1, x_2, x_3) := \frac{f(x_1 +_E x_2, x_3)}{f(x_1, x_3) f(x_2, x_3)}.$$

Proof. This is checked by calculating $\Pi_3 = (\mu_{12}^* - \pi_1^* - \pi_2^*)\Pi_2$.

With this now in hand, we have constructed the solid maps in the following diagram:

$$
\begin{array}{ccccc}
\text{Spec}\, E_0 BSU & \longrightarrow & \text{Spec}\, E_0 BU[6, \infty) & \longrightarrow & \text{Spec}\, E_0 \underline{H\mathbb{Z}}_3 \\
{\scriptstyle \cong}\downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_3} & & {\scriptstyle \cong}\downarrow \\
C^2(\mathbb{CP}_E^\infty; \mathbb{G}_m) & \xrightarrow{\ \delta\ } & C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m) & \xdashrightarrow{\ e_*\ } & \underline{\text{FormalGroups}}((\mathbb{CP}_E^\infty)^{\wedge 2}, \hat{\mathbb{G}}_m)
\end{array}
$$

We would like to prove enough about this diagram to show that is an isomorphism of short exact sequences.

Before we begin testing exactness, we first need a pair of sequences — i.e., we must construct the map e. There is a candidate construction, coming from the theory of θ–functions:

Cartier duality seems quite important to these examples. Could you say more about this construction, either here or on the day about Dieudonné modules?

This is not really a definition…

Is this bad notation?

Where did δ come from? A previous day? Talk about this some, probably.

Jun Hou asked if this map fits into a larger chain complex — presumably because of the name "δ". The problem is that δ is always bound to moving from 1–cocycles to 2–cocycles. But maybe "τ" is worth mentioning further on...

You wrote Π_2 and Π_3, but these are not exactly the maps Π_2 and Π_3. They're not even the adjoint maps $\hat{\Pi}_2$ and $\hat{\Pi}_3$. I don't think you have any other options since introducing even more notation would be cumbersome, but perhaps you could make a small remark when you write down the induced maps that you're also calling them Π_2 and Π_3.

The notation μ and π does not comport with that from the previous lecture.

You haven't said anything about why the rightmost vertical map is an isomorphism.

Was there any motivation or reason why one might look to elliptic function theory to supply this map?

Definition 79. Let A be an abelian variety equipped with a line bundle $\mathcal{L}$. Suppose that s is a symmetric, rigid section of $\Theta^3 \mathcal{L}$, i.e., a *cubical structure on* $\mathcal{L}$. This induces the structure of a *symmetric biextension* on $\Theta^2 \mathcal{L}$ by furnishing compatible multiplication maps

$$(\Theta^2 \mathcal{L})_{x,y} \otimes (\Theta^2 \mathcal{L})_{x',y} \to (\Theta^2 \mathcal{L})_{x+x',y}, \quad (\Theta^2 \mathcal{L})_{x,y} \otimes (\Theta^2 \mathcal{L})_{x,y'} \to (\Theta^2 \mathcal{L})_{x,y+y'}.$$

There is a canonical piece of gluing data on this biextension, in the form of an isomorphism of pullback bundles

$$e_{p^j} \colon (p^j \times 1)^* \mathcal{L}|_{A[p^j] \times A[p^j]} \cong (1 \times p^j)^* \mathcal{L}|_{A[p^j] \times A[p^j]},$$

$$(\ell, x, y) \mapsto \left(\ell \cdot \prod_{k=1}^{p^j-1} \frac{s(x, [k]x, y)}{s(x, [k]y, y)} \right).$$

This function e_{p^j} is called the *(p^{jth}) Weil pairing.*

> Hood wanted me to write out what the relevant pullbacks were, and I pretty well refused. We should here, at least. The point is that $(p_{12} - p_1 - p_2)^* \Theta^2 \mathcal{L} = \Theta^3 \mathcal{L}$.

> You might also want to give a second definition to the alternating-fraction one, using something like the tensor product of reduced line bundles, so that the link to the topology is more directly understood.

> Can you prove some parts of this more carefully?

Remark 72. In the case that A is an elliptic curve, this agrees with the usual definition of its "Weil pairing".

Lemma 71. *The Weil pairings assemble into a total Weil pairing on the p–divisible group associated to A. Together, the total Weil pairing is alternating and biexponential.* □

> Cite me: AS, I guess.

> What does biexponential mean?

We can use the same formula in the setting of a cubical structure on a line bundle over a finite height formal group $\widehat{\mathbb{G}}$ to produce the desired map

$$e_* \colon C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}(\widehat{\mathbb{G}}^{\wedge 2}, \mathbb{G}_m).$$

In fact, this is the right map:

> How? What is ℓ?

Lemma 72 ([AS01, Theorem 4.2, Corollary 4.4]). *The square commutes (up to sign):*

$$\begin{array}{ccc}
\mathrm{Spec}\, E_0 BU[6, \infty) & \longrightarrow & \mathrm{Spec}\, E_0 K(\mathbb{Z}, 3) \\
\downarrow {\scriptstyle \Pi_3} & & \downarrow {\scriptstyle b_*} \\
C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m) & \xrightarrow{\quad e \quad} & \mathrm{Weil}(\mathbb{CP}_E^\infty).
\end{array}$$

> You have not introduced the notation Weil.

Proof (Proof sketch). This is reasonably difficult. The main points are to show that the restriction

$$d_{p^j} \colon BC_{p^j}^{\times 2} \xrightarrow{\beta \circ \mu} \underline{H\mathbb{Z}}_3 \to BU[6, \infty)$$
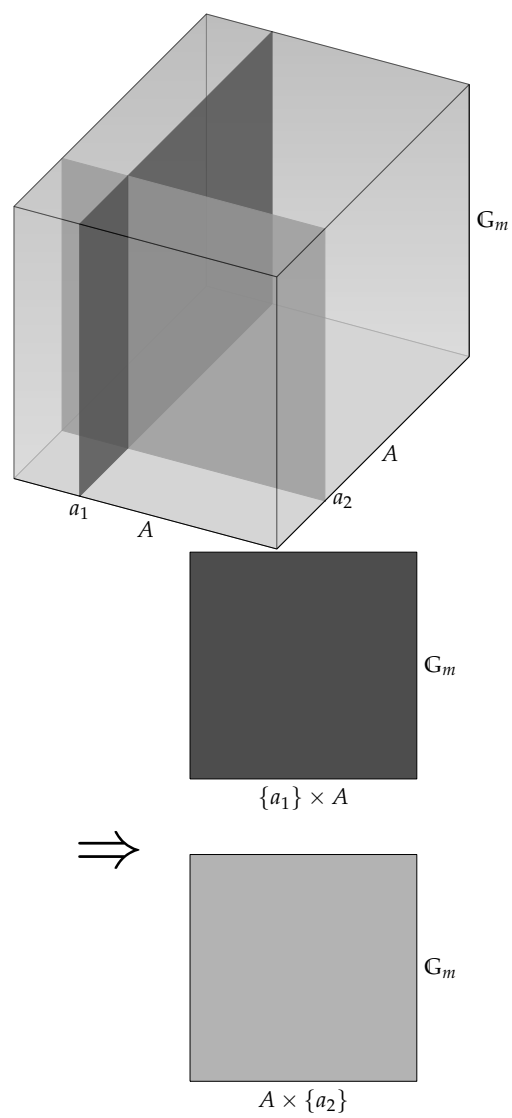
Fig. 5.1 Extensions contained in a biextension.

can be expressed by the Weil pairing formula:

$$d_{p^j} = \bigoplus_{k=1}^{p^j-1} \left((\mathcal{L}_1 - 1)(\mathcal{L}_1^{\otimes k} - 1)(\mathcal{L}_2 - 1) - (\mathcal{L}_1 - 1)(\mathcal{L}_2^{\otimes k} - 1)(\mathcal{L}_2 - 1) \right).$$

After this is accomplished, what's left is to use naturality properties of the adjoint construction to compute the clockwise and counterclockwise composites.

> Where does d_{p^j} appear in the diagram above?

This fills out the diagram we are considering. We now assemble just enough exactness results:

Lemma 73 ([AS01, Lemma 7.2]). *The map $\delta\colon C^2 \to C^3$ is injective for $\mathbb{CP}_E^\infty$ a finite height formal group.*

Proof. Being finite height means that the multiplication-by-p map of $\mathbb{CP}_E^\infty$ is fppf–surjective. The kernel of δ consists of alternating, biexponential maps $(\mathbb{CP}_E^\infty)^{\times 2} \to \mathbb{G}_m$. By restricting such a map f to

$$f\colon \mathbb{CP}_E^\infty[p^j] \times \mathbb{CP}_E^\infty \to \mathbb{G}_m,$$

we can calculate

$$f(x, p^j y) = f(p^j x, y) = f(0, y) = 1.$$

But since p^j is surjective on $\mathbb{CP}_E^\infty$, every point on the right-hand side can be so written, so at every left-hand stage the map is trivial. Finally, $\mathbb{CP}_E^\infty = \mathrm{colim}_j \mathbb{CP}_E^\infty[p^j]$, so this filtration is exhaustive and we conclude that the kernel is trivial.

> This doesn't sound like the proof that you and Hood gave in class? Is that a different proof?

Lemma 74 ([AS01, Lemma 7.3]). *In fact, the following sequence is exact*

$$0 \to C^2(\mathbb{CP}_E^\infty; \mathbb{G}_m) \xrightarrow{\delta} C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m) \to \mathrm{Weil}(\mathbb{CP}_E^\infty).$$

Proof. This is hard work. Breen's idea is to show that picking a preimage under δ is the same as picking a trivialization of the underlying symmetric biextension of the cubical structure. Then (following Mumford), one shows that the underlying symmetric biextension is trivial exactly if the Weil pairing is trivial.

> You could put in references here. AS gives them.

Finally, the top row falls quickly:

Lemma 75 ([AS01, Lemma 7.5]). *The top row of the main diagram is a short exact sequence of group schemes.*

> Remark: This proof also shows that the E–theory of $\underline{kU}_8$ fits into a sexseq.

Proof. This is easiest proved by considering the sequence of homology Hopf algebras instead. Since the integral homology of BSU and the E–homology of $\underline{H\mathbb{Z}}_3$ are both free and even, the Atiyah–Hirzebruch spectral sequence for $E_*BU[6,\infty)$ collapses.

Corollary 49. *The map*

$$\Pi_3 \colon \ \operatorname{Spec} E_0 BU[6,\infty) \to C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m)$$

is an isomorphism, and the map

$$e_* \colon \ C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m) \to \underline{\mathsf{FormalGroups}}((\mathbb{CP}_E^\infty)^{\wedge 2}, \mathbb{G}_m)$$

is a surjection.

Proof. This is a direct consequence of the 5–lemma.

5.5 Unstable additive cooperations for *kU*

Write $H = H\mathbb{F}_p$. Today we will study the effect of the map $\hat{\Pi}_3$ in ordinary homology. Many parts of the proof we explored for E–theory break. Topologically, the Serre spectral sequence for $H^*BU[6,\infty)$ is not even-concentrated and so is not forced to collapse. Algebraically, because $\widehat{\mathbb{G}}_a$ is not p–divisible the behavior of the model exact sequence is also suspect. Because the situation has fewer insulating good properties, we are forced to actually consider it carefully. The upside, however, is that the standard group law on $\widehat{\mathbb{G}}_a$ is simple enough that we can compute the problem to death.

We begin with the topological half of our tasks. The Serre spectral sequence

$$E_2^{*,*} = H\mathbb{F}_p^* BSU \otimes H\mathbb{F}_p^* \underline{H\mathbb{Z}}_3 \Rightarrow H\mathbb{F}_p^* BU[6,\infty)$$

is quite accessible, and we will recount the case of $p = 2$. In this case, the spectral sequence has E_2–page

$$E_2^{*,*} = H\mathbb{F}_2^* BSU \otimes H\mathbb{F}_2^* \underline{H\mathbb{Z}}_3 \cong \mathbb{F}_2[c_2, c_3, \dots] \otimes \mathbb{F}_2 \left[\mathrm{Sq}^I \iota_3 \,\middle|\, \begin{matrix} I_j \geq 2I_{j+1}, \\ 2I_1 - I_+ \end{matrix} \right].$$

Because the target is 6–connective, we must have the transgressive differential $d_4\iota_3 = c_2$, which via the Kudo transgression theorem spurs the much larger family of differentials

$$d_{4+I_+} \mathrm{Sq}^I \iota_3 = \mathrm{Sq}^I c_2.$$

This necessitates understanding the action of the Steenrod operations on the cohomology of BSU, which is due to Wu:

$$\mathrm{Sq}^{2^j} \cdots \mathrm{Sq}^4 \mathrm{Sq}^2 c_2 = c_{1+2^j}.$$

Accounting for the squares of classes left behind, this culminates in the following calculation:

This spectral sequence can be drawn in using Hood's package.

Theorem 68. *There is an isomorphism*

$$H\mathbb{F}_2^* BU[6, \infty) \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid j \neq 2^k + 1, j \geq 3)} \otimes F_2[\iota_3^2, (\mathrm{Sq}^2 \iota_3)^2, \ldots].\square$$

Theorem 69. *More generally, there is an isomorphism*

Cite me: Stong, Singer.

$$H\mathbb{F}_2^* \underline{kU}_{2k} \cong \frac{H\mathbb{F}_2^* BU}{(c_j \mid \sigma_2(j-1) < k-1)} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-1}],$$

where σ_2 is the dyadic digital sum and "Op" denotes the subalgebra of $H\mathbb{F}_2^* \underline{H\mathbb{Z}}_{2k-1}$ generated by the indicated class.

Proof (Remarks on proof). Stong specialized to $p = 2$ and carefully applied the Serre spectral sequence to the fibrations

$$\underline{kU}_{2(k+1)} \to \underline{kU}_{2k} \to \underline{H\mathbb{Z}}_{2k}.$$

Singer worked at an arbitrary prime and used the Eilenberg–Moore spectral sequence for the fibrations

$$\underline{H\mathbb{Z}}_{2k-1} \to \underline{kU}_{2(k+1)} \to \underline{kU}_{2k}.$$

Both used considerable knowledge of the interaction of these spectral sequences with the Steenrod algebra.

Remark 73. These methods and results generalize directly to odd primes. The necessary modifications come from understanding the unstable mod–p Steenrod algebra, using the analogues of Wu's formulas due to Shay, and employing Singer's Eilenberg–Moore calculation. Again, $H\mathbb{F}_p^* BU[6, \infty)$ is presented as a quotient by $H\mathbb{F}_p^* BU$ by certain Chern classes satisfying a p–adic sum condition, tensored up with the subalgebra of $H\mathbb{F}_p^* \underline{H\mathbb{Z}}_3$ generated by a certain element.

Cite me: Wu formulas, Shay's extension in mod–p Wu formulas for the Steenrod algebra and the Dyer–Lashof algebra.

Remark 74. We can already see from Theorem 68 that our map of short exact sequences in Lecture 5.4 does not have a full analogue in the setting of additive homology. By considering the edge homomorphism in the Serre spectral sequence, we see that

$$\mathrm{Spec}\, HP_0 BSU \to \mathrm{Spec}\, HP_0 BU[6, \infty)$$

is not a monomorphism.

Now, we turn to the algebra. The main idea, as already used in Lemma 48, is to first perform a tangent space calculation

$$T_0 C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \cong C^k(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a),$$

then study the behavior of the different tangent directions to determine the full object $C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$. As a warm-up, we will first consider the case $k = 2$:

Corollary 50 (cf. Lemma 40). *The unique symmetric 2–cocycle of homogeneous degree n has the form*

$$c_n(x, y) = \begin{cases} (x + y)^n - x^n - y^n & \text{if } n \neq p^j, \\ \frac{1}{p} \left((x + y)^n - x^n - y^n \right) & \text{if } n = p^j. \end{cases} \qquad \square$$

Our goal, then, is to select such a 2–cocycle f and study the minimal conditions needed on a symbol a to produce a multiplicative 2–cocycle of the form $1 + af + \cdots$. Since $c_n = \frac{1}{d_n} \delta(x^n)$ is itself produced by an additive formula, life would be easiest if we had access to an exponential, so that we could build

$$\text{"} \delta \exp(a_n x^n)^{1/d_n} = \exp(\delta a_n x^n / d_n) = \exp(a_n c_n). \text{"}$$

However, the existence of an exponential series is equivalent to requiring that a_n have all fractions, which turns out not to be minimal. In fact, *no* conditions on a_n are required *at all*, if we tweak the definition of an exponential series:

Definition 80. The *Artin–Hasse exponential* is the power series

$$E_p(t) = \exp\left(\sum_{j=0}^{\infty} \frac{t^{p^j}}{p^j} \right) \in \mathbb{Z}_{(p)}[\![t]\!].$$

This series has excellent properties, mimicking those of $\exp(t)$ as closely as possible while keeping coefficients in $\mathbb{Z}_{(p)}$ rather than in $\mathbb{Q}$. Writing $\delta \colon C^1 \to C^2$ and

$$d_n = \begin{cases} 1 & \text{if } n = p^j, \\ 0 & \text{otherwise,} \end{cases}$$

we set

$$g_n(x, y) := \delta E_p(a_n x^n)^{1/p^{d_n}} = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} \delta x^{n p^j}}{p^{j+d_n}} \right) = \exp\left(\sum_{j=0}^{\infty} \frac{a_n^{p^j} c_{n p^j}(x, y)}{p^j} \right).$$

This gives a point in $C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)(\mathbb{Z}_{(p)}[a_n])$, and exhaustion of the tangent spaceproves the following Lemma:

[margin note: Is this really the best example to reference?]

[margin note: The notation is a little odd, since you're conflating x^n with the function $x \mapsto x^n$.]

[margin note: Where does this come from? I've never learned a universal property for it. That bothers me. It must have something to do with p–typification.]

[margin note: Granting that this exhausts the tangent space, how do we recover the global scheme? Even if we had an inverse function theorem (like Thm. 2.1.4), I would only expect that to be at most a local isomorphism – unless these are all really formal schemes masquerading as ordinary schemes?]

Lemma 76 ([AHS01, Proposition 3.9]). *The map*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_n \mid n \geq 2] \xrightarrow{\Pi_{n \geq 2} g_2} C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}$$

is an isomorphism. □

The case $k = 3$ is similar, with one important new wrinkle. Over an $\mathbb{F}_2$–algebra there is an equality $c_n^2 = c_{2n}$. However, this relation does not generalize to trivariate 2–cocycles:

$$\frac{1}{2}\delta(c_6) = x^2y^2z^2 + x^4yz + xy^4z + xyz^4, \qquad \left(\frac{1}{2}\delta c_3\right)^2 = x^2y^2z^2.$$

This pattern is generic and exhaustive for $\mathbb{F}_p$–algebras:

Lemma 77 ([AHS01, Proposition 3.20, Proposition A.12]). *The p–primary residue of the scheme of trivariate symmetric 2–cocycles is presented by*

$$\operatorname{Spec} \mathbb{F}_p[a_d \mid d \geq 3] \times \operatorname{Spec} \mathbb{F}_p[b_d \mid d = p^j(1+p^k)] \xrightarrow{\cong} C^3(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_a) \times \operatorname{Spec} \mathbb{F}_p. \square$$

Similar juggling of the Artin–Hasse exponential yields the following multiplicative classification:

Theorem 70 ([AHS01, Proposition 3.28]). *There is an isomorphism*

$$\operatorname{Spec} \mathbb{Z}_{(p)}[a_d \mid d \geq 3, d \neq 1 + p^t] \times \operatorname{Spec} \Gamma[b_{1+p^t}] \to C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m) \times \operatorname{Spec} \mathbb{Z}_{(p)}.$$

Proof (Proof sketch). The main claim is that the Artin–Hasse exponential trick used in the case $k = 2$ works here as well, provided $d \neq 1 + p^t$ so that taking an appropriate p^{th} root works out. They then show that the remaining exceptional cases extend to multiplicative cocycles only when the pth power of the leading coefficient vanishes. Finally, a rational calculation shows how to bind these truncated generators together into a divided power algebra.

In our pursuit of the map of exact sequences of Lecture 5.4, we are missing one piece: a link from topology to the scheme of Weil pairings, $\operatorname{Weil}(\widehat{\mathbb{G}}_a)$. The object "$\operatorname{Spec} HP_0 H\mathbb{Z}_3$" is insuitable because it doesn't exist — the homology algebra $HP_0 H\mathbb{Z}_3$ is not even–concentrated. However, analyzing the edge homomorphism from our governing Serre spectral sequence shows that the map

$$HP^0 BU[6, \infty) \to HP^0 H\mathbb{Z}_3$$

factors through the subalgebra A generated by the *squares* of the polynomial generators. Accordingly, we aim to place $\operatorname{Spec} A^*$ in the top-right corner of our map of

Lemma 78 ([AHS01, Lemma 3.36, Proposition 4.13, Lemma 4.11]). *The scheme* $\operatorname{Spec} A^*$ *models* $\operatorname{Weil}(\widehat{\mathbb{G}}_a)$ *by an isomorphism* λ *commuting with* $e \circ \widehat{\Pi}_3$.

