Explanatory Notes for 6.390

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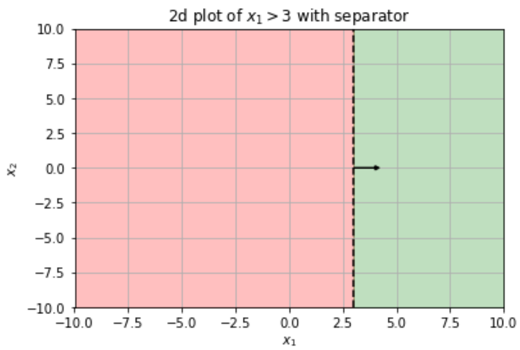
Introducing our offset

Now that we have handled the case where our linear separator is on the **origin**, we want to **shift** our separator **away** from it.

In our **1-D** case, we easily **shifted** away from the origin: any separator $x_1 > C$ where C **isn't zero**, we shift by C units.



By making our inequality $x_1 > 3$ **nonzero**, we moved away from the origin by 3 units!

We could make our inequality **nonzero**, then! That could move us **away** from the origin, just in a different **direction**.

Or, we could equivalently do this...

Note: $A \iff B$ means A and B are equivalent!

$$x_1 > 3 \iff x_1 - 3 > 0 \tag{1}$$

So, instead, we could just add a constant to our expression, which we will call θ_0.

We'll also switch out $\theta \cdot x = \theta^T x$.

Key Equation 1

A general **linear separator** can do **binary classification** using the hypothesis

$$h(x;\theta) = \text{sign}(\theta^T x + \theta_0) = \begin{cases} +1 & \text{if } \theta^T x + \theta_0 > 0 \\ -1 & \text{otherwise} \end{cases}$$

Notice that this looks very similar to what we did in regression! We'll get into that in a bit.

First, a quick look at the components of our equation:

Concept 2

For **binary classification**, θ and θ_0 entirely **define** our **linear separator**.

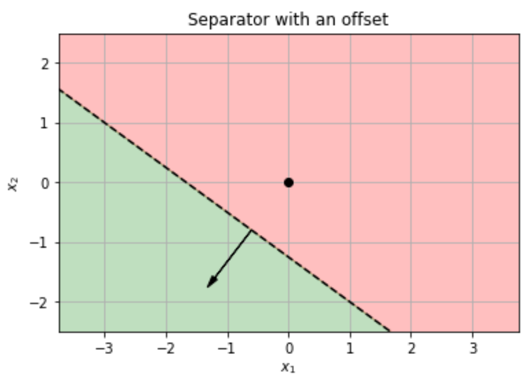
- θ gives us the **orientation** of our line.
- θ_0 **shifts** that line around in **space**.

How does the offset affect our classifier?

So, how exactly does our offset θ_0 affect our **classifier**? Well, we mark our classifier with our **normal vector** and the **boundary**.

Our **normal vector** is entirely captured by θ: it's unchanged by θ_0.

What about our **boundary**? We have its **orientation**, but we don't know where it has **shifted** to.



Note that the origin has been marked.

Well, let's use our equation: the **boundary** line is given by

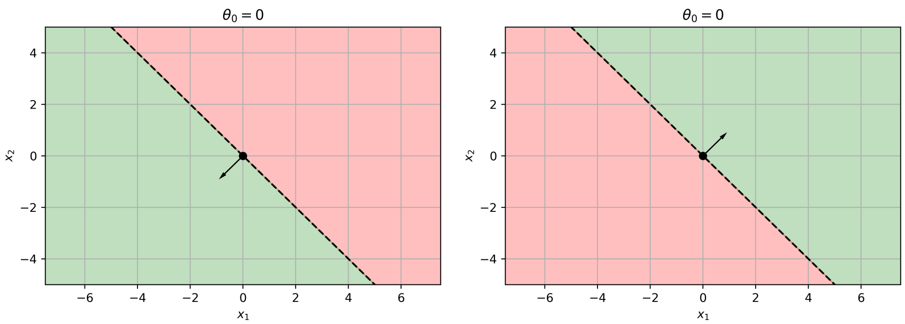
$$\theta^T x + \theta_0 = 0 \iff \theta^T x = -\theta_0 \tag{2}$$

We'll break the effects of θ_0 into three cases:

Note: the below statements are true no matter what θ we choose!

