1

(a) $A \cup B \cup C$, At least one of A, B, C occurs,

(b) $[(A \cap B) \cup (B \cap C) \cup (A \cap C)]^c$, At most one of the events occurs

(c) $A \cap B \cap C$, A, B, C all occur

(d) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$, exactly one of A, B, C occur

2

(a) $A \cup B = A \cup (B \cap A^c)$

Since A & $(B \cap A^c)$ are disjoint, we can apply axiom of probability # 3.

$$P(A \cup B) = P(A) + P(B \cap A^c) \quad \text{—} \; ①$$

Next, $B = (B \cap A) \cup (B \cap A^c)$

now again applying probability axiom on disjoint parts,

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

$$\therefore \quad P(B \cap A) = P(B) - P(B \cap A^c)$$

$$P(B \cap A^c) = P(B) - P(A \cap B) \quad \text{—} \; ②$$

② in ①,

$$P(A \cup B) = P(A) + [P(B) - P(A \cap B)]$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{—} \; ③$$

Q.E.D

2 (b) To show $P(A) = 1$, $P(B) = 1$ $\therefore$ $P(A \cap B) = 1$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$ from eqn ③ in 2(a)

$\therefore$ by substitution we get,

$$P(A \cap B) = 1 + 1 - P(A \cup B)$$

$$= 2 - P(A \cup B)$$

Since $P(A \cup B) = 1$

$$\therefore \quad P(A \cap B) = 2 - 1 = 1 \quad \text{[Answer]}$$

2.
(c) Exactly ~~A~~ one of A or B $\Rightarrow (A \cap B^c) \underline{\underline{\cup}} (B \cap A^c)$

$\therefore \; P(\text{exactly one of A or B}) = P(A \cap B^c) + P(B \cap A^c)$ since $A \cap B^c$ & $B \cap A^c$ are disjoints

Also,

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$P(B \cap A^c) = P(B) - P(A \cap B)$$

Since $A = (A \cap B) \cup (A \cap B^c)$. & union is disjoint

& $B = (B \cap A) \cup (B \cap A^c)$ & " " " " from 2(a)

adding them,

$$\therefore \; P(A \cap B^c) + P(B \cap A^c)$$

$$= \big(P(A) - P(A \cap B)\big) + \big(P(B) - P(A \cap B)\big)$$

$$= P(A) + P(B) - 2P(A \cap B) \qquad \text{QED.}$$

2.
(d) if $B = (B \cap A) \cup (B \cap A^c)$ shown earlier.

Given $A \subset B \quad \therefore A \cap B = A$

because every element of A is also an element of B.

$$\therefore \; A \subset B \Rightarrow A \cap B = A$$

$\therefore \quad B = A \cup (A^c \cap B)$

$\therefore \quad P(B) = P(A) + P(A^c \cap B)$ and since $P(A^c \cap B) \geq 0$

$\Rightarrow \quad P(B) \geq P(A)$ QED.

3. To prove

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

We know $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$.

$\therefore \quad P(A \cup B \cup C) = P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))]$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B) + P(A \cap B \cap C)$

Q.ED

5. To prove

$$P(A \cap B \cap C) = P(A \mid B \cap C)\, P(B \mid C)\, P(C)$$

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) > 0$ from conditional prob def^n.

$\therefore \quad P(A \mid B)\, P(B) = P(A \cap B)$

Proof $P(A \cap (B \cap C))$

$= P(A | B \cap C) \, P(B \cup C)$

$= P(A | B \cap C) \, P(B|C) \, P(C)$

Q.E.D

N.B

$P(B|C) = \dfrac{P(B \cap C)}{P(C)}$

$\therefore P(B|C) . P(C)$

$= P(B \cap C)$

6.

(a) To show if A & B are independent, then A^c & B are also independent

Here we need to show $P(A^c \cup B) = P(A^c) . P(B)$

expressing B as disjoint union.