Proof (Proof sketch). The $\mathbb{F}_p$–scheme $\text{Weil}(\widehat{\mathbb{G}}_a)$ is simple to describe:

$$(a_{mn})_{m,n} \longmapsto \prod_{m<n} \text{texp}\left(a_{mn}(x^{p^m}y^{p^n} - x^{p^n}y^{p^m})\right)$$

$$\text{Spec}\,\mathbb{F}_p[a_{mn} \mid m < n]/(a_{mn}^p) \xrightarrow{\;\cong\;} \text{Weil}(\widehat{\mathbb{G}}_a),$$

where $\text{texp}(t) = \sum_{j=0}^{p-1} t^j/j!$ is the truncated exponential series. It is easy to check that this ring of functions agrees with A^*, and it requires hard work (although not much creativity) to check the remainder of the statement: that $e \circ \widehat{\Pi}_3$ factors through $\text{Spec}\,A^*$ and that the factorization is an isomorphism.

It's maybe not obvious to a reader why this sequence is exact in the middle, although you have secretly proven this in the mess above.

We have now finally assembled our map of right-exact sequences:

$$\begin{array}{ccccccc}
\text{Spec}\,HP_0BSU & \longrightarrow & \text{Spec}\,HP_0BU[6,\infty) & \longrightarrow & \text{Spec}\,A^* & \longrightarrow & 0 \\
\cong \downarrow \widehat{\Pi}_2 & & \downarrow \widehat{\Pi}_3 & & \cong \downarrow \lambda & & \\
C^2(\widehat{\mathbb{G}}_a;\mathbb{G}_m) & \xrightarrow{\;\delta\;} & C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m) & \xrightarrow{\;e\;} & \text{Weil}(\widehat{\mathbb{G}}_a) & \longrightarrow & 0.
\end{array}$$

Our calculations now pay off:

Corollary 51. *The map $\widehat{\Pi}_3$ is an isomorphism:*

$$\widehat{\Pi}_3 : \text{ Spec}\,HP_0BU[6,\infty) \xrightarrow{\;\cong\;} C^3(\widehat{\mathbb{G}}_a;\mathbb{G}_m).$$

Proof (Proof sketch). The main point is that we don't actually have to compute much about the middle map. Because the squares commute and the sequences are exact as indicated, we at least learn that $\widehat{\Pi}_3$ is an epimorphism after base-change to $\text{Spec}\,\mathbb{Q}$ and $\text{Spec}\,\mathbb{F}_p$ for each prime p. But, since both source and target are affine schemes of graded finite type with equal Poincaré series in each case, our epimorphism is an isomorphism.

Corollary 52 ([AHS01, Theorem 2.31]). *The map $\widehat{\Pi}_3$ is an isomorphism for any complex-orientable E.*

Jay asked in class for a summary of exactly what exactness statements are true for a general E (especially relative to the extreme case of E–theory, where everything is exact and pleasant).

Proof (Proof sketch). This follows much along the lines of Corollary 37. We check that the statement holds for $E = MUP$ using a tangent space argument, and then an Atiyah–Hirzebruch argument gives the statement for any complex-oriented E.

Remark 75. Our analysis in Lecture 4.6 gives us full access to the Hopf ring structure of the *nonconnective* cooperations $H\mathbb{F}_{p*}\underline{KU}_{2*}$. Using a variety of techniques, Morton and Strickland calculated the Hopf ring structure of $H\mathbb{F}_{2*}\underline{K}_{2*}$ where K ranges among the nonconnective objects KO, KU, KSp, and the less common "KT", which is self–conjugated K–theory [Mor07, MS02, Stra].

This could be expanded some, as not all these references are relevant to what's written.

5.6 Covariance, Θ–structures on Thom sheaves

Today we will (despite appearances) mostly leave the algebraic geometry alone, instead attending to two lingering topological concerns. First, over the past few lectures we have been concerned with the homology schemes $\operatorname{Spec} E_0 \underline{kU}_{2*}$. We were originally motivated by a sequence of cohomological isomorphisms

$$(BU \times \mathbb{Z})_E \cong \operatorname{Sym}^0_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_0 \mathbb{CP}^\infty_E,$$

$$BU_E \cong \operatorname{Sym}^1_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_1 \mathbb{CP}^\infty_E,$$

$$BSU_E \cong \operatorname{Sym}^2_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E =: C_2 \mathbb{CP}^\infty_E,$$

along with identifications

$$BU \times \mathbb{Z} \simeq \underline{kU}_{2 \cdot 0}, \qquad BU \simeq \underline{kU}_{2 \cdot 1}, \qquad BSU \simeq \underline{kU}_{2 \cdot 2}.$$

Our analysis of Cartier duality in Remark 65 gave us isomorphisms like

$$\operatorname{Spec} E_0 BSU \cong \underline{\operatorname{GroupSchemes}}(BSU_E, \mathbb{G}_m) \cong \underline{\operatorname{FormalGroups}}(C_2 \mathbb{CP}^\infty_E, \widehat{\mathbb{G}}_m).$$

Following the universal property of this particular symmetric square, we were led to consider the scheme of symmetric bivariate functions on $\mathbb{CP}^\infty_E$ satisfying a 2–cocycle condition. Our next move was to show that $\operatorname{Spec} E_0 BU[6, \infty)$ was modeled by a similar scheme of *trivariate* functions — but we proved this directly, while avoiding the "predual" cohomological statement

$$BU[6, \infty)_E \cong C_3 \mathbb{CP}^\infty_E := \operatorname{Sym}^3_{\operatorname{Div} \mathbb{CP}^\infty_E} \operatorname{Div}_0 \mathbb{CP}^\infty_E.$$

This is because the homological statement is the low-hanging fruit: it is easy to demonstrate that the scheme of such functions exists as a closed subscheme of all functions. It is considerably harder to show that a symmetric cube exists at all.

Lemma 79. *$DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all formal varieties for $k \leq 3$.*

> **Cite me:** 4.41 in the AHS preprint.

Proof. We know that $\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ are all free $\mathbb{Z}$–modules of graded finite rank in the range $k \leq 3$, so we may write

> Cite this from a couple lectures ago.

$$\mathcal{O}(DC^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)) \cong (\mathcal{O}C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*.$$

We will show that this later Hopf algebra $\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m))^*$ is a power series ring, specializing for the moment to the case $k = 2$. It will suffice to show that it is a power series ring modulo p for every prime p. Such graded connected finite-type Hopf algebras over $\mathbb{F}_p$ were classified by Borel (and

exposited by Milnor–Moore [MM65, Theorem 7.11]) as either polynomial or truncated polynomial. These two cases are distinguished by the Frobenius operation: the Frobenius on a polynomial ring is injective, whereas the Frobenius on a truncated polynomial ring is not. It is therefore equivalent to show that the *Verschiebung* on the original ring $\mathcal{O}(C^2(\widehat{\mathbb{G}}_a; \mathbb{G}_m)) \otimes \mathbb{F}_p$ is *surjective*. Recalling that $c_n^p = c_{np}$ at the level of bivariate 2–cocycles, we compute

$$p^* a_n = a_{np}^p,$$

and since $F a_{np} = a_{np}^p$ and $FV = p^*$, we learn

$$V(a_{np}) = a_n.$$

Essentially the same proof handles the cases $k = 1$ and $k = 0$. The case $k = 3$ requires a small modification, to cope with the two classes of trivariate 2–cocycles. On the polynomial tensor factor of $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m))$ we can reuse the same Verschiebung argument to see that its dual Hopf algebra is polynomial. The dual of the divided power tensor factor is immediately a primitively generated polynomial algebra, without any further argument .

Theorem 71. *The scheme $C_3\mathbb{CP}_E^\infty$ exists, and it is modeled by $BU[6, \infty)_E$.*

Proof (Proof sketch). Let $\widehat{\mathbb{G}}$ be an arbitrary formal group. Note first that if $C^3(\widehat{\mathbb{G}}; \mathbb{G}_m)$ is coalgebraic, then $C_3\widehat{\mathbb{G}}$ exists and is its Cartier dual: the diagram presenting $\mathcal{O}(C^3(\widehat{\mathbb{G}}; \mathbb{G}_m))$ as a reflexive coequalizer of free Hopf algebras is also the diagram meant to present $C_3\widehat{\mathbb{G}}$ as a coalgebraic formal scheme. So, if the coequalizing Hopf algebra has a good basis, it will follow from Theorem 63 that the resulting diagram is a colimit diagram in formal schemes, with $C_3\widehat{\mathbb{G}}$ sitting at the cone point. It will additionally follow from there that the isomorphism from Corollary 52

$$\operatorname{Spec} E_0 BU[6, \infty) \xrightarrow{\cong} C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m)$$

will re-dualize to an isomorphism

$$BU[6, \infty)_E \xleftarrow{\cong} C_3\mathbb{CP}_E^\infty.$$

So, we reduce to checking that $\mathcal{O}(C^3(\widehat{\mathbb{G}}; \mathbb{G}_m))$ admits a good basis. By a base change argument, it suffices to take $\widehat{\mathbb{G}}$ to be the universal formal group over the Lazard ring. Noting that $\mathcal{O}(C^3(\widehat{\mathbb{G}}; \mathbb{G}_m))$ must be of graded finite type, we will work to show that it is free on a basis we have good control over.

Specializing from $\widehat{\mathbb{G}}$ over $\mathcal{M}_{\mathbf{fgl}}$ to $\widehat{\mathbb{G}}_a$ over $\operatorname{Spec} \mathbb{Z}$, we know from Lemma 79 that $\mathcal{O}(C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m))$ is a free abelian group, and we know from Theorem 25

that $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$ is as well. By picking a $\mathbb{Z}$–basis $\mathbb{Z}\{\beta_j\}_j$ of $\mathcal{O}C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$, we can choose a map of $\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})$–modules lifting it

$$
\begin{array}{ccc}
\mathcal{O}(\mathcal{M}_{\mathbf{fgl}})\{\tilde{\beta}_j\}_j & \xrightarrow{\ \alpha\ } & \mathcal{O}C^3(\widehat{\mathbb{G}}; \mathbb{G}_m) \\
\downarrow & & \downarrow \\
\mathbb{Z}\{\beta_j\}_j & \xrightarrow{\ \cong\ } & \mathcal{O}C^3(\widehat{\mathbb{G}}_a; \mathbb{G}_m).
\end{array}
$$

By induction on degree, one sees that α is surjective, and since the source is a free abelian group we need only check that the source and target have the same Poincaré series to conclude that α is an isomorphism.

We proceed to test this rationally: over $\mathrm{Spec}\,\mathbb{Q}$ we can use the logarithm to construct an isomorphism

$$
\mathrm{Spec}\,\mathbb{Q} \times (\mathcal{M}_{\mathbf{fgl}} \times C^k(\widehat{\mathbb{G}}; \mathbb{G}_m)) \to \mathrm{Spec}\,\mathbb{Q} \times (\mathcal{M}_{\mathbf{fgl}} \times C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)),
$$

hence the Poincaré series agree, so $\alpha \otimes \mathbb{Q}$ and α are both isomorphisms. Having established freenss, our other goal was to show that M has a sequence of good subcoalgebras. These come by considering the subcoalgebras, indexed on an integer d, spanned by the basis vectors of degree at most d.

Our second task today is to address the gap between $\underline{kU}_{2k}$ and its Thom spectrum $T(\underline{kU}_{2k})$. After all, our motivation for all of this algebraic geometry is to give a description to the set

$$
(\mathrm{Spec}\,E_0 MU[6, \infty))(E_0),
$$

but what we have done so far is describe the scheme $\mathrm{Spec}\,E_0 BU[6, \infty)$. Since these two spectra are related by a Thom construction, we should be able to deduce the description that we want by thinking about Thom sheaves. We now straighten this out. The place to start is with a construction:

Lemma 80. *For $\xi\colon G \to BGL_1\mathbb{S}$ a group map, the Thom spectrum $T\xi$ is a $(\Sigma_+^\infty G)$–cotorsor.*

Proof (Proof sketch). The Thom isomorphism $T\xi \wedge T\xi \simeq T\xi \wedge \Sigma_+^\infty G$ composes with the unit map $\mathbb{S} \to T\xi$ to give the *Thom diagonal*

> Which factor are we composing into? Does it matter?

$$
T\xi \to T\xi \wedge \Sigma_+^\infty G. \qquad \square
$$

Corollary 53. *In addition to our interpretation of $\mathbb{L}(\xi)$ as a $\mathbb{G}_m$–torsor over G_E, $\mathbb{L}(\xi)^{-1}$ is furthermore a $(G_E \times \mathbb{G}_m)$–torsor over S_E.* $\qquad\square$

> S_E vs. S_E, and how does this follow?

We expand this idea out in our situation. A morphism $MU[6, \infty) \to E$ produces a trivialization of $\mathbb{L}(\otimes_{j=1}^3 (\mathcal{L}_j - 1))$ over $(\mathbb{CP}_E^\infty)^{\times 3}$, and the associated trivializing section is symmetric and rigid. This prompts us to make the following definition:

> Cite me: Lemma 2.49 of AHS.

Definition 81. We write $C^k(\widehat{\mathbb{G}}; \mathcal{L})$ for the functor of trivializing sections of $\Theta^k \mathcal{L}$ over $\widehat{\mathbb{G}}^{\times k}$ which are symmetric and rigid. This construction has some nice properties:

- Note that taking the trivial sheaf $\mathcal{L} = \mathcal{O}_{\widehat{\mathbb{G}}}$ recovers the scheme $C^k(\widehat{\mathbb{G}}; \mathbb{G}_m)$ of $\mathbb{G}_m$–valued such functions from before.
- By consequence, if $\mathcal{L}$ is *trivializable* then this functor is an affine scheme.
- Affine or not, there is a pairing map

$$C^k(\widehat{\mathbb{G}}; \mathcal{L}) \times C^k(\widehat{\mathbb{G}}; \mathcal{H}) \to C^k(\widehat{\mathbb{G}}; \mathcal{L} \otimes \mathcal{H}).$$

In particular, this recovers the group structure on $C^k(\widehat{\mathbb{G}}; \mathbb{G}_m)$ and it makes $C^k(\widehat{\mathbb{G}}; \mathcal{L})$ into a $C^k(\widehat{\mathbb{G}}; \mathbb{G}_m)$–torsor.

Thus, we have constructed a map

$$\varphi \colon \operatorname{Spec} E_0 MU[6, \infty) \to C^3(\mathbb{CP}_E^\infty; \mathcal{I}(0)).$$

The following Lemma is a matter of fully expanding definitions:

Lemma 81 ([AHS01, Theorem 2.50]). *The map*

$$\operatorname{Spec} E_0 MU[6, \infty) \to C^3(\mathbb{CP}_E^\infty; \mathcal{I}(0))$$

is an isomorphism.

You shouldn't call this anything more than a "proof sketch" unless you expand out the reasoning for why this square commutes.

Proof (Proof sketch). This map is equivariant, in the sense that the following square commutes:

$$
\begin{array}{ccc}
\operatorname{Spec} E_0 MU[6,\infty) \times \operatorname{Spec} E_0 BU[6,\infty) & \longrightarrow & \operatorname{Spec} E_0 MU[6,\infty) \\
\downarrow{\varphi \times 52} & & \downarrow{\varphi} \\
C^3(\mathbb{CP}_E^\infty; \mathcal{I}(0)) \times C^3(\mathbb{CP}_E^\infty; \mathbb{G}_m) & \longrightarrow & C^3(\mathbb{CP}_E^\infty; \mathcal{I}(0)).
\end{array}
$$

Any equivariant map of torsors is automatically an isomorphism.

This, finally, gives us access to the analogue of Theorem 64, Theorem 65, and Corollary 47:

Corollary 54. *Take E to be complex–orientable. The functor $C^3(\mathbb{CP}_E^\infty; \mathcal{I}(0))$ defined by*

$$\mathsf{AffineSchemes}_{/\operatorname{Spec} E_0} \to \mathsf{Sets},$$

$$(\operatorname{Spec} T \xrightarrow{u} \operatorname{Spec} E_0) \mapsto \left\{ \begin{array}{l} \text{symmetric, rigid trivializations} \\ \text{of } u^* \Theta^3 \mathbb{L}(\mathcal{L}) \text{ over } u^*(\mathbb{CP}_E^\infty)^{\times 3} \end{array} \right\}$$

is isomorphic to the affine scheme $\operatorname{Spec} E_0 MU[6, \infty)$. Moreover, the E_0–points of this scheme biject with ring spectrum maps $MU[6, \infty) \to E$. $\square$

This is a leftover todo from the following section. We accomplish all of this here, but I haven't looked at the references tacked on at the end of this to see if they say anything I've missed. — — — — — I want to sketch the reduction for even-periodic elliptic cohomology theories to the case of *MUP*, then from there to *HkP* for the prime fields *K*, then from there to questions about additive cocycles. We certainly don't need to recall any of these calculations, but I think it's a nice example of the philosophy that the additive formal group is such a knotted point

5.7 Modular forms from $MU[6, \infty)$–manifolds

Our goal today is to actually leverage the arithmetic geometry in Corollary 48, rather than just using the body of results about θ–functions as inspiration. In order to do this, we need to place ourselves in a situation where algebraic topology is directly linked to abelian varieties.

Definition 82. An *elliptic spectrum* consists of a even–periodic ring spectrum E, a (generalized) elliptic curve C over Spec E_0, and a fixed isomorphism

$$\varphi \colon C_0^\wedge \xrightarrow{\cong} \mathbb{CP}_E^\infty.$$

A map among such spectra consists of a map of ring spectra $f \colon E \to E'$ together with a specified isomorphism of elliptic curves $\psi \colon f^*C \to C'$.

Remark 76. We have chosen to consider *isomorphisms* of elliptic curve rather than general homomorphisms because this is what algebraic topology suggests that we do. After all, the mixed cooperations of complex-oriented ring spectra are modeled by the isomorphisms of the associated formal groups. In the next Case Study, we will develop a theory (with an attendant notion of a "context") which incorporates isogenies of elliptic curves in addition to isomorphisms.

Coupling Definition 82 to Corollary 54 and Corollary 48, we learn the following result:

Corollary 55. *An elliptic spectrum* (E, C, φ) *receives a canonical map of ring spectra*

$$MU[6, \infty) \to E.$$

This map is natural in choice of elliptic spectrum: if $(E, C, \varphi) \to (E', C', \varphi')$ *is a map of elliptic spectra, then the triangle*

$$
\begin{array}{ccc}
 & MU[6, \infty) & \\
 \swarrow & & \searrow \\
E & \longrightarrow & E'
\end{array}
$$

commutes. □

Example 36. Our basic example of an elliptic curve was $E_\Lambda = \mathbb{C}/\Lambda$, with Λ a complex lattice. The projection $\mathbb{C} \to \mathbb{C}/\Lambda$ has a local inverse which defines an isomorphism of formal groups

$$\varphi \colon (E_\Lambda)_0^\wedge \xrightarrow{\cong} \widehat{\mathbb{G}}_a \otimes \mathbb{C}.$$

Accordingly, we define an elliptic spectrum $HE_\Lambda P$ whose underlying ring spectrum is $H\mathbb{C}$ and whose associated elliptic curve and isomorphism are E_Λ and φ. This spectrum receives a natural map

$$MU[6, \infty) \to HE_\Lambda P,$$

which to a bordism class $M \in MU[6, \infty)_{2n}$ assigns an element $\Phi_\Lambda(M) \cdot u_\Lambda^n \in HE_\Lambda P_{2n}$ for some $\Phi_\Lambda(M) \in \mathbb{C}$.

