ECE58000 FunWork1

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Solution to problem 1
Given a 4x5 matrix A as below.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & -1 & 3 & 0 & 1 \\ -3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$$

We can perform row reduction using Gauss elimination to obtain its row echelon form & find out the number of non-zero rows i.e. number of linearly independent row vectors to find its rank.

$$\xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ -3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 3R1} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 7 & -8 & 12 & 9 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - R1} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 7 & -8 & 12 & 9 \\ 0 & 0 & 4 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + 7/5 * R2} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 0 & -1 & 18/5 & 27/5 \\ 0 & 0 & 4 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 + 4 * R3} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 0 & -1 & 18/5 & 27/5 \\ 0 & 0 & 0 & 2/5 & 11/5 \end{bmatrix}$$

The number of non zero rows are 4 so Rank(A) here is 4.
(Answer)

Solution to problem 2
Given,

$$A = \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 2 & -1 & \gamma & -5 \\ 1 & 10 & -6 & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 0 & -1-2\gamma & \gamma+2 & -9 \\ 1 & 10 & -6 & 1 \end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 0 & -1-2\gamma & \gamma+2 & -9 \\ 0 & 10-\gamma & -5 & -1 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 - ((10-\gamma)/(-1-2\gamma))R2} \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 0 & -1-2\gamma & \gamma+2 & -9 \\ 0 & 0 & \frac{\gamma^2+2\gamma-15}{-2\gamma-1} & \frac{7(-\gamma+13}{-2\gamma-1} \end{bmatrix}$$

This is the final row echelon form. Clearly for all $\gamma \in R$, the number of non zero row vectors should be 3 since the last row cant be 0. So for all real values of γ the Rank is 3.
(Answer)

Solution to problem 3
Theorem 2.1 is for system of equations Ax = b, has a solution iff rank(A)= rank([A, b]). So we need to evaluate rank of A and rank([A,b]). Here
A = $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix}$ Performing row reduction we obtain,

$$\xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & -2 \end{bmatrix} \xrightarrow{R2 \to R2 = -\frac{1}{3}R2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Clearly two independent row vectors so rank is 2. Adding another column will not change the rank, so the Rank[A, b] or the augmented matrix has same rank as that of A. So based on Theorem 2.1, the system of linear equations has a solution.
Now for the next part of the problem. For the given $m \times n$ matrix A with $n > m$, we proved earlier that the system of equations has a solution from Theorem 2.1, so, the method of Theorem 2.2 is to assign arbitrary values for $n - m$ and solve for the remaining ones.
Let's from eqn (2) $x_1 = \alpha$ and $x_2 = \beta$, so we can get

$$x_4 = -(-2 - \alpha + 2\beta) = 2 + \alpha - 2\beta.$$

Substituting that in eqn (1)

$$x_3 = \frac{1 - \alpha - \beta - (2 + \alpha - 2\beta)}{2} = \frac{-1 - 2\alpha + \beta}{2}$$

note if we use particular values say set, $\alpha = \beta = 0$ so $\mathrm{x}_1 = 0, x_2 = 0$, then we get$x_3 = -\frac{1}{2}$ and $x_4 = 2$
(Answer)

Solution to problem4
Given,
A = $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix}$. To find the nullspace of a matrix, we need to solve the homogeneous system of linear equations given by Ax=0.
Setting up the augmented $\tilde{A} = \begin{bmatrix} 4 & -2 & 0 & | & 0 \\ -2 & 1 & -1 & | & 0 \\ 2 & -3 & 1 & | & 0 \end{bmatrix}$.
The RREF (Command in MATLAB rref(A)) for this augmented matrix gives,
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
To find the null space, solve the matrix equation, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Convert the matrix equation back to an equivalent system:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

This system has only the trivial solution and Thus, a basis for the null space is: $\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \}$.
(Answer)

Solution to problem 5a
Problem a) For the given function

$$f(x_1, x_2) = x_1 e^{-x_2} + x_2 + 1$$

with $x_0 = [1, 0]^T$
General formula for the Taylor series neglecting higher order terms H.O.T:

$$\begin{aligned} f(x_1, x_2) &= f(x_0) + \frac{\partial f(x_0)}{\partial x_1}(x_1 - x_{0,1}) + \frac{\partial f(x_0)}{\partial x_2}(x_2 - x_{0,2}) + \\ \frac{1}{2!}&\left(\frac{\partial^2 f(x_0)}{\partial x_1^2}(x_1 - x_{0,1})^2 + 2\frac{\partial^2 f(x_0)}{\partial x_1 \partial x_2}(x_1 - x_{0,1})(x_2 - x_{0,2}) + \frac{\partial^2 f(x_0)}{\partial x_2^2}(x_2 - x_{0,2})^2 \right) + H.O.T \end{aligned} \tag{1}$$

