ECE 600 Homework 1
COMER

1. This problem considers DeMorgan's laws.

   (a) Prove DeMorgan's laws, i.e., show that for two sets A and B, $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

   (b) Now use DeMorgan's laws to show that $\overline{A \cap (B \cup C)} = (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$.

   (c) Consider the countably infinite extension of DeMorgan's laws: For a collection of sets $\{A_1, A_2, A_3, ...\}$, show that $\overline{\bigcap_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n}$, and that $\overline{\bigcup_{n=1}^{\infty} A_n} = \bigcap_{n=1}^{\infty} \overline{A_n}$.

2. Prove that if $A \cup B = A$ and $A \cap B = A$, then $A = B$.

3. Let the universal set $\mathcal{S}$ be the real numbers $\mathbb{R}$.

   (a) For $n = 1, 2, \ldots$, let $A_n = (0, \frac{1}{n})$. Show that $\bigcup_{n=1}^{\infty} A_n = (0, 1)$.

   (b) Now let $A_n = [0, \frac{1}{n})$. Find $\bigcap_{n=1}^{\infty} A_n$.

4. Consider a countably infinite partition $C_1, C_2, C_3, ...$ of a sample space $\mathcal{S}$. Show that the collection

$$F = \{A : A = \bigcup_{i \in I} C_i \text{ for some } I \subset \mathbb{N}\},$$

   where $\mathbb{N}$ is the set of natural numbers, is a σ-field.

5. In this problem it is shown by example that two or more distinct collections can generate the same σ-field. Let the sample space be $\mathcal{S} = (0, 1]$, and consider the sets

$$F = \{(0, 1/3], (2/3, 1]\}$$
$$G = \{(0, 1/3], (1/3, 2/3], (2/3, 1]\}$$
$$H = \{(0, 1/3], (0, 2/3]\}$$

   Show that $\sigma(F) = \sigma(G) = \sigma(H)$.

6. A system is composed of five components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcomes of the experiment be given by all vectors $(x_1, x_2, x_3, x_4, x_5)$, where x_i is 1 if component i is working and 0 if component i is failed.

   (a) How many outcomes are in the sample space of this experiment?

   (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

   (c) Let A be the event that components 4 and 5 have both failed. How many outcomes are contained in the event A?

   (d) Write out all outcomes in the event $A \cap W$.