Example 37. The naturality of the $MU[6, \infty)$–orientation moves us to consider more than one elliptic spectrum at a time. If Λ' is another lattice with $\Lambda' = \lambda \cdot \Lambda$, then the multiplication map $\lambda \colon \mathbb{C} \to \mathbb{C}$ descends to an isomorphism $E_\Lambda \to E_{\Lambda'}$ and hence a map of elliptic spectra $HE_{\Lambda'} P \to HE_\Lambda P$ acting by $u_{\Lambda'} \mapsto \lambda u_\Lambda$. The commuting triangle in Corollary 55 then begets the *modularity relation*

$$\Phi(M; \lambda \cdot \Lambda) = \lambda^{-n} \Phi(M; \Lambda).$$

Example 38. This equation leads us to consider all curves E_Λ simultaneously — or, equivalently, to consider modular forms. The lattice Λ can be put into a standard form, by picking a basis and scaling it so that one vector lies at 1 and the other vector lies in the upper half-plane. This gives a cover

$$\mathfrak{h} \to \mathcal{M}_{\mathbf{ell}} \times \operatorname{Spec} \mathbb{C}$$

which is well-behaved away from the special points i and $e^{2\pi i/6}$. A *complex modular form of weight k* is an analytic function $\mathfrak{h} \to \mathbb{C}$ which satisfies a certain decay condition and which is quasi-periodic for the action of $SL_2(\mathbb{Z})$, i.e.,[3]

$$f\left(M; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(M; \tau).$$

Using these ideas, we construct a cohomology theory $H\mathcal{O}_\mathfrak{h} P$, where $\mathcal{O}_\mathfrak{h}$ is the ring of complex-analytic functions on the upper half-plane. The $\mathfrak{h}$–parametrized family of elliptic curves

$$\mathfrak{h} \times \mathbb{C}/(1, \tau) \to \mathfrak{h},$$

together with the logarithm, present $H\mathcal{O}_\mathfrak{h} P$ as an elliptic spectrum $H\mathfrak{h}P$. The canonical map $\Phi \colon MU[6, \infty) \to H\mathfrak{h}P$ specializes at a point to give the functions $\Phi(-; \Lambda)$ considered above, and hence $\Phi(M)$ is itself a complex modular form of weight k.

In fact, this is a ghost of Ochanine and Witten's modular genus from Theorem 1, as a bordism class in $MU[6, \infty)_{2n}$ is, in particular, a bordism class

---

[3] That is, for the action of change of basis vectors.

in $MString_{2n}$. However, they know more about this function than we can presently see: they claim that it has an integral q–expansion. In terms of the modular form, its q–expansion is given by building the Taylor expansion "at ∞" (using that unspoken decay condition). In order to use our topological methods, it would be nice to have an elliptic spectrum embodying these q–expansions in the same way that $H\hbar P$ embodied holomorphic functions, together with a comparison map that trades a modular form for its q–expansion. The main ideas leading to such a spectrum come from considering the behavior of E_Λ as τ tends to $i \cdot \infty$.

Definition 83. Note that as $\tau \to i \cdot \infty$, the parameter $q = \exp(2\pi i \tau)$ tends to 0. In the multiplicative model of Lecture 5.3, we considered D' the punctured complex disk with associated family of elliptic curves

$$C'_{\text{an}} = \mathbb{C}^\times \times D' / (u, q) \sim (qu, q).$$

The fiber of C' over a particular point $q \in D'$ is the curve $\mathbb{C}^\times / q^{\mathbb{Z}}$. The Weierstrass equations give an embedding of C'_{an} into $D' \times \mathbb{C}P^2$ described by

$$wy^2 + wxy = x^3 - 5\alpha_3 w^2 x + -\frac{5\alpha_3 + 7\alpha_5}{12} w^3$$

for certain functionsα_3 and α_5 of q. At $q = 0$, these curve collapses to the twisted cubic

$$wy^2 + wxy = x^3,$$

and over the whole open unit disc D we call this extended family C_{an}.

Now let $A \subseteq \mathbb{Z}[\![q]\!]$ by the subring of power series which converge absolutely on the open unit disk. It turns out that the coefficients of the Weierstrass cubic (i.e., $5\alpha_3$ and $\frac{1}{12}(5\alpha_3 + 7\alpha_5)$) lie in A, so it determines a generalized elliptic curve C over $\text{Spec } A$, and C_{an} is the curve given by base-change from A to the ring of holomorphic functions on D. The *Tate curve* C_{Tate} is defined to be the family over the intermediate object $D_{\text{Tate}} = \text{Spec } \mathbb{Z}[\![q]\!]$ basechanged from A.

The singular fiber at $q = 0$ prompts us to enlarge our notion of elliptic curve slightly.

Definition 84 ([AHS01, Definitions B.1-2]). A *Weierstrass curve* is *any* curve of the form

$$C(a_1, a_2, a_3, a_4, a_6) := \left\{ ([x : y : z], s) \in \mathbb{P}^2 \times S \left| \begin{array}{l} y^2 z + a_1(s)xyz + a_3(s)yz^2 = \\ x^3 + a_2(s)x^2 z + a_4(s)xz^2 + a_6(s) \end{array} \right. \right\}$$

A *generalized elliptic curve* over S is a scheme C equipped with maps

$$S \xrightarrow{0} C \xrightarrow{\pi} S$$

Do these have weights?

Cite me: Find a reference for this. You may be able to look in Morava's Section 5.

Cite me: Is this right? Check the published AHS to make sure.

Erick didn't like the name "generalized elliptic curve", which already means something else in arithmetic (disallowing cuspidal singularities and also allowing other funny behavior). He didn't have a second suggestion, though.

You use ws above but zs here.

such that C is Zariski–locally isomorphic to a system of Weierstrass curves (in a way preserving 0 and π).[4]

Remark 77. The singularities of a degenerate Weierstrass equation always occur outside of a formal neighborhood of the marked identity point, which in fact still carries the structure of a formal group. The formal group associated to the twisted cubic is the multiplicative group, and the isomorphism making the identification extends a family of such isomorphisms φ over the nonsingular part of the Tate curve.

The smooth locus of the twisted cubic is actually also the *informal* multiplicative group. The θ–function we're going to describe below also has the property that it specializes at $q = 0$ to the usual coordinate $(1 - u)$ on $\mathbb{G}_m$, which is nice. This degeneracy to honest-$\mathbb{G}_m$ is probably also related to the use of the Todd orientation further down.

Definition 85 ([Mor89, Section 5], [AHS01, Section 2.7]). The elliptic spectrum K_{Tate}, called *Tate K–theory*, has as its underlying spectrum $KU[\![q]\!]$. The associated elliptic curve is C_{Tate}, and the isomorphism $\mathbb{CP}^\infty_{KU[\![q]\!]} \cong (C_{\text{Tate}})_0^\wedge$ is φ from Remark 77.

The trade for the breadth of this definition is that theorems pulled from the study of abelian varities have to be shown to extend uniquely to those generalized elliptic curves which are not smooth curves.

Theorem 72 ([AHS01, Propositions 2.57 and B.25]). *For a generalized elliptic curve C, there is a canonicaltrivialization s of $\Theta^3 \mathcal{I}(0)$ which is compatible with change of base and with isomorphisms. If C is a smooth elliptic curve, then s agrees with that of Corollary 48.* $\square$

You should compare this with the unicity statement in the previous iteration of this theorem, plus how it's *not* unique at these singular points.

Corollary 56. *The trivializing section s associated to C_{Tate} is given by $\delta^{\circ 3}\theta$, where*

$$\theta_q(u) = (1 - u) \prod_{n > 0} (1 - q^n u)(1 - q^n u^{-1}).$$

This is a little funny, because this is *not* one of the usual θ–functions from Lecture 5.3. I think we should state its quasiperiodicity relation again, since it doesn't agree with the old one.

Proof. Even though θ is not a function on C_{Tate} because of quasiperiodicity, it does trivialize both $\pi^* \mathcal{I}(0)$ for $\pi \colon \mathbb{C}^\times \times D \to C_{\text{Tate}}$ and $\mathcal{I}(0)$ for $(C_{\text{Tate}})_0^\wedge$. Moreover, the quasiperiodicities in the factors in the formula defining $\delta^3\theta|_{(C_{\text{Tate}})_0^\wedge}$ cancel each other out, and the function does descend to give a trivialization of $\Theta^3 \mathcal{I}(0)$. By the unicity clause in Theorem 72, it must give a formula expressing s.

This use of unicity is a little opaque to me. I guess we're using that $\delta^3\theta$ is "obviously" the natural trivializing section for nonsingular values of q? Or maybe we just mean that the Tate curve covers much of $\mathcal{M}_{\text{ell}} \times \text{Spec}\,\mathbb{C}$.

Definition 86. The induced map

$$\sigma_{\text{Tate}} \colon MU[6, \infty) \to K_{\text{Tate}}$$

is called the *complex σ–orientation*.

Corollary 57. *Let $M \in \pi_{2n} MU[6, \infty)$ be a bordism class. The q–expansion of Witten's modular form $\Phi(M)$ has integral coefficients.*

---

[4] An elliptic curve in the usual sense turns out to be a generalized elliptic curve which is smooth, i.e., the discriminant of the Weierstrass equations is a unit.

Proof. The span of elliptic spectra equipped with $MU[6, \infty)$–orientations

$$
\begin{array}{ccc}
& MU[6, \infty) & \\
\sigma_{\text{Tate}} \swarrow & \downarrow & \searrow \Phi \\
K_{\text{Tate}} \longrightarrow & K_{\text{Tate}} \otimes \mathbb{C} \longleftarrow & H\mathfrak{h}P
\end{array}
$$

models q–expansion. The arrow $K_{\text{Tate}} \to K_{\text{Tate}} \otimes \mathbb{C}$ is injective on homotopy, which shows that the q–expansion of $\Phi(M)$ lands in the subring of integral power series.

In fact, Corollary 56 gives us access to a formula for $\sigma_{\text{Tate}} = \delta^3\theta$, where θ here is interpreted as a coordinate on $(C_{\text{Tate}})_0^\wedge$. This means that σ_{Tate} belongs to the commutative triangle

$$
\begin{array}{ccc}
MU[6, \infty) & & \\
\delta \downarrow & \xrightarrow{\hspace{2cm} \sigma_{\text{Tate}} \hspace{2cm}} & \\
MSU \xrightarrow{\delta} MU \xrightarrow{\delta} MUP & \xrightarrow{\theta} & KU[\![q]\!].
\end{array}
$$

To begin, the usual map $MUP \to KU$ selects the coordinate $f(u) = 1 - u$ on the formal completion of $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[u^{\pm}]$. The induced map

$$
MU \xrightarrow{\delta} MUP \to KU
$$

sends f to the rigid section δf of

$$
\Theta^1 \mathcal{I}(0) = \mathcal{I}(0)_0 \otimes \mathcal{I}(0)^{-1} \cong \omega \otimes \mathcal{I}(0),
$$

and in terms of the right-hand side,

$$
\delta f = \frac{f'(0)}{f(u)} Du = \frac{1}{1-u}\left(-\frac{du}{u}\right),
$$

where Du is the invariant differential. We can augment this to a calculation of $\delta\theta$ by considering the composite

$$
\delta\theta \colon MU \xrightarrow{\eta_R} MU \wedge MU \simeq MU \wedge BU_+ \xrightarrow{\delta(1-u)\wedge\theta'} K_{\text{Tate}}
$$

where θ' is the element of $BU^{K_{\text{Tate}}} \cong C^1(\widehat{C}_{\text{Tate}}; \mathbb{G}_m)$ given by the formula

$$
\theta' = \prod_{n \geq 1} \frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}.
$$

This means that its effect on a line bundle is determined by

Erick asked whether these are cusp forms (i.e., vanishing to first order at $i \cdot \infty$) or not. They aren't: the spectrum *tmf* has homotopy starting in degree 0, with no serious gaps toward ∞, and the fiber of the evaluation map to *KO* "has homotopy starting in degree 24", ignoring some low–dimensional phenomena. So, you need to know that $S \to MString \to tmf$ is nonzero a little ways above degree 3 to conclude nontriviality, but that is the case, since $S[0,7) \to MString[0,7)$ is an equivalence.

This first map is one of the units. Which is it?

I'm confused. Where did this class come from? Is it the derivative of θ? The δ– derivative of θ? Something else? Actually, it sort of looks like $(1-u) \cdot \theta(1)/\theta(u)$...?

$$\theta'(1-\mathcal{L}) = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n\mathcal{L})(1-q^n\mathcal{L}^{-1})},$$

and its effect on vector bundles in general is determined by the splitting principle and an exponential law. In fact, one can work this exponential law out to mean

$$\theta'(\dim V \cdot 1 - V) = \bigotimes_{n\geq 1}\bigoplus_{j\geq 0}\mathrm{Sym}^j(\dim V \cdot 1 - V \otimes_{\mathbb{R}} \mathbb{C})q^{jn} =: \bigotimes_{n\geq 1}\mathrm{Sym}_{q^n}(-\bar{V}_{\mathbb{C}}).$$

Knowing that $(\eta_R)_*: MU_* \to \pi_*(MU \wedge \Sigma_+^\infty BU)$ sends a manifold M with stable normal bundle ν to the pair (M, ν), we compute the composite on homotopy to be

$$\sigma_{\mathrm{Tate}}(M \in \pi_{2n}MU[6,\infty)) = (\delta(1-u) \wedge \theta')(M,\nu)$$

$$=: \mathrm{Td}\left(M; \bigotimes_{n\geq 1}\mathrm{Sym}_{q^n}(\bar{\tau}_{\mathbb{C}})\right).$$

This is exactly Witten's formula for his genus, as applied to complex manifolds with first two Chern classes trivialized.

This is missing something. We have written $\mathrm{Td}(M)(-du/u)^n$ for $MU_{2n} \to KU_{2n}$, the effect in homotopy of the usual coordinate on KU. You can also think of this as $p_!1$ for $1 \in K^{2n}M$ and $p: M \to *$. More generally, then, $\mathrm{Td}(M;V) := p_!V$. But, also, we did describe pushforwards earlier... maybe we can make use of this?

5.8 Odd-primary real bordism orientations

This is under construction.

A different approach to this is to work through the $p \neq 2$ case, since you can totally, correctly work it out and see the Σ–structure relation get imposed. That's kind of attractive. Too bad you had this thought after the class was complete. —— We might also try to understand the extension problem of orientations across $MU \to MSO$ for spectra which are local away from 2. It seems like some juggling of the complex-conjugation idempotents could give you access to BSO information in terms of BU, which would be satisfying. Compare with Neil's response at http://mathoverflow.net/questions/123958/a-formal-group-law-over-oriented-bordism. Hood also points out that the usual "$\tanh^{-1}$" equation for the L–genus has an expansion in terms of the usual logarithm: it's an averaging between the positive and negative logarithmic series.

Hood also points out that the mysterious series $x/(1-e^{-x})$ occurs as $\exp_{\widehat{\mathbb{G}}_a} / \exp_{\widehat{\mathbb{G}}_m}$.

5.9 Chromatic *Spin* and *String* orientations

This is under construction. The stuff here is correct, but it didn't turn into a super coherent lecture, and I have an idea for something better to say.

In the previous Lecture, we proved that elliptic spectra receive canonical $MU[6, \infty)$–orientations, that complex elliptic spectra collectively give rise to a genus valued in modular forms, and that the q–expansions of these modular forms are integral. However, the original Theorem 1 of Ochanine and Witten claimed to describe a genus on *Spin*– and *String*–manifolds, which we have only managed to approximate with our study of $MU[6, \infty)$–orientations. Our last goal for this Case Study is to show that the chromatic formal schemes associated to spaces like *BString* are somewhat accessible, and so chromatically-amenable elliptic spectra receive canonical *MString*–orientations.

Fix a formal group Γ of finite height d, and write $K = K_\Gamma$ for the associated Morava K–theory. We will start with the more modest goal of understanding the bottom few layers of the Postnikov tower for $\underline{kO}_0 \simeq BO \times \mathbb{Z}$, which have the names

$$BO[2, \infty) := BSO, \qquad BO[4, \infty) := BSpin, \qquad BO[8, \infty) := BString.$$

> The := should be =: perhaps?

Remark 78. Unlike kU, there is *not* an equivalence

$$\underline{kO}_n \not\simeq (BO \times \mathbb{Z})[n, \infty),$$

unless n happens to take the form $n = 8k$ for a nonnegative integer k. The reasoning for this stray equivalence is similar to that for kU: the homotopy ring of kO has a polynomial factor of degree 8, and the other elements lie in a band of dimensions smaller than 8. Otherwise, other things happen — for instance,

$$\underline{kO}_1 \simeq O/U, \qquad\qquad \underline{kO}_6 \simeq Spin/SU.$$

Remark 79 ([KLW04, Section 5.2]). We may as well take the ground field of our Morava K–theory to have characteristic $p = 2$, since at odd characteristics there is little distinction between kO and kU, owing to the fiber sequence

$$\Sigma kO \xrightarrow{\cdot \eta = 0} kO \to kU.$$

> The way you wrote this fiber sequence doesn't show why $p = 2$ is special.

However, this reveals a disadvantage of Morava K–theory that will finally cause us real consternation: Morava K–theories at the prime 2 are not commutative ring spectra. Accordingly, $(K_\Gamma)_* G$ for a commutative H–group G may fail to give a commutative algebra. Luckily, Remark 40 tells us that if $(K_\Gamma)_* G$ happens to be even-concentrated, then the obstructions to commutativity identically vanish. So, we can be somewhat indelicate about this noncommutativity issue, provided that we continually check that the algebras we are forming are even-concentrated.

In order to get off the ground, we will need the following Lemma about the behavior of the Atiyah–Hirzebruch spectral sequence for a Morava K–theory:

Lemma 82 ([Yag80, Lemma 2.1]). *Let k_Γ be the connective cover of the Morava K–theory K_Γ. In the Atiyah–Hirzebruch spectral sequence*

$$E_2^{*,*} = Hk^*X \otimes_k k_\Gamma^* \Rightarrow k_\Gamma^*X,$$

the differentials are given by

$$d_r(x) = \begin{cases} 0 & \text{if } r \leq 2(p^d - 1), \\ \lambda Q_d x \otimes v_d & \text{if } r = 2(p^d - 1) + 1 \end{cases}$$

where $\lambda \neq 0$ and Q_d is the dth Milnor primitive. $\square$

Corollary 58 ([RWY98, Section 2.5], [KLW04, Equation 3.1]). *There is a bi-Cartesian square of coalgebraic formal schemes*

$$
\begin{array}{ccc}
\mathrm{Div}_0\,\overline{G}[2] & \longrightarrow & \mathrm{Div}_0\,\widehat{G}[2] \\
& \searrow & \qquad \searrow \\
\mathrm{Div}_0\,\overline{G} & \longrightarrow & BO_K.
\end{array}
$$

Proof. We apply Lemma 82 to the analysis of the spectral sequence of Hopf algebras

$$H\mathbb{F}_{2*}BO \otimes (K_\Gamma)_* \Rightarrow (K_\Gamma)_*BO.$$

We have $H\mathbb{F}_{2*}BO \cong \mathbb{F}_2[b_1, b_2, \ldots]$ and

$$Q_d b_{2^{d+1}+2j} = b_{2j+1},$$

from which it follows that all the odd generators are killed, all their squares survive , and only the even generators of low degree are permanent cycles. This results in a decomposition

$$(K_\Gamma)_*BO \cong (K_\Gamma)_*[b_2, b_4, b_{2^{d+1}-2}] \underset{(K_\Gamma)[b_{2j}^2 | j < 2^d]}{\otimes} (K_\Gamma)_*[b_{2j}^2],$$

and so we are tasked with assigning names to the coalgebraic formal schemes appearing in this formula.

The left-hand factor is the free Hopf algebra on the coalgebra determined by the 2–torsion in the formal group Γ . The right-hand factor is the free Hopf algebra on the formal curve $\overline{G} := \mathcal{H}P_K^\infty$, using the isogeny

$$\mathcal{H}P_K^\infty \to \mathbb{C}P_K^\infty$$
$$y \mapsto x \cdot [-1](x)$$

induced by desymplectification. Because $\mathcal{H}^\times$ is not commutative, $\overline{G}$ is not a formal group, but we pull back the multiplication-by-2 isogeny from $\widehat{G}$ to $\overline{G}$ and define the subscheme $\overline{G}[2]$ of points mapping to zero.

Suppose that the Postnikov sections

$$X(n, \infty) \to X[n, \infty] \to X[n, n]$$

induce short exact sequences of formal groups. A presentation of X_K as a bi-Cartesian square then acquires value via the following algebraic Lemma:

Lemma 83. *Consider the cube of formal group schemes constructed by taking pointwise fibers of the composite to G:*

If the bottom face is bi-Cartesian, then so is the top. □

Corollary 59. *There is a bi-Cartesian square*

Proof (Proof sketch). The fibration $BSO \to BO \to BO(1)$ gives a short exact sequence of Hopf algebras , so using Corollary 58 we are in the situation of Lemma 83. To compute the pointwise kernels, begin by considering the commuting square of Postnikov sections

$$
\begin{array}{ccc}
BO_K & \longrightarrow & BU_K \\
\downarrow & & \downarrow \\
\widehat{G}[2] & \longrightarrow & \widehat{G}.
\end{array}
$$

Both horizontal maps are injections. Since $\mathrm{Div}\,\overline{G} \to BU_K \cong \mathrm{Div}_0\,\widehat{G} \to \widehat{G}$ is null, the composite $\mathrm{Div}_0\,\overline{G} \to \widehat{G}[2]$ is null. Similarly, the composite $\mathrm{Div}\,\widehat{G}[2] \to \widehat{G}[2]$ acts by summation, and its kernel is $\mathrm{SDiv}_0\,\widehat{G}[2]$.