$B = (A \cap B) \cup (A^c \cap B)$

$\therefore P(B) = P(A \cap B) + P(A^c \cap B)$

since A & B are independent, $P(A \cap B) = P(A) . P(B)$

$\therefore P(B) = P(A) P(B) + P(A^c \cap B)$

$\therefore P(A^c \cap B) = P(B) - P(A) . P(B)$

$= P(B) [1 - P(A)]$

$= P(B) . P(A^c)$

hence A^c & B are independent

Q.E.D.

$\therefore$

6. (b) Mutually exclusive $\Rightarrow$ $A \cap B = \phi$ ie. disjoint $\therefore P(A \cap B) = 0$

& independent $\Rightarrow$ $P(A \cap B) = P(A) . P(B)$ — (1)

if A & B are both independent & mutually exclusive, then they must satisfy $P(A) . P(B) = P(A \cap B) = 0$

which requires $P(A) = 0$ or $P(B) = 0$ or both are zero.

So, independent doesn't generally mutually exclusive unless one of the events have zero probability.

if $P(A) > 0$ & $P(B) > 0$, independence & mutually exclusive cannot happen at the same time.

6 (c) to show,
if $P(A|B) > P(A)$ then $P(B|A) > P(B)$.

From definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} > P(A)$$

multiply both sides by $P(B) > 0$

$$P(A \cap B) > P(A)\, P(B) \quad \text{---} \; (1)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

divide each side by $P(A) > 0$ on eqn (1)

$$\frac{P(A \cap B)}{P(A)} > P(B)$$

$$\therefore \; P(B|A) > P(B)$$

$$\therefore \; P(A|B) > P(A) \Rightarrow P(B|A) > P(B) \quad \text{Q.E.D.}$$

□ □

(7) We need to show

$$P(A \cap B) = P(A)\, P(B)$$

Using total prob formula

$$P(B) = P(B|A) . P(A) + P(B|A^c)\, P(A^c)$$

or $P(B) = \frac{1}{2} P(B|A) + \frac{1}{2} P(B|A^c)$ since $P(A) = \frac{1}{2}$ & $P(A^c) = \frac{1}{2}$ —— (2)

$$P(B|A) = \cancel{P(B) P(A)} \frac{P(B \cap A)}{P(A)} \quad \text{—— (3)}$$

$$\& \; P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{—— (4)}$$

(2) $\times P(A)$ both sides

$$P(A).P(B) = \frac{1}{2} P(B|A).P(A) + \frac{1}{2} P(B|A^c) P(A)$$

$$= \frac{1}{2}\left[P(B \cap A)\right] + \frac{1}{2}\left[P(B|A^c).P(A)\right]$$

iff $P(B|A^c) = P(B|A)$

then RHS $\Rightarrow \frac{1}{2}\left[P(B \cap A)\right] + \frac{1}{2}\left[P(B|A) P(A)\right]$

$$= \frac{1}{2}\left[P(B \cap A) + P(B \cap A)\right]$$

LHS $\Rightarrow P(A).P(B) = P(B \cap A)$

$\therefore$ only condition A & B are independent iff

$$P(B|A^c) = P(B|A)$$

ie. black & white balls are in same ratios.

Ans

S1 S2

$1 \leftrightarrow 5$ $\quad 1 \leftrightarrow 10$

$$P(T_s \leqslant t \mid \text{Server } 1) = \begin{cases} 0 & t \leqslant 1 \\ \frac{t-1}{4} & 1 < t \leqslant 5 \\ 1 & t > 5 \end{cases}$$

$$P(T_s \leqslant t \mid \text{Server } 2) = \begin{cases} 0 & t \leq 1 \\ \frac{t-1}{9} & 1 < t \leq 10 \\ 1 & t > 10 \end{cases}$$

