

Explanatory Notes for 6.390

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X.14 Derivative: matrix/scalar

Now, we have our general form for creating derivatives.

We'll get our derivative of the form

$$\frac{\partial(\text{Matrix})}{\partial(\text{Scalar})} = \frac{\partial \mathbf{M}}{\partial \mathbf{s}} \quad (1)$$

We have a matrix \mathbf{M} in the shape $(r \times k)$ and a scalar \mathbf{s} . Our **input** is a **scalar**, and our **output** is a **matrix**.

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1r} \\ m_{21} & m_{22} & \cdots & m_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kr} \end{bmatrix} \quad (2)$$

This may seem concerning: before, we divided **inputs** across **rows**, and **outputs** across **columns**. But in this case, we have **no** input axes, and **two** output axes.

Well, let's try to make this work anyway.

What did we do before, when we didn't know how to handle a **new** derivative? We compared it to **old** versions: we built our vector/vector case using the vector/scalar case and the scalar/vector case.

We did this by **compressing** one of our *vectors* into a *scalar* temporarily: this works, because we want to treat each of these objects the **same way**.

We don't know how to work with Matrix/Scalar, but what's the **closest** thing we do know? **Vector/Scalar**.

How do we accomplish that? As we saw above, a matrix is a **vector** of **vectors**. We could turn it into a **vector** of **scalars**.

Concept 1

A **matrix** can be thought of as a **column vector** of **row vectors** (or vice versa).

So, we can use our earlier technique and convert the **row vectors** into **scalars**.

We'll replace the **row vectors** in our matrix with **scalars**.

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \vdots \\ \mathbf{M}_k \end{bmatrix} \quad (3)$$

Now, we can pretend our matrix is a vector! We've got a derivative for that:

$$\frac{\partial \mathbf{M}}{\partial \mathbf{s}} = \begin{bmatrix} \frac{\partial \mathbf{M}_1}{\partial \mathbf{s}} & \frac{\partial \mathbf{M}_2}{\partial \mathbf{s}} & \cdots & \frac{\partial \mathbf{M}_r}{\partial \mathbf{s}} \end{bmatrix} \quad (4)$$

Aha - we have the same form that we did for our vector/vector derivative! Each derivative is a column vector. Let's expand it out:

$$\frac{\partial \mathbf{M}}{\partial \mathbf{s}} = \left[\begin{array}{c|c|c|c} \left[\begin{array}{c} \frac{\partial \mathbf{m}_{11}}{\partial \mathbf{s}} \\ \frac{\partial \mathbf{m}_{12}}{\partial \mathbf{s}} \\ \vdots \\ \frac{\partial \mathbf{m}_{1r}}{\partial \mathbf{s}} \end{array} \right] & \left[\begin{array}{c} \frac{\partial \mathbf{m}_{21}}{\partial \mathbf{s}} \\ \frac{\partial \mathbf{m}_{22}}{\partial \mathbf{s}} \\ \vdots \\ \frac{\partial \mathbf{m}_{2r}}{\partial \mathbf{s}} \end{array} \right] & \cdots & \left[\begin{array}{c} \frac{\partial \mathbf{m}_{k1}}{\partial \mathbf{s}} \\ \frac{\partial \mathbf{m}_{k2}}{\partial \mathbf{s}} \\ \vdots \\ \frac{\partial \mathbf{m}_{kr}}{\partial \mathbf{s}} \end{array} \right] \\ \hline \end{array} \right] \quad \begin{array}{l} \text{Column } j \text{ matches } \mathbf{m}_{j?} \\ \text{Row } i \text{ matches } \mathbf{m}_{?i} \end{array} \quad (5)$$

Definition 2

If \mathbf{M} is a matrix in the shape $(r \times k)$ and \mathbf{s} is a scalar,