From here, the computation gets harder.

Corollary 60 ([KLW04, Section 5.3]). *There is a bi-Cartesian square*

$$
\begin{array}{ccc}
\mathrm{Div}_0\,\overline{\mathrm{G}}[2] & \longrightarrow & \ker\omega \\
\swarrow & & \searrow \\
\mathrm{Div}_0\,\overline{\mathrm{G}} & \longrightarrow & BSpin_K,
\end{array}
$$

where $\omega\colon C_2\widehat{\mathrm{G}} \to \widehat{\mathrm{G}}[2]^{\wedge 2}$ is the map $([a]-[0])([b]-[0]) \mapsto a \wedge b$.

Proof. This goes similarly to Corollary 59, once you know that the Postnikov section induces a short exact sequence of formal groups. The composite $\mathrm{Div}\,\overline{\mathrm{G}} \to \widehat{\mathrm{G}}[2]^{\wedge 2}$ is shown to be zero using an identical technique. To identify the behavior on the other factor, we need the following diagram of exact sequences of Hopf algebras from Kitchloo, Laures, and Wilson [KLW04, Theorem 6.4]:

> **Cite me:** Cite the relevant part of KLW.

> Really? This is what I have written down in my notes, but I can't expand it out.

$$
\begin{array}{ccccccc}
 & & & & & & K_* \\
 & & & & & & \downarrow \\
 & & K_* & & & \dashrightarrow & K_*K(\mathbb{Z},3) \\
 & & \downarrow & & & & \downarrow \\
K_* & \longrightarrow & K_*BSpin & \longrightarrow & K_*BSU & \xrightarrow{\ \tau\ } & K_*BU[6,\infty) \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle\delta} \\
K_* & \longrightarrow & K_*BSO & \xrightarrow{\ i\ } & K_*BSU & \xrightarrow{1-\xi} & K_*BSU \\
 & & \downarrow & & & & \downarrow \\
 & & K_*K(C_2,2) & \cdots & & & K_* \\
 & & \downarrow & & & & \\
 & & K_*, & & & &
\end{array}
$$

where $\tau\colon C_2\widehat{\mathrm{G}} \to C_3\widehat{\mathrm{G}}$ is specified at the level of formal schemes by

$$
\tau\colon ([a]-[0])([b]-[0]) \mapsto ([a]-[0])([b]-[0])([-a-b]-[0]).
$$

Since $(1-\xi)\circ i = 0$, we have that $\delta\circ\tau\circ i = 0$ and hence that $\tau\circ i$ lifts to $K_*K(\mathbb{Z},3)$. Identifying $\mathrm{SDiv}_0\,\widehat{\mathrm{G}}[2]$ with $C_2\widehat{\mathrm{G}}[2]$, we check that the composites

$$
C_2\widehat{\mathrm{G}}[2] \xrightarrow{\ \omega\ } \widehat{\mathrm{G}}[2]^{\wedge 2} \xrightarrow{\ \varepsilon\ } C_3\widehat{\mathrm{G}}
$$

and

$$
C_2\widehat{\mathrm{G}}[2] \to C_2\widehat{\mathrm{G}} \xrightarrow{\ \tau\ } C_3\widehat{\mathrm{G}}
$$

agree. For a point $[a, b] \in C_2\widehat{G}$, this is the claim

$$
\begin{aligned}
0 &= \varepsilon(a \wedge b) - \tau[a, b] \\
&= [a, a, b] - [b, a, b] - [-a - b, a, b] \\
&= [a, a, b] - [b + a, a, b] + [b, a + a, b] - [b, a, b],
\end{aligned}
$$

and this is forced null in $C_3\widehat{G}$, as it looks like a 2–cocycle shuffle.

> I'm a little fuzzy on the coherence of this with the Bockstein: this computes the lift of $\tau \circ f$ into $K(\mathbb{Z}, 3)_K$, and it does happen to factor through the subscheme $K(\mathbb{Z}/2, 2)_K$ determined by the Bockstein. However, I don't immediately see why this agrees with the bottom Postnikov section of BSO: that's a map off of BSO and this is a rotated map into $BU[6, \infty)$, so it's not an immediate consequence of naturality. It has to do with rotating the Wood cofiber sequence just right, and in particular where the horizontal sequences come from: they're stitched-together from two consecutive Wood cofiber sequences.

> Where is $\widehat{G}[2]^{\wedge 2}$ in the diagram? What is ξ? What is ε? Is the dotted arrow the Bockstein? In the diagram you wrote $K_* K(C_2, 2)$, but I think we already have too many C_2's. I'm also confused about the appearances of all the symmetric powers C_2 and C_3. Where do they appear in the diagram?

Ideally, we would use this presentation of $BSpin_K$ to say something about $MSpin$–orientations. I just realized I don't know how, though!

Remark 80. This computation becomes almost unfeasible for $BString$, but we will sketch two approaches. One is that the sequence

$$
\underline{HC_{22}} \to \underline{HC_{2^\infty 2}} \to BString \to BSpin \to \underline{HC_{2^\infty 4}} \xrightarrow{2} \underline{HC_{2^\infty 4}}
$$

induces an exact sequence of group schemes. The other avenue of access is the pair of fiber sequences

$$
\underline{H\mathbb{Z}_3} \to \widetilde{BSpin} \to BSpin, \qquad BString \to \widetilde{BSpin} \to \underline{HC_{23}},
$$

formed by considering the pullback of the corner

$$
BSpin \to \underline{H\mathbb{Z}_4} \leftarrow \underline{HC_{23}}.
$$

Both of these fibrations induce short exact sequences of Hopf algebras.

However, since we are specifically interested in *MString*–orientations, there is an alternative approach that avoids describing the formal scheme $BString_K$. Again appealing to results of Kitchloo, Laures, and Wilson, we find that the sequence

$$
K_* Spin/SU \to K_* BU[6, \infty) \to K_* BString
$$

is exact and right-exact. The kernel of the map $K_* Spin/SU \to K_* BU[6, \infty)$ is a Hopf algebra they call "CK_3", where

> **Cite me:** KLW.

> **Cite me:** Theorem 2.3.5.vi of KLW.

$$
CK_j = \bigoplus_{k=j}^{\infty} K_* K(\mathbb{Z}/2, k).
$$

More than that, KLW even say where the polynomial and nonpolynomial parts of $K_* Spin/SU$ land inside of $K_* BU[6, \infty)$. I think that this means that

> But I have not checked!

$K_*BU[6,\infty)$ is a flat K_*Spin/SU–module at heights $d \leq 2$.

Applying the Thom spectrum functor to the fiber square gives the pushout diagram

$$\begin{array}{ccc} \Sigma_+^\infty Spin/SU & \longrightarrow & MU[6,\infty) \\ \downarrow & & \downarrow \\ S & \longrightarrow & MString, \end{array}$$

or, equivalently, an equivalence

$$MString \simeq MU[6,\infty) \wedge_{\Sigma_+^\infty Spin/SU} S.$$

This in turn gives a Tor spectral sequence of signature

$$\mathrm{Tor}_{*,*}^{K_*Spin/SU}(K_*MU[6,\infty), K_*) \Rightarrow K_*MString.$$

So, under the flatness hypothesis above, there are no higher Tor terms so the spectral sequence collapses to give

$$K_*MString \cong K_*MU[6,\infty)/\!\!/ K_*Spin/SU.$$

So, what remains to be shown is that K_*Spin/SU picks out the correct extra relation for Σ–structures . Then, we need a density argument to show that this handles all of the at-a-point cases of elliptic cohomology.

Remark 81. However, the spectra $HE_\Lambda P$ do not qualify as "chromatically amenable" from the perspective of this last argument, and so we lose access to our genus valued in modular forms. Additionally, K_{Tate} does not qualify, essentially because it is integral rather than p–adic. The methods described here give rise to putative p–adic q–expansions of modular forms, but we have not been able to check that they satisfy the modularity condition, nor that they assemble into a single, integral object. Amazingly, such theorems are achievable, and they are the impetus for the study of topological modular forms and algebraic geometry done with E_∞–rings.

It seems that this approach, which I presume is the one that ultimately works, is independent of all the earlier work involving *BSpin*.

You have not introduced Σ–structures yet.

Expand this last sentence. Point to the Appendix, or cite the *TMF* book, or idk.

The Atiyah–Bott–Shapiro orientation and the fibration *BSU* → *BSpin*. This is possibly somewhere around Theorem 2.3.5.iv in KLW, the last fibration in 2.3.2 at $k = -2$, and sections 5.3 and 5.13

Cite Kitchloo–Laures's "Real structures and Morava *K*–theories"

What follows is the analysis for *MString*. Is the one for *MSpin* analogous and doable? Does it involve CK_2 and maybe a clever choice of *MSU*–orientation?

If you managed to get all this to work, you could try to understand the *MSpin*–orientation of K_{Tate}, as in page 628 of the published AHS.

Some other things that might belong in this chapter

The cubical structure on a singular (generalized) elliptic curve is not unique, but (published) AHS has an argument showing that the unicity of the choice on the nonsingular "bulk" extends to a unique choice on the "boundary" of the compactified moduli too.

There's also the work of Ando–French–Ganter on factorized / iterated Θ structures and how they give rise to the "two–variable Jacobi genus".

Case Study 6
Power operations

This is completely under construction.

There should be a context-based presentation of this chapter's material too. What do contexts for structured ring spectra look like? Why would you consider them — what object are you trying to approximate? How do you guess that the algebraic model is reasonable until you're aware of something like Strickland's theorem?

Since you spend so much time talking about descent in other parts of these notes, maybe you should also read the end of the AHS H_∞ paper where they claim to recast their results in the usual language of descent.

Conversation with Nat on 2/9 suggests taking the following route in this chapter: contexts for E_∞ mapping spaces in general; Subgroups and level structures; the Drinfel'd ring and the universal level structure; the isogenies pile; power operations and Adams operations, after Ando (naturally indexed vs indexed on subgroups; have a look at the Screenshot you took on this day); comparison of comodules M for the isogenies pile with the action of $M_n(\mathbb{Z}_p)$ on $M \otimes_{E_n^*} D_\infty$ (this is a modern result due to Tomer, Tobi, Lukas, and Nat); H_∞ MU–orientations and Matt's thesis; the analogous results for Θ^k–structures. In particular, leave character theory, p–divisible groups, and rational phenomena for spillover at the end of the year. They aren't strictly necessary to telling the story; you just need to know a little about the Drinfel'd ring to construct Matt's maps. (If you have time, though, the point is that the rationalized Drinfel'd ring carries the universal level structure which is also an isomorphism.)

The stuff around 4.3.1-2 of Matt's published thesis talks about H_∞–maps being determined by their values on $*$ and $\mathbb{CP}^\infty$, which is an interesting result. You might also compare with Butowiez–Turner.

Work in height 1 (and height 2??) examples through this? K–theory is pretty accessible, and the height 2 examples are somewhat understood (Charles, Yifei), and they're both relevant for the elliptic *MString* story. (There's also the pile of elliptic curves with isogenies...)

Nat warns that the very end of Matt's thesis uses character theory for S^1, which you have to be very careful about to pull off correctly. (S^1 is not a finite group, but in certain contexts it can be approximated by its torsion subgroups...)

Yifei warned me that Matt's "there exists a unique coordinate..." Lemma is specifically about lifting the *Honda* formal group law over $\mathbb{F}_q$. If you want to do this with elliptic cohomology or something, then you need a stronger statement (and it's clear what this statement should be, but no one has proven it).

—— Here are various notes from conversations with Nat, recorded and garbled well after they happened. ——

We could try to understand Matt's thesis's Section 4.2. It identifies the action of the internal power operation on E_n using the internal theory of quotient isogenies to the Lubin–Tate deformation problem (2.5.4). Conditions 1 and 3 of 4.2.1 are easy to verify: they are 4.2.3 (evaluate on a point) and 4.2.4

(the power operation does raise things to a power) respectively. Condition 2 takes more work, and it's about identifying the divisor associated to the isogeny granted by Condition 1. It's worked out in 4.2.5, which is not very hard, and 4.2.6, which shows that *the* Thom class associated to a vector bundle is sent under a power operation to *some* Thom class. 4.2.5 then uses that the quotient of *some* Thom classes has to be a unit in the underlying ring.

(Q: Can 4.2.5 be phrased about two coordinates on the same formal group, rather than two presentations of the same divisor? There's a comparison between functions on the quotient with invariant functions on the original group — and perhaps with functions invariant by pulling back along the isogeny?)

Prop 8.3 in "Character of the Total Power Operation" provides an algebro-geometric proof of something in AHS04, using the fact that for R a nice complete local ring and G, G' p–divisible groups over R, there is an injection

$$\mathrm{Isog}_R(G, G') \hookrightarrow \mathrm{Isog}_{R/\mathfrak{m}}(G, G').$$

Nat thinks that using the power operation internal to MU is what gets you Lubin's product formula for the quotient (cf. the calculation in Quillen's theorem), and using the power operation internal to E–theory gives you *something* called ψ^H, which you separately calculate on Euler classes. The point (cf. Ando's thesis's Theorem 1) is to pick a coordinate so that these coincide. (Lubin–Tate theory and Lubin's theory of isogenies says that they always coincide up to unique $\star$–isomorphism — after all, automorphisms (and isogenies) don't deform — and the point is that for particular coordinates on particular formal groups, you can take the $\star$–isomorphisms to all be the identity.)

Ando's thesis only deals with power operations internal to E–theory starting in Section 4. Before then, he shows that the pushforward of the power operations internal to MU can be lifted through maps on E–theory (although these maps may not be topologically induced). It's not clear to me what the value of this is — if you're constructing the operations on the E–theory side, then surely you're going to construct them so that they're on-the-nose equal to the MU–operations?

The meaty part of AHS04 is Theorem 6.1, that the necessary condition is sufficient. It falls into steps: first, we can restrict attention to Σ_p, and even inside of there we can restrict attention to C_p. Then, the two directions around the H_∞ square give two trivializations (cf. 4.2.6 of Ando's thesis) $g_{cl(ockwise)}$ and $g_{c(ounter)c(lockwise)}$ of $\Theta^k \mathcal{I}(0)$. The fact that they're both trivializations means there's an equation $g_{cl} = r g_{cc}$ for $r \in E^0 D_{C_p} BU[2k, \infty)^\times$. Then, he wants to study the map

$$E^0 D_{C_p} BU[2k, \infty)_+ \xrightarrow{\Delta^* \times i^*} E^0 BC_p^* \times BU[2k, \infty) \times E^0 BU[2k, \infty)^{\times p},$$

which they know to be an injection by work of McClure, but for some reason they can restrict attention to just the left-hand factor. The left-hand factor is the ring of functions on $\underline{\mathsf{FormalGps}}(A, \widehat{\mathbb{G}}_F) \times BU[2k, \infty)_E$, and they can further restrict attention to level structures, where there are only two: the injective one and the null map. They then check these two cases by hand, and it follows that $r = 0$, so the two ways of navigating the diagram agree at the level of topology.

(Section 8 of Hopkins–Lawson has an injectivity proof that smells similar to the above injectivity trick with McClure's map.)

Just working in the case $k = 1$ (or $k = 0$), which is supposed to recover the "classical" results of Ando's thesis, we can try to recursively expand the various arguments and definitions. The counterclockwise map appears to be the easy one, and it's discussed around 4.11. The clockwise map appears to be the hard one, and it's discussed in 3.21. For $\chi_\ell = \chi_\ell \times \widehat{\mathbb{G}}$ given by

$$T \times \widehat{\mathbb{G}} \xrightarrow{\chi_\ell} \underline{\mathrm{Hom}}(A, \widehat{\mathbb{G}}) \times \widehat{\mathbb{G}},$$

the main content of 3.21 is an equality

$$\chi_\ell^* s_{cl} = \psi_\ell^{\mathcal{L}}(s_g) = (\psi_\ell^{\widehat{\mathbb{G}}/E})^* (\psi_\ell^E)^* s_g,$$

where ψ_ℓ^E is defined in 3.9, $\psi_\ell^{\widehat{\mathbb{G}}/E}$ is defined in 3.14 and the preceding remarks, s_g is the section describing the source coordinate (cf. part 2 of 3.21), and $\psi_\ell^{\mathcal{L}}$ is described between the paragraph before 3.16 and Definition 3.20. Trying to rewrite $\psi_\ell^{\mathcal{L}}$ into the form required for 3.21 requires pushing through 8.11 and 10.15.

We spent a lot of time just writing out the definitions of things, trying to get them straight in the universal case (which AHS04 wants to avoid for some reason — maybe they didn't yet have a good form of Strickland's theorem?). It was helpful in the moment, but hard to read now.

All of this rests, most importantly, on how a quotient of the Lubin–Tate universal deformation by a subgroup still gives a Lubin–Tate universal deformation. This is Section 12.3 of AHS04, and it's Section 9 of Neil's Finite Subgroups paper. (Nat says there's something to look out for in here. Watch where they say they have E_0–algebra maps versus ring maps.)

6.1 E_∞ ring spectra and their contexts

Mike has suggested looking at the paper *The K–theory localization of an unstable sphere*, by Mahowald and Thompson. In it, they manually construct a resolution of S^{2n+1} suitable for computing the unstable Adams spectral sequence for K–theory, but the resolution that they build

is also exactly what you would use to compute the mapping spectral sequence for $E_\infty(K^{S^{2n-1}}, K)$. Additionally, because the unstable K–theoretic operations are exhausted by the power operations, these two spectral sequences converge to the same target.

Purely in terms of the E_∞ version, one can consider the composition of spectral sequences

$$\mathrm{Ext}_{\mathbb{Z}[\theta]}(\mathbb{Z}, \mathrm{Der}_{K_*-alg}(K^*X, K^*)) \Rightarrow \mathrm{Der}_{K_*-Dyer-Lashof-alg}(K^*X, K^*) \Rightarrow E_\infty(\widehat{S0}^X, K_p^\wedge)$$

and

$$E_\infty(\widehat{S0}^X, K_p^\wedge)^{h\mathbb{Z}_p^*} = E_\infty(\widehat{S0}^X, \widehat{S0})$$

where the first spectral sequence is a composition spectral sequence for derivations in K_*–algebras and then derivations respecting the Mandell's θ-operation. If X is an odd sphere, then K^*X has no derivations and this composite spectral sequence collapses, making the composition possible.

This is also related to recent work of Behrens–Rezk on the Bousfield–Kuhn functor...

Another unpublished theorem of Hopkins and Lurie is that the natural map $Y = F(*, Y) \to E_\infty(E_n^Y, E_n)$ is an equivalence when Y is a finite Postnikov tower in the range of degrees that E_n can see.

6.2 Subgroups and level structures

Something that these notes routinely fail to do is to lead into the algebraic geometry in a believable way. "Today we're going to talk about isogenies" — and then, lo' and behold, isogenies appear the next day in algebraic topology. This book would read much better if it showed how these structures were guessed to exist to begin with.

Here's a definition of an isogeny. Weierstrass preparation can be phrased as saying that a Weierstrass map is a coordinate change and a standard isogeny.

Definition 87. Take C and D to be formal curves over X. A map $f : C \to D$ is an *isogeny* when the induced map $C \to C \times_X D$ exhibits C as a divisor on $C \times_X D$ as D–schemes.

Be sure to compare this definition with the usual one for formal groups: surjections with finite cokernel. These are easy to come up with lots of examples for! (Don't ditch this definition, though, since it's the one that lets you prove something about Weierstrass preparation geometrically.)

In fact, every map in positive characteristic can be factored as a coordinate change and an isogeny, which is a weak form of preparation.

Lubin's finite quotients of formal groups. (Interaction with the Lubin–Tate moduli problem? Or does this belong in the next day?)

Write out isogenies of the additive formal group, note that you just get the unstable Steenrod algebra again. This is a remarkable accident.

