

2. $\binom{n}{k}$ ways for k successes to occur in n Bernoulli trials, to show $P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}$

Let n independent trials with prob of a success p & a failure $1-p$. Since Bernoulli trials are independent, we multiply individual probabilities. So for sequence of n trials with k success & $n-k$ fails

$$\underbrace{p \cdot p \cdots p}_{k \text{ times}} \underbrace{(1-p) \cdot (1-p) \cdots (1-p)}_{n-k \text{ times}} = p^k (1-p)^{n-k}$$

The event k success in n trials include all possible sequence that contain exactly k successes & regardless of the order. \therefore total number of distinct combination is $\binom{n}{k}$ is the number of ways to choose exactly k positions for successes out of n total trials.

Since prob of each

By multiplying the number of possible ways by the probability of any one of those ways, we get

$$P(k \text{ success in } n \text{ trial}) = \binom{n}{k} p^k (1-p)^{n-k}$$

[Ans]

3.

(b) There are total $\binom{n}{k}$ ways to draw k balls without replacement from total n balls

to pick largest num here drawn to be less than or equal to m , $\binom{m}{k}$ is the ways we can do it.

$$\therefore P(\text{draw} \leq m) = \frac{\binom{m}{k}}{\binom{n}{k}} \text{ for } k < m$$

3.

(a) To find prob that m is the largest number drawn.

if we choose largest ball m , we choose the remaining $k-1$ balls from numbers smaller than m

$$P(\text{draw largest} = m) = \frac{\binom{m-1}{k-1}}{\binom{n}{k}} \quad \text{Answer}$$

1.



(a) Find $P(6 \leq t \leq 8)$

$$\text{interval} = 8 - 6 = 2$$

$$\text{Total} = 10$$

$$\therefore P(6 \leq t \leq 8) = \frac{2}{10} = 0.2$$

|||



(b) $P(6 \leq t \leq 8 | t > 5)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = 0.2$$

$$P(t > 5) = \frac{5}{10} = 0.5$$

$$\therefore P(6 \leq t \leq 8 | t > 5) = \frac{0.2}{0.5} = 0.4$$

- ④ To find conditional prob that the i th trial in n Bernoulli trials is a success given that there are a total of k successes. ③

Let A be an event that i -th trial is a success, let B be the event that there are k success in n trials

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) = \binom{n}{k} p^k (1-p)^{n-k} \text{ i.e. prob of } k \text{ successes in } n \text{ trials}$$

now if i -th trial is success, in remaining $n-1$ trials there are $k-1$ successes

$$\therefore P(A \cap B) = P \left[\binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \right]$$

$$= \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$\therefore P(A|B) = \frac{\binom{n-1}{k-1} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}$$

$$= \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{(n-1)!}{(k-1)!} \times \frac{k!}{n!} = \frac{(n-1)!}{n!} \times \frac{k!}{(k-1)!}$$

$$P(A|B) = \frac{k}{n}$$

(Any)

(6)

$$S = \{1, 2, 3, 4\} \text{ and } \mathcal{F} = \{\emptyset, S, \{1\}, \{2, 3, 4\}\}$$

(4)

$X(\omega) = 1 + \omega$, looking into all possible values of X based on sample space $S = \{1, 2, 3, 4\}$

$$\omega = 1, X(1) = 2,$$

$$\omega = 2, X(2) = 3$$

$$\omega = 3, X(3) = 4$$

$$\omega = 4, X(4) = 5$$

To be random variable w.r.t \mathcal{F} , $X^{-1}(A)$ will be \emptyset unless A contains at least one value from $\{2, 3, 4, 5\}$

Let $A = 5$, $X^{-1}(\{5\}) = \{4\}$, but $\{4\} \notin \mathcal{F}$ so we have

found a $A \in \mathcal{B}(\mathbb{R})$ for which $X^{-1}(A) \notin \mathcal{F}$. $\Rightarrow X(\omega) = 1 + \omega$ is not a random variable w.r.t \mathcal{F} .

