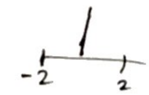
1. $X(\omega) = \begin{cases} 1 & \omega \in A \\ -2 & \omega \in B \\ 2 & \text{otherwise. i.e } \omega \notin A \cup B \end{cases}$ A & B are disjoint events in $\mathcal{F}$.

To find cdf of X if $P(A) = \frac{1}{3}$ & $P(B) = \frac{1}{2}$



$$1 - P(A) - P(B) = 1 - \frac{1}{3} - \frac{1}{2}$$

$$= 1 - \frac{2+3}{6}$$

$$= 1 - \frac{1}{6} = \frac{5}{6} = P(X=2)$$

x	$P(X = x)$
-2	$1/2$
1	$1/3$
2	$1/6$ ✓

By definition of cdf $F_X(x) = P(X \leq x)$

when $x < -2$: $F_X(x) = 0$

$-2 \leq x < 1$, $F_X(x) = P(X = -2) = \frac{1}{2}$

$1 \leq x < 2$, $F_X(x) = P(X = -2) + P(X = 1) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

& $x \geq 2$ $F_X(x) = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

$$\therefore F_X(x) = \begin{cases} 0 & x < -2 \\ 1/2 & -2 \leq x < 1 \\ 5/6 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Ans.

2. $a < b$, $a, b \in \mathbb{R}$

Rewrite prob in form of distribution f^n of X

(a) $P_X((a,b])$ ie $X > a$ & $X \le b$

from definition of cdf $F_X(x) = P(X \le x)$

here $\cancel{P(X}$ $P_X((a,b]) = P(a < X \le b) = P(X \le b) - P(X \le a)$

$= F_X(b) - F_X(a)$

(b) $P_X((a,b))$

So $X > a$ and $X < b$

$\therefore$ Prob $P_X((a,b)) = P(a < \cancel{b} X < b)$

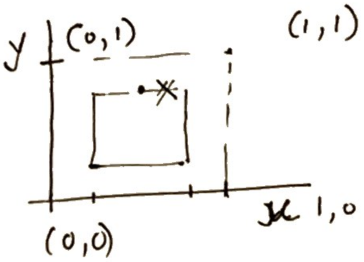
$P(a < X < b) = \cancel{P(a < X \le b)} = F_X(b^-) - F_X(a)$

where $F_X(b^-) = \lim_{t \to b^-} F_X(t)$ is the value of CDF approaches as we come upto b from left

(c) $P_X([a,b]) = P(a \le X \le b) = F_X(b) - F_X(a^-)$

where $F_X(a^-) = \lim_{t \to a^-} F_X(t)$

3.



Outcome $\omega = (u, v)$ is a point uniformly chosen within the square $[0,1] \times [1,0]$. The distance from (u,v) to the four edges are $\{u, 1-u, v, 1-v\}$

So $X(u,v) = \min(u, 1-u, v, 1-v)$, distance of the nearest edge.

Since by definition,

cdf $F_X(t) = P(X \le t)$, for $t < 0$, $F_X(t) = 0$ since distances cannot be negative. For $t \geq 1/2$, $F_X(t) = 1$.

This is because max value of $X = 0.5$, at the center of side 1, the farthest any point can be from an edge

For $0 \le t \le 0.5$, the event $\{X > t\}$ means point is at least t away from every side

ie. $t < u < 1-t$, $t < v < 1-t$

Inner square $(t, 1-t) \times (t, 1-t)$ whose length is $1-2t$

so the area is $P(X > t) = (1-2t)^2$ for $0 \le t \le \frac{1}{2}$

$\therefore \forall \quad 0 \le t \le 0.5$

$$F_X(t) = 1 - (1-2x)^2 = 4x - 4x^2$$

when $t < 0$, $F_X(t) = 0$

& $t \geq 0.5$, $F_X(t) = 1$

$$\therefore \quad \text{cdf } F_X(t) = \begin{cases} 0 & t < 0 \\ 4x - 4x^2 & 0 \le x \le 0.5 \\ 1 & t \geq 0.5 \end{cases}$$