Push and pull maps for divisor schemes

Moduli of subgroup divisors

The Drinfel'd moduli ring, level structures

—-

Lemma 84. *The following conditions on a homomorphism*

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

are equivalent:

1. *For all $\alpha \neq 0$ in Λ_r^*, $\varphi(\alpha)$ is a unit (resp., not a zero-divisor).*
2. *The Hopf algebra homomorphism*

$$R[\![x]\!]/[p^r](x) \to R^{\Lambda_r^*}$$

is an isomorphism (resp., a monomorphism).   □

Lemma 85. *Let $\mathcal{L}_r(R)$ be the set of all group homomorphism*

$$\varphi : \Lambda_r^* \to F[p^r](R)$$

satisfying either of the conditions 1 or 2 above. This functor is representable by a ring

$$L_r(E^*) := S^{-1}E^*(B\Lambda_r)$$

that is finite and faithfully flat over $p^{-1}E^*$. (Here S is generated by the $\varphi(\alpha)$ with $\alpha \neq 0$, $\varphi : \Lambda_r^* \to F[p^r](E^*B\Lambda_r)$ the canonical map.)

—

Section 2: complete local rings

"Galois" means $R \to S$ a finite extension of integral domains has R as the fixed subring for $\mathrm{Aut}_R(S)$ and S is free over R. Galois extension of rings implies the extension of fraction fields is Galois. The converse holds for finite (finitely generated as a module) dominant (kernel of f is nilpotent) maps of smooth (regular local ring) schemes.

Section 3: basic facts about formal groups
definition of height
Section 4: basic facts about divisors
Since $x -_F a \doteq x - a$, you can treat the divisor $[a]$ (defined in a coordinate by the ideal sheaf generated by $x - x(a)$) as generated just by $x - a$.

Lemma 86. *Let D and D' be two divisors on $\widehat{G}$ over X. There is then a closed subscheme $Y \leq X$ such that for any map $a : Z \to X$ we have $a^*D \leq a^*D'$ if and only if a factors through Y.*   □

Section 5: quotient by a finite sbgp is again a fml gp

Definition 88. A *finite subgroup* of $\widehat{G}$ will mean a divisor K on $\widehat{G}$ which is also a subgroup scheme. Let $\mathcal{O}_{\widehat{G}/K}$ be the equalizer

$$\mathcal{O}_{\widehat{G}/K} \longrightarrow \mathcal{O}_{\widehat{G}} \begin{array}{c} \xrightarrow{\mu^*} \\ \xrightarrow{\pi^*} \end{array} \mathcal{O}_K \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{G}}.$$

Lemma 87. *Write $y = N_\pi \mu^* x \in \mathcal{O}_{\widehat{G}}$.[1] Then $y \equiv x^{p^m} \pmod{\mathfrak{m}_X}$ and $\mathcal{O}_{\widehat{G}/K} = $*

$\mathcal{O}_X[\![y]\!]$. *Moreover, the projection $\widehat{G} \to \widehat{G}/K$ is the categorical cokernel of $K \to \widehat{G}$.*
This all commutes with base change: given $f : Y \to X$ we have $f^*\widehat{G}/f^*K = f^*(\widehat{G}/K)$.   □

Section 6: coordinate-free lubin-tate theory
nothing you haven't already seen. in fact, most of it is done in coordinates, with only passing reference to the decoordinatization.
Section 7: level–A structures: smooth, finite, flat

As discussed long ago, for finite abelian p–groups there's a scheme

$$\underline{\mathsf{FormalGroups}}(A, \widehat{G})(Y) = \underline{\mathsf{Groups}}(A, \widehat{G}(Y)).$$

If $\widehat{G}$ were a discrete group, we could decompose this as

$$\text{``}\underline{\mathsf{FormalGroups}}(A, \widehat{G}) = \coprod_{B \leq A} \mathrm{Mono}(A/B, \widehat{G})\text{''}$$

along the different kernel types of homomorphisms, but Mono does not exist as a scheme. Level structures approximate this as best one can be approximating $\widehat{G}$ by something essentially discrete: an étale group scheme.

For a map $\varphi : A \to \widehat{G}(Y)$, we write $[\varphi A] = \sum_{a \in A}[\varphi(a)]$. We also write $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, so that $\Lambda[p^m] = (\mathbb{Z}/p^m)^{\times n}$. Note

$$|\mathsf{AbelianGroups}(A, \Lambda)| = |A|^n = \mathrm{rank}\left(\underline{\mathsf{FormalGroups}}(A, \widehat{G}) \to X\right).$$

Definition 89. A *level–A structure* on $\widehat{G}$ over an X–scheme Y is a map $\varphi : A \to \widehat{G}(Y)$ such that $[\varphi A[p]] \leq G[p]$ as divisors. A *level–m structure* means a level–$\Lambda[p^m]$ structure.

Lemma 88. *The functor from schemes over X to sets given by*

$$Y \mapsto \{\text{level–}A \text{ structures on } \widehat{G} \text{ over } Y\}$$

is represented by a finite flat scheme $\mathrm{Level}(A, \widehat{G})$ over X. It is contravariantly functorial for monomorphisms of abelian groups. Also, if $\varphi : A \to \widehat{G}$ is a level structure then $[\varphi A]$ is a subgroup divisor and $[\varphi A[p^k]] \leq \widehat{G}[p^k]$ for all k. In fact, if $A = \Lambda[p^m]$ then $[\varphi A] = \widehat{G}[p^m]$.   □

In Section 26 of FPFP Neil says there's a decomposition into irreducible components

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[1] Remember that if $f : X \to Y$ is a finite flat map, then $N_f : \mathcal{O}_X \to \mathcal{O}_Y$ is the nonadditive map sending u to the determinant of multiplication by u, considered as an $\mathcal{O}_Y$–linear endomorphism of $\mathcal{O}_X$.

$$\mathrm{Hom}(A, \widehat{G}) = \mathrm{Hom}(A, \widehat{G}_{\mathrm{red}}) = \bigcup_B \mathrm{Level}(A/B, \widehat{G})$$

and this $\bigcup$ turns into a $\coprod$ after inverting p. He also mentions this as motivation in Finite Subgroups, but he doesn't appear to prove it?

Section 8: maps among level–A schemes, their Galois behavior

Theorem 73. *Let A, B be finite abelian p–groups of rank at most n, and let $u : A \to B$ be a monomorphism. Then:*

1.

$$\mathrm{FormalSchemes}_X(\mathrm{Level}(B, \widehat{G}), \mathrm{Level}(A, \widehat{G})) = \mathrm{Mono}(A, B).$$

2. Such homomorphisms are detected by the behavior at the generic point.
3. The map $u^! : \mathrm{Level}(B, \widehat{G}) \to \mathrm{Level}(A, \widehat{G})$ is finite and flat.
4. If $B \simeq \Lambda[p^m]$, then $u^!$ is a Galois covering.
5. The torsion subgroup of $\widehat{G}(\mathrm{Level}(A, \widehat{G}))$ is A.   □

Section 9: epimorphisms of groups become maps of level schemes, quotients by level structures

Let $\widehat{G}_0$ be a formal group of height n over $X_0 = \mathrm{Spec}\,\kappa$. For every m, the divisor $p^m[0]$ is a subgroup of $\widehat{G}_0$. We write $\widehat{G}_0\langle p^m \rangle$ for the quotient group $\widehat{G}_0/p^m[0]$ and $\widehat{G}\langle m \rangle \to X\langle m \rangle$ for the universal deformation of $\widehat{G}_0\langle m \rangle \to X_0$. Note that $\widehat{G}_0[p] = p^n[0]$, which induces isomorphisms $\widehat{G}_0\langle m + n \rangle \to \widehat{G}_0\langle m \rangle$, and we use this to make as many identifications as we can.

Lemma 89. *Let $u : A \to B$ be an epimorphism of abelian p–groups wit kernel $|\ker(u)| = p^\ell$. Then u induces a map*

$$u_! : \mathrm{Level}(A, \widehat{G}\langle m \rangle) \to \mathrm{Level}(B, \widehat{G}\langle m + \ell \rangle).$$

Also, if $A = \Lambda[p^m]$, then $u_!$ is a Galois covering with Galois group

$$\Gamma = \{\alpha \in \mathrm{Aut}(A) \mid u\alpha = u\}.\,\square$$

Corollary 61. *In particular, the map $A \to 0$ induces a map*

$$0_! : \mathrm{Level}(A, \widehat{G}\langle m \rangle) \to \mathrm{Level}(0, \widehat{G}\langle m + \ell \rangle) = X\langle m + \ell \rangle$$

which extracts quotient formal groups from level structures. In the case $A = \Lambda[p^\ell]$, $0_!$ is just the projection $0^!$.   □

Section 10: moduli of subgroup schemes

Theorem 74. *The functor*

$$Y \mapsto \{\text{subgroups of } \widehat{G} \times_X Y \text{ of degree } p^m\}$$

is represented by a finite flat scheme $\mathrm{Sub}_{p^m}(\widehat{G})$ *over X of degree* $|\mathrm{Sub}_{p^m}(\Lambda)|$. *The formation commutes with base change.*   □

We can at least give the construction: let D be the universal divisor defined over $Y = \mathrm{Div}_{p^m}(\widehat{G})$ with equation $f_D(x) = \sum_{k=0}^{p^m} c_k x^k$. There are unique elements $a_{ij} \in \mathcal{O}_Y$ such that

$$f(x +_F y) = \sum_{i,j=0}^{p^m-1} a_{ij} x^i y^j \quad (\mathrm{mod}\ f(x), f(y)).$$

Define

$$\mathrm{Sub}_{p^m}(\widehat{G}) = \mathrm{Spf}\,\mathcal{O}_Y/(c_0, a_{ij} \mid 0 \le i, j < p^m).$$

Finiteness, flatness, and rank counting are what take real work, starting with an arithmetic fracture square.

Section 13: deformation theory of isogenies

Definition 90. Suppose we have a morphism of formal groups

$$\begin{array}{ccc}
\widehat{G}_0 & \xrightarrow{q_0} & \widehat{G}'_0 \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{f_0} & X'_0
\end{array}$$

such that the induced map $\widehat{G}_0 \to f_0^* \widehat{G}'_0$ is an isogeny of degree p^m. By a deformation of q_0 we mean a prism

where the middle face is the pullback of the left face, the back-right and front-right faces are pullbacks, so that q is also an isogeny of degree p^m.

Let $\widehat{G}/X$ be the universal deformation of $\widehat{G}_0$, let $a : \mathrm{Sub}_{p^m}(\widehat{G}) \to X$ be the usual projection, and let $K < a^*\widehat{G}$ be the universal example of a subgroup of degree p^m. As $\mathrm{Sub}_{p^m}(\widehat{G})$ is a closed subscheme of $\mathrm{Div}_{p^m}(\widehat{G})$ and $\mathrm{Div}_{p^m}(\widehat{G})_0 = X_0$, we see that $\mathrm{Sub}_{p^m}(\widehat{G})_0 = X_0$. There is a unique subgroup

of order p^m of $\widehat{\mathbb{G}}_0$ defined over X_0, viz. the divisor $p^m[0] = \mathrm{Spf}\, \mathcal{O}_{\widehat{\mathbb{G}}_0}/x^{p^m}$. In particular, $K_0 = p^m[0] = \ker(q_0)$. It follows that there is a pullback diagram as shown below:

$$
\begin{array}{ccccc}
(a^*\widehat{\mathbb{G}}/K)_0 & \xrightarrow{\simeq} & \widehat{\mathbb{G}}_0/p^m[0] & \xrightarrow{\bar{q}_0,\simeq} & \widehat{\mathbb{G}}'_0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})_0 & \xrightarrow{a_0,\simeq} & X_0 & \xrightarrow{f_0,\simeq} & X'_0.
\end{array}
$$

We see that $a^*\widehat{\mathbb{G}} \to a^*\widehat{\mathbb{G}}/K$ is a deformation of q_0, and it is terminal in the category of such.

Now let $\widehat{\mathbb{G}}'/X'$ be the universal deformation of $\widehat{\mathbb{G}}'_0/X'_0$. The above construction also exhibits $a^*\widehat{\mathbb{G}}/K$ as a deformation of $\widehat{\mathbb{G}}'_0$, so it is classified by a map $b : \mathrm{Sub}_{p^m}(\widehat{\mathbb{G}}) \to X'$ extending the map $b_0 = f_0 \circ a_0 : \mathrm{Sub}_{p^m}(\widehat{\mathbb{G}})_0 \to X'_0$.

Theorem 75. *b is finite and flat of degree $|\mathrm{Sub}_{p^m}(\Lambda)|$.* $\qquad\square$

Cf. Matt's thesis's Prop 2.5.1: Φ is a formal group over $\mathbb{F}_p$, F a lift of Φ to E_n, H a finite subgroup of $F(D_k)$, then F/H is a lift of Φ to D_k. (This is because the quotient map to F/H reduces to $t \mapsto t^{p^r}$ for some r over $\mathbb{F}_p$, which is an endomorphism of Φ, so the quotient map over the residue field doesn't do anything!) See also Prop 2.5.4, where he characterizes all isogenies of this sort as arising from this construction.

Section 14: connections to AT

Neil's *Finite Subgroups of Formal Groups* has (in addition to lots of results) a section 14 where he talks about the action of a generalized Hecke algebra on the E–theory of a space. Let a and b be two points of X, with fibers $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_b$, and let $q : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_b$ be an isogeny. Then there's an induced map $(Z_E)_a \to (Z_E)_b$, functorial in q and natural in Z. "Certain Ext groups over this Hecke algebra form the input to spectral sequences that compute homotopy groups of spaces of maps of strictly commutative ring spectra, for example." **This sounds like the beginning of an answer to my context question.**

Section 11: flags of controlled rank ascending to $\widehat{\mathbb{G}}[p]$ and a map $\mathrm{Level}(1, \widehat{\mathbb{G}}) \to \mathrm{Flag}(\lambda, \widehat{\mathbb{G}})$. Section 12: the orbit scheme $\mathrm{Type}(A, \widehat{\mathbb{G}}) = \mathrm{Level}(A, \widehat{\mathbb{G}})/\mathrm{Aut}(A)$: smooth, finite, flat Section 15: formulas for computation Section 16: examples

———

Theorem 76. *Let R be a complete local domain with positive residue characteristic p, and let F be a formal group of finite height d over R. If $\mathcal{O}$ is the ring of integers in the algebraic closure of the fraction field of R, then $F(\mathcal{O})[p^k] \cong (\mathbb{Z}/p^k)^d$ and $F(\mathcal{O})_{\mathrm{tors}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^d$.* $\qquad\square$

———

Section 20 of FPFP is about "full sets of points" and the comparison with the cohomology of the flag variety of a vector bundle.

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Talk with Nat:

- Definitions in terms of divisors.
- Equalizer diagram for quotients by finite subgroups.
- The image of a level structure ℓ is a subgroup divisor.
- The schemes classifying subgroups and level structures (which are hard and easy respectively, and which have hard and easy connections to topology respectively).
- It's easy to give explicit examples of the behavior of level structures based on cyclic groups.
- Galois actions on the rings of level structures.

6.3 The Drinfel'd ring and the universal level structure

Talk with Nat:

- Recall the Lubin–Tate moduli problem.
- Show that quotients of deformations by finite subgroups give deformations again.
- Define the Drinfel'd ring.
- As an E^0–algebra, it carries the universal level structure.
- As an ind–(complete local ring), it corepresents deformations (by precomposition with the map $E^0 \to D_n$) *equipped with level structures*.
- Describe the action by $GL_n(\mathbb{Z}_p)$. (Hint at the action by $M_{n\times n}(\mathbb{Z}_p)$ with $\det \neq 0$.)
- Describe the isogenies pile and its relation to all this? (This doesn't really fit precisely, but it may be good to put here, on an algebraic day.)

6.4 Descending coordinates along level structures

> It's not clear to me what theorems about level structures and so forth are best included on this day and which belong back in the lecture above. We should be able to split things apart into stuff desired for character theory and stuff desired for descent.

Ando's Theorem 3.4.4: Let D_j be the ring extension of E_n which trivializes the p^j–torsion subgroup of $\widehat{\mathbb{G}}_{E_n}$. Let H be a finite subgroup of $\widehat{\mathbb{G}}_{E_n}(D_k)$. There is an unstable transformation of ring-valued functors

$$E_n X \xrightarrow{\Psi^H} D_j \otimes E_n X,$$

and if F is an Ando coordinate then for any line bundle $\mathcal{L} \to X$ there is a formula
$$\psi^H(e\mathcal{L}) = \prod_{h \in H} (h +_F e\mathcal{L}) \in D_j \otimes E_n(X).$$

D_j is Galois over E_n with Galois group $GL_n(\mathbb{Z}/p^j)$. If ρ is a collection of finite subgroups weighted by elements of E_n which is stable under the action of the Galois group, then Ψ^ρ descends to take values in just E_n. (For example, the entire subgroup has this property.)

This is built by a character map. Take $H \subseteq F(D_j)[p^j]$ to be a finite subgroup again; then there is a map

$$\chi^H : E_n(D_{H^*}X) \to D_j \otimes E_n(X),$$

where D_{H^*} denotes the extended power construction on X using the Pontryagin dual of H. This composes to give an operation

$$Q^H : MU^{2*}(X) \xrightarrow{P_{H^*}} MU^{2|H|*}(D_{H^*}X) \to E_n^{2|H|*}(D_{H^*}X) \xrightarrow{\chi^H} D_j \otimes E_n^{2|H|*}(X).$$

Then Q^H is a ring homomorphism with effects

$$Q^H F^{MU} = F/H, \qquad Q^H(e_{MU}\mathcal{L}) = \prod_{h \in H} h +_F e\mathcal{L}.$$

Then we need to factor $Q^H : MU(X) \to D_j \otimes E_n(X)$ across the orienting map $MU \to E_n$. Since E_n is Landweber flat and Q^H is a ring map, it suffices to do this for the one–point space, i.e., to construct a ring homomorphism

$$\Psi^H : E_n \to D_j$$

so that $\Psi^H = \Psi^H(*) \otimes Q^H$. The first condition above then translates to $\Psi^H F^{MU} = F/H$.

Theorem 77. *For each $\star$–isomorphism class of lift F of Φ to E_n, there is a unique choice of coordinate x on F, lifting the preferred coordinate on Φ, such that $\alpha_*^H F_x = F_x/H$, or equivalently that $l_H^x = f_H^x$, for all finite subgroups H. (These morphisms are arranged in the following diagram:)*

$$
\begin{array}{ccccc}
& & l_H^x & & \\
& \overset{f_H^x}{\overbrace{\qquad}} & & \overset{g_H^x}{\nearrow} & \\
F_x & \xrightarrow{\quad} & F_x/H & \xrightarrow{\quad} & \alpha_*^H F_x \\
x \uparrow & & x_H \uparrow & & \alpha_* x \uparrow \\
F & \xrightarrow{\quad} & F/H & \xrightarrow{g_H} & \alpha_*^H F,
\end{array}
$$

where $\alpha_H : E_n \to D_k$ is the unique ring homomorphism such that there is a $\star$– isomorphism $g_H : F/H \to \alpha_*^H F$. □

Lemma 90. $P_r(x + y) = \sum_{j=0}^r \mathrm{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$.

Proof. Represent x and y by maps

$$U \xrightarrow{f} X, \qquad\qquad V \xrightarrow{g} Y.$$

Then $P_r(x + y)$ is represented by

$$D_r(U \sqcup V) \xrightarrow{D_r(f \sqcup g)} D_r X.$$

There is a decomposition

$$D_r(U \sqcup V) = \coprod_{j=0}^r E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j}),$$

and on the j factor the map $D_r(f \sqcup g)$ restricts to

$$E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (U^j \times V^{r-j})^j \xrightarrow{E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} (f^j \times g^{r-j})} E\Sigma_r \times_{\Sigma_j \times \Sigma_{r-j}} X^r$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$D_r(U \sqcup V) \xrightarrow{D_r(f \sqcup g)} D_r X,$$

where the vertical maps are projections. The counterclockwise composite represents the j summand of $P_r(x + y)$ coming from the decomposition above; the clockwise composite represents the class $\mathrm{Tr}_{j,r}^{MU} d^*(P_j x \times P_{r-j} y)$. $\square$

Lemma 91. *Write* $\Delta : B\pi \times X \to D_\pi X$ *and let* $\mathcal{L}$ *be a complex line bundle on* X.

$$\Delta^* P_\pi(e\mathcal{L}) = \prod_{u \in \pi^*} \left(e \begin{pmatrix} E\pi \times_u \mathbb{C} \\ \downarrow \\ B\pi \end{pmatrix} +_{MU} e(\mathcal{L}) \right).$$

Now, suppose that we pass down to the kth Drinfel'd ring, so that the p^k–torsion in the formal group is presented as a discrete group $\Lambda^*[p^k]$. Pick such a subgroup $H \subseteq \Lambda^*[p^k]$ with $|H| = r$, and consider also the dual map $\pi : \Lambda[p^k] \to H^*$. We define the character map associated to H to be the composite

$$\chi^H\colon E_n(D_{H^*}X) \xrightarrow{\Delta^*} E_n(BH^*)\otimes_{E_n} E_n(X) \xrightarrow{\chi_\pi\otimes 1} D_k\otimes_{E_n} E_n(X) =: D_k(X).$$

This definition is set up so that

$$\chi^H\left(e\begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix}\right).$$

In the presence of a coordinate x, this sews together to give a cohomology operation:

$$Q^H\colon MU^{2*}(X) \xrightarrow{P_G^{MU}} MU^{2r*}(D_{H^*}X)$$

$$\xrightarrow{\Delta^*} MU^{2r*}(BH^* \times X)$$

$$\xrightarrow{t_x} E_n(BH^* \times X)$$

$$\xrightarrow{\cong} E_n BH^* \otimes_{E_n} E_n X$$

$$\xrightarrow{\chi^H\otimes 1} D_k X.$$

It turns out that Q^H is a ring homomorphism (cf. careful manipulation of HKR's Theorem C, which may not be worth it to write out, but it seems like the main manipulation is the last line of Proof of Theorem 3.3.8 on pg. 466), so each choice of H (and x) determines a new coordinate on D_k.

