HW1 SHAUNAK MUKHERJEE

1 a) $\overline{A \cup B} = \{x : x \in \overline{A \cup B}\}$ let x is an element

$= \{x : x \notin A \cup B\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \notin A\} \cap \{x : x \notin B\} = \overline{A} \cap \overline{B}$

Q.E.D

$\overline{A \cap B} = \{x : x \in \overline{A \cap B}\} = \{x : x \notin A \cap B\}$

$= \{x : x \notin A \text{ or } x \notin B\}$

$= \{x : x \notin A\} \cup \{x : x \notin B\}$

$= \{x : x \in \overline{A}\} \cup \{x : x \in \overline{B}\}$

$= \overline{A} \cup \overline{B}$ Q.E.D.

1. b) $\underset{\text{LHS}}{\overline{A \cap (B \cup C)}} = \{x : x \notin A \cap (B \cup C)\}$

$= \{x : x \notin A \text{ or } x \notin (B \cup C)\}$

$= \{x : x \notin A \text{ or } (x \notin B \text{ and } x \notin C)\}$

$= \{x : (x \notin A \text{ or } x \notin B) \text{ and } (x \notin A \text{ or } x \notin C)\}$

$= \{x : x \notin A \text{ or } x \notin B\} \cap \{x : x \notin A \text{ or } x \notin C\}$

$= (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$ Q.E.D.

1(c) To show

$$\overline{\bigcap_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n}$$

LHS $\overline{\bigcap_{n=1}^{\infty} A_n} = \{x : x \in A_i \ \forall i \in \mathbb{N}\}^c$

$= \{x : x \notin A_i \text{ for some } i \in \mathbb{N}\}$

$= \{x : x \in \overline{A_i} \text{ for some } i \in \mathbb{N}\}$

$= \bigcup_{n=1}^{\infty} \overline{A_n} = \text{RHS}$ QED

To show,

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \bigcap_{n=1}^{\infty} \overline{A_n}$$

LHS $\overline{\bigcup_{n=1}^{\infty} A_n} = \{x : x \in A_i \text{ for some } i \in \mathbb{N}\}^c$

$= \{x : x \notin A_i \ \forall i \in \mathbb{N}\}$

$= \{x : x \in \overline{A_i} \ \forall i \in \mathbb{N}\} = \bigcap_{n=1}^{\infty} \overline{A_n} = \text{RHS}$

Q.E.D.

(2) if $A \cup B = A$ & $A \cap B = A$ then $A = B$

① Let $x \in A$, then

since, $A = A \cap B$ then we can say $x \in A \cap B$

by def^n of intersection it must saisfy the condition

$x \in A$ and $x \in B$ ∴ if x is in A it must be in B ∴ $x \in B$

② next let $x \in B$. then since $A = A \cup B$, by def^n of union if x in B then x is automatically in set $A \cup B$.

This is because the union will contain all elements of both sets. Now since $A \cup B = A$, so $x \in A$

∴ $x \in A$

$\therefore$ from ① we show there are no elements in A that not in B
② " " " " " " B " not in A.

Since every element x belongs to both sets or neither the sets are identical $A = B$

④ $F = \{ A : A = \bigcup_{i \in I} C_i$ for some $I \subset \mathbb{N} \}$

$\mathbb{N} \to$ Nat num set

Need to show

① F is non empty. → Given partition elements C_i are member of F for any $i \in \mathbb{N}$ so it def not empty.