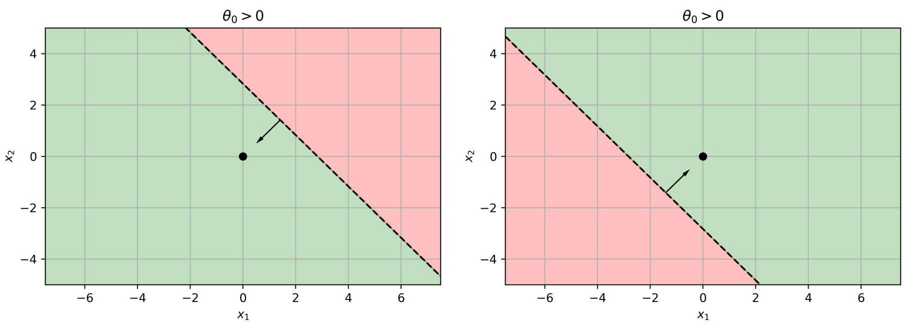
For each, we'll show two different θ values.

- If $\theta_0 = 0$, then $x = (0, 0)$ is **on the line**.
  - Without an **offset**, our line goes through the **origin**.



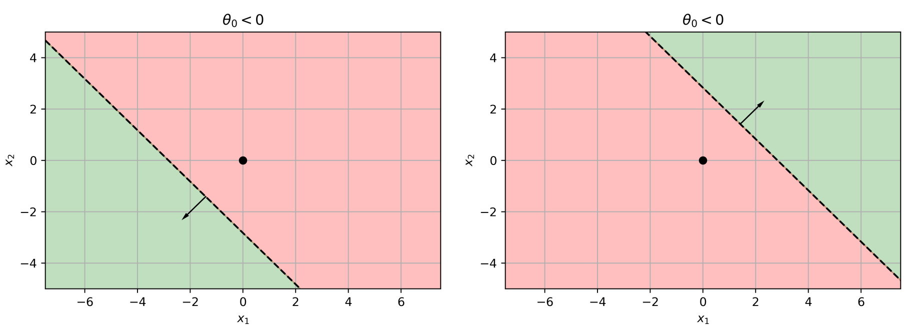
The boundary is on the origin.

- If $\theta_0 > 0$, then $x = (0, 0)$ is in the **positive** region.
  - That means the positive region is **larger**: the line must have moved in the $-\theta$ direction.



If we have a **positive** constant, it's "easier" to get a positive **result**: more positive space.

- If $\theta_0 < 0$, then $x = (0, 0)$ is in the **negative** region.
  - That means the positive region is **smaller**: the line must have moved in the $+\theta$ direction.



If we have a **negative** constant, it's "harder" to get a positive **result**: more negative space.

This can be a bit confusing, so we'll summarize:

Concept 3

The **sign** of our θ_0 and the **direction** we move away from the origin are **opposite**.

If $\theta_0 > 0$ (positive), our boundary moves in the $-\theta$ **direction**.

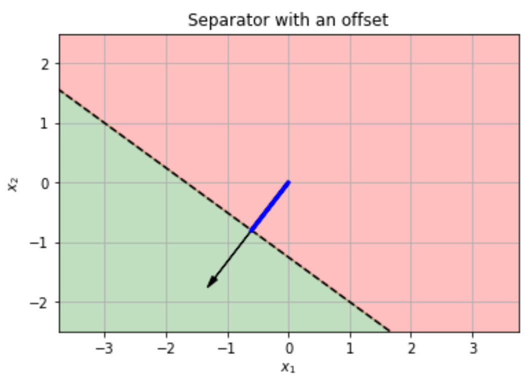
If $\theta_0 < 0$ (negative), our boundary moves in the $+\theta$ **direction**.

This gives us a general idea of how the offset affects it, but what is the **exact** effect of θ_0 on the line?

We'll focus on one point on the line: the **closest point to the origin** We want to look at this **point** because it's **unique**.

Points that aren't unique are hard to keep track of!