Theorem 78. *The effect of Q^H on Euler classes is*

$$Q^H e_{MU}\mathcal{L} = f_H^x e_x\mathcal{L} \in D_k(X),$$

and its effect on coefficients is

$$Q_*^H F_{MU} = F_x/H.$$

Proof. We chase through results established so far:

$$Q^H(e_{MU}\mathcal{L}) = (\chi^H\otimes 1)\circ t_x\circ \Delta^*\circ P_G^{MU}(e_{MU}\mathcal{L})$$

$$= (\chi^H\otimes 1)\circ t_x\left(\prod_{u\in H^**=H} e_{MU}\begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{MU} e_{MU}\mathcal{L}\right)$$

$$= (\chi^H\otimes 1)\left(\prod_{u\in H} e_{E_n}\begin{pmatrix} EH^* \times_u \mathbb{C} \\ \downarrow \\ BH^* \end{pmatrix} +_{F_x} e_{E_n}\mathcal{L}\right)$$

$$= \prod_{u\in H}(\varphi_{univ}(u) +_{F_x} e_{E_n}\mathcal{L}) = f_H^x(e_{E_n}\mathcal{L}).$$

Then, "since D_k is a domain, F_x/H is completely determined by the functional equation"

$$f_H^x(F_x(t_1, t_2)) = F_x/H(f_H^x(t_1, f_H^x(t_2))).$$

Take t_1 and t_2 to be the Euler classes of the two tautological bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, so that

$$Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) = Q^H \left(e_{MU} \begin{pmatrix} \mathcal{L}_1 \otimes \mathcal{L}_2 \\ \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \end{pmatrix} \right)$$

$$= f_H^x \left(e_{E_n} \begin{pmatrix} \mathcal{L}_1 \otimes \mathcal{L}_2 \\ \downarrow \\ \mathbb{CP}^\infty \times \mathbb{CP}^\infty \end{pmatrix} \right) = f_H^x(t_1 +_{F_x} t_2).$$

On the other hand, Q^H is a ring homomorphism, so we can also split it over the sum first:

$$Q^H(e_{MU}\mathcal{L}_1 +_{MU} e_{MU}\mathcal{L}_2) = Q^H(e_{MU}\mathcal{L}_1) +_{Q_*^H F^{MU}} Q^H(e_{MU}\mathcal{L}_2)$$

$$= f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2),$$

hence $f_H^x(t_1) +_{Q_*^H F^{MU}} f_H^x(t_2) = f_H^x(t_1 +_{F_x} t_2)$ and $Q_*^H F^{MU} = F_x/H$.

Finally, we would like to produce a factorization

$$MU \xrightarrow{\Psi^H} E_n \to D_k$$

of the long natural transformation Q^H. Since E_n was built by Landweber flatness, it suffices to do this on coefficient rings, i.e., when applying the functors in the diagram to the one-point space. On a point, our calculations above show that Ψ^H exists exactly when $\alpha_*^H F_x = F_x/H$. We did this algebraic calculation earlier: given any coordinate, there is a unique coordinate P that is $\star$–isomorphic to it and through which the operations Q^H factor to give ring operations Ψ^H for all subgroups $H \subseteq \Lambda_k^* = F_P(D_k)[p^k]$. This solves the problem of giving the operations the right *source*.

> Leave a remark in here about this: McClure in BMMS works along similar lines to show that the Quillen idempotent is not H_∞, but he doesn't get any positive results (and, in particular, he can't complete his analysis as we do because he doesn't have access to the BP–homology of finite groups and to HKR character theory). One wonders whether the stuff here does say something about BP as the height tends toward ∞. So far as I know, no one has written much about this. Surely it remains a bee in Matt's bonnet.

Now we focus on giving the operations the right *target*. This is considerably easier. The group $\mathrm{Aut}(\Lambda_k^*)$ acts on the set of subgroups of Λ_k^*, and we define a ring Op^k by the fixed points of $\mathrm{Aut}(\Lambda_k^*)$ acting on the polynomial ring $E_n[\text{subgroups of } \Lambda_k^*]$. Note that $Op^k \subseteq Op^{k+1}$, and define $Op = \mathrm{colim}_k Op^k$, which consists of elements $\rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} H$, I a finite set, $a_i \in E_n$, and α_i are certain $\mathrm{Aut}(\Lambda_k^*)$–stable lists of subgroups of

Λ_k^*, $k \gg 0$, with possible repetitions. For such a ρ, we define the associated operations

$$Q^\rho: \ MU^{2*}(X) \xrightarrow{\ \sum_{i \in I} a_i \prod_{H \in \alpha_i} Q^H\ } D_k(X),$$

$$\Psi^\rho: \ E_n(X) \xrightarrow{\ \sum_{i \in I} a_i \prod_{H \in \alpha_i} \Psi^H\ } D_k(X).$$

The theorem is that these actually land in $E_n(X)$, as they definitely land in $D_k^{\mathrm{Aut}(\Lambda_k^*)} \otimes_{E_n} E_n(X)$, and Galois descent for level structures says that left–hand factor is just E_n.

6.5 The moduli of subgroup divisors

Following... the original? Following Nat?

Continuing on from the above, if we expected E_n to be E_∞ (or even H_∞) so that it had power operations, then we would want to understand $E_n B\Sigma_{p^j}$ and match that with the operations we see.

—

There are union maps

$$B\Sigma_j \times B\Sigma_k \to B\Sigma_{j+k},$$

stable transfer maps

$$B\Sigma_{j+k} \to B\Sigma_j \times B\Sigma_k,$$

and diagonal maps

$$B\Sigma_j \to B\Sigma_j \times B\Sigma_j.$$

These induce a coproduct ψ as well as products $\times$ and $\bullet$ on $E^0 \mathbb{P}S^0$, where $\mathbb{P}S^0 = \coprod_{j=0}^\infty B\Sigma_j$ is the free E_∞–ring on S^0. This is a Hopf ring, and under $\times$ alone it is a formal power series ring. The $\times$–indecomposables (which, I guess, are analogues of considering additive unstable cooperations) are

$$Q^\times E^0 \mathbb{P}S^0 = \prod_{k \geq 0} \left(E^0 B\Sigma_{p^k} / \operatorname{tr} E^0 B\Sigma_{p^{k-1}}^p \right),$$

where the kth factor in the product is naturally isomorphic to $\mathcal{O}_{\mathrm{Sub}_{p^k}(\hat{\mathbb{G}})}$. The primitives are also accessible as the kernel of the dual restriction map.

Theorem 3.2 shows that $E^0 B\Sigma_k$ is free over E^0, Noetherian, and of rank controlled by generalized binomial coefficients. Prop 3.4 is the only place where work gets done, and it's all in terms of K–theory and HKR characters.

There's actually an extra coproduct, coming from applying D to the fold map $S^0 \vee S^0 \to S^0$.

The main content of Prop 5.1 (due to Kashiwabara) is that $K_0 \mathbb{P}\mathbb{S}^0$ injects into $K_0 \underline{BP}_0$. Grading $K_0 \mathbb{P}\mathbb{S}^0$ using the k–index in $B\Sigma_k$, you can see that it's of graded finite type, so we need only know it has no nilpotent elements to see that $K_0 \mathbb{P}\mathbb{S}^0$ is $*$–polynomial. This follows from our computation that $K_0 \underline{BP}_0$ is a tensor of power series and Laurent series rings. Corollary 5.2 is about $K_0 QS^0$, which is the group completion of $K_0 \mathbb{P}\mathbb{S}^0$, so it's the tensor of $K_0 \mathbb{P}\mathbb{S}^0$ with a graded field.

Prop 5.6, using a double bar spectral sequence method, shows that $K^0 QS^2$ is a formal power series algebra. Tracking the spectral sequences through, you'll find that $Q^\times K^0 QS^0$ agrees with $PK^0 QS^2$. (You'll also notice that $K^0 QS^2$ only has one product on it, cf. Remark 5.4.)

Snaith's theorem says $\Sigma^\infty QX = \Sigma^\infty \mathbb{P}X$ for connected spaces X. You can also see (just after Theorem 6.2) the nice equivalences

$$\mathbb{P}_k S^2 \simeq B\Sigma_k^{V_k} \simeq \mathbb{P}_k(S^0)^{V_k},$$

where superscript denotes Thom complex. So, for a complex-orientable co-homology theory, you can learn about $\mathbb{P}_k S^0$ from $\mathbb{P}_k S^2$. In particular, we finally learn that $E^0 \mathbb{P}\mathbb{S}^0$ is a formal power series $\times$–algebra (once checking that the Thom isomorphism is a ring map). (We already knew the homological version of this claim.)

Section 8 has a nice discussion about indecomposables and primitives, to help move back and forth between homology and cohomology. It probably helps most with the dimension count argument below that we aren't going to get into.

Start again with $D_{p^k} S^2 \simeq B\Sigma_{p^k}^{V_{p^k}}$. We can associate to this a divisor $\mathbb{D}(V_{p^k})$ on $(B\Sigma_{p^k})_E$, which we know little about, but it is classified by a map to $\mathrm{Div}_{p^k} \mathbb{CP}_E^\infty$. This receives a closed inclusion from $\mathrm{Sub}_{p^k} \mathbb{CP}_E^\infty$, so their pullback Z_k is the largest subscheme of $(B\Sigma_{p^k})_E$ over which $\mathbb{D}(V_{p^k})$ is a subgroup divisor.

We will show the existence of the dashed map, implying that the restricted divisor H_k is a subgroup divisor on $Y_k = \mathrm{Spf}\, E^0 B\Sigma_{p^k}/\mathrm{tr}$.

(Prop 9.1:) This proof falls into two parts: first we construct a family of maps to $(B\Sigma_{p^k})_E$ on whose image $\mathbb{D}(V_{p^k})$ restricts to a subgroup divisor, and then we show that the union of their images is exactly Y_k. Let A be an abelian p–subgroup of Σ_{p^k} that acts transitively on $\{1,\ldots,p^k\}$ (i.e., it is not boosted from some transfer). The restriction of V_{p^k} to A is the regular representation, which splits as a sum of characters $V_{p^k}|_A = \bigoplus_{\mathcal{L}\in A^*}\mathcal{L}$. Identifying $BA_E = \underline{\mathrm{FormalGroups}}(A^*, \mathbb{CP}_E^\infty)$, $\mathbb{D}(V_{p^k})$ restricts all the way to $\sum_{\mathcal{L}\in A^*}[\varphi(\mathcal{L})]$, with $\varphi : A^* \to \text{“}\Gamma(\mathrm{Hom}(A^*,\widehat{G}),\widehat{G})\text{”}$. In Finite Subgroups of Formal Groups (see Props 22 and 32), we learned that the restriction of $\mathbb{D}(V_{p^k})$ further to $\mathrm{Level}(A^*, \mathbb{CP}_E^\infty)$ is a subgroup divisor. So, our collection of maps are those of the form

$$\mathrm{Level}(A^*, \mathbb{CP}_E^\infty) \to \underline{\mathrm{FormalGroups}}(A^*, \mathbb{CP}_E^\infty) = BA_E \to (B\Sigma_{p^k})_E.$$

Here, finally, is where we have to do some real work involving Chern classes and commutative algebra, so I'm inclined to skip it in the lectures. Finally, you do a dimension count to see that Z_k and $\mathrm{Spf}\, E^0 B\Sigma_{p^k}/\mathrm{tr}$ have the same dimension (which requires checking enough commutative algebra to see that "dimension" even makes sense), and so you show the map is injective and you're done.

———

Here's Neil's proof of the joint images claim. It seems like a clear enough use of character theory that we should include it, if we can make character theory itself clear.

Recall from [18, Theorem 23] that $\mathrm{Level}(A^*, \widehat{G})$ is a smooth scheme, and thus that $D(A) = \mathcal{O}_{\mathrm{Level}(A^*,\widehat{G})}$ is an integral domain. Using [18, Proposition 26], we see that when $\mathcal{L} \in A^*$ is nontrivial, we have $\varphi(\mathcal{L}) \neq 0$ as sections of $\widehat{G}$ over $\mathrm{Level}(A^*, \widehat{G})$, and thus $e(\mathcal{L}) = x(\varphi(\mathcal{L})) \neq 0$ in $D(A)$. It follows that that $c_{p^k} = \prod_{\mathcal{L}\neq 1} e(\mathcal{L})$ is not a zero-divisor in $D(A)$. On the other hand, if A' is an Abelian p-subgroup of Σ_{p^k} which does not act transitively on $\{1,\ldots,p^k\}$, then the restriction of $V_{p^k}1$ to A' has a trivial summand, and thus c_{p^k} maps to zero in $D(A')$. Next, we recall the version of generalised character theory described in [8, Appendix A].

$$p^{-1}E^0 BG = \left(\prod_A p^{-1}D(A)\right)^G$$

where A runs over all Abelian p-subgroups of G. As $\overline{R}_k = E^0(B\Sigma_{p^k})/ann(c_{p^k})$ and everything in sight is torsion-free, we see that $p^1\overline{R}_k$ is the quotient of $p^1 E^0 B\Sigma_{p^k}$ by the annihilator of the image of c_{p^k}. Using our analysis of the

images of c_{p^k} in the rings $D(A)$, we conclude that

$$p^{-1}\overline{R}_k = \left(\prod_A p^1 D(A)\right)^{\Sigma_{p^k}},$$

where the product is now over all transitive Abelian p-subgroups. This implies that for such A, the map $E^0 B\Sigma_{p^k} \to D(A)$ factors through $\overline{R}_k$, and that the resulting maps $\overline{R}_k \to D(A)$ are jointly injective. This means that $Y_k = \operatorname{Spf} \overline{R}_k$ is the union of the images of the corresponding schemes $\operatorname{Level}(A^*, \widehat{G})$, as required.

6.6 Interaction with Θ–structures

The Ando–Hopkins–Strickland result that the σ–orientation is an H_∞–map

The main classical point is that an $MU\langle 0\rangle$–orientation is H_∞ when the following diagram commutes for every choice of A:

$$\begin{array}{ccccc} (BA^* \times \mathbb{CP}^\infty)^{V_{reg} \otimes \mathcal{L}} & \longrightarrow & D_n MU\langle 0\rangle & \longrightarrow & D_n E \\ & & \downarrow & & \downarrow \\ & & MU\langle 0\rangle & \longrightarrow & E \end{array}$$

(This is equivalent to the condition given in the section on Matt's thesis. In fact, maybe I should try writing this so that Matt's thesis uses the same language?) If you write out what this means, you'll see that a given coordinate on E pulls back to give two elements in the E–cohomology of that Thom spectrum (or: sections of the Thom sheaf), and the orientation is H_∞ when they coincide.

Similarly, an $MU\langle 6\rangle$–orientation corresponds to a section of the sheaf of cubical structures on a certain Thom sheaf. Using the H_∞ structures on $MU\langle 6\rangle$ and on E give two sections of the pulled back sheaf of cubical structures, and the H_∞ condition is that they agree for all choices of group A.

Then you also need to check that the σ–orientation actually satisfies this.

> The AHS document really restrictions attention to E_2. Is there a version of this story that gives non-supersingular orientations too, or even the K_{Tate} orientation? I can't tell if the restriction in AHS's exposition comes from not knowing that K_{Tate} has an E_∞ structure or if it comes from a restriction on the formal group. (At one point it looks like they only need to know that p is regular on $\pi_0 E$, cf. 16.5...)

Section 3.1: Intrinsic description of the isogenies story for an H_∞ *complex orientable* ring spectrum, without mention of a specific orientation / coordinate. This is nice: it means that a complex orientation has to be a coordinate which is compatible with the descent picture already extant on the level of formal groups, which is indeed the conclusion of Matt's thesis.

Section 3.2: They define an abelian group indexed extended power construction

$$D_A(X) = \mathcal{L}(U^{A^*}, U) \wedge_{A^*} X^{(A^*)},$$

where $\mathcal{L}(U^{A^*}, U)$ is the space of linear isometries from the $A^{*\text{th}}$ power of a universe U down to itself. Yuck. Then, given a level structure $(i\colon \operatorname{Spf} R \to S_E, \ell\colon A_{\operatorname{Spf} R} \to i^*\widehat{G})$, they construct a map

$$\psi_\ell^E\colon \pi_0 E \xrightarrow{D_A} \pi_0 \operatorname{Spectra}(D_A S^0, E) = \pi_0 E^{BA_+^*} \to \mathcal{O}((BA^*)_E) \xrightarrow{\chi_\ell} R,$$

where χ_ℓ is the map classifying the homomorphism ℓ. This is a continuous map of rings: it's clearly multiplicative, it's additive up to transfers (but those vanish for an abelian group), and it's continuous by an argument in Lemma 3.10. (You don't actually need an abelian group here; you can work in the scheme of subgroups — i.e., in the cohomology of $B\Sigma_k$ modulo transfers — and this will still work.) This construction is natural in H_∞ maps $f\colon E \to F$:

begetting the relative map $\psi_\ell^{F/E}\colon i^*S_F \to (\psi_\ell^E)^*S_F$ as indicated. For example, take $F = E^{\mathbb{CP}_+^\infty}$, so that $\widehat{G} = S_F$, giving the (group) map

$$\psi_\ell^{\widehat{G}/E}\colon i^*\widehat{G} \to (\psi_\ell^E)^*\widehat{G}.$$

One of the immediate goals is to show that this is an isogeny. A different construction we can do is take V to be a virtual bundle over X and set $F = E^{X_+}$. Given $m \in \pi_0 \operatorname{Spectra}(X^V, E)$ applying the construction of D_A above gives an element

$$\psi_\ell^V(m) \in R \underset{\chi_\ell, \hat{\pi}_0 E^{BA_+^*}}{\widehat{\otimes}} \hat{\pi}_0 \operatorname{Spectra}((BA^* \times X)^{V_{reg} \otimes V}, E).$$

This map is additive and also $\psi_\ell^V(xm) = \psi_\ell^F(x)\psi_\ell^V(m)$, so we can interpret this as a map

$$\psi_\ell^V\colon (\psi_\ell^F)^*\mathbb{L}(V) \to \chi_\ell^*\mathbb{L}(V_{reg} \otimes V)$$

of line bundles over $i^*S_F = i^*X_E$.

Lemma 92. *The map ψ_ℓ^V has the following properties:*

1. If m trivializes $\mathbb{L}(V)$ then $\psi_\ell^V(m)$ trivializes $\chi_\ell^*\mathbb{L}(V_{reg} \otimes V)$.
2. $\psi_\ell^{V_1 \oplus V_2} = \psi_\ell^{V_1} \otimes \psi_\ell^{V_2}$.
3. For $f : Y \to X$ a map, $\psi_\ell^{f^*V} = f^*\psi_\ell^V$. $\square$

In particular, we can apply this to $X = \mathbb{CP}^\infty$ and $\mathbb{L}(\mathcal{L} - 1) = \mathcal{I}(0)$. Then 8.11 gives

$$\psi_\ell^{\mathcal{L}-1} : (\psi_\ell^F)^*\mathcal{I}_{\widehat{G}}(0) \to \chi_\ell^*\mathbb{L}(V_{reg} \otimes (\mathcal{L} - 1)) = \mathcal{I}_{i^*\widehat{G}}(\ell).$$

Theorem 79. *The map $\psi_\ell^{\widehat{G}/E} : i^*\widehat{G} \to (\psi_\ell^E)^*\widehat{G}$ of 3.15 is an isogeny with kernel $[\ell(A)]$. Using $\psi_\ell^{\widehat{G}/E}$ to make the identification*

$$(\psi_\ell^{\widehat{G}/E})^*\mathcal{I}_{(\psi_\ell^E)^*\widehat{G}}(0) \cong \mathcal{I}_{i^*\widehat{G}}(\ell),$$

the map $\psi_\ell^{\mathcal{L}-1}$ sends a coordinate x on $\widehat{G}$ to the trivialization $(\psi_\ell^{\widehat{G}/E})^*(\psi_\ell^E)^*x$ of $\mathcal{I}_{i^*\widehat{G}}(\ell)$. $\square$

3.24 might be interesting.

