Seminar on Algebraic Geometry at Bois-Marie

Étale Cohomology

(SGA $4\frac{1}{2}$)

by P. Deligne

in collaboration with J.F. Boutot,
A. Grothendieck, L. Illusie, and J.L. Verdier

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Typesetters note

This is a LaTeX rendition of Deligne's *Cohomologie Étale*, Lecture Notes in Mathematics, 569, Springer-Verlag. The typesetting was done by Daniel Miller. The source code may be found at GitHub (`https://www.github.com/dkmiller/sga4.5`). It is an essentially unmodified version of the original. Most changes are slight, e.g. using script instead of roman letters for sheaves. The biggest is that $\mathscr{H}om$ is used to denote "sheaf hom," and $\mathsf{R}\mathscr{H}om$ its derived functor, while $\mathsf{R}\hom$ is used to denote the derived functor of the regular hom-functor. Any comments or corrections should be sent to `dkmiller@math.cornell.edu`.

Translation note

This is a literal English translation of SGA $4\frac{1}{2}$ by Hao Xiao, which was possible via a Python program connecting to the OpenAI API. See OpenAI Examples (`https://platform.openai.com/examples`) for details. Thanks to the power of artificial intelligence nowadays, the translation program directly deals with source codes and keeps almost all LaTeX commands unchanged. This preliminary version surely contains many typos, and it will be updated frequently in the future.

The translation process took less than three hours without much human intervention. Currently, the OpenAI package in Python cannot translate an entire book. One has to split the book into many small fragments. OpenAI GPT somehow deleted `\end{...}` several times during the translation. It won't change even if one additionally requires OpenAI GPT not to delete this specific command. This happens probably because there is no corresponding `\begin{...}` in a small book fragment and OpenAI GPT identifies `\end{...}` as an unnecessary command. Consequently, one has to manually fix this issue.

In case of comments and corrections, one is welcomed to contact `xiao@uni-bonn.de`.

Introduction

The purpose of this volume is to facilitate the use of ℓ-adic cohomology for the non-expert. I hope it will often allow him to avoid the need for the dense exposition of SGA 4 and SGA 5. It also contains some new results.

The first exposure, written by J.F. Boutot, overviews SGA 4. It gives the main results - with a minimum of generality, often insufficient for applications - and an idea of their demonstration. For complete results, or detailed demonstrations, SGA 4 remains essential.

The "II" contains a completed demonstration of the trace formula for the Frobenius endomorphism. The demonstration is that given by Grothendieck in SGA 5, pruned of any unnecessary detail. This report should allow the user to forget SGA 5, which can be considered as a series of digressions, some of them very interesting. Its existence will allow the immediate publication of SGA 5 as is. It is completed by the exposure "VI" which explains how the trace formula allows the study of trigonometric sums, and gives examples.

The intended audience for the other lectures is more limited, and this is reflected in their style. The lecture "III" is a "modular" generalization of the Report, based on the study SGA 4 XVII 5.5 of symmetric powers. The lecture "IV" defines this class in various contexts, and gives compatibility between intersections and cup products. In "V" are gathered some known results, for which was lacking a reference, and some compatibility. The lecture "VII" is new. It gives, among other things, in cohomology without support, finiteness theorems analogous to those known in cohomology with compact support.

For more details on the lectures, I refer to their respective introductions.

Finally, I thank J.L. Verdier to have allowed me to reproduce here his notes "VIII". They remain, I think, very useful, and were: become unobtainable.

In the internal references to this volume, the lectures are cited by an abbreviated title, indicated between [ ] in the table of contents.

Bures-sur-Yvette, le 20 Septembre 1976

Pierre Deligne

Contents

Chapter 1

Etale Cohomology: The Starting Points

1 Grothendieck Topologies

At the beginning, Grothendieck topologies appeared as underlying to his theory of descent (cf. SGA 1 VI, VIII); the use of the corresponding cohomology theories is more recent. The same approach is followed here: by formalizing the classical notions of localization, of local property and of gluing (1.1, 1.2, 1.3), we derive the general concept of Grothendieck topology (1.6); to justify its introduction in algebraic geometry, we prove a theorem of faithfully flat descent (1.4), generalization of the classical Hilbert theorem (1.5).

The reader will find a more complete, but concise, exposition of the formalism in Giraud [18]. M. Artin's notes: "Grothendieck topologies" [1] (chapters I to III) are also useful. The 866 pages of SGA 4's exposés I to V are precious when considering exotic topologies, such as the one giving birth to crystalline cohomology; to use the étale topology so close to the classical intuition, it is not necessary to read them.

1.1 Cribles

Let X be a topological space and $f : X \to \mathbb{R}$ be a function with real values on X. The continuity of f is a local property; that is, if f is continuous on every open set small enough of X, f is continuous on the whole of X. To formalize the notion of "local property," we will introduce some definitions.

We say that a set $\mathcal{U}$ of open sets of X is a *crible* if for every $U \in \mathcal{U}$ and $V \subset U$, we have $V \in \mathcal{U}$. We say that a crible is *covering* if the union of all open sets belonging to this crible is equal to X.

Given a family $\{U_i\}$ of open sets of X, the crible generated by $\{U_i\}$ is by definition the set of open sets U of X such that U is contained in one of the U_i.

by P. Deligne, written by J.F. Boutot

We say that a property $P(U)$, defined for every open set U of X, is *local* if, for every covering crible $\mathcal{U}$ of every open set U of X, $P(U)$ is true if and only if $P(V)$ is true for every $V \in \mathcal{U}$. For example, given $f : X \to \mathbb{R}$, the property "f is continuous on U" is local.

1.2 Sheaves

Let us specify the notion of a function given locally on X.

1.2.1 The sieve point of view

Let $\mathcal{U}$ be a sieve of open sets of X. We call a function given $\mathcal{U}$-locally on X the data for each $U \in \mathcal{U}$ of a function f_U on U such that, if $V \subset U$, we have $f_V = f_U|V$.

1.2.2 The Čech point of view

If the sieve $\mathcal{U}$ is generated by a family of open sets U_i of X, giving a function $\mathcal{U}$-locally is equivalent to giving a function f_i on each U_i, such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$.

In other words, if $Z = \coprod U_i$, giving a function $\mathcal{U}$-locally is equivalent to giving a function on Z which is constant on the fibers of the natural projection $Z \to X$.

1.2.3

Continuous functions form a sheaf; this means that for any crible covering $\mathcal{U}$ of an open V of X and any given function $\mathcal{U}$-locally $\{f_U\}$ such that each f_U is continuous on U, there exists a unique continuous function f on V such that $f|U = f_U$ for all $U \in \mathcal{U}$.

1.3 Fields

Let us now specify the notion of a vector bundle given locally on X.

1.3.1 The point of view of the covers

Let $\mathcal{U}$ be a cover of open sets of X. We call a vector bundle given $\mathcal{U}$-locally on X the data of

a) a vector bundle E_U on each $U \in \mathcal{U}$,

b) if $V \subset U$, an isomorphism $\rho_{U,V} : E_V \xrightarrow{\sim} E_U|V$, satisfying

c) if $W \subset V \subset U$, the diagram

$$\begin{tikzcd} E_W \arrow[r, "\rho_{U,W}"] \arrow[dr, "\rho_{V,W}"'] & E_U|W \\ & E_V|W \arrow[u, "\rho_{U,V}|W"'] \end{tikzcd}$$

commutes, that is $\rho_{U,V} = (\rho_{U,V}$ restricted to $W) \circ \rho_{V,W}$.

1.3.2 Cech's perspective

If the sieve $\mathcal{U}$ is generated by a family of open sets U_i of X, giving a $\mathcal{U}$-local vector bundle is equivalent to giving:

a) a vector bundle E_i on each U_i,

b) if $U_{ij} = U_i \cap U_j = U_i \times_X U_j$, an isomorphism $\rho_{ji} : E_i|U_{ij} \xrightarrow{\sim} E_j|U_{ij}$, such that

c) if $U_{ijk} = U_i \times_X U_j \times_X U_k$, the diagram

$$\begin{tikzcd} E_i|U_{ijk} \arrow[r, "\rho_{ki}|U_{ijk}"] \arrow[dr, "\rho_{ji}|U_{ijk}"'] & E_k|U_{ijk} \\ & E_j|U_{ijk} \arrow[u, "\rho_{kj}|U_{ijk}"'] \end{tikzcd}$$

commutes, that is, $\rho_{ki} = \rho_{kj} \circ \rho_{ji}$ on U_{ijk}.

In other words, if $Z = \coprod U_i$ and $\pi : Z \to X$ is the natural projection, giving a locally $\mathcal{U}$-vector bundle is equivalent to giving:

a) a vector bundle E on Z,

b) if x and y are two points of Z such that $\pi(x) = \pi(y)$, an isomorphism $\rho_{yx} : E_x \xrightarrow{\sim} E_y$ between the fibers of E at x and y, depending continuously on (x, y) and such that,

c) if x, y and z are three points of Z such that $\pi(x) = \pi(y) = \pi(z)$, we have $\rho_{zx} = \rho_{zy} \circ \rho_{yx}$.

1.3.3

A vector bundle E on X defines a locally $\mathcal{U}$-given vector bundle $E_{\mathcal{U}}$; the system of restrictions E_U of E to the objects of $\mathcal{U}$. The fact that the notion of vector bundle is of a local nature can be expressed as follows: for any covering of $\mathcal{U}$ of X, the functor $E \mapsto E_{\mathcal{U}}$, from vector bundles on X to locally $\mathcal{U}$-given vector bundles, is an equivalence of categories.

1.3.4

If in 1 we replace "open of X" with "part of X," we obtain the notion of a space sieve. In this context there are also gluing theorems. For example: let X be a space and $\mathcal{C}$ a space sieve of X generated by a closed locally finite covering of X, then the functor $E \mapsto E_{\mathcal{C}}$, of vector bundles on X to the vector bundles given $\mathcal{C}$-locally is an equivalence of categories.

In algebraic geometry, it is also useful to consider "sieves of spaces over X"; this is what we will see in the next paragraph.

1.4 Faithfully flat descent

1.4.1

In the context of schemes, the Zariski topology is not fine enough for the study of nonlinear problems and we are led to replace in the previous definitions the open immersions with more general morphisms. From this point of view, descent techniques appear as localization techniques. Thus the following descent statement can be expressed by saying that the properties considered are of a local nature for the faithfully flat topology (we say that a morphism of schemes is faithfully flat if it is flat and surjective).

Proposition 1.4.2. *Let A be a ring and B a faithfully flat A-algebra. Then:*

(i) A sequence $\Sigma = (M' \to M \to M'')$ of A-modules is exact whenever the sequence $\Sigma_{(B)}$ obtained by extension of scalars to B is exact.

(ii) An A-module M is of finite type (resp. of finite presentation, flat, locally free of finite rank, invertible (i.e. locally free of rank one)) whenever the B-module $M_{(B)}$ is.

Proof. (i) The functor $M \mapsto M_{(B)}$ being exact (flatness of B), it is enough to show that, if an A-module N is nonzero, $N_{(B)}$ is nonzero. If N is nonzero, N contains a nonzero monogenic submodule $A/\mathfrak{a}$; then $N_{(B)}$ contains a monogenic submodule $(A/\mathfrak{a})_{(B)} = B/\mathfrak{a}B$, nonzero by surjectivity of the structural morphism $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (if $V(\mathfrak{a})$ is nonempty, $\varphi^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}B)$ is nonempty).

(ii) For any family (x_i) of elements of $M_{(B)}$, there exists a finite-type submodule M' of M such that $M'_{(B)}$ contains the x_i. If $M_{(B)}$ is of finite type and if the x_i generate $M_{(B)}$, then $M'_{(B)} = M_{(B)}$, so $M' = M$ and M is of finite type. □

If $M_{(B)}$ is of finite presentation, we can, according to what precedes, find a surjection $A^n \to M$. If N is the kernel of this surjection, the B-module $N_{(B)}$ is of finite type, so N is as well, and M is of finite presentation. The assertion for “flat” follows immediately from (i); “locally free of finite rank” means “flat and of finite presentation” and the rank is tested by extension of scalars to fields.

1.4.3

Let X be a scheme and $\mathcal{S}$ a class of X-schemes stable by fiber product over X. A class $\mathcal{U} \subset \mathcal{S}$ is a *sieve* on X (relative to $\mathcal{S}$) if, for any morphism $\varphi : V \to U$ of X-schemes, with $U, V \in \mathcal{S}$ and $U \in \mathcal{U}$, we have $V \in \mathcal{U}$. The sieve *generated* by a family $\{U_i\}$ of X-schemes in $\mathcal{S}$ is the class of $V \in \mathcal{S}$ such that there exists a morphism of X-schemes from V to one of the U_i.

1.4.4

Let $\mathcal{U}$ be a covering on X. We call a quasi-coherent module given $\mathcal{U}$-locally on X the data of

a) a quasi-coherent module E_U on each $U \in \mathcal{U}$,

b) for each $U \in \mathcal{U}$ and for each morphism $\varphi : V \to U$ of X-schemas in $\mathcal{S}$, an isomorphism $\rho_\varphi : E_V \xrightarrow{\sim} \varphi^* E_U$, these being such that

c) if $\psi : W \to V$ is a morphism of X-schemas in $\mathcal{S}$, the diagram

$$\begin{tikzcd} E_W \arrow[r, "\rho_{\varphi\circ\psi}"] \arrow[dr, "\rho_\psi"'] & \psi^*\varphi^* E_U \\ & \psi^* E_V \arrow[u, "\psi^*\rho_\varphi"'] \end{tikzcd}$$

commutes, that is $\rho_{\varphi\circ\psi} = (\psi^*\rho_\varphi) \circ \rho_\psi$.

If E is a quasi-coherent module on X, we denote by $E_{\mathcal{U}}$ the module given $\mathcal{U}$-locally equal to $\varphi_U^* E$ on $\varphi : U \to X$ and such that, for any morphism $\psi : V \to U$ the restriction isomorphism ρ_ψ is the canonical isomorphism $E_V = (\varphi_U \circ \psi)^* E \xrightarrow{\sim} \psi^* \varphi_U^* E = \psi^* E_U$.

Théorème 1.4.5. *Let $\{U_i\} \in \mathcal{S}$ be a finite family of flat X-schemes such that X is the union of the images of the U_i, and let $\mathcal{U}$ be the sieve generated by $\{U_i\}$. Then the functor $E \mapsto E_{\mathcal{U}}$ is an equivalence of the category of quasi-coherent modules on X with the category of quasi-coherent modules given $\mathcal{U}$-locally.*

Proof. We will only treat the case where x is affine and where $\mathcal{U}$ is generated by an affine X-scheme U faithfully flat over X. The reduction to this case is formal. We set $X = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$.

If the morphism $U \to X$ has a section, X belongs to the filter $\mathcal{U}$ and the assertion is evident. We will reduce to this case.

Given a quasi-coherent module $\mathcal{U}$-locally defines modules M', M'' and M''' on U, $U \times_X U$ and $U \times_X U \times_X U$, and isomorphisms $\rho : p^* M^\bullet \simeq M^\bullet$ for any projection morphism p between these spaces; this is a *cartesian diagram*

$$M^* : M' \rightrightarrows M'' \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; M'''$$

over

$$U_* : U \leftleftarrows U \times_X U \;\substack{\longleftarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; U \times_X U \times_X U.$$

Conversely M^* determines the given module $\mathcal{U}$-locally: for $V \in \mathcal{U}$, there is $\varphi : V \to U$ and we set $M_U = \varphi^* M'$; for $\varphi_1, \varphi_2 : V \to U$, we have a natural identification $\varphi_1^* M' \simeq (\varphi_1 \times \varphi_2)^* M'' \simeq \varphi_2^* M'$, and we see using M''' that these identifications are compatible, so that the definition is legitimate. In short, it is the same to give a module $\mathcal{U}$-locally or a cartesian diagram M^* on U_*.

Translating into algebraic terms: giving M^* is the same as giving a Cartesian diagram of modules

$$M' \;\substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}}\; M'' \;\substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}}\; M'''$$

over the diagram of rings

$$B \;\substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}}\; B \otimes_A B \;\substack{\xrightarrow{\partial_0}\\ \xrightarrow{\partial_1}\\ \xrightarrow{\partial_2}}\; B \otimes_A B \otimes_A B$$

(to be specific: we have $\partial_i(bm) = \partial_i(b) \cdot \partial_i(m)$, the usual identities such as $\partial_0 \partial_1 = \partial_0 \partial_0$ are true, and "Cartesian" means that the morphisms $\partial_i : M' \otimes_{B, \partial_i} (B \otimes_A B) \to M''$ and $M'' \otimes_{B \otimes_A B, \partial_i} (B \otimes_A B \otimes_A B) \to M'''$ are isomorphisms).

The functor $E \mapsto E_{\mathcal{U}}$ becomes the functor that, to an A-module M, associates

$$M^* = (M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; M \otimes_A B \otimes_A B \otimes_A B).$$

It has as right adjoint the functor

$$(M' \rightrightarrows M'' \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; M''') \longmapsto \ker(M' \rightrightarrows M''').$$

We need to prove that the adjunction arrows

$$M \to \ker(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B)$$

and

$$\ker(M' \rightrightarrows M'') \otimes_A B \to M'$$

are isomorphisms. According to (1.4.2.i), it is enough to prove it after a faithfully flat base change $A \to A'$ (B becoming $B' = B \otimes_A A'$). Taking $A' = B$, this reduces us to the case where $U \to X$ has a section. □

1.5 A special case: Hilbert's Theorem 90

1.5.1

Let k be a field, k' a Galois extension of k, and $G = \mathrm{Gal}(k'/k)$. Then the homomorphism

$$k' \otimes_k k' \to \oplus_{\sigma \in G} k'$$
$$x \otimes y \mapsto \{x \cdot \sigma(y)\}_{\sigma \in G}$$

is bijective.

It follows that it is equivalent to give a locally for the sieve generated by $\mathrm{Spec}(k')$ on $\mathrm{Spec}(k)$ or to give a k'-vector space with a semi-linear action of G, that is:

a) a k'-vector space V',

b) for all $\sigma \in G$, an endomorphism of the group structure of V' such that $\varphi_\sigma(\lambda v) = \sigma(\lambda)\varphi_\sigma(v)$, for all $\lambda \in k'$ and $v \in V'$, satisfying the condition

c) for all $\sigma, \tau \in G$, we have $\varphi_{\tau\sigma} = \varphi_\tau \circ \varphi_\sigma$.

Let $V = {V'}^G$ be the group of invariants under this action of G; it is a k-vector space and, according to theorem 1.4.5, we have:

Proposition 1.5.2. *The inclusion of V in V' defines an isomorphism $V \otimes_k k' \xrightarrow{\sim} V'$.*

In particular, if V' is one-dimensional and if $v' \in V$ is nonzero, φ_σ is determined by the constant $c(\sigma) \in {k'}^\times$ such that $\varphi_\sigma(v') = c(\sigma)v'$ and condition c) is written

$$c(\tau\sigma) = c(\tau) \cdot \tau(c(\sigma)).$$

According to proposition there exists an invariant nonzero vector $v = \mu v'$, $\mu \in {k'}^\times$. So for all $\sigma \in G$,

$$c(\sigma) = \mu \cdot \sigma(\mu^{-1}).$$

In other words, every 1-cocycle of G with values in ${k'}^\times$ is a cobord.

Corollaire 1.5.3. *On a* $\mathrm{H}^1(G, {k'}^\times) = 0$.

1.6 Topologies de Grothendieck

We now transcribe the definitions from the previous paragraphs in an abstract framework encompassing both the case of topological spaces and that of schemes.

1.6.1

Let $\mathcal{S}$ be a category and U an object of $\mathcal{S}$. A *covers* on U is a subset $\mathcal{U}$ of $\mathrm{Ob}(\mathcal{S}/U)$ such that if $\varphi : V \to U$ belongs to $\mathcal{U}$ and if $\psi : W \to V$ is a morphism in $\mathcal{S}$, then $\varphi \circ \psi : W \to U$ belongs to $\mathcal{U}$.

If $\{\varphi_i : U_i \to U\}$ is a family of morphisms, the covers generated by the U_i is by definition the set of morphisms $\varphi : V \to U$ which factorize through one of the φ_i.

If $\mathcal{U}$ is a covers on U and if $\varphi : V \to U$ is a morphism, the restriction $\mathcal{U}_V$ of $\mathcal{U}$ to V is by definition the covers on V consisting of the morphisms $\psi : w \to V$ such that $\varphi \circ \psi : W \to U$ belongs to $\mathcal{U}$.

1.6.2

The data of a *Grothendieck topology* on $\mathcal{S}$ consists of the data, for every object U of $\mathcal{S}$, of a set $C(U)$ of sieves on U, called covering sieves, such that the following axioms are satisfied:

a) The sieve generated by the identity of U is covering.

b) If $\mathcal{U}$ is a covering sieve on U and if $V \to U$ is a morphism, the sieve $\mathcal{U}_V$ is covering.

c) A locally covering sieve is covering. In other words, if $\mathcal{U}$ is a covering sieve on U and if $\mathcal{U}'$ is a sieve on U such that, for every $V \to U$ belonging to $\mathcal{U}$, the sieve $\mathcal{U}'_V$ is covering, then $\mathcal{U}'$ is covering.

One calls *site* the data of a category equipped with a Grothendieck topology.

1.6.3

Given a site $\mathcal{S}$, one calls a presheaf on $\mathcal{S}$ a contravariant functor $\mathscr{F}$ from $\mathcal{S}$ to the category of sets. For every object U of $\mathcal{S}$, one calls a section of $\mathscr{F}$ over U the elements of $\mathscr{F}(U)$. For every morphism $V \to U$ and for every $s \in \mathscr{F}(U)$, one denotes $s|V$ (s restricted to V) the image of s in $\mathscr{F}(V)$.

If $\mathcal{U}$ is a sieve on U, we call a section given $\mathcal{U}$-locally the given data, for every $V \to U$ belonging to $\mathcal{U}$, of a section $s_V \in \mathscr{F}(V)$ such that, for every morphism $W \to V$, we have $s_V|W = s_W$. We say that $\mathscr{F}$ is a *sheaf* if, for every object U of $\mathcal{S}$, for every sieve covering $\mathcal{U}$ on U and for every section given $\mathcal{U}$-locally $\{s_V\}$, there exists a unique section $s \in \mathscr{F}(U)$ such that $s|V = s_V$, for every $V \to U$ belonging to $\mathcal{U}$.

We define abelian sheaves similarly by replacing the category of sets with the category of abelian groups. We show that the category of abelian sheaves on $\mathcal{S}$ is an abelian category possessing enough injectives. A sequence $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ of sheaves is exact if, for every object U of $\mathcal{S}$, and for every $s \in \mathscr{G}(U)$ such that $g(s) = 0$, there exists locally t such that $f(t) = s$; i.e. if there exists a sieve covering $\mathcal{U}$ on U and for every $V \in \mathcal{U}$, a section t_V of $\mathscr{F}$ on V such that $f(t_V) = s|V$.

1.6.4 Examples

We saw two of them earlier.

a) Let X be a topological space and $\mathcal{S}$ the category whose objects are the open sets of X and the morphisms the natural inclusions. The Grothendieck topology on $\mathcal{S}$ corresponding to the usual topology of X is the one for which a filter $\mathcal{U}$ on an open set U of X is covering if the union of the open sets belonging to this filter is equal to U. It is clear that the category of sheaves on $\mathcal{S}$ is equivalent to the category of sheaves on X in the usual sense.

b) Let X be a scheme and $\mathcal{S}$ the category of schemes over X. We call the fpqc (faithfully flat and quasi-compact) topology on $\mathcal{S}$ the Grothendieck topology for which a filter on an X-scheme U is covering if it is generated by a finite family of flat morphisms whose images cover U.

1.6.5 Cohomology

We will always assume that the category $\mathcal{S}$ has a final object X. Then we call global sections of an abelian sheaf $\mathscr{F}$, and we denote $\Gamma\mathscr{F}$ or $\mathrm{H}^0(X,\mathscr{F})$, the group $\mathscr{F}(X)$. The functor $\mathscr{F} \mapsto \Gamma\mathscr{F}$ is an exact left functor from the category of abelian sheaves on $\mathcal{S}$ to the category of abelian groups, or denote $\mathrm{H}^i(X,-)$ its derived functors (or satellites). These cohomology groups represent the obstructions to going from the local to the global. By definition, if $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of abelian sheaves, we have a long exact sequence of cohomology:

$$0 \longrightarrow \mathrm{H}^0(X,\mathscr{F}) \longrightarrow \mathrm{H}^0(X,\mathscr{G}) \longrightarrow \mathrm{H}^0(X,\mathscr{H}) \longrightarrow \mathrm{H}^1(X,\mathscr{F}) \longrightarrow \cdots$$

$$\ldots \longrightarrow \mathrm{H}^n(X,\mathscr{F}) \longrightarrow \mathrm{H}^n(X,\mathscr{G}) \longrightarrow \mathrm{H}^n(X,\mathscr{H}) \longrightarrow \mathrm{H}^{n+1}(X,\mathscr{F}) \longrightarrow \cdots$$

1.6.6

Given an abelian sheaf $\mathscr{F}$ on $\mathcal{S}$, we call a sheaf $\mathscr{G}$ with an action $\mathscr{F} \times \mathscr{G} \to \mathscr{G}$ of $\mathscr{F}$ a *$\mathscr{F}$-torsor* if locally (after restricting to all objects of a covering of the final object X) $\mathscr{G}$ with the action of $\mathscr{F}$ is isomorphic to $\mathscr{F}$ with the canonical action $\mathscr{F} \times \mathscr{F} \to \mathscr{F}$ by translations.

It can be shown that $\mathrm{H}^1(X,\mathscr{F})$ is interpreted as the set of classes of $\mathscr{F}$-torsors up to isomorphism.

2 The étale topology

We specialize the definitions of the previous chapter to the case of the étale topology of a scheme X (2.1, 2.2, 2.3). The corresponding cohomology coincides in the case where X is the spectrum of a field K with Galois cohomology of K (2.4).

2.1 Etale Topology

We will begin with a few reminders on the notion of an étale morphism.

Définition 2.1.1. *Let A be a ring (commutative). We say that an A-algebra B is étale if B is a finitely presented A-algebra and if the equivalent conditions below are satisfied:*

a) For every A-algebra C and for every nilpotent ideal J of C, the canonical map

$$\hom_{A\text{-alg}}(B,C) \to \hom_{A\text{-alg}}(B,C/J)$$

is a bijection.

b) *B is a flat A-module and $\Omega_{B/A} = 0$ (we denote $\Omega_{B/A}$ the module of relative differentials).*

c) *Let $B = A[X_1, \dots, X_n]/I$ be a presentation of B. Then for every prime ideal $\mathfrak{p}$ of $A[X_1, \dots, X_n]$ containing I, there exist polynomials $P_1, \dots, P_n \in I$ such that $I_{\mathfrak{p}}$ is generated by the images of $P_1, \dots, P_n$ and* $\det(\partial P_i/\partial X_j) \notin \mathfrak{p}$.

(cf. [22, I] or [29, V])

A morphism of schemes $f : X \to S$ is said to be *étale* if for every $x \in X$ there exists an open affine neighborhood $U = \mathrm{Spec}(A)$ of $f(x)$ and an open affine neighborhood $V = \mathrm{Spec}(B)$ of x in $X \times_S U$ such that B is an étale A-algebra.

2.1.2 Examples

a) If A is a field, an A-algebra B is étale if and only if it is a finite product of separable extensions of A.

b) If X and S are schemes of finite type over $\mathbb{C}$, a morphism $f : X \to S$ is étale if and only if its analyticity $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism.

2.1.3 Sorite

a) (change of base) If $f : X \to S$ is an étale morphism, the same is true of $f_{S'} : X \times_S S' \to S'$ for any morphism $S' \to S$.

b) If $f : X \to S$ and $g : Y \to S$ are two étale morphisms, any S-morphism from X to Y is étale.

c) (descent) Let $f : X \to S$ be a morphism. If there exists a faithfully flat morphism $S' \to S$, such that $f_{S'} : X \times_S S' \to S'$ is étale, then f is étale.

2.1.4

Let X be a scheme. Let $\mathcal{S}$ be the category of X-schemes étales; according to (2.1.3.c) any morphism of $\mathcal{S}$ is an étale morphism. We call the topology étale on $\mathcal{S}$ the topology for which a crible on U is couvrant if it is generated by a finite family of morphisms $\varphi_i : U_i \to U$ such that the union of the images of the φ_i covers U. We call the étale site of X, and note X_{et}, the site defined by $\mathcal{S}$ of the étale topology.

2.2 Examples of faisceaux

2.2.1 Faisceau constant

Let C be an abelian group and assume for simplicity that X is noethérian. We will denote $\underline{C}_X$ (or even C if there is no ambiguity) the faisceau defined by $U \mapsto C^{\pi_0(U)}$, where $\pi_0(U)$ is the (finite) set of connected components of U. The most important case will be $C = \mathbb{Z}/n$. We thus have by definition

$$\mathrm{H}^0(X, \mathbb{Z}/n) = (\mathbb{Z}/n)^{\pi_0(X)}.$$

In addition, $\mathrm{H}^1(X, \mathbb{Z}/n)$ is the set of isomorphism classes of $\mathbb{Z}/n$-torseurs (1.6.6), that is, of Galois étale covers of X with group $\mathbb{Z}/n$. In particular, if X is connected and if $\pi_1(X)$ is its fundamental group for a chosen base point, we have

$$\mathrm{H}^1(X, \mathbb{Z}/n) = \mathrm{hom}(\pi_1(X), \mathbb{Z}/n).$$

2.2.2 The multiplicative group

We will denote $\mathbb{G}_{m,X}$ (or $\mathbb{G}_m$ if there is no ambiguity) the sheaf defined by $U \mapsto \Gamma(U, \mathscr{O}_U^\times)$; it is indeed a sheaf thanks to the theorem of faithfully flat descent (1.4.5). By definition

$$\mathrm{H}^0(X, \mathbb{G}_m) = \mathrm{H}^0(X, \mathscr{O}_X)^\times;$$

in particular if X is reduced, connected and proper over an algebraically closed field k, then

$$\mathrm{H}^0(X, \mathbb{G}_m) = k^\times.$$

Proposition 2.2.3. *There is an isomorphism:*

$$\mathrm{H}^1(X, \mathbb{G}_m) = \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ *is the group of classes of invertible sheaves on* X*.*

Proof. Let $*$ be the functor that, to an invertible sheaf $\mathscr{L}$ on X, associates the following presheaf on X_{et}: for $\varphi : U \to X$ étale,

$$\mathscr{L}^*(U) = \mathrm{Isom}_U(\mathscr{O}_U, \varphi^*\mathscr{L}).$$

By (1.4.2.i) and (1.4.5) (full faithfulness), this presheaf is a sheaf; it is even a $\mathbb{G}_m$-torsor. We immediately verify that

a) the functor $*$ is compatible with localization (étale);

b) it induces an equivalence of the category of trivial invertible sheaves (i.e. to $\mathscr{O}_X$) with the category of trivial $\mathbb{G}_m$-torsors: $\mathscr{L}$ is trivial if and only if $\mathscr{L}^*$ is.

In addition, according to (1.4.2.ii) and (1.4.5),

c) the concept of an invertible sheaf is local for the étale topology.

It follows formally from a), b), c) that $*$ is an equivalence between the category of invertible sheaves on X and the category of $\mathbb{G}_m$-torsors over X_{et}; it induces the desired isomorphism. The inverse equivalence is constructed as follows: if $\mathscr{T}$ is a $\mathbb{G}_m$-torsor, there exists a finite étale covering $\{U_i\}$ of X such that the torsors $\mathscr{T}/U_i$ are trivial; $\mathscr{T}$ is then trivial over each V étale over X belonging to the sieve $\mathcal{U} \subset X_{\text{et}}$ generated by $\{U_i\}$. Over each $V \in \mathcal{U}$, $\mathscr{T}|V$ corresponds to an invertible sheaf $\mathscr{L}_V$ (by b)) and the $\mathscr{L}_V$ constitute an invertible sheaf given $\mathcal{U}$-locally $\mathscr{L}_\mathcal{U}$ (by a)). By c), this latter comes from an invertible sheaf $\mathscr{L}(\mathscr{T})$ on X, and $\mathscr{T} \mapsto \mathscr{L}(\mathscr{T})$ is the inverse sought of $*$. □

2.2.4 Roots of unity

For every integer $n > 0$, we call the sheaf of n-th roots of unity the kernel of the n-th power map in $\mathbb{G}_m$. If X is a scheme over a separably closed field k and if n is invertible in k, then the choice of a primitive n-th root of unity $\zeta \in k$ defines an isomorphism $i \mapsto \zeta^i$ from $\mathbb{Z}/n$ to $\boldsymbol{\mu}_n$.

The relation between cohomology with coefficients in $\boldsymbol{\mu}_m$ and cohomology with coefficients in $\mathbb{G}_m$ is given by the cohomology exact sequence deduced from the

Théorème 2.2.5 (Kummer Theory)**.** *If n is invertible on X, then the n-th power map on $\mathbb{G}_m$ is an epimorphism of sheaves. Therefore, we have an exact sequence*

$$0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

Proof. Let $U \to X$ be an étale morphism and $a \in \mathbb{G}_m(U) = \Gamma(U, \mathscr{O}_U^\times)$. Since n is invertible on U, the equation $T^n - a = 0$ is separable; in other words $U' = \operatorname{Spec}(\mathscr{O}_U[T]/(T^n - a))$ is étale over U. Furthermore, $U' \to U$ is surjective and admits an n-th root on U', which proves the result. □

2.3 Fibers, direct images

2.3.1

A *geometric point* of X is a morphism $\bar{x} \to X$, where $\bar{x}$ is the spectrum of a separably closed field $k(\bar{x})$. We will abbreviate it by $\bar{x}$, with the morphism $\bar{x} \to X$ understood. If x is the image of $\bar{x}$ in X, we say that $\bar{x}$ is centered at x. If the field $k(\bar{x})$ is an algebraic extension of the residue field $k(x)$, we say that $\bar{x}$ is an *algebraic geometric point* of X.

A *neighborhood étale* of $\bar{x}$ is a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \bar{x} \arrow[ur] \arrow[r] & X, \end{tikzcd}$$

where $U \to X$ is an étale morphism.

The *strict localization* of X at $\bar{x}$ is the ring $\mathscr{O}_{X,\bar{x}} = \varinjlim \Gamma(U, \mathscr{O}_U)$, the inductive limit being over the étale neighborhoods of $\bar{x}$. It is a strictly henselian local ring whose residue field is the separable closure of the residue field $k(x)$ of X at x in $k(\bar{x})$. It plays the role of a local ring for the étale topology.

2.3.2

Given a sheaf $\mathscr{F}$ on X_{et}, we call the *fiber* of $\mathscr{F}$ at $\bar{x}$ the set (resp. group, ...) $\mathscr{F}_{\bar{x}} = \varinjlim \mathscr{F}(U)$, the inductive limit always being taken over the étale neighborhoods of X.

For a homomorphism of sheaves $\mathscr{F} \to \mathscr{G}$ to be a mono-/epi-/isomorphism, it is necessary and sufficient that this be the case for the morphisms $\mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ induced on the fibers and at any point of X. If X is of finite type over an algebraically closed field, it is sufficient that this be the case at the rational points of X.

2.3.3

If $f : X \to Y$ is a morphism of schemes and $\mathscr{F}$ is a sheaf on X_{et}, the *direct image* $f_*\mathscr{F}$ of $\mathscr{F}$ by f is the sheaf on Y_{et} defined by $f_*\mathscr{F}(V) = \mathscr{F}(X \times_Y V)$ for all étale V on Y. The functor $f_* : \operatorname{Sh}(X_{\text{et}}) \to \operatorname{Sh}(Y_{\text{et}})$ is exact on the left. Its right derived functors $\mathsf{R}^q f_*$ are called direct higher images. If $\bar{y}$ is a geometric point of Y, we have

$$(\mathsf{R}^q f_* \mathscr{F})_{\bar{y}} = \varinjlim \mathrm{H}^q(V \times_Y X, \mathscr{F}),$$

the inductive limit being taken over étale neighborhoods V of $\bar{y}$.

Let $\mathscr{O}_{Y,\bar{y}}$ be the strict localization of Y at $\bar{y}$, $\widetilde{Y} = \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$ and $\widetilde{X} = X \times_Y \widetilde{Y}$. We can extend $\mathscr{F}$ to $\widetilde{X}_{\mathrm{et}}$ (it is a special case of the general notion of inverse image) in the following way: let $\widetilde{U}$ be an étale scheme over $\widetilde{X}$, then there exists a neighborhood étale V of $\bar{y}$ and an étale scheme U over $X \times_Y V$ such that $\widetilde{U} = U \times_V \widetilde{Y}$; we will set

$$\mathscr{F}(\widetilde{U}) = \varinjlim \mathscr{F}(U \times_V V'),$$

the inductive limit being taken over the étale neighborhoods V' of $\bar{y}$ which dominate V. With this definition, we have

$$(\mathsf{R}^q f_* \mathscr{F})_{\bar{y}} = \mathrm{H}^q(\widetilde{X}, \mathscr{F}).$$

The functor f_* has a left adjoint f^*, the "inverse image" functor. If $\bar{x}$ is a geometric point of X and $f(\bar{x})$ is its image in Y, we have $(f^*\mathscr{F})_{\bar{x}} = \mathscr{F}_{f(\bar{x})}$. This formula shows that f^* is an exact functor. The functor f_* therefore transforms injective sheaf into injective sheaf, and the spectral sequence of the composition functor $\Gamma \circ f_*$ (resp. $g_* f_*$) provides the

Théorème 2.3.4 (Leray's spectral sequence)**.** *Let $\mathscr{F}$ be an abelian sheaf on X_{et} and $f : X \to Y$ be a morphism of schemes (resp. the morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$). We have a spectral sequence*

$$E_2^{pq} = \mathrm{H}^p(Y, \mathsf{R}^q f_* \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

$$\textit{(resp.} \qquad E_2^{pq} = \mathsf{R}^p g_* \mathsf{R}^q f_* \mathscr{F} \Rightarrow \mathsf{R}^{p+q}(gf)_* \mathscr{F}\textit{).}$$

Corollaire 2.3.5. *If $\mathsf{R}^q f_* \mathscr{F} = 0$ for all $q > 0$, then $\mathrm{H}^p(Y, f_*\mathscr{F}) = \mathrm{H}^p(X, \mathscr{F})$ (resp. $\mathsf{R}^p g_*(\mathscr{F}_*\mathscr{F}) = \mathsf{R}^p(gf)_*\mathscr{F}$) for all $p \geqslant 0$.*

This applies in particular in the following case:

Proposition 2.3.6. *Let $f : X \to Y$ be a finite morphism (or, by passing to the limit, an integral morphism) and $\mathscr{F}$ an abelian sheaf on X. Then $\mathsf{R}^q f_* \mathscr{F} = 0$, for all $q > 0$.*

Indeed, let $\bar{y}$ be a geometric point of Y, $\widetilde{Y}$ the spectrum of the strict localization of Y at y and $\widetilde{X} = X \times_Y \widetilde{Y}$; according to what has been said before, it is enough to show that $\mathrm{H}^q(\widetilde{X}, \mathscr{F}) = 0$ for all $q > 0$. But $\widetilde{X}$ is the spectrum of a product of strict henselian local rings (cf. [29, I]), the functor $\Gamma(\widetilde{X}, -)$ is exact because any $\widetilde{X}$-scheme étale and surjective admits a section, whence the assertion.

2.4 Galois cohomology

For $X = \mathrm{Spec}(K)$ the spectrum of a field, we will see that étale cohomology is identified with Galois cohomology.

2.4.1

Let's start with a topological analogy. If K is the field of functions of an integral affine algebraic variety $Y = \mathrm{Spec}(A)$ over $\mathbb{C}$, we have $K = \varinjlim_{f \in A} A[1/f]$.

In other words, $X = \varprojlim U$, U ranging over the set of open sets of Y. We know that there exist arbitrarily small Zariski open sets which for the classical topology are $K(\pi, 1)$'s. We will not be surprised, then, if we consider $\mathrm{Spec}(K)$ itself as a $K(\pi, 1)$, π being the (algebraic) fundamental group of X, that is to say, the Galois group of $\bar{K}/K$, where $\bar{K}$ is the separable closure of K.

2.4.2

More precisely, let K be a field, $\bar{K}$ a separable closure of K, and $G = \mathrm{Gal}(\bar{K}/K)$ the topological Galois group. To every finite étale K-algebra A (a finite product of separable extensions of K), associate the finite set $\hom_K(A, \bar{K})$. The Galois group G operates on this set through a discrete (hence finite) quotient. If $A = K[T]/(F)$, it is identified with the set of roots in $\bar{K}$ of the polynomial F. Galois theory, in the form given to it by Grothendieck, says that:

Proposition 2.4.3. *The functor*

$$\left\{ \begin{array}{c} K\text{-algebras} \\ \text{finite étales} \end{array} \right\} \to \left\{ \begin{array}{c} \text{finite sets on which} \\ G \text{ operates continuously} \end{array} \right\}$$

which to an étale algebra A associates $\hom_K(A, \bar{K})$ *is an anti-equivalence of categories.*

We deduce an analogous description of sheaves for the étale topology on $\mathrm{Spec}(K)$:

Proposition 2.4.4. *The functor*

$$\left\{ \begin{array}{c} \text{Étale sheaves} \\ \text{on } \mathrm{Spec}(K) \end{array} \right\} \to \left\{ \begin{array}{c} \text{sets on which} \\ G \text{ operates continuously} \end{array} \right\}$$

which to a sheaf $\mathscr{F}$ associates its fibre $\mathscr{F}_{\bar{K}}$ at the geometric point $\mathrm{Spec}(\bar{K})$ *is an equivalence of categories.*

It is said that G continuously operates on a set E if the fixer of every element of E is an open subgroup of G. The functor in the inverse direction is described in an obvious way: let A be a finite étale K-algebra, $U = \mathrm{Spec}(A)$ and $U(\bar{K}) = \hom_K(A, \bar{K})$ the corresponding G-set to; then we have $\mathscr{F}(U) = \hom_{G\text{-set}}(U(\bar{K}), \mathscr{F}_{\bar{K}})$.

In particular, if $X = \mathrm{Spec}(K)$, we have $\mathscr{F}(X) = \mathscr{F}_{\bar{K}}^G$. If we restrict ourselves to abelian sheaves, we obtain by passing to derived functors the canonical isomorphisms

$$\mathrm{H}^q(X_{\mathrm{et}}, \mathscr{F}) = \mathrm{H}^q(G, \mathscr{F}_{\bar{K}})$$

2.4.5 Examples

a) To the constant sheaf $\mathbb{Z}/n$ corresponds $\mathbb{Z}/n$ with trivial action of G.

b) To the sheaf of n-th roots of unity $\boldsymbol{\mu}_n$ corresponds the group $\boldsymbol{\mu}_n(\bar{K})$ of n-th roots of unity in $\bar{K}$, with the natural action of G.

c) To the sheaf $\mathbb{G}_m$ corresponds the group $\bar{K}^\times$ with the natural action of G.

3 Cohomology of curves

In the case of topological spaces, unscrewing using the Künneth formula and simplicial decompositions allow us to reduce to the interval $I = [0, 1]$ for which we have $\mathrm{H}^0(I, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}^q(I, \mathbb{Z}) = 0$ for $q > 0$.

In our case, the unscrewing will lead to more complicated objects, namely curves over an algebraically closed field; we will calculate their cohomology in this chapter. The situation is more complex than in the topological case because the cohomology groups are only trivial for $q > 2$. The essential ingredient of the calculations is the triviality of the Brauer group of the function field of such a curve (Tsen's theorem, 3.2).

3.1 The Brauer group

First, let us recall the classical definition:

Définition 3.1.1. *Let K be a field and A a finite-dimensional K-algebra. We say that A is a central simple algebra over K if the equivalent conditions following are satisfied:*

a) A has no non-trivial two-sided ideal and its center is K.

b) There exists a finite Galois extension K'/K such that $A_{K'} = A \otimes_K K'$ is isomorphic to a matrix algebra over K'.

c) A is K-isomorphic to a matrix algebra over a left field of center K.

Two such algebras are said to be equivalent if the left fields associated to them by c) are K-isomorphic. If these algebras have the same dimension, this is equivalent to saying that they are K-isomorphic. The tensor product defined by passage to the quotient defines an abelian group structure on the set of equivalence classes. This group is traditionally called the *Brauer group* of K and is denoted by $\mathrm{Br}(K)$.

3.1.2

We will denote $\mathrm{Br}(n, K)$ the set of K-isomorphism classes of K-algebras A such that there exists a finite Galois extension K' of K for which $A_{K'}$ is isomorphic to the algebra $M_n(K')$ of square matrices $n \times n$ over K'. By definition $\mathrm{Br}(K)$ is the union of the subsets $\mathrm{Br}(n, K)$ for $n \in \mathbb{N}$. Let $\bar{K}$ be an algebraic closure of K and $G = \mathrm{Gal}(\bar{K}/K)$. The set $\mathrm{Br}(n, K)$ is the set of "forms" of $M_n(\bar{K})$, so it is canonically isomorphic to $\mathrm{H}^1\left(G, \mathrm{Aut}(M_n(\bar{K}))\right)$.

It is known that every automorphism of $M_n(\bar{K})$ is inner. Consequently, the group $\mathrm{Aut}(M_n(\bar{K}))$ is identified with the projective linear group $\mathrm{PGL}(n, \bar{K})$ and we have a canonical bijection:

$$\theta_n : \mathrm{Br}(n, K) \xrightarrow{\sim} \mathrm{H}^1\left(G, \mathrm{PGL}(n, \bar{K})\right).$$

On the other hand, the exact sequence:

$$1 \to \bar{K}^\times \to \mathrm{GL}(n, \bar{K}) \to \mathrm{PGL}(n, \bar{K}) \to 1, \tag{3.1.2.1}$$

defines a coboundary operator:

$$\Delta_n : \mathrm{H}^1\left(G, \mathrm{PGL}(n, \bar{K})\right) \to \mathrm{H}^2(G, \bar{K}^\times).$$

By composing θ_n and Δ_n, we obtain a map:

$$\delta_n : \mathrm{Br}(n, K) \to \mathrm{H}^2(G, \bar{K}^\times).$$

It is easily checked that the maps δ_n are compatible with each other and define a group homomorphism:

$$\delta : \mathrm{Br}(K) \to \mathrm{H}^2(G, \bar{K}^\times).$$

Proposition 3.1.3. *The homomorphism $\delta : \mathrm{Br}(K) \to \mathrm{H}^2(G, \bar{K}^\times)$ is bijective.*

This follows from the two lemmas below:

Lemme 3.1.4. *The application $\Delta_n : \mathrm{H}^1\left(G, \mathrm{PGL}(n, \bar{K})\right) \to \mathrm{H}^2(G, \bar{K}^\times)$ is injective.*

According to [34], cor. to the prop. I-44, it is enough to verify that each time we twist the exact sequence (3.1.2.1) by an element of $\mathrm{H}^1\left(G, \mathrm{PGL}(n, \bar{K})\right)$, the H^1 of the median group is trivial. This median group is the group of $\bar{K}$-points of the multiplicative group of a simple central algebra A of rank n^2 over K. To prove that $\mathrm{H}^1(G, A_{\bar{K}}^{\times}) = 0$, we interpret $A^{\times}$ as the group of automorphisms of the free A-module L of rank 1, and H^1 as the set of "forms" of L – A-modules of rank n^2 over K, automatically free.

Lemme 3.1.5. *Let $\alpha \in \mathrm{H}^2(G, \bar{K}^{\times})$, K' be a finite extension of K contained in $\bar{K}$, $n = [K' : K]$, and $G' = \mathrm{Gal}(\bar{K}/K')$. If the image of α in $\mathrm{H}^2(G', \bar{K}^{\times})$ is zero, then α belongs to the image of Δ_n.*

First of all, note that we have:

$$\mathrm{H}^2(G', \bar{K}^{\times}) \simeq \mathrm{H}^2\left(G, (\bar{K} \otimes_K K')^{\times}\right).$$

(Geometrically speaking, if we denote $x = \mathrm{Spec}(K)$, $x' = \mathrm{Spec}(K')$ and $\pi : x' \to x$ the canonical morphism, we have $\mathsf{R}^q\pi_*(\mathbb{G}_{m,x'}) = 0$ for $q > 0$ and as a result $\mathrm{H}^q(x', \mathbb{G}_{m,X'}) \simeq \mathrm{H}^q(x, \pi_*\mathbb{G}_{m,X'})$ for $q \geqslant 0$).

Furthermore, the choice of a basis of K' as a vector space over K defines a homomorphism

$$(\bar{K} \otimes_K K')^{\times} \to \mathrm{GL}(n, \bar{K})$$

which, to an element x, associates the endomorphism of multiplication by x of $\bar{K} \otimes_K K'$. We then have a commutative diagram with exact lines:

$$\begin{tikzcd}
1 \arrow[r] & \bar{K}^{\times} \arrow[r] \arrow[d, equal] & (\bar{K} \otimes_K K')^{\times} \arrow[r] \arrow[d] & (\bar{K} \otimes_K K')^{\times}/\bar{K}^{\times} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \bar{K}^{\times} \arrow[r] & \mathrm{GL}(n, \bar{K}) \arrow[r] & \mathrm{PGL}(n, \bar{K}) \arrow[r] & 1
\end{tikzcd}$$

The lemma follows from the commutative diagram that we deduce by passing to cohomology:

$$\begin{tikzcd}
\mathrm{H}^1\left(G, (\bar{K} \otimes_K K')^{\times}/\bar{K}^{\times}\right) \arrow[r] \arrow[d] & \mathrm{H}^2(G, \bar{K}^{\times}) \arrow[r] \arrow[d, equal] & \mathrm{H}^2\left(G, (\bar{K} \otimes_K K')^{\times}\right) \\
\mathrm{H}^1\left(G, \mathrm{PGL}(n, \bar{K})\right) \arrow[r, "\Delta_n"] & \mathrm{H}^2(G, \bar{K}^{\times}). &
\end{tikzcd}$$

Proposition 3.1.6. *Let K be a field, $\bar{K}$ an algebraic closure of K, and $G = \mathrm{Gal}(\bar{K}/K)$. Suppose that, for every finite extension K' of K, we have $\mathrm{Br}(K') = 0$. Then we have:*

i) $\mathrm{H}^q(G, \bar{K}^{\times}) = 0$ *for all* $q > 0$.

ii) $\mathrm{H}^q(G, \mathscr{F}) = 0$ *for every G-module of torsion $\mathscr{F}$ and for all* $q \geqslant 2$.

(For the proof, see [34]).

3.2 Tsen's Theorem

Définition 3.2.1. *We say that a field K is C_1 if every non-constant homogeneous polynomial $f(x_1, \ldots, x_n)$ of degree $d < n$ has a non-trivial zero.*

Proposition 3.2.2. *If a field K is C_1, we have* $\mathrm{Br}(K) = 0$.

The goal is to show that every left body D with center K and finite over K is equal to K. Let r^2 be the degree of D over K and $\mathrm{Nrd} : D \to K$ the reduced norm.

(Locally for the étale topology on K, D is isomorphic–not canonically–to an algebra of matrices M_r and the reduced norm coincides with the determinant function. This is well defined, independently of the chosen isomorphism between D and M_r because every automorphism of M_r is inner and two similar matrices have the same determinant. This function defined locally for the étale topology descends, because of its local uniqueness, to a function $\mathrm{Nrd} : D \to K$).

The only zero of Nrd is the null element of D, because, if $x \neq 0$, then $\mathrm{Nrd}(x)\cdot\mathrm{Nrd}(x^{-1}) = 1$. On the other hand, if $\{e_1, \dots, e_{r^2}\}$ is a basis of D over K and if $x = \sum x_i e_i$, the function $\mathrm{Nrd}(x)$ is written as a homogeneous polynomial $\mathrm{Nrd}(x_1, \dots, x_{r^2})$ of degree r (this is clear locally for the étale topology). Since K is C_1, then $r^2 \leqslant r$, that is, $r = 1$ and $D = K$.

Théorème 3.2.3 (Tsen)**.** *Let k be an algebraically closed field and K an extension of degree of transcendence* 1 *of k. Then K is C_1.*

First suppose that $K = k(X)$. Let

$$f(T) = \sum a_{i_1,\dots i_n} T_1^{i_1} \cdots T_n^{i_n}$$

be a homogeneous polynomial of degree $d < n$ with coefficients in $k(X)$. By multiplying the coefficients by a common denominator we can assume that they are in $k[X]$. Let then $\delta = \sup \deg(a_{i_1\dots i_n})$. We seek a non-trivial zero in $k[X]$ by the method of indeterminate coefficients by writing each T_i $(i = 1, \dots, n)$ as a polynomial of degree N in X. Then the equation $f(T) = 0$ becomes a system of homogeneous equations in the $n \times (N+1)$ coefficients of the polynomials $T_i(X)$ expressing the nullity of the coefficients of the polynomial in X obtained by replacing T_i with $T_i(X)$. This polynomial is of degree at most $\delta + ND$, so there are at most $\delta + Nd + 1$ equations in $n \times (N+1)$ variables. As k is algebraically closed this system has a non-trivial solution if $n(N+1) > Nd + \delta + 1$, which will be the case for N large enough if $d < n$.

It is clear that, to prove the theorem in the general case, it is enough to prove it when K is a finite extension of a pure transcendental extension $k(X)$ of k. Let $f(T) = f(T_1, \dots, T_n)$ be a homogeneous polynomial of degree $d < n$ with coefficients in K. Let $a = [K : k(X)]$ and $e_1, \dots, e_s$ be a basis of K over $k(X)$. Introduce new variables U_{ij}, in number sn, such that $T_i = \sum U_{ij} e_j$. For the polynomial $f(T)$ to have a non-trivial zero in K, it is enough that the polynomial $g(X_{ij}) = N_{K/k}(f(T))$ has a non-trivial zero in $k(X)$. But g is a polynomial of degree sd in sn variables, from which the result.

Corollaire 3.2.4. *Let k be an algebraically closed field and K an extension of degree of transcendence* 1 *of k. Then the étale cohomology groups* $\mathrm{H}^q(\mathrm{Spec}(K), \mathbb{G}_m)$ *are zero for all* $q > 0$.

3.3 Cohomology of smooth curves

From now on, and unless expressly stated otherwise, the cohomology groups considered are the étale cohomology groups.

Proposition 3.3.1. *Let k be an algebraically closed field and X a connected projective curve over k. Then we have:*

$$\begin{aligned}
&\mathrm{H}^0(X,\mathbb{G}_m) = k^{\times},\\
&\mathrm{H}^1(X,\mathbb{G}_m) = \mathrm{Pic}(X),\\
&\mathrm{H}^q(X,\mathbb{G}_m) = 0 \textit{ for } q \geqslant 2.
\end{aligned}$$

Let η be the generic point of X, $j : \eta \to X$ the canonical morphism and $\mathbb{G}_{m,X}$ the multiplicative group of the field of fractions $K(X)$. For every closed point x of X, let $i_x : x \to X$ be the canonical immersion and $\mathbb{Z}_x$ the constant sheaf of value $\mathbb{Z}$ on x. Thus $j_*\mathbb{G}_{m,\eta}$ is the sheaf of non-zero meromorphic functions on X and $\oplus_{x\in X} i_{x_I}\mathbb{Z}_x$ the sheaf of divisors, we have an exact sequence of sheaves:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow j_*\mathbb{G}_{m,\eta} \xrightarrow{\text{div}} \bigoplus_{x\in X} i_{x*}\mathbb{Z}_x \longrightarrow 0. \tag{3.3.1.1}$$

Lemme 3.3.2. *On a $\mathsf{R}^q j_*\mathbb{G}_{m,\eta} = 0$ for all $q > 0$.*

It is enough to show that the fiber of this sheaf at any closed point x of X is zero. If $\tilde{\mathscr{O}}_{X,x}$ is the henselization of X at x and K is the field of fractions of $\tilde{\mathscr{O}}_{X,x}$, then

$$\mathrm{Spec}(K) = \eta \times_X \mathrm{Spec}(\tilde{\mathscr{O}}_{X,x}),$$

so $(\mathsf{R}^q j_*\mathbb{G}_{m,\eta})_x = \mathrm{H}^q(\mathrm{Spec}(K),\mathbb{G}_m)$.

But K is an algebraic extension of $k(X)$, therefore an extension of transcendence degree 1 of k: the lemma follows from (3.2.4).

Lemme 3.3.3. *We have $\mathrm{H}^q(X, j_*\mathbb{G}_{m,\eta}) = 0$ for all $q > 0$.*

Indeed, from (3.3.2) and the Leray spectral sequence for j, we deduce:

$$\mathrm{H}^q(X, j_*\mathbb{G}_{m,\eta}) = \mathrm{H}^q(\eta,\mathbb{G}_{m,\eta})$$

for all $q \geqslant 0$ and the second member is zero for $q > 0$ according to (3.2.4).

Lemme 3.3.4. *On a $\mathrm{H}^q\left(X, \bigoplus_{x\in X} i_{x*}\mathbb{Z}_x\right) = 0$ for all $q > 0$.*

Indeed, for any closed point of X, we have $\mathsf{R}^q i_{x*}\mathbb{Z}_x 0$ for $q > 0$, because i_x is a finite morphism (2.3.6), and

$$\mathrm{H}^q(X, i_{x*}\mathbb{Z}_x) = \mathrm{H}^q(x,\mathbb{Z}_x).$$

The second member is null for all $q > 0$, because x is the spectrum of an algebraically closed field (We see that the lemma is true more generally for any “skyscraper” sheaf on X).

We deduce from the previous lemmas and the exact sequence (3.3.1.1) the equalities:

$$\mathrm{H}^q(X,\mathbb{G}_m) = 0 \text{ for } q \geqslant 2,$$

and an exact sequence of cohomology in low degree:

$$1 \to \mathrm{H}^0(X,\mathbb{G}_m) \to \mathrm{H}^0(X, j_*\mathbb{G}_{m,\eta}) \to \mathrm{H}^0\left(X, \bigoplus_{x\in X} i_{x*}\mathbb{Z}_x\right) \to \mathrm{H}^1(X,\mathbb{G}_m) \to 1$$

which is none other than the exact sequence:

$$1 \to k^\times \to k(X)^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 1.$$

From proposition 3.3.1 we deduce that the cohomology groups of X with values in $\mathbb{Z}/n$, n prime to the characteristic of k, have reasonable values:

Corollaire 3.3.5. *If X is of genus g and if n is invertible in k, the $\mathrm{H}^q(X, \mathbb{Z}/n)$ are null for $q > 2$, and free on $\mathbb{Z}/n$ of rank $1, 2g, 1$ for $q = 0, 1, 2$. Replacing $\mathbb{Z}/n$ with the isomorphic group $\boldsymbol{\mu}_n$, we have canonical isomorphisms*

$$\begin{aligned} \mathrm{H}^0(X, \boldsymbol{\mu}_n) &= \boldsymbol{\mu}_n \\ \mathrm{H}^1(X, \boldsymbol{\mu}_n) &= \mathrm{Pic}^0(X)_n \\ \mathrm{H}^2(X, \boldsymbol{\mu}_n) &= \mathbb{Z}/n. \end{aligned}$$

Since the field k is algebraically closed, $\mathbb{Z}/n$ is isomorphic (not canonically) to μ_n. From the Kummer exact sequence:

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

and from proposition 3.3.1, we deduce the equality:

$$\mathrm{H}^q(X, \mathbb{Z}/n) = 0 \text{ for } q > 2,$$

and, in low degree, the exact sequences:

$$0 \longrightarrow \mathrm{H}^0(X, \boldsymbol{\mu}_n) \longrightarrow k^\times \xrightarrow{n} k^\times \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^1(X, \boldsymbol{\mu}_n) \longrightarrow \mathrm{Pic}(X) \xrightarrow{n} \mathrm{H}^2(X, \boldsymbol{\mu}_n) \longrightarrow 0.$$

Moreover we have an exact sequence:

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

and $\mathrm{Pic}^0(X)$ is identified with the group of rational points on k of an abelian variety of dimension g, the jacobian of X. In such a group, multiplication by n is surjective and its kernel is a $\mathbb{Z}/n\mathbb{Z}$-module free of rank $2g$ (because n is invertible in k); whence the corollary.

A clever untwisting, using the "trace method," allows us to obtain as a corollary

Proposition 3.3.6 ([2, IX 5.7])**.** *Let k be an algebraically closed field, X an algebraic curve over k and $\mathscr{F}$ a torsion sheaf on X. Then:*

i) We have $\mathrm{H}^q(X, \mathscr{F}) = 0$ for $q > 2$.

ii) If X is affine, we even have $\mathrm{H}^q(X, \mathscr{F}) = 0$ for $q > 1$.

For the proof, as well as for the exposition of the "trace method," we refer to [2, IX 5].

3.4 Untwisting

To calculate the cohomology of varieties of dimension > 1, we use fibrations by curves, which allows us to reduce to studying morphisms whose fibres are of dimension $\leqslant 1$. This principle has several variants, let us mention a few.

3.4.1

Let A be a finite type k-algebra and $a_1, \dots, a_n$ generators of A. If we set $X_0 = \mathrm{Spec}(k)$, $X_i = \mathrm{Spec}(k[a_1, \dots, a_i])$, $X_n = \mathrm{Spec}(A)$, the canonical inclusions $k[a_1, \dots, a_i] \to k[a_1, \dots, a_i, a_{i+1}]$ define morphisms $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ whose fibres have dimension $\leqslant 1$.

3.4.2

In the case of a smooth morphism, we can be more precise. We call *elementary fibration* a morphism of schemes $f : X \to S$ which can be embedded in a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

i) j is an open dense immersion in each fibre and $X = \bar{X} \setminus Y$.

ii) $\bar{f}$ is smooth, projective, with irreducible and one-dimensional geometric fibres.

iii) g is an étale covering and no fibre of g is empty.

A *relative good neighborhood* of S is a S-scheme X such that there exist S-schemes $X = X_n, \dots, X_0 = S$ and elementary fibrations $f_i : X_i \to X_{i-1}$, $i = 1, \dots, n$. It can be shown [2, XI 3.3] that if X is a smooth scheme over an algebraically closed field k every rational point of X has an open neighborhood which is a good neighborhood (relative to $\mathrm{Spec}(K)$).

3.4.3

One can unravel a proper morphism $f : X \to S$ as follows. By the Chow lemma, there is a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & \bar{X} \arrow[l, "\pi"'] \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

where π and $\bar{f}$ are projective morphisms, π being moreover an isomorphism over a dense open of X. Locally on S, $\bar{X}$ is a closed subscheme of a projective space $\mathbb{P}^n_S$.

We unscrew the latter by considering the projection $\varphi : \mathbb{P}^n_S \to \mathbb{P}^1_S$ which sends the point of homogeneous coordinates $(x_0, x_1, \dots, x_n)$ to (x_0, x_1). It is a rational application defined outside the closed $Y \simeq \mathbb{P}^{n-2}_S$ of $\mathbb{P}^n_S$ of homogeneous equations $x_0 = x_1 = 0$. Let $u : P \to \mathbb{P}^n_S$ be the blow-up with center Y; the fibers of u are of dimension $\leqslant 1$. In addition there is a natural morphism $v : P \to \mathbb{P}^1_S$ which extends the rational φ and v makes P a $\mathbb{P}^1_S$-scheme locally isomorphic to the projective space type $\mathbb{P}^{n-1}$ which can in turn be projected onto a $\mathbb{P}^1$, etc.

3.4.4

We can sweep a projective and smooth variety X by a *Lefschetz brush*. The blown-up $\widetilde{X}$ of the intersection of the brush axis with X is projected onto $\mathbb{P}^1$ and the fibers of this projection are the sections hyperplanes of X by the hyperplanes of the brush."

4 Theorem of change of base for a proper morphism

4.1 Introduction

This chapter is dedicated to the demonstration and applications of the

Théorème 4.1.1. *Let $f : X \to S$ be a proper morphism of schemes and $\mathscr{F}$ an abelian sheaf of torsion on X. Then, for any $q \geqslant 0$, the fiber of $\mathsf{R}^q f_* \mathscr{F}$ at a geometric point s of S is isomorphic to the cohomology $\mathrm{H}^q(X_s, \mathscr{F})$ of the fiber $X_s = X \otimes_S \operatorname{Spec} k(s)$ of f at s.*

For $f : X \to S$ a proper and separated (separated means that the diagonal of $X \times_S X$ is closed) continuous map between topological spaces, and $\mathscr{F}$ an abelian sheaf on X, the analogous result is well known, and elementary: as f is closed, the $f^{-1}(V)$ for V a neighborhood of s form a fundamental system of neighborhoods of X_s, and it is verified that $\mathrm{H}^\bullet(X_s, \mathscr{F}) = \varinjlim_U \mathrm{H}^\bullet(U, \mathscr{F})$, for U ranging over the neighborhoods of X_s. In practice, X_s even has a fundamental system $\mathcal{U}$ of neighborhoods U of which it is a deformation retract, and, for $\mathscr{F}$ constant, we therefore have $\mathrm{H}^\bullet(X_s, \mathscr{F}) = \mathrm{H}^\bullet(U, \mathscr{F})$. In other words: the special fiber swallows the general fiber.

In the case of schemes, the demonstration is more delicate and it is essential to suppose that $\mathscr{F}$ is of torsion ([2, XII.2]). Given the description of the fibers of $\mathsf{R}^q f_* \mathscr{F}$ (2.3.3), the theorem (4.1.1) is essentially equivalent to

Théorème 4.1.2. *Let A be a strictly henselian local ring and $S = \operatorname{Spec}(A)$. Let $f : X \to S$ be a proper morphism and X_0 the closed fiber of f. Then, for every torsion abelian sheaf $\mathscr{F}$ on X and for every $q \geqslant 0$, we have $\mathrm{H}^q(X, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^q(X_0, \mathscr{F})$.*

By taking the limit, we see that it is enough to demonstrate the theorem when A is the strict henselization of a $\mathbb{Z}$-algebra of finite type in a prime ideal. We first treat the case $q = 0$ or 1 and $\mathscr{F} = \mathbb{Z}/n$ (4.2). An argument based on the notion of a constructible sheaf (4.3) moreover shows that it is enough to consider the case where $\mathscr{F}$ is constant. On the other hand, the devissage (3.4.3) allows us to suppose that X_0 is a curve; in this case, it only remains to demonstrate the theorem for $q = 2$ (4.4).

