

ECE 600 Homework 2

COMER

1. Express each of the following events in terms of the events A, B, C , and set operations:
 - (a) at least one of the events A, B, C occurs;
 - (b) at most one of the events A, B, C occurs;
 - (c) events A, B, C all occur;
 - (d) exactly one of the events A, B, C occurs.

2. Let A and B be two events in a probability space.
 - (a) Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
 - (b) Show that if $P(A) = P(B) = 1$, then $P(A \cap B) = 1$.
 - (c) Show that the probability that exactly one of the events A or B occurs is given by $P(A) + P(B) - 2P(A \cap B)$.
 - (d) Show that if A is a subset of B , then $P(A) \leq P(B)$.

3. Prove that for events A, B, C ,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C),$$
 then generalize this result to the union of n events.

4. If A_1, A_2, \dots are events in a probability space, show that if $P(A_i) = 1$ for all $i \geq 1$, then $P(\bigcap_{i=1}^{\infty} A_i) = 1$.

5. If A, B , and C are events in a probability space, show that $P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

6. Let A and B be two events in a probability space.
 - (a) Show that if A and B are independent, then A^C and B are also independent.
 - (b) If A and B are independent, are they mutually exclusive? Explain.
 - (c) Show that if $P(A|B) > P(A)$, then $P(B|A) > P(B)$.

7. An experiment consists of picking one of two urns at random, with the two urns being equally likely, and then selecting a ball from the urn and noting its color (black or white). Let A be the event "urn 1 is selected" and B the event "a black ball is selected". Under what conditions are the events A and B independent?

8. There are two servers at the grocery store. Server 1 can take anywhere from 1 to 5 minutes to complete an order. Server 2 can take anywhere from 1 to 10 minutes to complete an order. Let T_S denote the service time. You are given that

$$P(T_S \leq t | \text{Server 1}) = \begin{cases} 0 & t \leq 1 \\ \frac{t-1}{4} & 1 < t \leq 5 \\ 1 & t > 5 \end{cases}$$

$$P(T_S \leq t | \text{Server 2}) = \begin{cases} 0 & t \leq 1 \\ \frac{t-1}{9} & 1 < t \leq 10 \\ 1 & t > 10 \end{cases}$$

You are also given that $P(\{\text{Server 1}\}) = 0.4$ and $P(\{\text{Server 2}\}) = 0.6$.

- (a) Find $P(\{T_S \leq 5\})$.
 - (b) Find $P(\{\text{Server 1}\} | \{T_S \leq 5\})$.
 - (c) Find $a > 1$ such that $P(\{\text{Server 1}\} | \{T_S \leq a\}) = P(\{\text{Server 2}\} | \{T_S \leq a\})$. This means that given $T_S \leq a$, it is equally likely that either server performed this task.
9. You are a contestant on a TV game show. There are 3 identical closed doors leading to 3 rooms. Two of the rooms contain nothing, but the third contains a brand new Mercedes which is yours if you make the correct choice. You are asked to pick a door by the host of the show who knows which room contains the Mercedes. Once you have made your decision, he shows you a room (not the one you chose) that does not contain the car. Show that even without any further knowledge, you will greatly increase your chances of winning the Mercedes if you switch your choice to the other door still unopened.