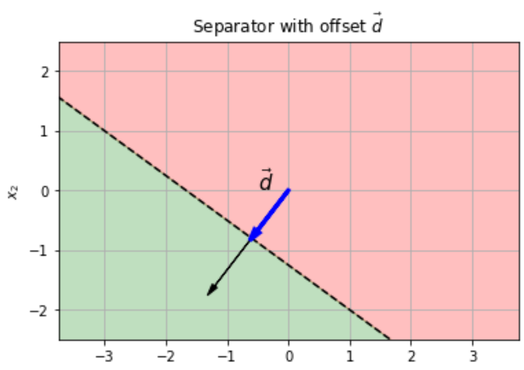
Distance from the Origin to the Plane



Notice that the **shortest** path from the origin to the line is **parallel** to θ!

So, we can think of our **line** as having been **pushed** in the θ direction. This **matches** what we did for 1-D separators: $x_1 > 3$ was moved in the x_1 direction.

So, we'll take the closest point on the line, $\vec{d}$. The **magnitude** d will give us the **distance** that the separator has been **shifted**.



Since $\vec{d}$ is in the direction of θ, the direction can be captured by the unit vector $\hat{\theta}$. Let's take a look at that:

Remember, a vector is direction (unit vector) times magnitude (scalar).

$$\theta = \|\theta\|\hat{\theta} \tag{3}$$

$$\vec{d} = d\hat{\theta} \tag{4}$$

They're in the same **direction**, so they have the same **unit vector** $\hat{\theta}$.

$\vec{d}$ is on the **line**, so it satisfies:

We'll use $\theta \cdot \vec{d}$ instead of $\theta^T\vec{d}$ here.

$$\theta \cdot \vec{d} + \theta_0 = 0 \tag{5}$$

We can plug our equations 4.8 and 4.9, where we've separated magnitude from unit vector:

$$\overbrace{\left(\|\theta\|\hat{\theta}\right)}^{\theta} \cdot \overbrace{\left(d\hat{\theta}\right)}^{\vec{d}} + \theta_0 \tag{6}$$

We can move the scalars $\|\theta\|$ and d out of the way of the dot product:

$$\|\theta\|d\left(\hat{\theta} \cdot \hat{\theta}\right) + \theta_0 \tag{7}$$

We know that $\hat{u} \cdot \hat{u} = 1$:

$$\|\theta\|d + \theta_0 = 0 \tag{8}$$

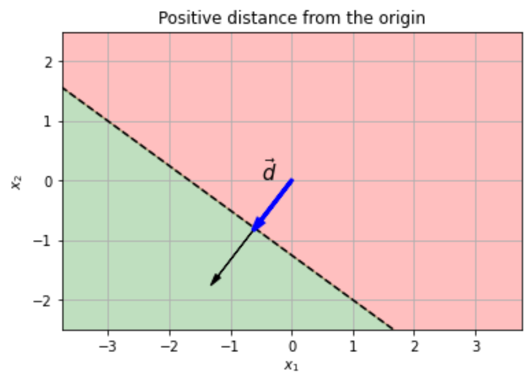
And now, we just solve for d:

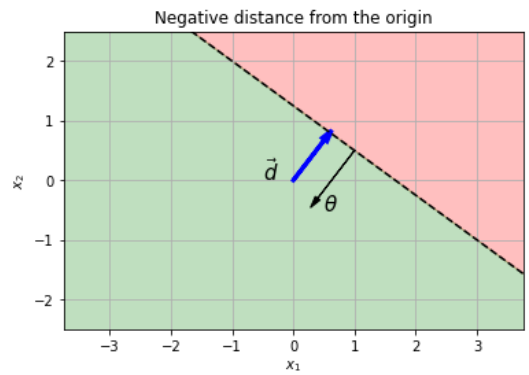
Concept 4

The **distance** d from the **origin** to our **linear separator** is

$$d = \frac{-\theta_0}{\|\theta\|} \tag{9}$$

A "negative" distance means $\vec{d}$ (the vector from the origin to the line) is pointed in the opposite direction of θ.





Notice, again, that this agrees with our **earlier** thought: the sign of θ_0 is the opposite (-1) of the θ direction we move in.