$P(\{\text{Server } 1\}) = 0.4$ & $P(\{\text{Server } 2\}) = 0.6$

8 (a) To find $P(\{T_s \leqslant 5\})$

Total probability $P(T_s \leqslant 5) = P(T_s \leqslant 5 \mid S1)\, P(S1) + P(T_s \leqslant 5 \mid S2)\, P(S2)$

$$= \left(\frac{t-1}{4}\right)(0.4) + \left(\frac{t-1}{9}\right)(0.6)$$

$$= \left(\frac{5-1}{4}\right) 0.4 + \left(\frac{5-1}{9}\right)(0.6)$$

$$= 0.4 + \frac{4}{9} \cdot 0.6 = \frac{4}{10} + \frac{24}{90} = \frac{36+24}{90}$$

$$= \frac{60}{90} = \frac{2}{3}$$

8 (b) $P(\{S_1\} \mid \{T_s \leqslant 5\}) = \dfrac{P(T_s \leqslant 5 \mid S_1)\, P(S_1)}{P(T_s \leqslant 5)}$. Bay's theorem

$$= \frac{1 \cdot 0.4}{(2/3)} = \frac{3}{2} \cdot \frac{\cancel{4}^{2}}{\cancel{10}5} = \frac{3}{5}$$

8 (c). To find $a > 1$ s.t $P(\{S_1\} \mid \{T_s \leqslant a\}) = P(\{S_2\} \mid \{T_s \leqslant a\})$

$$P(S_1 \mid T_s \leq a) = \left(\frac{a-1}{4} \cdot 0.4\right) \Big/ P(T_s \leq a) = \frac{a-1}{10} \cdot \frac{1}{P(T_s \leq a)}$$

$$P(S_2 \mid T_S \leq a) = \left(\frac{a-1}{9} \cdot 0.6\right) \Big/ P(T_s \leq a) = \frac{a-1}{15} \cdot \frac{1}{P(T_S \leq a)}$$

$\therefore \quad \frac{a-1}{10} = \frac{a-1}{15}$ no solution

$a > 5 \qquad P(S_1 \mid T_S \le a) = P(S_2 \mid T_S \le a)$

$$1 . 0.4 = \frac{a-1}{9} \, 0.6$$

$$\frac{4}{10} = \frac{a-1}{9} \, \frac{6}{10}$$

$$\frac{36}{6} = a-1 \qquad \therefore \; 7 = a \quad \text{Answer}$$

(4) if $P(A_i) = 1 \; \forall \; i \geq 1$ then to show, $P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$

From De Morgan's law, $\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$

we can show prob of complement is zero. ie $P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 0$

Since $P(A_i^c) = 1 - P(A_i) = 1 - 1 = 0$

Also $\quad 0 \leq P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \leq \sum_{i=1}^{\infty} P(A_i^c)$

Since $P(A_i^c) = 0 \rightarrow \sum_{i=1}^{\infty} 0 = 0$

$\therefore \quad 0 \leq P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \leq 0 \implies P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 0$

$$\therefore \quad P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right)$$

$$= 1 - 0 = 1$$

$$\therefore \quad P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1. \qquad \Box$$

⑨ C → initial choice of room have car event

W_s → event of winning by the switch.

E → event that initial choice is empty

W_{ns} → event of wining by not switching

Given $P(C) = \frac{1}{3}$ equally probable

$P(E) = \frac{2}{3}$

Using TPL we get

$$P(W_s) = P(W_s/C)P(C) + P(W_s/E)P(E)$$

we can consider two conditions. One where contestant switches the door and one where he doesn't

if switches ⇒ ~~P(Ws)~~ A

if first choice was correct event C, a switch will move away from winning car ∴ $P(W_s/C) = 0$

if first choice is incorrect, (event E) empty door is chosen and host reveals another empty door & unopened door will have car

$$P(W_s/E) = 1$$

$$P(W_s) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

if do not switch,

if first choice is correct & keeps car

$$P(W_{ns}/C) = 1.$$

if first choice is incorrect, → no car winnings

so $P(W_{ns}/E) = 0$

$\therefore$ $P(W_{ns}) = 1 \cdot \frac{1}{3} + 0 \cdot \left(\frac{2}{3}\right) = \frac{1}{3}$

$\therefore$ $P(\text{switch}) = \frac{2}{3}$ & $P(\text{not switch}) = \frac{1}{3}$

$\therefore$ switching door doubles probability.

—x—