② Show if $A \in F$ then $A^c \in F$

$A \in F$, $A = \bigcup_{i \in I} C_i$ for some $I \subset \mathbb{N}$ given.

now if set of C_i form partition left over partition pieces for A^c $\therefore$ $A^c = \bigcup_{i \in \mathbb{N}-I} C_i$ from def^n of F

③ if $A_1, \dots A_n \in F$, show $\bigcup_{k=1}^{N} A_i \in F$

for each k, $\exists$ $I_k \subset \mathbb{N}$ such that $A_k = \bigcup_{i \in I_k} C_i$

let $I = \bigcup_{k=1}^{n} I_k$ then $\bigcup_{k=1}^{n} A_k = \bigcup_{i \in I} C_i \in F$

④ if $A_1, A_2 \dots \in F$ then $\bigcup_{i=1}^{\infty} A_i \in F$

same as before, for each k there exists $I_k \subset \mathbb{N}$ such that $A_k = \bigcup_{i \in I_k} C_i$.

let $I = \bigcup_{k=1}^{\infty} I_k$ $\therefore$ $\bigcup_{k=1}^{\infty} A_k = \bigcup_{i \in I} C_i \in F$

4 Conditions satisfied.

Ans

5.

$S = (0, 1]$, $F = \{(0, \frac{1}{3}], (2/3, 1]\}$

$G = \{(0, \frac{1}{3}], (1/3, 2/3], (2/3, 1]\}$

$H = \{(0, 1/3], (0, 2/3]\}$

let $A = (0, 1/3]$, $B = (\frac{1}{3}, \frac{2}{3}]$ & $C = (\frac{2}{3}, 1]$

$\therefore F = \{A, C\}$, initial sets provided : ϕ, S, A, C

Complements : $A^c = B \cup C = (\frac{1}{3}, 1] \in \sigma(F)$ ✓

$C^c = A \cup B = (0, \frac{2}{3}] \in \sigma(F)$ ✓

Unions : $A \cup C = (0, \frac{1}{3}] \cup (\frac{2}{3}, 1] \in \sigma(F)$ ✓

also $(A \cup C)^c = B = (\frac{1}{3}, \frac{2}{3}] \in \sigma(F)$ ✓

$\sigma(F) = \{\cancel{\phi, S, A, B, C, A \cup B, A \cup C, B \cup C}\}$

$= \{\cancel{\phi, S, (0, \frac{1}{3}], (\frac{2}{3}, 1], (\frac{1}{3}, 1], (0, \frac{2}{3}]}$,

$\cancel{(\frac{1}{3}, \frac{2}{3}}$

Additional unions :

$A \cup (B \cup C) = (0, \frac{1}{3}] \cup (\frac{1}{3}, 1] = \sigma(F) = S$

$A \cup (A \cup B) = (0, \frac{1}{3}] \cup (0, \frac{2}{3}] = A \cup B \in \sigma(F)$

$C \cup (B \cup C) = (\frac{2}{3}, 1] \cup (\frac{2}{3}, 1] = (\frac{1}{3}, 1) = B \cup C \in \sigma(F)$

$C \cup (A \cup B) = (\frac{2}{3}, 1] \cup (0, \frac{2}{3}] = (0, 1] = S$

$(B \cup C) \cup (A \cup B) = (\frac{1}{3}, 1] \cup (0, \frac{2}{3}) = S$

$\therefore \sigma(F) = \{\phi, S, A, B, C, A \cup B, B \cup C, A \cup C\}$

$$\therefore \sigma(F) = \left\{ \phi, S, (0,\tfrac{1}{3}], (\tfrac{2}{3},1], (\tfrac{1}{3},1], (0,\tfrac{2}{3}], (\tfrac{1}{3},\tfrac{2}{3}], (0,\tfrac{1}{3}] \cup (\tfrac{2}{3},1] \right\}$$ 6

Next $\sigma(G)$

$$G = \{A, B, C\} = \left\{ (0, \tfrac{1}{3}], (\tfrac{1}{3}, \tfrac{2}{3}], (\tfrac{2}{3}, 1] \right\}$$

Complements

$$A^c = B \cup C = (\tfrac{1}{3}, 1] \in \sigma(G)$$

$$B^c = A \cup C = \cancel{(0,\tfrac{2}{3}] \in \sigma(G)}\ (0,\tfrac{1}{3}] \cup (\tfrac{2}{3},1] \in \sigma(G)$$

$$C^c = A \cup B = (0, \tfrac{1}{3}] \cup (\tfrac{1}{3}, \tfrac{2}{3}] = (0, \tfrac{2}{3}] \in \sigma(G)$$