Ans

4. Median of X defined as real number x_m

$$P(X \leq x_m) = 0.5$$

To find median of the r.v with pdf

$$f_X(x) = \begin{cases} b e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

by defn of prob density function, $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

by defn here $\int_0^{\infty} b e^{-x} dx = 1$

or $b[-e^{-x}]_0^{\infty} = 1$ or $b(0 - (-e^0)) = 1$

or $b = 1$

$\therefore$ pdf is $F_X(x) = \int_0^x e^{-t} dt = 1 - e^{-x} \quad \forall x \geq 0$

$$P(X \leq x_m) = 0.5$$

$$F_X(x_m) = 0.5$$

or $1 - e^{-x_m} = 0.5$

$e^{-x_m} = 0.5$

$\therefore -x_m = \ln(0.5)$

$\therefore x_m = -\ln(0.5) = \ln(2)$

Answer

5. pdf $f_X(x) = \begin{cases} 10/x^2 & x > 10 \\ 0 & x \leq 10 \end{cases}$ ← Given

Prob p that a single device function for at least 15 hours

$$p = P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} dx = \left[-\frac{10}{x}\right]_{15}^{\infty} = +\frac{10}{15} = \frac{2}{3}$$

Let Y = number that last ≥ 15

we want $P(Y \geq 3)$

$Y \sim \text{Binomial}\left(n=6,\ p=\frac{2}{3}\right)$, since devices are independent

$$P(Y \geq 3) = 1 - P(Y \leq 2)$$
$$= 1 - [P(Y=0) + P(Y=1) + P(Y=2)]$$

$$P(Y=k) = \binom{n}{k}(p)^k(1-p)^{n-k}$$
$$= \binom{6}{k}\left(\frac{2}{3}\right)^k\left(1-\frac{2}{3}\right)^{6-k} = \binom{6}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{6-k}$$

$$P(Y=0) = \binom{6}{0}\cdot 1\cdot\left(\frac{1}{3}\right)^6 = \frac{1}{3^6}$$

$$P(Y=1) = \binom{6}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^5 = 6\cdot\frac{2}{3}\ \frac{1}{3^5} = \frac{12}{3^6}$$

$$P(Y=2) = \binom{6}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 = 15\cdot\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4$$
$$= 15\cdot\frac{4}{3^6} = \frac{60}{3^6}$$

$$\therefore P(Y \leq 2) = \frac{1+12+60}{3^6} = \frac{73}{3^6}$$

$$\therefore P(Y \geq 3) = 1 - \frac{73}{3^6} = \frac{656}{729} \approx 0.9 \quad \underline{\text{Ans}}$$

5 Prob density function Pdf of an exponential random variable with parameter $f_X(x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$

$$= 0, \quad \forall x < 0$$

$$F(X=x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big]_0^x = 1-e^{-\lambda x}$$

cdf $F_X(x) = 1 - e^{-\lambda x}$

we want to find two points x_1 & x_2 creating three intervals $[0, x_1)$, $[x_1, x_2)$ and $[x_2\ \infty)$ so that prob of X in those interval must be $1/3$

$$P(0 \leq X \leq x_1) = \frac{1}{3}$$

$$1 - e^{-\lambda x_1} = \frac{1}{3} \quad \text{or} \quad e^{-\lambda x_1} = \frac{2}{3} \quad \text{or} \quad \lambda x_1 = \ln\left(\frac{3}{2}\right)$$

$$x_1 = \frac{1}{\lambda} \ln(3/2)$$

$$P(X \leq x_2) = \frac{2}{3}$$

$$1 - e^{-\lambda x_2} = \frac{2}{3} \quad \text{or} \quad e^{-\lambda x_2} = \frac{1}{3} \quad \therefore \quad x_2 = \frac{1}{\lambda} \ln(3)$$