Example of valid r.v. $Y(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ 2 & \text{if } \omega \in \{2, 3, 4\} \end{cases}$

To verify we consider cases for $A \in \mathcal{B}(\mathbb{R})$

$$\text{if } 1 \in A, 2 \notin A, X^{-1}(A) = \{1\} \in \mathcal{F}$$

$$\text{if } 1 \in A, 2 \in A, X^{-1}(A) = \{2, 3, 4\} \in \mathcal{F}$$

$$\text{if } 1 \notin A, 2 \notin A, X^{-1}(A) = \emptyset \in \mathcal{F}$$

$$\text{if } 1 \in A, 2 \in A, X^{-1}(A) = S \in \mathcal{F}$$

Since every $A \in \mathcal{B}(\mathbb{R})$ satisfies one of

these four conditions $X^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathbb{R})$

(Ans)

5.

(5)

A fn $X: S \rightarrow \mathbb{R}$ is a random variable w.r.t σ field F if for every Borel set $B \subset \mathbb{R}$, the pre image belongs to F

$$\{\omega \in S : X(\omega) \leq x\} \in F \quad \forall x \in \mathbb{R}$$

Given $X(\omega) = c \quad \forall \omega \in S$ where $c \in \mathbb{R}$ a constant
we need to show $X^{-1}(A) \in F$ for every $A \in \mathcal{B}(\mathbb{R})$
for any event space F .

By definition, pre image of set A is the set of all outcomes ω in S that X maps into A

$$X^{-1}(A) = \{\omega \in S : X(\omega) \in A\}$$

Given $X(\omega) = c$ for every outcome of sample space,
 $X(\omega) \in A$ is satisfied only if constant c is an element of A
Since function is constant, there are only two possible outcomes of $X^{-1}(A)$

\rightarrow if $c \in A$: Every $\omega \in S$ is mapped to c , and since c is in A , every outcome satisfies the condition $\therefore X^{-1}(A) = S$

\rightarrow if $c \notin A$, so no value ω in S is mapped into A
because only value X produces is c , which is not in A

$$\therefore X^{-1}(A) = \emptyset$$

\therefore for any $A \in \mathcal{B}(\mathbb{R})$, we found that $X^{-1}(A)$ is either the empty set (\emptyset) or the entire sample space S . By definition of event space, \emptyset & S , are guaranteed to be in every event space F . Thus $X^{-1}(A) \in F \quad \forall A \in \mathcal{B}(\mathbb{R})$
proving that X is random variable under any event space

7. R.V. X with CDF $F_X(x)$

To show if $b > a$ then $F_X(b) \geq F_X(a)$

if $b > a$

$$\Rightarrow \{X \leq a\} \subset \{X \leq b\}$$

$$\therefore P(A) \leq P(B)$$

$$\therefore P(X \leq a) \leq P(X \leq b) \Rightarrow F_X(a) \leq F_X(b)$$

8. (a)
$$F_X(x) = \begin{cases} 1 & x > 1 \\ x^2 & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$$

Need to verify 4 conditions

- ① $F_X(x)$ is non decreasing, ② $F_X(x)$ is continuous from the right ③ $\lim_{x \rightarrow -\infty} F_X(x) = 0$ ④ $\lim_{x \rightarrow \infty} F_X(x) = 1$.

$$F_X(x) = \begin{cases} 1 & x > 1 \\ x^2 & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$$

③ as $x \rightarrow -\infty$ $\forall x < 0$, $F_X = 0$ $\therefore F_X(x) = 0$ satisfied

④ as $x \rightarrow +\infty$ $\forall x > 1$ $F_X = 1$ $\therefore F_X(x) = 1$ satisfied.

② when $(-\infty, 0)$, it is const at 0
on $[0, 1]$ the fcn is x^2 & increasing

on $(1, \infty)$ it is constant at 1.

A boundary $x = 0$, $F_X(0) = 0^2 = 0$ non decreasing

& $x = 1$ $F_X(1) = 1^2 = 1 \leq F_X(x)$ again non decreasing

② At $x = 0$: $\lim_{x \rightarrow 0^+} x^2 = 0$ & defined value $F_X(0) = 0$

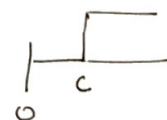
At $x = 1$: $\lim_{x \rightarrow 1^+} F_X(x) = 1^2 = 1$ & $F_X(1) = 1$ \therefore all condition satisfied.