Unions

$$A \cup B = (0, \tfrac{2}{3}] \in \sigma(G)$$

$$B \cup C = (\tfrac{1}{3}, 1] \in \sigma(G)$$

$$A \cup C = (0, \tfrac{1}{3}] \cup (\tfrac{2}{3}, 1] \in \sigma(G)$$

$$\therefore \sigma(G) = \left\{ \phi, S, (0,\tfrac{1}{3}], (\tfrac{1}{3},\tfrac{2}{3}], (\tfrac{2}{3},1], (\tfrac{1}{3},1], (0,\tfrac{2}{3}], (0,\tfrac{1}{3}] \cup (\tfrac{2}{3},1] \right\}$$

Next $\sigma(H)$ $\quad H = \{A, A \cup B\}$ $\quad A = (0, \tfrac{1}{3}]$ & $A \cup B = (0, \tfrac{2}{3}]$

Complements

$$A^c = B \cup C = (\tfrac{1}{3}, 1] \in \sigma(H)$$

$$(A \cup B)^c = C = (\tfrac{2}{3}, 1] \in \sigma(H)$$

$$(A \cup B) \cap A^c = B = [\tfrac{1}{3}, \tfrac{2}{3}] \in \sigma(H)$$

Unions

$$A \cup C = (0, \tfrac{1}{3}] \cup (\tfrac{2}{3}, 1] \in \sigma(H)$$

$$B \cup C = (\tfrac{1}{3}, \tfrac{2}{3}] \cup (\tfrac{2}{3}, 1] = (\tfrac{1}{3}, 1] \in \sigma(H)$$

$$\sigma(H) = \left\{ \phi, S, (0,\tfrac{1}{3}], (0,\tfrac{2}{3}], (\tfrac{1}{3},1], (\tfrac{2}{3},1], (0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1], (\tfrac{1}{3},1], (\tfrac{1}{3},\tfrac{2}{3}] \right\}$$

$\therefore \sigma(F) = \sigma(G) = \sigma(H)$ Same elements in set Q.E.D.

5 (a) for $n = 1, 2, \dots$ let $A_n = (0, \frac{1}{n})$

To show $\bigcup_{n=1}^{\infty} A_n = (0, 1)$

Let $x \in \bigcup_{n=1}^{\infty} (0, \frac{1}{n})$, then $\exists\, n \in \mathbb{N}$ s.t $x \in (0, \frac{1}{n})$

so $0 < x < 1/n$

$\because 1/n \leq 1 \ \forall n \geq 1 \Rightarrow 0 < x < 1$ $\therefore x \in (0, 1)$

$\Rightarrow \bigcup_{n=1}^{\infty} (0, \frac{1}{n}) \subset (0, 1)$

Conversely let $x \in (0, 1)$, then $0 < x < 1$, $\frac{1}{x} > 1$

if n s.t $n > \frac{1}{x}$ or $\frac{1}{n} > x$ $\therefore 0 < x < \frac{1}{n}$ $\Rightarrow n \in (0, \frac{1}{n})$

$\therefore x \in \bigcup_{n=1}^{\infty} (0, \frac{1}{n})$ $\Rightarrow (0, 1) \subset \bigcup_{n=1}^{\infty} (0, \frac{1}{n})$

$\therefore \bigcup_{n=1}^{\infty} (0, \frac{1}{n}) = (0, 1)$ Q.E.D.

3 (b) $\bigcap_{n=1}^{\infty} [0, \frac{1}{n})$ or $\bigcap_{n=1}^{\infty} A_n$

$\therefore A_1 = [0, 1)$

$A_2 = [0, 1/2)$

$A_3 = [0, 1/3)$ as $n \uparrow$ interval $\downarrow$

...

let x is in $\bigcap_{n=1}^{\infty} A_n$ iff $x \in A_n \ \forall n$ $\therefore 0 \leq x < \frac{1}{n} \ \forall n \in \mathbb{N}$

if $x = 0$, $0 \in [0, 1/n) \ \forall n$ $\therefore 0$ is ok & in the intersection.