Theorem 80. *Let $g : MU\langle 0 \rangle \to E$ be a homotopy multiplicative map, and let $s = s_g$ be the corresponding trivialization of $\mathcal{I}_{\widehat{G}}(0)$. If the map g is H_∞, then for any level structure $\ell : A \to i^*\widehat{G}$ the section s satisfies the identity*

$$N_{\psi_\ell^{\widehat{G}/E}}i^*s = (\psi_\ell^E)^*s,$$

in which the isogeny $\psi_\ell^{\widehat{G}/E}$ has been used to make the identification

$$N_{\psi_\ell^{\widehat{G}/E}}i^*\mathcal{I}_{\widehat{G}}(0) \cong \mathcal{I}_{(\psi_\ell^E)^*\widehat{G}}(0).\square$$

Lemma 93. *For V a vector bundle on a space X and V_{reg} the (vector bundle over BA^* induced from) the regular representation on A, there is an isomorphism of sheaves over $(BA^* \times X)_E$*

$$\mathbb{L}(V_{reg} \otimes V) \cong \bigotimes_{a \in A} \widetilde{T}_a\mathbb{L}(V).$$

Eqn 5.4 claims to use 5.3 but seems to be using something about the behavior of the norm map on line bundles vs the translated sum of divisors appearing in 5.3.

The beginning of the proof of 6.1 appears to be a simplification of some of the descent arguments appearing in the algebraic parts of Matt's thesis's main calculations. On the other hand, I can't even read what the McClure reference in 6.1 is doing. What's Δ^*??

Lemma 94. *Take $\pi_0 E$ to be a complete local ring and $\widehat{\mathbb{G}}_E$ to be of finite height. If $B^* \subset A^*$ is a proper subgroup, then the following composite map of $\pi_0 E$–modules is zero:*

$$\pi_0 E^{BB_+^*} \xrightarrow{transfer} \pi_0 E^{BA_+^*} \xrightarrow{\chi_\ell} \mathcal{O}(T).$$

Proof. It suffices to consider the tautological level structure over $\mathrm{Level}(A, \widehat{\mathbb{G}})$. We may take A to be a p–group, and indeed for now we set $A = \mathbb{Z}/p$, $B = 0$. For $t \in \pi_0 E^{\mathbb{CP}_+^\infty}$ a coordinate with formal group law F, we have

$$\pi_0 E^{BA_+^*} \cong \pi_0 E[\![t]\!]/[p]_F(t)$$

and $\tau : \pi_0 E^{BB_+^*} = \pi_0 E \to \pi_0 E^{BA_+^*}$ is given by $\tau(1) = \langle p \rangle_F(t)$, where $\langle p \rangle_F(t) = [p]_F(t)/t$ is the "reduced p–series". The result then follows from the isomorphism $\mathcal{O}(\mathrm{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}_E)) \cong \pi_0 E[\![t]\!]/\langle p \rangle_F(t)$. The result then follows in general by induction: B^* can be taken to be a *maximal* proper subgroup of A^*, with cokernel $\mathbb{Z}/p$.

Example 39. Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group with coordinate x so that the group law is

$$x +_{\widehat{\mathbb{G}}_m} y = x + y - xy, \quad [p](x) = 1 - (1-x)^p.$$

The monomorphism $\mathbb{Z}/p \to \widehat{\mathbb{G}}_m(\mathbb{Z}[\![y]\!]/[p](y))$ given by $j \mapsto [j](y)$ becomes the zero map under the base change

$$\mathbb{Z}[\![y]\!]/[p](y) \to \mathbb{Z}/p,$$
$$y \mapsto 0.$$

Remark 82. If R is a domain of characteristic 0, then a level structure over R actually induces a monomorphism on points.

Lemma 95. *The natural map*

$$\mathcal{O}(\underline{\mathrm{FormalGroups}}(\mathbb{Z}/p, \widehat{\mathbb{G}})) \to R \times \mathcal{O}(\mathrm{Level}(\mathbb{Z}/p, \widehat{\mathbb{G}}))$$

is injective.

Proof. _____

—— Descent along level structures, simplicially (Section 11) ——

Write $\mathrm{Level}(A) \to \mathrm{FGps}$ for the parameter space of a formal group equipped with a level–A structure, together with its structure map (to the *coarse moduli of formal groups!!!*). We define a sequence of schemes by: $\mathrm{Level}_0 = \mathrm{FGps}$, $\mathrm{Level}_1 = \coprod_{A_0} \mathrm{Level}(A_0)$ for finite abelian groups A_0, and most generally

$$\mathrm{Level}_n = \coprod_{0 = A_n \subseteq \cdots \subseteq A_0} \mathrm{Level}(A_0).$$

There are two maps $\mathrm{Level}_1 \to \mathrm{Level}_0$. One is the structural one, where we simply peel off the formal group and forget the level structure. The other comes from the quotient map: $\ell \colon A \to \widehat{G}$ yields a quotient isogeny $q \colon \widehat{G} \to \widehat{G}/\ell$, and we take the second map $\mathrm{Level}_1 \to \mathrm{Level}_0$ to send ℓ to $\widehat{G}/\ell$. Then, consider the following Lemma:

Lemma 96. *For $\ell \colon A \to \widehat{G}$ a level structure and $B \subseteq A$ a subgroup, the induced map $\ell|_B \colon B \to \widehat{G}$ is a level structure and the quotient $\widehat{G}/\ell|_B$ receives a level structure $\ell' \colon A/B \to \widehat{G}/\ell|_B$.* $\qquad\square$

This gives us enough compatibility among quotients to use the two maps above to assemble the Level_* schemes into a simplicial object. Most face maps just omit a subgroup, except for the last face map, since the zero subgroup is not permitted to be omitted. Instead, the last face map sends the string of subgroups $0 = A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_0$ and level structure $\ell \colon A_0 \to \widehat{G}$ to the quotient string $0 = A_{n-1}/A_{n-1} \subseteq \cdots \subseteq A_0/A_{n-1}$ and quotient level structure $\ell \colon A_0/A_{n-1} \to \widehat{G}/\ell|_{A_{n-1}}$. The degeneracy maps come from lengthening one of these strings by an identity inclusion.

Definition 91. Let $\widehat{G} \colon F \to \mathrm{FGps}$ be a functor over formal groups, and define schemes $\mathrm{Level}(A, F) = \mathrm{Level}(A) \times_{\widehat{G}} F$ and $\mathrm{Level}_n(F) = \mathrm{Level}_n \times_{\widehat{G}} F$. Then, *descent data for level structures on F* is the structure of a simplicial scheme on $\mathrm{Level}_*(F)$, together with a morphism of simplicial schemes $\mathrm{Level}_*(F) \to \mathrm{Level}_*$. It is enough to specify a map $d_1 \colon \mathrm{Level}_1(F) \to F$, use that to build the simplicial scheme structure as in the above Lemma, and assert that the following square commutes:

$$
\begin{array}{ccc}
\mathrm{Level}_1(F) & \longrightarrow & \mathrm{Level}_1 \\
\downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} \\
F & \longrightarrow & \mathrm{FGps.}
\end{array}
$$

Example 40. Let $\widehat{G} \colon S \to \mathrm{FGps}$ be a formal group of finite height over a p–local formal scheme S. The functor $\mathrm{Level}(A, \widehat{G})$ is exactly the functor defined in Section 9 (see above), and in particular it is represented by an S–scheme. The maps ψ_ℓ and f_ℓ from Definition 3.1 amount to giving a map

$d_1\colon \mathrm{Level}_1(\widehat{\mathsf{G}}) \to S$ and an isogeny $q\colon d_0^*\widehat{\mathsf{G}} \to d_1^*\widehat{\mathsf{G}}$ whose kernel on $\mathrm{Level}(A,\widehat{\mathsf{G}})$ is A. The other conditions on Definition 3.1 exactly ensure that $(\mathrm{Level}_*(\widehat{\mathsf{G}}), d_*, s_*)$ is a simplicial functor and over $\mathrm{Level}_2(\widehat{\mathsf{G}})$ the relevant hexagonal diagram commutes:

$$
\begin{array}{ccc}
 & d_0^* d_0^* \widehat{\mathsf{G}} & \\
 & & \\
d_1^* d_0^* \widehat{\mathsf{G}} & & d_0^* d_1^* \widehat{\mathsf{G}} \\
\Big\downarrow{\scriptstyle d_1^* q} & & \Big\| \\
d_1^* d_1^* \widehat{\mathsf{G}} & & d_2^* d_0^* \widehat{\mathsf{G}} \\
 & & \\
 & d_2^* d_1^* \widehat{\mathsf{G}}. &
\end{array}
$$

with arrows $d_0^* q$ from $d_0^* d_0^*\widehat{\mathsf{G}}$ to $d_0^* d_1^*\widehat{\mathsf{G}}$ and $d_2^* q$ into $d_2^* d_1^*\widehat{\mathsf{G}}$.

Example 41. We now further package this into a single object. Let $\underline{\widehat{\mathsf{G}}}$ be the functor over FGps whose value on R is the set of pullback diagrams

$$
\begin{array}{ccc}
\widehat{\mathsf{G}}' & \xrightarrow{\ f\ } & \widehat{\mathsf{G}} \\
\downarrow & & \downarrow \\
\mathrm{Spf}\,R & \xrightarrow{\ i\ } & S
\end{array}
$$

such that the map $\widehat{\mathsf{G}}' \to i^*\widehat{\mathsf{G}}$ induced by f is a homomorphism (hence isomorphism) of formal groups over $\mathrm{Spf}\,R$. For a finite abelian group A, write $\mathrm{Level}(A,\underline{\widehat{\mathsf{G}}})(R)$ for the set of diagrams

$$
\begin{array}{ccccc}
A_{\mathrm{Spf}\,R} & \xrightarrow{\ \ell\ } & \widehat{\mathsf{G}}' & \xrightarrow{\ f\ } & \widehat{\mathsf{G}} \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mathrm{Spf}\,R & \xrightarrow{\ i\ } & S
\end{array}
$$

where the square forms a point in $\underline{\widehat{\mathsf{G}}}(R)$ and ℓ is a level–A structure. Giving a map of functors $d_1\colon \mathrm{Level}_1(\underline{\widehat{\mathsf{G}}}) \to \underline{\widehat{\mathsf{G}}}$ making the above square commute is to give a pullback diagram

$$
\begin{array}{ccc}
\widehat{\mathsf{G}}/\ell & \longrightarrow & \widehat{\mathsf{G}} \\
\downarrow & & \downarrow \\
\mathrm{Level}_1(\widehat{\mathsf{G}}) & \longrightarrow & S,
\end{array}
$$

or equivalently a map of formal schemes $\mathrm{Level}_1(\widehat{\mathsf{G}}) \to S$ and an isogeny $q\colon d_0^*\widehat{\mathsf{G}} d_1^*\widehat{\mathsf{G}}$ whose kernel on $\mathrm{Level}(A,\widehat{\mathsf{G}})$ is A. Therefore, descent data for

level structures on the formal group $\widehat{G}$ (in the sense of Section 3) are equivalent to descent data for level structures on the functor $\underline{\widehat{G}}$.

—— Section 12: Descent for level structures on Lubin–Tate groups ——

Let k be perfect of positive characteristic p, and let Γ be a formal group of finite height over k. Recall that this induces a relative Frobenius

$$
\begin{array}{ccc}
& & \overset{\varphi_\Gamma}{} \\
\Gamma \xrightarrow{\;F\;} \varphi_k^*\Gamma & \longrightarrow & \Gamma \\
\downarrow & \downarrow & \downarrow \\
& \operatorname{Spec} k \xrightarrow{\;\varphi_k\;} & \operatorname{Spec} k.
\end{array}
$$

The map F is an isogeny of degree p, with kernel the divisor $p \cdot [0]$. Recall also that a deformation H of Γ to T induces a map $\underline{H} \to \operatorname{Def}(\Gamma)$, and there is a universal such $\widehat{G}$ over the ground scheme $S \cong \operatorname{Spf} W(k)[\![u_1, \ldots, u_{d-1}]\!]$ such that $\underline{\widehat{G}} \to \operatorname{Def}(\Gamma)$ is an isomorphism of functors over FGps.

Now consider a point in $\operatorname{Level}(A, \operatorname{Def}\Gamma)$:

$$
\begin{array}{ccccc}
A_T \xrightarrow{\;\ell\;} H & \longleftarrow & H_0 & \xrightarrow{\;f\;} & \Gamma \\
\searrow \downarrow & & \downarrow & & \downarrow \\
T & \longleftarrow & T_0 & \xrightarrow{\;j\;} & \operatorname{Spec} k.
\end{array}
$$

The level structure ℓ gives rise to a quotient isogeny $q \colon H \to H'$. Since A is sent to 0 in $\mathcal{O}_{T_0}$, there is a canonical map $\bar{q}$ fitting into the diagram

The map $\bar{q}$ combines with the rest of the maps to exhibit H' as a deformation of Γ, and hence we get a natural transformation

$$
d_1 \colon \ \operatorname{Level}_1(\operatorname{Def}(\Gamma)) \to \operatorname{Def}(\Gamma).
$$

Since $\varphi^r \varphi^s = \varphi^{r+s}$, this gives descent data for level structures on $\operatorname{Def}(\Gamma)$. Identifying this functor with $\underline{\widehat{G}}$ using Lubin–Tate theory, we equivalently have shown the existence of descent data for level structures on $\underline{\widehat{G}}$.

Incidentally, the descent data constructed here is also the descent data that would come from the structure of an E_∞–orientation on the Morava E–theory E_d, essentially because the divisor associated to the kernel of the relative Frobenius on the special fiber is forced to be $p[0]$, and everything is dictated by how the deformation theory *has* to go (and the fact that the topological operations we're studying induce deformation-theoretic-describable operations on algebra).

——Section 15: Level structures on elliptic curves, and the relation to the σ–orientation / the corresponding section of the Θ^3–sheaf——

Tyler's argument

There's an important injectivity result used by Ando and Ando–Hopkins–Strickland (though Matt blames it on Hopkins and Strickland both times) about the injectivity of a certain p^{th} power map. They cite the McClure chapter of BMMS, but McClure's proof requires finite type hypotheses on the cohomology theory involved, which Morava E–theory does not satisfy. There is a similar proof in the recent paper of Hopkins–Lawson, and so Nat and I wrote to Tyler about whether there was a common generalization of the two theorems that would give a good replacement argument. Here is his reply:

—

Here are my current thought processes, which may be a bit messy at present. Fix a space X and take $X^{(p)}$ for its smash power, as McClure does.

Let's write $M = F(\Sigma^\infty X^{(p)}, E)$ for the function spectrum which is now C_p–equivariant, and $N = F(\Sigma^\infty X, E)$. Let's assume that E has an E_∞ multiplication and that X is nice in the following sense: E^X is a wedge of copies of E (unshifted). This is satisfied when E is E–theory and X is finite type with $\mathbb{Z}_{(p)}$-homology only in even degrees.

We get two maps:

$$M^{hC_p} \to M$$

This will realize our "forgetful" map $E^*(DX) \to E^*(X^{(p)})$.

$$M^{hC_p} \to N^{hC_p}$$

This will realize the "other" map $E^*(DX) \to E^*((BC_p)_+ \wedge X)$.

We want to prove that these are jointly monomorphisms.

The assumptions on X actually imply that $E^{X^{(p)}} = (E^X)^{(p)}$ where the latter smash is taken over E. This decomposes, C_p–equivariantly, into a wedge of copies of E with trivial action and a bunch of regular representations $E[C_p]$. Since M is E–dual to this, we find that the map $M^{hC_p} \to M$ is a monomorphism on all the $E[C_p]$ components; on the E parts with triv-

ial action it decomposes as a product of projections $E_*[\![x]\!]/[p](x) \to E_*$. The kernel of this consists of the multiples of x. So if we want to prove a monomorphism, all we have to do now is show that these multiples of x map monomorphically into the homotopy of N^{hC_p}.

I now want to consider the composite to the Tate spectra

$$M^{hC_p} \to N^{hC_p} \to N^{tC_p}$$

or equivalently

$$M^{hC_p} \to M^{tC_p} \to N^{tC_p}.$$

The first composition shows that, if we can show that this composite is a monomorphism on the multiples of x, we will be done. The second composition has, as its first map, inverting x, and it's a monomorphism on the desired classes. So we just have to check that the second map preserves that.

This has the following benefit: instead of being born out of the unstable diagonal map $X \to X^{(p)}$, the constructions

$$M^{tC_p} = F(X^{(p)}, E)^{tC_p}$$

and

$$N^{tC_p} = F(X, E)^{tC_p}$$

take cofiber sequences in (finite) X to fiber sequences of spectra. I think that this means that, instead of being functions of the unstable diagonal map on X, they are constructions that only require knowledge of the *stable* homotopy type of X. I believe in fact that, by checking the case $X = S^0$, we then find that the map $M^{tC_p} \to N^{tC_p}$ is an equivalence for any finite X, and that should hopefully be enough to buy us a monomorphism on any of the X's that we're describing.

The stability of the Tate construction comes about for the following reason.

Say we have a cofiber sequence $X \to Y \to Y/X$. Then the p–fold smash power $Y^{(p)}$ has a natural, equivariant filtration: Y is, for dumb reasons, the homotopy colimit of $(X \to Y)$, then $Y^{(p)}$ is the hocolim of a p–fold smash power of this diagram (now indexed on $\{0 \to 1\}^{\times p}$). We can filter this diagram, equivariantly, according to "distance from the initial vertex", and get an equivariant filtration of $Y^{(p)}$ whose associated graded in degree k consists of all ways to smash k copies of Y/X with $(p-k)$ copies of X in some order. In grading 0 this is $X^{(p)}$, and in grading p this is $(Y/X)^{(p)}$; in all the gradings in between you get a wedge of terms where C_p acts by permuting the wedge factors.

The Tate construction preserves C_p-equivariant cofiber sequences, and destroys anything where C_p acts by permuting wedge factors. As a result,

I don't understand this. I guess this has been a recurring theme in the Thursday seminar, and also in Chapters 2 and 6 of these notes (in some guise). I can ask Mike to explain it to me.

the only parts that survive are the bottom $(X^{(p)})^{tC_p}$ and the top $((Y/X)^{(p)})^{tC_p}$ in a cofiber sequence.

The standard reference should probably be Greenlees–May's *Generalized Tate cohomology theories* but I'm not as closely familiar with the contents there. Sometimes this is called the topological Singer construction and I originally learned about it from Lunøe–Nielsen–Rognes. This talk about the stability properties is present in section 2 of DAG XIII somewhere, and possibly also in Jacob's notes on the Sullivan conjecture. Charles pointed out to me recently that these properties of the Tate construction are really why the Steenrod operations and certain power operations are stable: it's because they don't come out of a homotopy orbit construction on a smash power, but instead out of a Tate construction.

Other stuff that goes in this chapter

Dyer–Lashof operations, the Steenrod operations, and isogenies of the formal additive group

> **Cite me:** See Neil's *Steenrod algebra* note, maybe? Talk to Mike?.

Another augmentation to the notion of a context: working not just with E_*X but with $E_*(X \times BG)$ for finite G.

Charles's *The congruence criterion* paper codifies the Hecke algebra picture Neil is talking about, and in particular it talks about sheaves over the pile of isogenies.

If we're going to talk about that Hecke algebra, then maybe we can also talk about the period map, since one of the main points of it is that it's equivariant for that action.

Section 3.7 of Matt's thesis also seems to deal with the context question: he gives a character-theoretic description of the total power operation, which ties the behavior of the total power operation to a formula of type "decomposition into subgroups". Worth reading.

The rational story: start with a sheaf on the isogenies pile. Tensor everything with $\mathbb{Q}$. That turns this thing into a rational algebra under the Drinfel'd ring together with an equivariant action of $GL_n\mathbb{Q}_p$.

> This is Nat's claim. Check back with him about how this is visible.

Matt's Section 4 talks about the E_∞ structure on E_n and compatibility with his power operations. It's not clear how this doesn't immediately follow from the stuff he proves in Section 3, but I think I'm just running out of stream in reading this thesis. One of the neat features of this later section is that it relies on calculations in $E_n D_\pi \underline{MU}_{2*}$, which is an interesting way to mix operations coming from instability and from an H_∞–structure. This is yet another clue about what the relevant picture of a context should look like. He often cites VIII.7 of BMMS.