where $x_{0,1}$ and $x_{0,2}$ are the coordinates of the expansion point x_0.
Let's calculate the derivatives:

$$f(1,0) = 1 + 1 = 2$$

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = e^{-x_2}\Big|_{[1,0]^T} = 1$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = -x_1 e^{-x_2} + 1\Big|_{[1,0]^T} = -1 + 1 = 0$$

$$\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2)\Big|_{[1,0]^T} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2)\Big|_{[1,0]^T} = -e^{-0} = -1$$

$$\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2)\Big|_{[1,0]^T} = 1$$

Now, substituting these into the Taylor series formula:

$$f(x_1, x_2) \approx 1 + x_1 + (1 - x_1)x_2 + \frac{1}{2}(x_1)^2 + h.o.t$$

Matlab code to verify expansion

```
syms x y
f = x*exp(-y) + y + 1;
T = taylor(f,[x,y],'Order',3,'ExpansionPoint',[1,0])
```

```
Result of matlab run
T =
y^2/2 + (1 - x)*y + x + 1
```

Hence proved & Answer

Solution to problem 5b
Problem b) For the given function

$$f(x_1, x_2) = x_1^4 + 2x_1^2x_2^2 + x_2^4$$

where $x_0 = [1, 1]^T$
General formula for multivariate Taylor series expansion upto second order only:

$$f(x_1, x_2) \approx f(x_0) + \nabla f(x_0) \cdot \mathbf{h} + \frac{1}{2!}\mathbf{h}^T H_f(x_0)\mathbf{h}$$

Here, $f(x_0)$ is the value of the function at the point x_0, $\nabla f(x_0)$ is the gradient of the function at x_0, $H_f(x_0)$ is the Hessian matrix of the function at x_0, $\mathbf{h}$ is the vector $[x_1 - 1, x_2 - 1]^T$.
Value of the $f(x_1, x_2)$ at x_0:

$$f(1, 1) = 1^4 + 2 \cdot 1^2 \cdot 1^2 + 1^4 = 4$$

Gradient of the function at x_0

$$\nabla f(x_0) = [4x_1^3 + 4x_1x_2^2, 4x_2^3 + 4x_1^2x_2]\Big|_{[1,1]^T}$$

$= [8, 8]$, (at $x_0 = [1, 1]^T$)
Hessian matrix at x_0:

$$H_f(x_0) = \begin{bmatrix} 12x_1^2 + 4x_2^2 & 8x_1x_2 \\ 8x_1x_2 & 12x_2^2 + 4x_1^2 \end{bmatrix}\Big|_{[1,1]^T}$$

$= \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}$, (at $x_0 = [1, 1]^T$)
Now, we substitute these into the Taylor series expansion formula:

$$f(x_1, x_2) \approx 4 + [8, 8] \cdot \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix}\begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

or simplify more as

$$f(x_1, x_2) \approx -12 + 8x_1 + 8x_2 + 8(x_1 - 1)^2 + 8(x_1 - 1)(x_2 - 1) + 8(x_2 - 1)^2$$

(Answer)
Matlab code to verify expansion

```
syms x y
f = x^4 + 2*x^2*y^2 + y^4;
T = taylor(f,[x,y],'Order',3,'ExpansionPoint',[1,1])

 Result of matlab run

T =
8*x + 8*y + 8*(x - 1)*(y - 1) + 8*(x - 1)^2 + 8*(y - 1)^2 - 12
```

Hence proved & Answer

Solution to problem 5c
Problem c) Given function $f(x_1, x_2) = e^{x_1 - x_2} + e^{x_1 + x_2} + x_1 + x_2 + 1$ around the point $x_0 = [1, 0]^T$, we'll use the general formula for a Taylor series expansion up to

$$f(x_1, x_2) \approx f(x_0) + \nabla f(x_0) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^T H_f(x_0)\mathbf{h}$$

as previous problem $f(x_0)$ at the point x_0, $\nabla f(x_0)$ gradient at x_0, $H_f(x_0)$ Hessian matrix at x_0, $\mathbf{h}$ vector $[x_1 - 1, x_2 - 0]^T$.
Value of the Function at x_0:

$$f(1, 0) = e^{(1+0)} + e^{(1-0)} + 1 + 0 + 1 = 2e + 2$$

Gradient at x_0:

$$\nabla f(x_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$x_0 = [1, 0]^T$:

$$\frac{\partial f}{\partial x_1} = e^{(1-0)} + e^{(1+0)} + 1 = 2e + 1$$

$$\frac{\partial f}{\partial x_2} = -e^{(1-0)} + e^{(1+0)} + 1 = 1$$

$\nabla f(x_0) = \begin{bmatrix} 2e + 1 \\ 1 \end{bmatrix}$.
Hessian matrix at x_0

$$H_f(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

values at $x_0 = [1, 0]^T$:

$$\frac{\partial^2 f}{\partial x_1^2} = e^{1-0} + e^{1+0} = 2e$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -e^{1-0} + e^{1+0} = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = -e^{1-0} + e^{1+0} = 0$$

So, $H_f(x_0) = \begin{bmatrix} 2e & 0 \\ 0 & 2e \end{bmatrix}$.
Now, we substitute these values into the Taylor series expansion formula:

$$f(x_1, x_2) \approx (2e+2) + \begin{bmatrix} 2e + 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 - 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 - 0 \end{bmatrix} \begin{bmatrix} 2e & 0 \\ 0 & 2e \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 0 \end{bmatrix}$$

or simplifying

$$f(x_1, x_2) \approx (2e + 2) + (2e + 1)(x_1 - 1) + x_2 + e(x_1 - 1)^2 + ex_2^2$$

Answer
Matlab code to verify expansion

```
syms x1 x2
f = exp(x1-x2)+exp(x1+x2)+x1+x2+1;
T = taylor(f,[x1,x2],'Order',3,'ExpansionPoint',[1,0])
\textbf{Result of matlab run
}T =
x2 + 2*exp(1) + (2*exp(1) + 1)*(x1 - 1) + x2^2*exp(1) + exp(1)*(x1 - 1)^2 + 2
```

Answer

Solution to problem 6
Provided

$$f(x_1, x_2) = x_1^3 + x_1x_2 - x_1^2x_2^2$$

to be approximated at $x_0 = [1, 1]^T$
Linear approximation $l(x_1, x_2)$ of $f(x_1, x_2)$ at a point $x_0 = [a, b]^T$ can be derived from below

$$l(x_1, x_2) = f(a, b) + f'_{x1}(a, b) \cdot (x_1 - a) + f'_{x2}(a, b) \cdot (x_2 - b)$$

where f'_{x1} and f'_{x2} partial derivatives of f w.r.t x_1 and x_2, at $x_0 = [a, b]^T$.

$$\begin{aligned} f'_{x1} &= \frac{\partial f}{\partial x_1} = (3x_1^2 + x_2 - 2x_1x_2^2)\Big|_{[1,1]^T} \\ &= 3(1)^2 + 1 - 2(1)(1)^2 = 2 \\ f'_{x2} &= \frac{\partial f}{\partial x_2} = x_1 - 2x_1^2x_2\Big|_{[1,1]^T} \\ &= 1 - 2(1)^2(1) = -1 \end{aligned}$$

The linear approximation $l(x_1, x_2)$ at $x_0 = [1, 1]^T$ is:

$$l(x_1, x_2) = f(1, 1) + f'_{x1}(1, 1) \cdot (x_1 - 1) + f'_{x2}(1, 1) \cdot (x_2 - 1)$$

Substitute $f(1, 1) == 1$ & other values we get,

$$l(x_1, x_2) = 1 + 2 \cdot (x_1 - 1) - 1 \cdot (x_2 - 1)$$

or,

$$l(x_1, x_2) = 2x_1 - x_2$$

So, the linear approximation is $l(x_1, x_2) = 2x_1 - x_2$ at the point $x_0 = [1, 1]^T$.
(Answer)

Quadratic approximation $q(x_1, x_2)$ at the point $x_0 = [1,1]^T$ can be expressed as below:

$$q(x_1,x_2) = f(x_0) + f'_{x}1(x_0)\cdot(x_1-1) + f'_{x}2(x_0)\cdot(x_2-1)+ \\ \frac{1}{2}\left[f''_{x_1x_1}(x_0)\cdot(x_1-1)^2 + 2f''_{x_1x_2}(x_0)\cdot(x_1-1)(x_2-1) + f"_{x_2x_2}(x_0)\cdot(x_2-1)^2\right] \tag{2}$$