Among other applications (4.6), the theorem allows us to define the notion of proper cohomology (4.5).

4.2 Demonstration for $q = 0$ or 1 and $\mathscr{F} = \mathbb{Z}/n$

The result for $q = 0$ and $\mathscr{F}$ constant is equivalent to the following proposition (Zariski connection theorem):

Proposition 4.2.1. *Let A be a noetherian local henselian ring and $S = \operatorname{Spec}(A)$. Let $f : X \to S$ be a proper morphism and X_0 the closed fiber of f. Then the sets of connected components $\pi_0(X)$ and $\pi_0(X_0)$ are in bijection.*

It is the same to show that the sets of both open and closed parts $\mathrm{Of}(X)$ and $\mathrm{Of}(X_0)$ are in bijection. We know that the set $\mathrm{Of}(X)$ corresponds bijectively to the set of idempotents of $\Gamma(X, \mathscr{O}_X)$, similarly $\mathrm{Of}(X_0)$ corresponds bijectively to the set of idempotents of $\Gamma(X_0, \mathscr{O}_{X_0})$. It remains to show that the canonical map

$$\mathrm{Idem}\,\Gamma(X, \mathscr{O}_X) \to \mathrm{Idem}\,\Gamma(X_0, \mathscr{O}_{X_0})$$

is bijective.

We note $\mathfrak{m}$ the maximal ideal of A, $\Gamma(X, \mathscr{O}_X)^\wedge$ the completion of $\Gamma(X, \mathscr{O}_X)$ for the $\mathfrak{m}$-adic topology and, for any integer $n \geqslant 0$, $X_n = X \otimes_A A/\mathfrak{m}^{n+1}$. According to the finiteness theorem for proper morphisms [19, III.3.2], $\Gamma(X, \mathscr{O}_X)$ is a finite A-algebra; as A is henselian, it follows that the canonical map

$$\mathrm{Idem}\,\Gamma(X, \mathscr{O}_X) \to \mathrm{Idem}\,\Gamma(X, \mathscr{O}_X)^\wedge$$

is bijective. In particular the canonical map

$$\mathrm{Idem}\,\Gamma(X, \mathscr{O}_X)^\wedge \to \varprojlim \mathrm{Idem}\,\Gamma(X_n, \mathscr{O}_{X_n})$$

is bijective. But, since X_n and X_0 have the same underlying topological space, the canonical map

$$\mathrm{Idem}\,\Gamma(X_n, \mathscr{O}_{X_n}) \to \mathrm{Idem}\,\Gamma(X_0, \mathscr{O}_{X_0})$$

is bijective for all n, which ends the proof.

Since $\mathrm{H}^1(X, \mathbb{Z}/n)$ is in bijection with the set of isomorphism classes of Galois étale covers of X with group $\mathbb{Z}/n$, the theorem for $q = 1$ and $\mathscr{F} = \mathbb{Z}/n$ follows from the following proposition.

Proposition 4.2.2. *Let A be a Noetherian local henselian ring and $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be a proper morphism and X_0 the closed fiber of f. Then the restriction functor*

$$\mathrm{Rev.\,et}(X) \to \mathrm{Rev.\,et}(X_0)$$

is an equivalence of categories.

(If X_0 is connected and if we have chosen a geometric point of X_0 as a base point, this is equivalent to saying that the canonical $\pi_1(X_0) \to \pi_1(X)$ map on fundamental groups (pro-finite) is bijective).

Proposition (4.2.1) shows that this functor is fully faithful. In fact, if X' and X'' are two étale covers of X, a X-morphism from X' to X'' is determined by its graph which is an open and closed subset of $X' \times_X X''$.

So it is enough to show that any étale covering X'_0 of X_0 extends to an étale covering of X. We know that étale coverings do not depend on nilpotent elements [22, ch.1], so X'_0 uniquely extends to an étale covering X'_n of X_n for all $n \geqslant 0$, that is, an étale covering $\mathfrak{X}'$ of the formal scheme $\mathfrak{X}$ completed along X from X_0. By Grothendieck's algebraization theorem for coherent formal sheaves (existence theorem, [19, III.5]), $\mathfrak{X}'$ is the formal completion of an étale covering $\bar{X}'$ of $\bar{X} = X \otimes_A \hat{A}$.

By taking the limit, it is enough to prove the proposition in the case where A is the henselization of a $\mathbb{Z}$-algebra of finite type. We can then apply Artin's approximation theorem to the functor $\mathscr{F}$: (A-algebras) $\to$ (sets) which, to an A-algebra B, associates the set of isomorphism classes of étale coverings of $X \otimes_A B$. Indeed this functor is locally of finite presentation: if B_i is an inductive system of filtered A-algebras and if $B = \varinjlim B_i$, then $\mathscr{F}(B) = \varinjlim \mathscr{F}(B_i)$. By Artin's theorem, given an element $\xi \in \mathscr{F}(\hat{A})$, in this case the isomorphism class of $\bar{\xi}$, there exists $\xi \in \mathscr{F}(A)$ with the same image as $\bar{\xi}$ in $\mathscr{F}(A/\mathfrak{m})$. In other words, there exists an étale covering X' of X whose restriction to X_0 is isomorphic to X'_0.

4.3 Constructible sheaves

In this paragraph, we consider a *Noetherian* scheme X and call a sheaf on X an *Abelian* sheaf on X_{et}.

Définition 4.3.1. *We say that a sheaf $\mathscr{F}$ on X is* locally constant constructible *(abbreviated l.c.c.) if it is represented by an étale covering of X.*

Définition 4.3.2. *We say that a sheaf $\mathscr{F}$ on X is* constructible *if it satisfies the equivalent conditions below:*

(i) There exists a finite surjective family of sub-schemes X_i of X such that the restriction of $\mathscr{F}$ to X_i is l.c.c..

(ii) There exists a finite family of finite morphisms $p_i : X'_i \to X$, for each i a constant constructible sheaf (= defined by a finite Abelian group) C_i on X'_i, and a monomorphism $\mathscr{F} \to \prod p_{i*}C_i$.

It is easily verified that the category of constructible sheaves on X is an Abelian category. Furthermore, if $u : \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves and if $\mathscr{F}$ is constructible, the sheaf $\mathrm{im}(u)$ is constructible.

Lemme 4.3.3. *Every torsion sheaf $\mathscr{F}$ is the inductive limit of constructible sheaves.*

Indeed, if $j : U \to X$ is a finite type étale scheme over X, an element $\xi \in \mathscr{F}(U)$ such that $n\xi = 0$ defines a homomorphism of sheaves $j_!\underline{\mathbb{Z}/n} \to \mathscr{F}$ whose image (the smallest sub-sheaf of $\mathscr{F}$ for which ξ is a local section) is a constructible sub-sheaf of $\mathscr{F}$. It is clear that $\mathscr{F}$ is the inductive limit of such sub-sheaves.

Définition 4.3.4. *Let $\mathcal{C}$ be an Abelian category and T a functor defined on $\mathcal{C}$ with values in the category of Abelian groups. We will say that T is* effaceable *in $\mathcal{C}$ if, for every object A of $\mathcal{C}$ and every $\alpha \in T(A)$, there exists a monomorphism $u : A \to M$ in $\mathcal{C}$ such that $T(u)\alpha = 0$.*

Lemme 4.3.5. *The functors $\mathrm{H}^q(X, -)$ for $q > 0$ are effaceable in the category of constructible sheaves on X.*

It is enough to notice that, if $\mathscr{F}$ is a constructible sheaf, there necessarily exists an integer $n > 0$ such that $\mathscr{F}$ is a sheaf of $\mathbb{Z}/n$-modules. Then there exists a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$, where $\mathscr{G}$ is a sheaf of $\mathbb{Z}/n$-modules and $\mathrm{H}^q(X, \mathscr{G}) = 0$ for all $q > 0$. For example, we can take for $\mathscr{G}$ the Godement resolution $\prod_{x \in X} i_{x*}\mathscr{F}_{\bar{x}}$, where x ranges over the points of X and $i_x : \bar{x} \to X$ is a geometric point centered at X. According to (4.3.3) $\mathscr{G}$ is an inductive limit of constructible sheaves, whence the lemma, because the functors $\mathrm{H}^q(X, -)$ commute with inductive limits.

Lemme 4.3.6. *Let $\varphi^\bullet : T^\bullet \to T'^\bullet$ be a morphism of cohomological functors defined on an abelian category $\mathcal{C}$ and with values in the category of abelian groups. Suppose that T^q is effaceable for $q > 0$ and let $\mathcal{E}$ be a subset of objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is contained in an object belonging to $\mathcal{E}$. Then the following conditions are equivalent:*

(i) $\varphi^q(A)$ *is bijective for all* $q \geqslant 0$ *and all* $A \in \mathrm{Ob}\,\mathcal{C}$.

(ii) $\varphi^0(M)$ *is bijective and* $\varphi^q(M)$ *is surjective for all* $q > 0$ *and all* $M \in \mathcal{E}$.

(iii) $\varphi^0(A)$ *is bijective for all* $A \in \mathrm{Ob}\,\mathcal{C}$ *and* T'^q *is effaceable for all* $q > 0$.

The demonstration is done by recurrence on q and does not present any difficulties.

Proposition 4.3.7. *Let* X_0 *be a sub-scheme of* X*. Suppose that, for all* $n \geqslant 0$ *and for all finite scheme* X' *over* X*, the canonical application*

$$\mathrm{H}^q(X', \mathbb{Z}/n) \to \mathrm{H}^q(X'_0, \mathbb{Z}/n),$$

where $X'_0 = X' \times_X X_0$*, is bijective for* $q = 0$ *and surjective for* $q > 0$*. Then, for all torsion sheaf* $\mathscr{F}$ *on* X *and for all* $q \geqslant 0$*, the canonical application*

$$\mathrm{H}^q(X, \mathscr{F}) \to \mathrm{H}^q(X_0, \mathscr{F})$$

is bijective.

By passage to the limit, it is enough to demonstrate the assertion for $\mathscr{F}$ constructible. We apply the lemma (4.3.6) by taking for $\mathcal{C}$ the category of constructible sheaves on X, $T^q = \mathrm{H}^q(X, -)$, $T'^q = \mathrm{H}^q(X_0, -)$, and $\mathcal{E}$ the set of constructible sheaves of the form $\prod p_{i*}C_i$, where $p_i : X'_i \to X$ is a finite morphism and C_i is a finite constant sheaf on X'_i.

4.4 The end of the demonstration

By the method of fibration by curves (3.4.3), we are reduced to proving the theorem in relative dimension $\leqslant 1$. According to the previous paragraph, it will suffice to show that, if S is the spectrum of a local noetherian strictly henselian ring, $f : X \to S$ a proper morphism whose closed fiber X_0 has dimension $\leqslant 1$ and n an integer $\geqslant 0$, the canonical homomorphism

$$\mathrm{H}^q(X, \mathbb{Z}/n) \to \mathrm{H}^q(X_0, \mathbb{Z}/n)$$

is bijective for $q = 0$ and surjective for $q > 0$.

The cases $q = 0$ and 1 were seen above and we have $\mathrm{H}^q(X_0, \mathbb{Z}/n) = 0$ for $q \geqslant 3$; it is therefore enough to treat the case $q = 2$. We can obviously assume that n is a power of a prime number. If $n = p^r$, where p is the characteristic of the residue field of S, the Artin-Schreier theory shows that a $\mathrm{H}^2(X_0, \mathbb{Z}/p^r) = 0$. If $n = \ell^r$, $\ell \neq p$, we deduce from Kummer theory a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "\alpha"] \arrow[d] & \mathrm{H}^2(X, \mathbb{Z}/\ell^r) \arrow[d] \\ \mathrm{Pic}(X_0) \arrow[r, "\beta"] & \mathrm{H}^2(X_0, \mathbb{Z}/\ell^r) \end{tikzcd}$$

where the application β is surjective (We saw it in chapter 3 for a smooth curve over an algebraically closed field, but similar arguments apply to any curve over a separably closed field).

To conclude, it will therefore suffice to show:

Proposition 4.4.1. *Let S be the spectrum of a noetherian henselian local ring and $f : X \to S$ a proper morphism whose closed fiber X_0 has dimension $\leqslant 1$. Then the canonical restriction map*

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$$

is surjective (It is even enough that the morphism f is separated *of finite type).*

To simplify the demonstration, we will assume that X is integral, although this is not necessary. Any invertible sheaf on X_0 is associated to a Cartier divisor (because X_0 is a curve, thus quasi-projective), so it suffices to show that the canonical map $\mathrm{Div}(X) \to \mathrm{Div}(X_0)$ is surjective.

Any divisor on X_0 is a linear combination of divisors whose support is concentrated at a single non-isolated closed point of X_0. Let x be such a point, $t_0 \in \mathscr{O}_{X_0,x}$ a regular non-invertible element of $\mathscr{O}_{X_0,x}$, and D_0 the divisor concentrated at x with local equation t_0. Let U be an open neighborhood of x in X such that there exists a section $t \in \Gamma(U, \mathscr{O}_U)$ lifting t_0. Let Y be the closed subset of U with equation $t = 0$; by taking U small enough, we can assume that x is the only point of $Y \cap X_0$. Then Y is quasi-finite over S at x; since S is the spectrum of a henselian local ring, it follows that $Y = Y_1 \amalg Y_2$, where Y_1 is finite over S and where Y_2 does not meet X_0. Moreover, since X is separated over S, Y_1 is closed in X.

We can suppose that $Y = Y_1$, that is, that Y is closed in X, by replacing U with a smaller open neighborhood of x. We then define a divisor D on X relevant to D_0 by setting $D|X \smallsetminus Y = 0$ and $D|U = \mathrm{div}(t)$, which makes sense because t is invertible on $U \smallsetminus Y$.

4.4.2 Remark

In the case where f is proper, we could also give a demonstration of the same style as that of proposition (4.2.2). Indeed, as X_0 is a curve, there is no obstruction to lifting an invertible sheaf on X_0 to the infinitesimal neighborhoods X_n of X_0, and therefore to the formal completion $\mathfrak{X}$ of X along X_0. We then conclude by successively applying the Grothendieck existence theorem and the Artin approximation theorem.

4.5 Proper support cohomology

Définition 4.5.1. *Let X be a separated scheme of finite type over a field k. According to a theorem of Nagata, there exists a proper scheme $\bar{X}$ over k and an open immersion $j : X \to \bar{X}$. For any torsion sheaf $\mathscr{F}$ on X, we denote by $j_!\mathscr{F}$ the extension by* 0 *of $\mathscr{F}$ to $\bar{X}$, and we define the* proper support cohomology groups

$$\mathrm{H}^q_c(X, \mathscr{F}) = \mathrm{H}^q(\bar{X}, j_!\mathscr{F}).$$

We show that this definition is independent of the compactification $j : X \to \bar{X}$ chosen. Let $j_1 : X \to \bar{X}_1$ and $j_2 : X \to \bar{X}_2$ be two compactifications. Then X is mapped to $\bar{X}_1 \times \bar{X}_2$ by $x \mapsto (j_1(x), j_2(x))$ and the closed image $\bar{X}_3$ of X by this application is a compactification of X. We thus have a commutative diagram

$$\begin{tikzcd} & \bar{X}_1 & \\ X \arrow[ur, "j_1"] \arrow[rr, "(j_1,j_2)"] \arrow[dr, "j_2"'] & & \bar{X}_3 \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \\ & \bar{X}_2 & \end{tikzcd}$$

where p_1 and p_2, the restrictions of the natural projections to $\bar{X}_3$, are proper morphisms. It is therefore enough to consider the case where we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_2"] \arrow[dr, "j_1"'] & \bar{X}_2 \arrow[d, "p"] \\ & \bar{X}_1 \end{tikzcd}$$

with p a proper morphism.

Lemme 4.5.2. *On a $p_*(j_{2!}\mathscr{F}) = j_{1!}\mathscr{F}$ et $\mathsf{R}^q p_*(j_{2!}\mathscr{F}) = 0$, for $q > 0$.*

We note immediately that the lemma is sufficient to conclude. Using the Leray spectral sequence of the morphism p, we deduce that we have, for all $q \geqslant 0$,

$$\mathrm{H}^q(\bar{X}_2, j_{2!}\mathscr{F}) = \mathrm{H}^q(\bar{X}_1, j_{1!}\mathscr{F}).$$

To prove the lemma, we reason fiber by fiber using the change of base theorem (4.1.1) for p. The result is immediate, because, over a point of X, p is an isomorphism and, over a point of $\bar{X}_1 \smallsetminus X$, $j_{2!}\mathscr{F}$ is zero on the fiber of p.

4.5.3

Similarly, if $f : X \to S$ is a morphism of noetherian schemes of finite type, there exists a proper morphism $\bar{f} : \bar{X} \to S$ and an open immersion $j : X \to \bar{X}$. We then define the higher direct images with proper support $\mathsf{R}^q f_!$ by setting for any torsion sheaf $\mathscr{F}$ on X

$$\mathsf{R}^q f_! \mathscr{F} = \mathsf{R}^q f_*(j_!\mathscr{F}).$$

We check as before that this definition is independent of the compactification chosen.

Théorème 4.5.4. *Let $f : X \to S$ be a separated finite type morphism of noetherian schemes and $\mathscr{F}$ a torsion sheaf on X. Then the fiber of $\mathsf{R}^q f_! \mathscr{F}$ at a geometric point s of S is isomorphic to the cohomology with compact support $\mathrm{H}^q_c(X_s, \mathscr{F})$ of the fiber X_s of f at s.*

This is a simple variant of the base change theorem for a proper morphism (4.1.1). More generally, if

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ S & S' \arrow[l, "g"] \end{tikzcd}$$

is a cartesian diagram, there is a canonical isomorphism

$$g^*(\mathsf{R}^q f_! \mathscr{F}) \simeq \mathsf{R}^q f'_!(g'^* \mathscr{F}). \tag{4.5.4.1}$$

4.6 Applications

Théorème 4.6.1 (d'annulation)**.** *Let $f : X \to S$ be a separated finite type morphism whose fibers have dimension $\leqslant n$ and $\mathscr{F}$ a torsion sheaf on X. Then we have $\mathsf{R}^q f_! \mathscr{F} = 0$ for $q > 2n$.*

According to the change of base theorem, we can assume that S is the spectrum of a separably closed field. If $\dim X = n$, there is an affine open U of X such that $\dim(X \smallsetminus U) < n$; we then have an exact sequence $0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_{X \smallsetminus U} \to 0$ and, by induction on n, it is enough to prove the theorem for $X = U$ affine. Then the method of fibration by curves (3.4.1) and the change of base theorem allow us to reduce to a curve over a separably closed field for which we deduce the desired result from the Tsen theorem (3.3.6).

Théorème 4.6.2 (de finitude)**.** *Let $f : X \to S$ be a separated finite type morphism and $\mathscr{F}$ a constructible sheaf on X. Then the sheaves $\mathsf{R}^q f_! \mathscr{F}$ are constructible.*

We only consider the case where $\mathscr{F}$ is annihilated by an invertible integer on X.

To prove the theorem, we reduce to the case where $\mathscr{F}$ is a constant sheaf $\underline{\mathbb{Z}/n}$ and where $f : X \to S$ is a proper and smooth morphism whose fibers are geometrically connected curves of genus g. For n invertible on X, the sheaves $\mathsf{R}^q f_* \mathscr{F}$ are then locally finitely generated and free, and vanish for $q > 2$ (4.6.1). Replacing $\mathbb{Z}/n$ by the locally isomorphic sheaf (on S) $\boldsymbol{\mu}_n$, we have canonically

$$\begin{aligned} \mathsf{R}^0 f_* \boldsymbol{\mu}_n &= \boldsymbol{\mu}_n \\ \mathsf{R}^1 f_* \boldsymbol{\mu}_n &= \underline{\mathrm{Pic}}(X/S)_n \\ \mathsf{R}^2 f_* \boldsymbol{\mu}_n &= \mathbb{Z}/n. \end{aligned} \tag{4.6.2.1}$$

Théorème 4.6.3 (comparison with classical cohomology)**.** *Let $f : X \to S$ be a separated morphism of finite type schemes over $\mathbb{C}$, and $\mathscr{F}$ a torsion sheaf on X. Denote by an exponent $(-)^{\mathrm{an}}$ the functor of passage to the usual topological spaces, and by $\mathsf{R}^q f_!^{\mathrm{an}}$ the derived functors of the direct image functor with proper support by f^{an}. We have*

$$(\mathsf{R}^q f_! \mathscr{F})^{\mathrm{an}} \simeq \mathsf{R}^q f_!^{\mathrm{an}} \mathscr{F}^{\mathrm{an}}.$$

In particular, for $S =$ a point and $\mathscr{F}$ the constant sheaf $\mathbb{Z}/n$,

$$\mathrm{H}^q_c(X, \mathbb{Z}/n) \simeq \mathrm{H}^q_c(X^{\mathrm{an}}, \mathbb{Z}/n).$$

Unscrewings using the change of basis theorem bring us back to the case where X is a smooth, proper curve, where $S =$ a point, and where $\mathscr{F} = \mathbb{Z}/n$. The cohomology groups considered are then null for $q \neq 0, 1, 2$, and we invoke [30]: indeed, if X is proper over $\mathbb{C}$, we have $\pi_0(X) = \pi_0(X^{\mathrm{an}})$ and $\pi_1(X) =$ profinite completion of $\pi_1(X^{\mathrm{an}})$, from which the assertion for $q = 0, 1$ follows. For $q = 2$, we use the Kummer exact sequence and the fact that, by [30] again, $\mathrm{Pic}(X) = \mathrm{Pic}(X^{\mathrm{an}})$.

Théorème 4.6.4 (dimension cohomologique des schémas affines)**.** *Let X be a finite type scheme over a separably closed field and $\mathscr{F}$ a torsion sheaf on X. Then we have $\mathrm{H}^1(X, \mathscr{F}) = 0$ for $q > \dim(X)$.*

For the very pretty proof we refer to [2], XIV §2 and 3.

4.6.5 Remarque

This theorem is in some sense a substitute for the Morse theory. Consider for example the classical case where X is smooth and affine over $\mathbb{C}$ embedded in an affine space of type $\mathbb{C}^N$. Then, for almost every point $p \in \mathbb{C}^N$, the function "distance to p" on X is a Morse function and the indices of its critical points are smaller than $\dim(X)$. Thus X is obtained by gluing handles of index smaller than $\dim(X)$, from which the classical analogue of (4.6.4) follows.

5 Local acyclicity of smooth morphisms

Let X be a complex analytic variety and $f : X \to D$ be a morphism from X to the disk. We denote $[0, t]$ the closed right segment with endpoint 0 and t in D and $(0, t]$ the open right segment. If f is *smooth*, the inclusion

$$j : f^{-1}\left((0,t]\right) \hookrightarrow f^{-1}\left([0,t]\right)$$

is a homotopy equivalence; we can push the special fiber $X_0 = f^{-1}(0)$ into $f^{-1}([0, t])$.

In practice, for t small enough, $f^{-1}((0, t])$ will be a fiber bundle over $(0, t]$ so that the inclusion

$$X_t = f^{-1}(t) \hookrightarrow f^{-1}\left((0,t]\right)$$

will also be a homotopy equivalence. We then call the homotopy class of applications the *cospacialization morphism*:

$$\mathrm{cosp} : X_0 \hookrightarrow f^{-1}\left([0,t]\right) \xleftarrow{\sim} f^{-1}\left((0,t]\right) \xleftarrow{\sim} X_t.$$

We can express this construction in terms of images by saying that, for a smooth morphism, the general fiber swallows the special fiber.

Do not assume that f is necessarily smooth (but assume that $f^{-1}((0, t])$ is a fiber bundle over $(0, t]$). We can still define a morphism $\mathrm{cosp}^\bullet$ in cohomology as long as $j_*\mathbb{Z} = \mathbb{Z}$ and $\mathsf{R}^q j_*\mathbb{Z} = 0$ for $q > 0$. Under these hypotheses, the Leray spectral sequence for j shows that we have

$$\mathrm{H}^\bullet\left(f^{-1}\left([0,t]\right), \mathbb{Z}\right) \xrightarrow{\sim} \mathrm{H}^\bullet\left(f^{-1}\left((0,t]\right), \mathbb{Z}\right)$$

and $\mathrm{cosp}^\bullet$ is the composed morphism:

$$\mathrm{cosp}^\bullet : \mathrm{H}^\bullet(X_t, \mathbb{Z}) \xleftarrow{\sim} \mathrm{H}^\bullet\left(f^{-1}\left((0,t]\right), \mathbb{Z}\right) \xleftarrow{\sim} \mathrm{H}^\bullet\left(f^{-1}\left([0,t]\right), \mathbb{Z}\right) \to \mathrm{H}^\bullet(X_0, \mathbb{Z})$$

The fiber of $\mathsf{R}^q j_*\mathbb{Z}$ at a point $x \in X_0$ is calculated as follows. We take in an ambient space a ball B_ε of center x and radius ε small enough, and for η small enough, we set $E = X \cap B_\varepsilon \cap f^{-1}(\eta t)$; this is the *vanishing cycle variety* at x. We have

$$(\mathsf{R}^q j_*\mathbb{Z})_x \xleftarrow{\sim} \mathrm{H}^q\left(X \cap B_\varepsilon \cap f^{-1}\left((0,\eta t]\right), \mathbb{Z}\right) \xrightarrow{\sim} \mathrm{H}^q(E, \mathbb{Z})$$

and the cospacialization morphism is defined in cohomology as soon as the vanishing cycle varieties are acyclic ($\mathrm{H}^0(E, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{H}^q(E, \mathbb{Z}) = 0$ for $q > 0$), which is expressed by saying that f is *locally acyclic*.

This chapter is dedicated to the analogue of this situation for a smooth morphism of schemes for étale cohomology. However, it is essential in this context to limit ourselves to torsion coefficients and *prime to the residual characteristics*. Paragraph 5.1 is dedicated to generalities on locally acyclic morphisms and specialization arrows. In paragraph **??**, we prove that a smooth morphism is locally acyclic. In paragraph 5.2, we join this result to those of the previous chapter to deduce two applications: a specialization theorem for cohomology groups (the cohomology of the geometric fibers of a proper and smooth morphism is locally constant) and a change of base theorem by a smooth morphism.

Throughout, we fix an integer n and "scheme" means "scheme on which n is invertible." "Geometric point" will always mean "algebraic geometric point" (2.3.1) $x : \mathrm{Spec}(k) \to X$, with k algebraically closed.

5.1 Morphisms locally acyclic

5.1.1 Notation

Given a scheme S and a geometric point s of S, we denote by $\widetilde{S}^s$ the spectrum of the strict localization of S at s.

Définition 5.1.2. *We say that a geometric point t of S is a* generalization *of s if it is defined by an algebraic closure of the residue field of a point of $\widetilde{S}^s$. We also say that s is a* specialization *of t and we call the* specialization arrow *the S-morphism $t \to \widetilde{S}^s$.*

Définition 5.1.3. *Let $f : X \to S$ be a morphism of schemes. Let s be a geometric point of S, t a generalization of s, x a geometric point of X over s and $\widetilde{X}^x_t = \widetilde{X}^x \times_{\widetilde{S}^s} t$. Then we say that $\widetilde{X}^x_t$ is a* variety of vanishing cycles *of f at the point x.*

It is said that f is locally acyclic *if, the reduced cohomology of every vanishing cycle variety $\widetilde{X}^x_t$ is trivial:*

$$\hat{\mathrm{H}}^\bullet(\widetilde{X}^x_t, \mathbb{Z}/n) = 0, \tag{5.1.3.1}$$

i.e. $\mathrm{H}^0\,(X^x_t, \mathbb{Z}/n) = \mathbb{Z}/n$ *and* $\mathrm{H}^q(\widetilde{X}^x_t, \mathbb{Z}/n) = 0$ *for* $q > 0$.

Lemme 5.1.4. *Let $f : X \to S$ be a locally acyclic morphism and $g : S' \to S$ be a quasi-finite (or projective limit of quasi-finite morphisms). Then the morphism $f' : X' \to S'$ deduced from f by change of base is locally acyclic.*

It is checked that every vanishing cycle variety of f' is a vanishing cycle variety of f.

Lemme 5.1.5. *Let $f : X \to S$ be a locally acyclic morphism. For every geometric point t of S, giving rise to a Cartesian diagram*

$$\begin{tikzcd} X_t \arrow[r, "\varepsilon'"] \arrow[d] & X \arrow[d, "f"] \\ t \arrow[r, "\varepsilon"] & S \end{tikzcd}$$

We have $\varepsilon'_*\mathbb{Z}/n = f^*\varepsilon_*\mathbb{Z}/n$ and $\mathsf{R}^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$.

Let $\bar{S}$ be the closure of $\varepsilon(t)$, S' the normalization of $\bar{S}$ in $k(t)$, and the Cartesian diagram

$$\begin{tikzcd} X_t \arrow[r, "f'"] \arrow[d] & X' \arrow[r, "\alpha'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ t \arrow[r, "i"] & S' \arrow[r, "\alpha"] & S \end{tikzcd}$$

The local rings of S' are normal with separable closure fields. They are therefore strictly Henselian, and the local acyclicity of f' (5.1.4) provides $f'_*\mathbb{Z}/n = \mathbb{Z}/n$, $\mathsf{R}^q i'_*\mathbb{Z}/n = 0$ for $q > 0$. Since α is integral, we then have

$$\mathsf{R}^q\varepsilon'_*\mathbb{Z}/n = \alpha'_*\mathsf{R}^q i'_*\mathbb{Z}/n = \alpha'_* f'^*\mathsf{R}^q i_*\mathbb{Z}/n = f^*\alpha_*\mathsf{R}^q i_*\mathbb{Z}/n = f^*\mathsf{R}^q\varepsilon_*\mathbb{Z}/n,$$

and the lemma.

5.1.6

Given a locally acyclic morphism $f : X \to S$ and a specialization arrow $t \to \widetilde{S}^s$, we will define canonical homomorphisms, called specialization arrows

$$\mathrm{cosp}^\bullet : \mathrm{H}^\bullet(X_t, \mathbb{Z}/n) \to \mathrm{H}^\bullet(X_s, \mathbb{Z}/n),$$

relating the cohomology of the general fiber $X_t = X \times_S t$ and that of the special fiber $X_s = X \times_X s$.

Consider the cartesian diagram

$$\begin{tikzcd} X_t \arrow[r, "\varepsilon'"] \arrow[d] & \widetilde{X} \arrow[d, "f'"] & X_s \arrow[l] \arrow[d] \\ t \arrow[r, "\varepsilon"] & \widetilde{S}^s & s \arrow[l] \end{tikzcd}$$

derived from f by base change. By (5.1.4), f' is still locally acyclic. From the definition of local acyclicity, we immediately deduce that the restriction to X_s of the sheaves $\mathsf{R}^q\varepsilon'_*\mathbb{Z}/n$ is $\mathbb{Z}/n$ for $q = 0$, and 0 for $q > 0$. By (5.1.5) we even know that $\mathsf{R}^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$. We define $\mathrm{cosp}^\bullet$ as the composite arrow

$$\mathrm{H}^\bullet(X_t, \mathbb{Z}/n) \simeq \mathrm{H}^\bullet(X, \varepsilon'_*\mathbb{Z}/n) \to \mathrm{H}^\bullet(X_s, \mathbb{Z}/n). \tag{5.1.6.1}$$

Variants Let $\bar{S}$ be the closure of $\varepsilon(t)$ in $\widetilde{S}^s$, S' the normalization of $\bar{S}$ in $k(t)$ and X'/S' the one deduced from X/S by base change. The diagram (5.1.6.1) can also be written

$$\mathrm{H}^\bullet(X_t, \mathbb{Z}/n) \simeq \mathrm{H}^\bullet(X', \mathbb{Z}/n) \to \mathrm{H}^\bullet(X_s, \mathbb{Z}/n).$$

Théorème 5.1.7. *Let S be a locally noetherian scheme, s a geometric point of S and $f : X \to S$ a morphism. We assume*

a) the morphism f is locally acyclic,

b) for any specialization morphism $t \to \widetilde{S}^s$ and for any $q \geqslant 0$, the cosepecialization arrows $\mathrm{H}^q(X_t, \mathbb{Z}/n) \to \mathrm{H}^q(X_s, \mathbb{Z}/n)$ are bijective.

Then the canonical homomorphism $(\mathsf{R}^q f_*\mathbb{Z}/n)_s \to \mathrm{H}^q(X_s, \mathbb{Z}/n)$ is bijective for any $q \geqslant 0$.

To demonstrate the theorem, it is clear that we can assume $S = \widetilde{S}^s$. We will show that, for any sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules $\mathscr{F}$ on S, the canonical homomorphism $\varphi^q(\mathscr{F}) : (\mathsf{R}^q f_* f^*\mathscr{F})_s \to \mathrm{H}^q(X_s, f^*\mathscr{F})$ is bijective.

Any sheaf of $\mathbb{Z}/n\mathbb{Z}$ is the inductive limit of a filtered system of constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (4.3.3). Furthermore, any constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules can be embedded in a sheaf of the form $\prod i_{\lambda *}C_\lambda$, where $i_\lambda : t_\lambda \to S$ is a finite family of points specializing to s and C_λ is a free $\mathbb{Z}/n\mathbb{Z}$-module of finite rank over t_λ. According to the definition of the specialization maps, condition b) means that the homomorphisms $\varphi^q(\mathscr{F})$ are bijective if $\mathscr{F}$ is of this form.

We conclude using a variant of lemma (4.3.6):

Lemme 5.1.8. *Let $\mathcal{C}$ be an abelian category in which filtered inductive limits exist. Let $\varphi^\bullet : T^\bullet \to T'^\bullet$ be a morphism of cohomological functors commuting with filtered inductive limits, defined on $\mathcal{C}$ and with values in the category of abelian groups. Suppose that there exist two subsets $\mathcal{D}$ and $\mathcal{E}$ of objects of $\mathcal{C}$ such that:*

a) every object of $\mathcal{C}$ is a filtered inductive limit of objects belonging to $\mathcal{D}$,

b) every object belonging to $\mathcal{D}$ is contained in an object belonging to $\mathcal{E}$.

Then the following conditions are equivalent:

(i) $\varphi^q(A)$ is bijective for all $q \geqslant 0$ and all $A \in \mathrm{Ob}\,\mathcal{C}$.

(ii) $\varphi^q(M)$ is bijective for all $q \geqslant 0$ and all $M \in \mathcal{E}$.

The demonstration of the lemma is done by induction, recurrence on q and repeated application of the five lemma to the cohomology exact sequence derived from an exact sequence $0 \to A \to M \to A' \to 0$, with $A \in \mathcal{D}$, $M \in \mathcal{E}$, $A' \in \mathrm{Ob}\,\mathcal{C}$.

Corollaire 5.1.9. *Let S be the spectrum of a local noetherian strictly henselian ring and $f : X \to S$ a locally acyclic morphism. Suppose that, for every geometric point t of S, we have $\mathrm{H}^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $\mathrm{H}^q(X_t, \mathbb{Z}/n) = 0$ for $q > 0$ (in other words, the geometric fibers of f are acyclic). Then we have $f_*\mathbb{Z}/n\mathbb{Z}/n$ and $\mathsf{R}^q f_*\mathbb{Z}/n = 0$ for $q > 0$.*

Corollaire 5.1.10. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms of locally noetherian schemes. Then, if f and g are locally acyclic, so is $g \circ f$.*

We can assume that X, Y and Z are strictly local and that f and g are local morphisms. We need to show that, if z is an algebraic geometric point of Z, we have $\mathrm{H}^0(X_z, \mathbb{Z}/n) = \mathbb{Z}/n$ and $\mathrm{H}^q(X_z, \mathbb{Z}/n) = 0$ for $q > 0$.

Since g is locally acyclic, we have $\mathrm{H}^0(Y_z, \mathbb{Z}/n) = \mathbb{Z}/n$, and $\mathrm{H}^q(Y_z, \mathbb{Z}/n) = 0$ for $q > 0$. Furthermore, the morphism $f_z : X_z \to Y_z$ is locally acyclic (5.1.4) and its geometric fibers are acyclic, because they are vanishing cycle varieties of f. According to (5.1.9), we therefore have $\mathsf{R}^q f_{z*}\mathbb{Z}/n = 0$ for $q > 0$. Moreover $f_{z*}\mathbb{Z}/n$ is constant of fiber $\mathbb{Z}/n$ on Y_z. We conclude using the Leray spectral sequence of f_z.

Théorème 5.1.11. *A smooth morphism is locally acyclic.*

Let $f : X \to S$ be a smooth morphism. The assertion is local for the étale topology on X and S, so we can assume that X is the affine space type of dimension d over S. By passage to the limit, we can assume that S is noetherian and the transitivity of the local acyclicity (5.1.10) shows that it is enough to treat the case $d = 1$.

Let s be a geometric point of S and x be a geometric point of X centered at a closed point of X_s. It is enough to show that the geometric fibers of the morphism $\widetilde{X}^x \to \widetilde{S}^s$ are acyclic. We will now set $S = \widetilde{S}^s = \mathrm{Spec}(A)$ and $X = \widetilde{X}^x$. We have $X \simeq \mathrm{Spec}\, A\{T\}$, where $A\{T\}$ is the henselization of $A[T]$ at the point $T = 0$ over s.

If t is a geometric point of S, the fiber X_t is the projective limit of affine and smooth curves over t. So we have $\mathrm{H}^q(X_t, \mathbb{Z}/n) = 0$ for $q \geqslant 0$ and it is enough to show that $\mathrm{H}^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $\mathrm{H}^1(X_t, \mathbb{Z}/n) = 0$ for n prime to the residual characteristic of S. This follows from the two following propositions.

Proposition 5.1.12. *Let A be a strictly henselian local ring, $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}\, A\{T\}$. Then the geometric fibers of $X \to S$ are connected.*

We can by going to the limit to the case where A is a strict henselization of a $\mathbb{Z}$-algebra of finite type.

Let $\bar{t}$ be a geometric point of S, localized at t, and k' a finite separable extension of $k(t)$ in $k(\bar{t})$. We set $t' = \mathrm{Spec}(k')$ and $X_{t'} = X \times_S \mathrm{Spec}(k')$. We must check that, for any $\bar{t}$ and t', $X_{t'}$ is connected (by which we mean connected and nonempty). Let A' be the normalization of A in k', i.e. the ring of elements of k' integral over the image of A in $k(t)$. We have $A\{t\} \otimes_A A' \xrightarrow{\sim} A'\{T\}$: the left member is in fact henselian local (because A' is finite over A, and local) and limit of local étale algebras over $A'\{T\} = A\{T\} \otimes_A A'$. The scheme $X_{t'}$ is thus still the fiber over t' of $X = \mathrm{Spec}\,(A'\{T\})$ over $S' = \mathrm{Spec}(A')$. The local scheme X' is normal, so it is integral; its localization $X_{t'}$ is still integral, hence connected.

Proposition 5.1.13. *Let A be a strictly henselian local ring, $S = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}\,(A\{T\})$. Let $\bar{t}$ be a geometric point of S and $X_{\bar{t}}$ the corresponding geometric fiber. Then every Galois étale covering of $X_{\bar{t}}$ of prime order to the characteristic of the residue field of A is trivial.*

Lemme 5.1.14 (Zariski-Nagata purity theorem in dimension 2)**.** *Let C be a regular local ring of dimension 2 and C' a finite étale C-algebra normal over the complement of the closed point of $\mathrm{Spec}(C)$. Then C' is étale over C.*

Indeed C' is normal of dimension 2, so $\mathrm{prof}(C') = 2$. Since $\mathrm{prof}(C') + \dim \mathrm{proj}(C') = \dim(C) = 2$, it follows that C' is free over C. Then the set of points of C where C' is ramified is defined by an equation, the discriminant; since it contains no point of height 1, it is empty.

Lemme 5.1.15 (a special case of Abhyankar's lemma)**.** *Let $S = \mathrm{Spec}(V)$ be a line, π a uniformizer, η the generic point of S, X smooth over S, irreducible, of relative dimension 1, $\widetilde{X}_\eta$ a Galois étale covering of X_η, of degree n invertible on S, and $S_1 = \mathrm{Spec}\left(V[\pi^{1/n}]\right)$. We denote by $(-)_1$ the change of base from S to S_1. Then, $\widetilde{X}_{1\eta}$ extends to an étale covering of X_1.*

Let $\widetilde{X}_1$ be the normalization of X_1 of $\widetilde{X}_{1\eta}$. Given the structure of the moderate inertia groups of the local rings of X at the generic points of the special fiber X_s, $\widetilde{X}_1$ is étale over X_1 on the generic fiber, and at the generic points of the special fiber. By (5.1.14), it is étale everywhere.

5.1.16

Let t be the point where $\bar{t}$ is located. We can replace A with the normalized of A in a separable finite extension $k(t')$ of $k(t)$ in $k(\bar{t})$ (cf. 5.1.12). This, and a preliminary limit, allow us to assume that

a) A is normal noetherian, and t is the generic point of S.

b) The étale covering considered of $X_{\bar{t}}$ comes from an étale covering of X_t.

c) It comes from an étale covering $\widetilde{X}_U$ of the inverse image X_U of a nonempty open U of S (results from b): t is the limit of the U).

d) The complement of U is of codimension $\geqslant 2$ (this by enlarging $k(t)$, by applying (5.1.15) to the discrete valuation rings localized at points of $S \smallsetminus U$ of codimension 1 in S; these points are in finite number).

e) The covering $\widetilde{X}_U$ is trivial over the sub-scheme $T = 0$. (This by further enlarging $k(t)$).

When these conditions are met, we will see that the covering $\widetilde{X}_U$ is trivial.

Lemme 5.1.17. *Let A be a strictly henselian normal and noetherian local ring, U an open subset of* $\mathrm{Spec}(A)$ *whose complement has codimension $\geqslant 2$, V its image in $X = \mathrm{Spec}\,(A\{T\})$, and V' an étale covering of V. If V' is trivial over $T = 0$, then V' is trivial.*

Let $B = \Gamma(V', \mathscr{O})$. Since V' is the inverse image of V in $\mathrm{Spec}(B)$, it suffices to see that B is finite étale over $A\{T\}$ (hence decomposed, since $A\{T\}$ is strictly henselian). Let $\widehat{X} = A[\![T]\!]$, and denote by $(-)^\wedge$ the change of base of X to $\widehat{X}$. The scheme $\widehat{X}$ is faithfully flat over X. We therefore have $\Gamma(\widehat{V}', \mathscr{O}) = B \otimes_{A\{T\}} A[\![T]\!]$, and it suffices to see that this ring $\widehat{B}$ is finite étale over $A[\![T]\!]$.

Let V_m (resp. V'_m) be the sub-scheme of $\widehat{V}$ (resp. $\widehat{V}'$) with equation $T^{m+1} = 0$. By hypothesis, V'_0 is a trivial covering of V_0: a sum of n copies of V_0. The same is true for V'_m/V_m, since étale covers are insensitive to nilpotents. We deduce that

$$\varphi : \Gamma(\widehat{V}', \mathscr{O}) \to \varprojlim_m \Gamma(V'_m, \mathscr{O}) = \left(\varprojlim_m \Gamma(V, \mathscr{O})\right)^n .$$

By hypothesis, the complement of U has depth $\geqslant 2$: we have $\Gamma(V_m, \mathscr{O}) = A[T]/(T^{m+1})$, and φ is a homomorphism of $\widehat{B}$ into $A[\![T]\!]^n$. Over U, it provides n distinct sections of $\widehat{V}'/\widehat{V}$: $\widehat{V}'$ is therefore trivial, a sum of n copies of $\widehat{V}$. The complement of $\widehat{V}$ in $\widehat{X}$ being of codimension $\geqslant 2$ (thus of depth $\geqslant 2$), we deduce that $\widehat{B} = A[\![T]\!]^n$, from which the lemma follows.

5.2 Applications

Théorème 5.2.1 (specialization of cohomology groups)**.** *Let $f : X \to S$ be a proper and locally acyclic map, for example a proper and smooth map. Then the sheaves $\mathsf{R}^q f_* \mathbb{Z}/n$ are locally constant constructible and for any specialization arrow $t \to \widetilde{S}^s$, the cospécialization arrows $\mathrm{H}^q(X_t, \mathbb{Z}/n) \to \mathrm{H}^q(X_s, \mathbb{Z}/n)$ are bijective.*

This follows immediately from the definition of cospécialization arrows and the finiteness and change of base theorems for proper maps.

Théorème 5.2.2 (change of base by a smooth map)**.** *Consider a cartesian diagram*

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ S' @>{g}>> S \end{CD}$$

with g smooth. For any sheaf $\mathscr{F}$ on X, of torsion and prime to the residual characteristics of S, we have

$$g^*\mathsf{R}^q f_*\mathscr{F} \xrightarrow{\sim} \mathsf{R}^q f'_*(g'^*\mathscr{F}).$$

By taking an open cover of $X<$ we may assume that X is affine, and then by taking a limit we may assume that X is of finite type over S. Then f factors as an open immersion $j : X \to \bar{X}$ and a proper morphism $\bar{f} : \bar{X} \to S$. From the Leray spectral sequence for $\bar{f} \circ j$ and the change of base theorem for proper morphisms, we deduce that it is enough to prove the theorem in the case where $X \to S$ is an open immersion.

In this case, if $\mathscr{F}$ is of the form $\varepsilon_* C$, where $\varepsilon : t \to X$ is a geometric point of $X<$ the theorem is a corollary of (5.1.5). The general case follows from the lemma (5.1.8).

Corollaire 5.2.3. *Let K/k be a separably closed extension of fields, X a k-scheme and n an integer prime to the characteristic of k. Then the canonical map $\mathrm{H}^q(X, \mathbb{Z}/n) \to \mathrm{H}^q(X_K, \mathbb{Z}/n)$ is bijective for all $q \geqslant 0$.*

It is enough to notice that $\bar{K}$ is the inductive limit of $\bar{k}$-smooth algebras.

Théorème 5.2.4 (relative purity)**.** *Let there be a commutative diagram*

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr] & X \arrow[d, "f"] & Y \arrow[l, hook', "i"'] \arrow[dl, "h"] \\ & S & \end{tikzcd} \tag{5.3.4.1}$$

with f purely of relative dimension N, h purely of relative dimension $N-1$, i a closed immersion, and $U = X \smallsetminus Y$. For n prime to the residual characteristics of S, we have

$$\begin{aligned} j_*\mathbb{Z}/n &= \mathbb{Z}/n \\ \mathsf{R}^1 j_*\mathbb{Z}/n &= \mathbb{Z}/n(-1)_Y \\ \mathsf{R}^q j_*\mathbb{Z}/n &= 0 \qquad \textit{for } q \geqslant 2 \end{aligned}$$

In these formulas, $\mathbb{Z}/n(-1)$ denotes the $\mathbb{Z}/n$-dual of $\boldsymbol{\mu}_n$. If t is a local equation for Y, the isomorphism $\mathsf{R}^1 j_*\mathbb{Z}/n \simeq \mathbb{Z}/n(-1)_Y$ is defined by the map $a : \mathbb{Z}/n \to \mathsf{R}^1 j_*\boldsymbol{\mu}_n$ which sends 1 to the class of the $\boldsymbol{\mu}_n$-torsor of n-th roots of t.

The question is of a local nature. This allows us to replace (X, Y) with a locally isomorphic pair, for example

$$\begin{tikzcd} \mathbb{A}^1_T \arrow[r, hook, "j"] \arrow[dr, "g"'] & \mathbb{P}^1_T \arrow[d, "f"] & T \arrow[l, hook', "i"'] \arrow[dl] \\ & T & \end{tikzcd}$$

with $T = \mathbb{A}^{n-1}_S$ and $i =$ section at infinity. Corollary (5.1.9) applies to g, and provides $\mathsf{R}^q g_{\mathbb{Z}}/n = \mathbb{Z}/n$ for $q = 0$, 0 for $q > 0$. For f, we also have (4.6.2.1)

$$\mathsf{R}^q f_*\mathbb{Z}/n = \mathbb{Z}/n, 0, \mathbb{Z}/n(-1), 0 \text{ for } q > 2.$$

We easily check that $j_*\mathbb{Z}/n = \mathbb{Z}/n$, and that the $\mathsf{R}^q j_*\mathbb{Z}/n = 0$ are concentrated on $i(T)$ for $q > 0$. The Leray spectral sequence $E_2^{pq} = \mathsf{R}^p f_* \mathsf{R}^q j_* \mathbb{Z}/n \Rightarrow \mathsf{R}^{p+q} g_* \mathbb{Z}/n$ therefore reduces to

$$\begin{tikzcd}
i^*\mathsf{R}^q j_*\mathbb{Z}/n & 0 & \cdots & & \\
i^*\mathsf{R}^1 j_*\mathbb{Z}/n \arrow[drr, "d_2"] & 0 & \cdots & & \\
\mathbb{Z}/n & 0 & \mathbb{Z}/n(-1) & 0 & \cdots
\end{tikzcd}$$

$\mathsf{R}^q j_*\mathbb{Z}/n = 0$ for $q \geqslant 0$, and $\mathsf{R}^q j_*\mathbb{Z}/n$ is the zero extension of a locally free sheaf of rank one on T (isomorphic, via d_2, to $\mathbb{Z}/n(-1)$). The map a, defined above, being injective (as we check fiber by fiber), it is an isomorphism, and this proves (5.2.4).

5.2.5

We refer to [2, XVI.4,5] for the proof of the following applications of the acyclicity theorem (5.1.11).

Théorème 5.2.6. *Let $f : X \to S$ be a morphism of schemes of finite type over $\mathbb{C}$ and $\mathscr{F}$ a constructible sheaf on X. Then*

$$(\mathsf{R}^q f_* \mathscr{F})^{\mathrm{an}} \sim \mathsf{R}^q f_*^{\mathrm{an}}(\mathscr{F}^{\mathrm{an}})$$

(cf. 4.6.3: in ordinary cohomology, it is necessary to assume $\mathscr{F}$ is constructible and not just of torsion).

Théorème 5.2.7. *Let $f : X \to S$ be a morphism of schemes of finite type over a field k of characteristic 0 and $\mathscr{F}$ a constructible sheaf on X. Then, the $\mathsf{R}^q f_* \mathscr{F}$ are also constructible.*

The proof uses the resolution of singularities and (5.2.4). It is generalized to the case of a morphism of finite type of excellent schemes of characteristic 0 in [2, XIX.5]. Another proof, independent of the resolution, is given in this volume (Finiteness theorems in ℓ-adic cohomology, 1.1) it applies to a morphism of schemes of finite type over a field or over a Dedekind ring.

6 Poincaré duality

6.1 Introduction

Let X be an oriented topological space, purely of dimension N, and assume that X admits a finite open cover $\mathcal{U} = (U_i)_{1 \leqslant i \leqslant K}$, such that the non-empty intersections of open sets U_i are homeomorphic to balls. For such a space, the Poincaré duality theorem can be presented as follows:

A The cohomology of X is the Čech cohomology corresponding to the cover $\mathcal{U}$. It is the cohomology of the complex

$$0 \to \mathbb{Z}^{A_0} \to \mathbb{Z}^{A_1} \to \cdots \tag{1}$$

where $A_k = \{(i_0, \ldots, i_k) : i_0 < \cdots < i_k \text{ and } U_{i_0} \cap \cdots \cap U_{i_k} \neq \varnothing\}$.

B For $a \in A_k$, $a = (i_0, \dots, i_k)$, let $U_a = U_{i_1} \cap \cdots \cap U_{i_k}$ and let j_a be the inclusion of U_a into X. The constant sheaf $\mathbb{Z}$ on X admits the resolution (on the left)

$$\begin{tikzcd} \cdots \arrow[r] & \bigoplus_{a \in A_1} j_{a!}\mathbb{Z} \arrow[r] & \bigoplus_{a \in A_0} j_{a!}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ & & \mathbb{Z}. & \end{tikzcd} \tag{2}$$

The compactly supported cohomology $\mathrm{H}_c^\bullet(X, j_{a!}\mathbb{Z})$ is none other than the proper cohomology of the (oriented) ball U_a:

$$\mathrm{H}_c^i(X, j_{a!}\mathbb{Z}) = \begin{cases} 0 & \text{if } i \neq N \\ \mathbb{Z} & \text{if } i = N \end{cases}$$

The spectral sequence of hypercohomology, for the complex (2), and the cohomology with compact support, therefore affirms that $\mathrm{H}_c^i(X)$ is the $(i-N)$-th cohomology group of a complex

$$\cdots \to \mathbb{Z}^{A_1} \to \mathbb{Z}^{A_0} \to 0. \tag{3}$$

This complex is the dual of the complex (1), from which the Poincare duality.

The key points of this construction are

a) the existence of a cohomology theory with compact support:

b) the fact that every point x of a variety X purely of dimension N has a fundamental system of open neighborhoods U for which

$$\mathrm{H}_c^i(U) = \begin{cases} 0 & \text{for } i \neq N, \\ \mathbb{Z} & \text{for } i = N. \end{cases} \tag{4}$$

The Poincaré duality in étale cohomology can be constructed on this model. For X a smooth, pure N-dimensional variety over an algebraically closed field k, n invertible on X, and x a closed point of X, the key point is to calculate the projective limit, extended to étale neighborhoods U of x

$$\varprojlim \mathrm{H}_c^i(U, \mathbb{Z}/n) = \begin{cases} 0 & \text{if } i \neq 2N \\ \mathbb{Z}/n & \text{if } i = 2N. \end{cases} \tag{5}$$

Just as for topological varieties one must first directly treat the case of an open ball (or even just the case of the interval $(0,1)$), here one must first directly treat the case of curves (6.2). The local acyclicity theorem for smooth morphisms then allows us to reduce (5) to this particular case (6.3).

The isomorphisms (4) and (5) are not canonical: they depend on the choice of an *orientation* of X. For n invertible on a scheme X, $\boldsymbol{\mu}_n$ is a sheaf of free $\mathbb{Z}/n$-modules of rank one. We denote by $\mathbb{Z}/n(N)$ its N-th tensor power ($N \in \mathbb{Z}$). The intrinsic form of the second line of (5) is

$$\varprojlim \mathrm{H}_c^{2N}(U, \mathbb{Z}/n(N)) = \mathbb{Z}/n, \tag{6}$$

and $\mathbb{Z}/n(N)$ is called the *orientation sheaf* of X; the sheaf $\mathbb{Z}/n(N)$ being constant, isomorphic to $\mathbb{Z}/n$, we can take it out of the N sign and write rather

$$\varprojlim \mathrm{H}_c^{2N}(U,\mathbb{Z}/n) = \mathbb{Z}/n(-N), \tag{7}$$

and the Poincaré duality will take the form of a perfect duality, with values in $\mathbb{Z}/n(-N)$, between $\mathrm{H}^i(X,\mathbb{Z}/n)$ and $\mathrm{H}_c^{2N-i}(X,\mathbb{Z}/n)$.

6.2 The case of curves

6.2.1

Let $\bar{X}$ be a projective and smooth curve over an algebraically closed field k, and n invertible on $\bar{X}$. The proof of (3.3.5) provides, for a connected $\bar{X}$, a canonical isomorphism

$$\mathrm{H}^2(\bar{X},\boldsymbol{\mu}_n) = \mathrm{Pic}(\bar{X})/n\,\mathrm{Pic}(\bar{X}) \xrightarrow[\sim]{\deg} \mathbb{Z}/n$$

Let D be a reduced divisor of $\bar{X}$ and $X = \bar{X} \smallsetminus D$

$$X \overset{j}{\hookrightarrow} \bar{X} \overset{i}{\hookleftarrow} D.$$

The exact sequence $0 \to j_!\boldsymbol{\mu}_n \to \boldsymbol{\mu}_n \to i_*\boldsymbol{\mu}_n \to 0$ and the fact that $\mathrm{H}^i(\bar{X}, i_*\boldsymbol{\mu}_n) = \mathrm{H}^i(D,\boldsymbol{\mu}_n) = 0$ for $i > 0$ provide an isomorphism

$$\mathrm{H}_c^2(X,\boldsymbol{\mu}_n) = \mathrm{H}^2(\bar{X}, j_!\boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{H}^2(\bar{X},\boldsymbol{\mu}_n) \xrightarrow{\sim} \mathbb{Z}/n.$$

For a disconnected $\bar{X}$, we have similarly

$$\mathrm{H}_c^2(X,\boldsymbol{\mu}_n) = (\mathbb{Z}/n)^{\pi_0(X)}$$

and we define the *trace map* as the sum

$$\mathrm{tr} : \mathrm{H}_c^2(X,\boldsymbol{\mu}_n) \simeq (\mathbb{Z}/n)^{\pi_0(X)} \xrightarrow{\Sigma} \mathbb{Z}/n.$$

Théorème 6.2.2. *The form* $\mathrm{tr}(a \smile b)$ *identifies each of the two groups* $\mathrm{H}^i(X,\mathbb{Z}/n)$ *and* $\mathrm{H}_c^{2-i}(X,\boldsymbol{\mu}_n)$ *with the dual (with values in* $\mathbb{Z}/n$*) of the other.*

Transcendental proof. If $\bar{X}$ is a projective and smooth curve over a line S, and $j : X \hookrightarrow \bar{X}$ is the inclusion of the complementary of a divisor D étale over S< the cohomologies (resp. the cohomologies with compact support) of the special and generic geometric fibers of X/S are "the same," i.e. the fibers of locally constant sheaves on S. This follows from the analogous facts for $\bar{X}$ and D, via the exact sequence $0 \to j_!\mathbb{Z}/n \to \mathbb{Z}/n \to \mathbb{Z}/n_D \to 0$ (for cohomology with compact support) and the formulas $j_*\mathbb{Z}/n = \mathbb{Z}/n$, $\mathsf{R}^1 j_*\mathbb{Z}/n \simeq \mathbb{Z}/n_D(-1)$, $\mathsf{R}^i j_*\mathbb{Z}/n = 0$ $(i \geqslant 2)$ (for ordinary cohomology) (5.2.4).

This principle of specialization reduces (6.2.2) to the case where k has characteristic 0. By (5.2.3), this case reduces to the case where $k = \mathbb{C}$. Finally, for $k = \mathbb{C}$, the groups $\mathrm{H}^\bullet(X,\mathbb{Z}/n)$ and $\mathrm{H}_c^\bullet(X,\boldsymbol{\mu}_n)$ coincide with the groups of the same name, calculated for the classical topological space X_{cl} and, via the isomorphism $\mathbb{Z}/n \to \boldsymbol{\mu}_n : x \mapsto \exp\left(\frac{2\pi i x}{n}\right)$, the trace map is identified with "integration over the fundamental class," so that (6.2.2) results from the Poincaré duality for X_{cl}.

6.2.3 Proof algebraic

For a very economical proof, see (Duality, §2). Here is another one, related to the self-duality of the Jacobian.

We keep the notations of (6.2.1). We can assume – and we assume – that X is connected. The cases $i = 0$ and $i = 2$ being trivial, we also assume that $i = 1$. We define ${}_D\mathbb{G}_m$ by the exact sequence $0 \to_D \mathbb{G}_m \to \mathbb{G}_m \to i_*\mathbb{G}_m \to 0$ (sections of $\mathbb{G}_m$ congruent to $1 \mod D$). The group $\mathrm{H}^1(\bar{X},_D\mathbb{G}_m)$ classifies the invertible sheaves on $\bar{X}$ trivialized on D. It is the group of points of $\mathrm{Pic}_D(\bar{X})$, an extension of $\mathbb{Z}$ (the degree) by the group of points of a generalized Rosenlicht Jacobian (corresponding to conductor 1 at each point of D) $\mathrm{Pic}^0_D(\bar{X})$, itself an extension of the abelian variety $\mathrm{Pic}^0(\bar{X})$ by the torus $\mathbb{G}_m^D/(\mathbb{G}_m$ diagonal).

a) The exact sequence $0 \to j_!\boldsymbol{\mu}_n \to_D \mathbb{G}_m \xrightarrow{n}_D \mathbb{G}_m \to 0$ provides an isomorphism

$$\mathrm{H}^1_c(X, \boldsymbol{\mu}_n) = \mathrm{Pic}^0_D(\bar{X})_n. \tag{6.2.3.1}$$

b) The map which sends $x \in X(k)$ to the class of the invertible sheaf $\mathscr{O}(x)$ on $\bar{X}$, trivialized by 1 on D, comes from a morphism

$$f : X \to \mathrm{Pic}_D(\bar{X}).$$

For the sequel, we fix a base point 0 and set $f_0(x) = f(x) - f(0)$. For every homomorphism $v : \mathrm{Pic}^0_D(\bar{X})_n \to \mathbb{Z}/n$, let $\bar{v} \in \mathrm{H}^1\left(\mathrm{Pic}^0_D(\bar{X}), \mathbb{Z}/n\right)$ be the image by v of the class in $\mathrm{H}^1\left(\mathrm{Pic}^0_D(\bar{X}), \mathrm{Pic}^0_D(\bar{X})_n\right)$ of the torsor defined by the extension

$$0 \longrightarrow \mathrm{Pic}^0_D(\bar{X})_n \longrightarrow \mathrm{Pic}^0_D(\bar{X}) \xrightarrow{\ n\ } \mathrm{Pic}^0_D(\bar{X}) \longrightarrow 0$$

The theory of the geometric class field (as exposed in Serre [31]) shows that the map $v \mapsto f_0^*(\bar{v})$:

$$\hom\left(\mathrm{Pic}^0_D(\bar{X})_n, \mathbb{Z}/n\right) \to \mathrm{H}^1(X, \mathbb{Z}/n) \tag{6.2.3.2}$$

is an isomorphism. To deduce (6.2.2) from (6.2.3.1) and (6.2.3.2), it remains to know that

$$\mathrm{tr}\,(u \smile f_0^*(\bar{v})) = -v(u). \tag{6.2.2.3}$$

6.3 The general case

Let X be a smooth, pure-dimensional algebraic variety over an algebraically closed field k. To state the Poincaré duality theorem, we first need to define the trace map

$$\mathrm{tr} : \mathrm{H}^{2N}_c\left(X, \mathbb{Z}/n(N)\right) \to \mathbb{Z}/n.$$

The definition is a tedious unwinding from the case of curves [2, XVIII §2]. We then have

Théorème 6.3.1. *The form* $\mathrm{tr}(a \smile b)$ *identifies each of the groups* $\mathrm{H}^i_c\left(X, \mathbb{Z}/n(N)\right)$ *and* $\mathrm{H}^{2N-i}(X, \mathbb{Z}/n)$ *with the dual of the other.*

Let $x \in X$ be a closed point and X_x the strict localization of X at x. We set, for U ranging over the étale neighborhoods of x

$$\mathrm{H}^\bullet_c(X_x, \mathbb{Z}/n) = \varprojlim \mathrm{H}^\bullet_c(U, \mathbb{Z}/n). \tag{6.3.1.1}$$

It would be preferable to consider instead the pro-object “$\varprojlim$” $\mathrm{H}_c^\bullet(U, \mathbb{Z}/n)$ but, since the groups in question are finite, the difference is unimportant. As we tried to explain in the introduction, (6.3.1) follows from the fact that

$$\begin{gathered} \mathrm{H}_c^i(X_x, \mathbb{Z}/n) = 0 \text{ for } i \neq 2N \text{ and that} \\ \mathrm{tr} : \mathrm{H}_c^{2N}(X_x, \mathbb{Z}/n(N)) \xrightarrow{\sim} \mathbb{Z}/n \text{ is an isomorphism.} \end{gathered} \tag{6.3.1.2}$$

The case where $N = 0$ is trivial. If $N > 0$, let Y_y be the strict localization at a closed point of a smooth scheme Y of pure dimension $N-1$, and let $f : X_x \to Y_y$ be a morphism that is essentially smooth (of relative dimension one). The proof uses the Leray spectral sequence in proper cohomology for f, to reduce to the case of curves. The "proper cohomology" considered here is defined as the limits (6.3.1.1), the existence of such a spectral sequence poses various problems of passage to the limit, treated with too much detail in [2, XVIII]. For any geometric point z of Y_y, we have

$$(\mathsf{R}^i f_! \mathbb{Z}/n)_z = \mathrm{H}_c^i(f^{-1}(z), \mathbb{Z}/n).$$

The geometric fiber $f^{-1}(z)$ is a projective limit of smooth curves over an algebraically closed field. It satisfies Poincaré duality. Its ordinary cohomology is given by the local acyclicity theorem for smooth morphisms:

$$\mathrm{H}^i(f^{-1}(z), \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & \text{for } i = 0 \\ 0 & \text{for } i > 0. \end{cases}$$

By duality, we have

$$\mathrm{H}_c^i(f^{-1}(z), \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n(-1) & \text{for } i = 2 \\ 0 & \text{for } i \neq 2. \end{cases}$$

and the Leray spectral sequence is written

$$\mathrm{H}_c^i(X_x, \mathbb{Z}/n(N)) = \mathrm{H}_c^{i-2}(Y_y, \mathbb{Z}/n(N-1)).$$

We conclude by induction on N.