Mike says that Mahowald–Thompson analyzed $L_{K(n)}\Omega S^{2n+1}$ by writing down some clever finite resolution. The resolution that they produce by hand is actually exactly what you would get if you tried to understand the mapping spectral sequence for $E_\infty(E_n^{\Omega S^{2n+1}}, E_n)$.

Mike also says that a consequence of the unpublished Hopkins–Lurie ambidexterity follow-up is that the comparison map $\mathsf{Spaces}(*, Y) \to E_\infty(E_n^Y, E_n^*)$ is an equivalence if Y is a finite Postnikov tower living in the range of degrees visible to Morava E–theory.

The final chapter of Matt's thesis has never really been published, where he investigates power operations on elliptic cohomology theories. That might belong in this chapter as an example of the techinques, since we've already defined elliptic cohomology theories.

Barry's description of the image of

$$E_\infty(A, B) \to \mathsf{Spaces}(\Omega^\infty A, \Omega^\infty B)$$

for $K(1)$–local A and B using p–adic moments is pretty digestable. That might belong in here, or at least it could be referenced. (I guess it didn't ever get published??)

Appendix A
Loose ends

A.1 E_∞ geometry

Example 38 is an inspiration for considering *tmf* as well.

The modularity of the MString orientation

E_∞ orientations by *MString*
   tmf, *TMF*, and *Tmf* in terms of $\mathcal{M}_{\mathbf{ell}}$
   Thom spectra and ∞–categories
   The Bousfield–Kuhn functor and the Rezk logarithm

A.2 Rational phenomena: character theory for Lubin–Tate spectra

There's a sufficient amount of reliance on character theory in Matt's thesis that we should talk about it. You should write that action and then backtrack here to see what you need for it.

   See Morava's *Local fields* paper

Remark 83. Theorem 2.6 of Greenlees–Strickland for a nice transchromatic perspective. See also work of Stapleton and Schlank–Stapleton, of course. Flesh this out.

———

Theorem 81. *Let E be any complex-oriented cohomology theory. Take G to be a finite group and let Ab_G be the full subcategory of the orbit category of G built out of abelian subgroups of G. Finally, let X be a finite G–CW complex. Then, each of the natural maps*

$$E^*(EG \times_G X) \to \lim_{A \in \mathsf{Ab}_G} E^*(EG \times_A X) \to \int_{A \in \mathsf{Ab}_G} E^*(BA \times X^A)$$

becomes an isomorphism after inverting the order of G. In particular, there is an isomorphism

$$\frac{1}{|G|} E^* BG \to \lim_{A \in \mathsf{Ab}_G} \frac{1}{|G|} E^* BA. \square$$

This is an analogue of Artin's theorem:

Theorem 82. *There is an isomorphism*

$$\frac{1}{|G|} R(G) \to \lim_{C \in \mathsf{Cyclic}_G} \frac{1}{|G|} R(C). \square$$

———

HKR intro material connecting Theorem A to character theory:

Recall that classical characters for finite groups are defined in the following situation: take $L = \mathbb{Q}^{\mathrm{ab}}$ to be the smallest characteristic 0 field containing all roots of unity, and for a finite group G let $Cl(G; L)$ be the ring of class functions on G with values in L. The units in the profinite integers $\widehat{\mathbb{Z}}$ act on L as the Galois group over $\mathbb{Q}$, and since $G = \mathsf{Groups}(\widehat{\mathbb{Z}}, G)$ they also act naturally on G. Together, this gives a conjugation action on $Cl(G; L)$: for $\varphi \in \widehat{\mathbb{Z}}$, $g \in G$, and $\chi \in Cl(G; L)$, one sets

$$(\varphi \cdot \chi)(g) = \varphi(\chi(\varphi^{-1}(g))).$$

The character map is a ring homomorphism

$$\chi : R(G) \to Cl(G; L)^{\widehat{\mathbb{Z}}},$$

and this induces isomorphisms

$$\chi : L \otimes R(G) \xrightarrow{\simeq} Cl(G; L)$$

and even

$$\chi : \mathbb{Q} \otimes R(G) \xrightarrow{\simeq} Cl(G; L)^{\widehat{\mathbb{Z}}}.$$

Now take $E = E_\Gamma$ to be a Morava E–theory of finite height $d = \mathrm{ht}(\Gamma)$. Take $E^*(B\mathbb{Z}_p^d)$ to be topologized by $B(\mathbb{Z}/p^j)^d$. A character $\alpha : \mathbb{Z}_p^d \to S^1$

will induce a map $\alpha^* : E^* \mathbb{CP}^\infty \to E^* B\mathbb{Z}_p^d$. We define $L(E^*) = S^{-1}E^*(B\mathbb{Z}_p^d)$, where S is the set of images of a coordinate on $\mathbb{CP}_E^\infty$ under α^* for nonzero characters α. Note that this ring inherits an $\mathrm{Aut}(\mathbb{Z}_p^d)$ action by E^*–algebra maps.

The analogue of $Cl(G; L)$ will be $Cl_{d,p}(G; L(E^*))$, defined to be the ring of functions $\chi : G_{d,p} \to L(E^*)$ stable under G–orbits. Noting that

$$G_{d,p} = \mathrm{Hom}(\mathbb{Z}_p^d, G),$$

one sees that $\mathrm{Aut}(\mathbb{Z}_p^d)$ acts on $G_{d,p}$ and thus on $Cl_{d,p}(G; L(E^*))$ as a ring of E^*–algebra maps: given $\varphi \in \mathrm{Aut}(\mathbb{Z}_p^d)$, $\alpha \in G_{d,p}$, and $\chi \in Cl_{d,p}(G; L(E^*))$ one lets

$$(\varphi \cdot \chi)(\alpha) = \varphi(\chi(\varphi^{-1}(\alpha))).$$

Now we introduce a finite G–CW complex X. Let

$$\mathrm{Fix}_{d,p}(G, X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)} X^{\mathrm{im}\,\alpha}.$$

This space has commuting actions of G and $\mathrm{Aut}(\mathbb{Z}_p^d)$. We set

$$Cl_{d,p}(G, X; L(E^*)) = L(E^*) \otimes_{E^*} E^*(\mathrm{Fix}_{d,p}(G, X))^G,$$

which is again an E^*–algebra acted on by $\mathrm{Aut}(Z_p^d)$. We define the character map "componentwise": a homomorphism $\alpha \in \mathrm{Hom}(\mathbb{Z}_p^d, G)$ induces

$$E^*(EG \times_G X) \to E^*(B\mathbb{Z}_p^d) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}) \to L(E^*) \otimes_{E^*} E^*(X^{\mathrm{im}\,\alpha}).$$

Taking the direct sum over α, this assembles into a map

$$\chi_{d,p}^G : E^*(EG \times_G X) \to Cl_{d,p}(G, X; L(E^*))^{\mathrm{Aut}(Z_p^d)}.$$

Theorem 83. *The invariant ring is $L(E^*)^{\mathrm{Aut}(\mathbb{Z}_p^d)} = p^{-1}E^*$, and $L(E^*)$ is faithfully flat over $p^{-1}E^*$. The character map $\chi_{d,p}^G$ induces isomorphisms*

$$\chi_{d,p}^G : \; L(E^*) \otimes_{E^*} E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*)),$$

$$\chi_{d,p}^G : \; p^{-1}E^*(EG \times_G X) \xrightarrow{\simeq} Cl_{d,p}(G, X; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}.$$

In particular, when $X = *$, there are isomorphisms

Nat taught you how to say all these things with *p*–adic tori, which was *much* clearer.

Cite me: Theorem C.

Checking this invariant ring claim is easiest done by comparing the functors the two things corepresent.

$$\chi_{d,p}^G \colon\ L(E^*) \otimes_{E^*} E^*(BG) \xrightarrow{\cong} Cl_{d,p}(G; L(E^*)),$$

$$\chi_{d,p}^G \colon\ p^{-1} E^*(BG) \xrightarrow{\cong} Cl_{d,p}(G; L(E^*))^{\mathrm{Aut}(\mathbb{Z}_p^d)}. \square$$

———

Jack gives an interpretation of this in terms of formal $\mathcal{O}_L$–modules.

———

I also have this summary of Nat's of the classical case:

It's not easy to decipher if you weren't there for the conversation, but here's my take on it. First, the map we wrote down today was the non-equivariant chern character: it mapped non-equivariant $KU \otimes \mathbb{Q}$ to non-equivariant $H\mathbb{Q}$, periodified. The first line on Nat's board points out that if you use this map on Borel-equivariant cohomology, you get nothing interesting: $K^0(BG)$ is interesting, but $H\mathbb{Q}^*(BG) = H\mathbb{Q}^*(*)$ collapses for finite G. So, you have to do something more impressive than just directly marry these two constructions to get something interesting.

That bottom row is Nat's suggestion of what "more interesting" could mean. (Not really his, of course, but I don't know who did this first. Chern, I suppose.) For an integer n, there's an evaluation map of (forgive me) topological stacks

$$* /\!\!/ (\mathbb{Z}/n) \times \mathrm{Hom}(* /\!\!/ (\mathbb{Z}/n), * /\!\!/ G) \xrightarrow{\mathrm{ev}} * /\!\!/ G$$

which upon applying a global-equivariant theory like K_G gives

$$K_{\mathbb{Z}/n}(*) \otimes K_G \Big(\coprod_{\substack{\text{conjugacy classes of } g \text{ in } G}} * \Big) \xleftarrow{ev^*} K_G(*).$$

Now, apply the genuine G-equivariant Chern character to the K_G factor to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) \leftarrow K_{\mathbb{Z}/n}(*) \otimes K_G(\coprod *),$$

where the coproduct is again taken over conjugacy classes in G. Now, compute $K_{\mathbb{Z}/n}(*) = R(\mathbb{Z}/n) = \mathbb{Z}[x]/(x^n - 1)$, and insert this calculation to get

$$K_{\mathbb{Z}/n}(*) \otimes H\mathbb{Q}_G(\coprod *) = \mathbb{Q}(\zeta_n) \otimes \Big(\bigoplus_{\substack{\text{conjugacy classes}}} \mathbb{Q} \Big),$$

where ζ_n is an nth root of unity. As n grows large, this selects sort of the part of the complex numbers $\mathbb{C}$ that the character theory of finite groups cares about, and so following all the composites we've built a map

$$K_G(*) \to \mathbb{C} \otimes \Big(\bigoplus_{\substack{\text{conjugacy classes}}} \mathbb{C} \Big).$$

The claim, finally, is that this map sends a G-representation (thought of as a point in $K_G(*)$) to its class function decomposition.

———

A.3 The period map

Describe Dieudonné crystals and the Tapis de Cartier.

Show how Dieudonné crystals are used to give formulas for the action of the stabilizer group.

Give a sketch explanation of the Gross–Hopkins period map.

Draw the picture of the period map at $n = 2$, $p = 2$.

State the topological conclusion about $E_\Gamma \mathbb{I}_{\mathbb{Q}/\mathbb{Z}}$.

A.4 Knowns and unknowns

Higher orientations

TAF and friends

The $\alpha_{1/1}$ argument: Prop 2.3.2 of Hovey's v_n–elements of ring spectra

Equivariance

This is tied up with the theory of power operations in a way I've never really thought about. Seems complicated.

You should also mention the "rigidity" of the elliptic genus, which is about an S^1–equivariant version.

Index theorems

Connections with analysis

The Stolz–Teichner program

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Contexts for structured ring spectra

Difficulty in computing $\mathbb{S}_d \subsetQ E_d^*$. (Gross–Hopkins and the period map.)

Barry's p–adic measures

Fixed point spectra and e.g. $L_{K(2)} tmf$.

Blueshift, A–M–S, and the relationship to A–F–G?

Does E_n receive an E_∞ orientation? Does BP?

Remark 12.13 of published H_∞ AHS says their obstruction framework agrees with the E_∞ obstruction framework (if you take everything in sight to have E_∞ structures). This is almost certainly related to the discussion at the end of Matt's thesis about the MU–orientation of E_d.

> Section 12.4 compares doing H_∞ descent with doing E_∞ descent and shows that they're the same (in the case of interest?).

Hovey's paper on v_n–periodic elements in ring spectra. He has a nice (and thorough!) exposition on why one should be interested in bordism spectra and their splittings: for instance, a careful analysis of $MSpin$ will inexorably lead one toward studying KO. It would be nice if studying $MString$ (and potentially higher analogues) would lead one toward non-completed, non-connective versions of EO_n. Talk about BoP, for instance.

Matt's short resolutions of chromatically localized MU.

Appendix B
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Number of to-dos used: 548

Material for lecture

Mike's 1995 announcement is a nice read. There are many snippets you could pull out of it for use here. "$H\mathbb{Q}$ serves as the target for the Todd genus, but actually the Todd genus of a manifold is an integer and it turns out that KU refines the Todd genus." The end of section 3, with $\tau \mapsto 1/\tau$, is mysterious. In section 4, Mike claims that there's a $BU[6, \infty)$–structured splitting principle into sums of things of the form $(1 - \mathcal{L}_1)(1 - \mathcal{L}_2)(1 - \mathcal{L}_3)$. He then says that one expects the characteristic series of a $BU[6, \infty)$–genus to be a series of 3 variables, which is nice intuition. Could mention that Θ^k is a kind of kth difference operator, so that things in the kernel of Θ^k are "k^{th} order polynomials". (More than this, the theorem of the cube is reasonable from this perspective, since Θ^3 kills "quadratic things" and the topological object $H^2(-;\mathbb{Z})$ classifying line bundles is indeed "quadratic".) If the bundle admits a symmetry operation, then the fiber over $(x, y, -x - y)$ is canonically trivialized, so a Σ–structure on a symmetric line bundle is a Θ^3–structure that restricts to the identity on these canonical parts. Mike claims (Theorem 6.2) that if $1/2 \in E^0(*)$ or if E is $K(n)$–local, $n \leq 2$, then $BString^E$ is the parameter space of Σ–structures on the sheaf of functions vanishing at the identity on G_E. The map $MString \to KO_{\text{Tate}}$ actually factors through $MSpin$, so even though this produces the right q–series, you really need to know that $MString$ factors through tmf and $MSpin$ *doesn't* to deduce the modularity for *String*–manifolds. (You can prove modularity separately for $BU[6, infty)$–manifolds, though, by essentially the same technique: refer to the rest of the (complex!) moduli of elliptic curves, which exist as $MU[6, infty)$–spectra.)

Generally: if X is a space, then $X_{H\mathbb{F}_2}$ is a scheme with an $\operatorname{Aut}\widehat{\widehat{\mathbb{G}}}_a$–action. If X is a spectrum (so it fails to have a diagonal map) then $(H\mathbb{F}_2)_* X$ is just an $\mathbb{F}_2$–module, also with an $\operatorname{Aut}\widehat{\widehat{\mathbb{G}}}_a$–action.

The cohomology of a qc sheaf pushed forward from a scheme to a stack along a cover agrees with just the cohomology over the scheme. (In the case

of $* \to *//G$, this probably uses the cospan $* \to *//G \leftarrow *$ with pullback G...)

Akhil Mathew has notes from an algebraic geometry class (https://math.berkeley.edu/~amathew/232b.pd) where lectures 3–5 address the theorem of the cube.

Equivalences of various sorts of cohomologies: Ext in modules and quasi-coherent cohomology (goodness. Hartshorne, I suppose); Ext in comodules and quasicoherent cohomology on stacks (COCTALOS Lemma 12.4); quasicoherent cohomology on simplicial schemes (Stacks project 09VK).

Make clear the distinction between E_n and $\widehat{E(n)}$. Maybe explain the Devinatz–Hopkins remark that $r : \widehat{E(n)} \to E_n$ is an inclusion of fixed points and as such does not classify the versal formal group law.

when describing Quillen's model, he makes a lot of use of Gysin maps and Thom / Euler classes. at this point, maybe you can introduce what a Thom sheaf / Thom class is for a pointed formal curve?

> I think this theorem is motivated by Artin–Mazur formal groups, and the Crystals notes use it to extract a formal group from a Dieudonné module. Some motivation could go here.

Theorem 84. *Let A be a Noetherian ring and G : $\mathsf{AdicAlgebras}_A \to \mathsf{AbelianGroups}$ be a functor such that*

1. *$G(A) = 0$.*
2. *G takes surjective maps to surjective maps.*
3. *There is a finite, free A–module M and a functorial isomorphism*

$$I \otimes_A M \to G(B) \to G(B')$$

whenever I belongs to a square-zero extension of adic A–algebras

$$I \to B \to B'.$$

Then, $G \cong \widehat{\mathbb{A}}^n$ as a functor to sets, where $n = \dim M$.

Proof. This is 9.6.4 in the Crystals notes.

Lemma 97 ([Str99b, Proposition 4.6]). *A functor X: $\mathsf{Algebras} \to \mathsf{Sets}$ is a formal scheme exactly when*

1. *X preserves finite limits.*
2. *There exists a family of maps $X_i \to X$ from a set of S–schemes X_i such that the induced map*

$$\coprod_i X_i(T) \to X(T)$$

is jointly surjective for all test algebras T. $\square$

MUP happens to be the Thom spectrum of $BU \times \mathbb{Z}$.
–Formal groups in algebraic topology
—Day 1
+ Warning: noncontinuous maps of high-dimensional formal affine spaces.

—Day 2

+ Three definitions of complex orientable / oriented cohomology theories. + Some proofs: the splitting principle, Chern roots, diagrammatic Adam's condition, ...

—Day 3

+ Lemma and proof: homomorphisms $F \to G$ of $\mathbb{F}_p$-FGLs factor as $F \to G' \to G$, where $G' \to G$ is a Frobenius isogeny and $F \to G'$ is invertible. + Definition of height. Examples: $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$. + Logarithms for FGLs over torsion-free rings. The integral equation. Height as radius.

—Day 4

+ Plausibility argument for square-zero deformations being classified by "$\mathrm{Ext}^1(\widehat{\mathbb{G}}; M \otimes \widehat{\mathbb{G}}_a)$"

—Day 5

+ HondaâĂŹs theorem about ζ–functions as manufacturing integral genera. + Definition of forms of a module, map to Galois cohomology + Computation of the Galois cohomology for: $H\mathbb{F}_p$, MU/p, KU/p + Computation of the Galois cohomology for $\widehat{\mathbb{G}}_m$, explicit description of the invariant via the ζ–function + MoravaâĂŹs sheaf over $L_1(\mathbb{Z}_p^{nr})$, Gamma-equivariance and transitivity, ConnerâĂŤFloyd + Identification of L_1/Γ with $\mathcal{M}_{\mathbf{fg}}^{\leq 1}$, connection to LEFT.

—Future topics

+ The main theorem of class field theory + Lubin and Tate's construction of abelian extensions of local number fields + Description of the Lubin–Tate tower and the local Langlands correspondence + Uniqueness of $\mathcal{O}_K$-module structure in characteristic zero + Construction of the spectra MU, BP, $E(n)$, $K(n)$, E_n + Goerss–Hopkins–Miller and Devinatz–Hopkins + Gross–Hopkins period map and the calculation of the Verdier dualizing sheaf on LT_n + Kohlhaase's Iwasawa theory + Lubin's dynamical results on formal power series + Classification of field spectra

Ideas

1. Ando, Hopkins, Strickland on H_∞–orientations and the norm condition
2. The rigid, real σ–orientation: AHR. Its effect in homology.
3. The Rezk logarithm and the Bousfield–Kuhn functor
4. Statement of Lurie's characterization of *TMF*, using this to determine a map from *MString* by AHR
5. Dylan's paper on String orientations
6. Matt's calculation of E_∞–orientations of $K(1)$–local spectra using the short free resolution of *MU* in the $K(1)$–local category
   ————————————
7. *Forms of K–theory*, Elliptic spectra, Tate *K*–theory, *TMF*
8. What are Weil pairings for geometers?
9. The Atiyah–Bott–Shapiro orientation (Is there a complex version of this? I understand it as a splitting of *MSpin*...)
10. Sinkinson's calculation and $MBP\langle m\rangle$–orientations
11. Hovey–Ravenel on nonorientations of E_n by $MO[k, \infty)$. Other things in H–R?
12. The Serre–Tate theorem
13. The fundamental domain of π_{GH}
14. Orientations and the functor gl_1.

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Resources

Ando, Hopkins, Rezk
    Barry Walker's thesis
    Morava's *Forms of K–theory*
    Kitchloo, Laures, Wilson
    Akhil wrote a couple of blog posts about Ochanine's theorem: `https://amathew.wordpress.com/201`
and `https://amathew.wordpress.com/2012/05/31/the-other-direction-of-ochaines-theo`
Mentioning a more precise result might lend to a more beefy introduction.

What follows are notes from other talks I've given about quasi-relevant material which can probably be cannibalized for this class.