Explanatory Notes for 6.390

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7.X.14 Derivative: matrix/scalar

Now, we have our general form for creating derivatives.

We'll get our derivative of the form

$$\frac{\partial(Matrix)}{\partial(Scalar)} = \frac{\partial M}{\partial s}$$
 (1)

We have a matrix M in the shape $(r \times k)$ and a scalar s. Our **input** is a **scalar**, and our **output** is a **matrix**.

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1r} \\ m_{21} & m_{22} & \cdots & m_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kr} \end{bmatrix}$$
(2)

This may seem concerning: before, we divided **inputs** across **rows**, and **outputs** across **columns**. But in this case, we have **no** input axes, and **two** output axes.

Well, let's try to make this work anyway.

What did we do before, when we didn't know how to handle a **new** derivative? We compared it to **old** versions: we built our vector/vector case using the vector/scalar case and the scalar/vector case.

We did this by **compressing** one of our *vectors* into a *scalar* temporarily: this works, because we want to treat each of these objects the **same way**.

We don't know how to work with Matrix/Scalar, but what's the **closest** thing we do know? **Vector/Scalar**.

How do we accomplish that? As we saw above, a matrix is a **vector** of **vectors**. We could turn it into a **vector** of **scalars**.

Concept 1

A matrix can be thought of as a column vector of row vectors (or vice versa).

So, we can use our earlier technique and convert the row vectors into scalars.

We'll replace the **row vectors** in our matrix with **scalars**.

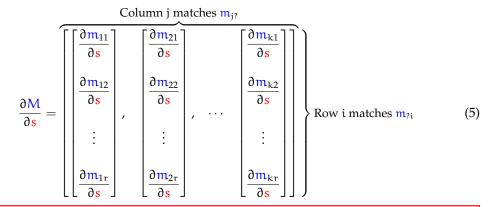
$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{bmatrix} \tag{3}$$

Now, we can pretend our matrix is a vector! We've got a derivative for that:

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$$\frac{\partial M}{\partial s} = \begin{bmatrix} \frac{\partial M_1}{\partial s} & \frac{\partial M_2}{\partial s} & \dots & \frac{\partial M_r}{\partial s} \end{bmatrix}$$
 (4)

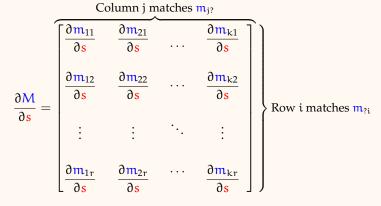
Aha - we have the same form that we did for our vector/vector derivative! Each derivative is a column vector. Let's expand it out:



Definition 2

If M is a matrix in the shape $(r \times k)$ and s is a scalar,

Then we define the **matrix derivative** $\partial M/\partial s$ as the $(k \times r)$ matrix:



This matrix has the transpose of the shape of M.

7.X.15 Derivative: scalar/matrix

We'll get our derivative of the form

$$\frac{\partial (Scalar)}{\partial (Matrix)} = \frac{\partial s}{\partial M}$$
 (6)

We have a matrix M in the shape $(r \times k)$ and a scalar s. Our **input** is a **matrix**, and our **output** is a **scalar**.

Let's do what we did last time: break it into row vectors.

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{bmatrix} \tag{7}$$

The gradient for this "vector" gives us a **column vector**:

$$\frac{\partial s}{\partial M} = \begin{bmatrix} \frac{\partial s}{\partial M_1} \\ \frac{\partial s}{\partial M_2} \\ \vdots \\ \frac{\partial s}{\partial M_k} \end{bmatrix}$$
(8)

This time, each derivative is a **row vector**. Let's **expand**:

$$\frac{\partial s}{\partial M} = \begin{bmatrix}
\frac{\partial s}{\partial m_{11}} & \frac{\partial s}{\partial m_{12}} & \cdots & \frac{\partial s}{\partial m_{1r}} \\
\frac{\partial s}{\partial m_{21}} & \frac{\partial s}{\partial m_{22}} & \cdots & \frac{\partial s}{\partial m_{2r}} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial s}{\partial m_{k1}} & \frac{\partial s}{\partial m_{k2}} & \cdots & \frac{\partial s}{\partial m_{kr}}
\end{bmatrix}$$
(9)

Definition 3

If M is a matrix in the shape $(r \times k)$ and s is a scalar,

Then we define the **matrix derivative** $\partial s/\partial M$ as the $(r \times k)$ matrix:

$$\frac{\partial s}{\partial M} =
\begin{bmatrix}
\frac{\partial s}{\partial m_{11}} & \frac{\partial s}{\partial m_{12}} & \cdots & \frac{\partial s}{\partial m_{1r}} \\
\frac{\partial s}{\partial m_{21}} & \frac{\partial s}{\partial m_{22}} & \cdots & \frac{\partial s}{\partial m_{2r}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial s}{\partial m_{2r}} & \frac{\partial s}{\partial m_{2r}} & \cdots & \frac{\partial s}{\partial m_{2r}}
\end{bmatrix}$$
Row i matches $m_{i?}$

This matrix has the same shape as M.