6.4 Variants and applications

In étale cohomology, one can construct a “duality formalism” (= functors $\mathsf{R}f_!$, $\mathsf{R}f_!$, $\mathsf{R}f^!$ satisfying various compatibilities and adjunction formulas) parallel to the one that exists in coherent cohomology. In this language, the results of the previous paragraph translate as follows: if $f : X \to S$ is smooth and purely of relative dimension N, and $S = \mathrm{Spec}(k)$, with k algebraically closed, then

$$\mathsf{R}f^! \mathbb{Z}/n = \mathbb{Z}/n[2N](N).$$

This result holds without any hypotheses on S. It has the

Corollaire 6.4.1. *If f is smooth and purely of relative dimension N, and the sheaves $\mathsf{R}^i f_! \mathbb{Z}/n$ are locally constant, then the sheaves $\mathsf{R}^i f_* \mathbb{Z}/n$ are also locally constant, and*

$$\mathsf{R}^i f_* \mathbb{Z}/n = \underline{\mathrm{hom}}\left(\mathsf{R}^{2N-i} f_! \mathbb{Z}/n(N), \mathbb{Z}/n\right).$$

In particular, under the hypotheses of the corollary, the sheaves $\mathsf{R}^i f_* \mathbb{Z}/n$ are constructible. From there, one can show that, if S is of finite type over the spectrum of a field or a Dedekind ring, then, for any finite type morphism $f : X \to S$ and any constructible sheaf $\mathscr{F}$ on X, the sheaves $\mathsf{R}^i f_* \mathscr{F}$ are constructible (Finiteness theorems in ℓ-adic cohomology, 1.1).

Chapter 2

Report on the trace formula

In this text, I have tried to present Grothendieck's theory of L-functions in as direct a way as possible. I closely follow some of the expositions given by Grothendieck at IHES in the spring of 1966. In the spirit of this volume, no reference will be made to SGA 5 - with the exception of two references to passages from lecture XI, which is independent of the rest of the seminar.

In §10 (p.81-90) of [13], the reader will find a summary of the theory, in the crucial case of curves.

1 A few reminders

1.1 Notations

$p, q, \mathbb{F}, \mathbb{F}_q$: p is a prime number, $q = p^f$ a power of p, and $\mathbb{F}$ an algebraic closure of the prime field $\mathbb{F}_p$. For any power p^n of p, we denote by $\mathbb{F}_{p^n}$ the subfield of $\mathbb{F}$ with p^n elements.

X_0, X: We will often use a symbol with a 0 index to represent an object over $\mathbb{F}_q$. The same symbol without the 0 then represents the object that is obtained by extension of scalars to $\mathbb{F}$. For example, if $\mathscr{F}_0$ is a sheaf on a scheme X_0 over $\mathbb{F}_q$, we denote by $\mathscr{F}$ its image by the reciprocal map on $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

F: If X_0 is a scheme over $\mathbb{F}_q$, we call the *Frobenius morphism* and denote by F the endomorphism of X_0 that "sends the point of coordinates x to the point of coordinates x^q": it is the identity on the underlying topological space and, for x a local section of the structural sheaf, $F^*x = x^q$.

We still denote by F the endomorphism of X that is obtained by extension of scalars. On the set $X(\mathbb{F}) = X_0(\mathbb{F})$ of rational points of X, F acts as the Frobenius substitution $\varphi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ defined by $\varphi(x) = x^q$. In particular, $X^F = X_0(\mathbb{F}_q)$.

In case of ambiguity, we will write $F_{(q)}$ rather than F.

1.2 Frobenius Correspondence

Let X_0 be a scheme over $\mathbb{F}_q$, and $\mathscr{F}_0$ a sheaf on X_0. I like to see the *Frobenius correspondence* (an F-endomorphism of $\mathscr{F}_0$) as follows:

$$F^* : F^*\mathscr{F}_0 \xrightarrow{\sim} \mathscr{F}_0 \tag{1.2.1}$$

We see $\mathscr{F}_0$ as the sheaf of local sections of a space $[\mathscr{F}_0]$ étale over X_0 ($[\mathscr{F}_0]$ is an algebraic space in the sense of M. Artin). The functoriality of F provides a commutative diagram

$$\begin{tikzcd} {[\mathscr{F}_0]} \arrow[r, "F"] \arrow[d] & {[\mathscr{F}_0]} \arrow[d] \\ X_0 \arrow[r, "F"] & X_0 \end{tikzcd}$$

Because $[\mathscr{F}_0]$ is étale over X_0, this diagram is cartesian; it provides an isomorphism $[\mathscr{F}_0] \xrightarrow{\sim} F^*[\mathscr{F}_0]$ with inverse (1.2.1). We still denote by F^* the correspondences or morphisms deduced from F by extension of scalars or by functoriality, such as $F^* : \mathrm{H}^\bullet_c(X, \mathscr{F}) \to \mathrm{H}^\bullet_c(X, \mathscr{F})$.

For a definition in shape, I refer to [20, XI.1,2]. There is only the case considered where $q = p$, but the general case is treated the same way.

1.3

The Frobenius correspondence is functorial in $(X_0, \mathscr{F}_0)$. If $f_0 : X_0 \to Y_0$ is a morphism of $\mathbb{F}_q$-schemes, $\mathscr{F}_0$ a sheaf on X_0, $\mathscr{G}_0$ a sheaf on Y_0 and $u \in \hom(f_0^*\mathscr{G}_0, \mathscr{F}_0)$ a f_0-morphism of $\mathscr{G}_0$ in $\mathscr{F}_0$, there is a commutative diagram of spaces $f_0F = Ff_0$, and above it a commutative diagram of sheaves

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "F"] & Y_0 \arrow[d, "F"] \\ X_0 \arrow[r, "f_0"] & Y_0 \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}_0 & \mathscr{G}_0 \arrow[l, "u"'] \\ \mathscr{F}_0 \arrow[u, "F^*"] & \mathscr{G}_0 \arrow[l] \arrow[u, "F^*"'] \end{tikzcd}$$

In particular, the correspondence on $f_{0*}\mathscr{F}_0$ deduced the Frobenius correspondence of $(X_0, \mathscr{F}_0)$ is the Frobenius correspondence of $(U_0, f_{0*}\mathscr{F}_0)$.

1.4

The natural morphism $Y \to Y_0$ is the limit of étale morphisms. It follows that for any abelian sheaf $\mathscr{F}_0$ on X_0, the inverse images on Y of the $\mathsf{R}^i f_0 \mathscr{F}_0$ are the $\mathsf{R}^i f_* \mathscr{F}$.

On $\mathsf{R}^i f_* \mathscr{F}$, we have

a) the correspondence $F^*\mathsf{R}^i f_*\mathscr{F} \xrightarrow{\sim} \mathsf{R}^i f_*\mathscr{F} \xrightarrow{F^*} \mathsf{R}^i f_*\mathscr{F}$ deduced by functoriality from the Frobenius correspondence;

b) the correspondence $F^*\mathsf{R}^i f_*\mathscr{F} \to \mathsf{R}^i f_*\mathscr{F}$ inverse image on Y of the Frobenius correspondence on $\mathsf{R}^i i f_{0*}\mathscr{F}_0$.

We deduce from (1.3) that these correspondences coincide. They induce the initial term of a Frobenius action on the whole spectral sequence

$$\mathrm{H}^p(Y, \mathsf{R}^q f_*\mathscr{F}) \to \mathrm{H}^{p+q}(X, \mathscr{F}).$$

Similarly in cohomology with proper support, and for the spectral sequence

$$\mathrm{H}^p_c(Y, \mathsf{R}^q f_*\mathscr{F}) \to \mathrm{H}^{p+q}_c(X, \mathscr{F})$$

1.5 Change of finite field

Let $(X_0, \mathscr{F}_0)$ be over $\mathbb{F}_q$, n an integer, and $(X_1, \mathscr{F}_1)$ over $\mathbb{F}_{q^n}$ be deduced from $(X_0, \mathscr{F}_0)$ by extension of scalars. On the one hand, we have the Frobenius correspondence $F_{(q)} : X_0 \to X_0)$, $F^*_{(q)} : F^*_{(q)}\mathscr{F}_0 \to \mathscr{F}_0$, on the other hand $F_{(q^n)} : X_1 \to X_1$, $F^*_{(q^n)} : F^*_{(q^n)}\mathscr{F}_1 \to \mathscr{F}_1$. It is easily verified that $F_{(q^n)}$ is deduced from the n-th power of $F_{(q)}$ by extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$; after extension of scalars to $\mathbb{F}$, we still have the same identity $F_{(q^n)} = F^n_{(q)}$ between endomorphisms of X and correspondences on X.

If x is a closed point of X_0, with $[k(x), \mathbb{F}_q] = n$, and $\bar{x} \in X_0(\mathbb{F}) = X(\mathbb{F})$ is localized at x, $\bar{x}$ is a fixed point of $F_{(q^n)} = F^n_{(q)}$. We denote by F^*_x the endomorphism of $\mathscr{F}_{\bar{x}}$ induced by $F^*_{(q^n)}$. Up to isomorphism, $(F^*_{\bar{x}}, \mathscr{F}_{\bar{x}})$ does not depend on the choice of $\bar{x}$. The trace, etc. of F^*_x is denoted by a notation such as $\mathrm{tr}(F^*_x, \mathscr{F})$.

1.6 Function L

Let X_0 be a finite type scheme over $\mathbb{F}_q$. We denote by $|X_0|$ the set of its closed points and, for $x \in |X_0|$, we set $\deg(x) = [k(x), \mathbb{F}_p]$. If $\mathscr{F}_0$ is a $\mathbb{Q}_\ell$-sheaf on X_0 (for this notion, see below) – or a constructible and flat sheaf of modules on a commutative Noetherian ring A, we denote $L(X_0, \mathscr{F}_0)$ the formal power series with constant term 1, in $\mathbb{Q}_\ell[\![t]\!]$ or in $A[\![t]\!]$,

$$L(X_0, \mathscr{F}_0) = \prod_{x \in |X_0|} \det\left(1 - F^*_x t^{\deg(x)}, \mathscr{F}\right)^{-1}.$$

1.7 Remark

In the definition (1.6), we have $\deg(x) = [k(x), \mathbb{F}_p]$ (absolute degree) and not $\deg(x) = [k(x), \mathbb{F}_q]$ (relative degree to $\mathbb{F}_q$). This has the effect that $L(X_0, \mathscr{F}_0)$ only depends on the scheme X_0 and the sheaf $\mathscr{F}_0$, not on q and the structural morphism $X_0 \to \mathrm{Spec}(\mathbb{F}_q)$. The indeterminate t is an avatar of p^{-s}.

1.8 The Galoisian point of view

This n° will not be used in the rest of the report. For $\mathscr{F}_0$ an abelian sheaf on X_0, the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on $\mathbb{F}$, and by transport of structure on $\mathrm{H}^i_c(X, \mathscr{F})$. In particular, the Frobenius substitution φ (1.1) induces an automorphism, still denoted φ, of $\mathrm{H}^i_c(X, \mathscr{F})$. The Frobenius correspondence defines on its side an endomorphism F^* of $\mathrm{H}^i_c(X, \mathscr{F})$. We have

$$F^{*-1} = \varphi \qquad (\text{on } \mathrm{H}^i_c(X, \mathscr{F})). \tag{1.8.1}$$

To see this, apply (1.4) to the case where Y_0 is a point: $Y_0 = \mathrm{Spec}(\mathbb{F}_q)$. We find that $\mathrm{H}^i_c(X, \mathscr{F})$ is the fiber, at the geometric point $\mathrm{Spec}(\mathbb{F})$ of Y_0, of $\mathsf{R}^i f_! \mathscr{F}_0$, and that F^* is the fiber of the Frobenius correspondence of $\mathsf{R}^i f_! \mathscr{F}_0$. Let $[\mathsf{R}^i f_! \mathscr{F}_0]$ be as in (1.2). We have $\mathrm{H}^i_c(X, \mathscr{F}) = [\mathsf{R}^i f_! \mathscr{F}_0](\mathbb{F})$ and

a) φ acts by transport of structure, via its action on $\mathbb{F}$,

b) F^* is the inverse of $F : [\mathsf{R}^i f_! \mathscr{F}_0](\mathbb{F}) \to [\mathsf{R}^i f_! \mathscr{F}_0](\mathbb{F})$.

We conclude with the identity $F = \varphi$ from (1.1).

One advantage of the Galoisian point of view is that it allows us to reason by transporting structure.

2 $\mathbb{Q}_\ell$-sheaves

The notion of $\mathbb{Q}_\ell$-sheaves is developed in detail in [20, V,VI]. We will only use the definitions and results below.

Définition 2.1. *Let X be a noetherian scheme. A $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$ on X is a projective system of sheaves $\mathscr{F}_n$ $(n \geqslant 0)$, with $\mathscr{F}_n$ a sheaf of $\mathbb{Z}/\ell^{n+1}$-modules constructible [2, IX.2], and such that the transition morphism $\mathscr{F}_n \to \mathscr{F}_{n-1}$ factors through an isomorphism*

$$\mathscr{F}_n \otimes_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \xrightarrow{\sim} \mathscr{F}_{n-1}.$$

We say that $\mathscr{F}$ is smooth *if the $\mathscr{F}_n$ are locally constant.*

The terminology of [20] is different, there it is called "$\mathbb{Z}_\ell$-faisceau constructible" for "$\mathbb{Z}_\ell$-faisceau," and "constant tordu constructible" for "lisse."

Similarly, below, for the $\mathbb{Q}_\ell$-faisceaux.

2.2

If k is a separably closed field, a sheaf of $\mathbb{Z}/\ell^n$-modules on $\operatorname{Spec}(k)$ is identified with a $\mathbb{Z}/\ell^n$-module, and the functor $\mathscr{F} \mapsto \varprojlim \mathscr{F}_n$ identifies the $\mathbb{Z}_\ell$-sheaves on $\operatorname{Spec}(k)$ with $\mathbb{Z}_\ell$-modules of finite type. This justifies the following definition: the *fiber* of the $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$ at the geometric point x of X is the $\mathbb{Z}_\ell$-module $\varprojlim (\mathscr{F}_n)_x$.

2.3

Suppose X is connected, and let x be a geometric point of X. We know that the "fiber at x" functor is an equivalence of the category of constructible locally constant sheaves of $\mathbb{Z}/\ell^n$-modules with the category of $\mathbb{Z}/\ell^n$-modules of finite type with a continuous action of $\pi_1(X, x)$. By passing to the limit, we deduce the following proposition.

Proposition 2.4. *Under the above assumptions, the "fiber at x" functor is an equivalence of the category of $\mathbb{Z}_\ell$-sheaves on X with that of $\mathbb{Z}_\ell$-modules of finite type equipped with a continuous action of $\pi_1(X, x)$.*

Proposition 2.5. *Let $\mathscr{F}$ be a $\mathbb{Z}_\ell$-sheaf on a noetherian scheme X. There exists a finite partition of X into locally closed parts X_i such that the $\mathscr{F}|X_i$ are smooth.*

Let $\operatorname{gr}^n_\ell \mathscr{F} = \ell^n \mathscr{F}_m / \ell^{m+1} \mathscr{F}_m$ for $m \geqslant n$, and $\operatorname{gr}_\ell \mathscr{F} = \bigoplus \operatorname{gr}^n_\ell \mathscr{F}$. It is a sheaf of modules on the noetherian ring $\operatorname{gr}_\ell \mathbb{Z} = \mathbb{Z}/\ell[t]$, graded from $\mathbb{Z}$ for the ℓ-adic filtration. It is a quotient of $\operatorname{gr}^0_\ell \mathscr{F} \otimes_{\mathbb{Z}/\ell} \mathbb{Z}/\ell[t]$, so it is a sheaf of $\operatorname{gr}_\ell \mathbb{Z}$-modules constructible [2, IX.2]. and there exists a finite partition of X into locally closed parts X_i, such that $\operatorname{gr}_\ell \mathscr{F}|X_i$ are locally constant. Each of the sheaves $\operatorname{gr}^n_\ell \mathscr{F}|X_i$ is then locally constant as well as the $\mathscr{F}_m|X_i$, which are successive extensions.

2.6

If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathbb{Z}_\ell$-sheaves on X, we define $\hom(\mathscr{F}, \mathscr{G}) = \varprojlim (\mathscr{F}_n, \mathscr{G}_n)$; it is a $\mathbb{Z}_\ell$-module. Since $\hom(\mathscr{F}_n, \mathscr{G}_n) = \hom(\mathscr{F}_m, \mathscr{G}_n)$ for $m \geqslant n$, we have

$$\hom(\mathscr{F}, \mathscr{G}) = \varprojlim_n \varinjlim_m \hom(\mathscr{F}_m, \mathscr{G}_n),$$

so that the functor $\mathscr{F} \mapsto$ "$\varprojlim$"$\mathscr{F}_n$ is fully faithful, from the category of $\mathbb{Z}_\ell$-sheaves to the category pro-sheaves on X (= pro-objects of the category of sheaves on X). We will use it to identify $\mathbb{Z}_\ell$-sheaves with certain pro-sheaves (those such that each $\mathscr{F}_n = \mathscr{F}/\ell^{n+1}\mathscr{F}$ is a sheaf, and that $\mathscr{F} \xrightarrow{\sim}$ "$\varprojlim$"$\mathscr{F}_n$). It is easily seen that if $\mathscr{F} +$ "$\varprojlim$"$\mathscr{F}_n$, with $\mathscr{F}_n$ a sheaf constructible of $\mathbb{Z}/\ell^n$-modules, for $\mathscr{F}$ to be a $\mathbb{Z}_\ell$-sheaf, it is necessary and sufficient that the projective systems (in n) $\mathscr{F}_n/\ell^{k+1}\mathscr{F}_n$ be essentially constant: the projective system of the $\mathscr{F}'_k = \varprojlim_n \mathscr{F}_n/\ell^{k+1}\mathscr{F}_n$ then satisfies the conditions of (2.1), and $\mathscr{F} =$ "$\varprojlim$"$\mathscr{F}'_k$.

In what follows, we abandon the notation $\mathscr{F}_n$ from (2.1); if $\mathscr{F}$ is a $\mathbb{Z}_\ell$-sheaf, the sheaf $\mathscr{F}_n$ from (2.1) will be denoted $\mathscr{F} \otimes \mathbb{Z}/\ell^{n+1}$.

2.7

In the abelian category of pro-sheaves, the subcategory of $\mathbb{Z}_\ell$-sheaves is stable by kernel, cokernel and extension. This is clear for the cokernels. For the kernels, if $f : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathbb{Z}_\ell$-sheaves, we deduce easily from (2.4, 2.5) and Artin-Rees that the projective system of $\ker(f : \mathscr{F} \otimes \mathbb{Z}/\ell^n \to \mathscr{G} \otimes \mathbb{Z}/\ell^n)$ satisfies the criterion given and (2.6) above. There even exists an integer r such that for all n

$$\ker(f) \otimes \mathbb{Z}/\ell^n = \ker(f : \mathscr{F} \otimes \mathbb{Z}/\ell^{n+r} \to \mathscr{G} \otimes \mathbb{Z}/\ell^{n+r}) \otimes \mathbb{Z}/\ell^n.$$

The case of extensions is left to the reader.

2.8

A $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$ is *torsion-free* if $\ker(\ell : \mathscr{F} \to \mathscr{F}) = 0$. Each $\mathscr{F} \otimes \mathbb{Z}/\ell^n$ is then flat over $\mathbb{Z}/\ell^n$ (i.e. with fibers of finite type $\mathbb{Z}/\ell^n$-modules). It follows from (2.4, 2.5) that for any $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$ there exists an integer n such that $\mathscr{F}/\ker(\ell^n)$ is torsion-free.

2.9

The abelian category of $\mathbb{Q}_\ell$*-sheaves* on X is the quotient of that of $\mathbb{Z}_\ell$-sheaves by the subcategory of $\mathbb{Z}_\ell$-sheaves of torsion. In equivalent terms, its objects are the $\mathbb{Z}_\ell$-sheaves and, denoting $\mathscr{F} \otimes \mathbb{Z}_\ell$ the $\mathbb{Z}_\ell$-sheaf $\mathscr{F}$, seen as an object of the category of $\mathbb{Q}_\ell$-sheaves, we have

$$\hom(\mathscr{F} \otimes \mathbb{Q}_\ell, \mathscr{G} \otimes \mathbb{Q}_\ell) = \hom(\mathscr{F}, \mathscr{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The *fiber* of a $\mathbb{Q}_\ell$-sheaf $\mathscr{F} \otimes \mathbb{Q}_\ell$ at a geometric point x of X is the finite dimensional $\mathbb{Q}_\ell$-vector space $\mathscr{F}_x \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

2.10

Let X be a separated scheme of finite type over an algebraically closed field k and $\mathscr{F}$ a $\mathbb{Q}_\ell$-sheaf constructible on X. Let $\mathscr{F}'$ be a constructible $\mathbb{Z}_\ell$-sheaf such that $\mathscr{F} \sim \mathscr{F}' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We will see in (4.11) that

$$\left(\varprojlim \mathrm{H}^q_c(X, \mathscr{F}' \otimes \mathbb{Z}/\ell^n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a finite dimensional $\mathbb{Q}_\ell$-vector space. It only depends on $\mathscr{F}'$, and is denoted $\mathrm{H}^q_c(X, \mathscr{F})$. The proof does not use that ℓ is prime to the characteristic, but only this case is of interest to us.

2.11

In this paper, all important calculations will be done at the level of $\mathbb{Z}/\ell^n$-sheaves, with the transition to $\mathbb{Q}_\ell$-sheaves only taking place in the very end. In practice, however, it is often more convenient to work systematically with $\mathbb{Q}_\ell$-sheaves. In papers V and VI of [20], Jouenolou provides the formalism that makes this possible. One of the essential results (VI 2.2) is that if $f : X \to Y$ is a separated finite type morphism of noetherian schemes, and $\mathscr{F}$ is a $\mathbb{Z}_\ell$-sheaf on Y, then, for each q, the pro-sheaf "$\varprojlim$"$\mathsf{R}^q f_!(\mathscr{F} \otimes \mathbb{Z}/\ell^n)$ is a $\mathbb{Z}_\ell$-sheaf on Y. He even proves a stronger regularity property (in the Artin-Rees sense) for the projective system of $\mathsf{R}^q f_!(\mathscr{F} \otimes \mathbb{Z}/\ell^n)$.

Jouanolou even places himself in a more general setting, encompassing the very useful case of E_λ-sheaves, for E_λ a finite extension of $\mathbb{Q}_\ell$. They are defined just as $\mathbb{Q}_\ell$-sheaves were, with $\mathbb{Q}_\ell$ replaced by E_λ, $\mathbb{Z}_\ell$ by the ring $\mathscr{O}_\lambda$ of the valuation λ, and $\mathbb{Z}/\ell^n$ by $\mathscr{O}_\lambda/\pi^n$, for π a uniformizer. It would amount to the same to define an E_λ-sheaf as a $\mathbb{Q}_\ell$-sheaf $\mathscr{F}$, equipped with a homomorphism $E_\lambda \to \mathrm{End}(\mathscr{F})$.

3 Trace formulas and functions L

We use the notation from §1, and ℓ is a prime number $\neq p$.

Let X_0 be a separated finite type scheme over $\mathbb{F}_q$ ($q = p^f$) and $\mathscr{F}_0$ a $\mathbb{Q}_\ell$-constructible sheaf on X_0. The cohomological interpretation of Grothendieck of L functions is the following theorem:

Théorème 3.1. $L(X_0, \mathscr{F}_0) = \prod_i \det\left(1 - F^* t^f, \mathrm{H}^i_c(X, \mathscr{F})\right)^{(-1)^{i+1}}$.

It is obtained as a consequence of the trace formula:

Théorème 3.2. *For every integer n,* $\sum_{x \in X^{F^n}} \mathrm{tr}\left(F^{n*}, \mathscr{F}_x\right) = \sum_i (-1)^i \,\mathrm{tr}\left(F^{n*}, \mathrm{H}^i_c(X, \mathscr{F})\right)$.

Let us briefly recall the classical method for deducing (3.1) from (3.2). Let $T = t^f$; both sides are formal power series in T, with constant term 1. Since $\mathbb{Q}_\ell$ has characteristic 0, it is enough to prove that the logarithmic derivatives $T\frac{d}{dT}\log$ of the two sides are equal.

To calculate these logarithmic derivatives, we will use the formula

$$T\frac{d}{dT}\log\det(1 - fT)^{-1} = \sum_{n>0} \mathrm{tr}(f^n) T^n,$$

whose proof (in a slightly more general context) is recalled below. At the first number, we found

$$\begin{aligned} T\frac{d}{dT}\log L(X_0, \mathscr{F}_0) &= \sum_{x \in |X_0|} \sum_n [k(x) : \mathbb{F}_q]\,\mathrm{tr}\left(F_x^n, \mathscr{F}\right) T^{n \cdot [k(x):\mathbb{F}_q]} \\ &= \sum_{n>0} T^n \sum_{x \in X^{F^n}} \mathrm{tr}\left(F^{n*}, \mathscr{F}_x\right) \end{aligned}$$

and at the second number

$$\sum_{n>0} T^n \sum (-1)^i \,\mathrm{tr}\left(F^{n*}, \mathrm{H}^i_c(X, \mathscr{F})\right) \;;$$

we compare terms term by term.

Proposition 3.3. *Let A be a commutative ring, M a finitely generated projective A-module, and f an endomorphism of M. For any invertible power series Q, we define $\frac{d}{dT}\log Q = Q^{-1}\frac{d}{dT}Q$. We have*

$$T\frac{d}{dT}\log\det(1-fT)^{-1} = \sum_{n>0}\mathrm{tr}(f^n)T^n. \tag{3.3.1}$$

Let M' be an A-module such that $M \oplus M'$ is of finite rank. Replacing (M, f) with $(M \oplus M', f \oplus 0)$ does not change the two sides of (3.3), and thus we may assume that M is free. For $M = A^d$, (3.3.1) is an algebraic identity involving the coefficients f_i^j of f. By the principle of prolongation of algebraic identities, it is enough to verify it for A an algebraically closed field of characteristic 0. Since the two sides are additive in M in an extension $0 \to M' \to M \to M'' \to 0$, it is even enough to consider the case where M is of rank 1. If f is multiplication by a, (3.3.1) then reduces to the formula

$$T\frac{d}{dT}\log\left((1-aT)^{-1}\right) = \sum_{n>0}a^nT^n.$$

Corollaire 3.4. *Let n be an integer, and $f^{(n)}$ the endomorphism $(x_1, \ldots, x_n) \mapsto (f(x_n), x_1, \ldots, x_{n-1})$ of M^n. We have*

$$\det(1-fT^n) = \det(1-f^{(n)}T). \tag{3.4.1}$$

Proceeding as in (3.3), we may assume that A is of characteristic 0. It is then enough to check that the logarithmic derivatives $-T\frac{d}{dT}\log$ of both sides are equal.

$$-T\frac{d}{dT}\log\det(1-fT^n) = -nT^n\frac{d}{dT^n}\log\det(1-fT^n) = n\sum_{m>0}\mathrm{tr}(f^m)T^{nm},$$
$$-T\frac{d}{dT}\log\det\left(1-f^{(n)}T\right) = \sum_{m>0}\mathrm{tr}\left(f^{(n)m}\right)T^m,$$

and we observe that

$$\mathrm{tr}(f^{(n)m}) = 0 \qquad \text{if } n \nmid m$$
$$\mathrm{tr}(f^{(n)nm}) = n\,\mathrm{tr}(f^m).$$

3.5 Remarque

For $A = \mathbb{Q}_\ell$, (3.4.1) is the particular case $X_0 = \mathrm{Spec}(\mathbb{F}_{q^n})$ of (3.1). Using this identity, it is easy to check directly that the second member of (3.1) is independent of q, and of the structural morphism $X_0 \to \mathrm{Spec}(\mathbb{F}_q)$.

3.6 Remarque

If $j : X_0 \hookrightarrow \bar{X}_0$ is a compactification of X_0, we have $L(X_0, \mathscr{F}_0) = L(\bar{X}_0, j_!\mathscr{F}_0)$ and $\mathrm{H}^i_c(X, \mathscr{F}) = \mathrm{H}^i(\bar{X}, j_!\mathscr{F})$. This reduces (3.1) to the proper case, and explains the use of cohomology with proper supports. For X proper, (3.2) has the form of a Lefschetz trace formula. We note that the local terms $\mathrm{tr}(F^{n*}, \mathscr{F}_x)$ $(x \in X^{F^n})$ only depend on the fibre of $\mathscr{F}$ at x, and are not affected by multiplicity. For X and $\mathscr{F}$ smooth, this corresponds to the fact that the graph of F is transverse to the diagonal (since $dF = 0$).

3.7 Remarque

If we admit the formalism of $\mathbb{Q}_\ell$-sheaves (cf. 2.11; we need the Leray spectral sequence and the theorem of change of base for the direct images $\mathsf{R}^q f_!$), it is easy to reduce the proof of (3.1, 3.2) to the case where X_0 is a smooth curve and where $\mathscr{F}_0$ is smooth. This is clearly explained in [21, §5] (for 3.1; 3.2 is treated similarly).

3.8

We have two methods to prove (3.2).

Lefschetz-Verdier If X_0 is proper (cf. 3.6), the general Lefschetz-Verdier trace formula allows us to express the second member of (3.2) as a sum of local terms, one for each point of X^{F^n}. In the original version of [20], this formula was only proven modulo the resolution of singularities. The reader will find an unconditional proof in the definitive version. In the case of curves, which we can reduce to (3.7), the ingredients of the demonstration were already available.

To deduce (3.2) from the Lefschetz-Verdier formula, one must be able to calculate the local terms. For a curve and the Frobenius endomorphism, this had been done by Artin and Verdier [38].

Nielson-Wecken A method inspired by the work of Nielsen and Wecken allows us to bring (3.2) back to a special case proven by Weil; this is what will be explained in the following paragraphs.

4 Reduction to theorems in $\mathbb{Z}/\ell^n$-cohomology

We will prove (3.2) by passage to the limit from an analogous theorem in $\mathbb{Z}/\ell^n$-cohomology. The statement presents the following difficulty. Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$, and $\mathscr{F}_0$ a flat and constructible sheaf of $\mathbb{Z}/\ell^n$-modules. Its fibers are free $\mathbb{Z}/\ell^n$-modules of finite type, and so the left-hand side of (3.2) makes sense (it is an element of $\mathbb{Z}/\ell^n$). However, in general the $\mathrm{H}^i_c(X, \mathscr{F})$ will not be free, so that the right-hand side is not defined. To circumvent this difficulty is one of the essential uses made by Grothendieck of the theory of derived categories.

4.1

For the definition of the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, and that of the subcategories $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^b(\mathcal{A})$, I refer the reader to Verdier's text in this volume. For Λ a ring, one writes $\mathsf{D}(\Lambda)$ instead of D(category of left Λ-modules); for X a site, one writes $\mathsf{D}(X, \Lambda)$ instead of D(category of left Λ-module sheaves). Similarly for D^-, etc. . .

The R sign (resp. L) indicates a right derived functor (resp. left). For example, $\overset{\mathsf{L}}{\otimes}$ indicates the derived functor of the tensor product: for K in $\mathsf{D}^-(\Lambda^\circ)$ and L in $\mathsf{D}^-(\Lambda)$ (Λ° the opposite ring to Λ), $K \overset{\mathsf{L}}{\otimes} L$ is calculated as follows: one takes quasi-isomorphisms $K' \to K$, $L' \to L$, with K' and L bounded above and one of them with flat components, and the simple complex $s(K' \otimes_\Lambda L)$ associated to the double complex $K' \otimes_\Lambda L'$.

4.2

Even to treat the case of $\mathbb{Z}/\ell^n$-cohomology alone, we will need the theory of traces of endomorphisms of modules over rings that are not necessarily commutative (of finite group algebras $\mathbb{Z}/\ell^n[G]$). These traces were introduced by Stallings and Hattori [3].

Let Λ be a ring (with unity, but not necessarily commutative). We denote by $\Lambda^\natural$ the quotient of the additive group of Λ by the subgroup generated by the commutators $ab - ba$. If f is an endomorphism of the left Λ-module free Λ^n, with matrix f_i^j, we denote by $\operatorname{tr}(f)$ the image of $\sum f_i^f$ in $\Lambda^\natural$. If f and g are two homomorphisms $f : \Lambda^n \leftrightarrows \Lambda^m : g$, we immediately verify that $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.

Let f be an endomorphism of a finitely generated projective Λ-module P. Write P as a direct factor of a free module Λ^n: this means choosing a module P', and an isomorphism $\alpha : P \oplus P' \xrightarrow{\sim} \Lambda^n$, or equivalently choosing a homomorphism $a : P \to \Lambda^n$, and a retraction $b : \Lambda^n \to P$ (take $P = \ker(b)$). Let $f' = \alpha(f \oplus 0)\alpha^{-1} = afb$. We set $\operatorname{tr}(f) = \operatorname{tr}(f')$. This element of $\Lambda^\natural$ only depends on f: if another choice $c : P \leftrightarrows \Lambda^m : d$ is made, we have $a = a(dc)(ba)$, whence

$$\operatorname{tr}(afb) = \operatorname{tr}(adcbafb) = \operatorname{tr}(cbafbad) = \operatorname{tr}(cfd).$$

If f is an endomorphism of a Λ-module projective of finite type $\mathbb{Z}/2$-graded, of components $f_i^j : P^j \to P^i$, we set

$$\operatorname{tr}(f) = \operatorname{tr}(f_0^0) - \operatorname{tr}(f_1^1).$$

If there is a risk of ambiguity, we will rather write $\operatorname{tr}(f; P)$. If the homomorphisms $f : P \leftrightarrows Q : g$ are homogeneous, we immediately verify by reduction to the free case that

$$\operatorname{tr}(fg) = \operatorname{tr}\left((-1)^{\deg f \deg g} gf\right). \tag{4.2.1}$$

If f is an endomorphism of a Λ-module $\mathbb{Z}/2$-graded filtered (P, F), of finite filtration, compatible with the graduation, and the $\operatorname{gr}_F^i(P)$ are projectives of finite type, then P is projective of finite type, and

$$\operatorname{tr}(f; P) = \sum \operatorname{tr}\left(f; \operatorname{gr}_F^i(P)\right) \tag{4.2.2}$$

(split the filtration to reduce to the case of a direct sum).

4.3

A $\mathbb{Z}$-graded module will be considered as a $\mathbb{Z}/2$-graded module by the parity of the degree. In particular, if f is an endomorphism of a complex bounded of projective Λ-modules of finite type, we set

$$\operatorname{tr}(f) = \sum (-1)^i \operatorname{tr}(f^i). \tag{4.3.1}$$

According to (4.2.1), if f is homotopic to zero, $\operatorname{tr}(f) = 0$:

$$f = dH + hD \to \operatorname{tr}(f) = 0. \tag{4.3.2}$$

Let $\mathsf{K}_{\text{parf}}(\Lambda)$ be the category of objects the complex bounded of projective Λ-modules of finite type, and of arrows the morphisms of complexes taken to homotopy near. The functor $\mathsf{K}_{\text{parf}}(\Lambda) \to \mathsf{D}(\Lambda)$ is fully faithful. We note $\mathsf{D}_{\text{parf}}(\Lambda)$ its essential image. The formulas (4.3.1) (4.3.2) allow to define $\operatorname{tr}(f)$ for f an endomorphism of $K \in \operatorname{Ob} \mathsf{D}_{\text{parf}}(\Lambda)$ (and this trace only depends on the class of isomorphism of (K, f)).

4.4

We recall that the filtered derived category $\mathsf{DF}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the derived category of that of filtered complexes of objects of $\mathcal{A}$ by inverting the filtered quasi-isomorphisms (= filtered complex morphisms inducing a quasi-isomorphism on the associated graded). We have the "underlying complex" functors: $\mathsf{DF}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}) : (K, F) \mapsto K$ and the "p-th graded component": $\mathsf{DF}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}) : (K, F) \mapsto \mathrm{gr}_F^p(K)$.

Let $\mathsf{KF}_{\mathrm{parf}}(\Lambda)$ be the category of objects the bounded filtered complexes of finite filtration (K, F) with the $\mathrm{gr}_F^p(K^q)$ projective of finite type, and of arrows the complex morphisms respecting the filtration, taken to a filtration respecting homotopy near. The functor $\mathsf{KF}_{\mathrm{parf}}(\Lambda) \to \mathsf{DF}(\Lambda)$ is fully faithful; its essential image $\mathsf{DF}_{\mathrm{parf}}(\Lambda)$ is almost all zero; if (K, F) is in $\mathsf{D}_{\mathrm{parf}}(\Lambda)$, then K is in $\mathsf{D}_{\mathrm{parf}}(\Lambda)$.

According to (4.2.2), if f is an endomorphism of $(K, F) \in \mathrm{Ob}\,\mathsf{DF}_{\mathrm{parf}}(\Lambda)$, we have

$$\mathrm{tr}(f, K) = \sum_p \mathrm{tr}(f, \mathrm{gr}_F^p(K)). \tag{4.4.1}$$

4.5

Let Λ be a ring, and $K \in \mathrm{Ob}\,\mathsf{D}^-(\Lambda)$. We say that K is of tor-dimension $\leqslant r$ if for any right Λ-module N, the hypertors $\mathrm{H}^i(N \overset{\mathsf{L}}{\otimes}_\Lambda K)$ are zero for $i < -r$. It is said to be of finite tor-dimension if this condition is satisfied for some r.

We use the same terminology for a complex of sheaves $K \in \mathrm{Ob}\,\mathsf{D}^-(X, \Lambda)$.

Lemme 4.5.1. *Let Λ be a left noetherian ring, and $K \in \mathrm{Ob}\,\mathsf{D}^-(\Lambda)$. For K to be in $\mathsf{DF}_{parf}(\Lambda)$, it is necessary and sufficient that it be of finite tor-dimension and that the $\mathrm{H}^i(K)$ be of finite type.*

We can suppose K is bounded above and has finitely generated free components (cf. 4.7, below). If K has tor-dimension $\leqslant r$, the $\mathrm{H}^i(K)$ are zero for $i < -r$, $K^{-r}/\,\mathrm{im}(d)$ has the free resolution

$$(\cdots \to K^{-r-1} \to K^{-r}) \to K^{-r}/\,\mathrm{im}(d),$$

and for any right Λ-module N, $\mathrm{H}^{-r-k}(N \overset{\mathsf{L}}{\otimes}_\Lambda K) = \mathrm{H}^{-r-k}(N \otimes_\Lambda K) = \mathrm{Tor}_k(N, K^r/\,\mathrm{im}\,d)$ for $k \geqslant 1$.

By hypothesis, these groups are zero; the module $K^r/\,\mathrm{im}\,d$ is therefore flat of finite presentation, i.e. projective of finite type. The quotient

$$0 \to K^r/\,\mathrm{im}(d) \to K^{r+1} \to K^{r+2} \to \cdots$$

of K is quasi-isomorphic to K, and this shows $K \in \mathrm{Ob}\,\mathsf{DF}_{\mathrm{parf}}(\Lambda)$.

Proposition-définition 4.6. *Let X be a noetherian scheme, and Λ be a noetherian left ring. We denote $\mathsf{D}^b_{\mathrm{ctf}}(X, \Lambda)$ the subcategory of $\mathsf{D}^-(X, \Lambda)$ consisting of complexes $\mathscr{K}$ satisfying the equivalent conditions below:*

- *i) $\mathscr{K}$ is isomorphic (in $\mathsf{D}^-(X, \Lambda)$) to a bounded complex of flat and constructible Λ-module sheaves.*
- *ii) $\mathscr{K}$ has finite tor-dimension and the sheaves $\mathrm{H}^i(K)$ are constructible.*

Same argument as in (4.5), starting from the following lemma.

Lemme 4.7. *If a complex $\mathscr{K}$ of Λ-module sheaves is such that the $\mathrm{H}^i(\mathscr{K}))$ are constructible, and vanish for i large, there exists a quasi-isomorphism $\mathscr{K}' \to \mathscr{K}$, with $\mathscr{K}'$ bounded above with constructible and flat components.*

$\mathscr{K}'$ is constructed by propagating from right to left. For an m such that the $\mathrm{H}^i(\mathscr{K})$ $(i \geqslant m)$ are zero, we take $\mathscr{K}'^i = 0$ for $i \geqslant m$. We then assume that the $\mathscr{K}'^i$ have already been constructed for $i > n$:

$$\begin{tikzcd} \cdots \arrow[r] & \mathscr{K}^{n-1} \arrow[r] & \mathscr{K}^n \arrow[r] & \mathscr{K}^{n+1} \arrow[r] & \cdots \\ & & & \mathscr{K}'^{n+1} \arrow[u] \arrow[r] & \cdots, \end{tikzcd}$$

with $\mathrm{H}^i(\mathscr{K}') \xrightarrow{\sim} \mathrm{H}^i(\mathscr{K})$ for $i > n+1$, and $\ker(d) \to \mathrm{H}^{n+1}(\mathscr{K})$ surjective in degree $n+1$. We define sheaves $\mathscr{A}$ and $\mathscr{B}$ by the Cartesian diagram

$$\begin{tikzcd} \mathscr{K}^n \arrow[r] & \mathscr{K}^n/\operatorname{im}(d) \arrow[r] & \ker(d) \\ \mathscr{A} \arrow[u] \arrow[r, "u"] & \mathscr{B} \arrow[u] \arrow[r] & \ker(d). \arrow[u] \end{tikzcd}$$

So, $\mathscr{B}$ is constructible, u is surjective, and to take a step forward, it is enough to construct K^n flat and constructible and $v : \mathscr{K}^n \to \mathscr{A}$ such that $u\ uv$ is surjective. That this is possible is guaranteed by the following lemma.

Lemme 4.8. *Let $v : \mathscr{A} \to \mathscr{B}$ be an epimorphism of sheaves of Λ-modules, with $\mathscr{B}$ constructible. Then there exists $u : \mathscr{K} \to \mathscr{A}$ with $\mathscr{K}$ flat and constructible and vu an epimorphism.*

The sheaf $\mathscr{A}$ is the quotient of a sum of sheaves of the form $\varphi_!\Lambda$, for $\varphi : U \to X$ étale. Since $\mathscr{B}$ is noetherian ([2, IX.2.10]), a finite sum of these already maps to $\mathscr{B}$, proving the lemma.

Théorème 4.9. *Let $f : X \to Y$ be a separated morphism of finite type of noetherian schemes. If $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_{\mathrm{ctf}}(X, \Lambda)$, then $\mathsf{R}f_!\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_{\mathrm{ctf}}(Y, \Lambda)$.*

Recall that if $j : X \hookrightarrow \bar{X} \xrightarrow{\bar{f}} Y$ is a relative compactification of X/Y, we have by definition $\mathsf{R}f_! = \mathsf{R}\bar{f}_* \circ j_!$, where $\mathsf{R}\bar{f}_*$ is the derived functor of the functor $\bar{f}_*$. This formula reduces the proof of (4.9) to the case where f is proper. For f proper, f_* has finite cohomological dimension; this allows us to define $\mathsf{R}f_*$ on the entire derived category, and also as a functor

$$\mathsf{R}f_* : \mathsf{D}^-(X, \Lambda) \to \mathsf{D}^-(Y, \Lambda).$$

In the same way, we define in general

$$\mathsf{R}f_! : \mathsf{D}^-(X, \Lambda) \to \mathsf{D}^-(Y, \Lambda).$$

For any right Λ-module N, we have (proof given below)

$$N \overset{\mathsf{L}}{\otimes} \mathsf{R}f_!\mathscr{K} \xrightarrow{\sim} \mathsf{R}f_!(N \overset{\mathsf{L}}{\otimes} \mathscr{K}). \tag{4.9.1}$$

Let us deduce (4.9) from (4.9.1).

a) The spectral sequence

$$E_2^{pq} = \mathsf{R}^p f_! \, \mathrm{H}^q(\mathscr{K}) \to \mathrm{H}^{p+q} \, \mathsf{R} f_! \mathscr{K}.$$

and the finiteness theorem for $\mathsf{R}f_!$, show that $\mathsf{R}f_!\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b(X,\Lambda)$ and has constructible cohomology. It remains to prove the tor-dimension finiteness (4.6).

b) If $\mathscr{K}$ has tor-dimension $\leqslant -r$, the same is true for $\mathsf{R}f_!\mathscr{K}$: we have $\mathrm{H}^i(N \overset{\mathsf{L}}{\otimes} \mathsf{R}f_!\mathscr{K}) = \mathrm{H}^i(\mathsf{R}f_!(N \overset{\mathsf{L}}{\otimes} \mathscr{K}))$, and, in the spectral sequence above for $N \overset{\mathsf{L}}{\otimes} \mathscr{K}$, the E_2^{pq} are zero for $q < r - s$ a fortiori for $p + q < r$.

Let us prove (4.9.1), staying at the level of complexes.

a) Using the definition formula $\mathsf{R}f_! = \mathsf{R}f_* \circ j_!$, we reduce to the case where f is proper.

b) Let us represent $\mathscr{K}$ by a complex bounded above with acyclic components for f_* : $\mathsf{R}^p f_* \mathscr{K}^q = 0$ for $p > 0$. We then have $\mathsf{R}f_* K \sim f_*\mathscr{K}$. It is only possible to work with complexes bounded above because f_* has finite cohomological dimension.

c) If L is a right Λ-module, we have $\mathsf{R}^p f_*(L \otimes \mathscr{K}^q) = L \otimes \mathsf{R}^p f_* \mathscr{K}^q$: it is trivial for Λ free of finite type, and the general case follows by passage to the limit. The sheaf $L \otimes \mathscr{K}^q$ is acyclic for f_*, and $f_*(L \otimes \mathscr{K}^q) = L \otimes f_*\mathscr{K}^q$.

d) Let $N_\bullet$ be a free resolution of N. We have $N \overset{\mathsf{L}}{\otimes} \mathscr{K} \sim s(N_\bullet \otimes \mathscr{K})$ (simple complex associated to the double complex $N_\bullet \otimes \mathscr{K}$). According to c),

$$\mathsf{R}f_*(N \overset{\mathsf{L}}{\otimes} \mathscr{K}) \sim \mathsf{R}f_*(s(N_\bullet \otimes \mathscr{K})) \sim f_*(s(N_\bullet \otimes \mathscr{K})) \sim s(N_\bullet \otimes f_*\mathscr{K}) \sim N \overset{\mathsf{L}}{\otimes} \mathsf{R}f_*\mathscr{K},$$

which is the announced formula.

When Y is the spectrum of a separably closed field, the functor $\mathsf{R}f_!$ is identified with a functor denoted $\mathsf{R}\Gamma_c : \mathsf{D}^b_{\mathrm{ctf}}(X,\Lambda) \to \mathsf{D}_{\mathrm{parf}}(\Lambda)$.

Théorème 4.10. *With the notation of §1, let X_0 be a finite type separated scheme over $\mathbb{F}_q$, and Λ a Noetherian ring of torsion, killed by a prime integer to p. Let $\mathscr{K}_0 \in \operatorname{Ob} \mathsf{D}^b_{\mathrm{ctf}}(X_0,\Lambda)$. With the definition (4.3) of traces, for all N*

$$\sum_{x \in X^{F^n}} \operatorname{tr}\left(F^{n*}, \mathscr{K}_x\right) = \operatorname{tr}\left(F^{n*}, \mathsf{R}\Gamma_c(X,\mathscr{K})\right).$$

In the end of this §, we will deduce 3.2 from 4.10, and prove 2.10.

4.11 Proof of 2.10

Let X be a separated scheme of finite type over an algebraically closed field k, and let $\mathscr{G}$ be a $\mathbb{Q}_\ell$-sheaf on X. According to 2.8, we can write it in the form $\mathscr{G} = \mathscr{F} \otimes \mathbb{Q}_\ell$, with $\mathscr{F}$ a $\mathbb{Z}_\ell$-sheaf without torsion. Let

$$K_n = \mathsf{R}\Gamma_c(X, \mathscr{F} \otimes \mathbb{Z}/\ell^{n+1}) \in \mathsf{D}(\mathbb{Z}/\ell^{n+1}).$$

According to 4.9, we have $K_n \in \operatorname{Ob} \mathsf{D}_{\text{parf}}(\mathbb{Z}/\ell^{n+1})$, and it follows from (4.9.1) that the natural arrow

$$K_n \overset{\mathsf{L}}{\otimes}_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \longrightarrow K_{n+1}, \tag{4.11.1}$$

is an isomorphism. The problem of the definition of this arrow is discussed in 4.12.

We can now invoke [20, XI.3.3] (p. 31, 1-2 at the end; this passage is independent of the rest of [20]) to realize each K_n by a bounded complex of finitely generated free $\mathbb{Z}/\ell^{n+1}$-modules, and the arrows (4.11.1) by *isomorphisms of complexes*

$$K_n \otimes \mathbb{Z}/\ell^n \xrightarrow{\sim} K_{n-1}.$$

Let K be the complex $\varprojlim K_n$. It is a bounded complex of free $\mathbb{Z}_\ell$-modules, and $K_n \simeq K \otimes_{\mathbb{Z}_\ell} \mathbb{Z}/\ell^{n+1}$. We have

$$\mathrm{H}^i(K) = \varprojlim \mathrm{H}^i(K_n),$$

(any projective system of finite groups satisfies the Mittag-Leffler condition), and $\mathrm{H}^i(K)$ is a finitely generated $\mathbb{Z}_\ell$-module. We have

$$\mathrm{H}^i_c(X, \mathscr{G}) = \left(\varprojlim \mathrm{H}^i(K_n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell:$$

it is a finite dimensional $\mathbb{Q}_\ell$-vector space.

4.12 The arrow (4.11.1)

To *define* the arrow (4.11.1), we cannot proceed as in 4.9, i.e. solving $\mathbb{Z}/\ell^n$ by a complex of free $\mathbb{Z}/\ell^{n+1}$-modules, since we want to construct an isomorphism in $\mathsf{D}_{\text{parf}}(\mathbb{Z}/\ell^n)$. The desired arrow is a special case of the following extension of scalars arrow.

(*) Let $\Lambda \to \Lambda'$ be a homomorphism of Noetherian rings of torsion. For X a separated scheme of finite type over an algebraically closed field k, and $\mathscr{K} \in \operatorname{Ob} \mathsf{D}_{\text{ctf}}(X, \Lambda)$, there is an isomorphism

$$\mathsf{R}\Gamma_c(X, \mathscr{K}) \overset{\mathsf{L}}{\otimes}_\Lambda \Lambda' \xrightarrow{\sim} \mathsf{R}\Gamma_c(X, \mathscr{K} \overset{\mathsf{L}}{\otimes}_\Lambda \Lambda').$$

More general arrows are constructed in [2, XVII 4.2.12]. The method is as follows:

a) The definition of $\mathsf{R}\Gamma_c$ is the formula $j_!(\mathscr{K} \otimes_\Lambda \Lambda') = (j_! \mathscr{K}) \otimes_\Lambda \Lambda'$ for an open immersion which reduces us to the case where X is proper.

b) For X proper, we need to replace $\mathscr{K}$ by a complex with bounded components which are acyclic for the functor Γ, and with fibers in every geometric point (or just in every closed point) homotopic to a complex of Λ-flat modules. For such a complex, we have $\Gamma(X, \mathscr{K}) \sim \mathsf{R}\Gamma(X, \mathscr{K})$ and $K \otimes_\Lambda \Lambda' \sim \mathscr{K} \overset{\mathsf{L}}{\otimes}_\Lambda \Lambda'$; the arrow (4.11.1) is defined as the composition

$$\mathsf{R}\Gamma(X, \mathscr{K}) \overset{\mathsf{L}}{\otimes}_\Lambda \Lambda' \to \Gamma(X, \mathscr{K}) \otimes_\Lambda \Lambda' \to \Gamma(X, \mathscr{K} \otimes_\Lambda \Lambda') \xleftarrow{\sim} \mathsf{R}\Gamma(X, \mathscr{K} \overset{\mathsf{L}}{\otimes}_\Lambda \Lambda').$$

c) It remains to show that it is possible to replace $\mathscr{K}$ by a complex of the desired type [2, XVII.4.2.10]. We first replace $\mathscr{K}$ by a complex with bounded flat components (4.6), then we take a canonical truncated flabby resolution (this does not change the homotopy type of the fiber complexes, and we truncate far enough so that the acyclicity condition for Γ is satisfied (i.e. beyond the cohomological dimension of Γ)).

4.13 Proof of (3.2)

To simplify the notation, we will only consider the case $n = 1$ of (3.2). The general case follows by replacing F with F^n below.

Let X_0 be a finite type separated scheme over $\mathbb{F}_q$, and $\mathscr{G}_0$ a $\mathbb{Q}_\ell$-sheaf on X_0. We have $\mathscr{G}_0 = \mathscr{F}_0 \otimes \mathbb{Q}_\ell$, for a suitable $\mathbb{Z}_\ell$-sheaf without torsion $\mathscr{F}_0$. Let $K_n = \mathsf{R}\Gamma_c(X, \mathscr{F} \otimes \mathbb{Z}/\ell^n)$. The Frobenius morphisms induce morphisms

$$F^* : K_n \to K_n \qquad (\text{in } \mathsf{D}_{\text{parf}}(\mathbb{Z}/\ell^{n+1})),$$

which are obtained from each other via the isomorphisms (4.11.1).

As in (4.11), we realize the K_n and the isomorphisms (4.11.1) at the level of complexes. The passage already cited in [20, XV.3.3] allows us to realize the morphisms F^* by endomorphisms of complexes, still denoted F^*, which are obtained from each other. We still denote by F^* their projective limit, and the endomorphisms which are obtained from it. Since $\mathrm{H}^i_c(X, \mathscr{G}) = \mathrm{H}^i(K) \otimes \mathbb{Q}_\ell = \mathrm{H}^i(K \otimes \mathbb{Q}_\ell)$, we have (with the sign convention of (?.1)

$$\operatorname{tr}(F^*, \mathrm{H}^\bullet_c(X, \mathscr{G})) = \operatorname{tr}(F^*, K^\bullet \otimes \mathbb{Q}_\ell) = \varprojlim_n \operatorname{tr}(F^*, K^\bullet_n) = \varprojlim_n \operatorname{tr}\left(F^*, \mathsf{R}\Gamma_c(X, \mathscr{F} \otimes \mathbb{Z}/\ell^{n+1})\right).$$

Apply (4.10) to $\mathscr{F}_0 \otimes \mathbb{Z}/\ell^{n+1}$ (viewed as a complex concentrated in degree 0), for $\Lambda = \mathbb{Z}/\ell^{n+1}$. We find that

$$\begin{aligned}\operatorname{tr}\left(F^*, \mathsf{R}\Gamma_c(X, \mathscr{F} \otimes \mathbb{Z}/\ell^{n+1})\right) &= \sum_{x \in X^F} \operatorname{tr}\left(F^*, \mathscr{F}_x \otimes \mathbb{Z}/\ell^{n+1}\right) \\ &= \sum_{x \in X^F} \operatorname{tr}(F^*, \mathscr{G}_x) \quad \bmod \ell^{n+1},\end{aligned}$$

and (3.2) follows by taking the limit.

5 The Nielsen-Wecken method

In this chapter, we prove two special cases of (4.10). The general case will be considered in the next chapter.

Lemme 5.1. *Theorem (4.10) is true for X_0 of dimension 0.*

In dimension 0, the formula (4.10) holds for any correspondence, and not only for the iterations of a Frobenius. It is a special case of the following statement (applied to $X(\mathbb{F})$, K and to the F^{*n}).

(*) Let X be a finite set, $F : X \to X$, $\mathscr{K} \in \operatorname{Ob} \mathsf{D}_{\text{parf}}(X, \Lambda)$ and $F^* : F^*\mathscr{K} \to \mathscr{K}$. We have

$$\sum_{X^F} \operatorname{tr}(F^*, \mathscr{K}_x) = \operatorname{tr}(F^*, \Gamma(X, K)).$$

We have $\Gamma(X, \mathscr{K}) = \bigoplus_{x \in X} \mathscr{K}_x$, and the formula results from the fact that for any morphism $u : \bigoplus_{x \in X} \mathscr{K}_x \to \bigoplus_{x \in X} \mathscr{K}_x$, with matrix u^y_x, we have $\operatorname{tr}(u) = \sum_{x \in X} \operatorname{tr}(u^x_x)$.

Lemme 5.2. *Let Λ be a Noetherian ring of torsion, killed by a power of the prime number $\ell \neq p$, X_0 be a projective, smooth and connected curve over $\mathbb{F}_q$, $j : U_0 \hookrightarrow X_0$ be a dense open of X_0 and $\mathscr{G}_0$ be a locally constant sheaf of Λ-modules over U_0. We have*

$$\sum_{x \in U^F} \operatorname{tr}(F^*, \mathscr{G}_x) = \operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U, \mathscr{G})).$$

More precisely, we will deduce (5.2) from

Théorème 5.3. *Let X be a non-singular projective curve over an algebraically closed field k, f be an endomorphism with isolated fixed points of X and $v(f)$ be the number of fixed points of f, each being counted with its multiplicity: $v(f)$ is the number of intersection of the graph of f with the diagonal of $X \times X$. Then, for ℓ prime to the characteristic exponent of k, we have*

$$v(f) = \sum (-1)^i \operatorname{tr}\left(f^*, \mathrm{H}^i(X, \mathbb{Q}_\ell)\right).$$

This theorem is due to Weil ([39], n° 43, 66 et 68) and the demonstration of Weil is reproduced in Lang [25] (VI §3 combined with VII §2 th.3). It can also be deduced from the formalism of the cohomology class associated with a cycle (The cohomology class associated to a cycle, 3.7).

Corollaire 5.4. *Let $j : U \hookrightarrow X$ be a dense open in X, such that $f^{-1}(U) = U$. If the fixed points of f in $X \smallsetminus U$ are of multiplicity one, then the number $v_U(f)$ of fixed points of f in U (each counted with its multiplicity) is*

$$v_U(f) = \sum (-1)^i \operatorname{tr}\left(f^*, \mathrm{H}^i_c(U, \mathbb{Q}_\ell)\right).$$

Let F be the finite set $X \smallsetminus U$. The cohomology sequence defined by the short exact sequence $0 \to j_!\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_{\ell F} \to 0$:

$$\xrightarrow{\partial} \mathrm{H}^i_c(U, \mathbb{Q}_\ell) \longrightarrow \mathrm{H}^i(X, \mathbb{Q}_\ell) \longrightarrow \mathrm{H}^i(F, \mathbb{Q}_\ell) \xrightarrow{\partial}$$

provides, with the sign convention of (3.1).

$$\operatorname{tr}(f^*, \mathrm{H}^\bullet_c(U, \mathbb{Q}_\ell)) = \operatorname{tr}(f^*, \mathrm{H}^\bullet(X, \mathbb{Q}_\ell)) - \operatorname{tr}(f^*, \mathrm{H}^\bullet(F, \mathbb{Q}_\ell)) = v(f) - \operatorname{tr}(f^*, \mathbb{Q}_\ell^F).$$

The trace of f^* on $\mathbb{Q}_\ell^F$ is the number $v_F(f)$ of fixed points of f in F; the hypothesis of multiplicity one therefore assures that

$$\operatorname{tr}(f^*, \mathrm{H}^\bullet_c(U, \mathbb{Q}_\ell)) = v_F(f) - v_U(f),$$

as promised.

The proof of (5.2) uses some lemmas that we will now develop for a particular case of non-commutative traces (4.2). *For brevity, we will say projective for a finite type projective.*

5.5

Let Λ be a ring, and H a finite group. The application $\sum a_h \cdot a \mapsto$ class of a_e, of the group algebra $\Lambda[H]$ in $\Lambda^\natural$, vanishes on the commutators $ab - ba$ of $\Lambda[H]$. It therefore factors through

$$\varepsilon : \Lambda[H]^\natural \to \Lambda^\natural.$$

If F is an endomorphism of a $\Lambda[H]$-module projective P, we define

$$\operatorname{tr}^H_\Lambda(F) = \varepsilon \operatorname{tr}_{\Lambda[H]}(F)$$

where, on the right hand side, is the trace (4.2). To avoid ambiguity, we will sometimes write this trace $\operatorname{tr}^H_\Lambda(F, P)$.

Proposition 5.6. $\operatorname{tr}_\Lambda(F) = |H| \operatorname{tr}^H_\Lambda(F)$.

The formal properties of traces bring us back to only considering the case where $P = \Lambda[H]$. The endomorphism F is then right multiplication by an element $\sum a_h \cdot h$ of $\Lambda[H]$, and $\operatorname{tr}_\Lambda(F) = |H| a_e$, $\operatorname{tr}^H_\Lambda(F) = a_e$.

5.7

Let A be a commutative ring and Λ an A-algebra. Multiplication by an element of A passes to the quotient to define an A-module structure on $A^\natural$. If H is a finite group, P an $A[H]$-module projective and M a $\Lambda[H]$-module, projective as a Λ-module, we know that the Λ-module $P \otimes_A H$, with the diagonal action of H, is a $\Lambda[H]$-module projective. Indeed:

a) It is enough to check it for $P = A[H]$.

b) The Λ-module $P \otimes_A M$, with the action of H defined by its action on the first factor, is clearly a $\Lambda[H]$-module projective; denote it by $(P \otimes_A M)'$.

c) The map $h \otimes m \mapsto h \otimes h^{-1}m$ induces an isomorphism

$$A[H] \otimes_A M \xrightarrow{\sim} (A[H] \otimes_A M)'. \tag{5.7.1}$$

Proposition 5.8. *With the previous notation, if u is an endomorphism of F and v is an endomorphism of M, we have*

$$\mathrm{tr}_\Lambda^H(u \otimes v) = \mathrm{tr}_A^H(u)\,\mathrm{tr}_\Lambda(v).$$

We may assume that $P = A[H]$; the endomorphism u is then right multiplication m_x by an element $x = \sum a_h h$ of $A[H]$. The isomorphism (5.7.1) transforms $u \otimes v$ into $\sum a_h m_h \otimes h^{-1}v$, of trace $a_e \cdot \mathrm{tr}_\Lambda(v)$.

5.9

Let G be an extension group of $\mathbb{Z}$ by a finite group H

$$1 \to H \to G \to \mathbb{Z} \to 1,$$

and G^+ the inverse image of $\mathbb{N}$. Let Λ be a ring, and P a $\Lambda[G^+]$-module, projective as a $\Lambda[H]$-module. If $g \in G^+$, we denote Z_g the centralizer of g in H; multiplication by g is an endomorphism of P, seen as a $\Lambda[Z_g]$-module. P being $\Lambda[Z_g]$-projective, a trace $\mathrm{tr}_\Lambda^{Z_g}(g)$ is defined. According to 5.6, we have

$$\mathrm{tr}_\Lambda(g) = |Z_g| \cdot \mathrm{tr}_\Lambda^{Z_g}(g). \tag{5.9.1}$$

Suppose that Λ is an algebra over a commutative ring A. For P a $A[G^+]$-module, projective as a $A[H]$-module and M a $\Lambda[G^+]$-module, projective as a Λ-module, the $\Lambda[G^+]$-module $P \otimes_A M$ (diagonal action of G^+) is projective as a $\Lambda[H]$-module and according to 5.8, for all $g \in G^+$, we have

$$\mathrm{tr}_\Lambda^{Z_g}(g, P \otimes_A M) = \mathrm{tr}_A^{Z_g}(g, P) \cdot \mathrm{tr}_\Lambda(g, M). \tag{5.9.2}$$

5.10

Let P be a $\Lambda[G^+]$-module projective as a $\Lambda[H]$-module, and P_H the coinvariants of H in P. It is a Λ-module projective. The action of $g \in G^+$ passes to the quotient, and defines an endomorphism of P_H which only depends on the image of g in $\mathbb{N}$. We denote F the action of the elements $g \in G^+$ with image 1.

Proposition 5.11. $\mathrm{tr}_\Lambda(F, P_H) = \sum'_{g\mapsto 1} \mathrm{tr}_\Lambda^{Z_g}(g, P)$, *where* $\sum'_{g\mapsto 1}$ *is a sum extended to the* H*-conjugacy classes of elements of* G^+ *with image* 1 *in* $\mathbb{N}$.

After multiplication by $|H|$, this formula can be verified as follows:

$$|H| \cdot \mathrm{tr}_\Lambda(F, P_H) = \mathrm{tr}_\Lambda\left(\sum_{g\mapsto 1} g, P\right) = \sum_{g\mapsto 1}{}' \frac{|H|}{|Z_g|} \mathrm{tr}_\Lambda(g, P) = \sum_{g\mapsto 1}{}' |H| \cdot \mathrm{tr}_\Lambda^{Z_g}(g, P).$$

Choose an element $g \in G^+$ with image 1 in $\mathbb{N}$, and let σ be the automorphism $\Lambda[H]$ induced by the automorphism $\mathrm{ad}\, g$ of H. It is the same to give a $\Lambda[G^+]$-module, or a $\Lambda[H]$–module with a σ-linear endomorphism γ: we let $g^n h$ act by $\gamma^n h$. In this language, formula 5.11 takes the form: for P a $\Lambda[H]$-module projective and for γ a σ-linear endomorphism of P, we have

$$\mathrm{tr}_\Lambda(\gamma, P_H) = \sum_h{}' \mathrm{tr}_\Lambda^{Z_g}(h\gamma, P),$$

where $\sum'$ is a sum over the $\mathrm{ad}\, g$-conjugacy classes in H (orbits of H acting on itself by $x \mapsto hx\, \mathrm{ad}\, g(h)^{-1}$).

In this form, the properties of the traces immediately bring us back to the case where $P = \Lambda[H]$. The endomorphism Y then has the form $x \mapsto \sigma(x)a$, with $a = \sum a_h h$ in $\Lambda[H]$. It is enough to treat the universal case where the a_h are indeterminates. In this case, $\Lambda = \mathbb{Z}\langle a_h\rangle_{h\in H}$ (non-commutative polynomials) and $\Lambda^\natural$ is torsion-free: we obtain 5.11 by the argument at the beginning and division by $|H|$. We can also calculate directly.

5.11.1 Remark

Everything that has been said above also applies to $\mathbb{Z}/2$-graded modules, or to complexes of modules (cf. 4.2, 4.3).

5.12

We take the notation from 5.2, and let $A = \mathbb{Z}/\ell^n$, n being large enough so that Λ is an A-algebra. Let $f : V_0 \to U_0$ be a finite étale and connected covering of U_0, Galoisian of Galois group H, and such that the sheaf $f^*\mathscr{G}_0$ is constant. We denote by M the value of the constant $\mathrm{H}^0(V_0, f^*\mathscr{G}_0)$, it is a $\Lambda[H]$-module, projective as a Λ-module.

The sheaf f_*A on U_0, equipped with the natural action of H, is a locally free (of rank one) sheaf of $A[H]$-modules. The complex $\mathsf{R}\Gamma_c(U, f_*A)$ is therefore an object of $\mathsf{D}_{\mathrm{parf}}(A[H])$. It is equipped with a Frobenius endomorphism F^*.