Then we define the **matrix derivative** $\partial \mathbf{M} / \partial \mathbf{s}$ as the $(k \times r)$ matrix:

$$\frac{\partial \mathbf{M}}{\partial \mathbf{s}} = \left[\begin{array}{c|c|c|c} \frac{\partial \mathbf{m}_{11}}{\partial \mathbf{s}} & \frac{\partial \mathbf{m}_{21}}{\partial \mathbf{s}} & \cdots & \frac{\partial \mathbf{m}_{k1}}{\partial \mathbf{s}} \\ \hline \frac{\partial \mathbf{m}_{12}}{\partial \mathbf{s}} & \frac{\partial \mathbf{m}_{22}}{\partial \mathbf{s}} & \cdots & \frac{\partial \mathbf{m}_{k2}}{\partial \mathbf{s}} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \frac{\partial \mathbf{m}_{1r}}{\partial \mathbf{s}} & \frac{\partial \mathbf{m}_{2r}}{\partial \mathbf{s}} & \cdots & \frac{\partial \mathbf{m}_{kr}}{\partial \mathbf{s}} \end{array} \right] \quad \begin{array}{l} \text{Column } j \text{ matches } \mathbf{m}_{j?} \\ \text{Row } i \text{ matches } \mathbf{m}_{?i} \end{array}$$

This matrix has the transpose of the shape of \mathbf{M} .

X.15 Derivative: scalar/matrix

We'll get our derivative of the form

$$\frac{\partial(\text{Scalar})}{\partial(\text{Matrix})} = \frac{\partial \mathbf{s}}{\partial \mathbf{M}} \quad (6)$$

We have a matrix \mathbf{M} in the shape $(r \times k)$ and a scalar \mathbf{s} . Our **input** is a **matrix**, and our **output** is a **scalar**.

Let's do what we did last time: break it into **row vectors**.

$$\mathbf{M} = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{bmatrix} \quad (7)$$

The gradient for this "vector" gives us a **column vector**:

$$\frac{\partial \mathbf{s}}{\partial \mathbf{M}} = \begin{bmatrix} \frac{\partial s}{\partial M_1} \\ \frac{\partial s}{\partial M_2} \\ \vdots \\ \frac{\partial s}{\partial M_k} \end{bmatrix} \quad (8)$$

This time, each derivative is a **row vector**. Let's **expand**:

$$\frac{\partial \mathbf{s}}{\partial \mathbf{M}} = \begin{bmatrix} \left[\frac{\partial s}{\partial m_{11}} & \frac{\partial s}{\partial m_{12}} & \cdots & \frac{\partial s}{\partial m_{1r}} \right] \\ \left[\frac{\partial s}{\partial m_{21}} & \frac{\partial s}{\partial m_{22}} & \cdots & \frac{\partial s}{\partial m_{2r}} \right] \\ \vdots \\ \left[\frac{\partial s}{\partial m_{k1}} & \frac{\partial s}{\partial m_{k2}} & \cdots & \frac{\partial s}{\partial m_{kr}} \right] \end{bmatrix} \quad (9)$$

Definition 3

If \mathbf{M} is a matrix in the shape $(r \times k)$ and s is a scalar,

Then we define the **matrix derivative** $\partial s / \partial \mathbf{M}$ as the $(r \times k)$ matrix:

$$\frac{\partial s}{\partial \mathbf{M}} = \begin{matrix} & \text{Column } j \text{ matches } \mathbf{m}_{\cdot j} \\ \left[\begin{array}{cccc} \frac{\partial s}{\partial m_{11}} & \frac{\partial s}{\partial m_{12}} & \cdots & \frac{\partial s}{\partial m_{1r}} \\ \frac{\partial s}{\partial m_{21}} & \frac{\partial s}{\partial m_{22}} & \cdots & \frac{\partial s}{\partial m_{2r}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial s}{\partial m_{k1}} & \frac{\partial s}{\partial m_{k2}} & \cdots & \frac{\partial s}{\partial m_{kr}} \end{array} \right] & \left. \vphantom{\begin{array}{c} \frac{\partial s}{\partial m_{11}} \\ \frac{\partial s}{\partial m_{21}} \\ \vdots \\ \frac{\partial s}{\partial m_{k1}} \end{array}} \right\} \text{Row } i \text{ matches } \mathbf{m}_{i \cdot} \end{matrix}$$

This matrix has the same shape as \mathbf{M} .