

2. $\binom{n}{k}$ ways for k successes to occur in n Bernoulli trials, to show $P(k \text{ successes in } n \text{ trial}) = \binom{n}{k} p^k (1-p)^{n-k}$

Let n independent trials with prob of a success p & a failure $1-p$. Since Bernoulli trials are independent, we multiply individual probabilities. So for sequence of n trials with k success & $n-k$ fails $p \cdot p \cdots p \underbrace{(1-p) \cdot (1-p) \cdots (1-p)}_{n-k \text{ times}} = p^k (1-p)^{n-k}$

The event k success in n trials include all possible sequence that contain exactly k successes & regard less of the order \geq total number of distinct combination is $\binom{n}{k}$ ie the number of ways to choose exactly k positions for successes out of n total trials.

Since prob of each

By multiplying the number of possible ways by the probability of any one of those ways, we get

$$P(k \text{ success in } n \text{ trial}) = \binom{n}{k} p^k (1-p)^{n-k}$$

(Ans)

(2)

3.

(b) There are total $\binom{n}{k}$ ways to draw k balls without replacement from total n balls

To pick largest number drawn to be less than or equal to m , $\binom{m}{k}$ is the ways we can do it.

$$\therefore P(\text{draw} \leq m) = \frac{\binom{m}{k}}{\binom{n}{k}} \text{ for } k \leq m$$

3.
(a) To find prob that m is the largest number drawn.

If we choose largest ball m , we choose the remaining $k-1$ balls from numbers smaller than m

$$P(\text{draw largest} = m) = \frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

Answer

1.



111

(a) Find $P(6 \leq t \leq 8)$

interval = $8 - 6 = 2$

Total = 10

$$\therefore P(6 \leq t \leq 8) = \frac{2}{10} = 0.2$$



(b) $P(6 \leq t \leq 8 | t > 5)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = 0.2$$

$$P(t > 5) = \frac{5}{10} = 0.5$$

$$\therefore P(6 \leq t \leq 8 | t > 5) = \frac{0.2}{0.5} = 0.4$$

(3)

④ To find conditional prob that the i -th trial in n Bernoulli trials is a success given that there are a total of k successes.

Let A be an event that i -th trial is a success & let B be the event that there are k success in n trials

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) = \binom{n}{k} p^k (1-p)^{n-k} \text{ - i.e. prob of } k \text{ successes in } n \text{ trials}$$

now if i -th trial is success, in remaining $n-1$ trials
there are $k-1$ successes

$$\therefore P(A \cap B) = P\left[\binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}\right]$$

$$= \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$\therefore P(A|B) = \frac{\binom{n-1}{k-1} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}$$

$$= \frac{\frac{(n-1)!}{(k-1)! (n-k)!}}{\frac{n!}{k! (n-k)!}} = \frac{(n-1)!}{(k-1)!} \times \frac{k!}{n!}$$

$$= \frac{(n-1)!}{n!} \times \frac{k!}{(k-1)!}$$

$$P(A|B) = \frac{k}{n}$$

(Ans)

$$⑥ S = \{1, 2, 3, 4\} \text{ & } F = \{\emptyset, S, \{1\}, \{2, 3, 4\}\}$$

(4)

$X(\omega) = 1 + \omega$, looking into all possible values of X based on sample space $\Omega = S = \{1, 2, 3, 4\}$

$$\omega=1, X(1)=2,$$

$$\omega=2, X(2)=3$$

$$\omega=3, X(3)=4$$

$$\omega=4, X(4)=5$$

To be random variable w.r.t F , $X^{-1}(A)$ will be \emptyset unless A contain at least one value from $\{2, 3, 4, 5\}$

let $A = \{5\}$, $X^{-1}(\{5\}) = \{4\}$, but $\{4\} \notin F$ so we have

found a $A \in B(\mathbb{R})$ for which $X^{-1}(A) \in F$. $\Rightarrow X(\omega) = 1 + \omega$ is not a random variable w.r.t F .