For the natural action of H on $f_*f^*\mathscr{G}_0$, the trace morphism $f_*f^*\mathscr{G}_0 \to \mathscr{G}_0$ factors through an isomorphism

$$(f_*f^*\mathscr{G}_0)_H \to \mathscr{G}_0.$$

Furthermore, we have $f_*f^*\mathscr{G}_0 = f_*A \otimes_A M$ (with diagonal action of H). The isomorphism

$$\mathscr{G}_0 \xleftarrow{\sim} (f_*A \otimes_A M)_H = (f_*A \otimes_A M) \otimes_{\Lambda[H]} \Lambda$$

is derived from (4.12) and an isomorphism

$$\mathsf{R}\Gamma_c(U, \mathscr{G}) \sim (\mathsf{R}\Gamma_c(U, f_*A) \otimes_A M) \otimes_{\Lambda[H]} \Lambda.$$

The endomorphism F^* of the left-hand side is deduced from the endomorphism F^* of $\mathsf{R}\Gamma_c(U, f_*A)$. The complex of $\Lambda[H]$-modules $\mathsf{R}\Gamma_c(U, f_*A)$, equipped with F^*, can be seen as a complex of $\Lambda[G^+]$-modules, for $G^+ = H \times \mathbb{N}$. The formulas 5.10 and (5.9.2), amplified by 5.11.1, therefore provide

$$\operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U, \mathscr{G})) = \sum_h{}' \operatorname{tr}_A^{Z_g}(hF^*, \mathsf{R}\Gamma_c(U, f_*A)) \cdot \operatorname{tr}_\Lambda(h, M)$$

where $\sum'$ is a sum over the conjugacy classes in H. Moreover,

$$|Z_h| \cdot \operatorname{tr}_A^{Z_h}(hF^*, \mathsf{R}\Gamma_c(U, f_A)) = \operatorname{tr}_A\left((Fh^{-1})^*, \mathsf{R}\Gamma_c(V, A)\right).$$

This formula remains true for $A = \mathbb{Z}/\ell^m$, with m increasingly large. Passing to the limit, it is found that $\operatorname{tr}_A^{Z_g}(hF^*, \mathsf{R}\Gamma_c(U, f^*A))$ is equal to

$$\frac{1}{|Z_h|} \sum (-1)^i \operatorname{tr}\left((Fh^{-1})^*, \mathrm{H}_c^i(V, \mathbb{Q}_\ell)\right)$$

(an element of $\mathbb{Z}_\ell$), and

$$\operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U, \mathscr{G})) = \sum_h{}' \frac{1}{|Z_h|} \operatorname{tr}\left((Fh^{-1})^*, \mathrm{H}_c^i(V, \mathbb{Q}_\ell)\right) \cdot \operatorname{tr}_\Lambda(h, M). \tag{5.12.1}$$

5.13

It remains to apply 5.4 (note that if $\bar{V}$ is the projective completion of V, the fixed points of Fh^{-1} are all of multiplicity one). We can do this without any calculation:

A. If $U^F = \varnothing$, $\operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U, \mathscr{G})) = 0$.

Indeed, Fh^{-1} has no fixed point on V, 5.4 shows that $\sum (-1)^i \operatorname{tr}\left((Fh^{-1})^*, \mathrm{H}_c^i(V, \mathbb{Q}_\ell)\right) = 0$ and we apply (5.12.1).

B. In the general case, let $j : U' = U \smallsetminus U^F \hookrightarrow U$. The filtration $j_! j^* \mathscr{G}_0 \subset \mathscr{G}_0$ of $\mathscr{G}_0$ induces a filtration of $\mathsf{R}\Gamma_c(U, \mathscr{G})$ with successive quotients $\mathsf{R}\Gamma_c(U', \mathscr{G})$ and $\mathsf{R}\Gamma_c(U^F, \mathscr{G})$. According to (4.4.1) and 5.1, we have

$$\operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U, \mathscr{G})) = \operatorname{tr}(F^*, \mathsf{R}\Gamma_c(U', \mathscr{G})) + \sum_{x \in U^F} \operatorname{tr}(F^*, \mathscr{G}_x),$$

and, since ${U'}^F = 0$, the first term on the right hand side is 0.

6 The end of the proof of 4.10

We prove 4.10, under the additional hypotheses that the ring Λ is annihilated by a power of a prime number $\ell \neq p$, and that $n = 1$. The general case will follow: decompose Λ into its ℓ-primary components (ℓ ranging over the prime numbers), and replace $\mathbb{F}_q$ by $\mathbb{F}_{q^n}$, X_0 by $X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$.

For X_0 separated of finite type over $\mathbb{F}_q$, and $\mathscr{K}_0 \in \operatorname{Ob} \mathsf{D}_{\text{ctf}}(X_0, \Lambda)$, we set

$$T'(X_0, \mathscr{K}_0) = \sum_{x \in X^F} \operatorname{tr}(F^*, \mathscr{K}_x)$$

and

$$T''(X_0, \mathscr{K}_0) = \operatorname{tr}(F^*, \mathsf{R}\Gamma_c(X, \mathscr{K})).$$

We need to prove the identity

$$(1; X_0, \mathscr{K}_0) \qquad T'(X_0, \mathscr{K}_0) = T''(X_0, \mathscr{K}_0).$$

We begin by observing that T' and T'' (generically denoted T) obey the same formalism:

(A) If $j : U_0 \hookrightarrow X_0$ is an open set of X_0, with complementary closed set Y_0, then

$$T(X_0, \mathscr{K}_0) = T(U_0, \mathscr{K}_0|U_0) + T(Y_0, \mathscr{K}_0|Y_0).$$

This is clear for T'. For T'', we observe that the filtration $0 \subset j_! j^* \mathscr{K}_0 \subset \mathscr{K}_0$ of $\mathscr{K}_0$ induces a filtration of $\mathsf{R}\Gamma_c(X, \mathscr{K})$, with successive quotients $\mathsf{R}\Gamma_c(U, \mathscr{K})$ and $\mathsf{R}\Gamma_c(Y, \mathscr{K})$, and we invoke (4.4.1) (more precisely stated: $\mathscr{K}_0$, equipped with the said filtration, defines an object of $\mathsf{DF}(X, \Lambda)$; by applying $\mathsf{R}\Gamma_c$, we deduced an object of $\mathsf{DF}_{\text{parf}}(\Lambda)$, to which $\mathsf{R}\Gamma_c(X, \mathscr{K})$ is subordinate, and of graded as said).

(B) If $\mathscr{K}_0$ is a complex with bounded components and flat,

$$T(X_0, \mathscr{K}_0) = \sum (-1)^i T(X_0, \mathscr{K}^i).$$

This is clear for T'; for T'', we apply the above argument to the stupid filtration of $\mathscr{K}_0$, to obtain

$$T''(X_0, \mathscr{K}_0) = \sum T''(X_0, \mathscr{K}_0^i[-i]) = \sum (-1)^i T''(X_0, \mathscr{K}_0^i).$$

(C) For $f : X_0 \to Y_0$ a morphism, we have

$$T''(X_0, \mathscr{K}_0) = T''(Y_0, \mathsf{R}f_! \mathscr{K}_0)$$

and the same identity holds for T' if the fibers of f at rational points of Y_0 satisfying $\left(1; f^{-1}(y), \mathscr{K}_0|f^{-1}(y)\right)$: we use that

$$T'(X_0, \mathscr{K}_0) = \sum_{y \in Y^F} T'\left(f^{-1}(y), \mathscr{K}_0|f^{-1}(y)\right)$$

and

$$\mathsf{R}\Gamma_c(Y, \mathsf{R}f_! \mathscr{K}) = \mathsf{R}\Gamma_c(X, \mathscr{K}).$$

Using this formalism, and 4.7, we bring the proof of 4.10 to the two particular cases 5.1 and 5.2.

Chapter 3

Functions L modulo ℓ^n and modulo p

In this paper, I prove a formula analogous to the theorem of the report on the trace formula (this volume; cited "Report" hereafter) for a commutative ring A which is of prime torsion to p, or a field of characteristic p. An essential use will be made of [2, XVII 5.5].

1 Symmetric tensors

1.1

Let A be a commutative ring. For a list of properties of the functors $\Gamma^n : (A\text{-modules}) \to (A\text{-modules})$, I refer to [2, XVII 5.5.1 and 2]. We only recall that for M flat (actually, only the case where M is projective of finite type is of interest to us), we have

$$\Gamma^n M = \left(M^{\otimes n}\right)^{S_n} \qquad \text{(symmetric tensors of degree } n\text{).}$$

1.2

Let X be an S-scheme, X^n/S the n-th fibered power of X, and π the projection of X^n/S onto $(X^n/S)/S_n = \mathrm{Sym}^n_S(X)$ (supposed to exist). If $\mathscr{G}$ is a sheaf of A-modules on X, the external tensor product of n copies of $\mathscr{G}$; $\mathscr{G}^{\boxtimes n}$, is a sheaf S_n-equivariant on X^n/S. We will make use of the functor Γ^n *external* of [2, XVII 5.5.7 à 9]. We only recall that for $\mathscr{G}$ a sheaf of A-modules flat (actually, only the case constructible and flat interests us), we have

$$\Gamma^n_{\text{ext}}(\mathscr{G}) = \left(\pi_* \mathscr{G}^{\boxtimes n}\right)^{S_n}.$$

Lemme 1.3. *For $S = \mathrm{Spec}(k)$, with k algebraically closed, X a finite sum of copies of S, and G a flat sheaf of A-modules on X, we have*

$$\Gamma\left(\mathrm{Sym}^n_S(X), \Gamma_{ext}(\mathscr{G})\right) = \Gamma^n \Gamma(X, \mathscr{G})$$

In fact,

$$\Gamma\left(\mathrm{Sym}^n_S(X), \Gamma^n_{\mathrm{ext}}(\mathscr{G})\right) = \Gamma\left(\mathrm{Sym}^n_S(X), \pi_*\mathscr{G}^{\boxtimes n}\right)^{S_n}$$
$$= \Gamma\left(X^n/S, \mathscr{G}^{\boxtimes n}\right)^{S_n} = \left(\Gamma(X,\mathscr{G})^{\otimes n}\right)^{S_n} = \Gamma^n\Gamma(X,\mathscr{G}).$$

1.4

We will need to use the derived functors of the non-additive functors Γ^n and Γ^n_{ext}.

The theory of derived functors is due to Dold and Puppe [17]. For more complete reminders than those given below, I refer to [2, XVII 5.5.3].

Let $\mathcal{A}$ be an abelian category (or an additive category where every idempotent endomorphism is the projection onto a direct factor). Let N be the normalization functor: N : (cosimplicial objects in $\mathcal{A}$) $\to$ (differentiable objects in $\mathcal{A}$, with K^i for $i < 0$). It is an equivalence of categories. As is its inverse, it transforms homotopes into homotopic morphisms. If Γ is a functor $\mathcal{A} \to \mathcal{B}$, and if K is a complex of objects of $\mathcal{A}$, in positive degrees ($K^i = 0$ for $i < 0$), we set $TK = NTN^{-1}K$.

1.5 Example

If K is reduced to M in degree 0, TK is reduced to TM in degree 0.

1.6

If K is a bounded complex of finitely generated projective A-modules, the complex of A-modules $\Gamma^n K$ has the same properties.

Let $\mathsf{D}^{\geqslant 0}_{\mathrm{parf}}(A)$ be the subcategory of $\mathsf{D}_{\mathrm{parf}}(A)$ (4.3) formed by complexes of tor-dimension $\leqslant 0$. We see that Γ^n induces a functor

$$\mathsf{L}\Gamma^n : \mathsf{D}^{\geqslant 0}_{\mathrm{parf}}(A) \to \mathsf{D}^{\geqslant 0}_{\mathrm{parf}}(A).$$

For the analogous, but more complicated, definition of $\mathsf{L}\Gamma^n_{\mathrm{ext}}$, I refer to [2, XVII 5.5.14].

1.7

Let K be a bounded complex of finitely generated projective A-modules, and u an endomorphism of K. We denote by $\det(1-ut, K)$ the formal power series with constant term 1

$$\det(1-ut, K) = \prod_i \det(1-ut, K^i)^{(-1)^i}.$$

We leave it to the reader to verify the following properties.

1.7.1

Let F be a finite filtration stable by u, such that the $\mathrm{gr}^p_F(K^q)$ are projective of finite type. Then

$$\det(1-ut, K) = \prod_p \det(1-ut, \mathrm{gr}^p_F(K)).$$

1.7.2

If we translate K by q steps to the left, we have

$$\det(1-ut, K[q]) = \det(1-ut, K)^{(-1)^q}.$$

Proposition 1.8. *Let u be an endomorphism of a bounded complex, with positive degrees, of A-modules projective of finite type. We have*

$$\det(1-ut, K)^{-1} = \sum_n \operatorname{tr}(u, \Gamma^n K)t^n. \tag{1.8.1}$$

Let $D(u, K)$ be the second member of (1.8.1).

Lemme 1.9. *Let u be an endomorphism of a short exact sequence of bounded complexes of finite type A-modules*

$$0 \to K' \to K \to K'' \to 0.$$

Then $D(u, K) = D(u, K') \cdot D(u, K'')$.

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of split A-modules, $\Gamma^n M$ has a natural filtration, with successive quotients $\Gamma^i M' \otimes \Gamma^j M''$ $(i+j=n)$. Therefore, $\Gamma^n K$ has a natural filtration, with successive quotients the complexes $N(N^{-1}\Gamma^i K' \otimes N^{-1}\Gamma^j K'')$ $(i+j=n)$, canonically homotopic to the simple complexes $s(\Gamma^i K' \otimes \Gamma^i K'')$ deduced from the double complexes $\Gamma^i K' \otimes \Gamma^i K''$, and

$$\operatorname{tr}(u, \Gamma^n K) = \sum_{i+j=n} \operatorname{tr}(u, \Gamma^i K') \cdot \operatorname{tr}(u, \Gamma^j K''),$$

from which the lemma follows. From 1.9, we deduce by induction.

Lemme 1.10. *Let F be a finite filtration of K that is stable under u, such that the $\operatorname{gr}_F^p(K^q)$ are projective of finite type. Then,*

$$D(u, K) = \prod_p D\left(u, \operatorname{gr}_F^p(K)\right).$$

Lemme 1.11. *Let u be an endomorphism of a projective A-module of finite type M, and $M[-q]$ the complex reduced to M in degree q. We have*

$$D\left(u, M[-q]\right) = \left(\sum \operatorname{tr}(u, \Gamma^n M)t^n\right)^{(-1)^q} \qquad (q \geqslant 0.)$$

According to 1.5, this is true for $q = 0$. We proceed by induction, it remains to show that

$$D\left(u, M[-q]\right) \cdot D\left(u, M[-(q+1)]\right) = 1.$$

This formula follows from 1.9, applied to the complex K reduced to M in degrees q and $q+1$ (with $d = \mathrm{id}$), and to its stupid filtration:

$$(\cdots 0 \to M \cdots) \to (\cdots M \to M) \to (\cdots M \to 0 \to 0 \cdots).$$

Indeed, K is homotopic to 0. In the derived category, we therefore have $\Gamma^n(K) = \Gamma^n(0)$; 0 for $n > 0$ and A for $n = 0$, and $D(u, K) = 1$.

1.12

Let's prove 1.8. According to 1.10 applied to the "stupid" filtration of K by the subcomplexes

$$0 \to \cdots \to 0 \to K^q \to K^{q+1} \to \cdots,$$

we have

$$D(u,K) = \prod_q D\left(u, K^q[-q]\right) =_{(1.10)} \prod_q \left(\sum \operatorname{tr}\left(u, \Gamma^n(K^q)\right) t^n\right)^{(-1)^q}.$$

It remains to prove that for u an endomorphism of a finitely generated projective module M, we have

$$\det\left(1-ut, M\right)^{-1} = \sum \operatorname{tr}(u, \Gamma^n M) t^n. \tag{1.12.1}$$

If $u = 0$, both sides equal 1. Adding to M a module N on which $u = 0$, and apply 1.9, we may assume that M is free. Take a basis. We then need to prove algebraic identities between the coordinates of u. The principle of prolongation of algebraic identities reduces us to assume that A is an algebraically closed field. The module M then admits a filtration stable by u with successive quotients of rank 1, and 1.10 reduces us to the case where M has rank 1. The formula (1.12.1) is then reduced to the identity

$$\frac{1}{1-at} = \sum a^n t^n.$$

Corollaire 1.13. *Let u and u' be endomorphisms of bounded complexes K and K' of A-modules projective of finite type. If, in the derived category, K, endowed with u, is isomorphic to K', endowed with u', then*

$$\det(1-ut, K) = \det(1-u't, K').$$

Apply 1.7.2, we may assume that K and K' are in positive degrees. We then observe that the second member of 1.8 has the desired invariance property.

1.14

This corollary allows us to define $\det(1-ut,K)$ for u an endomorphism of $K \in \operatorname{Ob} \mathsf{D}_{\mathrm{parf}}(A)$. This construction is ad hoc; in fact, for v an automorphism of $L \in \operatorname{Ob} \mathsf{D}_{\mathrm{parf}}(\Lambda)$, we can define $\det(v) \in \Lambda^\times$, and $\det(1-ut,K)$ is obtained for $v = 1-ut$, $\Lambda = A[\![t]\!]$ and $L = K \otimes_A \Lambda$.

2 The Theorem

2.1

We continue with the notations from (Report, §1). For A a Noetherian commutative ring of torsion, X_0 a finite type separated scheme over $\mathbb{F}_q$ and $\mathscr{K}_0 \in \operatorname{Ob} \mathsf{D}_{\mathrm{ctf}}(X_0; A)$, we set

$$L(X_0, \mathscr{K}_0) = \prod_{x \in |X_0|} \det\left(1 - F_x^* t^{\deg(x)}, \mathscr{K}\right)^{-1}$$

(1.7, 1.13 and Report, 1.5, 1.6).

Théorème 2.2. *In each of the following two cases*

a) A is of prime torsion to p;

b) A is reduced and of characteristic p

we have

$$L(X_0, \mathscr{K}_0) = \det\left(1 - F^* t^f, \mathsf{R}\Gamma_c(X, \mathscr{K})\right)^{-1}. \tag{2.2.1}$$

2.3 Remarque

In case a), the method of Rapport, §3 allows us to deduce from the formula of traces (Rapport 4.10) that the logarithmic derivatives of the two members of (2.2.1) are equal. The ring A being of torsion, this is not enough to prove 2.2.

2.4 Remarque

If A is a field, we have

$$L(X_0, \mathscr{K}_0) = \prod_i L\left(X_0, \underline{\mathrm{H}}^i(\mathscr{K}_0)\right)^{(-1)^i}$$

and the spectral sequence $E_2^{pq} = \mathrm{H}_c^p(X, \underline{\mathrm{H}}^q(\mathscr{K})) \Rightarrow \mathrm{H}_c^{p+q}(X, \mathscr{K})$ shows that

$$\begin{aligned}\det\left(1 - F^* t, \mathsf{R}\Gamma_c(X, \mathscr{K})\right) &= \prod_n \det\left(1 - F^* t^f, \mathrm{H}_c^n(X, \mathscr{K})\right)^{(-1)^n} \\ &= \prod_{p,q} \det\left(1 - F^* t, \mathrm{H}_c^p(X, \underline{\mathrm{H}}^q(\mathscr{K}))\right)^{(-1)^{p+1}}.\end{aligned}$$

This reduces 2.2 (for A a field) to the following particular case.

2.4.1

If $\mathscr{G}_0$ is a constructible sheaf of A-vector spaces, then

$$L(X_0, \mathscr{G}_0) = \prod_n \det\left(1 - F^* t^f, \mathrm{H}_c^n(X, \mathscr{G})\right)^{(-1)^{n+1}}.$$

2.5 Remark

It is easy to reduce case b) to the case where A is a field (or even a finite field) of characteristic p, cf. 4.2.

2.6 Remark

The special case of b) where $A = \mathbb{Z}/p$, and where X is reduced to a constant sheaf $\mathbb{Z}/p$ is of degree 0;

$$Z(X_0, t) = \prod_{x \in |X_0|} \left(1 - t^{\deg(x)}\right)^{-1} \equiv \prod_n \det\left(1 - F^* t^f, \mathrm{H}_c^n(X, \mathbb{Z}/p)\right)^{(-1)^{n+1}} \pmod p$$

had been proved by N. Katz [16, XXII 3.1]. His method, entirely different from the one followed here, starts from Dwork's p-adic theory of $Z(X_0, t)$.

2.7 Remarque

Contrary to what I carelessly affirmed to friends, in b) we cannot omit the hypothesis "A reduced." For a counterexample, see 4.5.

2.8 Plan of the proof of 2.2

A translation on the degrees 1.7.2 allows us to suppose that $\mathscr{K}_0$ is a bounded complex, in positive degrees, of constructible and flat A-module sheaves. We then have $\mathsf{R}\Gamma_c(X, \mathscr{K}) \in \operatorname{Ob} \mathsf{D}^{\geqslant 0}_{\mathrm{parf}}(A)$ (Rapport, proof 4.9, b). We will use 1.8 to develop both sides of (2.2.1) into a power series of $T = t^f$. The equality of the coefficients of T^n will follow from [2, XVII 5.5.21] and a trace formula on $\operatorname{Sym}^n(X)$; in case a), the formula (Rapport, 4.10), and in case b) a formula which will be proven in the next chapter. We may reduce to the case where the symmetric powers of X_0 exist (for example when X is affine) using the multiplicativity properties in X_0 of both sides of (2.2.1): for U_0 an open of X_0, of complementary closed Y_0, we have $L(X_0, \mathscr{K}_0) = L(U_0, \mathscr{K}_0) \cdot L(Y_0, \mathscr{K}_0)$ and the analogous formula for the right-hand side follows from 1.7.1 by the method of (Rapport §6A).

2.9 The case where X_0 is finite

The multiplicativity in X_0 of the two members of (2.2.1) reduces this case to the one where $X_0 = \operatorname{Spec}(\mathbb{F}_{q^k})$. Going back to the definitions, one sees that (2.2.1) is equivalent to Rapport 3.4.

2.10 The first member

According to 1.8, one has

$$L(X_0, \mathscr{K}_0) = \prod_{x \in |X_0|} \sum_n \operatorname{tr}(F_x^*, \Gamma^n \mathscr{K})\, t^{m \deg(x)}.$$

The coefficient of T^n is therefore the sum, extended to the formal linear combinations $\sum_{x \in |X_0|} n_x \cdot x$ ($n_x \geqslant 0$, the n_x almost all zero) of elements of $|X_0|$ such that $\sum n_x [k(x) : \mathbb{F}_q] = n$, of products

$$\prod_x \operatorname{tr}(F_x^*, \Gamma^{n_x} \mathscr{K}). \tag{2.10.1}$$

For $x \in |X_0|$, we denote by (x) the 0-cycle of the $[k(x) : \mathbb{F}_q]$ points of X above x (an orbit of F). The application $\sum n_x \mapsto \sum n_x \cdot (x)$ identifies formal linear combinations as above, with $\sum n_x [k(x) : \mathbb{F}_q] = n$, to effective 0-cycles of degree n of X fixed under F, that is, to the fixed points of F in $\operatorname{Sym}^n(X)$. The coefficient of T^n in $L(X_0, \mathscr{K}_0)$ thus appears as a sum of products (2.10.1), indexed by $\operatorname{Sym}^n(X)^F$. More precisely,

Lemme 2.11. $L(X_0, \mathscr{K}_0) = \sum_n \sum_{x \in \operatorname{Sym}^n(X)^F} \operatorname{tr}(F_x^*, \mathsf{L}\Gamma^n_{\mathrm{ext}} \mathscr{K})\, T^n$.

For Y_0 a finite sub-scheme of X_0 getting larger and larger, we have

$$L(X_0, \mathscr{K}_0) = \lim L(Y_0, \mathscr{K}_0),$$

i.e., for all m, there exists a finite sub-scheme $Z_0 \subset X_0$ such that if $Y_0 \supset Z_0$, $L(X_0, \mathscr{K}_0)$ and $L(Y_0, \mathscr{K}_0|Y_0)$ are congruent $\mod T^m$. It is enough that Z_0 contains the closed points of degree $\leqslant m$. The same being true for the right-hand side, it is enough to prove 2.11 for X_0 finite. We then have

$$\begin{aligned} L(X_0, \mathscr{K}_0) &=_{2.9} \det(1 - F^*T, \Gamma(X, \mathscr{K}))^{-1} \\ &=_{1.8} \sum_n \mathrm{tr}\,(F^*, \Gamma^n\Gamma(X, \mathscr{K}))\,T^n \\ &=_{1.3} \sum_n \mathrm{tr}\,(F^*, \Gamma(\mathrm{Sym}^n(X), \Gamma^n_{\mathrm{ext}}(\mathscr{K}))\,T^n \\ &= \sum_n \sum_{x \in \mathrm{Sym}^n(X)^F} \mathrm{tr}\,(F^*_x, \Gamma^n_{\mathrm{ext}}(\mathscr{K}))\,T^n \end{aligned}$$

(by the trace formula in the trivial case of a finite set).

2.12 The second term

According to 1.8 and [2, XVII 5.5.12], we have

$$\begin{aligned} \det(1 - F^*T, \mathsf{R}\Gamma_c(X, \mathscr{K}))^{-1} &= \sum_n \mathrm{tr}\,(F^*, \mathsf{L}\Gamma^n\mathsf{R}\Gamma_c(X, \mathscr{K}))\,T^n \\ &= \sum_n \mathrm{tr}\,(F^*, \mathsf{R}\Gamma_c(\mathrm{Sym}^n(X), \Gamma^n_{\mathrm{ext}}\mathscr{K}))\,T^n. \end{aligned}$$

2.13

The theorem is therefore equivalent to the trace formulas

$$\mathrm{tr}\,(F^*, \mathsf{R}\Gamma_c(\mathrm{Sym}^n(X, \Gamma^n_{\mathrm{ext}}\mathscr{K})) = \sum_{x \in \mathrm{Sym}^n(X)^F} \mathrm{tr}\,(F^*_x, \Gamma^n_{\mathrm{ext}}\mathscr{K})\,.$$

In case a), they follow from Report 4.10. The analogous formula in case b) will be proven in §4.

3 Artin-Scheier Theory

3.1

In what follows, a coherent sheaf on a scheme S will always be regarded as a sheaf on $S_{\text{ét}}$. Let S be a scheme of characteristic $p > 0$. If G is a locally constant sheaf of $\mathbb{Z}/p$-vector spaces of finite rank, $\mathscr{G} = G \otimes_{\mathbb{Z}/p} \mathscr{O}$ is a coherent locally free sheaf on S. We will denote Φ both the endomorphism $f \mapsto f^p$ of $\mathscr{O}$ and its tensor product with the identity of G: $\phi : \mathscr{G} \to \mathscr{G}$. Artin-Scheier Theory [2, IX 3.5] asserts that the sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathscr{O} \xrightarrow{\Phi - 1} \mathscr{O} \longrightarrow 0$$

is exact. By tensoring with G, we find an exact sequence

$$0 \longrightarrow G \longrightarrow \mathscr{G} \xrightarrow{\Phi - 1} \mathscr{G} \longrightarrow 0$$

3.2

Suppose that S is an open in a noetherian scheme $\bar{S}$. Let $j : S \hookrightarrow \bar{S}$ be the inclusion map, and I an ideal sheaf that defines the complementary closed $\bar{S} \smallsetminus S$. We still denote by Φ the extension by functoriality of Φ to $j_*\mathscr{G}$.

Lemme 3.3. *There exist coherent extensions $\bar{\mathscr{G}} \subset j_*\mathscr{G}$ of $\mathscr{G}$ such that*

$$\Phi(\bar{\mathscr{G}}) \subset I \cdot \bar{\mathscr{G}}. \tag{3.3.1}$$

If $\bar{\mathscr{G}}$ is such an extension, the sequence

$$0 \longrightarrow j_! G \longrightarrow \bar{\mathscr{G}} \xrightarrow{\Phi - 1} \bar{\mathscr{G}} \longrightarrow 0 \tag{3.3.2}$$

is exact.

a) *Existence*: Let $\mathscr{G}' \subset j_*\mathscr{G}$ be a coherent extension of $\mathscr{G}$ to $\bar{S}$. For all n, we have $\Phi(I^n\mathscr{G}') \subset I^{p^n}\Phi(\mathscr{G}')$. Let n be large enough so that $I^{n(p-1)-1}\Phi(\mathscr{G}') \subset \mathscr{G}'$, and set $\bar{\mathscr{G}} = I^n\mathscr{G}'$. We have $\Phi(\bar{\mathscr{G}}) \subset I^{pn}\Phi(\mathscr{G}') = I \cdot I^n \cdot I^{n(p-1)-1}\Phi(\mathscr{G}') \subset I \cdot I^n\mathscr{G}' = I\bar{\mathscr{G}}$.

b) *Kernel of* $\Phi - 1$: On S, the assertion follows from 3.1; it remains to check that a section x of $\ker(\Phi - 1)$, on U étale over $\bar{S}$, vanishes in a neighborhood of the inverse image of $\bar{S} \smallsetminus S$. Since $x = \Phi(x)$, we have by (3.3.1) $x \in I\bar{\mathscr{G}}$; moreover, $x \in I^n\bar{\mathscr{G}} \Rightarrow x = \Phi(x) \in \Phi(I^n\bar{\mathscr{G}}) \subset I^{np}\bar{\mathscr{G}}$, and x is in all the $I^n\bar{\mathscr{G}}$, so it is null in a neighborhood of $\bar{S} \smallsetminus S$.

c) *Surjectivity of* $\Phi - 1$: We can assume $\bar{S}$ is affine, write $\mathscr{G}$ as a quotient of $\mathscr{O}^n$; Φ then lifts to a p-linear map $\widetilde{\Phi} : \mathscr{O}^n \to \mathscr{O}^n$, and it is enough to check the surjectivity of $\widetilde{\Phi} - 1$:

$$\begin{tikzcd} \mathscr{O}^n \arrow[r, "\widetilde{\Phi}-1"] \arrow[d] & \mathscr{O}^n \arrow[d] \\ \mathscr{G} \arrow[r, "\Phi-1"] & \mathscr{G} \end{tikzcd}$$

The sheaf $\mathscr{O}^n$ is the sheaf of local sections of the $\mathbb{G}_a^n$ group scheme, and $\widetilde{\Phi} - 1$ is induced by a $\mathbb{G}_a^n$ homomorphism into $\mathbb{G}_a^n$. This homomorphism is étale. To check that it is surjective, it is enough to see this after any base change $\mathrm{Spec}(k) \to \bar{S}$ (k algebraically closed) and over k algebraically closed, any étale homomorphism between connected groups of the same dimension is surjective. Finally, an étale and surjective morphism of $\bar{S}$-schemes induces an epimorphism between the corresponding sheaves on $\bar{S}_{\mathrm{et}}$.

3.4 Frobenius and Frobenius

The absolute Frobenius morphism $F_{\mathrm{abs}} : S \to S$ is the identity on the underlying topological space of S, and $f \mapsto f^p$ on the structural sheaf. For an étale U/S, we have a canonical isomorphism $F_{\mathrm{abs}}^* U = U$: the diagram

$$\begin{tikzcd} U \arrow[r, "F_{\mathrm{abs}}"] \arrow[d] & S \arrow[d] \\ S \arrow[r, "F_{\mathrm{abs}}"] & S \end{tikzcd}$$

is cartesian. The endomorphism of the sheaf of rings $(\bar{S}_{\text{et}}, \mathscr{O}) \to (\bar{S}_{\text{et}}, \mathscr{O})$ defined by F_{abs} can thus be described as follows:

$$F_{\text{abs}}^* \text{ is the identity on } S_{\text{et}}\text{; the homomorphism } F_{\text{abs}}^*\mathscr{O} = \mathscr{O} \to \mathscr{O} \text{ is } \Phi.$$

With this description, for any étale sheaf $\mathscr{G}$ on S, the Frobenius correspondence $F_{\text{abs}}^*\mathscr{G} \to \mathscr{G}$ is the identity.

Giving a p-linear morphism of quasi-coherent sheaves $u : \mathscr{M} \to \mathscr{N}$ on S is the same as giving a linear morphism $u' : F_{\text{abs}}^*\mathscr{M} \to \mathscr{N}$. In particular, with the notation of 3.5, the homomorphism $\Phi : \mathscr{G} \to \mathscr{G}$ corresponds to $\Phi' : F_{\text{abs}}^*\mathscr{G} \to \mathscr{G}$.

Lemme 3.5. *With the notation of 3.4, the diagram*

$$\begin{tikzcd} F_{\text{abs}}^* G \arrow[r] \arrow[d] & G \arrow[d] \\ F_{\text{abs}}^*\mathscr{G} \arrow[r, "\Phi'"] & \mathscr{G} \end{tikzcd}$$

is commutative.

It is enough to check it locally. So we can assume G is constant, in which case the lemma is evident.

The same theory holds for the powers of F_{abs}.

3.6

Now suppose $\bar{S}$ is a scheme over $\mathbb{F}_q$ $(q = p^f)$ and in accordance with the conventions of Report §1, write $S_0, \bar{S}_0, G_0, \mathscr{G}_0, \bar{\mathscr{G}}_0$ instead of $S, \ldots$. The lemma 3.5, for $F_{\text{abs}}^F : S_0 \to S_0$, gives a commutative diagram

$$\begin{tikzcd} F^* G_0 \arrow[r] \arrow[d] & G_0 \arrow[d] \\ F^*\mathscr{G}_0 \arrow[r, "(\Phi^f)'"] & \mathscr{G}_0 \end{tikzcd} \quad \text{, then} \quad \begin{tikzcd} F^*(j_! G_0) \arrow[r] \arrow[d] & j_! G_0 \arrow[d] \\ F^*\bar{\mathscr{G}}_0 \arrow[r, "(\Phi^f)'"] & \bar{\mathscr{G}}_0 \end{tikzcd} ,$$

then, by extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$ and passage to cohomology

$$\begin{tikzcd} F^*(j_! G) \arrow[r, "F_*"] \arrow[d] & j_! G \arrow[d] \\ F^*\bar{\mathscr{G}} \arrow[r] & \bar{\mathscr{G}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{H}^\bullet(\bar{S}, j_! G) \arrow[r, "F^*"] \arrow[d] & \mathrm{H}^\bullet(\bar{S}, j_! G) \arrow[d] \\ \mathrm{H}^\bullet(\bar{S}, \bar{\mathscr{G}}) \arrow[r] & \mathrm{H}^\bullet(\bar{S}, \bar{\mathscr{G}}) \end{tikzcd} ,$$

where the second line is obtained by extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$ of the $\mathbb{F}_q$-linear endomorphism Φ^f of $\mathrm{H}^\bullet(\bar{S}_0, \bar{G}_0)$.

Artin-Scheier theory provides a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(\bar{S}, j_! G) \longrightarrow \mathrm{H}^i(\bar{S}, \bar{\mathscr{G}}) \xrightarrow{\Phi - 1} \mathrm{H}^i(\bar{S}, \bar{\mathscr{G}}) \longrightarrow \cdots$$

where Φ is the p-linear extension to $\mathrm{H}^\bullet(\bar{S}, \bar{\mathscr{G}}) = \mathrm{H}^\bullet(\bar{S}_0, \bar{\mathscr{G}}_0) \otimes_{\mathbb{F}_q} \mathbb{F}$ of the p-linear endomorphism Φ of $\mathrm{H}^\bullet(\bar{S}_0, \bar{\mathscr{G}}_0)$.

3.7

If $\bar{S}_0$ is proper over $\mathbb{F}_q$, the $\mathrm{H}^i(\bar{S}, \bar{\mathscr{G}})$ are of finite dimension and $\Phi - 1$ is surjective (cf. 3.6b). We therefore have

$$\mathrm{H}^i_c(S, G) = \ker\left(\Phi - 1 : \mathrm{H}^i(\bar{S}, \bar{G}) \to \mathrm{H}^i(\bar{S}, \bar{\mathscr{G}})\right),$$

and F^* is induced by the $\mathbb{F}$-linear extension to $\mathrm{H}^i(\bar{S}, \bar{\mathscr{G}}) = \mathrm{H}^i(\bar{S}_0, \bar{\mathscr{G}}_0) \otimes_{\mathbb{F}_q} \mathbb{F}$ of the $\mathbb{F}_q$-linear endomorphism Φ^f of $\mathrm{H}^i(\bar{S}_0 m \bar{\mathscr{G}}_0)$.

By applying the lemma below, we deduce the identity

$$\operatorname{tr}\left(F^*, \mathrm{H}^i_c(S, G)\right) = \operatorname{tr}\left(\Phi^f, \mathrm{H}^i(\bar{S}_0, \bar{\mathscr{G}}_0)\right). \tag{3.7.1}$$

Lemme 3.8. *Let Φ be a p-linear endomorphism of a vector space V over $\mathbb{F}_q$: $\phi(\lambda x) = \lambda^p \phi(x)$. We set $F = \Phi^f$ (F is linear) and we still denote by Φ and F the respective p-linear extensions of Φ and F to $V \otimes_{\mathbb{F}_q} \mathbb{F}$. Then, F stabilizes* $\ker(\Phi - 1) \subset V \otimes_{\mathbb{F}_q} \mathbb{F}$ *and*

$$\operatorname{tr}\left(F, \ker(\Phi - 1)\right) = \operatorname{tr}(F, V) = \operatorname{tr}\left(F, V \otimes_{\mathbb{F}_q} \mathbb{F}\right).$$

On $V \otimes_{\mathbb{F}_q} \mathbb{F}$, we have $F\Phi = \Phi F$: the p-linear endomorphism $F\Phi - \Phi F$ vanishes on V, hence everywhere. We write $V = V' + V''$, with F invertible on V' and nilpotent on V''. The theory of p-linear endomorphisms assures that

$$\ker(\Phi - 1) \otimes_{\mathbb{F}_q} \mathbb{F} \xrightarrow{\sim} V' \otimes_{\mathbb{F}_q} \mathbb{F}.$$

Since $\operatorname{tr}(F, V'') = 0$, we therefore have

$$\begin{aligned} \operatorname{tr}\left(F, \ker(\Phi - 1)\right) &= \operatorname{tr}\left(F, \ker(\Phi - 1) \otimes_{\mathbb{F}_q} \mathbb{F}\right) = \operatorname{tr}\left(F, V' \otimes_{\mathbb{F}_q} \mathbb{F}\right) \\ &= \operatorname{tr}\left(F, V \otimes_{\mathbb{F}_q} \mathbb{F}\right) = \operatorname{tr}(F, V). \end{aligned}$$

4 Formulas for the traces modulo p

Théorème 4.1. *Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$, A a Noetherian commutative ring of characteristic p, and $\mathscr{K}_0 \in \operatorname{Ob} \mathsf{D}_{\mathrm{ctf}}(X_0, A)$. We have*

$$\sum_{x \in X^F} \operatorname{tr}\left(F^*_x, \mathscr{K}_x\right) = \operatorname{tr}\left(F^*, \mathsf{R}\Gamma_c(X, \mathscr{K})\right). \tag{4.1.1}$$

4.2 Reduction to the case where A is a finite field

A standard limit argument reduces us to the case where A is of finite type over $\mathbb{Z}/p$. For each maximal ideal $\mathfrak{m}$ of A, the image of each member of (4.1.1) in $A/\mathfrak{m}$ is given by the same formula, with $\mathscr{K}_0$ replaced by $\mathscr{K}_0 \overset{\mathsf{L}}{\otimes}_A A/\mathfrak{m} \in \operatorname{Ob} \mathsf{D}_{\mathrm{ctf}}(X_0, A/\mathfrak{m})$ (for the second member, cf. Rapport 4.12). The $A/\mathfrak{m}$ being finite, and the intersection of the maximal ideals of A being reduced to 0, we obtain the reduction announced.

4.3 Reducing to the case where $A = \mathbb{Z}/p$

Suppose that A is a finite field, and let $T'_A(\mathscr{K}_0)$ and $T''_A(\mathscr{K}_0)$ be the two members of (4.1.1). If, by restriction of scalars, we regard $\mathscr{K}_0$ as a complex of sheaves of $\mathbb{Z}/p$-modules, we have

$$T'_{\mathbb{Z}/p}(\mathscr{K}_0) = \mathrm{tr}_{A/(\mathbb{Z}/p)}\, T'_A(\mathscr{K}_0)$$

and similarly for T''. If (4.1.1) holds for $A = \mathbb{Z}/p$, we therefore have

$$\mathrm{tr}_{A/(\mathbb{Z}/p)}\, T'_A(\mathscr{K}_0) = \mathrm{tr}_{A/(\mathbb{Z}/p)}\, T''_A(\mathscr{K}_0). \tag{4.3.1}$$

I claim that, for every $\lambda \in A$, we even have

$$\mathrm{tr}_{A/(\mathbb{Z}/p)}\left(\lambda \cdot T'_A(\mathscr{K}_0)\right) = \mathrm{tr}_{A/(\mathbb{Z}/p)}\left(\lambda \cdot T''_A(\mathscr{K}_0)\right), \tag{4.3.2}$$

and therefore $T'_A(\mathscr{K}_0) = T''_A(\mathscr{K}_0)$. This is clear for $\lambda = 0$. For λ invertible, let $A(\lambda)$ be the image of X under the functor from the category of A-modules to the category of $\mathbb{Z}/p$-modules which sends a free A-module of rank one to the $\mathbb{Z}/p$-module for which $F^* = \lambda$. The formula (4.3.2) is nothing but (4.3.1) for $A(\lambda) \otimes_A \mathscr{K}_0$.

Théorème 4.4. *Let $\bar{S}_0$ be a projective and smooth curve, $j : S_0 \hookrightarrow \bar{S}_0$ a dense open of $\bar{S}_0$, and G_0 a locally constant sheaf of finite rank $\mathbb{Z}/p$-vector spaces on S_0. We have*

$$\sum_{x \in S^F} \mathrm{tr}\left(F_x^*, G_x\right) = \sum_i (-1)^i\, \mathrm{tr}\left(F^*, \mathrm{H}^i_c(S, G)\right). \tag{4.4.1}$$

Proof. Apply 3.3 to $(S_0, \bar{S}_0, G_0)$. The coherent extension $\bar{\mathscr{G}_0}$ of $\mathscr{G}_0$ whose existence 3.3 guarantees is locally free because it is torsion-free (on $\bar{\mathscr{G}_0} \subset j_*\mathscr{G}_0$) and $\bar{S}_0$ is regular of dimension one. The q-linear endomorphism Φ^f of $\bar{\mathscr{G}_0}$ can be interpreted as a morphism

$$u : F^*\bar{\mathscr{G}_0} \to \bar{\mathscr{G}_0}.$$

It factors through $I\bar{G}$, for I defining the closed $\bar{S} \smallsetminus S$. Let $v : j_!G_0 \to \bar{\mathscr{G}_0}$ be the unique extension of the natural map $G_0 \to \mathscr{G}_0 = G_0 \otimes_{\mathbb{Z}/p} \mathscr{O}$. The diagram

$$\begin{tikzcd} F^* j_! G_0 \arrow[r, "F^*"] \arrow[d, "F^* v"] & j_! G_0 \arrow[d, "v"] \\ F^* \bar{\mathscr{G}_0} \arrow[r, "u"] & \bar{\mathscr{G}_0} \end{tikzcd} \tag{4.4.2}$$

Finally, note uF^* the composed map

$$\mathrm{H}^\bullet(\bar{S}, \bar{\mathscr{G}}) \xrightarrow{F^*} \mathrm{H}^\bullet(\bar{S}, F^*\bar{\mathscr{G}}) \xrightarrow{u} \mathrm{H}^\bullet(\bar{S}, \bar{\mathscr{G}}),$$

by (3.7.1)

$$\mathrm{tr}\left(F^*, \mathrm{H}^i_c(S, G)\right) = \mathrm{tr}\left(uF^*, \mathrm{H}^i(\bar{S}, \bar{\mathscr{G}})\right). \tag{4.4.3}$$

For each closed point $i_x : x \hookrightarrow \bar{S}$ of $\bar{S}$, let $\bar{\mathscr{G}_x} = i_x^*\bar{\mathscr{G}}$. The Lefschetz trace formula in coherent cohomology ("Woodshole fixed point formula"), applied to F and u, says that

$$\sum_i (-1)^i\, \mathrm{tr}\left(u^*F^*, \mathrm{H}^i(\bar{S}, \bar{\mathscr{G}})\right) = \sum_{x \in \bar{S}^F} \frac{\mathrm{tr}(u_x, \mathscr{G}_x)}{\det(1 - dF_x)}. \tag{4.4.4}$$

On the right-hand side

a) since $dF = 0$, the denominators are 1;

b) for $x \in \bar{S}^F \setminus S^F$, the endomorphism u_x of $\bar{\mathscr{G}_x}$ is zero, so also $\mathrm{tr}(u_x, \bar{\mathscr{G}_x})$;

c) for $x \in S^F$, (4.4.4) provides a contribution

$$\mathrm{tr}(u_x, \bar{\mathscr{G}_x}) = \mathrm{tr}(F_x^*, G_x).$$

The identity (4.4.1) therefore results from (4.4.3) and (4.4.4).

□

4.5 A counterexample

An example is shown that in 4.1 it is not possible to omit the assumption that A is reduced.

Let Y_0 be the elliptic curve over $\mathbb{F}_2$ for which the Frobenius eigenvalues are $\pm\sqrt{-2}$. It has 3 rational points, and is supersingular. The involution $\sigma : x \mapsto -x$ therefore has only one fixed point, the point 0. The quotient of Y_0 by σ is a projective line; that of $Y_0 \setminus 0$ by σ is therefore an affine line $\mathbb{F}_0^1$, and $Y_0 \setminus 0$ is a double covering of $\mathbb{F}_0^1$ that is not ramified:

$$\pi : Y_0 \setminus 0 \to \mathbb{A}_0^1 \qquad , \pi = \pi\sigma.$$

The sheaf $\pi_*\mathbb{Z}/2$ is a free module of rank one over the algebra A of the group of two elements $\{1, \sigma\}$ over $\mathbb{Z}/2$. This algebra is isomorphic to the algebra of numbers over $\mathbb{Z}/2$.

The sheaf of A-modules $\pi_*\mathbb{Z}/2$ is our example:

a) We have $\mathrm{H}_c^\bullet(\mathbb{A}^1, \pi_*\mathbb{Z}/2) = \mathrm{H}_c^\bullet(Y \setminus 0, \mathbb{Z}/2)$. Since Y_0 is supersingular, we have $\mathrm{H}^i(Y, \mathbb{Z}/2) = 0$ for $i > 0$, and the long exact sequence

$$\cdots \to \mathrm{H}_c^i(Y \setminus 0, \mathbb{Z}/2) \to \mathrm{H}^i(Y, \mathbb{Z}/2) \to \mathrm{H}^i(\{0\}, \mathbb{Z}/2) \to \cdots$$

shows that $\mathrm{H}_c^\bullet(Y \setminus 0, \mathbb{Z}/2) = 0$. We therefore $\mathrm{H}_c^\bullet(\mathbb{A}^1, \pi_*\mathbb{Z}/2) = 0$ and

$$\mathrm{tr}_A\left(F^*, \mathsf{R}\Gamma_c(\mathbb{A}^1, \pi_*\mathbb{Z}/2)\right) = 0.$$

b) Since $Y_0 \setminus 0$ has two rational points, at one of the rational points of $\mathbb{A}_1^0$, the Frobenius substitution is the identity, and at the other, it is σ. We therefore

$$\sum_{x \in \mathbb{A}^{1F}} \mathrm{tr}\left(F^*, \pi_*\mathbb{Z}/2\right) = 1 + \sigma,$$

and $1 + \sigma \neq 0$.

The equation of the π coating is $y^2 - y = x^3$.

Chapter 4

The cohomology class associated to a cycle

This talk is inspired by Grothendieck's notes, which formed state 0 of [20, IV]. There one defines the cohomology class of a cycle X in a separated smooth scheme of finite type over a field and proves that the intersection corresponds to the cup product.

Chapter 1 contains some general sorites. In Chapter 2, we define the class of a cycle, in several more general situations than the one called above. The main compatibility not considered is that between the direct image of a cycle and the trace map in cohomology. In Chapter 3, we deduce from this formalism the Lefschetz trace formula for an endomorphism with isolated fixed points of a proper smooth scheme over k algebraically closed – and for an endomorphism of Frobenius of a curve.

We make the following conditions:

1) "scheme" means noetherian separated scheme (this is largely a convenience assumption).

2) In Chapter 2, we fix an integer n, and n is invertible on all schemes considered.

3) In Chapter 3, we fix a prime number ℓ, and ℓ is invertible on all schemes considered. The cohomology used is always ℓ-adic cohomology:

$$\mathrm{H}^\bullet(X) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{H}^\bullet(X, \mathbb{Z}/\ell^n).$$

Cohomology classes of cycles, trace maps, ... are defined by passing to the limit from the case of finite coefficients $\mathbb{Z}/\ell^n$ (cf. [20, VI]).

1 Cohomology with support and cup-products

This paragraph contains general topology reminders, which the reader is invited to consult only as needed.

by A. Grothendieck, written by P. Deligne

1.1 H^1 and torsors

1.1.1

Let $\mathscr{F}$ be an abelian sheaf on a site X. We know that $\mathrm{H}^1(X, \mathscr{F})$ classifies the $\mathscr{F}$-torsors on X. We will normalize (=choose the sign of) the isomorphism (=set of isomorphism classes of F-torsors) $\to \mathrm{H}^1(X, \mathscr{F})$ such that for any exact sequence $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ and any $h \in \mathrm{H}^0(X, \mathscr{H})$, the $\mathscr{F}$-torsor $\beta^{-1}(h) \subset \mathscr{G}$, on which F acts by $(f, x) \mapsto \alpha(f) + x$, is of class ∂h.

1.1.2

Let $\mathscr{P}$ be an $\mathscr{F}$-torsor. If (U_i) is an open covering of X, and p_i is a section of $\mathscr{P}$ over U_i, we associate to $\mathscr{P}$ the Čech cocycle

$$p_{ij} = p_j - p_i \qquad (p_{ij} \in \mathrm{H}^0(U_i \times U_j, \mathscr{F})).$$

If, according to the usual rule, we define the application $\check{\mathrm{H}}^\bullet(X, \mathscr{F}) \to \mathrm{H}^\bullet(X, \mathscr{F})$ so that it is a morphism of δ-functors, the image of $(p_{ij}) \in \check{\mathrm{H}}^1(X, \mathscr{F})$ in $\mathrm{H}^1(X, \mathscr{F})$ is the class of $\mathscr{P}$, such that normalized by 1.1.1.

1.1.3

The definition of $\mathrm{H}^\bullet(X, \mathscr{F})$ is the following: for $\mathscr{F}^\bullet$ an acyclic resolution of $\mathscr{F}$, $\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i\,\Gamma(X, \mathscr{F}^\bullet)$. The structure of δ-functor is obtained by associating to a short exact sequence of sheaves a short exact sequence of resolutions which remains exact after application of the functor Γ. If $\mathscr{F}^\bullet$ is a resolution of $\mathscr{F}$, the connection homomorphism ∂ associated to

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \xrightarrow{d} \ker(d) \longrightarrow 0$$

induces the opposite of the isomorphism of definition $\mathrm{H}^1(X, \mathscr{F}) = \Gamma\left(X, \ker(d)\right)/d\Gamma(X, \mathscr{F}^0)$.

1.1.4

Let U be an open part of X (a sub-sheaf of the final sheaf) and let D be the "complementary closed." We know that $\mathrm{H}^1_D(X, \mathscr{F})$ classifies the $\mathscr{F}$-torsors on X, trivialized on U. For any short exact sequence $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$, and any section with support in D $h \in \mathrm{H}^0_D(X, \mathscr{H})$, the torsor $\beta^{-1}(h)$, trivialized on U by the section 0, has class ∂h.

The long exact sequence of cohomology with support

$$\cdots \xrightarrow{\partial} \mathrm{H}^i_D(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(U, \mathscr{F}) \xrightarrow{\partial} \cdots$$

is defined from the sequence of functors

$$0 \longrightarrow \Gamma_D \longrightarrow \Gamma \longrightarrow \Gamma(U, -) \longrightarrow 0$$

(exact on injective sheaves). For any section $f \in \mathrm{H}^0(U, \mathscr{F})$, $\partial f \in \mathrm{H}^1_D(X, \mathscr{F})$ is the class of the trivial torsor $\mathscr{F}$, trivialized on U by the section f.

1.1.5

Let $j : U \hookrightarrow X$. If $\mathscr{F}$ injects into $j_* j^* \mathscr{F}$, the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow j_* j^* \mathscr{F} \longrightarrow j_* j^* \mathscr{F}/\mathscr{F} \longrightarrow 0$$

provides $\partial : \mathrm{H}^0(j_* j^* \mathscr{F}/\mathscr{F}) = \mathrm{H}^0_D(j_* j^* \mathscr{F}/\mathscr{F}) \to \mathrm{H}^1_D(X, \mathscr{F})$. The composed map

$$\mathrm{H}^0(U, \mathscr{F}) = \mathrm{H}^0(X, j_* j^* \mathscr{F}) \longrightarrow \mathrm{H}^0(X, j_* j^* \mathscr{F}/\mathscr{F}) \xrightarrow{\partial} \mathrm{H}^1_D(X, \mathscr{F})$$

is the opposite of the map considered in 1.1.4.

1.1.6

We recall that if $\mathscr{L}$ is an invertible sheaf on a scheme X, the corresponding $\mathbb{G}_m$-torsor is the sheaf $\mathrm{Isom}(\mathscr{O}, \mathscr{L})$, on which $\mathbb{G}_m$ acts by $(\lambda, f) \mapsto f \circ (\lambda \cdot) = \lambda f$. We also recall that if D is a Cartier divisor on X, and $j : U \hookrightarrow X$ the inclusion of complementary open, the invertible sheaf $\mathscr{O}(D)$ is the subsheaf of $j_* \mathscr{O}_U$ formed by local sections s such that sf is in $\mathscr{O}_X$, for f a local equation of D.

1.2 Cup-products

In this number, we develop some remarks on cup-products in cohomology with support, which we will use in [Duality], to relate the Poincaré duality of curves and the self-duality of the Jacobian.

1.2.1

Let X be a site, Y a closed part of X, and $\mathscr{F}, \mathscr{G}$ two abelian sheaves on X. For example: X a scheme, Y a closed sub-scheme and $\mathscr{F}, \mathscr{G}$ sheaves on X_{et}. Let us define a product

$$\Gamma_Y(X, \mathscr{F}) \otimes \Gamma(Y, \mathscr{G}) \to \Gamma_Y(X, \mathscr{F} \otimes \mathscr{G}). \tag{1.2.1.1}$$

The product of a section s, with support in Y, of $\mathscr{F}$ by a section t of $\mathscr{G}$ on Y is obtained as follows: locally on X, t is the restriction to Y of a section t' of $\mathscr{G}$ on X, and we form the product $s \otimes t'$. It has support in Y, and does not depend on the choice of t', which justifies and allows us to globalize the definition.

Let $i : Y \hookrightarrow X$ be the inclusion morphism. The local analogue of (1.2.1.1) is the product.

$$i^! \mathscr{F} \otimes i^* \mathscr{G} \to i^! (\mathscr{F} \otimes \mathscr{G}), \tag{1.2.1.2}$$

from which (1.2.1.1) is deduced by applying $\Gamma(Y, -)$.

1.2.2

Let's derive these arrows. Let $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{M}$ be in the derived category, and a bilinear map $\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{L} \to \mathscr{M}$. By deriving (1.2.1.1), we deduce

$$\mathsf{R}\Gamma_Y(X, \mathscr{K}) \overset{\mathsf{L}}{\otimes} \mathsf{R}\Gamma(Y, \mathscr{L}) \to \mathsf{R}\Gamma_Y(X, \mathscr{M}) \tag{1.2.2.1}$$

inducing

$$\mathrm{H}^i_Y(X,\mathscr{K}) \otimes \mathrm{H}^j(Y,\mathscr{L}) \to \mathrm{H}^{i+j}_Y(X,\mathscr{M}) \tag{1.2.2.2}$$

(the cup-product). The local arrow (1.2.1.2) provides

$$\mathsf{R}i^!\mathscr{K}\overset{\mathsf{L}}{\otimes} i^*\mathscr{L} \to \mathsf{R}i^!\mathscr{M}, \tag{1.2.2.3}$$

from which (1.2.2.1) is deduced by applying $\mathsf{R}\Gamma(Y,-)$.

1.2.3

Above, I have left out the conflicts that arise from the use of right derivations ($\mathsf{R}\Gamma$) and left derivations ($\overset{\mathsf{L}}{\otimes}$) in the same formula.

a) In the next section, the right derivations considered will all be of finite cohomological dimension. This allows us to work systematically in the derived categories D^-. To have complexes that are both flat and flasque, we use the canonical flasque resolutions as in [2, XVII].

b) For a more general theory, it is no longer possible to interpret the bilinear applications of $\mathscr{K}$ and $\mathscr{L}$ in $\mathscr{M}$ as morphisms of $\mathscr{K}\overset{\mathsf{L}}{\otimes}\mathscr{L}$ in $\mathscr{M}$. For example, $\mathsf{R}\Gamma_Y(X,\mathscr{K})\overset{\mathsf{L}}{\otimes}\mathsf{R}\Gamma(Y,\mathscr{L})$ is not defined if both factors are in D^+. One solution is to work in D^+, to define

$$\mathrm{Bil}(\mathscr{K},\mathscr{L};\mathscr{M}) = \varinjlim \hom(\mathscr{K}'\otimes\mathscr{L}',\mathscr{M}'),$$

where the limit is taken over quasi-isomorphisms $\mathscr{K}' \xrightarrow{\sim} \mathscr{K}$, $\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$, $\mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ ($\mathscr{K}'$, $\mathscr{L}'$, $\mathscr{M}'$ bounded below) and where hom is for "morphism of complexes, up to homotopy," and to use such bilinear applications systematically, without ever mentioning $\otimes$.

1.2.4 Second theme

Let U be an open set of X and $j : U \hookrightarrow X$ the inclusion morphism. For $\mathscr{K}$ in the derived category, on U, we have $\mathsf{R}\Gamma_!(U,\mathscr{K}) = \mathsf{R}\Gamma(X, j_!\mathscr{K})$. For $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{M}$ on U, and a bilinear map $\mathscr{K}\overset{\mathsf{L}}{\otimes}\mathscr{L} \to \mathscr{M}$, we want to define

$$\mathsf{R}\Gamma(U,\mathscr{K})\overset{\mathsf{L}}{\otimes}\mathsf{R}\Gamma_!(U,\mathscr{L}) \to \mathsf{R}\Gamma_!(U,\mathscr{M}). \tag{1.2.4.1}$$

At the level of sheaves, and of their global sections, such a product is deduced from the isomorphism $j_*\mathscr{F}\otimes j_!\mathscr{G} \xleftarrow{\sim} j_!(\mathscr{F}\otimes\mathscr{G})$, but one must take care of the fact that $\mathsf{R}\Gamma_!$ is not generally the derived functor of the functor $\Gamma(X, j_!-)$.

We start by defining

$$\mathsf{R}j_*\mathscr{K}\overset{\mathsf{L}}{\otimes} j_!\mathscr{L} \xleftarrow{\sim} j_!(\mathscr{K}\otimes\mathscr{L}) \longrightarrow j_!\mathscr{M}.$$

Applying $\mathsf{R}\Gamma(X,-)$, we find

$$\mathsf{R}\Gamma(U,\mathscr{K})\overset{\mathsf{L}}{\otimes}\mathsf{R}\Gamma_!(U,\mathscr{L}) = \mathsf{R}\Gamma(X,\mathsf{R}j_*\mathscr{K})\overset{\mathsf{L}}{\otimes}\mathsf{R}\Gamma(X,j_!\mathscr{L}) \longrightarrow \mathsf{R}\Gamma(X,j_!\mathscr{M}).$$

Once again, the conflict between left and right will be resolved in the next section by working in D^-.

1.2.5 Coda

Let $j : U \hookrightarrow X$ be an open part and $i : Y \hookrightarrow U$ be a closed part of U. Let $\bar{Y}$ be a closed subset of X such that $\bar{Y} \cap U = Y$, (for example, the complement of $U \smallsetminus Y$). Let $\mathscr{K}$ be a sheaf on U. We define $\mathsf{R}\Gamma_{Y!}(U, \mathscr{K}) = \mathsf{R}\Gamma_!(Y, \mathsf{R}i^!\mathscr{K})$ where $\mathsf{R}\Gamma_!$ is relative to the inclusion of Y in $\bar{Y}$. For any open set V of U, containing Y, $\mathsf{R}\Gamma_{Y!}(U, \mathscr{K})$ is mapped to $\mathsf{R}\Gamma_!(V, \mathscr{K})$: if we still denote by i the inclusion of $\bar{Y}$ in X, j the inclusion of Y in $\bar{Y}$ and k the inclusion of V in X, we have $\mathsf{R}i^!\mathscr{K} = j^*\mathsf{R}i^!k_!(k^*\mathscr{K})$, from which a morphism $j_!\mathsf{R}i^!\mathscr{K} \to \mathsf{R}i^!k_!(k^*\mathscr{K})$ is obtained. Applying $\mathsf{R}\Gamma(\bar{Y}, -)$ to this, we find $\mathsf{R}\Gamma_{Y!}(U, \mathscr{K}) \to \mathsf{R}\Gamma_{\bar{Y}}(k_!k^*\mathscr{K}) \to \mathsf{R}\Gamma(k_!k^*\mathscr{K}) = \mathsf{R}\Gamma_!(V, \mathscr{K})$.

For $\mathscr{K}, \mathscr{L}, \mathscr{M}$ on U, and a bilinear application $\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{L} \to \mathscr{M}$, we want to define

$$\mathrm{H}^n(U, \mathscr{K}) \otimes \mathrm{H}^m_!(Y, \mathscr{L}) \to \mathrm{H}^{n+m}_{Y!}(U, \mathscr{M}) \tag{1.2.5.1}$$

(the latter group itself being sent to $\mathrm{H}^{n+m}_!(V, \mathscr{M})$). In the derived category, it is a question of defining

$$\mathsf{R}\Gamma_Y(U, \mathscr{K}) \overset{\mathsf{L}}{\otimes} \mathsf{R}\Gamma_!(R, \mathscr{L}) \to \mathsf{R}\Gamma_{Y!}(U, \mathscr{M}). \tag{1.2.5.2}$$

We identify $\mathsf{R}\Gamma_Y$ to $\mathsf{R}\Gamma(Y, \mathsf{R}i^!\mathscr{K})$. The desired product is then of type (1.2.4.1) relative to the local product (1.2.2.3) on Y: $\mathsf{R}i^!\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{L} \to \mathsf{R}i^!\mathscr{M}$.

1.2.6

Above, we have unfolded the absolute sorite. We have a parallel relative sorite, with Γ replaced by f_* by f a morphism $X \to S$.

1.3 The Koszul rule

Let A be a commutative ring, and $(V_i)_{i \in I}$ a finite family of graded (or $\mathbb{Z}/2$-graded) A-modules. Recall the definition of the graded tensor product $\bigotimes_{i \in I} V_i$, in the sense of the Koszul rule (cf. [2, XVII 1.1]). For each total order a on I, we will define a module $V(a)$. We will also define a transitive system of isomorphisms $\varphi_{ab} : V(b) \xrightarrow{\sim} V(a)$ and the graded tensor product of the V_i will be the "common value" $\varprojlim V(a)$ of this system of modules. We take

a) $V(a) = \bigotimes_{i \in I} |V_i|$ (ordinary tensor product of the underlying ungraded modules of the V_i).

b) if the $x_i \in V_i$ are homogeneous, we take $\varphi_{ab}(\otimes x_i) = (-1)^N \otimes x_i$, where N is the sum of the $\deg(x_i)\deg(x_j)$ extended to pairs (i, j) such that $i <_a j$ and $i >_b j$.

Example 1

Let $I = \{1, 2\}$ and let a be the order in which $1 < 2$, b be the order in which $1 > 2$. Let $v_i \in V_i$, homogeneous, and denote $v_1 \otimes v_2$ (resp. $v_2 \otimes v_1$) the image in the graded tensor product of the product of v_i in $V(a)$ (resp. $V(b)$). We have

$$v_1 \otimes v_2 = (-1)^{\deg(v_1)\deg(v_2)} v_2 \otimes v_1$$

(Koszul rule).

Example 2

If the V_i are all of degree 1, we have, relating the module underlying the graded tensor product, and the ordinary tensor product of the modules $|V_i|$ underlying the V_i, a canonical isomorphism

$$\left|\bigotimes V_i\right| \simeq \bigotimes |V_i| \otimes_{\mathbb{Z}} \bigwedge^{|I|} \mathbb{Z}^I.$$

Example 3

For A a field, and X_i a finite family of spaces, the Künneth formula is written $\mathrm{H}^\bullet\left(\prod X_i, A\right) = \bigotimes \mathrm{H}^\bullet(X_i, A)$.

Let $V^\vee$ be the graded dual of V. The canonical application

$$V^\vee \otimes V = V \otimes V^\vee \to A \tag{1.3.1}$$

is $v' \otimes v \to v'(v)$.

We now suppose that A is a field, and we only consider finite-dimensional vector spaces. The canonical isomorphism

$$W \otimes V^\vee \to \hom(V, W) \tag{1.3.2}$$

is $w \otimes v' \mapsto (v \mapsto w \cdot v'(v))$. Via this isomorphism, the composition $\hom(Y, Z) \otimes \hom(X, Y) \to \hom(X, Z)$ is identified with the morphism induced by (1.3.1): $Z \otimes Y^\vee \otimes Y \otimes X^\vee \to Z \otimes X^\vee$.