8(b) $F_X(x) = \begin{cases} 0 & x \leq c \\ 1 & x > c \end{cases}$ (7)

1. limits at infinity

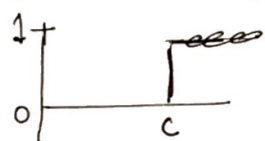
$x \rightarrow -\infty \quad F_X(x) = 0 \quad \therefore \lim_{x \rightarrow -\infty} F_X(x) = 0$

$x \rightarrow +\infty \quad F_X(x) = 1 \quad \therefore \lim_{x \rightarrow \infty} F_X(x) = 1.$



hence satisfied

2. Non decreasing for $x \leq c$, $F_X(x) = 0$ and at $x > c$ $F_X(x) = 1$



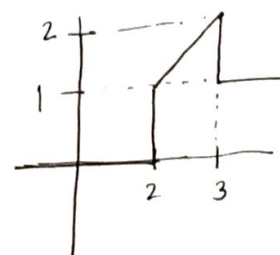
so for $x > c$ onward no change or decrease
hence satisfied.

3. Right continuity.

At $x = c$, $F_X(c) = 0$ and $F_X(c^+) = 1$ conflict

\therefore function is not right continuous

So not a valid CDF.



8(c) $F_X(x) = \begin{cases} 1 & x > 3 \\ x-1 & 2 \leq x \leq 3 \\ 0 & x < 2 \end{cases}$

1. limits at infinity

$x \rightarrow -\infty \quad \forall x < 2 \quad F_X(x) = 0 \quad \therefore \lim_{x \rightarrow -\infty} F_X(x) = 0$

$x \rightarrow \infty \quad \forall x > 3 \quad F_X(x) = 1 \quad \therefore \lim_{x \rightarrow \infty} F_X(x) = 1$

satisfied

2. Non decreasing properly check

$x < 2$, function yields 0

$2 \leq x \leq 3$ $f(x)$ is $x-1$ as $x \uparrow$ $f(x)$ increases

$x > 3$ $f(x)$ is constant 1.

At $x=2$, $2-1=1$ but before $x=2$ value was zero.

8

$x=3$ value is 1 before $x=3$ value was $3-1=2$

but if $x=2.5$ value $F_X(2.5) = 1.5 > F_X(2.5)$ but drops to 1 at $x=3$

so violates non decreasing rule

so not a valid CDF

$$8(d) \quad F_X(x) = 1 - e^{-x}$$

check limits

$$x \rightarrow -\infty \quad \lim (1 - e^{-x}) \neq 0$$

so not valid CDF Ans

8(e) $F_X(x) = (1 - e^{-x}) u(x)$ where c is constant $\geq u(x)$
is the unit step function. i.e. $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

① limits

$$\text{as } x \rightarrow -\infty \quad F_X(x) = 0$$

$$\text{as } x \rightarrow \infty \quad F_X(x) = 1 - e^{-x} \rightarrow 1 \quad \left. \vphantom{\text{as } x \rightarrow \infty} \right] \text{ satisfies}$$

② Non decreasing

$x < 0$ constant at 0

$$x \geq 0 \quad \text{at } x=0 \quad F_X(0) = (1-0) = 0$$

$$\text{at } x > 0, \quad \frac{d}{dx} (1 - e^{-x}) = e^{-x} > 0 \quad \text{so non decreasing}$$

③ Right continuity

function is continuous at every point including $x=0$

where it transitions from 0 to exponential curve
hence valid CDF. [Ans]

⑨ $P(H)$ for each coin tossed is $P(H) = \frac{1}{2} = p$
 $n = 10$ for 10 coins

Given X is discrete random variable.

The range of X is the possible X is the number of heads.

$$\therefore R_X = \{0, 1, 2, 3, \dots, 10\}$$

To build CDF (F_X) we first find probability of each specific outcome k in the range.

$$P(X = k) = \binom{10}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{10-k} = \binom{10}{k} \left(\frac{1}{2}\right)^{10}$$

by definition CDF $F_X(x)$ is the prob that X less than or equal to x . For a discrete variable

$$P(X \leq x) = \sum_{x_k \in R_X, x_k \leq x} P(X = x_k)$$

\therefore we express CDF as piecewise function to cover the entire real line $(-\infty \text{ to } +\infty)$:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} \binom{10}{k} \left(\frac{1}{2}\right)^{10} & x \in [0, 10] \\ 1 & x > 10 \end{cases}$$

$x < 0 \rightarrow$ impossible to find negative number of heads so prob is zero

$x \in [0, 10]$ we sum probabilities from 0 to floor of x

$x > 10$: no of heads cannot be more than 10 so prob reaches max at 1.

Ans