Project Report

Comparison of the Euler-Bernoulli beam theory with the Theory of Elastica

ES 221: Mechanics of Solids

Deepak Soni
22110068
Mechanical Engineering

Dharavath Mahesh
22110073
Mechanical Engineering

Shaury Kumar Patel
22110241
Mechanical Engineering

Piyush Singh
22110253
Mechanical Engineering

Sridhar Singh
22110257
Mechanical Engineering

Course Instructor:
Prof. Harmeet Singh

April 24, 2024

1 Objective

This report aims to compare the deflection shapes and stress distributions predicted by the Theory of Elastica and the Euler-Bernoulli Theorem for a cantilever beam subjected to a point load. By examining the distinct assumptions of each theory, we can understand how they influence beam shapes and stresses.

2 Problem Statement

The problem statement involves a detailed comparison of deflection shapes and stress distributions in cantilever beams as predicted by two fundamental theories in solid mechanics: the Theory of Elastica and the Euler-Bernoulli Theorem. The Theory of Elastica, considers large deflections and often yields different predictions compared to the Euler-Bernoulli beam theory, which assumes small deflections. Additionally, the report will compute and compare stress distributions predicted by the two theories, offering insights into their accuracy and limitations in capturing the complex behavior of cantilever beams under loading.

3 Methodology

3.1 Euler-Bernoulli Theory

$$EI(\frac{\partial^2 y}{\partial x^2}) = M \tag{1}$$

Where;

EI is Flexural rigidity
y is vertical displacement
M is Bending Moment

- **Stress Analysis**
  For this cantilever beam stress is defined as,

  $$\sigma = \frac{yM_b}{I}$$

  where,

  - σ represents the stress at the point
  - M_b is the bending moment at that point,
  - y is the perpendicular distance from the neutral axis to the point
  - I is the moment of inertia of the beam's cross-sectional area about the neutral axis.Stress in the theory of elastica is defined as, $\sigma = \frac{yM_b}{I}$

Calculating stress for P = 3N, EI = 1, Length = 1m, the stress comes out to be,

$$\sigma = \frac{0.0025 \times 3 \times 1}{0.25 \times 10^{-9}}$$

This gives us $\sigma_1 = 3 \times 10^7$ Pa

- **Deflection**
  Using Singularity function for this cantilever beam we get,

  $$M_b = -PL < x{-}0 >^0 + P < x{-}0 >^1$$

  Considering x will be always greater than 0, this equation will be simplified to

  $$M_b = P(x - L)$$

  Therefore,

  $$EI(\frac{d^2y}{dx^2}) = P(x - L)$$

  From this we get,

  $$y = \frac{-PLx^2}{2} + \frac{Px^3}{6}$$

  where, P is the Point Load applied at the end of beam

Figure 1: Cantilever beam under action of point load at its end.

3.2 Theory of Elastica

$$\theta''(s) + \frac{P}{EI} sin(\theta(s) + \alpha) = 0$$

- **Deflection**
  For the cantilever beam boundary conditions for the above equation are defined as,

$$\theta(0) = 0$$

$$\theta'(1) = 0$$

- **Stress Analysis**
  - Stress in the theory of elastica is defined as, $\sigma = -Ey\frac{d\theta}{ds}$
  - Where,
    E is Youngs Modulous,
    y is the distance from the centroid,

  - E=4Gpa,
  - The calculated value of $\frac{d\theta}{ds}$ is -0.30284 and distance from centroid is 0.25 mm
    Therefore σ_1=30.2 $\times 10^7$ Pa
    Note: Calculation of $\frac{d\theta}{ds}$ was done with the help of Python to calculate stress.

4 Numerical Implementation

- In this code we have find the deflection of cantilever beam under point load P, applied at the end of beam. The deflection is computed using the Euler-Bernoulli theory.

```
# Function to compute deflection of cantilever beam (Euler-
                                    Bernoulli)
def deflection_euler_bernoulli(load_force, beam_length):
    EI = 1  # Flexural rigidity
    load_position = beam_length  # Force at the end of the beam

    x = np.linspace(0, beam_length, 100)
    deflection = np.zeros_like(x)
    for i, xi in enumerate(x):
      deflection[i] = (load_force * xi**2) / (6 * EI) * (-3 *
                                         load_position + xi)

    return x, deflection
```

- In the below code code we are solving non linear differential equation derived from theory of elastica using python libraries. The point load is denoted by P and s is varied from 0 to length of the beam .

```
    # Assigning values to parameters
    P = -3
    alpha = -np.pi / 2

    # Solving the differential equations
    def equations(s, vars):
        x, y, theta = vars
        dxds = np.cos(theta)
        dyds = np.sin(theta)
        d2thetads2 = -P/EI * (np.sin(theta + alpha))
```

```
    return [dxds, dyds, d2thetads2]

# Initial conditions
initial_conditions = [0, 0, 0]

# Values of s from 0 to 1
s_values = np.linspace(0, beam_length, 100)

# Solve the differential equations with boundary conditions
solution = solve_ivp(equations, [0, 1], initial_conditions,
                                  t_eval=s_values)

# Extract the solution
xSol = solution.y[0]
ySol = solution.y[1]
thetaSol = solution.y[2]
```

5 Results and Discussions

- Experimental setup: A beam-like structure was created using MDF sheet with a length of 80 cm height of 0.5 cm and width of 2.4 cm
- A point load with a magnitude of 0.61803 N was applied at one end of the beam.
- Experimental deflection: The experimental deflection observed was 9.2cm cm. The Euler-Bernoulli theory deflection prediction came out to be 10.55 cm while the Theory of Elastica deflection prediction came out to be 19.02 cm.