The trace of $f : V \to V^\vee$ (null for f homogeneous of degree $\neq 0$) is the image of f by

$$\mathrm{tr} : \hom(V, V) \xleftarrow{\sim}_{(1.3.2)} V \otimes V^\vee = V^\vee \otimes V \to_{(1.3.1)} A. \tag{1.3.4}$$

It is easily checked that, for f of degree 0,

$$\mathrm{tr}(f, V) = \sum (-1)^i \,\mathrm{tr}(f, V^i). \tag{1.3.5}$$

If we express that the two morphisms composed of morphisms (1.3.1) $V^\vee \otimes V \otimes W^\vee \otimes W \to k$ commute, we find that, for $f : V \to W$ and $g : W \to V$ homogeneous, we have

$$\mathrm{tr}(fg) = (-1)^{\deg(f)\deg(g)} \,\mathrm{tr}(gf). \tag{1.3.6}$$

2 The cohomology class associated to a cycle

2.1 The class of a divisor

2.1.1

Let D be a Cartier divisor in a scheme X. Outside of D, the invertible sheaf $\mathscr{O}(D)$ is trivialized by the section 1. The class $\mathrm{cl}(D)$ of D, in $\mathrm{H}^1_D(X, \mathbb{G}_m)$, is the class of the $\mathbb{G}_m$-torsor trivialized over $X \setminus D$ corresponding (1.1.6 and 1.1.4).

Let $\partial : \mathrm{H}^i(X \setminus D, \mathbb{G}_m) \to \mathrm{H}^i_D(X, \mathbb{G}_m)$ be the morphism 1.1.4. If D has a global equation f, multiplication by f is an isomorphism of $\mathscr{O}(D)$, trivialized by 1 on $X \setminus D$, with $\mathscr{O}$, trivialized by f on $X \setminus D$. According to 1.1.4, we therefore have

$$\mathrm{cl}(D) = \partial f. \tag{2.1.1}$$

For any morphism $u : X' \to X$ such that u^*D is still a Cartier divisor (i.e., $u^{-1}D$ is disjoint from $\mathrm{Ass}(X')$), we have $\mathrm{cl}(u^*D) = u^* \mathrm{cl}(D)$. If we wanted such functoriality for any morphism u, we would have to consider not just Cartier divisors, but more generally invertible sheaves with a section.

Recall that n is now assumed to be invertible on the schemes under consideration. Let $\partial : \mathrm{H}^i_D(X, \mathbb{G}_m) \to \mathrm{H}^{i+1}_D(X, \boldsymbol{\mu}_n)$ be the coboundary for the Kummer exact sequence $0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$.

Définition 2.1.2. *The* classe $\mathrm{cl}_n(D)$ *of* D *in* $\mathrm{H}^2_D(X, \boldsymbol{\mu}_n)$ *is* $\partial\, \mathrm{cl}(D)$.

When there will be no risk of confusion, we will omit the mention of n.

2.1.3

The diagram

$$\begin{tikzcd} \mathrm{H}^0(X \smallsetminus D, \mathbb{G}_m) \arrow[r, "\partial"] \arrow[d, "\partial"] & \mathrm{H}^1_D(X, \mathbb{G}_m) \arrow[d, "\partial"] \\ \mathrm{H}^1(X \smallsetminus D, \boldsymbol{\mu}_n) \arrow[r, "\partial"] & \mathrm{H}^2_D(X, \boldsymbol{\mu}_n) \end{tikzcd}$$

is anticommutative. If D has a global equation f, $\mathrm{cl}_n(D)$ is therefore the opposite of the image by ∂ of the class in $\mathrm{H}^1(X \smallsetminus D, \boldsymbol{\mu}_n)$ of $\boldsymbol{\mu}_n$-torsor of n-th roots of f.

Proposition 2.1.4. *Let i be the inclusion of D in X. If D and X are regular, the cohomology sheaves with support $\mathsf{R}^p i^! \boldsymbol{\mu}_n$ are null for $p = 0, 2$, and $\mathsf{R}^2 i^! \boldsymbol{\mu}_n = \underline{\mathbb{Z}/n}$, generated by $\mathrm{cl}_n(D)$.*

It is enough to prove that, for X strictly local and D defined by a regular parameter, we have $\mathrm{H}^p_D(X, d\mu_n) = 0$ for $p = 0, 1$ and $\mathrm{H}^2(X, \boldsymbol{\mu}_n) = \mathbb{Z}/n$ generated by $\mathrm{cl}(D)$. Denoting by $\sim$ reduced cohomology, we have $\widetilde{\mathrm{H}}^{p-1}(X \smallsetminus D, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{H}^p_D(X, \boldsymbol{\mu}_n)$. The assertion for $p = 0, 1$ expresses that D does not disconnect X, and for $p = 2$ follows, via 2.1.3, from the Abhyankar lemma.

This is a partial analogue of the theorem relative (Etale Cohomology, 5.2.4). Grothendieck conjectured that the $\mathsf{R}^p i^! \boldsymbol{\mu}_n$ are null for $p \neq 2$, at least for X excellent (purity conjecture), but this is only known in characteristic 0 ([2, XIX]).

Théorème 2.1.5 (Fundamental Compatibility)**.** *Let X be a smooth curve over an algebraically closed field k, P a closed point of X and* tr *the composition $\mathrm{H}^2_P(X, \boldsymbol{\mu}_n) \to \mathrm{H}^2_c(X, \boldsymbol{\mu}_n) \xrightarrow{\mathrm{tr}} \mathbb{Z}/n$. We have*

$$\mathrm{tr}\, \mathrm{cl}(P) = 1.$$

Let $\bar{X}$ be the smooth projective curve completing X. The formula expresses that the invertible sheaf $\mathscr{O}(P)$ on $\bar{X}$ has degree 1.

2.2 Cohomological method

2.2.1

Let X be a (Noetherian) scheme. Recall that a subscheme Y of X is said to be of local complete intersection, of codimension c, if, locally (on Y), it is defined by a regular sequence of c equations in X. For X the spectrum of a local ring A with maximal ideal $\mathfrak{m}$ and Y with ideal $\mathfrak{a}$, this means that $\exp^i(A/\mathfrak{a}, A) = 0$ for $i < \dim(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) = c$, and any sequence of elements of $\mathfrak{a}$, with image in $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$ a basis of $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$, is a regular sequence of equations for Y.

2.2.2

Let $i : Y \hookrightarrow X$ be of local complete intersection, of codimension c. We propose to define a local fundamental class $\mathrm{cl}(Y)$ which is a global section of the cohomology sheaf with support $\mathsf{R}^{2c} i^! \mathbb{Z}/n(c)$ (recall that $\mathbb{Z}/n(c) = \boldsymbol{\mu}_n^{\otimes c}$).

Localement, Y is the intersection of a suite of c diviseurs D_i, and we define $\mathrm{cl}(Y)$ as the cup-product of the $\mathrm{cl}(D_i)$. Each $\mathrm{cl}(D_i)$ is supported in D_i, their product is supported in Y. That, locally, this product does not depend on the choice of the D_i, results from 2.2.3 below and the following invariance properties:

a) compatibility with localization;

b) independence of the order of the D_i (the $\mathrm{cl}(D_i)$ are of degree 2, even, so the cup-product is commutative);

c) the product only depends on the “flag” $D_1 \supset D_1 \cap D_2 \supset \cdots \supset Y$.

To prove c), we note the existence of a product (variant of 1.2.1)

$$\mathrm{H}^\bullet_{D_1}(X) \otimes \mathrm{H}^\bullet_{D_1 \cap D_2}(D_1) \otimes \cdots \otimes \mathrm{H}^\bullet_Y(D_1 \cap \cdots \cap D_{c-1}) \to \mathrm{H}^\bullet_Y(X);$$

the product of the $\mathrm{cl}(D_i)$ is still the product of the ($\mathrm{cl}(D_i)$ restricted to $D_1 \cap \cdots \cap D_{i-1}) =$ ($\mathrm{cl}(D_1 \cap \cdots \cap D_i)$ in $D_1 \cap \cdots \cap D_{i-1}$).

Lemme 2.2.3. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and $u = (u_1, \ldots, u_c)$, $v = (v_1, \ldots, v_c)$ two regular sequences generating the same ideal $\mathfrak{a}$. Then there exists a sequence w_i $(1 \leqslant i \leqslant N)$ of such sequences, linking them, such that w_{i+1} is deduced from w_i by a permutation, or by only changing the last element.*

Since we have permutations, it would amount to the same to allow ourselves to change only one element, rather than the last one. By 2.2.1, we then reduce to checking that, in the vector space $\mathfrak{a}/\mathfrak{m}\mathfrak{a}$, we can pass from one basis to another by a sequence of permutations and of changes of basis changing only one vector. The linear group is generated by the diagonal matrices, elementary matrices and permutation matrices.

2.2.4

The same methods allow us to define a local fundamental class for any $Y \subset X$ locally definable by c equations (where fewer equations suffice, the class is zero).

2.2.5

We can pass from such a local fundamental class, in $\mathrm{H}^0\left(Y, \mathsf{R}^{2c} i^! \mathbb{Z}/n(c)\right)$, to a global fundamental class, in $\mathrm{H}^{2c}_Y(X, \mathbb{Z}/n(c))$, when we have results of semi-purity:

Proposition 2.2.6. *(i) If $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p < 2c$, then*

$$\mathrm{H}^{2c}_Y(X, \mathbb{Z}/n(c)) \xrightarrow{\sim} \mathrm{H}^0\left(Y, \mathsf{R}^{2c} i^! \mathbb{Z}/n(c)\right).$$

(ii) Let Z be a closed subset of Y, with complement V in Y, and k the inclusion of Z in X. If $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p \leqslant 2c$, then

$$\mathrm{H}^{2c}_Y(X, \mathbb{Z}/n(c)) \hookrightarrow \mathrm{H}^{2c}_V(X, \mathbb{Z}/n(c)).$$

If $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p \leqslant 2c+1$, this arrow is an isomorphism.

The hypothesis $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ is equivalent to $\mathsf{R}^p i^! \mathbb{Z}/n(c) = 0$. This said, (i) is read on the spectral sequence $\mathrm{H}^p(Y, \mathsf{R}^q i^!) \Rightarrow \mathrm{H}^{p+q}_Y(X, -)$. If k_1 is the inclusion of Z in Y, we have $k = ik_1$, whence $\mathsf{R}k^! \mathbb{Z}/n = \mathsf{R}k_1^! \mathsf{R}i^!$, and the long exact sequence of cohomology for $Z \subset Y$ provides a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i\left(Z, \mathsf{R}k^! \mathbb{Z}/n(c)\right) \longrightarrow \mathrm{H}^i_Y\left(X, \mathbb{Z}/n(c)\right) \longrightarrow \mathrm{H}^i_V\left(X, \mathbb{Z}/n(c)\right) \longrightarrow \cdots$$

from which (ii) follows.

2.2.7 Amplification

La proposition 2.2.6 reste valable pour i un quelconque morphisme séparé de type fini, et $2c$ un entier (positif ou négatif) quelconque, pour autant qu'on y remplace $\mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right)$ par $\mathrm{H}^{2c}\left(Y, \mathsf{R}i^! \mathbb{Z}/n(c)\right)$, et de même pour H_V.

The following results of semi-purity come from [23, 1.8, 1.10, 1.15]. We will recall their proof.

Théorème 2.2.8. *Let S-schemes of finite type*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & S \end{tikzcd}$$

We assume X is smooth, purely of relative dimension N and that Y is fibre by fibre of dimension $\leqslant d$. Let $c = N - d$.

(i) We have $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p < 2c$; likewise for $\mathbb{Z}/n$ replaced by a sheaf $g^* \mathscr{F}$.

(ii) If, over a dense open U of S, Y is fibre by fibre of dimension $< d$, we have $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p \leqslant 2c$. If in addition the complement Z of U does not locally disconnect S, we have $\mathsf{R}^p i^! \mathbb{Z}/n = 0$ for $p \leqslant 2c + 1$.

Since $\mathsf{R}g^! \mathscr{F} = g^* \mathscr{F}(N)[2N]$, the transitivity formula $\mathsf{R}i^! \mathsf{R}g^! = \mathsf{R}f^!$ shows that $\mathsf{R}^{2c+q} i^!(g^* \mathscr{F}) = 0 \Leftrightarrow \mathsf{R}^{-2d+q} f^!(\mathscr{F})$. This reduces us to studying f, i.e. to assuming that $X = S$. The assertion (i) is then [2, XVIII 3.17].

Let Y' be the inverse image of Z in Y:

$$\begin{tikzcd} Y' \arrow[r, "v"] \arrow[d, "f"] & Y \arrow[d, "f"] \\ Z \arrow[r, "u"] & S \end{tikzcd}$$

According to (i), the $\mathsf{R}^p f^! \mathbb{Z}/n$ are supported in Y' for $p \leqslant -2d + 1$; and the spectral sequence $\mathsf{R}^a v^! \mathsf{R}^b f^! \Rightarrow \mathsf{R}^{a+b}(fv)^!$ shows that they coincide with the $\mathsf{R}^p(fv)^! \mathbb{Z}/n = \mathsf{R}^p(uf)^! \mathbb{Z}/n$. Applying (i) to Y'/Z and to the spectral sequence $\mathsf{R}^a f^! \mathsf{R}^b u^! \Rightarrow \mathsf{R}^{a+b}(uf)^!$, we find that (ii) follows from the vanishing of $\mathsf{R}^b u^! \mathbb{Z}/n$ for $b = 0$, or $b = 0$ and 1 according to the case.

2.2.9

Grothendieck conjecture the following absolute analogue of 2.2.8 (semi-purity conjecture - a consequence of the purity conjecture): for Y of codimension $\geqslant c$ in X regular, we have $\mathrm{H}^i_Y(X, \mathbb{Z}/n) = 0$ for $i < 2c$, at least if X is excellent.

2.2.10

We are now ready to define the class of a cycle Y of codimension c in a smooth k-variety X. We write $Y = \sum d_i Y_i$, where the Y_i are irreducible and reduced. An open U_i of Y_i, of complementary codimension $> c$, is then locally complete intersection in X. This allows us to define the local fundamental class of U_i. According to 2.2.6 and 2.2.8, this comes from a unique fundamental class $\mathrm{cl}(Y_i) \in \mathrm{H}^{2c}_{Y_i}(X, \mathbb{Z}/n(c))$, and we set

$$\mathrm{cl}(Y) = \sum d_i \,\mathrm{cl}(Y_i) \in \mathrm{H}^{2c}_{|Y|}(X, \mathbb{Z}/n(c)) .$$

2.2.11

In complex analytic geometry, a construction of Baum, Fulton and Mac Pherson [4] allows one to define the cohomology class of a local complete intersection $Y \subset X$ without any restrictions. Suppose Y is purely of codimension c, and let $\mathscr{N}$ be the vector bundle over Y which is the normal bundle of Y in X. Its local sections are those of $(\mathscr{I}/\mathscr{I}^2)^\vee$, where $\mathscr{I}$ is the sheaf of ideals of Y. Recall that the sheaf of smooth functions on X is defined locally, in terms of local embeddings of X in $\mathbb{C}^n$, as the restriction to X of the quotient of the sheaf of smooth functions on $\mathbb{C}^n$ by the ideal generated by the real and imaginary parts of the equations which define X. The bundle $\mathscr{N}$ extends to a complex vector bundle $C^\infty N$ over a neighborhood U of Y in X, and for U small enough, there exist sections f of N, with zeros at Y, and such that, on Y, $df : \mathscr{N} \to N$ is the identity. Two choices of N and f are homotopic on Y when Y is small enough.

The cohomology class $\mathrm{cl}(U)$ of the zero section 0 of N (denoted $z : U \to N$) is defined: U is a complete local intersection in N, and $\mathsf{R}^i z^! \mathbb{Z} = 0$ for $i \neq 2c$, $\mathsf{R}^{2c} z^! \mathbb{Z} = \mathbb{Z}$. We have $f^* : \mathrm{H}^\bullet_U(N, \mathbb{Z}) \to \mathrm{H}^\bullet_Y(X, \mathbb{Z})$ and we set

$$\mathrm{cl}(Y) = f^* \,\mathrm{cl}(U).$$

The same construction works whenever we have on an analytic subspace Y of X the following normal structure: a locally free sheaf $\mathscr{C}$ of rank c on Y_{red}, and an epimorphism $\mathscr{C} \to \mathscr{I}/\mathscr{I} \cdot \mathscr{I}_{\mathrm{red}}$.

2.3 Homological method

2.3.1

Let $f : Y \to X$ be a finite type flat morphism, with fibers of dimension $\leqslant d$. In [2, XVIII 2.9], we defined a trace morphism

$$\mathrm{tr}_f : \mathsf{R}^{2d} f_! \mathbb{Z}/n(d) \to \mathbb{Z}/n. \tag{2.3.1.1}$$

We have $\mathsf{R}^i f_! \mathbb{Z}/n(d) = 0$ for $i > 2d$, and $\mathsf{R}^i f^! \mathbb{Z}/n = 0$ for $i < -2d$. This, and the adjunction between $\mathsf{R} f_!$ and $\mathsf{R} f^!$, provide isomorphisms

$$\begin{aligned} \hom\left(\mathsf{R}^{2d} f_! \mathbb{Z}/n(d), \mathbb{Z}/n\right) &= \hom\left(\mathsf{R} f_! \mathbb{Z}/n(d), \mathbb{Z}/n[-2d]\right) \\ &= \hom\left(\mathbb{Z}/n, \mathsf{R} f^! \mathbb{Z}/n(-d)[-2d]\right) \\ &= \mathrm{H}^0\left(Y, \mathsf{R}^{-2d} f^! \mathbb{Z}/n\right). \end{aligned} \tag{2.1.3.2}$$

At least on S a point, the image of tr_f in the last two groups deserves the name of fundamental class of Y (in homology).

We still denote by tr_f the image of tr_f in the second group, and the morphisms that are deduced from it by functoriality. For example, for Y/S proper, the morphism

$$\mathrm{H}^i\left(Y, \mathbb{Z}/n(d)\right) = \mathrm{H}^i\left(S, \mathsf{R}f_!\mathbb{Z}/n(d)\right) \longrightarrow \mathrm{H}^{i-2d}(S, \mathbb{Z}/n). \tag{2.3.1.3}$$

Suppose Y is contained in X smooth over S, purely of relative dimension N, and let $c = N - d$.

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \arrow[d] \\ & S \end{tikzcd}$$

We have $\mathsf{R}f^! = \mathsf{R}i^!\mathsf{R}g^!$, and $\mathsf{R}g^!\mathbb{Z}/n = \mathbb{Z}/n(-N)[-2N]$. The fundamental class of Y is identified with an element of $\mathrm{H}^0\left(Y, \mathsf{R}i^!\mathbb{Z}/n(c)[2c]\right) = \mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right)$, the class $\mathrm{cl}(Y)$ of Y in X. We will see later that it only depends on $Y \subset X$, not on the projection of X on S, and that for Y of complete local intersection, it induces the local class of the previous number.

We explain the isomorphism

$$\mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right) \xrightarrow{\sim} \hom\left(\mathsf{R}^{2d}f_!\mathbb{Z}/n(d), \mathbb{Z}/n\right)$$

which transforms $\mathrm{cl}(Y)$ into tr_f: via the isomorphisms

$$\begin{aligned} \mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right) &= \mathrm{H}^{2c}_Y\left(\mathsf{R}i^!\mathbb{Z}/n(c)\right) \\ &= \hom_Y\left(\mathbb{Z}/n(d)[2d], \mathsf{R}i^!\mathbb{Z}/n(N)[2N]\right) \\ &= \hom_X\left(\mathbb{Z}/n(d)[2d]_Y, \mathbb{Z}/n(N)[2n]\right), \end{aligned}$$

it is the upper line of

$$\begin{tikzcd} \mathrm{H}^{2c}_Y \arrow[r] \arrow[dr, "(3)"'] & \hom\left(\mathsf{R}f_!\mathbb{Z}/n(d)[2d], \mathsf{R}f_!\mathsf{R}i^!\mathbb{Z}/n(N)[2n]\right) \arrow[r, "(1)"] \arrow[d, "\mathrm{tr}_i"] \arrow[dr, phantom, "(2)"] & \hom\left(\mathsf{R}f_!\mathbb{Z}/n(d)[2d], \mathbb{Z}/n\right) \arrow[d, equal] \\ & \hom\left(\mathsf{R}g_!\mathbb{Z}/n(d)[2d]_Y, \mathsf{R}g_!\mathbb{Z}/n(N)[2N]\right) \arrow[r, "\mathrm{tr}_g"] & \hom\left(\mathsf{R}g_!\mathbb{Z}/n(d)[2d]_Y, \mathbb{Z}/n\right), \end{tikzcd}$$

where (1) is the adjunction arrow $\mathsf{R}f_!\mathsf{R}f^! \to \mathrm{id}$. The commutativity (2) expresses that the isomorphism $\mathsf{R}f^! = \mathsf{R}i^!\mathsf{R}g^!$ is defined by adjunction. Finally, the arrow deduced from (3)

$$\mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right) \otimes \mathsf{R}^{2d}g_!\mathbb{Z}/n(d) \to \mathsf{R}^{2N}g_!\mathbb{Z}/n(N)$$

is interpreted as a cup-product (cf. 1.2).

Définition 2.3.2. *The class* $\mathrm{cl}(Y) \in \mathrm{H}^{2c}_Y\left(X, \mathbb{Z}/n(c)\right)$ *has the characteristic property that, for every local section* u *of* $\mathsf{R}^{2d}f_!\mathbb{Z}/n(d)$*, we have*

$$\mathrm{tr}_f(u) = \mathrm{tr}_g\left(\mathrm{cl}(Y) \smile u\right).$$

Because $\mathrm{cl}(Y)$ already has an image tr_f in $\hom\left(\mathsf{R}f_!\mathbb{Z}/n(d)[2d], \mathbb{Z}/n\right)$, the formula 2.3.2 holds for the arrows deduced from tr_f by functoriality. For example, for X over S proper and $u \in \mathrm{H}^\bullet(Y, \mathbb{Z}/n)$, the formula holds in $\mathrm{H}^\bullet\left(S, \mathbb{Z}/n(-d)\right)$. For $v \in \mathrm{H}^\bullet(X, \mathbb{Z}/n)$, it gives

$$\mathrm{tr}_f(i^*v) = \mathrm{tr}_g\left(\mathrm{cl}(Y) \smile v\right),$$

where the cup-product can be calculated in $\mathrm{H}^\bullet(X, \mathbb{Z}/n)$.

2.3.3

We will define the trace morphism, and thus the class $\mathrm{cl}(Y)$, under more general hypotheses. So let us consider a diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & S \end{tikzcd}$$

with g smooth, purely of relative dimension N, and Y closed in X, fiber by fiber of dimension $\leqslant d$. We also endow Y with a "weight" $\mathscr{K}$ of the following type: $\mathscr{K} \in \mathsf{D}^b_{\mathrm{parf},X}$ is a bounded complex of $\mathscr{O}$-modules on X, of Tor-dimension finite over S (or over X, which amounts to the same), and whose cohomology sheaves have coherent support in Y. We propose to define a morphism $\mathrm{tr}_{f,\mathscr{K}} : \mathsf{R}^{2d} f_! \mathbb{Z}/n(d) \to \mathbb{Z}/n$. For Y flat over S and $\mathscr{K} = \mathscr{O}_Y$, this will be the previous trace morphism. In general, it only depends on the lengths of $\mathscr{K}$ at the generic points y of the irreducible components of $Y(\mathrm{lg}_y(\mathscr{K}) = \sum(-1)^i \lg \mathrm{H}^i(\mathscr{K})_y)$. Finally, we will denote by $\mathrm{cl}(Y, \mathscr{K})$ the class with support in Y such that

$$\mathrm{tr}_g\left(\mathrm{cl}(Y, \mathscr{K}) \smile u\right) = \mathrm{tr}_{f,\mathscr{K}}(u). \tag{2.3.3.1}$$

2.3.4

The construction of $\mathrm{tr}_{f,\mathscr{K}}$ is parallel to that of [2, XVIII.2]; we will only indicate the main steps.

A $d = 0$ (f quasi-finite). Let x be a geometric point of Y. s is its image in S, $\mathscr{K}_{(x)}$ is the inverse image of $\mathscr{K}$ on the strict localization of X at x, $\mathscr{K}_{(x)s} = \mathscr{K}_{(x)} \overset{\mathsf{L}}{\otimes}_{\mathscr{O}^h_{S,s}} k(s)$ is its image on the geometric fiber at s, and $n(x) = \sum(-1)^i \dim_{k(s)} \mathrm{H}^i(\mathscr{K}_{(x)s})$. The function $x \mapsto n(x)$ is a weight of f, and we take the corresponding trace morphism [2, XVII.6.2.5]. The weight $n(x)$, and $\mathrm{tr}_{f,\mathscr{K}}$ are compatible with any base change.

B General case. If $u : Z \to \mathbb{A}^d_S$ is such that ui is quasi-finite, we set $\mathrm{tr}_{f,\mathscr{K}} = \mathrm{tr}_{\mathbb{A}^d_S} \circ \mathrm{tr}_{ui,\mathscr{K}}$. This trace morphism is clearly compatible with any base change S'/S. To prove that it does not depend on u, we can therefore assume that S is the spectrum of an algebraically closed field k. In this case, let Y_i be the irreducible components of Y, and lg_i the length of $\mathscr{K}$ at the generic point of Y_i. If α_i is the inclusion of $(Y_i)_{\mathrm{red}}$ in Y, we have

$$\mathrm{tr}_{f,\mathscr{K}} = \sum \mathrm{lg}_i \, \mathrm{tr}_{Y_{i,\mathrm{red}/S}} \, \alpha_i^* \tag{2.3.4}$$

(this formula follows from the similar and easy formula for tr_{ui}).

This defines $\mathrm{tr}_{f,\mathscr{K}}$ locally on Y; we then proceed as in [2, XVIII.2.9].

Lemme 2.3.5. *If $g = g'g'' : X \xrightarrow{g''} S' \xrightarrow{g'} S$, with g' and g'' smooth and purely of relative dimension N' and N'', and if Y is again, on S', fibre by fibre of codimension $\geqslant c$, then the class $\mathrm{cl}(Y, \mathscr{K})$ is the same, calculated in terms of g or of g''.*

Let $f' = g''i$ and $d' = d - N'$. The formula $\mathrm{tr}_g = \mathrm{tr}_{g'}\,\mathrm{tr}_{g''}$ ([2, XVIII.2.9] Var 3) assures the commutativity of

$$\begin{tikzcd}
\mathrm{H}^{2c}_X(Z, \mathbb{Z}/n(c)) \arrow[d] & \\
\hom_{S'}\left(\mathsf{R}f_!\mathbb{Z}/n(d')[2d'], \mathsf{R}g''_!\mathbb{Z}/n(N'')[2N'']\right) \arrow[r] \arrow[d, "\mathsf{R}g'_!(N')"] & \hom_{S'}\left(\mathsf{R}^{2d'}f'_!\mathbb{Z}/n, \mathbb{Z}/n\right) \arrow[d, "\mathrm{tr}_{g'}\circ \mathsf{R}^{2N'}g'_!"] \\
\hom_S\left(\mathsf{R}f_!\mathbb{Z}/n(d)[2d], \mathsf{R}g'_!\mathbb{Z}/n(N)[2N]\right) \arrow[r] & \hom_S\left(\mathsf{R}^{2d}f_!\mathbb{Z}/n, \mathbb{Z}/n\right)
\end{tikzcd}$$

while the analogous and easy to verify formula $\mathrm{tr}_{f,\mathscr{K}} = \mathrm{tr}_{g'}\,\mathrm{tr}_{f',\mathscr{K}}$ assures that the image of $\mathrm{tr}_{f',\mathscr{K}}$ is $\mathrm{tr}_{f,\mathscr{K}}$. The assertion follows.

Lemme 2.3.6. *If Y is a divisor in X, the class 2.3.2 coincides with the class 2.1.2.*

By semi-purity (2.2.6(i) and 2.2.8(i)), the problem is local on Y; using 2.3.5 to replace S by a suitable S', this reduces us to the case where $N = 1$. Using semi-purity again (2.2.6(ii) and 2.2.8(ii)), it is enough to prove 2.3.6 over the generic points of S. An étale localization then reduces us to the case where S is the spectrum of an algebraically closed field k. The classes 2.3.2 and 2.1.2 being each additive in Y, this reduces us to the case where Y is a closed point on a smooth curve over k. That the characteristic property 2.3.2 is satisfied is then the fundamental compatibility 2.1.5.

Lemme 2.3.7. *The formation of $\mathrm{cl}(Y, \mathscr{K})$ is compatible with any change of base S'/S.*

This is a consequence of the same assertion for trace morphisms.

Théorème 2.3.8. *(i) $\mathrm{cl}(Y, \mathscr{K})$ only depends on $Y \subset X$ and the lengths of $\mathscr{K}$ at the generic points of Y. This class is additive in $\mathscr{K}$. For $\mathscr{K} = \mathscr{O}_Y$ and Y of local complete intersection, it induces the local class 2.2.2.*

(ii) Let there be a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d] & X \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

with X' smooth over S'. If $u^{-1}(Y)$ is still fibre by fibre of codimension $\geqslant c$, then

$$\mathrm{cl}\left(u^{-1}(Y), \mathsf{L}u^*\mathscr{K}\right) = u^*\,\mathrm{cl}(Y, \mathscr{K}).$$

(iii) Let $Y' \subset X$ be fibre by fibre of codimension $\geqslant c'$, $\mathscr{K}'$ a weight on Y' and suppose that $Y \cap Y'$ is fibre by fibre of codimension $\geqslant c + c'$. Then

$$\mathrm{cl}\left(Y \cap Y', \mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{K}'\right) = \mathrm{cl}(Y, \mathscr{K}) \smile \mathrm{cl}(Y', \mathscr{K}').$$

(A) Given $Y \subset X/S$ as above, it follows from the semi-purity (2.2.6, 2.2.8) that to check that two classes in $\mathrm{H}^{2c}_Y(X, \mathbb{Z}/n(c))$ coincide, it is enough to check this locally at the generic points of Y, or even after any base change $s \to S$ (s a generic geometric point of S), at the generic points of Y.

(B) Proof of (ii) for $S = S'$ and u a closed immersion By localization (A) we can assume that X' is the inverse image of the section 0 by a smooth morphism $v : X \to \mathbb{A}_S^{N'}$

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d] & X \arrow[d, "v"] & \\ S \arrow[r, "0"] & \mathbb{A}_S^{N'} \arrow[r, "\pi"] & S \end{tikzcd}$$

We apply 2.3.5 to $g = \pi v$ and 2.3.7 to the base change $S \to \mathbb{A}_S^{N'}$.

(C) In a product situation: $X = X' \times_S X''$, $Y = Y' \times_S Y''$, $\mathscr{K} = \mathscr{K} \overset{\mathsf{L}}{\otimes}_S \mathscr{K}''$, we have $\mathrm{cl}(Y, \mathscr{K}) = \mathrm{cl}(Y, \mathscr{K}') \smile \mathrm{cl}(Y'', \mathscr{K}'')$ (external cup product, i.e. $\mathrm{pr}_1^* \mathrm{cl}(Y', \mathscr{K}') \smile \mathrm{pr}_2^* \mathrm{cl}(Y'', \mathscr{K}'')$).

From the functorial properties of tr_g [2, XVIII.2.12] it follows that the diagram

$$\begin{tikzcd} \mathrm{H}^{2c'}_{X'} \otimes \mathrm{H}^{2c''}_{X''} \arrow[r] \arrow[d] & \hom_S\left(\mathsf{R}f'_!\mathbb{Z}/n(d')[2d'], \mathsf{R}g'_!\mathbb{Z}/n(N')[2N']\right) \otimes \cdots \arrow[r] \arrow[d] & \hom_S\left(\mathsf{R}^{2d'} f'_!\mathbb{Z}/n, \mathbb{Z}/n\right) \otimes \cdots \arrow[d] \\ \mathrm{H}^{2c}_X \arrow[r] & \hom_S\left(\mathsf{R}f_!\mathbb{Z}/n(d)[2d], \mathsf{R}g_!\mathbb{Z}/n(N)[2N]\right) \arrow[r] & \hom_S\left(\mathsf{R}^{2d} f_!\mathbb{Z}/n, \mathbb{Z}/n\right) \end{tikzcd}$$

commutes and we see that tr_f is the image of $\mathrm{tr}_{f'} \otimes \mathrm{tr}_{f''}$ which proves the assertion.

Proof of (ii) (2.3.7) and a preliminary change of basis bring us back to the assumption that $S = S'$. Factorize u by its graph:

$$X' \xrightarrow{(\mathrm{id}, u)} X' \times X \xrightarrow{\mathrm{pr}_2} X.$$

This leads us to treat pr_2 separately, which is justifiable for $X' = X'$, $\mathscr{K}' = \mathscr{O}_{x'}$ (with $\mathrm{cl}(Y') = 1 \in \mathrm{H}^0(X, \mathbb{Z}/n)$), and (id, u), which is justifiable by (B).

Proof of (iii) Let $\delta : X \to X \times_S X$ be the diagonal inclusion. We use the formula $\delta^*(Y_1 \times_S Y_2) = Y_1 \cap Y_2$ to reduce to (B) and (C).

Proof of (i) The change of base $S_{\mathrm{red}} \to S$ replaces X with X_{red}; this reduces us to the case where X and S are reduced. A localization near the generic points of Y then further reduces to the case where Y_{red} is irreducible and the complete intersection of c divisors D_i. The same localization, and the definition of tr in the quasi-finite case, then shows that $\mathrm{cl}(Y, \mathscr{K}) = \lg \mathrm{cl}(Y_{\mathrm{red}}, \mathscr{O}_{Y_{\mathrm{red}}})$, where lg is the length of $\mathscr{K}$ at the generic point of Y. By (iv), we therefore have

$$\mathrm{cl}(Y, \mathscr{K}) = \lg \prod_i \mathrm{cl}(D_i, \mathscr{O}_{D_i})$$

and we conclude by 2.3.6.

2.3.9 Remarque

For S the spectrum of a field, and the class of a cycle being defined as in 2.2.10, assertion (iii) says that, if two cycles Y' and Y'' intersect with the right dimension, then

$$\mathrm{cl}(Y' \cap Y'') = \mathrm{cl}(Y') \smile \mathrm{cl}(Y''),$$

provided that the multiplicities of the components of $Y' \cap Y''$ are calculated as alternating sums of Tor.

2.3.10 Remarque

Let $X/\operatorname{Spec}(k)$, k algebraically closed. If two cycles of codimension c in X are algebraically equivalent, it follows from the existence of the relative theory above that the images in $\mathrm{H}^{2c}(X, \mathbb{Z}/n(c))$ of their classes are the fibres, in two points of connected suitable S, of a section over S of the constant sheaf $\mathrm{H}^{2c}(X, \mathbb{Z}/n(c))$ (the sheaf $\mathsf{R}^{2c} f_* \mathbb{Z}/n(c)$ for $f = \mathrm{pr}_2 : X \times S \to S$): two algebraically equivalent cycles have the same class in $\mathrm{H}^{2c}(X, \mathbb{Z}/n(c))$.

3 Application: the Lefschetz trace formula in the proper and smooth case

3.1

Let X and Y be proper and smooth algebraic varieties over an algebraically closed field k, purely of dimensions N and M. Via the Künneth isomorphism $\mathrm{H}^\bullet(X \times Y) = \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(Y)$, the trace map $\mathrm{H}^\bullet(X \times Y)(M) \to \mathrm{H}^\bullet(X)$ is none other than $\mathrm{H}^\bullet(X) \otimes$ (the trace map $\mathrm{H}^\bullet(Y)(M) \to \mathbb{Q}_\ell$). The map being homogeneous of even degree, there is no sign problem. We denote it $\int_Y$.

3.2

A class $\eta \in \mathrm{H}^{2q}(X \times Y)(q)$ defines a map of degree $2(q-N)$ $\eta_{XY} : \mathrm{H}^\bullet(Y)(N-q) \to \mathrm{H}^\bullet(X)$, by the formula

$$\beta \mapsto \int_Y \eta \cdot \mathrm{pr}_2^* \beta.$$

Proposition 3.3. *Let p and q be two integers, with $p+q = N+M$, and $\varepsilon \in \mathrm{H}^{2p}(X \times Y)(p)$, $\eta \in \mathrm{H}^{2q}(X \times Y)(p)$. The trace of $\eta_{XY}\varepsilon_{YX} : \mathrm{H}^\bullet(X) \to \mathrm{H}^\bullet(X)$ is understood in the sense of 1.3, then*

$$\mathrm{tr}_{X\times Y}(\eta \cdot \varepsilon) = \mathrm{tr}\left(\eta_{XY}\varepsilon_{YX}, \mathrm{H}^\bullet(X)\right).$$

To exorcise the signs, it is interesting to retain from the graduation of $\mathrm{H}^\bullet$ only the underlying $\mathbb{Z}/2$-graduation. To avoid dragging with us the Tate twist, we also fix an isomorphism $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell$. Let α_Y be the homomorphism $\mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(Y)^\vee$ (an isomorphism) making the diagram commutative

$$\begin{tikzcd}
\mathrm{H}^\bullet(Y) \otimes \mathrm{H}^\bullet(Y) \arrow[r] \arrow[d, "\alpha_Y \otimes 1"] & \mathrm{H}^\bullet(Y) \arrow[r, "\mathrm{tr}"] & \mathbb{Q}_\ell \arrow[d, equal] \\
\mathrm{H}^\bullet(Y)^\vee \otimes \mathrm{H}^\bullet(Y) \arrow[rr, "(1.3.1)"] & & \mathbb{Q}_\ell.
\end{tikzcd}$$

The application η_{XY} is the image of η by

$$\mathrm{H}^\bullet(X \times Y) = \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(Y) \xrightarrow{\alpha_Y} \mathrm{H}^\bullet(X) \otimes \mathrm{H}^\bullet(Y)^\vee \overset{(1.3.2)}{=\!=} \hom\left(\mathrm{H}^\bullet(X), \mathrm{H}^\bullet(Y)\right),$$

and similarly for ε. The definition 1.3.4 of the trace then reduces 3.3 to the formula $\mathrm{tr}_{X\times Y} = \mathrm{tr}_X \otimes \mathrm{tr}_Y$.

3.4 Remarque

The proof of 3.4 does not use that α_X and α_Y are isomorphisms. It is valid without assuming X and Y are smooth. However, outside of the smooth case, it is difficult to find *cohomology* classes to which we want to apply 3.3.

3.5 Remarque

If ε, η are the cohomology classes of algebraic cycles A and B on $X \times Y$, of codimension p and q, and $A \cap B$ is of dimension 0, then $\mathrm{tr}_{X\times Y}(\eta\varepsilon)$ is the intersection multiplicity $A \cdot B$, calculated by an alternating sum of Tor (2.3.9).

Proposition 3.6. *For $q = M$ and $\eta = \mathrm{cl}(B)$ the class of a closed subscheme $B \subset X \times Y$, finite over X, the application η_{XY} is the composition*

$$\eta_{XY} : \mathrm{H}^\bullet(Y) \xrightarrow{\mathrm{pr}_2^*} \mathrm{H}^\bullet(B) \xrightarrow{\mathrm{tr}_{B/X}} \mathrm{H}^\bullet(X).$$

According to 3.2 and (2.3.3.1) for $\mathscr{K} = \mathscr{O}_B$, we have indeed

$$\eta_{XY}(\beta) = \mathrm{tr}_{X\times Y/X}\left(\mathrm{cl}(B) \cdot \mathrm{pr}_2^*\beta\right) = \mathrm{tr}_{B/X}\left(\mathrm{pr}_2^*\beta\right).$$

The most important case is when B is the graph of an application $f : X \to Y$. In this case, 3.6 shows that $\eta_{XY} = f^*$.

Corollaire 3.7. *Let f be a proper and smooth endomorphism of X over an algebraically closed field k. We assume that the fixed points of f are isolated. Then, the trace $\sum(-1)^i \mathrm{tr}(f^*, \mathrm{H}^i(X))$ is the number of fixed points of f, each counted with its multiplicity.*

In 3.5, we take $A =$graph of the identity, $B =$graph of f and we apply 3.3, 3.6.

Corollaire 3.8. *Let X be a curve over k that is algebraically closed, deduced by extension of scalars from X_0 over $\mathbb{F}_q$, and let f be the Frobenius morphism. Then, $\sum(-1)^i \mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(X)\right)$ is the number of fixed points of f.*

We can replace X with X_{red}, so we can assume X is reduced. Let U be the open of X where X is smooth of dimension 1, and let $\bar{U}$ be the projective and smooth completion of U. The long exact cohomology sequences for (X, U) and $(\bar{U}, U)$ give

$$\mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(X)\right) = \mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(U)\right) + \mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(X \smallsetminus U)\right)$$

and

$$\mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(\bar{U})\right) = \mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(U)\right) + \mathrm{tr}\left(f^*, \mathrm{H}_c^\bullet(\bar{U} \smallsetminus U)\right).$$

The same formulas hold for the number of fixed points. The formula 3.8 is clear for a scheme of dimension 0, so this reduces to proving 3.8 for $\bar{U}$. This case is justifiable by 3.7. The fact that $df = 0$ guarantees that the fixed points are all of multiplicity one.

3.9 Remarque

For X of dimension $\leqslant 1$ over k algebraically closed, and $f : X \to X$ such that $df = 0$, it is not difficult to check that 3.8 is still valid.

Chapter 5

Duality

There will be found in this exposition some theorems and compatibilities, all relative to the Poincaré duality. In paragraph 1, the theorem of local biduality in dimension 1 [20, I.5.1] and some calculations of two. In paragraph 2, a very economical demonstration of the Poincaré duality on curves, which M. Artin taught me. In paragraph 3, a compatibility which makes the link between two definitions of the coupling which gives rise to the Poincaré duality for curves: by cup-product, or by self-duality of the Jacobian. In paragraph 4, finally, the proof of the compatibility of the title.

Throughout the exposition, the schemes will be noetherian and separated, and n is an invertible integer on all the schemes considered.

1 Local biduality, in dimension 1

1.1

Let S be a regular scheme of pure dimension 1. We want to show that the complex reduced to $\mathbb{Z}/n$ in degree 0 is dualisant, i.e. that for $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_c(S, \mathbb{Z}/n)$ ($(-)_c$ for constructible), if we set $D\mathscr{K} = \mathsf{R}\mathscr{H}om(\mathscr{K}, \mathbb{Z}/n)$, then $D\mathscr{K}$ is still constructible with bounded cohomology, and that the canonical morphism α of $\mathscr{K}$ into $DD\mathscr{K}$ is an isomorphism.

For $\mathscr{K}$ in D^-, and $\mathscr{L}$ any, we have

$$\hom(\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{L}, \mathbb{Z}/n) = \hom(\mathscr{L}, \mathscr{H}om(\mathscr{K}, \mathbb{Z}/n)).$$

The morphism of $\mathscr{K}$ into $DD\mathscr{K}$ is defined by assuming $D\mathscr{K}$ is in D^-; if $\beta : D\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{K} \to \mathbb{Z}/n$ is the canonical pairing, it is defined by the pairing $\mathscr{K} \overset{\mathsf{L}}{\otimes} D\mathscr{K} = D\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathscr{K} \xrightarrow{\beta} \mathbb{Z}/n$.

1.2

Let $f : X \to S$ be a separated finite type morphism. Let $\mathscr{K}_X = \mathsf{R}f^!\mathbb{Z}/n$. For $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^-(X, \mathbb{Z}/n)$, let $D\mathscr{K} = \mathsf{R}\mathscr{H}om(\mathscr{K}, \mathscr{K}_X)$. The adjunction between $\mathsf{R}f_!$ and $\mathsf{R}f^!$ ensures that

$$\mathsf{R}f_*\mathscr{K} \overset{\mathsf{L}}{\otimes} \mathsf{R}f_* D\mathscr{K} \longrightarrow \mathsf{R}f_!(\mathscr{K} \overset{\mathsf{L}}{\otimes} D\mathscr{K}) \longrightarrow \mathsf{R}f_!\mathsf{R}f^!\mathbb{Z}/n \longrightarrow \mathbb{Z}/n.$$

The symmetry of this description shows that, for f proper, the diagram

$$\begin{tikzcd} \mathsf{R}f_*\mathscr{K} \arrow[r] \arrow[d] & DD\mathsf{R}f_*\mathscr{K} \arrow[d, "\wr"] \\ \mathsf{R}f_*DD\mathscr{K} \arrow[r, "\sim"] & D\mathsf{R}f_*D\mathscr{K} \end{tikzcd} \tag{1.2.1}$$

is commutative.

If f is the inclusion of a closed point, we know that $\mathsf{R}f^!\mathbb{Z}/n = \mathbb{Z}/n(-1)[-2]$ – that is, up to torsion and shift, $\mathbb{Z}/n$. On the spectrum of a field, this complex is dualizing (Pontrjagin duality for $\mathbb{Z}/n$-modules). According to (1.2.1), we therefore have $\mathscr{K} \xrightarrow{\sim} DD\mathscr{K}$ for $\mathscr{K}$ of the form $\mathsf{R}f_!\mathscr{L}$ – and thus when the support of $\underline{\mathrm{H}}^\bullet(\mathscr{K})$ is finite.

Théorème 1.3. *Let $j : U \hookrightarrow S$ be a dense open in S and $\mathscr{F}$ a locally constant constructible sheaf of $\mathbb{Z}/n$-modules on U. We have $\mathsf{D}j_*\mathscr{F} = j_*D\mathscr{F}$, i.e. $\mathscr{H}om(j_*\mathscr{F}, \mathbb{Z}/n) = j_*\mathscr{H}om(\mathscr{F}, \mathbb{Z}/n)$ and $\mathscr{E}xt^i(j_*\mathscr{F}, \mathbb{Z}/n) = 0$ for $i > 0$.*

On U, $\mathscr{E}xt^i(j_*\mathscr{F}, \mathbb{Z}/n) = 0$ for $i > 0$, because $\mathscr{F}$ is locally constant (and $\mathbb{Z}/n$ is an injective $\mathbb{Z}/n$-module), while for $i = 0$ it is the dual $\mathscr{F}^\vee$ of $\mathscr{F}$. We check that $\hom(j_*\mathscr{F}, \mathbb{Z}/n) = j_*\mathscr{F}^\vee$, and it remains to check the nullity of the $\mathscr{E}xt^i$ $(i > 0)$ at the points of $S \smallsetminus U$. The problem is local at these points. This reduces us to assuming that S is a strictly local trait and that U is reduced to its generic point η. Let $I = \mathrm{Gal}(\bar{\eta}/\eta)$. The sheaf $\mathscr{F}$ is identified with the Galois module $\mathscr{F}_{\bar{\eta}}$, and the special fiber of $j_*\mathscr{F}$ with $\mathscr{F}_{\bar{\eta}}^I$.

Let i be the inclusion of the closed point s, and apply D to the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & j_!\mathscr{F} \arrow[r] & j_*\mathscr{F} \arrow[r] & i_*\mathscr{F}_{\bar{\eta}}^I \arrow[r] & 0 \\ 0 \arrow[r] & j_!\mathscr{F}_{\bar{\eta}}^I \arrow[r] \arrow[u] & \mathscr{F}_{\bar{\eta}}^I \arrow[r] \arrow[u] & i_*\mathscr{F}_{\bar{\eta}}^I \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

We obtain a morphism of triangles

$$\begin{tikzcd} i_*\left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee(-1)[-2] \arrow[r] \arrow[d, equal] & Dj_*\mathscr{F} \arrow[r] \arrow[d] & \mathsf{R}j_*\mathscr{F}^\vee \arrow[d] \\ i_*\left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee(-1)[-2] \arrow[r] & \left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee \arrow[r] & \mathsf{R}j_*\mathscr{F}_{\bar{\eta}}^I. \end{tikzcd}$$

The long exact sequence derived from the first line provides the vanishing of $\mathscr{E}xt^i$ $(i > 2)$ and, taking the fiber at s, we find

$$\begin{tikzcd} 0 \arrow[r] & \left(\underline{\mathrm{H}}^1(Dj_*\mathscr{F}\right)_s \arrow[r] & \mathrm{H}^1\left(I, \mathscr{F}_{\bar{\eta}}^\vee\right) \arrow[r, "\partial"] \arrow[d] & \left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee(-1) \arrow[r] \arrow[d, equal] & \left(\underline{\mathrm{H}}^2 Dj_*\mathscr{F}\right)_s \arrow[r] & 0 \\ & 0 \arrow[r] & \mathrm{H}^1\left(I, (\mathscr{F}_{\bar{\eta}}^I)^\vee\right) \arrow[r, "\partial"] & \left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee(-1) \arrow[r] & 0. \end{tikzcd}$$

Since $\left(\mathscr{F}_{\bar{\eta}}^I\right)^\vee = \left(\mathscr{F}_{\bar{\eta}}^\vee\right)_I$, setting $M = \mathscr{F}_{\bar{\eta}}^\vee$, we finally need to check that

$$\mathrm{H}^1(I, M) \xrightarrow{\sim} \mathrm{H}^1(I, M_I).$$

If p is the characteristic exponent, I is an extension of a group isomorphic to $\widehat{\mathbb{Z}}_{p'} = \varprojlim_{(m,p)=1} \mathbb{Z}/m$ by a p-group P. Since P is prime to the order of M, we have $\mathrm{H}^1(P, M) = 0$ $(i > 0)$ and $(M_I)^P = (M^P)_I$. This allows us to replace M by M^P and I by I/P. We finally have a functorial isomorphism $\mathrm{H}^1(\widehat{\mathbb{Z}}_{p'}, M) \sim$ (coinvariants of $\widehat{\mathbb{Z}}_{p'}$, in M), which proves the theorem.

Théorème 1.4. *For $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_c(S, \mathbb{Z}/n)$, we have $\mathscr{K} \xrightarrow{\sim} DD\mathscr{K}$.*

By unscrewing, we may assume that $\mathscr{K}$ is reduced to a constructible sheaf $\mathscr{F}$ in degree 0, and that $\mathscr{F}$ is either of finite support (1.2), or of the form $j_*\mathscr{F}_1$ as in 1.3. In the second case, 1.3 reduces us to the local duality for $\mathscr{F}$ locally constant on U.

2 La dualité de Poincaré pour les courbes, d'après M. Artin

2.1

Let X be a projective and smooth curve over an algebraically closed field k. We set $\mathscr{K}_X = \mathbb{Z}/n(1)[2]$ and, for $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_c(X, \mathbb{Z}/n)$, $D\mathscr{K} = \mathsf{R}\mathscr{H}om(\mathscr{K}, \mathscr{K}_X)$. For M a $\mathbb{Z}/n$-module, we also set $DM = \hom(M, \mathbb{Z}/n)$; similarly for complexes of modules. The trace map $\mathrm{H}^0(X, \mathscr{K}_X) \to \mathbb{Z}/n$, or $\mathsf{R}\Gamma(X, \mathscr{K}_X) \to \mathbb{Z}/n$, defines a pairing $\mathsf{R}\Gamma(X, \mathscr{K}) \overset{\mathsf{L}}{\otimes} \mathsf{R}\Gamma(X, D\mathscr{K}) \to \mathbb{Z}/n$. The Poincaré duality between cohomology and cohomology with proper supports of an open $j : U \hookrightarrow X$ of X says that, for $\mathscr{K} = j_!\mathbb{Z}/n$, this pairing identifies each factor with the dual of the other.

I am exposing below a demonstration, which was communicated to me by M. Artin, of what for $\mathscr{K} \in \operatorname{Ob} \mathsf{D}^b_c(X, \mathbb{Z}/n)$, this pairing is always perfect, i.e. defines an isomorphism

$$\mathsf{R}\Gamma(X, \mathscr{K}) \xrightarrow{\sim} D\mathsf{R}\Gamma(X, D\mathscr{K}). \tag{2.1.1}$$

For every constructible sheaf $\mathscr{F}$, let us set

$$'\,\mathrm{H}^i(X, \mathscr{F}) = \mathbb{Z}/n\text{-dual of } \mathrm{H}^{-i}(X, D\mathscr{F}).$$

That (2.1.1) is an isomorphism is equivalent to the

Théorème 2.2. *For $\mathscr{F}$ a constructible sheaf of $\mathbb{Z}/n$-modules, we have*

$$\mathrm{H}^i(X, \mathscr{F}) \xrightarrow{\sim} {}'\,\mathrm{H}^i(X, \mathscr{F}). \tag{2.2.1}$$

Lemme 2.3. *Let $f : X \to Y$ be a morphism that is generically étale between smooth curves over k. We have $\mathscr{K}_X = f^*\mathscr{K}_Y = \mathsf{R}f^!\mathscr{K}_Y$, with tr_f for the adjunction arrow $\mathsf{R}f_!\mathsf{R}f^!\mathscr{K}_Y \to \mathscr{K}_Y$.*

We can reduce to checking that for f finite, we have $f_*\mathsf{R}f^!\mathscr{K}_Y = \mathsf{R}\mathscr{H}om(f_*\mathbb{Z}/n, \mathscr{K}_Y)$ is $f_*\mathbb{Z}/n$, with tr_f for the adjunction arrow. The vanishing of the $\mathscr{E}xt^i$ $(i > 0)$ follows from 1.3, and the assertion follows.

We can say more explicitly that $\mathsf{R}f^!$ is the derived functor of the functor $f^! : f^!\mathscr{F}(U) = \hom(f_!\mathbb{Z}_U, \mathscr{F})$ and 1.3 implies the vanishing of the $\mathsf{R}^i f^!\mathbb{Z}/n$ for $i > 0$.

Corollaire 2.4. *Let $f : X' \to X$ be a morphism that is generically étale of projective and smooth curves over k. We have ${}'\operatorname{H}^i(X, f_*\mathscr{F}) = {}'\operatorname{H}^i(X', \mathscr{F})$; this isomorphism is functorial and the diagram*

$$\begin{tikzcd} \operatorname{H}^i(X, f_*\mathscr{F}) \arrow[r] \arrow[d, equal] & {}'\operatorname{H}^i(X, f_*\mathscr{F}) \arrow[d, equal] \\ \operatorname{H}^i(X', \mathscr{F}) \arrow[r] & {}'\operatorname{H}^i(X', \mathscr{F}) \end{tikzcd}$$

is commutative.

The isomorphism is given by 1.2: $Df_*\mathscr{F} = f_*D\mathscr{F}$.

2.5

We prove 2.2. If $\mathscr{F}$ is reduced to $\mathbb{Z}/n$ at one point, extended by 0, we have $\operatorname{H}^\bullet(D\mathscr{F}) = \operatorname{Ext}^\bullet(\mathscr{F}, \mathscr{K}_X) = \mathbb{Z}/n$ in degree 0, and in the pairing $\operatorname{H}^\bullet(\mathscr{F}) \otimes \operatorname{H}^\bullet(D\mathscr{F}) \to \mathbb{Z}/n$, we have $1 \otimes 1 \mapsto 1$ (Cycle, 2.1.5): the duality is perfect. The case where $\mathscr{F}$ has finite support is treated similarly.

For any constructible $\mathscr{F}$, $\mathscr{A}$ the subsheaf of its sections with finite support, and $\mathscr{G} = \mathscr{F}/\mathscr{A}$, the exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{2.5.1}$$

provides a diagram

$$\begin{tikzcd}[column sep=small] & & 0 \arrow[r] & \operatorname{H}^0(X, \mathscr{A}) \arrow[r] \arrow[d, "\wr"] & \operatorname{H}^0(X, \mathscr{F}) \arrow[r] \arrow[d] & \operatorname{H}^0(X, \mathscr{G}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {}'\operatorname{H}^{-1}(X, \mathscr{F}) \arrow[r] & {}'\operatorname{H}^{-1}(X, \mathscr{F}) \arrow[r] & {}'\operatorname{H}^0(X, \mathscr{A}) \arrow[r] & {}'\operatorname{H}^0(X, \mathscr{F}) \arrow[r] & {}'\operatorname{H}^0(X, \mathscr{F}) \arrow[r] & 0 \end{tikzcd} \tag{2.5.2}$$

and compatible isomorphisms $\operatorname{H}^i(X, \mathscr{F}) \xrightarrow{\sim} \operatorname{H}^i(X, \mathscr{G})$, ${}'\operatorname{H}^i(X, \mathscr{F}) \xrightarrow{\sim} {}'\operatorname{H}^i(X, \mathscr{G})$ $(i \neq 0, -1)$. If $j : U \to X$ is an open where $\mathscr{F}$ is locally constant, we have an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow j_*j^*\mathscr{F} \longrightarrow \mathscr{B} \longrightarrow 0 \tag{2.5.3}$$

with $\mathscr{B}$ of finite support. Since ${}'\operatorname{H}^i(X, j_*j^*\mathscr{F}) = D\operatorname{H}^{2-i}(X, j_*(j^*\mathscr{F})^\vee(1))$ according to 1.3, this ${}'\operatorname{H}^i = 0$ for $i < 0$ and (2.5.2) and the long exact sequence defined by (2.5.3) show that ${}'\operatorname{H}^i$ is always zero for $i < 0$.

If $\mathscr{F} = \mathbb{Z}/n$, the theorem is true for $i = 0$ and 2. This expresses that, for X connected, the trace morphism: $\operatorname{H}^2(X, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathbb{Z}/n$ is an isomorphism. According to 2.4, the theorem remains true for $i = 0$ and $\mathscr{F} = f_*\mathbb{Z}/n$.

For $\mathscr{F}$ any again and $\mathscr{G}$ as above, there exists $f : X' \to X$ as in 2.4 such that $\mathscr{G}$ is immersed in $f_*\mathbb{Z}/n$:

$$0 \longrightarrow \mathscr{G} \longrightarrow f_*\mathbb{Z}/n \longrightarrow \mathscr{Q} \longrightarrow 0.$$

Changing X' to a X''/X', we can suppose that $\operatorname{H}^i(X, \mathscr{F}) \to \operatorname{H}^i(X, f_*\mathbb{Z}/n) = \operatorname{H}^i(X', \mathbb{Z}/n)$ is null for $i > 0$. We then consider

$$\begin{tikzcd}[column sep=tiny] 0 \arrow[r] & \operatorname{H}^0(X, \mathscr{G}) \arrow[r] \arrow[d] & \operatorname{H}^0(S', \mathbb{Z}/n) \arrow[r] \arrow[d, "\wr"] & \operatorname{H}^0(X, \mathscr{Q}) \arrow[r] \arrow[d] & \operatorname{H}^1(X, \mathscr{G}) \arrow[r, "0"] \arrow[d] & \operatorname{H}^1(X', \mathbb{Z}/n) \arrow[r] \arrow[d] & \operatorname{H}^1(X, \mathscr{Q}) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & {}'\operatorname{H}^0(X, \mathscr{G}) \arrow[r] & {}'\operatorname{H}^0(X', \mathbb{Z}/n) \arrow[r] & {}'\operatorname{H}^0(X, \mathscr{Q}) \arrow[r] & {}'\operatorname{H}^1(X, \mathscr{G}) \arrow[r] & {}'\operatorname{H}^1(X', \mathbb{Z}/n) \arrow[r] & {}'\operatorname{H}^1(X, \mathscr{Q}) \arrow[r] & \cdots \end{tikzcd}$$

The isomorphism in $\mathrm{H}^0(X')$ provides $\mathrm{H}^0(X,\mathscr{G}) \hookrightarrow' \mathrm{H}^0(X,\mathscr{G})$, and the same injectivity for $\mathscr{F}$ (2.5.2). Applied to $\mathscr{Q}$, this gives $\mathrm{H}^1(X,\mathscr{G}) \hookrightarrow' \mathrm{H}^1(X,\mathscr{G})$. We apply this to $\mathrm{H}^1(X',\mathbb{Z}/n)$. This group having the same order as $'\,\mathrm{H}^1(X',\mathbb{Z}/n) = D(\mathrm{H}^1(X',\mathbb{Z}/n)(1))$, we find an isomorphism, and and $\mathrm{H}^1(X,\mathscr{G}) \xrightarrow{\sim}{}' \mathrm{H}^1(X,\mathscr{G})$. The same for $\mathscr{F}$. For $i=2$, we already know that $\mathrm{H}^2(X',\mathbb{Z}/n) \xrightarrow{\sim}{}' \mathrm{H}^2(X',\mathbb{Z}/n)$, and we proceed in the same way, and then stop, for lack of opponents.

3 Poincaré duality for curves

In this paragraph, we prove the compatibility (Arcata 6.2.2.3).

The notations are those of (Arcata 6.2.3): $\bar{X}$ is a projective, smooth, and connected curve over an algebraically closed field k, D is a reduced divisor on $\bar{X}$, 0 is a point of $X = \bar{X} \setminus D$

$$X \overset{j}{\hookrightarrow} \bar{X} \overset{i}{\hookleftarrow} D,$$

and $\mathrm{Pic}_D(\bar{X}) = \mathrm{H}^1(\bar{X}, {}_D\mathbb{G}_m)$ where ${}_D\mathbb{G}_m$ is the sheaf of sections of $\mathbb{G}_m$ congruent to 1 mod D. We recall that $\mathrm{H}^1(X,\boldsymbol{\mu}_n) = \mathrm{Pic}^0_D(\bar{X})_n$, and that $x \to \mathscr{O}(x)$ defines a canonical map $f : X \to \mathrm{Pic}_D(\bar{X})$. We set $f_0(x) = f(x) - f(0) : X \to \mathrm{Pic}^0_D(\bar{X})$.

We still denote by j the inclusion of $X \times X$ in $X \times \bar{X}$. The diagonal Δ of $X \times X$ is closed in $X \times \bar{X}$; it defines a class in $\mathrm{H}^2_\Delta(X \times X, \boldsymbol{\mu}_n) = \mathrm{H}^2_\Delta(X \times \bar{X}, j_!\boldsymbol{\mu}_n)$. We note c its image in $\mathrm{H}^2(X \times \bar{X}, j_!\boldsymbol{\mu}_n)$.

The Künneth formula assures that

$$\mathrm{H}^\bullet(X \times \bar{X}, j_!\boldsymbol{\mu}_n) = \mathrm{H}^\bullet(X,\mathbb{Z}/n) \otimes \mathrm{H}^\bullet(\bar{X}, j_!\boldsymbol{\mu}_n) = \mathrm{H}^\bullet(X,\mathbb{Z}/n) \otimes \mathrm{H}^\bullet_c(X,\boldsymbol{\mu}_n).$$

We propose to calculate the $(1,1)$-component of c,

$$c^{1,1} \in \mathrm{H}^1(X,\mathbb{Z}/n) \otimes \mathrm{H}^1_c(X,\boldsymbol{\mu}_n) = \mathrm{H}^1(X, \mathrm{H}^1_c(X,\boldsymbol{\mu}_n)) = \mathrm{H}^1\left(X, \mathrm{Pic}^0_D(\bar{X})_n\right).$$

The exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_D(\bar{X})_n \longrightarrow \mathrm{Pic}^0_D(\bar{X}) \xrightarrow{n} \mathrm{Pic}^0_D(\bar{X}) \longrightarrow 0$$

makes $\mathrm{Pic}^0_D(\bar{X})$ a $\mathrm{Pic}^0_D(\bar{X})_n$-torseur over $\mathrm{Pic}^0_D(\bar{X})$. Let u be the class, in $\mathrm{H}^1(X, \mathrm{H}^1_c(X,\boldsymbol{\mu}_n)) = \mathrm{H}^1(X, \mathrm{Pic}^0_D(\bar{X})_n)$, of its image by the inverse of f_0.

Proposition 3.1. *With the previous notation, $c^{1,1} = -u$.*

Let e be the difference in $\mathrm{H}^2(X \times \bar{X}, j_!\boldsymbol{\mu}_n)$ of the cohomology classes of Δ and $X \times \{0\}$, which is of Künneth type $(0,2)$. We have $c^{1,1} = e^{1,1}$. Let the Leray spectral sequence for $\mathrm{pr}_1 : X \times \bar{X} \to X$ and the sheaf $j_!\boldsymbol{\mu}_n$:

$$E_2^{pq} = \mathrm{H}^p(X, \mathsf{R}^q \mathrm{pr}_{1*} j_!\boldsymbol{\mu}_n) = \mathrm{H}^p(X, \mathrm{H}^q_c(X,\boldsymbol{\mu}_n)) \Rightarrow \mathrm{H}^{p+q}(X \times \bar{X}, j_!\boldsymbol{\mu}_n).$$

It degenerates: it is the tensor product of $\mathrm{H}^\bullet(X)$ by the Leray spectral sequence (trivial) for $\bar{X} \to \mathrm{Spec}(k)$ and the sheaf $j_!\boldsymbol{\mu}_n$. The divisor $\Delta - X \times \{0\}$ is fiberwise of degree 0, so that the image of e in E^{02} is zero. We can therefore talk about its image $\bar{e}$ in $E^{1,1}$, and we need to show that $\bar{e} = -u$.

Let $\mathscr{G}$ be the subsheaf of $\mathbb{G}_m$ on $X \times \bar{X}$ formed by the local sections whose restriction to the subscheme $X \times D$ is 1. We again have an exact sequence

$$0 \longrightarrow j_!\boldsymbol{\mu}_n \longrightarrow \mathscr{G} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{3.2.1}$$

and e is the image by ∂ of the class $e_1 \in \mathrm{H}^1(X \times \bar{X}, \mathscr{G})$ of the invertible sheaf $\mathscr{O}(\Delta - X \times \{0\})$ trivialized by 1 on $X \times D$.