Example of valid r.v. $Y(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ 2 & \text{if } \omega \in \{2, 3, 4\} \end{cases}$

To verify we consider cases for $A \in B(\mathbb{R})$

if $1 \in A, 2 \notin A, X^{-1}(A) = \{1\} \in F$

if $1 \notin A, 2 \in A, X^{-1}(A) = \{2, 3, 4\} \in F$

if $1 \notin A, 2 \notin A, X^{-1}(A) = \emptyset \in F$

if $1 \in A, 2 \in A, X^{-1}(A) = S \in F$

Since every $A \in B(\mathbb{R})$ satisfies one of

these four conditions $X^{-1}(A) \in F \quad \forall A \in B(\mathbb{R})$

(Ans)

(5)

5. A function $X: S \rightarrow \mathbb{R}$ is a random variable w.r.t. σ -field F if for every Borel set $B \subset \mathbb{R}$, the pre-image belongs to F

$$\{ \omega \in S : X(\omega) \leq x \} \in F \quad \forall x \in \mathbb{R}$$

Given $X(\omega) = c \quad \forall \omega \in S$ where $c \in \mathbb{R}$ a constant

we need to show $X^{-1}(A) \in F$ for every $A \in \mathcal{B}(\mathbb{R})$ for any event space F .

By definition, pre image of set A is the set of all outcomes ω in S that X maps onto A

$$X^{-1}(A) = \{ \omega \in S : X(\omega) \in A \}$$

Given $X(\omega) = c$ for every outcome of sample space, $X(\omega) \in A$ is satisfied only if constant c is an element of A . Since function is constant, there are only two possible outcomes of $X^{-1}(A)$

\rightarrow if $c \in A$: Every $\omega \in S$ is mapped to c , and since c is in A , every outcome satisfies the condition $\therefore X^{-1}(A) = S$

\rightarrow if $c \notin A$, so no value ω in S is mapped into A because only value X produces is c , which is not in A

$$\therefore X^{-1}(A) = \emptyset$$

\therefore for any $A \in \mathcal{B}(\mathbb{R})$, we found that $X^{-1}(A)$ is either the empty set (\emptyset) or the entire sample space S . By definition of event space, \emptyset , $\{S\}$, are guaranteed to be in every event space F . Thus $X^{-1}(A) \in F \quad \forall A \in \mathcal{B}(\mathbb{R})$ proving that X is random variable under any event space.

⑦ r.v X with CDF $F_X(x)$

To show if $b > a$ then $F_X(b) > F_X(a)$

if $b > a$

$$\Rightarrow \{X \leq a\} \subset \{X \leq b\}$$

$$\therefore P(A) \leq P(B)$$

$$\therefore P(X \leq a) \leq P(X \leq b) \Rightarrow F_X(a) \leq F_X(b)$$

8. (a) $F_X(x) = \begin{cases} 1 & x > 1 \\ x^2 & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$

Need to verify 4 conditions

- ① $F_X(x)$ is non decreasing, ② $F_X(x)$ is continuous from the right ③ $\lim_{x \rightarrow -\infty} F_X(x) = 0$ ④ $\lim_{x \rightarrow \infty} F_X(x) = 1$.

$$F_X(x) = \begin{cases} 1 & x > 1 \\ x^2 & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$$



③ as $x \rightarrow -\infty \quad \forall x < 0, F_X = 0 \quad \therefore F_X(x) = 0 \quad \text{satisfied}$

④ as $x \rightarrow +\infty \quad \forall x > 1 \quad F_X = 1 \quad \therefore F_X(x) = 1 \quad \text{satisfied.}$

② when $(-\infty, 0)$, it is const at 0

on $[0, 1]$ the $f(x) = x^2$ is increasing

on $(1, \infty)$ it is constant at 1.

A boundary $x = 0, F_X(0) = 0^2 = 0$ non decreasing

& $x = 1, F_X(1) = 1^2 = 1 \leq F_X(x)$ again non decreasing

② At $x = 0 : \lim_{x \rightarrow 0^+} F_X(x) = 0$ & defined value $F_X(0) = 0$

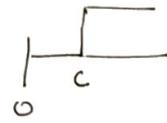
At $x = 1 : \lim_{x \rightarrow 1^-} F_X(x) = 1^2 = 1 \quad \& F_X(1) = 1 \quad \therefore \text{all conditions satisfied.}$

$$8(b) \quad F_X(x) = \begin{cases} 0 & x \leq c \\ 1 & x > c. \end{cases}$$

1. limits at infinity

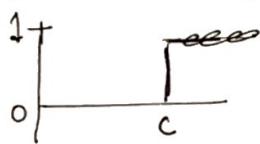
$$x \rightarrow -\infty \quad F_X(x) = 0 \quad \therefore \lim_{x \rightarrow -\infty} F_X(x) = 0$$

$$x \rightarrow +\infty \quad F_X(x) = 1 \quad \therefore \lim_{x \rightarrow \infty} F_X(x) = 1.$$



hence satisfied

2. Non decreasing for $x \leq c$, $F_X(x) = 0$ & at $x > c$ $F_X(x) = 1$



so for $x > c$ onward no change or decrease
hence satisfied.