First order terms we have already evaluated above. Second-order partial derivatives at $x_0 = [1,1]^T$:

$$f''_{x_1x_1}(1,1) = 6x_1 - 2(x_2)^2\Big|_{[1,1]^T} = 4$$

$$f''_{x_1x_2}(1,1) = 1 - 4(x_1)(x_2)\Big|_{[1,1]^T} = -3$$

$$f''_{x_2x_2}(1,1) = -2(x_1)^2\Big|_{[1,1]^T} = -2$$

So the quad approx is below, (some terms already obtained above)

$$q(x_1,x_2) = 2x_1 - x_2 + \frac{1}{2}[4(x_1-1)^2 - 2\cdot 3(x_1-1)(x_2-1) - 2(x_2-1)^2]$$

Simplify and we get final solution as,

$$q(x_1,x_2) = x_1 - 3x_1x_2 + 2x_1^2 - x_2^2 + 4x_2 - 2$$

(Answer)

Solution to problem 7.1

Part1 Calculating gradient

$$f(x) = x^\top \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} x - x^\top \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \pi$$

Let $\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ is matrix A and $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is vector b

Breaking it down, first term Derivative of the quadratic term: Now,

$\mathrm{A}^\top + A = \begin{bmatrix} 2 & 7 \\ 7 & 6 \end{bmatrix}$

$$\frac{\partial}{\partial x}(x^\top Ax) = (A + A^\top)x = \begin{bmatrix} 2x_1 + 7x_2 \\ 7x_1 + 6x_2 \end{bmatrix}$$

Next, second term, Derivative of the linear term:

$$\frac{\partial}{\partial x}(x^\top \begin{bmatrix} -2 \\ 3 \end{bmatrix}) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Final solution $\begin{bmatrix} 2x_1 + 7x_2 \\ 7x_1 + 6x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 2x_1 + 7x_2 + 2 \\ 7x_1 + 6x_2 - 3 \end{bmatrix}$ **(Answer)**

Solution to problem 7.2. Find Hessian

$$f(x) = \frac{1}{2}x^\top \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix} x + x^\top \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \log 3$$

Let $\begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$ is matrix A and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is vector b.

Taking Hessian of first term $\frac{1}{2}x^\top Ax$ we get,

$$\frac{1}{2}(A + A^T)$$

i.e. $\frac{1}{2}\left(\begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 4 & 10 \\ 10 & 2 \end{bmatrix}\right)$

Hessian of second term H $\left(x^\top \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right) = 0$

Final solution is $\begin{bmatrix} 2 & 5 \\ 5 & 1 \end{bmatrix}$ **Answer**

Solution to problem 8.1

Given $f(x_1, x_2) = e^{x_1 x_2^2}$ at the point $x = [1, 1]^\top$, partial derivatives:

$$\frac{\partial f}{\partial x_1} = x_2^2 \cdot e^{x_1 x_2^2}\Bigg|_{x=[1,1]^\top} = 1^2 \cdot e^{1 \cdot 1^2} = e$$

$$\frac{\partial f}{\partial x_2} = 2x_1 x_2 \cdot e^{x_1 x_2^2}\Bigg|_{x=[1,1]^\top} = 2 \cdot 1 \cdot 1 \cdot e^{1 \cdot 1^2} = 2e$$

Therefore, the gradient of f at $x = [1, 1]^\top$ is given by the column vector:

$\nabla f\Bigg|_{x=[1,1]^\top} = \begin{bmatrix} e \\ 2e \end{bmatrix}$ **Answer**

Solution to problem 8.2
To find rate of increase we need directional derivative of a function $f(x_1, x_2)$ at a point $x = [1, 1]^\top$ in the direction $d = [-3, 4]^\top$, which is given by the dot product of the gradient of f at that point and the direction vector

$$\nabla f \cdot d = \langle e, 2e\rangle \ \cdot \langle -3/5, 4/5\rangle$$

here ($\|d\| = \sqrt{(-3)^2 + (4)^2} = 5$).

$$\nabla f \cdot d = \langle -3e/5 + 8e/5\rangle \ = e$$

(**Answer**)

Solution to problem 8.3
The direction of maximum rate of increase of f($x_1, x_2) at x = [1, 1]^\top$ should be along unit vectors is the gradient of f($x_1, x_2) at x = [1, 1]^\top$

$$\nabla f(x_1, x_2)\Bigg|_{x=[1,1]^\top} = \begin{bmatrix} e \\ 2e \end{bmatrix},$$

the unit vector along the direction of max increase is $[\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$ & the rate of increase is calculated as

$$\nabla f \cdot d = \langle e, 2e\rangle \ \cdot \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle = \sqrt{5}e$$

(Answer)