The direct image by $\mathsf{R}\mathrm{pr}_{1*}$ of the distinguished triangle by (2.2.1) is a distinguished triangle

$$\xrightarrow{\partial} \mathsf{R}\mathrm{pr}_{1*}j_!\boldsymbol{\mu}_n \longrightarrow \mathsf{R}\mathrm{pr}_{2*}\mathscr{G} \longrightarrow \mathsf{R}\mathrm{pr}_{2*}\mathscr{G} \xrightarrow{\partial} \tag{3.2.2}$$

and the diagram

$$\begin{tikzcd}
\mathrm{H}^1(X \times \bar{X}, \mathscr{G}) \arrow[r, "\partial"] \arrow[d, equal, "(1)"] & \mathrm{H}^2(X \times \bar{X}, j_!\boldsymbol{\mu}_n) \arrow[d, equal] \\
\mathrm{H}^1(X, \mathsf{R}\mathrm{pr}_{1*}\mathscr{G}) \arrow[r, "\partial"] & \mathrm{H}^2(X, \mathsf{R}\mathrm{pr}_{1*}j_!\boldsymbol{\mu}_n)
\end{tikzcd}$$

is commutative: our problem becomes that of identifying the image in $E^{11} = \mathrm{H}^1(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n))$ (the image in E^{02} being zero) of the image by $\partial_{(3.2.2)} : \mathrm{H}^1(X, \mathsf{R}\mathrm{pr}_{1*}\mathscr{G}) \to \mathrm{H}^2(X, \mathsf{R}\mathrm{pr}_{1*}j_!\boldsymbol{\mu}_n)$ of the class, still denoted e_1, which corresponds to e_1 by (1).

The sheaf $\mathsf{R}^1\mathrm{pr}_{1*}\mathscr{G}$ is defined by the group scheme on X image reciprocal of the algebraic group $\mathrm{Pic}_D(\bar{X})$ on $\mathrm{Spec}(k)$, and the image of e_1 in $\mathrm{H}^0(X, \mathsf{R}^1\mathrm{pr}_*\mathscr{G}) = \mathrm{hom}(X, \mathrm{Pic}_D(\bar{X}))$ is f_0.

Let us represent the triangle (3.2.2) as defined by a short exact sequence of complexes of sheaves.

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0.$$

Let $\mathscr{B}'$ be the subsheaf of

$$\tau_{\leqslant 1}\mathscr{B} : \cdots \mathscr{B}^0 \to \ker(d) \to 0 \cdots$$

obtained by replacing $\mathscr{B}'^1 = \ker(d)$ by the image reciprocal, by $\ker(d) \to \underline{\mathrm{H}}^1(\mathscr{B}) = \mathrm{Pic}_D(\bar{X})$ on X, of $\mathrm{Pic}^0_D(\bar{X})$ on X. Let $\mathscr{A}' = \mathscr{A} \cap \mathscr{B}'$ and $\mathscr{C}'$ be the image of $\mathscr{B}'$. We have $\mathscr{A}' = \tau_{\leqslant 1}(\mathscr{A})$, and $\mathscr{C}'$ is deduced from $\mathscr{C}$ as $\mathscr{B}'$ from $\mathscr{B}$, whence a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A} \arrow[r] & \mathscr{B} \arrow[r] & \mathscr{C} \arrow[r] & 0 \\
0 \arrow[r] & \tau_{\leqslant 1}\mathscr{A} \arrow[r] \arrow[u] \arrow[d] & \mathscr{B}' \arrow[r] \arrow[u] \arrow[d] & \mathscr{C}' \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)[1] \arrow[r] & \mathrm{Pic}^0_D(\bar{X})[1] \arrow[r] & \mathrm{Pic}^0_D(\bar{X})[1] \arrow[r] & 0
\end{tikzcd} \tag{3.2.3}$$

where the last line should be seen as an exact sequence of complexes of reduced sheaves to degree 1 on X.

a) $e \in \mathsf{H}^2(X, \mathscr{A})$ comes from $\widetilde{e} \in \mathsf{H}^2(X, \tau_{\leqslant 1}\mathscr{A})$ (because its image in E^{02} is zero). We are looking for the image $\bar{e}$ of $\widetilde{e}$ in $\mathsf{H}^2(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)[-1]) = \mathrm{H}^1(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n))$.

b) $e_1 \in \mathsf{H}^1(X, \mathscr{C})$ comes from $\widetilde{e}_1 \in \mathsf{H}^1(X, \mathscr{C}')$, because its image f_0 in $\mathrm{H}^0(X, \mathrm{Pic}_D(\bar{X}))$ is in $\mathrm{H}^0(X, \mathrm{Pic}^0_D(\bar{X}))$. We can take $\widetilde{e} = \partial \widetilde{e}_1$.

c) So $\bar{e} = \partial f_0$, where ∂ is defined by (3.2.2), and where we use the usual isomorphisms $\mathsf{H}^2(X, \mathscr{F}[-1]) = \mathrm{H}^1(X, \mathscr{F})$. These isomorphisms anticommute to ∂, so that $\bar{e} = -\partial f_0$, for ∂ defined by the exact sequence

$$0 \longrightarrow \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \longrightarrow \mathrm{Pic}^0_D(\bar{X}) \longrightarrow \mathrm{Pic}^0_D(\bar{X}) \longrightarrow 0$$

We conclude by (Cycles, 1.1.1).

3.2

For every homomorphism $\varphi : \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \to \mathbb{Z}/n$, let $\varphi(u) \in \mathrm{H}^1(X, \mathbb{Z}/n)$ be the image of u by $\varphi : \mathrm{H}^1(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)) \to \mathrm{H}^1(X, \mathbb{Z}/n)$. If $x \in \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)$, the cup-product $\varphi(u) \smile x$ is in $\mathrm{H}^2_c(X, \boldsymbol{\mu}_n)$. We propose to prove that

Proposition 3.3. $\mathrm{tr}(\varphi(u) \smile x) = \varphi(x)$.

This identity follows, by application of φ, from

$$\mathrm{tr}(u \smile x) = x \tag{3.4.1}$$

where tr is this time the application

$$\mathrm{tr} : \mathrm{H}^2_c(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \otimes \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^1_c(X, \boldsymbol{\mu}_n),$$

so that $x \mapsto \mathrm{tr}(u \smile x)$ is the composed application

$$\begin{aligned}&\mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \xrightarrow{u \otimes x} \mathrm{H}^1(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)) \otimes \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \xleftarrow{\sim} \mathrm{H}^1(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \otimes \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)\\ &\xrightarrow{(1)} \mathrm{H}^2_c(X, \boldsymbol{\mu}_n) \otimes \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}} \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)\end{aligned} \tag{3.4.2}$$

where (1) is given by $a \otimes b \otimes c \mapsto (a \smile c) \otimes b$.

We express this morphism in terms of the trace morphism $\mathrm{tr} : \mathrm{H}^3_c(X \times X, \boldsymbol{\mu}_n^{\otimes 2}) \to \mathrm{H}^2_c(X, \boldsymbol{\mu}_n)$ relative to the second projection. For $x \in \mathrm{H}^3_c$, $\mathrm{tr}(x)$ is calculated as follows: from its Künneth components, we only keep the one of type $(2, 1)$, and we apply $\mathrm{tr} : \mathrm{H}^2_c(X, \boldsymbol{\mu}_n) \to \mathbb{Z}/n$.

Let u' be the image of u by the application

$$\begin{aligned}&\mathrm{H}^1(X, \mathrm{H}^1_c(X, \boldsymbol{\mu}_n)) \xleftarrow{\sim} \mathrm{H}^1(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_c(X, \boldsymbol{\mu}_n) = \mathrm{H}^1(X, \mathbb{Z}/n) \otimes \mathrm{H}^1(\bar{X}, j_! \boldsymbol{\mu}_n)\\ &\longrightarrow \mathrm{H}^2(X \times \bar{X}, \mathbb{Z}/n \boxtimes j_! \boldsymbol{\mu}_n).\end{aligned}$$

Given the relation between Künneth and cup-product, we have

$$\mathrm{tr}(u \smile x) = -\,\mathrm{tr}(u' \smile \mathrm{pr}_1^* x).$$

The second cup-product is the one from (Cycle 1.2.4):

$$\mathrm{H}^2(X \times \bar{X}, \mathbb{Z}/n \boxtimes j_! \boldsymbol{\mu}_n) \otimes \mathrm{H}^1(\bar{X} \times \bar{X}, j_! \boldsymbol{\mu}_n \boxtimes \mathbb{Z}/n) \longrightarrow \mathrm{H}^3(\bar{X} \times \bar{X}, j_! \boldsymbol{\mu}_n \boxtimes j_! \boldsymbol{\mu}_n).$$

The sign − comes from the permutation of two symbols of degree 1 in (1). Since $c \smile \mathrm{pr}_1^* x$ and $c^{1,1} \smile \mathrm{pr}_1^*$ have the same $(2,1)$-component of Künneth, and $u' = c^{1,1}$, we have

$$\mathrm{tr}(u \smile x) = \mathrm{tr}(c \smile \mathrm{pr}_1^* x).$$

The cup-product on the right can this time be interpreted as the cup-product (Cycle 1.2.5) of $\mathrm{cl}(\Delta) \in \mathrm{H}^2_\Delta(X \times X, \boldsymbol{\mu}_n)$ by $\mathrm{pr}_1^* x \in \mathrm{H}^1_c(\Delta, \boldsymbol{\mu}_n)$. On Δ, we have $\mathrm{pr}_1 = \mathrm{pr}_2$, whence

$$\mathrm{tr}(c \smile \mathrm{pr}_1^* x) = \mathrm{tr}(\mathrm{cl}(\Delta) \smile \mathrm{pr}_1^* x) = \mathrm{tr}(\mathrm{cl}(\Delta) \smile \mathrm{pr}_2^* x) = \mathrm{tr}(\mathrm{cl}(\Delta)) \smile x = x.$$

4 Compatibilité SGA 4 XVIII 3.1.10.3

In [2, XVIII], I was discouraged and did not check the compatibility of the title. Although it turns out to be only smoke, I feel obliged to repair this omission here. For $f : X \to S$ separated of finite type, it was about checking that the adjunction morphism $\mathsf{R}f_!\mathsf{R}f^!\mathscr{L} \to \mathscr{L}$ is compatible with any étale localization $k : V \to S$. Denoting by k and f several arrows, as below, it is about checking a commutativity:

$$\begin{tikzcd} X_V \arrow[r, "k"] \arrow[d, "f"] & X \arrow[d, "f"] \\ V \arrow[r, "k"] & S \end{tikzcd} \qquad \begin{tikzcd} k^*\mathsf{R}f_!\mathsf{R}f^!\mathscr{L} \arrow[r, "k^*(\mathrm{adj})"] \arrow[d, "\sim"] & k^*\mathscr{L} \arrow[d, equal] \\ \mathsf{R}f_!\mathsf{R}f^!k^*\mathscr{L} \arrow[r, "\mathrm{adj}"] & k^*\mathscr{L}. \end{tikzcd}$$

At the level of complexes, the desired commutativity is written (loc. cit.)

$$\begin{tikzcd} k^*f_!^\bullet f^!_\bullet\mathscr{L} \arrow[r] \arrow[d, "(1)"] & k^*\mathscr{L} \arrow[d, equal] \\ f_!^\bullet f^!_\bullet k^*\mathscr{L} \arrow[r] & k^*\mathscr{L}. \end{tikzcd}$$

We check it component by component, which reduces to a compatibility for $\mathscr{L}$ a sheaf and for each of the pairs of adjoint functors $(f_i^!, f_!^i)$, simply denoted $f^!$ and $f_!$. The arrow (1) is defined from the isomorphism (2) : $k^*f_! = f_!k^*$, and a morphism (3) : $k^*f^! \to f^!k^*$ which is deduced from it. In loc. cit., we first introduce a morphism (4) : $k_!f_! \to f_!k_!$, deduced from (2) by adjunction of $k^* = k^!$ and of $k_! : k_!f_! \to k_!f_!k^*k_! \to k_!k^*f_!k_! \to f_!k_!$, and deduce (3) from (4) by adjunction. It is the same to deduce (3) from (2) by adjunction of $f_!$ and $f^! : k^* \to f^!f_!k^*f^! \to f^!k^*f_!f^! \to f^!k^*$.

Writing k for k^* and omitting $\mathscr{L}$, commutativity is then that of the outer edge of

$$\begin{tikzcd} kf_!f^! \arrow[r] \arrow[d, equal] & f_!kf^! \arrow[r] & f_!f^!f_!kf^! \arrow[r] & f_!f^!kf_!f^! \arrow[d, equal] \\ kf_!f^! \arrow[d] & & & f_!f^!kf_!f^! \arrow[lll] \arrow[d] \\ k & & & f_!f^!k. \arrow[lll] \end{tikzcd}$$

On the first line there is a homomorphism $k_!f^! \to f_!f^!kf_!f^!$, defined because $kf_!f^!$ is of the form $f_!X$ $(X = kf^!)$; it has as a retraction the adjunction morphism in the second line, and the commutativity of the lower square the functoriality of this adjunction morphism.

Chapter 6

Applications of the trace formula to trigonometric sums

In this report, I explain how the trace formula can be used to calculate or study various trigonometric sums and how, combined with the Weil conjecture, it can be used to bound them.

The first two paragraphs give a "user's guide" to these tools. Paragraph 3 is an exposition, in cohomological language, of Weil's results on sums in one variable. Paragraphs 4 to 6 form a detailed study of Gauss and Jacobi sums - including Weil's old and recent results on Hecke characters defined by Jacobi sums. In paragraph 7, we study a generalization to several variables of Kloosterman sums. Finally, in paragraph 8, there are some indications on other uses that have been made or can be made of these methods.

Notations

0.1

We use the notation of Report, paragraph 1. We will often consider an extension $\mathbb{F}_{q^n} \subset \mathbb{F}$ of $\mathbb{F}_q$. We will denote by an index 0 an object over $\mathbb{F}_q$, and by an index 1 an object over $\mathbb{F}_{q^n}$. Replacing an index 0 with an index 1 (resp. removing the index) means that we extend the scalars to $\mathbb{F}_{q^n}$ (resp. to $\mathbb{F}$).

0.2

We denote by ℓ a prime number $\neq p$. We freely use the language of $\mathbb{Q}_\ell$-sheaves, as well as that of E_λ-sheaves, for E_λ a finite extension of $\mathbb{Q}_\ell$ (cf. Report, paragraph 2 and especially 2.11).

0.3

Let $H^\bullet$ be a graded vector space. If T is an endomorphism of $H^\bullet$, we set (cf. Cycle, 1.3.5)

$$\operatorname{tr}(T, H^\bullet) = \sum (-1)^i \operatorname{tr}(T, H^i).$$

1 Principles

1.1

Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$, E_λ a finite extension of $\mathbb{Q}_\ell$ and $\mathscr{F}_0$ an E_λ-sheaf on $X - 0$. The trace formula says that

$$\sum_{x \in X^F} \mathrm{tr}(F_x^*, \mathscr{F}) = \sum (-1)^i \, \mathrm{tr}\left(F^*, \mathrm{H}_c^i(X, \mathscr{F})\right). \tag{1.1.1}$$

With the notation 0.3, the right-hand side is simply $\mathrm{tr}(F^*, \mathrm{H}_c^\bullet(X, \mathscr{F}))$.

This trace formula for E_λ-sheaves can either be deduced from Report 4.10 by taking the limit (cf. Report 4.11 to 4.13), or deduced from the trace formula for $\mathbb{Q}_\ell$-sheaves (Report 3.2) by the method of (Functions L mod. ℓ^n, 4.3).

We will interpret various trigonometric sums as the left-hand side of (1.1.1), for suitable X_0 and $\mathscr{F}_0$.

1.2

Let A be a finite group. A *A-torsor* on a scheme X is a sheaf T on X, equipped with a right action of A, which, locally (for the étale topology) on X, is isomorphic to the constant sheaf A on which A acts by right translation.

If $\tau : A \to B$ is a homomorphism, and T a A-torsor, there is a unique up to isomorphism B-torsor $\tau(T)$, equipped with a sheaf morphism $\tau : T \to \tau(T)$ such that

$$\tau(ta) = \tau(t)\tau(a). \tag{1.2.1}$$

If T is an A-torsor on X, and ρ is a linear representation of A: $\rho : A \to \mathrm{GL}(V)$ (V a finite dimensional vector space over E_λ), there is a unique up to isomorphism only one E_λ-sheaf $\mathscr{F}$, smooth of rank $\dim(V)$, with a morphism of sheaves

$$\rho : T \to \underline{\mathrm{Isom}}(V, \mathscr{F}). \tag{1.2.2}$$

such that $\rho(ta) = \rho(t)\rho(a)$. We denote it $\rho(T)$. For $\tau : A \to B$ and ρ a representation of B, we have a canonical isomorphism $\rho(\tau(T)) = (\rho\tau)(T)$.

In this paragraph (except for the appendix), we only consider the case where A is commutative and where $V = E_\lambda$: ρ is a character $A \to E_\lambda^\times$, and $\rho(T)$ is a smooth E_λ-sheaf of rank one. The morphism (1.2.2) is interpreted as a morphism everywhere nonzero

$$\rho : T \to \rho(T) \tag{1.2.3}$$

such that $\rho(ta) = \rho(a)\rho(t)$.

1.3

Let S be a scheme, G a commutative group scheme over S, and G' an extension of commutative group schemes over S

$$0 \longrightarrow A \longrightarrow G' \xrightarrow{\pi} G \longrightarrow 0.$$

The sheaf T of local sections of π is then an A-torsor on G.

If X is a scheme over S, and f, g are two S-morphisms of X into G, because T is defined by an extension, we have a canonical isomorphism

$$(f+g)^*T = f^*T + g^*T. \tag{1.3.1}$$

On the left, $f+g$ is the sum of f and g, in the sense of the group law G: on the right, $+$ denotes a sum of torsors.

If $f : X \to G$ is factored by G', f^*T is trivial (and a factorisation gives a trivialisation); up to isomorphism (not unique), the A-torseur f^*T only depends on the image of f in $\hom_S(X, G)/\pi \hom_S(X, G')$: it is given by the morphism ∂ in the exact sequence

$$\hom(X, G') \longrightarrow \hom(X, G) \xrightarrow{\partial} \mathrm{H}^1(X, A).$$

We will apply these constructions to the Kummer exact sequences $0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$, and to the Lang torseurs.

1.4 The Lang torseur

Let G_0 be a connected algebraic group over $\mathbb{F}_q$ (for the non-commutative case, see the appendix). The group law is denoted *multiplicatively*. The Lang isogeny

$$\mathfrak{L} : G_0 \to G_0 : x \mapsto Fx \cdot x^{-1}$$

is étale; its image, a subgroup open of G_0, can only be G_0 itself; its kernel is the finite group $G_0(\mathbb{F}_q)$. The *Lang torseur* L is the $G_0(\mathbb{F}_q)$-torseur over G_0 defined by the exact sequence

$$0 \longrightarrow G_0(\mathbb{F}_q) \longrightarrow G_0 \xrightarrow{\mathfrak{L}} G_0 \longrightarrow 0. \tag{1.4.1}$$

Notation We still denote L and $\mathfrak{L}$ (or $L_{(q)}$ and $\mathfrak{L}_{(q)}$) the objects deduced from L and $\mathfrak{L}$ by extension of the base field. If necessary, we will specify with an index 0 or 1.

Examples For $G_0 = \mathbb{G}_a$ or $\mathbb{G}_m$, the sequences (1.4.1) are written

$$0 \longrightarrow \mathbb{F}_q \longrightarrow \mathbb{G}_a \xrightarrow{x^q - x} \mathbb{G}_a \longrightarrow 0 \tag{1.4.2}$$

$$0 \longrightarrow \boldsymbol{\mu}_{q-1} \longrightarrow \mathbb{G}_m \xrightarrow{x^{q-1}} \mathbb{G}_m \longrightarrow 0 \tag{1.4.3}$$

1.5

Calculate the endomorphism F^* of the fiber of the Lang torseur at $\gamma \in G^F = G_0(\mathbb{F}_q)$. If $g \in \mathfrak{L}^{-1}(\gamma) \subset G$, then $Fg = Fg \cdot g^{-1} \cdot g = \gamma g$. Therefore (Report, 1.2)

$$\text{on } L_0(G_0)_\gamma \simeq \mathfrak{L}^{-1}(\gamma),\ F^* \text{ is } g \mapsto g\gamma^{-1}. \tag{1.5.1}$$

1.6

On $\mathbb{F}_{q^n}$, the identity $F_{(q^n)} = F_{(q)}^n$ between endomorphisms of G_1 implies that $\mathfrak{L}_{(q^n)} = \mathfrak{L}_{(q)} \circ \prod_{i=0}^{n-1} F_{(q)}^i$. On $G_0(\mathbb{F}_{q^n})$, $F_{(q)}^i$ acts as the element $x \mapsto x^{q^i}$ of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. On $G_0(\mathbb{F}_{q^n})$, $\prod F_{(q)}^i$ is therefore the composition of the norm $G_0(\mathbb{F}_{q^n}) \to G_0(\mathbb{F}_q)$ and of the inclusion $G_0(\mathbb{F}_q) \subset G_0(\mathbb{F}_{q^n})$: the diagram

$$\begin{tikzcd}
0 \arrow[r] & G_0(\mathbb{F}_{q^n}) \arrow[r] \arrow[d, "N"] & G_1 \arrow[r, "\mathfrak{L}_{(q^n)}"] \arrow[d, "\prod_{i=0}^{n-1} F_{(q)}^i"] & G_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_0(\mathbb{F}_q) \arrow[r] & G_q \arrow[r, "\mathfrak{L}_{(q)}"] & G_1 \arrow[r] & 0
\end{tikzcd} \tag{1.6.1}$$

is commutative. In the language of torseurs, this provides a canonical isomorphism between $G_0(\mathbb{F}_q)$-torseurs on G_1

$$NL_{(q^n)} = L_{(q)}. \tag{1.6.2}$$

Définition 1.7. *Let $f_0 : X_0 \to G_0$ be a morphism and $\chi : G_0(\mathbb{F}_q) \to E_\lambda^\times$ be a character. We define $\mathscr{F}(\chi, f_0) = \chi^{-1}(f_0^*(L_0(G_0))) = f_0^*\chi^{-1}(L_0(G_0))$.*

We have the following functorial properties:

a) $\mathscr{F}(\chi, f_0)$ is bimultiplicative in χ and f_0:

$$\mathscr{F}(\chi, f_0' \cdot f_0'') = \mathscr{F}(\chi, f_0') \otimes \mathscr{F}(\chi, f_0''), \tag{1.7.1}$$

$$\mathscr{F}(\chi'\chi'', \mathscr{F}_0) = \mathscr{F}(\chi', f_0) \otimes \mathscr{F}(\chi'', f_0). \tag{1.7.2}$$

This follows from the definition of $\chi(T)$, for T a torseur, combined with (1.3.1) for (1.7.1).

b) For $g_0 : Y_0 \to X_0$ a morphism, we have

$$\mathscr{F}(\chi, f_0 \circ g_0) = g_0^*\mathscr{F}(\chi, f_0). \tag{1.7.3}$$

In particular, $\mathscr{F}(\chi, f_0) = f_0^*\mathscr{F}(\chi)$, where $\mathscr{F}(\chi)$ denotes $\mathscr{F}(\chi, \mathrm{id}_{G_0})$,

c) For $u_0 : G_0 \to H_0$ a morphism, and $\chi : H_0(\mathbb{F}_q) \to E_\lambda^\times$, we have

$$\mathscr{F}(\chi, u_0 f_0) = \mathscr{F}(\chi u_0, f_0). \tag{1.7.4}$$

d) For $G_0 = \prod_{i\in I} G_0^i$, χ of coordinates χ_i $(i \in I)$ and f_0 of coordinates f_0^i, it follows from (1.7.1) to (1.7.4) that

$$\mathscr{F}(\chi, f_0) = \bigotimes_{i\in I} \mathscr{F}(\chi_i, f_0^i). \tag{1.7.5}$$

Note the χ^{-1} in the definition 1.7. It ensures that, for $x \in X^F$, we have on $\mathscr{F}(\chi, f_0)_x$

$$F_x^* = \chi f_0(x) \tag{1.7.6}$$

(use that the fiber at x of the morphism (1.2.2), for $\rho = \chi^{-1}$, commutes with F_x^*, and (1.5.1)).

If $\mathscr{F}(\chi, f_0)_1$ is the sheaf deduced from $\mathscr{F}(\chi, f_0)$ by extension of the base field to $\mathbb{F}_{q^n}$, we deduce from (1.6.2) that

$$\mathscr{F}(\chi, f_0)_1 = \mathscr{F}(\chi \circ N, f_1). \tag{1.7.7}$$

1.8 Abuse of notation

(i) If Ξ is a notation for the composed map $\chi f_0 : X_0(\mathbb{F}_q) \to E_\lambda^\times$, we will sometimes write $\mathscr{F}(\Xi)$ instead of $\mathscr{F}(\chi, f_0)$. Thanks to (1.7.1) to (1.7.5), there is little risk of ambiguity. For example:

a) we write $\mathscr{F}(\chi)$ for $\mathscr{F}(\chi, \mathrm{id}_{G_0})$ (notations already used in 1.7b));

b) we write $\mathscr{F}(\chi f_0)$ for $\mathscr{F}(\chi, f_0)$;

c) with the notation of 1.7d), we write $\mathscr{F}(\prod \chi_i f_0^i)$ for $\mathscr{F}(\chi, f)$).

With this notation, (1.7.4) expresses that the writing $\mathscr{F}(\chi u_0 f_0)$ is not ambiguous, (1.7.1) (1.7.2) (1.7.5) express a multiplicativity of $\mathscr{F}(\Xi)$ in Ξ, and (1.7.6) is rewritten as $F_x^* = \Xi(x)$ on $\mathscr{F}(\Xi)_x$."

(ii) If X is a scheme over an extension k of $\mathbb{F}_q$, and f is a morphism from X to $G_0 \otimes_{\mathbb{F}_q} k$, we will still denote by $\mathscr{F}(\chi, f)$, $\mathscr{F}(\chi f)$, or $\mathscr{F}(\chi)$ (for f an inclusion) the image by f of the E_λ-sheaf deduced from $\chi^{-1}(L_0(G))$ by extension of the scalars from $\mathbb{F}_q$ to k. If f_0 is the composition $X \to G_0 \otimes_{\mathbb{F}_q} k \to G_0$, it is still the $\mathscr{F}(\chi, f_0)$ of 1.7. For k is a finite extension of $\mathbb{F}_q$, we have $\mathscr{F}(\chi f) = \chi(\chi N_{k/\mathbb{F}_q} f)$.

Applying (1.1.1) to (1.7.6) and (1.7.7), we find:

Théorème 1.9. *Let S_0 be a separated scheme of finite type over $\mathbb{F}_q$, G_0 a commutative connected algebraic group over $\mathbb{F}_q$, $f_0 : S_0 \to G_0$ a morphism, and $\chi : G_0(\mathbb{F}_q) \to E_\lambda^\times$. We have*

$$\sum_{s \in S_0(\mathbb{F}_q)} \chi f_0(s) = \mathrm{tr}(F^*, \mathrm{H}_c^\bullet(S, \mathscr{F}(\chi, f_0))) \tag{1.9.1}$$

and, for any integer $n \geqslant 1$

$$\sum_{s \in S_0(\mathbb{F}_{q^n})} \chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q} f_0(s) = \mathrm{tr}(F^{*n}, \mathrm{H}_c^\bullet(S, \mathscr{F}(\chi, f_0)))\,. \tag{1.9.2}$$

1.9.3 Remark

Let us take G_0 to be a product. With the notations of 1.7d) and 1.8c), the formula (1.9.2) becomes

$$\sum_{s \in S_0(\mathbb{F}_{q^n})} \prod_i \chi_i \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(f_0^i(s)) = \mathrm{tr}\left(F^{*n}, \mathrm{H}_c^\bullet\left(S, \mathscr{F}\left(\prod_i \chi_i f_0^i\right)\right)\right).$$

1.9.4 Remarque

Let $G_0 = \{e\}$. The formula (1.9.1) becomes

$$|S_0(\mathbb{F}_q)| = \sum_{s \in S_0(\mathbb{F}_q)} 1 = \mathrm{tr}(F^*, \mathrm{H}_c^\bullet(S, \mathbb{Q}_\ell)).$$

It is rare that one can explicitly calculate the right-hand side of (1.9.1). Here is an amusing example, with $G_0 = \{e\}$.

1.10 Example

A non-singular quadric projective of odd dimension over $\mathbb{F}_q$ has the same number of rational points as the projective space over $\mathbb{F}_q$ of the same dimension.

If, over $\mathbb{C}$, X is a non-singular quadric hypersurface in the projective space $\mathbb{P}^{2N}$, and Y is a hyperplane, we know that for $i \leqslant 2\dim(X) = 2\dim(Y)$, the inclusions $X \hookrightarrow \mathbb{P}^{2N} \hookleftarrow Y$ induce isomorphisms (in ordinary cohomology)

$$\mathrm{H}^i(X, \mathbb{Q}) \xleftarrow{\sim} \mathrm{H}^i(\mathbb{P}^{2N}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(Y, \mathbb{Q}).$$

By specializing, it follows that if X_0' is a non-singular quadric hypersurface in the projective space $\mathbb{P}_0^{2N}$ over $\mathbb{F}_q$, and Y_0' is a hyperplane, the inclusions $X_0' \hookrightarrow \mathbb{P}_0^{2N} \hookleftarrow Y_0'$ induce isomorphisms

$$\mathrm{H}^i(X', \mathbb{Q}_\ell) \xleftarrow{\sim} \mathrm{H}^i(\mathbb{P}^{2N}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i(Y', \mathbb{Q}_\ell).$$

These isomorphisms commute with F^*, and we apply the trace formula.

1.11

We can also use the trace formula to understand the classical formulas giving the number of rational points of linear groups over finite fields, and those of certain homogeneous spaces (see paragraph 8).

1.12

We have a dictionary that allows us to translate various types of classical manipulations on trigonometric sums into cohomological terms. This dictionary will be given in paragraph 2. The cohomological statement being more “geometric,” it can sometimes be applied to situations where the classical argument only applies after an extension of the finite base field. Here is an example (a special case of a theorem of Kazhdan).

Théorème 1.13 (Kazhdan)**.** *Let X_0 be a scheme over $\mathbb{F}_q$. Suppose there exists an action ρ of $\mathbb{G}_a$ on X, and a morphism $f : X \to Y$ of schemes over $\mathbb{F}$, making X a $\mathbb{G}_a$-torsor over Y. Then, the number of rational points of X_0 is divisible by q.*

First suppose that ρ, Y and f are defined over $\mathbb{F}_q$.

Classical argument The fibers of $f : X_0(\mathbb{F}_q) \to Y_0(\mathbb{F}_q)$ all have q elements: $|X_0(\mathbb{F}_q)| = q \cdot |Y_0(\mathbb{F}_q)|$, whence the divisibility.

Cohomological translation The higher direct image sheaves $\mathsf{R}^i f_! \mathbb{Q}_\ell$ are

$$\mathsf{R}^i f_! \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell(-1) & \text{if } i = 2 \\ 0 & \text{for } i \neq 2 \end{cases}$$

The Leray spectral sequence of f therefore degenerates into an isomorphism

$$\mathrm{H}_c^i(X, \mathbb{Q}_\ell) = \mathrm{H}_c^{i-2}(Y, \mathbb{Q}_\ell)(-1).$$

To understand the effect of a Tate twist on the eigenvalues of Frobenius, the most convenient approach is the Galois one (Report 1.8), and to note that the Frobenius substitution φ acts on $\mathbb{Q}_\ell(1)$ by multiplication by q. The eigenvalues of F^* on $\mathrm{H}^p_c(X, \mathbb{Q}_\ell)$ are therefore the products by q of the eigenvalues of F^* acting on $\mathrm{H}^{i-2}_c(Y, \mathbb{Q}_\ell)$. We know that these are algebraic integers [16, XXI 5.2.2]; therefore

$$\begin{array}{c}\text{the eigenvalues of } F^* \text{ acting on } \mathrm{H}^i_c(X, \mathbb{Q}_\ell) \text{ are} \\ \text{algebraic integers that are divisible by } q.\end{array} \tag{$*$}$$

Descent Let's go back to the hypotheses of the theorem. For a suitable n, we can suppose that ρ, Y and f are defined over $\mathbb{F}_{q^n}$. The Frobenius morphism relative to $\mathbb{F}_{q^n}$ being the n-th power of the one relative to $\mathbb{F}_q$, the statement $(*)$ for the scheme over $\mathbb{F}_{q^n}$ deduced from X_0 by extension of scalars gives us:

$$\begin{array}{c}\text{the } n\text{-th powers of the eigenvalues of } F^* \text{ acting on} \\ \mathrm{H}^i_c(X, \mathbb{Q}_\ell) \text{ are algebraic integers that are divisible by } q^n.\end{array} \tag{$**$}$$

This assertion implies $(*)$. The number of rational points of X_0, that is $\sum(-1)^i \operatorname{tr}(F^*, \mathrm{H}^i(X, \mathbb{Q}_\ell))$, is therefore an algebraic integer that is divisible by q. This implies that it is divisible by q as a rational integer.

1.14

The formula 1.9.3 controls the dependence on n of the trigonometric sum on the left-hand side. It implies identities between a trigonometric sum and those that can be deduced from it by "extension of scalars." The most famous of these identities is the one of Hasse-Davenport: let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be characters, ψ non-trivial, and define the Gauss sum $\tau(\chi, \psi)$ by

$$\tau(\chi, \psi) = -\sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi^{-1}(x). \tag{1.14.1}$$

Théorème 1.15 (Hasse-Davenport)**.** *On a*

$$\tau\left(\chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}, \psi \circ \operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}\right) = \tau(\chi, \psi)^n. \tag{1.15.1}$$

Let $E \subset \mathbb{C}$ be a number field containing the values of χ and ψ, and E_λ the completion of E at a place of characteristic $\ell \neq p$. The identity (1.15.1) is an identity in E. It is the same to prove it in $\mathbb{C}$, or in E_λ. We will prove it in E_λ, by looking at χ and ψ as having values in $E_\lambda^\times$.

We apply 1.9.3 for $X_0 = \mathbb{G}_m$ over $\mathbb{F}_q$ and $G_0 = \mathbb{G}_m \times \mathbb{G}_a$. The $\mathrm{H}^i_c(\mathbb{G}_m, \mathscr{F}(\chi^{-1}\psi))$ are zero for $i \neq 1$, and the H^1_c is of dimension 1 (3.2). According to 1.9.3, $\tau(\chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}, \psi \circ \operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ is the unique eigenvalue of $(F^*)^n$ on H^1_c, and (1.15.1) follows.

Similar examples are given in Weil [41, App V].

1.16

An E_λ-sheaf $\mathscr{F}_0$ on $X - 0$ is said to be *punctually of weight* n if for every closed point $x \in |X_0|$, the eigenvalues of F_x^* (Rapport 1.2) are algebraic numbers all of whose complex conjugates have absolute value $q_x^{n/2}$, where q_x is the number of elements of $k(x)$. The following theorem will be proved in [15].

Théorème 1.17. *If $\mathscr{F}_0$ is punctually of weight n, for every eigenvalue α of the endomorphism F^* of $\mathrm{H}^i_c(X, \mathscr{F})$, there exists an integer $m \leqslant n+i$ such that all complex conjugates of α are of absolute value $q^{m/2}$.*

The sheaves considered in 1.9 are of weight 0. The theorem therefore provides for the sum 1.9.1 (or rather, for all its complex conjugates) the upper bound

$$\left| \sum_{s \in S_0(\mathbb{F}_q)} \prod_i \chi_i(f^i(s)) \right| \leqslant \sum_i \dim \mathrm{H}^i_c \left(S, \mathscr{F} \left(\prod \chi_i f^i \right) \right) \cdot q^{i/2}. \tag{1.17.1}$$

1.18 Remarques

a) If $\dim S = n$, for every E_λ-sheaf $\mathscr{F}$ on S, we have

$$\mathrm{H}^i_c(S, \mathscr{F}) = 0 \qquad \text{for } i \notin [0, 2n].$$

b) The group $\mathrm{H}^0_c(X, \mathscr{F})$ is the group of global sections of $\mathscr{F}$ on S whose support is proper. If S is connected, non-complete and $\mathscr{F}$ is smooth, then if S is connected, $\mathscr{F}$ is smooth of rank one, and not constant, this group is zero.

c) If S is purely of dimension n, and $\mathscr{F}$ is smooth, Poincaré duality says that the vector spaces $\mathrm{H}^i_c(X, \mathscr{F})$ and $\mathrm{H}^{2n-i}(X, \mathscr{F}^\vee)(2n)$ are dual to each other (for the effect of a Tate twist on the eigenvalues of Frobenius, see the second part of the proof of 1.13).

d) If $\dim S = n$, Poincaré duality allows us to compute $\mathrm{H}^{2n}_c(S, \mathscr{F})$ as follows: we take an open U of S_{red}, whose complement is of dimension $< n$, purely of dimension n, and on which $\mathscr{F}$ is smooth. We then have $\mathrm{H}^{2n}_c(U, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{2n}_c(S, \mathscr{F})$, because in the long exact sequence of cohomology, $\mathrm{H}^i_c(S \smallsetminus U, \mathscr{F}) = 0$ for $i = 2n, 2n-1$. By Poincaré, we therefore have

$$\mathrm{H}^{2n}_c(S, \mathscr{F}) \xrightarrow{\sim} \left(\mathrm{H}^0(U, \mathscr{F}^\vee)(2n) \right)^\vee.$$

Suppose U is connected, and let u be a geometric point of U. The "local system" $\mathscr{F}$ corresponds to a representation of $\pi_1(U, u)$ on $\mathscr{F}_u$, and $\mathrm{H}^0(U, \mathscr{F}^\vee)$ is $(\mathscr{F}^\vee_u)^{\pi_1(U,u)} = ((\mathscr{F}_u)_{\pi_1(U,u)})^\vee$ (the dual of coinvariants). We therefore have

$$\mathrm{H}^{2n}_c(S, \mathscr{F}) \simeq (\mathscr{F}_u)_{\pi_1(U,u)}(-2n).$$

e) These remarks often allow the calculation of b_0 and b_{2n}. Calculating the other b_i can be difficult. If $b_{2n} = 0$ and $b_{2n-1} = 0$, the bound (1.17.1) is a Lang-Weil bound, with, for q large, a gain in $\sqrt{q}$ compared to the trivial bound in $O(q^n)$. However, note that the bound (1.17.1) does not formally imply the trivial bound

$$\left| \sum_{s \in S_0(\mathbb{F}_q)} \prod_i \chi_i(f^i(s)) \right| \leqslant |S_0(\mathbb{F}_q)|,$$

so it can be useful to take into account (cf. the proof of 2.16).

I explain below a method for proving that $b_i = 0$ for $i \neq n$. When it applies, it provides an estimate in $O(q^{n/2}$. The implicit constant in 0 is $b_n = (-1)^n \chi(S, \mathscr{F}(\prod \chi_i f^i))$ which can be difficult to calculate.

Proposition 1.19. *Let $\bar{X}$ be a proper scheme over an algebraically closed field k, $j : X \hookrightarrow barX$ an open of X and $\mathscr{F}$ an E_λ-sheaf on X. We assume that*

a) $(j_*\mathscr{F})_x = 0$ *for* $x \in \bar{X} \smallsetminus X$, *i.e.* $j_!\mathscr{F} \xrightarrow{\sim} j_*\mathscr{F}$;

b) $\mathsf{R}^i j_*\mathscr{F} = 0$ *for* $i > 0$.

Then, the maps

$$\mathrm{H}^i_c(X, \mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F})$$

are isomorphisms.

The hypotheses a) b) mean that $j_!\mathscr{F} \xrightarrow{\sim} \mathsf{R}j_*\mathscr{F}$, whence

$$\mathrm{H}^i_c(X, \mathscr{F}) = \mathrm{H}^i(\bar{X}, j_!\mathscr{F}) \xrightarrow{\sim} \mathsf{H}^i(\bar{X}, \mathsf{R}j_*\mathscr{F}) = \mathrm{H}^i(X, \mathscr{F}).$$

In other words, the Leray spectral sequence for j

$$E_2^{pq} = \mathrm{H}^p(\bar{X}, \mathsf{R}^q j_*\mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

degenerates, by b) into isomorphisms

$$\mathrm{H}^i(\bar{X}, j_*\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X, \mathscr{F}),$$

while, by a)

$$\mathrm{H}^i_c(X, \mathscr{F}) = \mathrm{H}^i(\bar{X}, j_!\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(\bar{X}, j_*\mathscr{F}).$$

1.19.1 Example

If X is a curve, or if $\bar{X}$ is smooth, X is the complement of a divisor with normal crossings and $\mathscr{F}$ is moderately ramified, the hypothesis a) of 1.19 (no invariants under local monodromy) implies hypothesis b).

Proposition 1.20. *Let X_0 be a smooth affine scheme of pure dimension n over $\mathbb{F}_q$ and $\mathscr{F}_0$ be a locally constant sheaf of weight m on X_0. We assume that the maps $\mathrm{H}^i_c(X, \mathscr{F}) \to \mathrm{H}^i(X, \mathscr{F})$ are isomorphisms (cf. 1.17). Then, $\mathrm{H}^i_c(X, \mathscr{F}) = 0$ for $i \neq n$, and the complex conjugates of the eigenvalues of F^* on $\mathrm{H}^n_c(X, \mathscr{F})$ are of absolute value $q^{(m+n)/2}$.*

Poincaré duality puts in duality $\mathrm{H}^i_c(X, \mathscr{F})$ (resp. $\mathrm{H}^i_c(X, \mathscr{F}^\vee)$) with $\mathrm{H}^{2n-i}(X, \mathscr{F}^\vee(n))$ (resp. $\mathrm{H}^{2n-i}(X, \mathscr{F}(n))$). According to [2, XIV 3.2], the H^i without support are zero for $i > n$. By duality, the H^i_c are zero for $i < n$; since $\mathrm{H}^i_c(X, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X, \mathscr{F})$, these groups are zero for $i \neq n$. Apply 1.15 to $\mathscr{F}_0$ and to $\mathscr{F}_0^\vee(n)$ (of weight $-m-2n$). We find that the complex conjugates α of the eigenvalues of F^* on $\mathrm{H}^n_c(X, \mathscr{F}) = \mathrm{H}^n_c(X, \mathscr{F}^\vee(n))^\vee$ satisfy

$$|\alpha| \leqslant q^{(m+n)/2} \qquad \text{and}$$
$$|\alpha^{-1}| \leqslant q^{((-m-2n)+n)/2} = q^{-(m+n)/2},$$

from which the assertion follows.

1.21 Remarque

The last argument shows that if X_0 is a separated smooth scheme over $\mathbb{F}_q$, $\mathscr{F}_0$ is a locally constant smooth sheaf of weight m on X_0, and $\mathrm{H}^i_c(X, \mathscr{F}) \hookrightarrow \mathrm{H}^i(X, \mathscr{F})$, then the complex conjugates of the eigenvalues of F^* on $\mathrm{H}^i_c(X, \mathscr{F})$ are of absolute value $q^{(m+i)/2}$.

Appendix. The Lang isogeny in the non-commutative case

1.22

Let $G - 0$ be a connected algebraic group over $\mathbb{F}_q$ and $\mathfrak{L}$ the map $x \mapsto Fx \cdot x^{-1}$ from $G - 0$ to itself. It is an isomorphism of homogeneous G_0-spaces, if at the source we let $G - 0$ act by left translation $x \mapsto (y \mapsto xy)$, at the target by $x \mapsto (y \mapsto Fx \cdot y \cdot x^{-1})$. We have $\mathfrak{L}(xy) = \mathfrak{L}(x)$ for $y \in G_0(\mathbb{F}_q)$, and $\mathfrak{L}$ induces an isomorphism $G_0/G_0(\mathbb{F}_q) \xrightarrow{\sim} G_0$.

As in the commutative case, $\mathfrak{L}$ therefore makes G_0 a $G_0(\mathbb{F}_q)$-torsor over G_0, the *Lang torsor L*. If $\rho : G_0(\mathbb{F}_q) \to \mathrm{GL}(n, E_\lambda)$ is a linear representation of $G_0(\mathbb{F}_q)$, we denote by $\mathscr{F}(\rho)$ the E_λ-sheaf $\rho(L)$ (1.2).

1.23

Let $\gamma \in G^F = G_0(\mathbb{F}_q)$, and calculate the endomorphism F^* of $\mathscr{F}(\rho)_\gamma$. Let $g \in G$ be such that $\mathfrak{L}(g) = \gamma$, hence a basis $\rho(g) \in \mathrm{Isom}(E_\lambda^n, \mathscr{F}(\rho)_\gamma)$. For $e \in E_\lambda^n$, we have

$$F(\rho(g)(e)) = \rho(Fg)(e) = \rho(\gamma g)(e) = \rho(g \cdot g^{-1}\gamma g)(e).$$

We will see that $g^{-1}\gamma g \in G_0(\mathbb{F}_q)$, hence

$$F(\rho(g)(e)) = \rho(g)\rho(g^{-1}\gamma g)(e)$$

and

$$\rho(g)^{-1}(F_\gamma^*)^{-1}\rho(g) = \rho(g^{-1}\gamma g): \tag{1.23.1}$$

$\rho(g)$ identifies the inverse of F^* at γ with the automorphism $\rho(g^{-1}\gamma g)$ of E_λ^n. It remains to understand what $g^{-1}\gamma g$ is.

Lemme 1.24. *(i) If $Fg \cdot g^{-1} \in G_0(\mathbb{F}_q)$, then $g^{-1}(Fg \cdot g^{-1})g = g^{-1}Fg \in G_0(\mathbb{F}_q)$.*

(ii) Let $\gamma \in G_0(\mathbb{F}_q)$, and g such that $Fg \cdot g^{-1} = \gamma$. The conjugacy class of $\gamma' = g^{-1}Fg$ in $G_0(\mathbb{F}_q)$ only depends on that of γ.

(iii) The map induced by $\gamma \mapsto \gamma'$, from the set $G_0(\mathbb{F}_q)^\natural$ of conjugacy classes of $G_0(\mathbb{F}_q)$ in itself, is bijective. Its inverse is the map for the opposite group.

(i) If $\gamma = Fg \cdot g^{-1} \in G_0(\mathbb{F}_q)$, the identities $Fg = \gamma g$ and $F\gamma = \gamma$ give $F(g^{-1}Fg) = (Fg)^{-1}FFg = g^{-1}\gamma^{-1}F(\gamma g) = g^{-1}\gamma^{-1}F\gamma Fg = g^{-1}Fg$, so that $g^{-1}Fg \in G_0(\mathbb{F}_q)$.

(ii) For $\alpha, \gamma \in G_0(\mathbb{F}_q)$, the g such that $Fg \cdot g^{-1} = \gamma$ form a right class under $G_0(\mathbb{F}_q)$, and if $Fg \cdot g^{-1} = \gamma$, then $F(\alpha g\alpha^{-1})(\alpha g\alpha^{-1})^{-1} = \alpha\gamma\alpha^{-1}$. As γ runs through a conjugacy class in $G_0(\mathbb{F}_q)$, the g such that $Fg \cdot g^{-1} = \gamma$ therefore run through a double class under $G_0(\mathbb{F}_q)$, and $\gamma' = g^{-1}Fg$ runs through a conjugacy class.

(iii) Finally, the inverse of $Fg \cdot g^{-1} \mapsto g^{-1}Fg$ is $g^{-1}Fg \mapsto Fg \cdot g^{-1}$.

1.25

The formula (1.23.1) can be read briefly as: F^* on γ is $\rho(\gamma')^{-1}$. If γ belongs to the neutral component of its centralizer, we can take g in $Z(\gamma)$. We then have $\gamma = \gamma'$. If this condition is not met, γ and γ' may not be conjugate.

1.26 An example

For $G_0 = \mathrm{SL}(2)$, q odd, $\alpha \in F_q^\times \smallsetminus \mathbb{F}_q^{\times 2}$ and $\bar{\gamma}$ the conjugacy class of $-1(1+N)$, with $N^2 = 0$, $\bar{\gamma}'$ is the conjugacy class of $-1 \cdot (1 + \alpha N)$, so that $\bar{\gamma}' \neq \bar{\gamma}$ if $N \neq 0$.

2 Dictionary

A elementary manipulation of trigonometric sums can often be seen as the reflection, via the trace formula, of a cohomological statement. In this paragraph, I list such statements, and their reflection; they have the same number, with a $*$ added for the cohomological statement.

2.1

"A sum of algebraic integers is an algebraic integer" has for cohomological analogue the following theorem, proved in [16, XXI 5.2.2] (cf. the use of 2.1^* in 1.13).

Théorème (2.1*)**.** *Let X_0 be a finite type separated scheme over $\mathbb{F}_q$ and $\mathscr{F}_0$ a E_λ-sheaf on X_0. If, for all $x \in |X_0|$, the eigenvalues of F_x^* on $\mathscr{F}_0$ are algebraic integers, then the eigenvalues of F^* on $\mathrm{H}_c^i(X, \mathscr{F})$ are algebraic integers.*

2.2

Other results of integrals are proven in [16, XXI, 5.2.2 and 5.4]: if $\dim(x_0) \leqslant n$, and α is an eigenvalue of F^* on $\mathrm{H}_c^i(X, \mathscr{F})$, then:

a) under the hypotheses of 2.1^*, and if $i > n$, then q^{i-n} divides α;

b) if the inverses of the eigenvalues of F_x^* are algebraic integers, then α^{-1} is an integer, except at p. More precisely, $q^{\inf(n,i)}\alpha^{-1}$ is an algebraic integer.

For the sheaves considered in 1.9, the eigenvalues of the F_x^* are roots of unity, so the hypotheses of the theorem and those of b) above are verified.

2.3

Let $f : A \to B$ be a map from a finite set to another, and ε a function on A. We have

$$\sum_{a \in A} \varepsilon(a) = \sum_{b \in B} \sum_{f(a)=b} \varepsilon(a). \tag{2.3.1}$$

2.3*

Let $f : X \to Y$ be a morphism of separated schemes of finite type over k algebraically closed, and $\mathscr{F}$ an E_λ-sheaf on X. The Leray spectral sequence of f in cohomology with proper support is

$$E_2^{pq} = \mathrm{H}_c^p(Y, \mathsf{R}^q f_! \mathscr{F}) \Rightarrow \mathrm{H}_c^{p+q}(X, \mathscr{F}). \tag{2.3.1*}$$

Let $f_0 : X_0 \to Y_0$ be a morphism of separated schemes of finite type over $\mathbb{F}_q$ and $\mathscr{F}_0$ an E_λ-sheaf on X_0. We have the following compatibility between (2.3.1) and (2.3.1*).

a) The sheaf $\mathsf{R}^q f_!\mathscr{F}$ is obtained by extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$ of the sheaf $\mathsf{R}^q f_{0!}\mathscr{F}_0$ on Y_0. As such, it is equipped with a Frobenius correspondence $F^* : F^*\mathsf{R}^q f_!\mathscr{F} \to \mathsf{R}^q f_!\mathscr{F}$. It is obtained by functoriality of $\mathsf{R}f_!$ from the Frobenius correspondence of $\mathscr{F}$: it is the composition

$$F^*\mathsf{R}^q f_!\mathscr{F} \xrightarrow{\sim} \mathsf{R}^q f_! F^*\mathscr{F} \xrightarrow{\sim} \mathsf{R}^q f_!\mathscr{F}.$$

b) The trace formula for $\mathscr{F}_q$ is written

$$\sum_{x\in X_0(\mathbb{F}_q)} \operatorname{tr}(F_x^*, \mathscr{F}_0) = \operatorname{tr}(F^*, \mathrm{H}^\bullet(X, \mathscr{F})); \tag{1}$$

that for $\mathsf{R}^q f_!\mathscr{F}$ is written

$$\sum_{y\in Y_0(\mathbb{F}_q)} \operatorname{tr}(F_y^*, \mathsf{R}^q f_!\mathscr{F}) = \operatorname{tr}(F^*, \mathrm{H}^\bullet(Y, \mathsf{R}^q f_!\mathscr{F})). \tag{2_q}$$

For $y \in Y(\mathbb{F})$, $(\mathsf{R}^q f_!\mathscr{F})_y = \mathrm{H}_c^q(f^{-1}(y), \mathscr{F})$, and if $y \in Y_0(\mathbb{F}_q)$,

$$\operatorname{tr}(F_y^*, \mathsf{R}^q f_!\mathscr{F}) = \operatorname{tr}(F^*, \mathrm{H}_c^q(f^{-1}(y), \mathscr{F})),$$

from which by the trace formula on the fiber

$$\sum_{\substack{x\in X_0(\mathbb{F}_q)\\ f(x)=y}} \operatorname{tr}(F_x^*, \mathscr{F}_0) = \sum (-1)^q \operatorname{tr}(F_y^*, \mathsf{R}^q f_!\mathscr{F}).$$

Taking the alternating sum of the identities (2_q), we thus find

$$\sum_{y\in Y_0(\mathbb{F}_q)} \sum_{\substack{x\in X_0(\mathbb{F}_q)\\ f(x)=y}} \operatorname{tr}(F_x^*, \mathscr{F}_0) = \operatorname{tr}(F^*, \mathrm{H}^\bullet(Y, \mathsf{R}^\bullet f_!\mathscr{F})). \tag{2}$$

The equality of the left members of (1) and (2) results from (2.3.1), that of the right members of (2.3.1*) (taking into account that F^* is an endomorphism of the whole spectral sequence).

2.4

Let $(A_i)_{i\in I}$ be a finite family of finite sets, $A = \prod_{i\in I} A_i$, ε_i a function on A_i, and ε the function $a \mapsto \prod \varepsilon_i(a_i)$ on A. We have

$$\sum_{a\in A} \varepsilon(a) = \prod_{i\in I} \sum_{a_i\in A_i} \varepsilon_i(a_i). \tag{2.4.1}$$

2.4*

Let $(X_i)_{i\in I}$ be a finite family of separated schemes of finite type over an algebraically closed field k, $X = \prod_{i\in I} X_i$, $\mathscr{F}_i$ an E_λ-sheaf on X_i and $\mathscr{F}$ the exterior product $\boxtimes_{i\in I} \mathscr{F}_i = \bigotimes_{i\in I} \mathrm{pr}_i^*(\mathscr{F}_i)$ of the $\mathscr{F}_i$. We have the Künneth formula

$$\mathrm{H}_c^\bullet(X, \mathscr{F}) = \bigotimes_{i\in I} \mathrm{H}_c^\bullet(X_i, \mathscr{F}_i). \tag{2.4.1*}$$

If we want a canonical isomorphism (2.4.1*), and one that is independent of the choice of a total order on I, we need to take on the right hand side the graded tensor product according to the Koszul rule (Cycle, 1.3).

2.5

Let A be a finite set, B a subset of A and ε a function on A. We have

$$\sum_{a\in A}\varepsilon(a)=\sum_{a\in A\smallsetminus B}\varepsilon(a)+\sum_{a\in B}\varepsilon(a). \tag{2.5.1}$$

2.5*

Let x be a separated scheme of finite type over an algebraically closed field k, Y a closed sub-scheme, and $\mathscr{F}$ an E_λ-sheaf on X. We have a long exact sequence of cohomology

$$\cdots\xrightarrow{\partial}\mathrm{H}^i_c(X\smallsetminus Y,\mathscr{F})\longrightarrow\mathrm{H}^i_c(X,\mathscr{F})\longrightarrow\mathrm{H}^i_c(Y,\mathscr{F})\xrightarrow{\partial}\cdots \tag{2.5.1*}$$

More generally, if we start with a finite filtration of X by closed (resp. open) sub-schemes X_p ($p\in\mathbb{Z}$; we assume that $X_p\supset X_{p+1}$, that $X_p=X$ for p small enough, and that $X_p=\varnothing$ for p large enough), we have a spectral sequence

$$E_1^{pq}=\mathrm{H}_c^{p+1}(X_p\smallsetminus X_{p+1},\mathscr{F})\Rightarrow\mathrm{H}_c^{p+q}(X,\mathscr{F}). \tag{2.5.2*}$$

2.6

Let A be a finite set, $(A_i)_{i\in I}$ a finite cover of A, and ε a function on A. For $J\subset I$, let A_J be the intersection of the A_j ($j\in J$). We have

$$\sum_{J\subset I}(-1)^{|J|}\sum_{a\in A_J}\varepsilon(a)=0. \tag{2.6.1}$$

2.6*

Let X be a separated scheme of finite type over an algebraically closed field k, $(X_i)_{i\in I}$ a closed (resp. open) cover of X, and $\mathscr{F}$ an E_λ-sheaf on X. For $J\subset I$, let X_J be the intersection of the X_j ($j\in J$). We have spectral sequences

$$E_1^{pq}=\bigoplus_{|J|=p+1>0}\mathrm{H}^q_c(X_J,\mathscr{F})\otimes_{\mathbb{Z}}\bigwedge^{|J|}\mathbb{Z}^J\Rightarrow\mathrm{H}_c^{p+q}(X,\mathscr{F}) \tag{2.6.1*}$$

$$E_1^{pq}=\bigoplus_{|J|=1-p>0}\mathrm{H}^q_c(X_J,\mathscr{F})\otimes_{\mathbb{Z}}\bigwedge^{|J|}\mathbb{Z}^J\Rightarrow\mathrm{H}_c^{p+q}(X,\mathscr{F}) \tag{2.6.2*}$$

If $J=\varnothing$ was not excluded, we would have similar spectral sequences converging to 0. The factors $\bigwedge^{|J|}\mathbb{Z}^J$ are there to save us from having to choose a total order on I.

2.7

Let A be a finite commutative group, and $\chi: A\to E_\lambda^\times$ a non-trivial character. We have

$$\sum_{a\in A}\chi(a)=0. \tag{2.7.1}$$

We will prove the cohomological analogue of (2.7.1) by transposing the following proof: if $x \in A$, $a \mapsto xa$ is a permutation of A, and

$$\sum_{a\in A}\chi(a) = \sum_{a\in A}\chi(xa) = \chi(x)\sum_{a\in A}\chi(a) \qquad , \text{ so that}$$
$$(\chi(x)-1)\sum_{a\in A} = 0,$$

and there exists by hypothesis x such that $\chi(x) - 1 \neq 0$.

Théorème (2.7*). *Let G_0 be a connected commutative algebraic group over $\mathbb{F}_q$ and $\chi : G_0(\mathbb{F}_q) \to E_\lambda^\times$ a non-trivial character. We have*

$$\mathrm{H}_c^\bullet(G, \mathscr{F}(\chi)) = 0. \tag{2.7.1*}$$

For x a rational point of G, let t_x be the translation $t_x(g) = xg$ of G. The formula $\mathfrak{L}t_x = t_{\mathfrak{L}(x)}\mathfrak{L}$ expresses that $(t_x, t_{\mathfrak{L}(x)})$ is an automorphism of the diagram $G \xrightarrow{\mathfrak{L}} G$ (G equipped with the Lang torsor). Let $\rho(g)$ be the automorphism of $(G, \mathscr{F}(\chi))$ which results from it. For $g \in G_0(\mathbb{F}_q)$, i.e. for $\mathfrak{L}(g) = e$, it is the identity on G, and the multiplication by $\chi(g)^{-1}$ on $\mathscr{F}(\chi)$.

Let $\rho_H(g)$ be the automorphism of $\mathrm{H}_c^\bullet(G, \mathscr{F}(\chi))$ deduced from $\rho(g)$. A homotopy argument (Lemma 2.8 below) shows that $\rho_H(g) = \rho_H(e)$, so it is the identity. On the other hand, for $g \in G_0(\mathbb{F}_q)$, $\rho_H(g)$ is the multiplication by $(\chi^{-1}(g):$ on $\mathrm{H}^\bullet(G, \mathscr{F}(\chi))$, the multiplication by $(\chi^{-1}(g) - 1)$ is zero. Taking g such that $\chi(g) \neq 1$, we obtain 2.7.

Lemme 2.8. *Let X and Y be two schemes over k that are algebraically closed, with X being of finite type and Y being connected. Let $\mathscr{F}$ be a sheaf on X and (ρ, ε) a family of endomorphisms of $(X, \mathscr{F})$ parametrized by Y:*

$$\begin{aligned} &\rho : Y \times_k X \to Y \times_k X && \text{is a } Y\text{-morphism, and} \\ &\varepsilon : \rho^*\mathrm{pr}_2^*\mathscr{F} \to \mathrm{pr}_2^*\mathscr{F} && \text{is a morphism of sheaves.} \end{aligned}$$

We assume that ρ is proper. For $y \in Y(k)$, let $\rho_H(y)^*$ be the endomorphism of $\mathrm{H}_c^\bullet(X, \mathscr{F})$ induced by $\rho_y : X \to X$ and $\varepsilon_y : \rho_y^*\mathscr{F} \to \mathscr{F}$. Then, $\rho_H(y)^*$ is independent of y.

Indeed, $\mathsf{R}^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathscr{F}$ is the sheaf constant on Y with value $\mathrm{H}^p(X, \mathscr{F})$, and $\rho_H(y)^*$ is the fiber at y of the endomorphism

$$\mathsf{R}^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathscr{F} \xrightarrow{\rho^*} \mathsf{R}^p\mathrm{pr}_{1!}\rho^*\mathrm{pr}_2^*\mathscr{F} \xrightarrow{\varepsilon} \mathsf{R}^p\mathrm{pr}_{1!}\mathrm{pr}_2^*\mathscr{F}$$

of this sheaf.

2.9

Weil is the first to have applied "cohomological" methods to the study of trigonometric sums with one variable; since he had proven the analogue of the Riemann hypothesis for Artin L-functions on function fields, these methods provided him with excellent estimations (in O(square root of the number of terms)), and the behavior of these sums by "extension of the base field."

The essentials of this paragraph are exposed in a cohomological language, of its results.

2.10

Cohomological methods lead here to calculate groups $\mathrm{H}^i_c(X, \mathscr{F})$, for $\mathscr{F}_0$ an E_λ-sheaf on a curve $X - 0$ on $\mathbb{F}_q$. These groups are null for $i \neq 0, 1, 2$ and for $i = 0, 2$, they have a simple interpretation (1.8 a,b,c). In addition, the Euler-Poincaré characteristic $\chi_c(\mathscr{F}) = \sum(-1)^i \dim \mathrm{H}^i_c(X, \mathscr{F})$ (also equal to $\chi(\mathscr{F}) = \sum(-1)^i \dim \mathrm{H}^i(X, \mathscr{F})$ can be calculated in terms of x, the rank of $\mathscr{F}$ at the generic points of X, and the ramification properties of $\mathscr{F}$. The essential result is the following.

Let $\bar{X}$ be a smooth, connected, projective curve over an algebraically closed field k, X a dense open subset of $\bar{X}$, the complement of a finite set S of points and $\mathscr{F}$ a smooth E_λ-sheaf on X. Let $\chi(X) = 2 - 2g - \#S$ be the Euler-Poincaré characteristic of X, and $\mathrm{rg}(\mathscr{F})$ the rank of $\mathscr{F}$. For each point $s \in S$, we define an integer $\mathrm{Sw}_s(\mathscr{F})$, the Swan conductor, measuring the wild ramification of $\mathscr{F}$, and

$$\chi_c(\mathscr{F}) = \mathrm{rg}(\mathscr{F}) \cdot \chi(X) - \sum_{s \in S} \mathrm{Sw}_s(\mathscr{F}) \tag{3.2.1}$$

[28].

2.11

Let $\bar{X}$, $j : X \hookrightarrow \bar{X}$ and $\mathscr{F}$ be as in 2.10. The Poincaré duality theorem takes the very manageable form: $\mathrm{H}^i(\bar{X}, j_*\mathscr{F})$ and $\mathrm{H}^{2-i}(\bar{X}, j_*(\mathscr{F}^\vee(1)))$ are dual to each other (Dualité, 1.3).

2.12

Let $k = \mathbb{F}$, and suppose that X, $\bar{X}$, and $\mathscr{F}$ come from X_0, $\bar{X}_0$ and $\mathscr{F}_0$ on $\mathbb{F}_q$. If $\mathscr{F}_0$ becomes trivial on a finite covering of X_0, Weil proved that the complex conjugates of the eigenvalues of F^* on $\mathrm{H}^i(\bar{X}, j_*\mathscr{F})$ are of absolute value $q^{i/2}$. Such sheaves $\mathscr{F}_0$ correspond to Artin L-functions on the function field of $\bar{X}_0$.

3.5

Under the same hypotheses, or more generally if E_λ is the completion at a place λ of a number field E and if $\mathscr{F}_0$ belongs to an infinite compatible system of v-adic representations (v a place of E), one can compute

$$\prod_i \det(-F^*, \mathrm{H}^i(X, \mathscr{F}))^{(-1)^i}$$

from *local* information on $\mathscr{F}_0$. This is explained in [13]. For $\mathscr{F}_0$ trivial on a finite covering of X_0, this is the expression of the constant term of the functional equation of an Artin L-function as a product of local constants.

2.13 Example

Let ψ be the character $\exp\left(\frac{2\pi i}{p}\,\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}\right)$ of $\mathbb{F}_q$; we have $\psi(a^p - a) = 0$. Let X_0 be a smooth absolutely irreducible projective curve of genus g over $\mathbb{F}_q$, and let f be a rational function on

X_0, i.e. a morphism $f : X_0 \to \mathbb{P}^1$, not identically equal to ∞. We are interested in the sum

$$S'_f = \sum_{\substack{x \in X_0(\mathbb{F}_q) \\ f(x) \neq \infty}} \psi(f(x)).$$

This sum is zero for f of the form $g^p - g$. This suggests modifying the sum S'_f as follows:

a) For every closed point x of X_0, we define $v_x(f)$ = order of the pole of f at x if $f(x) = \infty$, $v_x(f) = 0$ otherwise, and $v^*_x(f) = \inf v_x(f + g^p - g)$ (lower bound on g).

b) If $v^*_x(f) = 0$, $x \in X_0(\mathbb{F}_q)$, and $f + g^p - g$ is regular at x, we define $\psi(f(x)) = \psi((f + g^p - g)(x))$. We finally define

$$S_f = \sum_{x \in X_0(\mathbb{F}_q)}{}' \psi(f(x)), \tag{3.5.1}$$

where $(-)'$ indicates that the sum is extended to those x such that $v^*_x(f) = 0$.