$\therefore$ the three intervals are

$$[0, x_1) \Rightarrow \left[0, \frac{\ln(3/2)}{\lambda}\right)$$

$$[x_1, x_2) \Rightarrow \left[\frac{\ln(3/2)}{\lambda}, \frac{\ln(3)}{\lambda}\right)$$

and $[x_2, \infty) \Rightarrow \left[\frac{\ln(3)}{\lambda}, \infty\right)$

each interval will prob $1/3$. <u>Ans</u>

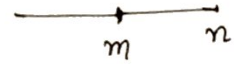
(7) Discrete random variable X is geometric random variable if

$$P(X = k) = p\, q^{k-1} \quad \text{(1)} \quad k = 1, 2, 3, \ldots$$

where $p, q > 0$ & $p + q = 1$.

To show, for natural numbers m & n

$$P(X > m+n \mid X > m) = P(X > n)$$



$$P(X > m+n \mid X > m) = \frac{P(X > m+n \overset{\nearrow \text{and}}{,} X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

$$P(X > m+n \mid X > m) = \frac{(1-p)^{m+n}}{(1-p)^m} =$$

Given (1) $\quad P(X = k) = p q^{k-1} = p(1-p)^{k-1}$

$$P(X > k) = P(X = k+1) + P(X = k+2) + \cdots$$

$$= \sum_{i=k+1}^{\infty} p q^{i-1} = pq^k + pq^{k+1} + pq^{k+2} + \cdots$$

$$= pq^k (1 + q + q^2 + \cdots)$$

$$= pq^k \frac{1}{1-q} \quad \text{geometric series}$$

$$= pq^k \frac{1}{p}$$

$$= q^k$$

$$\therefore P(X > m+n \mid X > m) = \frac{q^{m+n}}{q^m} = q^n$$

$$= P(X > n)$$

$$\therefore P(X > m+n \mid X > m) = P(X > n) \quad \square$$

7(b) Reverse statement,

X is +ve integer-valued random variable

$$P(X > m+n \mid X > m) = P(X > m) \quad \forall m, n \geq 1$$

we want to show X must be geometric.

we know $P(X > m+n \mid X > m) = \dfrac{P(X > m+n)}{P(X > m)} = P(X > n)$

shown earlier

$$\therefore \quad P(X > m+n) = P(X > m) \cdot P(X > n) \quad \forall m, n \geq 1 \quad \text{--- (1)}$$

let $P(X > k) = f(k)$

$\therefore$ $k = 1$, $f(\underset{=1}{k}) = P(X > 1)$ say q

$k = 2$, $f(2) = \cancel{P}(1+1) = f(1) \cdot f(1) = q \cdot q$

$k = 3$, $f(3) = f(2+1) = f(2)\, f(1) = q^2 \cdot q = q^3$

$\therefore$ ~~ccc~~ $f(k) = q^k$

$\therefore$ ~~(1) becomes~~

~~$P(X > m+n) =$~~

$$P(X = k) = P(X > k-1) - P(X > k)$$

$$= q^{k-1} - q^k = (1-q)\, q^{k-1}$$

$$= p\, q^{k-1}$$

$\rightarrow$ geometric distribution.

eq^n (1) $\square$

(8) X have a binomial dist

to show

$$P(X = k) = \frac{p}{1-p} \frac{n-k+1}{k} P(X = k-1)$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad , k = 0, 1, 2, \ldots, n$$

$$P(X = k-1) = \binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}$$

$$\frac{P(X = k)}{P(X = k-1)} = \frac{\binom{n}{k}}{\binom{n}{k-1}} \cdot \frac{p^k}{p^{k-1}} \frac{(1-p)^{n-k}}{(1-p)^{n-k+1}}$$

$$= \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} \cdot p \cdot \frac{1}{(1-p)}$$

$$= \frac{\cancel{n!}(k-1)!(n-k+1)!}{k!(n-k)!\cancel{n!}} p \frac{1}{(1-p)}$$

$$= \frac{n-k+1}{k} \cdot p \cdot \frac{1}{(1-p)}$$

$$\therefore \quad P(X = k) = \frac{p}{(1-p)} \cdot \frac{n-k+1}{k} P(X = k-1)$$