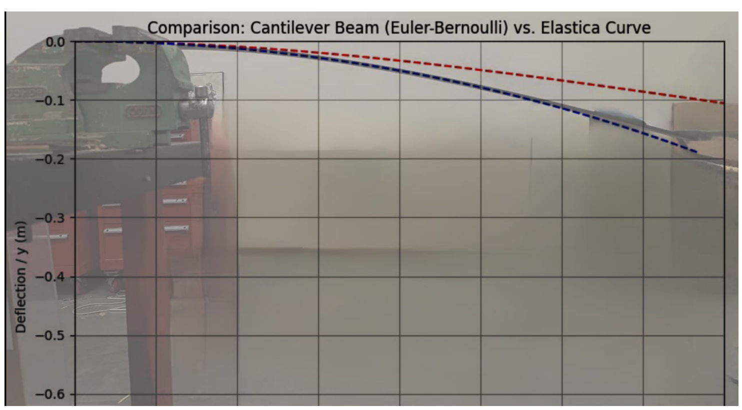
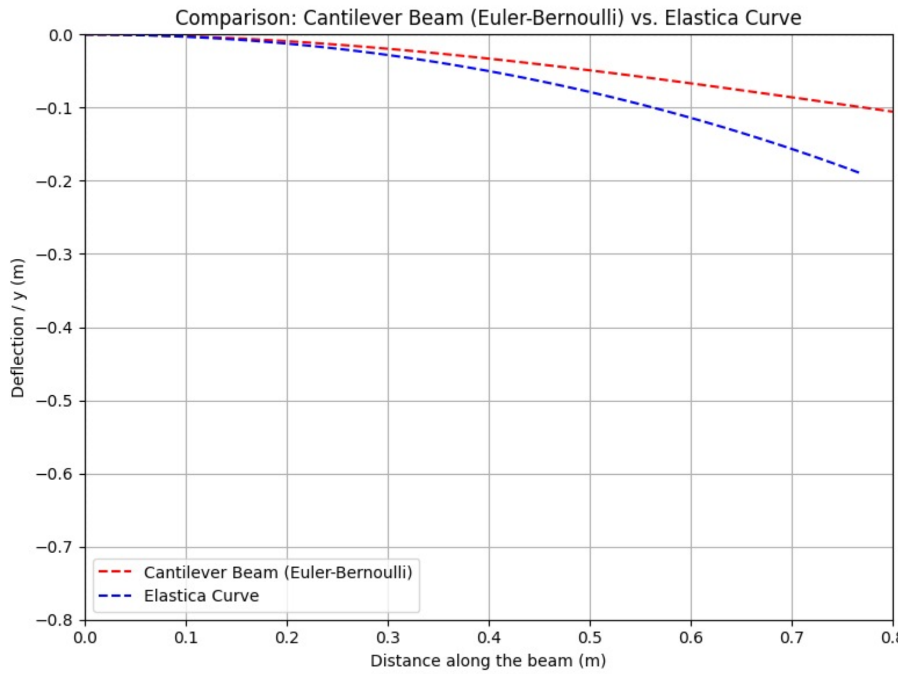


Figure 2: Experimental Deflection vs Theoretical Deflection

- Subsequently, we superimposed the plots of the theoretical and experimental deflections as shown above. This analysis revealed that the deflected shape exhibited the best overlap with the theory of the elastica curve



6 Learning Outcomes

- Understanding fundamental beam theories: Theory of Elastica and Euler-Bernoulli Theorem and explored the variation in deflected shapes using two different theories
- Applying Euler-Bernoulli Theory for computing small deflections, utilizing Python code and libraries like 'RK45' and 'solve_ivp'.
- Exploring Theory of Elastica, its conditions, and practical applications..

Bibliography

[1] S. H. Crandall, N. C. Dahl, and T. J. Lardner, An Introduction to the Mechanics of Solids, 3rd ed., McGraw-Hill,Âă2012.

[2] Levien, R. (2008). The elastica: a mathematical history. EECS Department, University of California, Berkeley. Retrieved from http://www2.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-103.html http://www2.eecs.berkeley.edu/Pubs/TechRpts/2008/EECS-2008-103.html

[3] https://colab.research.google.com/drive/1-jA97HsoWEoZRtwiJuzsbCPx2tK7MCvn?usp=sharing