It is proposed to verify that if f is not of the form $g^p - g + c^{te}$, then

$$|S_f| \leqslant \left(2g - 2 + \sum_{v^*_x(f) \neq 0} [k(x) : \mathbb{F}_q](1 + v^*_x(f)) \right) q^{1/2}. \tag{3.5.2}$$

We begin by going from the complex to the ℓ-adic, by replacing ψ with a character of the form $\psi_0 \circ \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}$, with $\psi_0 : \mathbb{F}_p \hookrightarrow E^\times_\lambda$. For $j : U_0 \hookrightarrow X_0$ the inclusion of the open where $f \neq \infty$, we then prove that

$$S_f = \sum (-1)^i \,\mathrm{tr}(F^*, \mathrm{H}^i(X, j_*\mathscr{F}(\psi f))). \tag{3.5.3}$$

The trace formula reduces this statement to the following (for $x \in X_0(\mathbb{F}_q)$):

a) If $v^*_x(f) = 0$, then F^*_x on $j_*\mathscr{F}(\psi f)$ is $\psi(f(x))$;

b) If $v^*_x(f) \neq 0$, then $j_*\mathscr{F}(\psi f)$ ramifies in x: $(j_*\mathscr{F}(\psi f))_{\bar{x}} = 0$.

The formula a) follows from (1.7.6) if $v_x(f) = 0$, and we may assume this case by replacing f with $f + g^p - g$: up to isomorphism, this does not change $\mathscr{F}(\psi f)_0$ on the open where f and g are regular (1.3).

The formula b) follows from

$$\mathrm{Sw}_x\, \mathscr{F}(f, \psi) = v^*_x(f). \tag{3.5.4}$$

After reducing to the case where $v_x(f) = v^*_x(f)$ (hence $p \nmid v_x(f)$) and extending scalars to $\mathbb{F}$, this formula is in [32, 4.4].

Finally, we check that the sheaf $\mathscr{F}(\psi f)$ on U is non-constant if f is not of the form $g^p - g + c^{te}$: since ψ_0 is injective, the triviality of $\mathscr{F}(\psi f) = \mathscr{F}(\psi_0 f)$ (1.8.ii) is equivalent to that of the $\mathbb{F}_q$-torsor of equation would be the inverse image of a $\mathbb{F}_q$-torsor on $\mathrm{Spec}(\mathbb{F}_q)$, of equation $T^p - T - \lambda = 0$, the $\mathbb{F}_p$-torsor of equation $T^p - T - (f - \lambda) = 0$ on U_0 would be trivial and f would therefore be of the form $g^p - g + \lambda$. The $\mathrm{H}^i(\bar{X}, j_*\mathscr{F}(\psi f))$ are therefore zero for $i \neq 1$, the formula (3.5.3) reduces to

$$S_f = -\,\mathrm{tr}(F^*, \mathrm{H}^1(\bar{X}, j_*\mathscr{F}(\psi f))) \tag{3.5.5}$$

and, according to (3.2.1) and (3.5.4), of H^1 is of dimension $2g - 2 + \sum_{v^*_x(f) > 0} [k(x) : \mathbb{F}_q](1 + v^*_x(f))$ and $-S_f$ is the sum of this number of eigenvalues of F^*, each with absolute value complex $q^{1/2}$.

2.14

Suppose that, for an automorphism σ of X_0, we have

$$f(\sigma x) = -f(x).$$

Then the sum S_f is real. If σ is involutive, Poincaré duality allows us to say a little more: if we set $\mathscr{F}_0 = j_*\mathscr{F}(\psi f)$, $\mathscr{G}_0 = j_*\mathscr{F}(\psi(-f))$, we have

a) $\mathscr{F}_0$ and $\mathscr{G}_0$ are in duality,

b) $\sigma^*\mathscr{F}_0 \xrightarrow{\sim} \mathscr{G}_0$ and $\sigma^*\mathscr{G}_0 \xrightarrow{\sim} \mathscr{F}_0$, for natural isomorphisms such that the composition $\mathscr{F}_0 = (\sigma^2)^*\mathscr{F}_0 \xrightarrow{\sim} \sigma^*\mathscr{G}_0 \xrightarrow{\sim} \mathscr{F}_0$ is the identity.

c) The pairing $\mathscr{F}_0 \otimes \mathscr{G}_0 \to E_\lambda$ satisfies $\sigma^*(f \cdot g) = \sigma^*(f) \cdot \sigma^*(g)$.

Moving to cohomology, we find that $\mathrm{H}^1(X, \mathscr{F})$ and $\mathrm{H}^1(X, \mathscr{F})$ are in perfect duality with values in $E_\lambda(1)$, and that the bilinear form $\alpha \cdot \sigma^*\beta$ on $\mathrm{H}^1(X, \mathscr{F})$ is *alternating*: σ^* acts trivially on $E_\lambda(1)$, and

$$\alpha \cdot \sigma^*\beta = \sigma^*(\alpha \cdot \sigma^*\beta) = \sigma^*\alpha \cdot \beta = -\beta \cdot \sigma^*\alpha.$$

We conclude that $\mathrm{H}^1(X, \mathscr{F})$ is of even dimension (we also easily verify that each $v_x^*(f)$ which is not zero is odd) and that the eigenvalues of F^* on $\mathrm{H}^1(X, \mathscr{F})$ come in pairs α and q/α.

2.15 Example

This applies to Kloosterman sums $\sum_{x \in \mathbb{F}_q^\times} \psi\left(x + \frac{a}{x}\right)$ (take $X_0 = \mathbb{P}^1$, $f = x + \frac{a}{x}$, $\sigma x = -x$; the poles of f are 0 and ∞, and in each of them $v_x^*(f) = 1$; the H^1 is of dimension $-2+2+2 = 2$). We therefore have (for $a \neq 0$)

$$\sum_{\substack{xy=a \\ x,y \in \mathbb{F}_{q^n}}} \exp\left(\frac{2\pi i}{p} \operatorname{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_p}(x+y)\right) = (-\alpha^n + \alpha^{-n}) \qquad , \alpha\bar{\alpha} = q.$$

This result is due to L. Carlitz [9].

Here now is an application of a method of Lang-Weil.

Proposition 2.16. *Let $P \in \mathbb{F}_q[X_1, \ldots, X_n]$ be a polynomial in n variables, of degree d, which we assume is not of the form $Q^p - Q + C^{te}$. Again setting $\psi(x) = \exp\left(\frac{2\pi i}{p} \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)\right)$ we have*

$$\left| \sum_{x_i \in \mathbb{F}_q} \psi P(x_1, \ldots, x_n) \right| \leqslant (d+1) q^{n-\frac{1}{2}}.$$

For the left-hand side, we have the trivial estimate q^n. So it is enough to prove 2.16 when $d - 1 < q^{1/2}$; let us only assume that $d < q+1$, and let P_d be the homogeneous part of degree d of P. Recall the

Lemme 2.17. *A hypersurface of degree d in $\mathbb{P}^r$ $(r \geqslant i)$ cannot pass through all rational points on $\mathbb{F}_q$ that are $d \geqslant q+1$.*

We proceed by induction: if there exists a rational hyperplane not entirely contained in the hypersurface (such is not the case for $r = 1$), we apply the induction hypothesis to the trace of the hypersurface on this hyperplane. Otherwise, the degree d is $\geqslant$ the number of rational hyperplanes, $\geqslant q+1$.

Now let us distinguish between two cases.

Case 1: $p \nmid d$. Applying the lemma to P_d, we see that, by making a linear change of variables, we can assume that $P_d(1, 0, \ldots, 0) \neq 0$, i.e. that the coefficient of X_1^d in P is nonzero. A polynomial in one variable

$$S(X) = \sum_{i=0}^{d} a_i x^i,$$

with $a_d \neq 0$ $(p \nmid d)$ is never of the form $Q^p - Q + C^{te}$, and the estimate (3.5.2) reduces to

$$\left|\sum \psi(S(x))\right| \leqslant (d-1)q^{1/2}.$$

Applying this estimate to the partial sums obtained by only making x_1 vary, we find

$$\left|\sum \psi P(x_1, \ldots, x_n)\right| = \left|\sum_{x_2,\ldots,x_n} \sum_{x_1} \psi P(x_1, \ldots, x_n)\right| \leqslant q^{n-1}(d-1)q^{1/2}$$

as promised.

Case 2: $p \mid d$ $(d > 0)$. If P_d is a power of p, replacing P by $P - (P_d - P_d^{1/p})$ does not change the sum under consideration, and lowers the degree: we get rid of this case by proceeding by induction on d. Otherwise, the differential of P_d is not identically zero; applying 2.17, we can assume, by making a linear change of variables, that it is not zero at the point $(1, 0, \ldots, 0)$. If we write

$$P_d = \sum_{i=0}^{d} X_1^{d-i} S_i(X_2, \ldots, X_n),$$

this means that the linear form S_1 is not identically zero.

The form S_0 is a constant, and by replacing P with $P - (S_0 X_1^d - S_0^{1/p} X_1^{d/p})$, we can assume that it is zero. Let finally $-\lambda$ be the coefficient of X_1^{d-1} in P. For fixed $x_2, \ldots, x_n$, we have

$$P(X_1, x_2, \ldots, x_n) = (S_1(x_2, \ldots, x_n) - \lambda)X_1^{d-1} + \text{terms of lower degree in } X_1$$

and if $S_1(x_2, \ldots, x_n) \neq \lambda$, we therefore have

$$\left|\sum_{x_1} \psi P(x_1, \ldots, x_n)\right| \leqslant (d-2)q^{1/2}.$$

In total,

$$\begin{aligned}
\left|\sum \psi P(x_1, \ldots, x_n)\right| &\leqslant \left|\sum_{S=\lambda} \psi P(x_1, \ldots, x_n)\right| + \sum_{S(X_2,\ldots,X_n)\neq\lambda} \left|\sum_{x_1} \psi P(x_1, \ldots, x_n)\right| \\
&\leqslant q^{n-1} + (q^{n-1} - q^{n-2})(d-2)q^{1/2} \\
&< (d-2)q^{n-1/2} + q^{n-1} \\
&< (d-1)q^{n-1/2}.
\end{aligned}$$

Another result of this nature is given by R.A. Smith [35].

3 Sommes de Gauss et sommes de Jacob

3.1

Let k be a finite field of characteristic p, χ a character of $k^\times$, and ψ a non-trivial character of the additive group of k. We will take the definition of Gauss sums:

$$\tau(\chi, \psi) = -\sum_{x \in k^\times} \psi(x)\chi^{-1}(x) \tag{4.1.1}$$

(note the sign). Classically, χ and ψ take values in complex numbers. We will take them to values in $E_\lambda^\times$ (cf. the proof of 1.15). Consider $-\tau(\chi, \psi)$ as a sum over the rational points of the $\mathbb{G}_m$ scheme over k. According to paragraph 1, we have $-\tau(\chi, \psi) = \mathrm{tr}(F^*, \mathrm{H}_c^*(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1}))$. In addition

Proposition 3.2. *The cohomology of $\mathbb{G}_m$ with coefficients in $\mathscr{F}(\psi\chi^{-1})$ satisfies*

(i) $\mathrm{H}_c^i = 0$ *for* $i \neq 1$, *and* $\dim \mathrm{H}_c^1 = 1$.

(ii) F^*, *acting on* H_c^1, *is multiplication by* $\tau(\chi, \psi)$.

(iii) If χ is non-trivial, then $\mathrm{H}_c^\bullet \xrightarrow{\sim} \mathrm{H}^\bullet$.

Preuve. For every prime n not equal to p, let K_n be the $\boldsymbol{\mu}_n$-torseur over $\mathbb{G}_m$ defined by the Kummer exact sequence $0 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \xrightarrow{x^n} \mathbb{G}_m \to 0$. If k has q elements, then on k: $\boldsymbol{\mu}_{q-1} = k^\times$, and the Lang torseur on $\mathbb{G}_m/k$ is K_{q-1}. Therefore, $\mathscr{F}(\chi^{-1}) = \chi(K_{q-1})$. Assertion 3.2(ii) follows from (i,iii), themselves contained in the following geometric statement, where $\mathscr{F}(\psi)$ denotes the sheaf on $\mathbb{G}_m/\mathbb{F}$ obtained by extension of scalars from k to $\mathbb{F}$ of $\mathscr{F}(\psi)$ on $\mathbb{G}_m/k$ (1.8.b). ☐

Proposition 3.3. *Let $\chi : \boldsymbol{\mu}_n \to E_\lambda^\times$. The cohomology of $\mathbb{G}_m$ with coefficients in $\mathscr{F}(\psi) \otimes \chi(K_n)$ satisfies*

(i) $\mathrm{H}_c^i = 0$ *for* $i \neq 1$, *and* $\dim \mathrm{H}_c^1 = 1$.

(ii) If χ is non-trivial, then $\mathrm{H}_c^\bullet \xrightarrow{\sim} \mathrm{H}^\bullet$.

Preuve. The sheaf $\mathscr{F}(\psi)$ is the restriction to $\mathbb{G}_m$ of a locally constant sheaf on $\mathbb{G}_a$, wildly ramified at infinity, with Swan conductor 1. The sheaf $\chi(K_n)$ is constant if $\chi = 1$; if $\chi \neq 1$, it is ramified at 0 and ∞, mildly.

The beam $\mathscr{F}(\psi) \otimes \chi(K_n)$ is thus ramified at ∞; applying 1.18(b,c), we find that $\mathrm{H}_c^i(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \chi(K_n)) = 0$ for $i \neq 1$. If $\chi \neq 1$, it is ramified at 0 and ∞ and (ii) results from 1.19.a.

The conductors of Swan are 0 at 0 and 1 at ∞. According to (3.2.1), the Euler-Poincaré characteristic is therefore -1 and this completes the proof. ☐

3.4 Remark

If χ is non-trivial, 3.2(iii) and Poincaré duality show that $\mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1}))$ and $\mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\psi^{-1}\chi))$ are in duality (duality with values in $E_\lambda(-1)$). We therefore have $\tau(\chi, \psi) \cdot \tau(\chi^{-1}, \psi^{-1}) = q$, i.e. $|\tau(\chi, \psi)| = 1$.

3.5

If k is an extension of degree N of $\mathbb{F}_q$, we can also look at (4.1.1) as a sum with N variables over $\mathbb{F}_q$. More generally, let k be an étale algebra over $\mathbb{F}_q$, of degree N over $\mathbb{F}_q$. It is a product of fields k_i, of degree N_i, and we set

$$\varepsilon(k) = (-1)^{\sum(N_i+1)}. \tag{4.5.1}$$

This is the signature of the permutation of $S = \hom_{\mathbb{F}_q}(k, \mathbb{F})$ induced by the substitution of Frobenius $\varphi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$.

Let $\psi : \mathbb{F}_q \to E_\lambda^\times$ be a non-trivial character, and $\chi : k^\times \to E_\lambda^\times$. We set

$$\tau_{\mathbb{F}_q}(\chi, \psi) = (-1)^N \sum_{x \in k^\times} \psi\, \mathrm{tr}_{k/\mathbb{F}_q}(x) \cdot \chi^{-1}(x). \tag{4.5.2}$$

If χ has coordinates the $\chi_i : k_i^\times \to E_\lambda^\times$, we have the trivial identity

$$\tau_{\mathbb{F}_q}(\chi, \psi) = \varepsilon(k) \prod \tau(\chi_i, \psi \circ \mathrm{tr}_{k_i/\mathbb{F}_q}). \tag{4.5.3}$$

After some preliminaries, we will give in 3.10 a cohomological interpretation of the sums (4.5.2). In 3.12, we will interpret the Hasse-Davenport identity as a twisted (cf. 1.12) special case of (4.5.3): for $k = \mathbb{F}_q^N$, and χ a character of $\mathbb{F}_q^\times$, we have $\tau_{\mathbb{F}_q}(\chi \circ N_{k/\mathbb{F}_q}, \psi) = \tau(\chi, \psi)^N$.

3.6

Recall that for M a finitely generated projective module over a ring A, the functor $\mathrm{Spec}(B) \mapsto M \otimes_A B$, from affine schemes over $\mathrm{Spec}(A)$ to Ens is represented by the affine scheme $\mathbb{V}(M^\vee)$ (notations of EGA), the spectrum of $\mathrm{Sym}_A^\bullet(M^\vee)$. If $M = A^n$, this is the affine space of dimension n.

Suppose that M is a unitary A-algebra, and let $V = \mathbb{V}(M^\vee)$. By definition, for any extension B of A, the set $V(B)$ of points of V with coordinates in B is $M \otimes_A B$. The functor $B \mapsto M \otimes_A B$ is valued in the unital rings, so V is a scheme in unital rings over $\mathrm{Spec}(A)$. The norm and trace morphisms: $M \otimes_A B \to B$ are functorial in B; they correspond therefore to morphisms of schemes N and T from V to $\mathbb{G}_a$. We will consider the following derived schemes of V:

a) V^* is the open of invertible elements of V (a group scheme for $\cdot$);

b) W is the hyperplane of equation $T = 0$ and $W^* = W \cap V^*$;

c) P is the hyperplane at infinity $V \setminus \{0\}/\mathbb{G}_m$ of the affine space V over $\mathrm{Spec}(A)$. If Q is the hyperplane at infinity of W, $W^*/\mathbb{G}_m$ and $V^*/\mathbb{G}_m$ are open subsets of the projective spaces Q and P over $\mathrm{Spec}(A)$.

$$\begin{tikzcd} W^* \arrow[r, hook] \arrow[d, "\pi"] & V^* \arrow[d, "\pi"] \\ W^*/\mathbb{G}_m \arrow[r, hook] & V^*/\mathbb{G}_m \end{tikzcd} \quad \subset \quad \begin{tikzcd} W \setminus \{0\} \arrow[r, hook] \arrow[d, "\pi"] & V \setminus \{0\} \arrow[d, "\pi"] \\ Q \arrow[r, hook] & P. \end{tikzcd} \tag{4.6.1}$$

If $M = A^I$, I a finite set, then $V \simeq \mathbb{G}_a^I$, $V^* = \mathbb{G}_m^I$, $V^*/\mathbb{G}_m$ is the torus $\mathbb{G}_m^I/(\mathbb{G}_m$ diagonal), N and T are written $\prod x_i$ and $\sum x_i$, and Q is therefore the projective hyperplane of equation $\sum x_i = 0$ in P.

The case that interests us is when M is finite étale over A. The previous description then holds locally on $\mathrm{Spec}(A)$ (for the étale topology) and we can write $V^* = \mathbb{G}_m^I$, for I a locally constant sheaf of finite sets on $\mathrm{Spec}(A)$. For example, if A is a field, with algebraic closure $\bar{A}$, and M is a separable A-algebra, we set $I = \hom_A(M, \bar{A})$, and, over $\bar{A}$, we have $V \sim \mathbb{G}_a^I$. Via this isomorphism, N and T are written $\prod x_i$ and $\sum x_i$.

3.7 Kummer's torseur

Let S be a scheme, n an integer that is invertible on S, and G a scheme with commutative group fibers that are connected (multiplicatively noted) over S. The sequence

$$0 \longrightarrow G_n \longrightarrow G \xrightarrow{x^n} G \longrightarrow 0$$

is exact. It defines a G_n-torseur $K_n(G)$ (or simply K_n) over G. For $\chi : G_n \to E_\lambda^\times$ a homomorphism of the S-étale sheaf G_n into the constant sheaf $E_\lambda^\times$, we denote $\mathscr{K}_n(\chi)$ the E_λ-sheaf $\chi^{-1}(K_n)$.

The torseurs K_n form a projective system of torseurs under the projective system of groups G_n (transition morphisms $x \mapsto x^d : G_{nd} \to G_n$). This expresses the commutativity of the diagrams

$$\begin{tikzcd} 0 \arrow[r] & G_{nd} \arrow[r] \arrow[d, "x^d"] & G \arrow[r, "x^{nd}"] \arrow[d, "x^d"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G_n \arrow[r] & G \arrow[r, "x^n"] & G \arrow[r] & 0. \end{tikzcd}$$

We therefore have $\mathscr{K}_n(\chi) = \mathscr{K}_{nd}(\chi \circ x^d)$.

3.8

Apply the construction 3.6 for $A = \mathbb{F}_q$, and $M = k$ an étale algebra over $\mathbb{F}_q$. In accordance with the general conventions, we will denote with an index 0 the $\mathbb{F}_q$-schemes noted $V, V^*, \ldots$ in 3.6. The same letters without an index denote the schemes over $\mathbb{F}$ that are deduced from them by extension of scalars.

We set $I = \hom(k, \mathbb{F})$, from which $V \sim \mathbb{G}_a^I$ and T is written $(x_i) \mapsto \sum x_i$.

$$\mathscr{F}(\psi \circ \mathrm{tr}_{k/\mathbb{F}_q}) = T^*\mathscr{F}(\psi) = \bigotimes_i \mathrm{pr}_i^* \mathscr{F}(\psi). \tag{4.8.1}$$

Since $V^* \sim \mathbb{G}{+}m^I$, a character χ of $V_n^* \sim \boldsymbol{\mu}_n^I$, with values in $E_\lambda^\times$, is written as a family $(\chi_i)_{i\in I}$ of characters of $\boldsymbol{\mu}_n$ that are unramified by I. It is defined over $\mathbb{F}_q$ if, for all $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, we have $\chi_{\sigma_i} = \chi_i \circ \sigma^{-1}$. It then defines an E_λ-sheaf $\mathscr{K}_n((\chi_i)_{i\in I})$ over V_0^*. If the product of the χ_i is trivial, χ factors through a character of $(V_0^*/\mathbb{G}_m)_n$ and $\mathscr{K}_n((\chi_i)_{i\in I})$ is the inverse image of a sheaf, denoted in the same way, over $V - 0^*/\mathbb{G}_m$. This construction is compared as follows to Lang's torseur.

Lemme 3.9. *(i) For n divisible enough, we have a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & (V_0^*)_n \arrow[r] \arrow[d, "\tau"] & V_0^* \arrow[r, "x^n"] \arrow[d, "\tau"] & V_0^* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k^\times \arrow[r] & V_0^* \arrow[r, "\mathfrak{L}"] & V_0^* \arrow[r] & 0; \end{tikzcd}$$

if χ is a character $\chi : k^\times \to E_\lambda^\times$, we have $\mathscr{F}(\chi) = \mathscr{K}_n(\chi \circ \tau)$.

(ii) For n divisible enough, τ identifies $k^\times$ with the coinvariants of $\mathrm{Gal}(\bar{\mathbb{F}}/F)$ acting on $V_n^* \sim \boldsymbol{\mu}_n^I$.

(iii) For k a field, $N = [k : \mathbb{F}_q]$, $\omega \in I$ an embedding of k in $\mathbb{F}$, and $n = q^N - 1$, the ω-component of $\chi \circ \tau$ is $\chi \circ \omega^{-1}$.

(iv) If χ is non-trivial on each factor of k, the $(\chi \circ \tau)_i$ are all non-trivial.

It is enough to prove the lemma when k is a field, of degree N over $\mathbb{F}_q$. In this case, n is "divisible enough" if $q^N - 1 \mid n$. Choose an embedding ω of k in $\mathbb{F}_q$; I is then identified with $\mathbb{Z}/N$: to $i \in \mathbb{Z}/N$ corresponds $\omega_i = \omega^{q^i}$. Via the isomorphism $V_0^*(\mathbb{F}) = F^{\times I}$, we have

a) $x \in k^\times$ corresponds to $(\omega_i(x)) \in \mathbb{F}^{\times I}$;

b) $F((x_i)_{i\in\mathbb{Z}/N}) = (x_{i-1}^q)_{i\in\mathbb{Z}/N}$.

A character $\chi = (\chi_i)$ of $V_n^* \sim \boldsymbol{\mu}_n^I$ will be defined on $\mathbb{F}_q$ if $\chi_{-i} = \chi_0(x^{q^i})$ $(i \in \mathbb{Z})$. If $q^N - 1 \mid n$, there are $q^N - 1$ such characters: χ_0 factors through $\boldsymbol{\mu}_{q^N-1}$, and determines the χ_i. To prove (ii), it is therefore enough to verify (i) and (iii) (or its corollary (iv)) which ensures that $\chi \mapsto \chi \circ \tau$ is injective.

We prove (i) and (iii), for $n = q^N - 1$. Let us additively group the endomorphisms of $V^* \sim \mathbb{G}_m^I$; in particular, let us note n the operator $x \mapsto x^n$. If α is the operator of circular permutations $(x_i) \mapsto (x_{i-1})$, we have $q^N - 1 = (q\alpha)^N - 1 = (q\alpha - 1)((q\alpha)^{N-1} + \cdots + 1)$. This determines τ: we have $\tau((x_i)) = \prod_{0\leqslant j<N} x_{i-j}^{q^j}$, and the induced application of $(V_0^*)_n$ in $k^\times$ is written [a) above] $(x_i) \mapsto \prod x_{-i}^{q^i}$; from there results (iii).

Proposition 3.10. *The cohomology of V^* with coefficients in $\mathscr{F}(\chi^{-1} \cdot \psi \operatorname{tr}_{k/\mathbb{F}_q})$ satisfies*

(i) $\mathrm{H}_c^i = 0$ for $i \neq N$, and $\dim \mathrm{H}_c^N = 1$.

(ii) On H_c^N, F^* is multiplication by $\tau_{\mathbb{F}_q}(\chi, \psi)$.

(iii) If χ is non trivial on each factor of k, we have $\mathrm{H}_c^\bullet \xrightarrow{\sim} \mathrm{H}^\bullet$.

We apply 3.9(i): on $\mathbb{F}$, if $\chi \circ \tau = (\chi_i)_{i\in I}$, we have $\mathscr{F}(\chi) = \mathscr{K}_n((\chi_i)_{i\in I})$. Points (i) and (iii) therefore result from the more geometric statement below (for (iii), apply 3.9(iv)) and (ii) follows from the trace formula:

Proposition 3.11. *Let $(\chi_i)_{i\in I}$ be a family of characters of $\boldsymbol{\mu}_n(\mathbb{F})$. The cohomology of $V^* \simeq \mathbb{G}_m^I$ with coefficient in $\mathscr{K}_n((\chi_i)_{i\in I}) \otimes \mathscr{F}(\psi \operatorname{tr}_{k/\mathbb{F}_q})$ satisfies*

(i) $\mathrm{H}_c^i = 0$ for $i \neq N$, and $\dim \mathrm{H}_c^N = 1$.

(ii) If the χ_i are all non-trivial, we have $\mathrm{H}_c^\bullet \xrightarrow{\sim} \mathrm{H}^\bullet$.

We have $V^* \sim \mathbb{G}_m^I$, $\mathscr{K}((\chi_i)) = \bigotimes_i \mathrm{pr}_i^* \chi_i(K_n(\mathbb{G}_m))$ and $\mathscr{F}(\psi \circ \operatorname{tr}_{\mathbb{F}_q/k}) = T^*\mathscr{F}(\psi) = \bigotimes_i \mathrm{pr}_i^* \mathscr{F}(\psi)$. This allows us to apply the Künneth formula, and 3.11 follows from 3.3.

3.12

From 2.4* and (Cycle, 1.3 example 2), we also deduce that the group of permutations σ of I such that $\chi_i = \chi_{\sigma i}$ $(i \in I)$ acts on H_c^N by multiplication by the signature $\varepsilon(\sigma)$. We will deduce a second proof of the Hasse-Davenport identity from this.

If χ is a character of $\mathbb{F}_q^\times$, the fact that, on $\mathbb{F}$, N is written $(x_i) \mapsto \prod x_i$ provides: $\mathscr{F}(\chi \circ N) = N^*\mathscr{F}(\chi) = \bigotimes_i \mathrm{pr}_i^* \mathscr{F}(\psi)$ and the Künneth formula provides:

$$\mathrm{H}_c^\bullet(V^*, \mathscr{F}(\chi^{-1} \circ N, \psi \circ \mathrm{tr})) \sim \mathrm{H}_c^\bullet(\mathbb{G}_m, \mathscr{F}(\chi^{-1}\psi))^{\otimes I},$$

where, on the right-hand side, the tensor product is taken in the sense of 2.4*. Since $\mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1}))$ is of dimension 1, its tensor power $\otimes I$, in the ordinary sense, only depends on the cardinal n of I. Applying (Cycle, 1.3 example 2), we obtain a canonical isomorphism

$$\mathrm{H}_c^N\left(V, \mathscr{F}(\chi^{-1} \circ N) \otimes T^*\mathscr{F}(\psi)\right) \sim \mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1})) \otimes_{\mathbb{Z}} \bigwedge^N \mathbb{Z}^I. \tag{4.12.1}$$

To calculate the action of F^*, it is most convenient to adopt the Galois point of view and say that the isomorphism (4.12.1) is canonical, it is compatible with the action by transport of structure of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. If $\varepsilon(k)$ is the signature of the permutation φ of I, we find that

$$\begin{aligned}
\tau_{\mathbb{F}_q}(\chi \circ N_{k/\mathbb{F}_q}, \psi) &= \mathrm{tr}\left(\varphi^{-1}, \mathrm{H}_c^N\left(V^*, N^*\mathscr{F}(\chi^{-1}) \otimes T^*\mathscr{F}(\psi)\right)\right) \\
&= \varepsilon(k)\,\mathrm{tr}\left(\varphi^{-1}, \mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\chi^{-1}\psi))\right)^N \\
&= \varepsilon(k)\tau(\chi, \psi)^N,
\end{aligned}$$

that is

$$\tau_{\mathbb{F}_q}\left(\chi \circ N_{k/\mathbb{F}_q}, \psi\right) = \varepsilon(k)\tau(\chi, \psi)^N. \tag{4.12.2}$$

If k is a field, φ is a circular permutation of I, $\varepsilon(k) = (-1)^{N+1}$, and we recover the Hasse-Davenport identity:

$$\tau\left(\chi \circ N_{k/\mathbb{F}_q}, \psi \circ \mathrm{tr}_{k/\mathbb{F}_q}\right) = (-1)^{N+1}\tau_{\mathbb{F}_q}\left(\chi \circ N_{k/\mathbb{F}_q}, \psi\right) = \tau(\chi, \psi)^N.$$

Lemme 3.13. *For $\chi \circ \tau = (\chi_i)_{i \in I}$ as in 3.9(i), the following conditions are equivalent*

(i) $\chi|\mathbb{F}_q^\times$ is trivial

(ii) The product of the χ_i is trivial

(iii) $\mathscr{F}(\chi) = \mathscr{K}_n((\chi_i))$ is the inverse image of a (unique) sheaf on $V^*/\mathbb{G}_m$.

(iv) The inverse image of $\mathscr{F}(\chi)$ on $\mathbb{G}_m$ (sent to V^* from the structural morphism $\mathbb{F}_q \to k$) is trivial.

(i)⇒(ii). We regard χ as a character of $V^*/\mathbb{G}_m(\mathbb{F}_q) = k^\times/\mathbb{F}_q^\times$, and take $\mathscr{F}(\chi)$ on $V^*/\mathbb{G}_m$.

(ii)⇒(iii). Similarly, we regard (χ_i) as a character of $(V^*/\mathbb{G}_m)_n$. The uniqueness in (iii) follows from the fact that V^* is a fiber bundle with connected fibers over $V^*/\mathbb{G}_m$, and (iii)⇒(iv) is trivial.

(iv)⇒(i), (ii). This inverse image is $\mathscr{F}(\chi|\mathbb{F}_q^\times)$ and $\mathscr{K}_n(\prod \chi_i)$.

3.14

Let k be an étale algebra over $\mathbb{F}_q$, of dimension $N+1$ and χ a non-trivial character of $k^\times$, trivial on $\mathbb{F}_q^\times$. The Jacobi sum $J(\chi)$ is defined by

$$J(\chi) = (-1)^{N-1} \sum_{\substack{x\in k^\times/\mathbb{F}_q^\times \\ \operatorname{tr}(x)=0}} \chi^{-1}(x). \tag{4.14.1}$$

We have the following identity between Gauss sums and Jacobi sums.

Proposition 3.15. *For χ as above and ψ a non-trivial additive character of $\mathbb{F}_q$, we have*

$$qJ(\chi) = \tau_{\mathbb{F}_q}(\chi,\psi).$$

In the particular case where $k = \mathbb{F}_q^n$, this formula is rewritten by (4.5.3)

$$qJ(\chi) = \prod_i \tau(\chi_i,\psi), \tag{4.15.1}$$

more written in the form

$$\chi_0(-1)J(\chi) = (-1)^{N-1} \sum_{\substack{x_1,\dots,x_N\in\mathbb{F}_q^\times \\ \sum x_i=1}} \prod_{i=1}^N \chi_i^{-1}(x_i) = \tau(\chi_0^{-1},\psi)^{-1}\prod_{i=1}^N \tau(\chi_i,\psi)$$

$$\chi_0^{-1} = \prod_{i=1}^N \chi_i$$

Preuve de 3.15.

$$\tau_{\mathbb{F}_q}(\chi,\psi) = (-1)^{N+1}\sum_{x\in k^\times}\chi(x)^{-1}\psi\operatorname{tr}(x) = (-1)^{N+1}\sum_{x\in k^\times/\mathbb{F}_q^\times}\chi(x)^{-1}\sum_{\lambda\in\mathbb{F}_q^\times}\psi(\lambda\operatorname{tr}x).$$

The sum $\sum_{\lambda\in\mathbb{F}_q^\times}\psi(\lambda\operatorname{tr}x)$ is equal to $q-1$ if $\operatorname{tr}(x)=0$, and -1 if $\operatorname{tr}(x)\neq 0$. Therefore,

$$\tau_{\mathbb{F}_q}(\chi,\psi) = (-1)^{N+1}\left(q\sum_{x\in k^\times/\mathbb{F}_q^\times}\chi(x)^{-1} - \sum_{x\in k^\times/\mathbb{F}_q^\times}\chi(x)^{-1}\right).$$

The second term in the second member is zero, because it is the sum of the values of a character, and 3.15 follows. □

Proposition 3.15 can be translated into cohomology in this way:

Proposition 3.16. *The cohomology of $W^*/\mathbb{G}_m$ (3.6, 3.8) with coefficient in $\mathscr{F}(\chi^{-1})$ satisfies*

(i) $\mathrm{H}_c^i = 0$ *for $i \neq N-1$, and* $\dim \mathrm{H}_c^{N-1} = 1$.

(ii) On H_c^{N-1}, *F^* is multiplication by $J(\chi)$.*

(iii) H_c^{N-1} *is canonically isomorphic to* $\mathrm{H}_c^{N+1}\left(V^*, \mathscr{F}(\chi^{-1},\psi\circ\operatorname{tr}_{k/\mathbb{F}_q})\right)(1)$.

Assertion (ii) follows from (i) and the trace formula; (i) and (iii) follow from 3.11 and the following more geometric statement.

Proposition 3.17. *Let $(\chi_i)_{i\in I}$ be a family of non-trivial trivial characters of $\boldsymbol{\mu}_n(\mathbb{F})$, with product* 1*. Then,* $\mathrm{H}_c^{i-1}\left(W^*/\mathbb{G}_m, \mathscr{K}_m((\chi_i)_{i\in I})\right)$ *is canonically isomorphic to* $\mathrm{H}_c^{i+1}\left(V^*, \mathscr{K}_m((\chi_i)_{i\in I}) \otimes \mathscr{F}(\psi\,\mathrm{tr}_{k/\mathbb{F}_q})\right)(1)$.

The first line of 3.15 becomes (π as in (4.6.1)).

$$\mathsf{R}^i\pi_!\left(\mathscr{K}_n((\chi_i)_{i\in I}) \otimes \mathscr{F}(\psi\,\mathrm{tr}_{k/\mathbb{F}_q})\right) = \mathscr{K}_n((\chi_i)_{i\in I}) \otimes \mathsf{R}^i\pi_!\mathscr{F}(\psi\,\mathrm{tr}_{k/\mathbb{F}_q}) \tag{4.17.1}$$

(over $V^*/\mathbb{G}_m$). Let's calculate the sheaves $\mathsf{R}^i\pi_!\mathscr{F}(\psi\,\mathrm{tr}_{k/\mathbb{F}_q}) = \mathsf{R}^i\pi_! T^*\mathscr{F}(\psi)$. Let $v_0 : \widetilde{V}_0 \to V_0$ be the blow-up of V_0 at $\{0\}$. In the diagram

$$\begin{tikzcd} V_0 \smallsetminus \{0\} \arrow[r, hook] \arrow[dr, "\pi"'] & \widetilde{V}_0 \arrow[r, "v_0"] \arrow[d, "\bar\pi"] & V_0 \\ & P_0 & \end{tikzcd} \tag{4.17.2}$$

π is a fibration with fiber the pointed lines, $\widetilde{V}_0$ is the corresponding line bundle, and $Z_0 = v_0^{-1}(0)$ is its section 0. The sheaf $T^*\mathscr{F}(\psi)$ on $V - 0 \smallsetminus \{0\}$ extends to $(Tv_0)^*\mathscr{F}(\psi)$ on $\widetilde{V}_0$, and the restriction of this sheaf to Z_0 is the constant sheaf E_λ, because T_0v_0 is zero on Z_0. The long exact sequence in cohomology with compact support therefore writes

$$\longrightarrow \mathsf{R}^i\pi_! T^*\mathscr{F}(\psi) \longrightarrow \mathsf{R}^i\bar\pi_!(Tv)^*\mathscr{F}(\psi) \longrightarrow (E_\lambda \text{ for } i=0,\ 0 \text{ otherwise}) \longrightarrow \tag{4.17.3}$$

The reciprocal image $\bar\pi^{-1}(Q_0)$ is the blow-up $\widetilde{W}_0$ of W_0 at 0; T_0V_0 vanishes on $\widetilde{W}_0$; we therefore have $(Tv)^*\mathscr{F}(\psi) = E_\lambda$ on $\pi^{-1}(Q_0)$ and, $\bar\pi$ being a fiber bundle of lines

$$\mathsf{R}^i\bar\pi_!(Tv)^*\mathscr{F}(\psi)|Q = \begin{cases} 0 & \text{for } i \neq 2 \\ E_\lambda(-1) & \text{for } i = 2. \end{cases} \tag{4.17.4}$$

If $x \in P$, $x \notin Q$, the line $D = \pi^{-1}(x)$ is sent isomorphically onto $\mathbb{G}_a$ by Tv; we therefore have 2.7*)

$$\mathrm{H}_c^\bullet(D, (Tv)^*\mathscr{F}(\psi)) = \mathrm{H}^\bullet(\mathbb{G}_a, \mathscr{F}(\psi)) = 0 \qquad \text{and}$$

$$\mathsf{R}^i\bar\pi_!(Tv)^*\mathscr{F}(\psi) = \begin{cases} 0 & \text{for } i \neq 2 \\ E_\lambda(-1)_Q & \text{for } i = 2 \end{cases} \tag{4.17.5}$$

Conjugating (4.17.3) and (4.17.5), we finally find

$$\mathsf{R}^i\pi_! T^*\mathscr{F}(\psi) = \begin{cases} 0 & \text{for } i \neq 1, 2 \\ E_\lambda & \text{for } i = 1 \\ E_\lambda(-1)_Q & \text{for } i = 2. \end{cases} \tag{4.17.6}$$

3.18

Calculating the cohomology with proper support of the sheaf $\mathscr{K}_n((\chi_i)) \otimes T^*\mathscr{F}(\psi)$ on V^* using the Leray spectral sequence of $\pi : V^* \to V^*/\mathbb{G}_m$. Applying (4.17.1) and (4.17.6), we find as initial terms the $E_2^{p,1} = \mathrm{H}_c^p(V^*/\mathbb{G}_m, \mathscr{K}_n(\chi_i)) = 0$ (because $(\chi_i) \neq 0$) and the $E_2^{p,2} = \mathrm{H}_c^p(W^*/\mathbb{G}_m, \mathscr{K}(\chi_i))(-1)$.

The spectral sequence reduces to an isomorphism, and 3.17 follows.

3.19

The sheaves $\mathscr{K}_n((\chi_i)_{i\in I})$ are defined on any field, or even any base scheme. This will allow us to generalize 3.16(i). With the notations of 3.6, suppose M is finite étale everywhere of rank N over A, and let n be an integer invertible in A. We denote a the projection of $W^*/\mathbb{G}_m$ onto Spec(A). Also let χ be a character (sheaf morphism) $\chi : (V^*/\mathbb{G}_m)_n \to E_\lambda^\times$, and $\mathscr{K}_n(\chi)$ the corresponding sheaf. Locally for the étale topology, we can regard χ as a family of characters $(\chi_i)_{i\in I}$ of $\boldsymbol{\mu}_n$, of trivial product.

Proposition 3.20. *(i) If χ is (everywhere) non-trivial, then $\mathsf{R}^i a_!(\mathscr{K}_n(\chi)) = 0$ for $i \neq N-1$, and $\mathsf{R}^{N-1}a_!(\mathscr{K}_n(\chi))$ is smooth, of rank 1.*

(ii) An automorphism σ of M that respects χ acts on this $\mathsf{R}^{N-1}a_!$ by multiplication with the signature $\varepsilon(\sigma)$ of σ, seen as a permutation of I.

(iii) If the χ_i are all non-trivial, then $\mathsf{R}a_! \xrightarrow{\sim} \mathsf{R}a_*$.

Preuve. The scheme $W^*/\mathbb{G}_m$ is the complement of a relative normal crossing divisor in the scheme Q, proper and smooth over Spec(A), and $\mathscr{K}_n(\chi)$ is locally constant on $W^*/\mathbb{G}_m$, with moderate ramification at infinity. It follows that the $\mathsf{R}^i a_!$ and $\mathsf{R}^i a_*$ are smooth, with formation compatible to any base change. A standard argument then reduces us to assuming that A is a finite field, and (i) follows from 3.17 and 3.11(i). To prove (ii), we use that the action of σ is compatible with the isomorphism 3.17, and 3.12. For (iii), we note that if the χ_i are all non-trivial, then $\mathscr{K}_n(\chi)$ on $W^*/\mathbb{G}_m \subset Q$ is ramified along each divisor at infinity, and 1.19.1. □

4 Hecke Characters

4.1

Let F be a number field (of finite degree over $\mathbb{Q}$) and k a field of characteristic 0. Here are various equivalent ways of saying what an *algebraic* homomorphism $\alpha : F^\times \to k^\times$ is.

(i) Let $\{e_i\}$ be a basis of F over $\mathbb{Q}$. The homomorphism α is algebraic if it is given by a formula $\alpha(\sum x^i e_i) = A(x^i)$, $A \in k(X^i)$.

This means that α coincides on $F^\times$ with a rational function defined on k of $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m)$ in $\mathbb{G}_m$. By Zariski density of $F^\times$, and the fact that a biregular homomorphism is defined everywhere, such a function is a homomorphism of group schemes:

(ii) α is induced by a homomorphism of k-schemes of groups

$$\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m) \otimes_{\mathbb{Q}} k \to \mathbb{G}_m.$$

If k is a number field, the adjunction property of the restriction of scalars R shows that this is equivalent to

(iii) (k a number field) α is induced by a homomorphism of $\mathbb{Q}$-schemes of groups: $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m) \to \mathrm{R}_{k/\mathbb{Q}}(\mathbb{G}_m)$.

Let $\bar{k}$ be an algebraic closure of k, and $I = \hom(F, \bar{k})$. On $\bar{k}$, the group of characters of $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m)$ is $\mathbb{Z}^I$, with basis the embeddings of F in $\bar{k}$. The characters defined on k are those invariant by $\mathrm{Gal}(\bar{k}/k)$; they can be described either as those sending $F^\times$ in $k^\times \subset \bar{k}$, or in terms of the orbits of $\mathrm{Gal}(\bar{k}/k)$ in I, corresponding itself to the factors of $F \otimes k$.

(iv) α is of the form $\alpha = \prod_{\omega \in I} \omega^{n_\omega}$. The families of exponents (n_ω) allowed are those such that $n_\omega = n_{\omega'}$ for ω and ω' in the same orbit of $\mathrm{Gal}(\bar{k}/k)$. They are also those such that $\prod_\omega (x)^{n_\omega} \in k$ for all $x \in F$.

(v) Let $F \otimes k = \prod_{j \in J} F_j$, the F_j being fields. Then, α is written $\alpha = \prod N_{F_j/k}^{m_j}$.

In the particular case where k contains a normal closure of F, these expressions become: $\alpha = \prod \omega^{n_\omega}$, where ω goes through the $[F : \mathbb{Q}]$ embeddings of F in k.

4.2

Suppose that k is a number field, and let α be an algebraic homomorphism of $F^\times$ into $k^\times$. There is then a unique homomorphism, still denoted α, from the group $I(F)$ of fractional ideals of F into that of k, such that $\alpha((x)) = (\alpha(x))$. The uniqueness follows from the fact that every ideal a has a power that is a principal ideal, and from the fact that $I(k)$ is torsion-free. To prove existence, one uses for example 4.1(v): we have $\alpha(a) = \prod N_{F_j/k}^{m_j}((a))$.

4.3

We recall the definition of Hecke characters, called Hecke characters of Weil type A_0. Let F be a number field, $\mathfrak{m}$ an ideal of F (i.e., an ideal of the ring of integers of F), $I_\mathfrak{m}$ the group of fractional ideals of F prime to $\mathfrak{m}$, and k a field of characteristic 0. A homomorphism $\chi : I_\mathfrak{m} \to k^\times$ is a *Hecke character* (of conductor $\leqslant \mathfrak{m}$) if there exists an algebraic homomorphism $\chi_{\mathrm{alg}} : F^\times \to k^\times$ satisfying

$$\text{For } x \in F^\times\text{, prime to } \mathfrak{m}\text{, totally positive and } \equiv 1 \pmod{\mathfrak{m}}\text{, we have } \chi((x)) = \chi_{\mathrm{alg}}(x). \tag{$*$}$$

By density of the set of x of $(*)$, χ_{alg} is entirely determined by χ. This is the *algebraic part* of χ. If $\chi((x)) = \chi_{\mathrm{alg}}(x)$ for x totally positive and $\equiv 1 \pmod{\mathfrak{m}'}$, χ extends to a Hecke character of conductor $\leqslant \inf(\mathfrak{m}, \mathfrak{m}')$: $I_{\mathfrak{m}+\mathfrak{m}'} \to k^\times$. We will identify Hecke characters which coincide on their common domain of definition, and call the *conductor* of χ the smallest $\mathfrak{m}'$ such that χ is of conductor $\leqslant \mathfrak{m}'$.

4.4 Remarque

If an algebraic Hecke character χ takes its values in a subfield k' of k, it is already a Hecke character with values in k': it is enough to see that $\chi_{\mathrm{alg}} : \mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m) \to \mathbb{G}_m$ is already defined on k', and this follows from the Zariski density in $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of the set of totally positive $x \equiv 1 \pmod{\mathfrak{m}}$. We can always take k' to be a subfield of k of finite degree over $\mathbb{Q}$.

4.5

If ε is a totally positive unit $\equiv 1 \pmod{\mathfrak{m}}$, we have $\chi_{\mathrm{alg}}(\varepsilon) = \chi((\varepsilon)) = 1$. The homomorphism χ_{alg} therefore factors through the quotient $T_{\mathfrak{m}}$ of $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m)$ by the Zariski closure of the group $E_{\mathfrak{m}} \subset F^{\times}$ of totally positive units $\equiv 1 \pmod{\mathfrak{m}}$.

According to Serre [33, II.3], if k is an algebraic closure of $\mathbb{Q}$, and $\mathfrak{m}$ is large enough, the characters $\prod \omega^{n_\omega}$ of $\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of the form χ_{alg} are characterized as follows: there must exist an integer N, the *weight* of χ (or of χ_{alg}), such that for any element σ of $\mathrm{Gal}(k/\mathbb{Q})$ conjugate to complex conjugation, we have $n_\omega + n_{\sigma\omega} = N$. If F_1 is the largest subfield of F that is a totally imaginary quadratic extension of a totally real field F_1', this is equivalent to saying that χ_{alg} is of the form $\chi_1 \circ N_{F/F_1}$, and that $\chi_1|F_1 = (N_{F_1/\mathbb{Q}})^N$.

If χ is of weight N, for any ideal $\mathfrak{a}$ of F, $\chi(\mathfrak{a})$ is an algebraic number all of whose conjugates have absolute value $N(\mathfrak{a})^{N/2}$ [33, II.3 prop.2].

4.6

For a fixed number field k, additional conditions are imposed on χ_{alg}. For every ideal $\mathfrak{a}$ of F prime to the conductor, we in fact have

$$\chi_{\mathrm{alg}}(\mathfrak{a}) = (\chi(\mathfrak{a})), \tag{5.6.1}$$

so that $\chi_{\mathrm{alg}}(\mathfrak{a})$ is principal. Since the group of fractional ideals of k is of torsion, it suffices to prove the n-th power of (5.6.1), for some appropriate $n \neq 0$; this allows us to replace $\mathfrak{a}$ by $\mathfrak{a}^n$, so we may assume $\mathfrak{a} = (x)$, with x totally positive $\equiv 1 \pmod{\mathfrak{m}}$. All that remains is to use the definitions.

This, combined with 4.5, shows that χ_{alg} determines the norm of the $\chi(\mathfrak{a})$ in all places of k.

4.7

Serre's group $S_{\mathfrak{m}}$ could be characterized as being the multiplicative group whose group of characters (over any field) is the group of Hecke algebraic characters of conductor $\leqslant \mathfrak{m}$. (cf. [33, II 2.1 and 2.2]). Its relation with ℓ-adic representations is explained in [33, II 2.3].

Théorème 4.8 ([33])**.** *Let χ be a Hecke algebraic character of F in E_λ, for E_λ a finite extension of $\mathbb{Q}_\ell$. Then there exists a (unique) homomorphism $\chi_\lambda : \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to E_\lambda$, such that*

(i) χ_λ is unramified outside the conductor $\mathfrak{f}$ of χ and ℓ.

(ii) For $\mathfrak{p}$ a prime ideal of F prime to $\mathfrak{f}$ and ℓ, and $F_{\mathfrak{p}} \in \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$ the geometric Frobenius at $\mathfrak{p}$, we have

$$\chi_\lambda(F_{\mathfrak{p}}) = \chi(\mathfrak{p}).$$

4.9

At the end of this paragraph, we will give a criterion for a λ-adic representation to come from a Hecke character, and, in the next paragraph, we will apply it to Jacobi sums. We will thus recover, in a slightly generalized form, results of Weil [40, 42], with the second part of Weil's proof replaced by an argument of ℓ-adic cohomology.

Théorème 4.10. *Let F and k be two number fields, λ a place of k of characteristic ℓ, and $\chi_\lambda : \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to k_\lambda^\times$ a homomorphism, unramified outside ℓ and a finite set S of places. We assume that there exists a set T of places (containing S and the places above ℓ), of density 0, and an algebraic homomorphism $\chi_0 : F^\times \to k^\times$ such that*

(i) For $\bar{k}$ an algebraic closure of k, $\chi_0 : F^\times \to k^\times \hookrightarrow \bar{k}^\times$ is of the type considered in 4.5, of weight N;

(ii) For $\mathfrak{p}$ a prime not in T, $\chi_\lambda(F_\mathfrak{p})$ is in $k^\times$, $(\chi_\lambda(F_\mathfrak{p})) = \chi_0(\mathfrak{p})$, and all complex conjugates of $\chi_\lambda(F_\mathfrak{p})$ are of absolute value $(N\mathfrak{p})^{N/2}$.

Then, χ_λ is defined (4.8) by a Hecke character algebraic of F with values in k, of algebraic part χ_0.

Let χ' be a Hecke character algebraic with values in a finite Galois extension k' of k, of algebraic part χ_0 (4.5). By enlarging T if necessary, we can assume that the conductor of χ' is supported in T. Let λ' be a place of k' above λ, $\chi_{\lambda'}$ the composition $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to k_\lambda^\times \to {k'}_{\lambda'}^\times$, and $\chi'_{\lambda'} : \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to {k'}_{\lambda'}^\times$ defined by χ' (4.8). Let $\varepsilon = \chi_{\lambda'} \cdot {\chi'}_{\lambda'}^{-1}$.

Hypothesis (ii), and 4.5, 4.6, 4.8 ensure that for $\mathfrak{p} \notin T$, we have $\varepsilon(F_\mathfrak{p}) \in {k'}^\times$, and that, in all the completions of k', this number has norm 1. It follows that $\varepsilon(F_\mathfrak{p})$ belongs to the (finite) group $\boldsymbol{\mu}$ of roots of unity of k'. According to Cebotarev's density theorem, and the density hypothesis made on T, the $F_\mathfrak{p}$ ($\mathfrak{p} \notin T$) are dense in $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$, so that $\varepsilon(\sigma) \in \mu$ for all $\sigma \in \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$: the character ε is a character of finite order $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to \mu$. According to the theory of class fields, it corresponds to a character of finite order of the group of idèle classes of F, i.e. to an algebraic Hecke character with trivial algebraic part. Correcting χ' by this character, we may assume that $\chi_{\lambda'} = \chi'_{\lambda'}$, and it remains to show that χ', a priori with values in k', is actually with values in k. If $\sigma \in \mathrm{Gal}(k'/k)$, χ' and ${\chi'}^\sigma$ coincide on any prime ideal $\mathfrak{p} \notin T$, since $\chi'(\mathfrak{p}) = \chi'_{\lambda'}(F_\mathfrak{p}) = \chi_{\lambda'}(F_\mathfrak{p}) \in k^\times$. The characters $\chi'_{\lambda'}$, and ${\chi'_{\lambda'}}^\sigma$, coincide therefore on a dense part of $\mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$, so they are equal, so that $\chi' = {\chi'}^\sigma$: χ' is with values in k.

5 The Hecke characters defined by the sums of Jacobi

The idea of this paragraph is the following. Let $\tau_{\mathbb{F}_q}(\chi, \psi)$ be a sum of Gauss. It is itself, and the corresponding sheaf $\mathscr{F}(\chi^{-1}\psi)$ are both twice linked to the characteristic p, and to the base field $\mathbb{F}_q$:

a) because the additive character ψ appears in it, and the sheaf (ψ);

b) because $\mathscr{F}(\chi)$ is defined from the exact sequence of Lang

$$0 \longrightarrow k^\times \longrightarrow V^* \longrightarrow V^* \longrightarrow 0.$$

Two difficulties if we want to lift these cohomological constructions to a number field F, and find ℓ-adic representations of $\mathrm{Gal}(\bar{F}/F)$ where the eigenvalues of Frobenius are sums of Gauss.

If χ is trivial on $\mathbb{F}_q^\times$, the sum $\tau_{\mathbb{F}_q}(\chi, \psi)$ is independent of ψ, and q times a sum of Jacobi. This is related to the cohomology of $W^*/\mathbb{G}_m$, with values in a sheaf derived from χ: ψ, and difficulty a), have disappeared. To solve b), it is enough to describe the sheaves used using

Kummer exact sequences (which make sense in any characteristic). This is possible thanks to 3.9.

Once the ℓ-adic representations are found, we use 4.10 and the Stickelberger theorem to show that they come from Hecke algebraic characters.

5.1

Let $n > 0$ be a multiple of the d_α, Σ the set of places of F where E/F ramifies or which divide n, μ a place of k with residual characteristic ℓ, and A the ring of elements of F that are integral outside of Σ and ℓ. The integral closure M of A is a finite étale algebra of rank $N = |I|$ over A. With the notation of 3.6, 3.8, the ${}_n\lambda_i$ define

$$ {}_n\lambda : (V^*/\mathbb{G}_m)_n \to k^\times \subset k_\mu^\times . $$

Let a be the projection of $W^*/\mathbb{G}_m$ onto $\mathrm{Spec}(A)$. According to 3.20, the sheaf $\mathsf{R}^{N-2}a_!\mathscr{K}_n({}_n\lambda)$ is a smooth rank one ℓ-adic

$$ j_\mu[\lambda] : \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} \to k_\mu^\times $$

(or simply j_μ) representation that is unramified outside of σ and ℓ.

5.2

We will now calculate $j_\mu(F_\mathfrak{p})$. According to 3.20, it is enough to do this after reducing modulo $\mathfrak{p}$. So let $f = A/\mathfrak{p}$, $e = M/\mathfrak{p}M$, and $\bar{f}$ be the algebraic closure of f defined by a place of $\bar{F}$ above $\mathfrak{p}$. We still have $I = \hom_f(e, \bar{f})$. After reduction, ${}_n\lambda$ still has a description **??**:

a) Let D be the set of orbits of $\mathrm{Gal}(\bar{f}/f)$ in I. The decomposition of e into a product of fields is $e = \prod_{\beta \in D} e_\beta$, where the σ_i $(i \in \beta)$ are identified with the embeddings of e_β into F. For $\beta \in D$, we note $\alpha(\beta)$ the element of C containing β and $d_\beta = d_{\alpha(\beta)}$. Let

$$ \lambda_\beta : \mu_{d_\beta}(e_\beta) \xleftarrow{\sim} \mu_{d_\beta}(E_{\alpha(\beta)}) \xrightarrow{\lambda_{d(\beta)}} k^\times . $$

b) If $i \in I$ corresponds to the embedding σ_i of e_β into $\bar{f}$, the reduction ${}_{d\beta}\bar{\lambda}_i : \mu_{d\beta}(\bar{f}) \to k^\times$ of ${}_{d\beta}\lambda_i$ of ${}_{d\beta}\lambda_i$ is $\lambda_\beta \circ \sigma_i^{-1}$.

Let q_β be the number of elements of e_β, $\chi_\beta =_{(q_\beta - 1)} \lambda_\beta$, and $\chi : e^\times \to k^\times$ of coordinates the χ_β. On the affine space on $A/\mathfrak{p}$ defined by e (3.6), or rather on the torus $e_\beta^\times$, there are isomorphisms $\mathscr{K}_n(({}_n\bar{\lambda}_i)_{i\in\beta}) = \mathscr{K}_{d_\beta}((\lambda_\beta \circ \sigma_i^{-1})_{i\in\beta}) = \mathscr{K}_{q_{\beta-1}}((\chi_\beta \circ \sigma_i^{-1})_{i\in I}) = \mathscr{F}(\chi_\beta)$ (3.9). On $e^\times$, we have $\mathscr{K}_n(({}_n\bar{\lambda}_i)_{i\in\beta}) = \mathscr{F}(\chi)$ and this isomorphism descends to the reduction mod $\mathfrak{p}$ of $V^*/\mathbb{G}_m$. Therefore

Proposition 5.3. *With the notation of 5.1 and 5.2, $j_\mu(F_\mathfrak{p})$ is the Jacobi sum of χ.*

Rewrite the formulas linking $(\lambda_i)_{i\in I}$ to $(\chi_\beta)_{\beta\in B}$: for $i \in \beta \subset \alpha$,

$$
\begin{array}{ll}
{}_{d_\alpha}\lambda_i = \lambda_\alpha \circ \sigma_i^{-1} & \lambda_\alpha : \mu_{d_\alpha}(E_\alpha) \to k^\times \\
\lambda_\beta = \text{reduction of } \lambda_\alpha & : \mu_{d_\alpha}(e_\beta) \to k^\times \\
\chi_\beta =_{q_{\beta-1}} \lambda_\beta : x \mapsto \lambda_\beta(x^{(q_\beta - 1)/d_\alpha}) & : e_\beta^\times \to k^\times
\end{array}
$$

Théorème 5.4. *The representation j_μ is defined by an algebraic Hecke character of F with values in $k^\times$.*

We will verify it using 4.10. We here join the demonstration of Weil [40, 42], by the use of the theorem of Stickelberger, and I will only give some indications.

5.5

Given $j : \mathrm{Gal}(\bar{F}/F) \to k_\lambda^\times$, and a finite extension F'/F, we verify using 4.10 that j is defined by a Hecke character if and only if $j|\,\mathrm{Gal}(\bar{F}/F')$ is. This remark allows us in 5.4 to reduce to the case where Galois acts trivially on I, and where F contains the n-th roots of unity. We can then reduce to the case where $F = \mathbb{Q}(\sqrt[n]{1})$, and then take $k = F$. In this case, each ${}_n\lambda_i$ is defined by $a_i \in \mathbb{Z}/n$: it is $x \mapsto x^{a_i} : \mu_n \to k^\times$. We have $\sum a_i = 0$, and $a_i \neq 0$.

5.6

In a number field, the places of degree 1 always form a set of density 0. By applying 4.10, we can therefore neglect the others, and only use Stickelberger for a prime field. So let p be prime, $0 < a < p-1$, $\widetilde{x} : \mathbb{F}_p^\times \to \mathbb{Q}_p^\times$ the "multiplicative lift" character and $g = -\sum_{x \in \mathbb{F}_p^\times} \widetilde{x}^{-a} \zeta^z \in \mathbb{Q}_p(\zeta)$, for ζ a p-th root of 1. Let x also denote the integer in $[0, p-1]$ relevant to x, and expand $\zeta^x = (1+\pi)x$ by the binomial formula. We find $g = -\sum_{0=0}^{p-1} A_i \pi^i$ with $A_i \in \mathbb{Z}_p$, of reduction $\mod p$ the sum $\sum_{x \in \mathbb{F}_p^\times} x^{-a}\binom{x}{i}$. For b not divisible by $p-1$, we have $\sum_{x \in \mathbb{F}_p^\times} x^b = 0$. If $i < a$, $\binom{x}{i}$ is a polynomial in x of degree $i < a$, whence $A_i \equiv 0 \pmod p$. Similarly, $A_a \equiv \frac{p-1}{a!} \pmod p$, and $g \sim \frac{\pi^a}{a!}$: the valuation of g is given by $v(g)/v(p) = \frac{a}{p-1}$.

5.7

Let σ_i $(i \in (\mathbb{Z}/n)^\times)$ be the element of $\mathrm{Gal}(\mathbb{Q}(\sqrt[n]{1})/\mathbb{Q})$ such that $\sigma_i(\zeta) = \zeta^i$ for $\zeta^n = 1$, and let us denote the group of algebraic homomorphisms $\mathbb{Q}(\sqrt[n]{1})^\times \to \mathbb{Q}(\sqrt[n]{1})^\times$ additively. For λ as in 5.5, defined by a family $(a_i)_{i \in I}$ of integers $\mod n$, not zero and with sum zero 4.10, (4.15.1) and 5.6 show that $j_\mu[\lambda]$ is defined by an algebraic Hecke character $j[a]$ and that, if N is the norm, its algebraic part is given by

$$(Nj[a])_{\mathrm{alg}} = \sum_{i \in (\mathbb{Z}/n)^\times} \sum_{j \in I} \left\lfloor \frac{ia_j}{n} \right\rfloor \sigma_i^{-1} \tag{6.8.1}$$

where $\lfloor \cdot \rfloor$ is the fractional part.

5.8

The most interesting case is when $F \subset \mathbb{Q}(\sqrt[n]{1})$, and the family of ${}_n\lambda_i$ is constructed as follows:

a) We take a finite family $(Aj)_{j \in J}$ of orbits of $\mathrm{Gal}(\sqrt[n]{1}/F) = H \subset (\mathbb{Z}/n)^\times$ in $\mathbb{Z}/n$. We assume that $A_j \neq \{0\}$ and that $\sum_{j \in J} \sum_{a \in A_j} a = 0$.

b) We take $I = \sqcup_{j \in J} A_j$; Galois acts on each A_j.

c) We take $\bar{F} \supset \mathbb{Q}(\sqrt[n]{1})$, $k = \mathbb{Q}(\sqrt[n]{1})$ and, if $i \in I$ corresponds to $a \in A_j \subset (\mathbb{Z}/n)^\times$, ${}_n\lambda_i(x) = x^a$.

We use 3.20 to verify that a general j_μ (relative to F_1) can be deduced from a j_μ as above (for $F \subset \mathbb{Q}(\sqrt[n]{1})$, F_1 an extension of F) by restriction to $\mathrm{Gal}(\bar{F}/F_1)$ and multiplication by a character of order two.

For $a \in \mathrm{Gal}(k/\mathbb{Q}) = (\mathbb{Z}/n)^\times$, a transforms the family of A_j into the family of aA_j. If $a \in H$, the family of A_j is therefore preserved, and the $j_\mu(F_\mathfrak{p})$ are invariant under H:

Proposition 5.9. *Under the hypotheses of 5.8, j_μ is defined by an algebraic Hecke character of F with values in F.*

Let us again take $F = \mathbb{Q}(\sqrt[n]{1})$, and consider the Hecke characters $j[a]$ defined by a family $(a_i)_{i\in I}$, $i \in \mathbb{Z}/n$, $a_i \neq 0$, $\sum a_i = 0$. The following theorem was obtained independently by Vishik.

Théorème 5.10. *Any algebraic Hecke character of $\mathbb{Q}(\sqrt[n]{1})$, a a power in the group generated by the $j[a]$ and the norm.*

Since we are working "modulo torsion," only the algebraic part of the Hecke characters considered matters. Applying (6.8.1) to the $j[a]$, with $a =$ an element of $\mathbb{Z}/n$ repeated n times, we are reduced to the following lemma.

Lemme 5.11. *Let $\sum x_j\sigma_j$ be the elements of the group algebra $\mathbb{Q}[(\mathbb{Z}/n)^\times]$, and let x be the subgroup of those such that $x_j + x_{-j} = C^{\underline{te}}$. Then, X is generated over $\mathbb{Q}$ by $N = \sum \sigma_j$ and by the $g_a = \sum \left\lfloor \frac{ia}{n} \right\rfloor \sigma)i^{-1}$ ($a \in \mathbb{Z}/n$, $a \neq 0$).*

Let X and $Y = \langle N, (g_a)\rangle$ be ideals of the group algebra, and it is enough to check that they are annihilated by the same complex characters χ of $(\mathbb{Z}/n)^\times$. This is equivalent to saying that an odd character ($\chi(j) = -\chi(-j)$) never annihilates all the g_a. If χ is primitive, we have

$$\chi(g_1) = \sum \frac{i}{n}\chi^{-1}(i) = -L(\chi, 0) \neq 0.$$

In the general case, if χ comes from a primitive character of $(\mathbb{Z}/n)^\times$, $\chi(g_{n/d})$ is reduced to a similar sum on $(\mathbb{Z}/n)^\times$, from which the lemma follows.