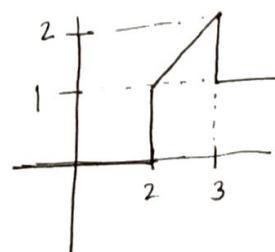
3. Right continuity.

At $x = c$, $F_X(c) = 0$ & $F_X(c^+) = 1$ conflict

\therefore function is not right continuous

So not a valid CDF.

$$8(c) \quad F_X(x) = \begin{cases} 1 & x > 3 \\ x-1 & 2 \leq x < 3 \\ 0 & x < 2 \end{cases}$$



1. limits at infinity

$$x \rightarrow -\infty \quad \forall x < 2 \quad F_X(x) = 0$$

$$x \rightarrow \infty \quad \forall x \geq 3 \quad F_X(x) = 1 \quad \therefore \lim_{x \rightarrow \infty} F_X(x) = 1$$

$\therefore \lim_{x \rightarrow -\infty} F_X(x) = 0$] satisfied

2. Non decreasing property check

$x < 2$, function yields 0

$2 \leq x < 3$ f^x is $x-1$ as $x \uparrow f^x$ increases

$x \geq 3$ f^x is constant 1.

(8)

At $x=2$, $2-1=1$ but before $x=2$ value was zero.

$x=3$ value is 1 before $x=3$ value was $3-1=2$

but if $x=2.5$ value $F_X(2.5) = 1.5 > F_X(2) = 1$ but drops to 1 at $x=3$, so violates non decreasing rule

so not a valid CDF

$$8(d) \quad F_X(x) = 1 - e^{-x}$$

check limits

$$x \rightarrow -\infty \quad \text{Lt } (1 - e^{-x}) \neq 0 \quad \text{so not valid CDF} \quad \underline{\text{Ans}}$$

$$8(e) \quad F_X(x) = (1 - e^{-x}) u(x) \quad \text{where } c \text{ is constant} \geq u(x)$$

is the unit step function. i.e. $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

$$u(x) = \begin{cases} 0 & x < 0 \\ 1-e^{-x} & x \geq 0 \end{cases}$$

① limits

$$\text{as } x \rightarrow -\infty \quad F_X(x) = 0 \quad \boxed{\text{satisfies}}$$

$$\text{as } x \rightarrow \infty \quad F_X(x) = 1 - e^{-x}, \quad x \rightarrow 1$$

② Non decreasing

$x \leq 0$ constant at 0

$$x \geq 0 \quad \text{at } x=0 \quad F_X(0) = (1-1)1 = 0$$

~~$$\text{at } x > 0, \quad \frac{d}{dx}(1 - e^{-x}) = e^{-x} > 0 \quad \text{so non decreasing}$$~~

③ Right continuity

function is continuous at every point including $x=0$

where it transitions from 0 to exponential curve hence valid CDF. [Ans]

(9)

(9) $P(H)$ for each coin tossed is $P(H) = \frac{1}{2} = p$
 $n = 10$ for 10 coins

given X is discrete random variable.

The range of X is the possible X is the number of heads.
 $\therefore R_X = \{0, 1, 2, 3, \dots, 10\}$

To build CDF (F_X) we first find probability of each specific outcome k in the range.

$$P(X=k) = \binom{10}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{10-k} = \binom{10}{k} \left(\frac{1}{2}\right)^{10}$$

by definition CDF $F_X(x)$ is the prob that X less than or equal to x . For a discrete variable

$$P(X \leq x) = \sum_{x_k \in R_X, x_k \leq x} P(X = x_k)$$

\therefore we express CDF as piecewise function to cover the entire real line $(-\infty \text{ to } +\infty)$:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} \binom{10}{k} \left(\frac{1}{2}\right)^{10} & x \in [0, 10] \\ 1 & x > 10 \end{cases}$$

$x < 0 \rightarrow$ impossible to find negative number of heads
 \therefore prob is zero

$x \in [0, 10]$ we sum probabilities from 0 to floor of x

$x > 10$: no of heads cannot be more than 10 so

prob reaches max at 1.

Ans