if $x > 0$, even for small +ve x, at large n we get $\frac{1}{n} < x$

so $x \not< \frac{1}{n}$ $\Rightarrow x \notin A_n$

$\therefore$ no positive numbers can belong to A_n

So only number belong to every A_n is 0 $\therefore \bigcap_{i=1}^{\infty} A_n = \{0\} \neq \phi$

6. working = 1
a) failed : 0

Sample space $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{0,1\}$

so 5 components 2 outcome each

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

6 (b) ① Components 1 & 2 works so $x_1 = 1, x_2 = 1$

② 3 & 4 " so $x_3 = 1, x_4 = 1$

③ 1, 3 & 5 " $x_1 = 1, x_3 = 1, x_5 = 1$

① when $x_1 = 1, x_2 = 1, \rightarrow (1, 1, x_3, x_4, x_5)$ where $x_3, x_4, x_5 \in \{0,1\}$

$\therefore$ (1, 1, 0, 0, 0), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1), (1, 1, 0, 0, 1)
(1, 1, 1, 0, 0), (1, 1, 1, 0, 1), (1, 1, 1, 1, 0), (1, 1, 1, 1, 1) $2^3 = 8$ outcome

② $x_3 = 1, x_4 = 1 \rightarrow (x_1, x_2, 1, 1, x_5)$

(0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 1, 1, 0), (0, 1, 1, 1, 1)
(1, 0, 1, 1, 0), (1, 0, 1, 1, 1), (1, 1, 1, 1, 0) (1, 1, 1, 1, 1) $2^3 = 8$

③ $x_1 = 1, x_3 = 1, x_5 = 1 \rightarrow (1, x_2, 1, x_4, 1)$, $x_1, x_4 \in \{0,1\}$

so
(1, 0, 1, 0, 1), (1, 0, 1, 1, 1), (1, 1, 1, 0, 1), (1, 1, 1, 1, 1) $2^2 = 4$

hence $W = \{(x_1, x_2, x_3, x_4, x_5) : (x_1 = 1 \wedge x_2 = 1) \vee (x_3 = 1 \wedge x_4 = 1) \vee (x_1 = 1 \wedge x_3 = 3 \wedge x_5 = 1)\}$

Total 15 outcomes of W are below (no duplicate)

{(1, 1, 0, 0, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1)
(1, 1, 1, 0, 0), (1, 1, 1, 0, 1), (1, 1, 1, 1, 0), (1, 1, 1, 1, 1), (0, 0, 1, 1, 0),
(0, 0, 1, 1, 1), (0, 1, 1, 1, 0), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0) (1, 0, 1, 1, 1)
(1, 0, 1, 0, 1

6(c) 4 & 5 failed $\therefore x_4 = 0,\ x_5 = 0,\ (x_1, x_2, x_3, 0, 0)$

$\rightarrow (\cancel{x_1, x_2, x_3, x_4}, 0, \cancel{x_5})$ where $x_1, x_2\ x_3 \in \{0, 1\}$

$\therefore$ Total $2^3 = 8$ outcome in A

$\{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0), (1, 0\ 0, 0, 0), (0, 0, 0, 0, 0)$
$(0, 0, 1, 0, 0), (1, 0, 1, 0, 0), (0, 1, 0, 0, 0), (0, 1, 1, 0, 0)\}$

6(d) $\underline{A \cap W}$

① $x_1 = 1,\ x_2 = 1$
② $x_3 = 1,\ x_4 = 1$
③ $x_1 = 1,\ x_3 = 1,\ x_5 = 1$

were total 15 from 4(b).

Ⓐ $x_4 = 0,\ x_5 = 0$, we put 0 in x_4 & x_5 & look at those only from 4(b) & 4(c)

i.e. $(1, 1, 1, 0, 0)$
& $(1, 1, 0, 0, 0)$

(Answer)

$\therefore A \cap W = \{(1, 1, 0, 0, 0), (1, 1, 1, 0, 0)\}$

—x—