Solution to problem 9
Given,

$$f(x_1, x_2, x_3) = 2x_1x_3 - x_1^2 - x_2^2 - 5x_3^2 - 2\xi x_1x_2 - 4x_2x_3$$

to find range of ξ for negative semi definite
Converting the quadratic equation as below where x is column vector,

$$f = x^\top \begin{bmatrix} -1 & -\xi & 1 \\ -\xi & -1 & -2 \\ 1 & -2 & -5 \end{bmatrix} x$$

As per lecture notes, a symmetrized quadratic form is negative-semidefinite if and only if its negative is positive-semidefinite, a symmetric matrix is positive semidefinite if and only if all its principal minors, not just the leading principal minors, are non-negative so we need to determine them.

$$-f = x^\top \begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} x = x^\top A x$$

by inspection we see first order principle minors (i.e. diagonal elements of A) are all positive. Also three second-order principal minors starting with below,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \det \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} = 1 - \xi^2$$

, is non-negative if and only if $\xi \in [-1, 1]$. Other minors,

$$\det A \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \quad \text{and} \quad \det A \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

are positive. Next, third order minor of A

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - \xi \det \begin{bmatrix} \xi & 2 \\ -1 & 5 \end{bmatrix} - \det \begin{bmatrix} \xi & 1 \\ -1 & 2 \end{bmatrix} \\ &= 1 - \xi(5\xi + 2) - (2\xi + 1) \\ &= 1 - 5\xi^2 - 2\xi - 2\xi - 1 \\ &= -5\xi^2 - 4\xi \end{aligned}$$

This principal minor is non-negative if and only if

$$-\xi(5\xi + 4) \geq 0$$

i.e.

$$\xi \in \left[-\frac{4}{5}, 0\right]$$

Combining the above, we conclude that the function is negative-semidefinite if and only if,

$$\xi \in \left[-\frac{4}{5}, 0\right]$$

(Answer)

Solution to problem 10.1
Objective is to find points that satisfy the first-order necessary condition (FONC) for the extremum; Given function,

$$f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 5$$

FONC condition, $\nabla F(x_1, x_2) = 0$ i.e. $\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{vmatrix} = 0$
Partial derivatives,

$$\frac{\partial f}{\partial x_1} = x_1^2 + x_1 + 2x_2$$

$$\frac{\partial f}{\partial x_2} = 2x_1 + x_2 - 1$$

Next, we set partial derivative equal to zero and solve the resulting system of equations:

$$x_1^2 + x_1 + 2x_2 = 0$$

$$2x_1 + x_2 - 1 = 0$$

or

$$x_2 = 1 - 2x_1$$

Substitute x_2 into the first equation

$$x_1^2 + x_1 + 2(1 - 2x_1) = 0$$

or,

$$x_1^2 - 3x_1 + 2 = 0$$

$$(x_1 - 1)(x_1 - 2) = 0$$

$x_1 = 1$ or $x_1 = 2$ For $x_1 = 1$:

$$x_2 = 1 - 2(1) = -1$$

& For $x_1 = 2$:

$$x_2 = 1 - 2(2) = -3$$

solutions: $x_1 = 1, x_2 = -1$, $x_1 = 2, x_2 = -3$ So, the critical points are (1, -1) and (2, -3) i.e. points that satisfy FONC for an extremum. **Answer**

Solution to problem 10.2
which point is a strict local minimizer? Justify your answer.
To determine whether a critical point is a strict local minimizer, we need to apply the Second-Order Necessary Condition (SONC). The SONC involves evaluating the Hessian of the function at the critical points and examining their signs.
The Hessian matrix H is given by:

$$H(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

For the given function $f(x_1, x_2)$:

$$\frac{\partial^2 f}{\partial x_1^2} = 2x_1 + 1$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 1$$

The Hessian matrix at a critical point $(1, -1)$ and (2, -3)is:

$$H_{(1,-1)} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

and

$$H_{(2,-3)} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

[To check if x* is strict local minimizer, we need to check, the Hessian at x* if its, positive definite.]
For the point (1,-1), Hessian $H_{(1,-1)}$ is indefinite since $\begin{vmatrix} 3 \end{vmatrix} > 0$ and the determinant $\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$ = -1 $<$ 0 so (1,-1) is not strict local minimizer.
For the point (2,-3), Hessian $H_{(2,-3)} \succ 0$ since $\begin{vmatrix} 5 \end{vmatrix} > 0$ and the determinant $\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1 > 0$ so (2,-3) is the strict local minimizer.
(Answer)