6 Generalized Kloosterman sums

6.1

Let $\mathbb{F}_q$ be a finite field with q elements, and ψ be a non-trivial additive character. In this paragraph, we study the cohomological properties of the generalized Kloosterman sums

$$\mathrm{K}_{n,a} = \sum_{x_1\cdots x_n = a} \psi(x_1 + \cdots + x_n) \qquad (n \geqslant 1). \tag{7.1.1}$$

For $a = 0$, this sum is elementary (cf. 6.7):

$$\mathrm{K}_{n,0} = (-1)^{n-1}. \tag{7.1.2}$$

The interesting case is when $a \neq 0$. The x_i are then in $\mathbb{F}_q^\times$. We will prove in this case an upper bound

$$|K_{n,a}| \leqslant nq^{\frac{n-1}{2}}. \tag{7.1.3}$$

Our essential tools will be the cohomological statements of the following identities

$$\begin{aligned} \mathrm{K}_{n,a} &= \sum_{x_1} \psi(x_i) \sum_{x_2 \cdots x_n = a/x_1} \psi(x_2 + \cdots + x_n) \\ &= \sum_{x \in \mathbb{F}_q^\times} \psi(x) \mathrm{K}_{n-1,a/x} \qquad (a \neq 0, n \geqslant 2) \end{aligned} \tag{7.1.4}$$

$$\sum_a \mathrm{K}_{n,a} = \sum_a \sum_{x_1 \cdots x_n = a} \psi\left(\sum x_i\right) = \sum_{x_1,\ldots,x_n} \psi\left(\sum x_i\right) = 0 \tag{7.1.5}$$

For χ a non-trivial character of $\mathbb{F}_q^\times$,

$$\sum \chi(a) \mathrm{K}_{n,a} = \sum_{x_1,\ldots,x_n} \chi\left(\prod x_i\right) \cdot \psi\left(\sum x_i\right) = (-\tau(\chi^{-1}, \psi))^n \tag{7.1.6}$$

(Gauss sum).

6.2

Let k be an étale algebra of degree n over $\mathbb{F}_q$, and $\varepsilon(k)$ as in (4.5.1). Let

$$\mathrm{K}_{k,a} = \sum_{N_{k/\mathbb{F}_q}(x) = a} \psi\,\mathrm{tr}(x).$$

We still have

$$\mathrm{K}_{k,0} = (-1)^{n-1} \cdot \varepsilon(k) \tag{7.2.1}$$

$$\sum_a \mathrm{K}_{k,a} = 0 \tag{7.2.2}$$

$$\sum_a \chi(a) \mathrm{K}_{k,a} = (-1)^n \tau_{\mathbb{F}_q}\left(\chi^{-1} \circ N_{k/\mathbb{F}_q}, \psi\right). \tag{7.2.3}$$

By Fourier inversion on $\mathbb{F}_q^\times$, we deduce an expression of $\mathrm{K}_{k,a}$ in terms of Gauss sums:

$$\mathrm{K}_{k,a} = \frac{1}{q-1} \sum_\chi \chi(a) \sum_{b \in \mathbb{F}_q^\times} \chi^{-1}(b) \mathrm{K}_{k,b}$$

$$(-1)^n \mathrm{K}_{k,a} = \frac{1}{q-1} \left(\varepsilon(k) + \sum_{\chi \neq 1} \chi(a) \tau_{\mathbb{F}_q}\left(\chi \circ N_{k/\mathbb{F}_q}, \psi\right) \right). \tag{7.2.4}$$

The Hasse-Davenport identity now allows us to compare $\mathrm{K}_{k,a}$ to $\mathrm{K}_{n,a} = \mathrm{K}_{\mathbb{F}_{q^n},a}$:

$$\mathrm{K}_{k,a} = \varepsilon(k) \mathrm{K}_{n,a}. \tag{7.2.5}$$

I do not know of a cohomological interpretation of (7.2.4), but I will give one of (7.2.5). Let us return to the sums $\mathrm{K}_{n,a}$.

6.3

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$ and, for $a \in \mathbb{F}$, let V_a^{n-1}, or simply V_a, be the hypersurface of $\mathbb{A}^n$ with equation $x_1 \cdots x_n = a$. As usual, we will regard ψ as being ℓ-adic rather than complex-valued. With the notation 1.8(ii), (7.1.3) follows from the

Théorème 6.4. *The cohomology of V_a^{n-1} with values in $\mathscr{F}(\psi(\sum x_i))$ satisfies*

(i) $\mathrm{H}_c^i = 0$ *for* $i \neq n-1$*;*

(ii) $\mathrm{H}_c^\bullet \xrightarrow{\sim} \mathrm{H}^\bullet$ *;*

(iii) for $a \neq 0$*,* $\dim \mathrm{H}_c^{n-1} = n$*;*

(iv) for $a = 0$*,* H_c^{n-1} *is canonically isomorphic to* E_λ*.*

6.5

As usual, this theorem also controls the dependence of $\mathrm{K}_{n,a}$ on q: for each $a \in \mathbb{F}_q^\times$, there are n eigenvalues of Frobenius $\alpha_1, \ldots, \alpha_n$, of complex absolute value $q^{\frac{n-1}{2}}$, such that

$$\sum_{\substack{x_1 \cdots x_n = a \\ x_i \in \mathbb{F}_{q^m}}} \psi \circ \mathrm{tr}\left(\sum x_i\right) = (-1)^{n-1}\left(\alpha_1^m + \cdots + \alpha_n^m\right).$$

For even n, $x \mapsto -x$ is an involution of V_a, which transforms $\mathscr{F}(\psi)$ into its dual. Reasoning as in 2.14, we deduce a canonical alternating non-degenerate form on $\mathrm{H}_c^{n-1}(V_a, \mathscr{F}(\psi))$, with values in $\mathbb{Q}_\ell(-(n-1))$ and the

Corollaire 6.6. *With the notation of 6.5, if n is even, the α_i are grouped into $\frac{n}{2}$ pairs of roots $\alpha, \bar{\alpha}$ of product equal to q^{n-1}.*

6.7

Let us check the case $a = 0$ of 6.5. The hypersurface V_0 is the union of the coordinate hyperplanes $x_i = 0$. Let us compute the Leray spectral sequence (2.6.1*) of this covering of V_0, for cohomology with or without support in $\mathscr{F}(\psi)$. According to 2.7*, all initial terms are zero, except the cohomology of the intersection $\{0\}$ of all coordinate hyperplanes. There remains an isomorphism $\mathrm{H}_c^0 \xrightarrow{\sim} \mathrm{H}^0 = E_\lambda$ which justifies 6.4.

We will prove 6.4 and 6.8 below by a simultaneous recurrence on n, and relying on the spectral sequence of which (7.1.4) is the reflection. This requires a control of the dependence on a of $\mathrm{K}_{n,a}$.

Let $\pi : \mathbb{A}^n \to \mathbb{A}$ be the "product of coordinates" map, and σ the sum of coordinates. The hypersurfaces V_a are the fibers of π.

Théorème 6.8. *(i) The sheaf $\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is lisse of rank n on $\mathbb{A}^1 \setminus \{0\}$.*

(ii) its extension by 0 over $\mathbb{P}^1$ is the direct image of its restriction to $\mathbb{A}^1 \setminus \{0\}$.

(iii) in 0, the monodromy is unipotent, with a single Jordan block

(iv) in ∞, the wild inertia acts without fixed point $\neq 0$, and the Swan conductor is 1,

(v) we have $\mathsf{R}^i\pi_!\mathscr{F}(\psi\sigma) \xrightarrow{\sim} \mathsf{R}^i\pi_*\mathscr{F}(\psi\sigma)$ (zero for $i \neq n-1$).

Preuve. of what 6.4 (for a given value of n) implies 6.8 (for the same value of n).

The fiber of $\mathsf{R}^i\pi_!\mathscr{F}(\psi\sigma)$ at a is $\mathrm{H}^i_c(V_a, \mathscr{F}(\psi\sigma))$, and on the dense open $\mathbb{A}^1$ where the $\mathsf{R}^i\pi_!\mathscr{F}(\psi^{-1}\sigma)$ are locally constant, that of $\mathsf{R}^i\pi_*\mathscr{F}(\psi\sigma)$ is $\mathrm{H}^i(V_a, \mathscr{F}(\psi\sigma))$ (Th. Finitude 2.1). The hypothesis 6.4(n) provides us with the

Lemme 6.9. *(i)* $\mathsf{R}^i\pi_!\mathscr{F}(\psi\sigma) = 0$ *for* $i \neq n-1$.

(ii) For $i = n-1$*, the fiber of this sheaf at any point* $a \neq 0$ *is of constant rank* n*. At* 0*, it is of rank* 1.

(iii) On a dense open U*, we have* $\mathsf{R}^i\pi_!\mathscr{F}(\psi) \xrightarrow{\sim} \mathsf{R}^i\pi_*\mathscr{F}(\psi)$.

6.10

We will now use the Leray spectral sequence of π, for cohomology with or without support, with coefficients in $\mathscr{F}(\psi\sigma)$. The outcome is known: $\mathrm{H}^\bullet_c(\mathbb{A}^n, \mathscr{F}(\psi\sigma)) = \mathrm{H}^\bullet(\mathbb{A}^n, \mathscr{F}(\psi\sigma)) = 0$ (2.7*). In proper support cohomology, 6.9(i) ensures that $E_2^{pq} = 0$ for $q \neq n-1$. The E_2 terms are therefore all zero:

$$\mathrm{H}^p_c(\mathbb{A}^1, \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)) = 0. \tag{7.10.1}$$

For $p = 0$, this means that $\mathsf{R}^{n-1}\pi+!\mathscr{F}(\psi\sigma)$ has no section with punctual support. Given the constant rank 6.9(ii), we deduce

$$\text{The sheaf } \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma) \text{ on } \mathbb{A}^1 \text{ is smooth on } \mathbb{A}^1 \smallsetminus \{0\}, \text{ and is a subsheaf of the direct image } \mathscr{G} \text{ of its restriction to } \mathbb{A}^1 \smallsetminus \{0\}. \tag{7.10.2}$$

The exact long cohomology sequence of

$$0 \longrightarrow \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma) \longrightarrow \mathscr{G} \longrightarrow \mathscr{Q} \longrightarrow 0.$$

where the support of $\mathscr{Q}$ is concentrated in 0, gives

$$\mathrm{H}^0_c(\mathbb{A}^1, \mathscr{G}) \longrightarrow \mathscr{Q}_0 \longrightarrow \mathrm{H}^1_c(\mathbb{A}^1, \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)).$$

The extreme terms being zero, $\mathscr{Q} = 0$ and

$$\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma) \text{ is the direct image of its restriction to } \mathbb{A}^1 \smallsetminus \{0\}. \tag{7.10.3}$$

Let j be the inclusion of $\mathbb{A}^1$ in $\mathbb{P}^1$. The "derived category" version of the Leray spectral sequences is

$$\mathsf{H}^\bullet(\mathbb{P}^1, j_!\mathsf{R}\pi_!\mathscr{F}(\psi\sigma)) = \mathsf{H}^\bullet_c(\mathbb{A}^1, \mathsf{R}\pi_!\mathscr{F}(\psi\sigma)) = \mathrm{H}^\bullet_c(\mathbb{A}^n, \mathscr{F}(\psi\sigma)) = 0$$
$$\mathsf{H}^\bullet(\mathbb{P}^1, \mathsf{R}j_*\mathsf{R}\pi_*\mathscr{F}(\psi\sigma)) = \mathsf{H}^\bullet(\mathbb{A}^1, \mathsf{R}\pi_*\mathscr{F}(\psi\sigma)) = \mathrm{H}^\bullet(\mathbb{A}^n, \mathscr{F}(\psi\sigma)) = 0.$$

Let Δ be the mapping cylinder of $j_!\mathsf{R}\pi_!\mathscr{F}(\psi\sigma) \to \mathsf{R}j_*\mathsf{R}\pi_*\mathscr{F}(\psi\sigma)$. According to 6.9(iii), the cohomology sheaves of this complex of sheaves have finite support. On the other hand, the exact long cohomology sequence of the triangle $(j_!\mathsf{R}\pi_!\mathscr{F}(\psi\sigma), \mathsf{R}j_*\mathsf{R}\pi_*\mathscr{F}(\psi\sigma), \Delta)$ shows that $\mathsf{H}^\bullet(\mathbb{P}^1, \Delta) = 0$. We have $\mathsf{H}^\bullet(\mathbb{P}^1, \Delta) = \mathrm{H}^0(\mathbb{P}^1, \mathscr{H}^\bullet(\Delta))$, and finally $\Delta = 0$:

$$j_!\mathsf{R}\pi_!\mathscr{F}(\psi\sigma) \xrightarrow{\sim} \mathsf{R}j_*\mathsf{R}\pi_*\mathscr{F}(\psi\sigma).$$

In particular, $\mathsf{R}\pi_!\mathscr{F}(\psi\sigma) \xrightarrow{\sim} \mathsf{R}\pi_*\mathscr{F}(\psi\sigma)$ and, these complexes of sheaves having only one non-zero cohomology sheaf,

$$j_!\mathsf{R}\pi_!\mathscr{F}(\psi\sigma) \xrightarrow{\sim} j_*\mathsf{R}\pi_!\mathscr{F}(\psi\sigma). \tag{7.10.4}$$

This completes the proof of (i) (ii) (v).

6.11

Let Sw_0 and Sw_∞ be the Swan conductors at 0 and at ∞ of $\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)$. We have $\chi_c(\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)) = 0$ (7.10.1) and the Euler-Poincaré formula (3.2.1) reduces to

$$\mathrm{Sw}_0 + \mathrm{Sw}_\infty = 1: \tag{7.11.1}$$

one of these conductors is 0 (moderate ramification), the other is 1.

6.12

Let $\mathscr{G}$ be a sheaf of rank 1 on $\mathbb{G}_m$, of Kummer type 3.7 ($\mathscr{G} = \mathscr{K}_n(\chi)$) and not constant ($\chi \neq 1$). The sheaf $\mathscr{F}(\psi\sigma) \otimes \pi^*\mathscr{G}$ on $V^* = \pi^{-1}(\mathbb{G}_m) = \mathbb{A}^n \smallsetminus \pi^{-1}(0)$ is then of the type encountered in the study of Gauss sums, and 3.11(ii) gives us

$$\mathrm{H}_c^\bullet(V^*, \pi^*\mathscr{G} \otimes \mathscr{F}(\psi\sigma)) \xrightarrow{\sim} \mathrm{H}^\bullet(V^*, \pi^*\mathscr{G} \otimes \mathscr{F}(\psi\sigma)). \tag{7.12.1}$$

The Leray spectral sequence of π transforms this statement into

$$\mathrm{H}_c^\bullet(\mathbb{G}_m, \mathscr{G} \otimes \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathbb{G}_m, \mathscr{G} \otimes \mathsf{R}^{n-1}\pi_*\mathscr{F}(\psi\sigma)). \tag{7.12.2}$$

Let i be the inclusion of $\mathbb{G}_m$ in $\mathbb{P}^1$, and Δ the mapping cylinder of $\delta : i_!(\mathscr{G}\otimes\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi)) \to \mathsf{R}i_*(\mathscr{G} \otimes \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma))$ (where $\mathsf{R}^{n-1}\pi_! = \mathsf{R}^{n-1}\pi_*$). Reasoning as in 6.10, we deduce from (7.12.2) that $\Delta = 0$. In particular

$$\text{In 0 and in } \infty\text{, the inertia acts without fixed points} \neq 0 \text{ on } \mathscr{G} \otimes \mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma). \tag{7.12.3}$$

In ∞, this is even true if $\mathscr{G}$ is trivial (7.10.4), so that the wild inertia acts without fixed points. In particular, the ramification is wild: according to 6.11, we have $\mathrm{Sw}_\infty = 1$, and the ramification in 0 is moderate.

In 0, (7.12.3) imposes on the ramification – moderate – of $\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ to be unipotent. According to (7.10.3) and 6.9(ii), we have only one Jordan block. This finishes the proof of 6.8(n).

6.13

For $n = 1$, 6.4 is trivial. To finish the proof of 6.4 and 6.8, it remains to prove that 6.8, for a given value of n, implies 6.4, for $n + 1$. The case $a = 0$ having been treated (6.7), we assume that $a = 0$. We will write $x_0, \ldots, x_n$ for the coordinates of $\mathbb{A}^{n+1}$. Let $g = x_0 : V_a \to \mathbb{G}_m$, τ the involution $x \mapsto ax^{-1}$ of $\mathbb{G}_m$, and let π be the product of the coordinates: $\mathbb{A}^{n+1} \to \mathbb{A}^1$, and σ their sum. We will denote $\mathscr{F}(\psi\sigma)$ the sheaf $\mathscr{F}(\psi(\sum x_i))$ both on $\mathbb{A}^{n+1}$ and on $\mathbb{A}^n$.

6.14

Let's write the Leray spectral sequence of g. The sheaf $\mathscr{F}(\psi\sigma)$ on $\mathbb{A}^{n+1}$ being the exterior tensor product of the sheaves $\mathscr{F}(\psi)$ on $\mathbb{A}^1$ and $\mathscr{F}(\psi\sigma)$ on $\mathbb{A}^n$, we find (cf. (7.1.4)).

$$\begin{aligned} {}'E_2^{pq} &= \mathrm{H}_c^p(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \tau^*\mathsf{R}^q\pi_!\mathscr{F}(\psi\sigma)) \Rightarrow \mathrm{H}_c^{p+q}(V_a, \mathscr{F}(\psi\sigma)) \\ {}''E_2^{pq} &= \mathrm{H}^p(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \tau^*\mathsf{R}^q\pi_*\mathscr{F}(\psi\sigma)) \Rightarrow \mathrm{H}^{p+q}(V_a, \mathscr{F}(\psi\sigma)). \end{aligned}$$

By hypothesis, $\mathsf{R}^q\pi_!\mathscr{F}(\psi\sigma) \xrightarrow{\sim} \mathsf{R}^q\pi_*\mathscr{F}(\psi\sigma)$, and this sheaf is null for $q \neq n-1$. The sheaf $\mathscr{F}(\psi)$ on $\mathbb{G}_m$ is wildly ramified at ∞ and unramified at 0, while $\tau^*\mathsf{R}^q\pi_!\mathscr{F}(\psi\sigma)$ is wildly ramified at 0 (without invariant under the wild inertia, of Swan conductor 1) and mildly ramified at ∞. Their tensor product is therefore

a) wildly ramified at 0 and ∞, without invariant under the wild inertia;

b) of Swan conductor 1 at 0 and n (the rank of $\mathsf{R}^q\pi_!\mathscr{F}(\psi)$) at ∞.

We have ${}'E_2^{pq} \xrightarrow{\sim} {}''E_2^{pq}$, and ${}'E_2^{pq} = 0$ except for $p = 1$ and $q = n-1$. The Euler-Poincaré formula finally gives

$$\dim {}'E_2^{pq} = 1 + n;$$

this completes the proof. □

6.15 Remark

The isomorphism obtained in 6.14

$$\mathrm{H}_c^n(V_a^n, \mathscr{F}(\psi\sigma)) = \mathrm{H}_c^1(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \tau^*\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi\sigma)) \tag{7.15.1}$$

allows, with the notation of 6.5, to calculate the product $\det(F)$ of the eigenvalues α_i of Frobenius. The formalism of [13] applies, since $\mathscr{F}(\psi)$ and $\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi)$ belong to compatible infinite systems of ℓ-adic representations. This reduces the problem to a local one, which can be simplified by noting that (cf. [13, 9.5])

a) the global constants for $\mathscr{F}(\psi)$ and $\tau^*\mathsf{R}^{n-1}\pi_!\mathscr{F}(\psi)$ are easy to calculate, the cohomology of these sheaves being of trivial nature.

b) at any point, for either of these sheaves, the corresponding ℓ-adic representation of the decomposition group is unramified and semisimple.

The result, for the sum in n variables, is that

$$\det\left(F, \mathrm{H}_c^{n-1}(V_a^{n-1}, \mathscr{F}(\psi))\right) = q^{\frac{n(n-1)}{2}}. \tag{7.15.2}$$

6.16 Remark

For $p = 2$, the sheaf (ψ) is orthogonal. For $a \neq 0$ and n odd, we deduce from 6.4(ii) and Poincaré duality a symmetric bilinear form on $\mathrm{H}_c^{n-1}(V_a^{n-1}, \mathscr{F}(\psi))$, with values in $\bar{\mathbb{Q}}_\ell(1-n)$. Relative to this form, F is an orthogonal similarity of multiplier q^{n-1}, and $q^{\frac{1-n}{2}}F$ belongs to the special orthogonal group, according to (7.15.2). One of the eigenvalues of F is therefore $q^{\frac{1-n}{2}}$, and the others are arranged in pairs of roots α, $\bar{\alpha}$ of product q^{n-1}.

For $n = 3$, L. Carlitz [8] obtained a more precise result, equivalent to the following proposition.

Proposition 6.17. *For $p = 2$, and $\psi(x) = (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_q}(x)}$, there is an isomorphism*

$$\mathrm{H}_c^2(V_a^2, \mathscr{F}(\psi\sigma)) = \mathrm{Sym}^2\, \mathrm{H}_c^1(V_a^1, \mathscr{F}(\psi\sigma)).$$

The second tensor power of $\mathrm{H}_c^1(V_a^1, \mathscr{F}(\psi\sigma))$ is $\mathrm{H}_c^1(V_a^1 \times V_a^1, \mathscr{F}(\psi\sigma) \boxtimes \mathscr{F}(\psi\sigma))$ (Künneth), the symmetry $x \otimes y \mapsto y \otimes x$ is represented geometrically as $-\tau^*$, where τ is the automorphism $(x, y) \mapsto (y, x)$ of $V_a \times V_a$. If X' is the quotient of $V_a^1 \times V_a^1$ by $\{\mathrm{id}\,, \tau\}$ and π is the projection of $V_a^1 \times V_a^1$ onto X', the group $\mathrm{Sym}^2\, \mathrm{H}_c^1(V_a^1, \mathscr{F}(\psi\sigma))$ is therefore identified with the cohomology of X' with coefficients in the anti-invariant part by τ of $\pi_*(\mathscr{F}(\psi\sigma) \boxtimes \mathscr{F}(\psi\sigma))$. Let s be the function on X' such that $s\pi = \sigma\mathrm{pr}_1 + \sigma\mathrm{pr}_2$. We have $\mathscr{F}(\psi\sigma) \boxtimes \mathscr{F}(\psi\sigma) = \mathscr{F}(\psi(\sigma\mathrm{pr}_1 + \sigma\mathrm{pr}_2)) = \mathscr{F}(\psi s\pi) = \pi^*\mathscr{F}(\psi s)$ (isomorphism compatible with τ). On the image $\pi(\Delta)$ of the diagonal, the sought anti-invariant sheaf is therefore zero, while on the complement $X = X' \smallsetminus \pi(\Delta)$, it is the tensor product of $\mathscr{F}(\psi s)$ by $\varepsilon(P)$, for ε the unique character of order 2 of $\{1, \tau\}$, and P the $\{1, \tau\}$-torsor over X that is the double covering induced by $V_a^1 \times V_a^1$. We have

$$\mathrm{Sym}^2\, \mathrm{H}_c^1(V_a^1, \mathscr{F}(\psi\sigma)) = \mathrm{H}_c^2(X, \mathscr{F}(\psi\sigma) \otimes \varepsilon(P)). \tag{7.17.1}$$

We calculate X, s, and $\varepsilon(P)$. On $V_a^1 \times V_a^1$, we have $x_1 = x_1 \circ \mathrm{pr}_1$, $x_2 = x_2 \circ \mathrm{pr}_2$, $x_1' = x_1 \circ \mathrm{pr}_2$, $x_2' = x_2 \circ \mathrm{pr}_2$. We also have $x_1 x_2 = x_1' x_2' = a$. We identify functions on X' and functions on X that are invariant under τ. Let $s_1 = x_1 + x_1'$, $s_2 = x_2 + x_2'$, $p_1 = x_1 x_1'$, $p_2 = x_2 x_2'$. We have $s_1 p_2 = a s_2$, $s_2 p_1 = a s_1$ and X is defined in X' by $s_1 \neq 0$, $s_2 \neq 0$. So on X, we have $p_1 = a s_1 s_2^{-1}$, and (s_1, s_2) identifies x with $\mathbb{G}_m \times \mathbb{G}_m$. On X, the covering P has the equation $T^2 - s_1 T + a s_1 s_2^{-1}$ (take for T the coordinate x_1). If $T = s_1 t$, this can be rewritten as $t^2 - t + a s_1^{-1} s_2^{-1}$. So we have $\varepsilon(P) = \mathscr{F}(\psi(a s_1^{-1} s_2^{-1}))$. Since $s = s_1 + s - 2$, we have $\mathscr{F}(\psi s) \otimes \varepsilon(P) = \mathscr{F}(\psi(s_1 + s_2 + a s_1^{-1} s_2^{-1}))$ and the second term of (7.17.1) is identified with the first term of 6.17.

Remark $\mathrm{H}_c^1(V_a^1, \mathscr{F}(\psi\sigma))$ is also the first cohomology group of the completed elliptic curve of the affine curve with equation

$$\begin{aligned} y^2 - y &= x_1 + x_2 \\ x_1 x_2 &= a; \end{aligned}$$

its modular invariant is $j = a^{-2}$.

6.18 Remark

For χ a multiplicative character of $\mathbb{F}_q^\times$, and $a \neq 0$, the method of 6.14 can be applied to the study of the sum

$$s = \sum_{x_0 \cdots x_n = a} \chi(x_0)\psi(x_0 + \cdots + x_n)$$

(replace the sheaf $\mathscr{F}(\psi)$ on $\mathbb{G}_m$ with $\mathscr{F}(\chi \cdot \psi)$). We find cohomology of dimension $n + 1$ again, and $|S| \leqslant (n + 1)q^{n/2}$. Such sums were considered by Salie and Mordell [27].

We can hope that our methods allow us to study the sums

$$S_{k,\chi,a} = \sum_{N(x) = a} \chi(x)\psi\, \mathrm{tr}(x)$$

(k an étale algebra over $\mathbb{F}_q$, $a \in \mathbb{F}_q^\times$, χ a character of $k^\times$). The analogue of 6.4 (for $a \neq 0$) should be true.

6.19

The symmetric group S_n acts on $\mathbb{A}^n$ respecting V_a and the map σ, so also the sheaf $(\psi\sigma)$, the image by σ of the sheaf (ψ) on $\mathbb{G}_a$. So it acts on $\mathrm{H}_c^{n-1}(V_a, \mathscr{F}(\psi\sigma))$. The formula (7.2.5) has the following cohomological interpretation

Proposition 6.20. *The action of S_n on $\mathrm{H}_c^{n-1}(V_a, \mathscr{F}(\psi\sigma))$ is multiplication by the sign character.*

This is equivalent to showing that a transposition τ acts by multiplication by -1. Since $\tau^2 = 1$, τ can only have eigenvalues ± 1. Let V_a/τ be the quotient of V_a by $\{\mathrm{id}, \tau\}$. The map σ, so the sheaf $(\psi\sigma)$, pass to the quotient, and the subspace of $\mathrm{H}_c^{n-1}(V_a, \mathscr{F}(\psi))$ fixed by τ is $\mathrm{H}_c^{n-1}(V_a/\tau, \mathscr{F}(\psi\sigma))$. We need to prove that this cohomology is zero.

Suppose that $a \neq 0$. For $\tau = (1, 2)$ the quotient V_a/τ is identified with $\mathbb{G}_a \times \mathbb{G}_m^{n-2}$, with the map of passage to the quotient $(x_1, x_2, x_3, \ldots, x_n) \mapsto (x_1 + x_2, x_3, \ldots, x_n)$. Indeed, x_1 and x_2 are determined up to order by $x_1 + x_2$ and $x_1 x_2 = a/x_3 \cdots x_n$. In these coordinates $(t, x_3, \ldots, x_n)$, the sheaf is written $\mathscr{F}(\psi(t + x_3 + \cdots + x_n)) = \mathscr{F}(\psi(t)) \otimes \mathscr{F}(\psi(x_3 + \cdots + x_n))$. The Künneth formula, and $\mathrm{H}_0^\bullet(\mathbb{G}_a, \mathscr{F}(\psi)) = 0$, provide therefore the desired vanishing.

For $a = 0$, we can deduce 6.20 from calculation 6.7.

6.21

The sums $\mathrm{K}_{n,a}$ from this paragraph were studied by S. Sperber, in his thesis, and in [36] by p-adic methods inspired by Dwork and Bombieri's article [6]. For $p \neq 2$, these methods give him the q-dependence in 6.5, the functional equation $\alpha_i = q^{n-1}\beta_i^{-1}$ between the eigenvalues of Frobenius for the sums $\mathrm{K}_{n,a}$ and $\mathrm{K}_{n,b}$ if $b = (-1)^n a$ (cf. 6.6), and the value in 6.15 of the product of the roots. For $p \geqslant n + 3$, he also determines the p-adic valuations of the roots. The result is very beautiful: if we arrange the roots α_i in a convenient order, with $0 \leqslant i \leqslant n$, the $\alpha_i q^{-i}$ are units. The restrictions on p are probably not essential.

7 Other applications

7.1

The identity 1.9.4 can sometimes be used to calculate the number of rational points of an algebraic variety over $\mathbb{F}_q$. In most cases where an exact result has been obtained, the cohomology is expressed in terms of algebraic cycles; this is the case for rational surfaces, quadrics, smooth intersections of two odd-dimensional quadrics, flag varieties. . . . However, this is not always the case; in the very beautiful article by Lusztig: [26], algebraic cycles do not appear.

7.2

For reductive groups, cohomology can be calculated by lifting to characteristic 0: if B is a Borel subgroup of G, G is a B-torsor over G/B; the cohomology of the projective and smooth variety G/B is invariant under specialization, and we use the Leray spectral sequence of $G \to G/B$ to prove the same result for G. It is conveniently expressed in terms of the *maximal* torus T of G and the action of the *Weil group* W on T (for the definition of the maximal torus, see [7, VIII §2 Rem.2] or [10, p.105]). It is the exterior algebra of its primitive part, which coincides up to a shift with the quotient $I(\mathrm{H}^\bullet(BG, \mathbb{Q}_\ell))$ of the cohomology of BG formed by the "indecomposable elements," and $\mathrm{H}^\bullet(BG, \mathbb{Q}_\ell) = \mathrm{H}^\bullet(BT, \mathbb{Q}_\ell)^W$. If X is the group of characters of T, we have in total

$$\mathrm{H}^\bullet(G, \mathbb{Q}_\ell) = \bigwedge \left(I\left(\mathrm{Sym}^\bullet(X \otimes \mathbb{Q}_\ell(-1))^W\right)[-1]\right).$$

If G is defined over a finite field $\mathbb{F}_q$, $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on X and this isomorphism is compatible with Galois. Passing from there to the cohomology with compact support, we find the classical formulas for the number of rational points of G. If $F^* : X \to X$ is defined by the Frobenius morphism $F : T \to T$, and $\deg_G(F)$ is the degree $q^{\dim G}$ of $F : G \to G$, we have

$$|G_0(\mathbb{F}_q)| = \deg_G(F) \cdot \det\left(1 - F^{*-1}, I(\mathrm{Sym}^\bullet(X \otimes \mathbb{Q})^W)\right). \tag{8.2.1}$$

The same formula applies to the Ree and Suzuki groups; these groups are of the form G^F (G reductive, F^2 a Frobenius), and it is enough to check that the Lefschetz trace formula is true for F acting on G. If B is a Borel subgroup stable by F, we check it directly for F acting on B, and for the action on G/B, proper and smooth, we can invoke the general theorems.

7.3

The trace formula was used by Deligne-Lusztig [10], Kazhdan [24] and Springer [37] in the study of the complex representations of the finite groups $G_0(\mathbb{F}_q)$, for G_0 reductive over $\mathbb{F}_q$. The works of Kazhdan and Springer contain admirable examples of how the trace formula allows the "analytic extension" of a deployed situation to a non-deployed situation (see 1.13).

Chapter 7

Finiteness theorems in ℓ-adic cohomology

1 Statement of theorems

Throughout this paper, S will be a Noetherian scheme and A a Noetherian left ring, with torsion annihilated by an invertible integer on S. Our main result is the following.

Théorème 1.1. *We assume S is regular of dimension 0 or 1. Let $f : X \to Y$ be a morphism of S-schemes of finite type and $\mathscr{F}$ a constructible left A-module sheaf on X. Then, the sheaves $\mathsf{R}^i f_* \mathscr{F}$ are constructible.*

The proof will be given in paragraph 2 and in 3.10.

1.2 Remark

In characteristic 0, this result is less general than [2, XIX paragraph 5], which proves the conclusion of the theorem for any finite type morphism of excellent schemes of characteristic 0. The proof of [2, XIX] uses on the one hand the resolution of singularities, on the other hand that the schemes are of equal characteristic (to be able to deduce from the resolution the "purity theorem").

1.3 Remark

We will say that a complex $K \in \operatorname{Ob} \mathsf{D}(X, A)$ is *constructible* if its cohomology sheaves are constructible and we will denote with an index c the subcategory of $\mathsf{D}(X, A)$ (or D^+, D^-, D^b) formed by constructible complexes. In the language of derived categories, which we will freely use, 1.1 says that $\mathsf{R}f_* : \mathsf{D}^+(X, A) \to \mathsf{D}^+(Y, A)$ sends D^+_c to D^+_c. Let us explain:

a) For K reduced to a sheaf $\mathscr{F}$ in degree 0, we have $\mathscr{H}^i \mathsf{R}f_* K = \mathsf{R}^i f_* \mathscr{F}$; the derived statement implies therefore the theorem.

b) In the other direction, we invoke the spectral sequence

$$E_1^{pq} = \mathsf{R}^q f_* \mathscr{H}^p(K) \Rightarrow \mathscr{H}^{p+q} \mathsf{R} f_* K.$$

Speaking the language of derived categories has the advantage of replacing a simple transitivity formula $\mathsf{R}(fg)_* = \mathsf{R}f_*\mathsf{R}g_*$ with a Leray spectral sequence $E_2^{pq} = \mathsf{R}^p f_* \mathsf{R}^q g_* \Rightarrow \mathsf{R}^{p+q}(fg)_*$.

Under the hypotheses of 1.1, it follows from [2] that $\mathsf{R}f_*$ has finite cohomological dimension. This allows us to replace D^+ with D, D^- or D^b above.

1.4 The ideas of the proof

a) Poincaré duality allows us to deal with the case where X is smooth, $\mathscr{F}$ locally constant, and $Y = S$. The dimension hypothesis appears when calculating $\mathsf{R}\mathscr{H}om$ on S.

b) We factorize f as gj: g proper and j open immersion. We have $\mathsf{R}f_* = \mathsf{R}g_*\mathsf{R}j_*$ and the finiteness theorem for proper morphisms controls $\mathsf{R}g_*$. Unscrewing, we may assume x is smooth, $\mathscr{F}$ locally constant, f is an open immersion, and Y is proper over S.

c) A recurrence on $\dim X$ (with a change of S) allows, roughly speaking, to assume the theorem is true outside a finite part of Y over S. Noting $a : X \to S$ and $b : Y \to S$ are the structural morphisms, we then show that if the theorem were false for $\mathscr{F}$ and $\mathsf{R}f_*$, it would also be false for $\mathscr{F}$ and $\mathsf{R}b_*\mathsf{R}f_* = \mathsf{R}a_*$: contradiction.

Corollaire 1.5. *Under the hypotheses of the theorem, the categories $\mathsf{D}_c(X, A)$ and $\mathsf{D}_c(Y, A)$ are transformed into each other by the 4 operations $\mathsf{R}f_*$, $\mathsf{R}f_!$, f^*, $\mathsf{R}f^!$.*

$\mathsf{R}f_*$ is treated above, $\mathsf{R}f_!$ in [2, XVII 5.3], f^* is clear. Remains $\mathsf{R}f^!$. The problem is local, which allows us to suppose that f factors as $X \xrightarrow{i} Z \xrightarrow{g} Y$ (i closed immersion and g smooth, purely of relative dimension n). The Poincaré duality $\mathsf{R}g_!K(n)[2n]$ and the transitivity $\mathsf{R}f^! = \mathsf{R}i^!\mathsf{R}g^!$ bring us back to proving 1.5 for i. For all $L \in \mathsf{D}(Z, A)$, if j is the inclusion in Z of $U = Z \setminus X$, $i_*\mathsf{R}i^!L$ is the mapping cylinder of $K \to \mathsf{R}j_*j^*K$ and 1.5 for i follows from 1.1 for j.

Corollaire 1.6. *Let X be as in the theorem, and suppose A is commutative. Then, if $\mathscr{F}$ and $\mathscr{G}$ are sheaves of constructible A-modules on X, the $\mathscr{E}xt^i_A(\mathscr{F}, \mathscr{G})$ are constructible: $\mathsf{R}\mathscr{H}om$ sends $\mathsf{D}_c^-(X, A) \times \mathsf{D}_c^+(X, A)$ to $\mathsf{D}_c^+(X, A)$.*

By unravelling $\mathscr{F}$, we may assume $\mathscr{F}$ is of the form $j_!\mathscr{F}_1$ ($j : Y \hookrightarrow X$) a locally closed immersion and $\mathscr{F}_1$ locally constant on Y). We then have

$$\mathsf{R}\mathscr{H}om(j_!\mathscr{F}_1, \mathscr{G}) = \mathsf{R}j_*\, \mathscr{H}om(\mathscr{F}_1, \mathsf{R}j^!\mathscr{G}):$$

according to 1.3, we are reduced to the case where $\mathscr{F}$ is locally constant – even constant since the problem is local. For $\mathscr{F}$ locally constant constructible, we have $\mathscr{H}om(\mathscr{F}, \mathscr{G})_x = \hom(\mathscr{F}_x, \mathscr{G}_x)$, and similarly for $\mathsf{R}\mathscr{H}om$. For $\mathscr{F}$ constant, we can therefore calculate the $\mathscr{E}xt^i$ using a finite projective resolution of its constant value, and the $\mathscr{E}xt^i$ are constructible if $\mathscr{G}$ is – from which the corollary follows.

1.7 Remarque

Since $\mathsf{R}f_*$ has finite cohomological dimension, it transforms complexes of tor-dimension finite (resp. $\leqslant d$) into complexes of tor-dimension finite (resp. $\leqslant d$) [2, XVII 5.2.11]. [2, XVII 5.2.10] and the proof of 1.5 then show that the tor-dimension finite is stable by the 4 operations $\mathsf{R}f_*$, $\mathsf{R}f_!$, f^*, $\mathsf{R}f^!$. A variant of the one of 1.6 (unscrew K according to a partition of x to suppose the cohomology sheaves locally constant, then localize to replace K by a finite complex of locally free sheaves of finite type) then shows that, for A commutative, $\mathsf{R}\mathscr{H}om$ induces

$$\mathsf{R}\mathscr{H}om : \mathsf{D}^b_{\mathrm{ctf}}(X, A) \times \mathsf{D}^b_{\mathrm{tf}}(X, A) \to \mathsf{D}^+_{\mathrm{tf}}(X, A)$$

(t.f.: tor-dimension finite).

1.8

Under the hypotheses of the theorem, the same method allows us to prove a local duality theorem $K \xrightarrow{\sim} DDK$ (paragraph 4). We also prove a finiteness theorem for vanishing cycle sheaves. For any S, we still obtain a "generic" theorem:

Théorème 1.9. *Let $f : X \to Y$ be a morphism of S-schemes of finite type and $\mathscr{F}$ a constructible sheaf of A-modules on X. There exists a dense open U of S such that*

(i) Above U, the $\mathsf{R}^i f_* \mathscr{F}$ are constructible, and null except for a finite number of them.

(ii) The formation of the $\mathsf{R}^i f_* \mathscr{F}$ is compatible to any change of base $S' \to U \subset S$.

For S the spectrum of a field, we have $U = S$.

If S is the spectrum of a field, a limit argument provides the compatibility to S-base changes of the $\mathsf{R}^i f_*$ for any quasi-compact quasi-separated morphism of S-schemes and any sheaf $\mathscr{F}$.

Corollaire 1.10. *Let x be a finite type scheme over a separably closed field k and $\mathscr{F}$ a constructible sheaf of A-modules. Then, the $\mathrm{H}^i(X, \mathscr{F})$ are of finite type.*

This is the special case $S = \mathrm{Spec}(k)$, $Y = S$.

Corollaire 1.11. *Let X and Y be two finite type schemes over a separably closed field k and $K \in \mathrm{Ob}\,\mathsf{D}^-(X, A^\circ)$, $L \in \mathrm{Ob}\,\mathsf{D}^-(Y, A)$ complexes of right and left sheaves of modules. The Künneth map*

$$\mathsf{R}\Gamma(X, K) \overset{\mathsf{L}}{\otimes} \mathsf{R}\Gamma(Y, L) \to \mathsf{R}\Gamma(X \times Y, \mathrm{pr}_1^* K \overset{\mathsf{L}}{\otimes} \mathrm{pr}_2^* L)$$

is an isomorphism.

We proceed as in [2, XVII 5.4.3]:

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "a"] \\ Y \arrow[r, "b"] & \mathrm{Spec}(k) \end{tikzcd}$$

changing base by b, we find that $\mathsf{R}\mathrm{pr}_{2,*}\mathrm{pr}_1^* K$ is $b^* \mathsf{R}\Gamma(X, K)$. We then have (cf. [2, XVII 5.2.11] and proof)

$$\begin{aligned} \mathsf{R}\Gamma(X \times Y, \mathrm{pr}_1^* K \overset{\mathsf{L}}{\otimes} \mathrm{pr}_2^* L) &= \mathsf{R}\Gamma(Y, \mathsf{R}\mathrm{pr}_{2,*}(\mathrm{pr}_1^* K \overset{\mathsf{L}}{\otimes} \mathrm{pr}_2^* L)) \\ &= \mathsf{R}\Gamma(Y, (\mathsf{R}\mathrm{pr}_{2,*}\mathrm{pr}_1^* K) \overset{\mathsf{L}}{\otimes} L) \\ &= \mathsf{R}\Gamma(Y, b^* \mathsf{R}\Gamma(X, K) \overset{\mathsf{L}}{\otimes} L) \\ &= \mathsf{R}\Gamma(X, K) \overset{\mathsf{L}}{\otimes} \mathsf{R}\Gamma(Y, L) \end{aligned}$$

1.12

Finally, as the local counterpart of 1.9(ii), we will prove a generic local acyclicity theorem (2.13).

2 Generic Theorems

In this paragraph, we prove 1.9 and the generic local acyclicity theorem 2.13. To prove 1.9, we immediately reduce to the assumption that S is integral. Let η be its generic point.

2.1

We will begin by proving 1.9 under the following additional assumptions: X is smooth over S, purely of relative dimension n, $A = \mathbb{Z}/m$, $\mathscr{F}$ is locally constant and $Y = S$. Let $\mathscr{F}' = \mathscr{H}om(\mathscr{F}, \mathbb{Z}/m)$. We will show that if the $\mathsf{R}^i f_! \mathscr{F}'$ are locally constant, the conclusions of 1.9 hold for $U = S$. Recall that if $\mathscr{L}$ is a locally constant constructible sheaf, the $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G})$ are calculated fiber by fiber, and are constructible if $\mathscr{G}$ is. Recall also that $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$-module, and is the dualizing module, so that $\mathscr{F} = \mathsf{R}\mathscr{H}om(\mathscr{F}', \mathbb{Z}/m)$. Poincaré duality

$$\mathsf{R}f_* \, \mathsf{R}\mathscr{H}om(K, \mathsf{R}f^! L) = \mathsf{R}\mathscr{H}om(\mathsf{R}f_! K, L),$$

for $K = \mathscr{F}'$, $L = \mathbb{Z}/m$, $\mathsf{R}f^! L = \mathbb{Z}/m[2n](n)$, therefore provides

$$\mathsf{R}^{2n-i} f_* \mathscr{F} = \mathscr{H}om(\mathsf{R}^i f_! \mathscr{F}', \mathbb{Z}/m)(-n).$$

The sheaf on the right hand side is locally constant, with formation compatible to any change of base – whence the assertion.

2.2

We prove 1.9 under the following additional assumptions: X is smooth over S, $\mathscr{F}$ is locally constant and $Y = S$.

a) Decomposing X into connected components, we may assume that it is purely of relative dimension n over S.

b) Decomposing A into a product, we may assume that $\ell^m A = 0$, with ℓ an invertible prime on S; we then filter $\mathscr{F}$ by the $\ell^k \mathscr{F}$ to reduce, by the corresponding spectral sequence, to the case where $\ell \mathscr{F} = 0$. We may then replace A by $A/\ell A$ and assume that $\ell A = 0$.

c) By replacing S with a suitable U, we can assume that there exists a finite étale Galois covering X_1/X, with Galois group G, such that the pullback $\mathscr{F}_1$ of $\mathscr{F}$ to X_1 is a constant sheaf, with constant value F. Denoting f_1 the projection of X_1 onto S, we then have

$$\mathsf{R}^i f_{1*} \mathscr{F}_1 = (\mathsf{R}^i f_{1*} \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} F.$$

We also have the Hochschild-Serre spectral sequence (or: Leray's for the covering X_1/X)

$$\underline{\mathrm{H}}^p(G, \mathsf{R}^q f_{1*} \mathscr{F}_1) \Rightarrow \mathsf{R}^{p+q} f_* \mathscr{F}.$$

By 2.1, we can assume, by shrinking U, that the $\mathsf{R}^i f_{1*}\mathbb{Z}/\ell$ are locally constant, with formation compatible to any change of base. The same property then holds for the $\mathsf{R}^i f_*\mathscr{F}$. Finally, if $\bar{\eta}$ is a geometric point localized at the generic point η of S, the $\mathsf{R}^i f_*\mathscr{F}$ are almost all zero, because the $(\mathsf{R}^i f_*\mathscr{F})_{\bar{\eta}} = \mathrm{H}^i(X_{\bar{\eta}}, \mathscr{F})$ are.

2.3

Prouvons par récurrence sur n que

$$\begin{array}{c}\text{Les conclusions de 1.9 sont vraies lorsque } \dim X_\eta \leqslant n \\ \text{et que } f \text{ est un plongement ouvert d'image dense.}\end{array} \tag{$*_n$}$$

Pour $n = 0$, quitte à rétrécir S, on a $X = Y$: $(*_0)$ est évident. Supposons $(*_{n-1})$, et prouvons $(*_n)$. Dans $(*_{n-1})$, on peut remplacer "plongement ouvert" par "plongement" comme on le voit en factorisant en plongement ouvert et plongement fermé.

Lemme 2.4. *By shrinking S, the conclusions of 1.9 hold over $Y' \subset Y$, with Y_1, the complement of Y', being finite over S.*

The assertion is local on Y, which we can assume is affine: $Y \subset \mathbb{A}^n_S$. The induction hypothesis $(*_{n-1})$ applies to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "\mathrm{pr}_i"] \\ & \mathbb{A}^1_S \end{tikzcd}$$

Therefore, there exists for each i an open dense subset U_i of $\mathbb{A}^1_S$ such that the conclusions of 1.9 hold over $\mathrm{pr}_i^{-1}(U_i)$; they hold over the union of the $\mathrm{pr}_i^{-1}(U_i)$, and 2.4 follows.

2.6

We prove $(*_n)$ for X smooth over S and $\mathscr{F}$ locally constant on X. The problem being local on Y, we can assume Y is affine, then projective (replace Y by its closure in a projective space). Let $i : Y_1 \hookrightarrow Y$ and $j : Y' \to Y$ be as guaranteed by 2.4.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "a"'] & Y \arrow[d, "b"] & Y_1 \arrow[l, "i"'] \arrow[dl, "b_1"] \\ & S & \end{tikzcd}$$

By shrinking S, we know that $j^*\mathsf{R}f_*\mathscr{F}$ is constructible, with formation compatible to any base change in S; we also know that $\mathsf{R}a_*\mathscr{F} = \mathsf{R}b_*\mathsf{R}f_*\mathscr{F}$ is constructible, with formation compatible to any base change.

Apply $\mathsf{R}b_*$ to the triangle defined by the exact sequence

$$0 \longrightarrow j_!j^*\mathsf{R}f_*\mathscr{F} \longrightarrow \mathsf{R}f_*\mathscr{F} \longrightarrow i_!i^*\mathsf{R}f_*\mathscr{F} \longrightarrow 0: \tag{1}$$

we obtain a triangle

$$\longrightarrow \mathsf{R}b_*j_!j^*\mathsf{R}f_*\mathscr{F} \longrightarrow \mathsf{R}a_*\mathscr{F} \longrightarrow b_{1*}i^*\mathsf{R}f_*\mathscr{F} \longrightarrow \tag{2}$$

in which the first two terms are constructible, with formation compatible to any base change in S (for the first, by the finiteness theorem for the proper morphism b). Therefore, the same is true for the third. Since b_1 is finite, we deduce that $i^*\mathsf{R}f_*\mathscr{F}$ is constructible with formation compatible to any base change in S, and the same is true for $\mathsf{R}f_*\mathscr{F}$ by (1).

2.7

Let us prove $(*_n)$ in general. We begin by reducing to the case where in X there exists a dense open set V that is smooth over S. For S the spectrum of a perfect field, it is enough to replace X by X_{red} (and Y by Y_{red}). In general, we need to shrink S, make a finite and surjective radicial base change $S' \to S$, and replace X and Y by X_{red} and Y_{red}. The étale topology being insensitive to finite and surjective radicial morphisms, this is harmless. After shrinking V, we can assume $\mathscr{F}$ is locally constant on V

$$V \xhookrightarrow{i} X \xhookrightarrow{f} Y.$$

We define Δ by the triangle

$$\longrightarrow \mathscr{F} \longrightarrow \mathsf{R}j_*j^*\mathscr{F} \longrightarrow \Delta \longrightarrow . \tag{1}$$

The cohomology sheaves of Δ are supported in $X \smallsetminus V$, and $\dim(X \smallsetminus V)_\eta < n$. The induction hypothesis therefore allows us to assume that $\mathsf{R}f_*\Delta$ is constructible. Applying $\mathsf{R}f_*$ to the triangle (1) gives us a triangle

$$\longrightarrow \mathsf{R}f_*\mathscr{F} \longrightarrow \mathsf{R}(fj)_*j^*\mathscr{F} \longrightarrow \mathsf{R}f_*\Delta \longrightarrow$$

in which two of the vertices are constructible and compatible with any base change on S. The third is therefore also constructible.

2.8

Let us prove 1.9. The problem is local on Y, which we can assume is affine. Taking an affine cover of X and invoking the Leray spectral sequence, we can reduce to the case where X is also affine. We do all of this to ensure that we can factorize f as an open immersion followed by a proper morphism: $f = gj$, from which $\mathsf{R}f_* = \mathsf{R}g_*\mathsf{R}j_*$. The open immersion is justifiable by an $(*_n)$, the proper morphism by the finiteness theorem.

The following corollaries are proven as in paragraph 1.

Corollaire 2.9. *Under the hypotheses of the theorem, for K in $\mathsf{D}^b_c(X, A)$ or $\mathsf{D}^b_c(Y, A)$ respectively, there exists a nonempty open U of S over which $\mathsf{R}f_*K$, $\mathsf{R}f_!K$, f^*K, $\mathsf{R}f^!K$ are in $\mathsf{D}^b_c(Y, A)$ or $\mathsf{D}^b_c(X, A)$, and of compatible formation to any base change $S' \to U \subset S$.*

Corollaire 2.10. *Let X be as in the theorem, and suppose A is commutative. Then, if $\mathscr{F}$ and $\mathscr{G}$ are constructible sheaves of A-modules on X, over a dense open U of S, the $\mathscr{E}xt^i_A(\mathscr{F}, \mathscr{G})$ are still constructible, of compatible formation to any base change $S' \to U \subset S$.*

2.11

If x is a geometric point of a scheme X, we will denote X_x the strict henselization of X at x. For $f : X \to S$ and t a geometric point of S, we will denote X_t the geometric fiber of X at t. Finally, for x a geometric point of X, and t a geometric point of $S_{f(x)}$, $(X_x)_t$ is the fiber at t of $X_x \to S_{f(x)}$.

Définition 2.12. *Let $f : X \to S$ and $K \in \operatorname{Ob} \mathsf{D}^+(X, A)$. We say that f is* locally acyclic at x, res. K*, if for all geometric points t of $S_{f(x)}$ we have $K_x = \mathsf{R}\Gamma(X_x, K) \xrightarrow{\sim} \mathsf{R}\Gamma(X_{x,t}, K)$. We say that f is* locally acyclic, rel. K*, if this is true for all x, and* universally locally acyclic, rel. K*, if this remains true after any base change $S' \to S$.*

Théorème 2.13. *Let $f : X \to S$ be a finite type morphism and $\mathscr{F}$ a constructible sheaf on X. There exists a dense open U of S over which f is universally locally acyclic, rel. $\mathscr{F}$.*

We will assume the following result, proven in the appendix.

Lemme 2.14. *Let $X \xrightarrow{f} S_1 \xrightarrow{g} S_2$. If g is smooth, and f universally locally acyclic rel. K, then gf is also.*

We may assume S is integral, with generic point η, and proceed by induction on $\dim X_\eta$. We begin by deducing from the induction hypothesis that

$$\begin{array}{c}\text{By shrinking } S\text{, there exists } T \subset X\text{, finite over } S\text{, such that } f \\ \text{is universally locally acyclic outside of } T.\end{array} \tag{A}$$

This question being local, we localize on X and factorize f as $X \xrightarrow{u} \mathbb{A}^1_S \xrightarrow{S}$. The induction hypothesis applies to u and we conclude by 2.14 and the usual arguments.

To prove the theorem, we may assume X is proper, since the problem is local. We shrink S so that the $\mathsf{R}^i f_* \mathscr{F}$ are locally constant and (A) is applicable, and use the following lemma.

Lemme 2.15. *Let $f : X \to S$, proper, and $\mathscr{F}$ constructible. We assume the $\mathsf{R}^i f_* \mathscr{F}$ are locally constant and that, outside of $T \subset X$ finite over S, f is locally acyclic rel. to $\mathscr{F}$. Then, f is locally acyclic rel. to $\mathscr{F}$.*

Let s be a geometric point of S, S_s the strict localization of S at s, and t a geometric point of S_s. Let $X_{(s)}$ be the inverse image of X on S_s, $\bar{j} : X_t \to X_{(s)}$ and $i : X_s \hookrightarrow X_{(s)}$. Denoting again $\mathscr{F}$ the inverse image of $\mathscr{F}$ on $X_{(s)}$ or X_t, we must prove that $i^* \mathscr{F} \xrightarrow{\sim} i^* \mathsf{R}\bar{j}_* \mathscr{F}$. Let Δ be the mapping cylinder of this application. Its cohomology sheaves are supported in T_s. To prove that $\Delta = 0$, it is therefore enough to prove that $\mathsf{R}\Gamma(X_s, \Delta) = 0$. This is the mapping cylinder of $(\mathsf{R}f_* \mathscr{F})_s = \mathsf{R}\Gamma(X_s, \mathscr{F}) \to \mathsf{R}\Gamma(X_s, i^* \mathsf{R}\bar{j}_* \mathscr{F}) = \mathsf{R}\Gamma(X_{(s)}, \mathsf{R}\bar{j}_* \mathscr{F}) = \mathsf{R}\Gamma(X_t, \mathscr{F}) = (\mathsf{R}f_* \mathscr{F})_t$. This specialization morphism is, by hypothesis, an isomorphism, and $\Delta = 0$.

Corollaire 2.16. *For S the spectrum of a field, every S-scheme X is universally locally acyclic, rel. K, for any $K \in \operatorname{Ob} \mathsf{D}^+(X, A)$.*

Follows from 2.13 by taking the limit (possible, because U from 2.13 is always equal to S.

3 Proof of 1.1 and constructibility of vanishing cycle sheaves

3.1

Let S be a strict local trait (the spectrum of a henselian discrete valuation ring with separably closed residue field). We denote by s and η its closed and generic points, and by $\bar{\eta}$ a geometric point localized at η (a separable closure of $k(\eta)$). Let X be a scheme over S and $\mathscr{F}$ a sheaf on X_η. We recall the definition of the vanishing cycle sheaves $\mathsf{R}^i \Psi_\eta(\mathscr{F})$ (sheaves on X_s, with an action of $\mathrm{Gal}(\bar\eta/\eta)$). Let $i : X_s \hookrightarrow X$, $j : X_\eta \hookrightarrow X$, and $\bar j$ be the composition $X_{\bar\eta} \to X_\eta \hookrightarrow X$; we have

$$\mathsf{R}^i\Psi_\eta(\mathscr{F}) = i^*\mathsf{R}^i\bar j_*(\bar j^*\mathscr{F}).$$

Here is the main theorem of this number. It improves [16, XIII 2.3.1, 2.4.2].

Théorème 3.2. *We assume that X is of finite type over S and $\mathscr{F}$ is constructible. Then, the vanishing cycle sheaves $\mathsf{R}^i\Psi_\eta(\mathscr{F})$ are constructible.*

We proceed by induction on the dimension n of X_η. We can moreover assume – and we assume – X_η is dense in X (replacing X by $\bar X_\eta$ does not change the $\mathsf{R}^i\Psi$, which are supported in $\bar X_\eta$).

Lemme 3.3. *If 3.2 is true in dimension of $X_\eta < n$, and $\dim X_\eta = n$, there exist constructible sub-sheaves $\mathscr{G}_i$ of the $\mathsf{R}^i\Psi_\eta(\mathscr{F})$ such that the supports of the local sections of $\mathsf{R}^i\Psi_\eta(\mathscr{F})/\mathscr{G}$ are finite.*

Let s' be a generic geometric point of the affine line over s, and let S' be the strict localization at s' of the affine line $\mathbb{A}^1_S$ over S. It is still a strict local line, and the uniformizers for S are uniformizers on S'.

$$\begin{tikzcd}
 & \mathrm{Spec}(k') \arrow[rr] \arrow[dd] \arrow[dr] & & (S',\eta',s') \arrow[dd] \arrow[dr] & \\
\bar\eta' \arrow[ur] \arrow[dr] & & \mathbb{A}^1_{\bar S} \arrow[d] & & \mathbb{A}^1_S \arrow[dl] \\
 & \bar\eta \arrow[r, hook] & \bar S \arrow[r] & (S,\eta,s) &
\end{tikzcd}$$

Let $k' = k(\bar\eta)\otimes_{k(\eta)} k(\eta')$. If $\bar S$ is the normalization of S in $\bar\eta$, k' is the field of fractions of the strict localization of $\mathbb{A}^1_{\bar S}$ at s'; it is therefore a field, and $\mathrm{Gal}(\bar\eta/\eta) = \mathrm{Gal}(k'/\eta')$. Let $\bar\eta'$ be the spectrum of an algebraic closure of k'. The Galois group $P = \mathrm{Gal}(\bar\eta'/k')$ is a pro-p-group, for p the characteristic exponent of $k(s)$.

Lemme 3.4. *For any scheme X' over S', and any sheaf $\mathscr{F}$ over $X'_\eta = X'_{\eta'}$, there is a relation between the vanishing cycles sheaves for X'/S' and for X'/S given by $\mathsf{R}^i\Psi_\eta(\mathscr{F}) = \mathsf{R}^i\Psi_{\eta'}(\mathscr{F})^P$.*

The following diagram compares the morphisms used in the definition of $\mathsf{R}\Psi$ for X'/S' and X'/S:

$$\begin{tikzcd}
X_{\bar\eta'} \arrow[rr] \arrow[d] & & X'_{\eta'} \arrow[dd, equal] \arrow[dr, hook] & & S' \\
X_{\eta'}\times_{\eta'}\mathrm{Spec}(k') \arrow[urr] \arrow[d, equal] & & & X \arrow[ur] \arrow[dr] & \\
X'_\eta \arrow[rr] & & X'_\eta \arrow[ur, hook] & & S
\end{tikzcd}$$

and the lemma follows from the Hochschild-Serre spectral sequence, taking into account that P is a pro-p-group, with p invertible in A.

Let us prove 3.3. The question is local; this allows us to assume X affine, $X \subset \mathbb{A}^n_S$. Let f be one of the projection $X \subset \mathbb{A}^n_S \to \mathbb{A}^1_S$. X' is the "localized" $X_{\mathbb{A}^1_S} S'$ of x, and $\mathscr{F}'$ is the inverse image of $\mathscr{F}$ on x'

$$\begin{tikzcd}
\mathscr{F}' \text{ on } X' \arrow[r] \arrow[d, "\lambda"] & S' \arrow[d, "\lambda"] & \\
\mathscr{F} \text{ on } X \arrow[r, "f"] & \mathbb{A}^1_S \arrow[r] & S.
\end{tikzcd}$$

We have, on $X'_s = X'_{s'}$,

$$\lambda^* \mathsf{R}^i \Psi_\eta(\mathscr{F}) = \mathsf{R}^i \Psi_\eta(\mathscr{F}') = \mathsf{R}^i \Psi_{\eta'}(\mathscr{F}')^P:$$

this sheaf is constructible, because the induction hypothesis applies to X'/S', and we use the following lemma applied to $X_s \subset \mathbb{A}^n_S$ and to vanishing cycle sheaves.

Lemme 3.5. *Let $\mathbb{A}^n$ be the affine space of dimension n over a field k, $X \subset \mathbb{A}^n$, $]sF$ be a sheaf of A-modules on X and $\bar\eta$ be a generic geometric point of $\mathbb{A}^1$. Let $X_{\bar\eta,i}$ be the generic geometric fiber of $X \subset \mathbb{A}^n \xrightarrow{\mathrm{pr}_i} \mathbb{A}^1$ and $\mathscr{F}_{\bar\eta,i}$ be the restriction of $\mathscr{F}$ to $X_{\bar\eta,i}$. If the $\mathscr{F}_{\bar\eta,i}$ are constructible, there exists $\mathscr{F}' \subset \mathscr{F}$, constructible, such that the local sections of $\mathscr{F}/\mathscr{F}'$ have finite support.*

By taking the limit for each i

a) There exists an étale neighborhood U of $\bar\eta$ and a constructible sheaf $\mathscr{H}$ on $X_{U,i} = X \times_{\mathbb{A}^1, \mathrm{pr}_i} U$ with restriction $\mathscr{F}_{\bar\eta,i}$ to $X_{\bar\eta,i}$.

b) For a suitable U, the isomorphism $\mathscr{H}_{\bar\eta,i} \xrightarrow{\sim} \mathscr{F}_{\bar\eta,i}$ comes from $u : \mathscr{H} \to \mathscr{F}_{U,i}$ where $\mathscr{F}_{U,i}$ is the inverse image of $\mathscr{F}$ on $X_{U,i}$. Let $\varphi : X_{U,i} \to X$. The map u defines $v_i : \varphi_! \mathscr{H} \to \mathscr{F}$, with image $\mathscr{F}'_i$. We have $(\mathscr{F}/\mathscr{F}'_i)_{\bar\eta,i} = 0$ and we take for $\mathscr{F}'$ the sum of the $\mathscr{F}'_i$.

3.6

Let us prove 3.2 in dimension n. We can assume X is affine (because the problem is local on X_s), then projective over S (take the closure of X in a projective embedding). We then have a spectral sequence [16, I 2.2.3] (the Leray spectral sequence for $\bar{j}$, taking into account the proper morphism $X \to S$).

$$E_2^{pq} = \mathrm{H}^p(X_s, \mathsf{R}^i \Psi_\eta(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X_{\bar\eta}, \mathscr{F}). \tag{1}$$

Let $\mathscr{G}^q \subset \mathsf{R}^q \Psi_\eta(\mathscr{F})$ be as in 3.3, and let $\mathscr{H}^q$ be the quotient sheaf. We have $\mathrm{H}^i(X_s, \mathscr{H}^q) = 0$ for $i \neq 0$, and for $\mathscr{H}^q$, and hence $\mathsf{R}^q \Psi_\eta(\mathscr{F})$, to be constructible, it is enough that $\mathrm{H}^0(X_s, \mathscr{H}^q)$ is of finite type.

Calculate (1) modulo modules of finite type (i.e. in the quotient category of the category of A-modules by the thick subcategory of A-modules of finite type). The long exact sequence of cohomology deduced from $0 \to \mathscr{G}^q \to \mathsf{R}^q \Psi_\eta(\mathscr{F}) \to \mathscr{H}^q \to 0$ provides, by 1.10 applied to $\mathscr{G}^q$, that $E_2^{pq} \sim \mathrm{H}^q(X_s, \mathscr{H}^q)$. In particular, $E_2^{pq} \sim 0$ for $p \neq 0$, $E_2^{0q} \sim \mathrm{H}^q(X_{\bar\eta}, \mathscr{F}) \sim 0$, and this completes the proof.

Parallel arguments provide the following result, which improves [16, XIII 2.1.12, 2.4.2].

Proposition 3.7. *The formation of vanishing cycles sheaves is compatible with changes of traits.*

Let $g : (S', \eta', s') \to (S, \eta, s)$ be a morphism (surjective) of strictly local traits, X/S, $\mathscr{F}$ on X_η, of torsion prime to the residual characteristic and $(X', \mathscr{F}')$ their reciprocal images on S'. Denoting again g the morphisms parallel to g, such as $X'_{s'} \to X_s$, we need to prove that

$$g^* \mathsf{R}\Psi_\eta(\mathscr{F}) \xrightarrow{\sim} \mathsf{R}\Psi_{\eta'}(\mathscr{F}'). \tag{3.7.1}$$

This morphism is defined by the commutative diagram

$$\begin{tikzcd} X'_{s'} \arrow[r, hook, "i'"] \arrow[d, "g"] & X' \arrow[d, "g"] & X'_{\eta'} \arrow[l, "\bar{j}'"'] \arrow[d, "g"] \\ X_s \arrow[r, hook, "i"] & X & X_{\bar{\eta}} \arrow[l, "\bar{j}"'] \end{tikzcd}$$

A limit passage reduces to assuming X is of finite type over S, and we proceed by induction on $\dim X_\eta$. The question being local, we can assume X is affine, then projective. Assume (3.7.1) for $\dim X_\eta < n$. For $\dim X_\eta = n$, it then follows from the two assertions below, where Δ is the mapping cylinder of (3.7.1).

(A) The support of local sections of the cohomology sheaves of Δ is finite.

(B) $\mathsf{R}\Gamma(X_s, \Delta) = 0$, so $\Delta = 0$, according to (A).

3.8

The proof of (A) is parallel to that of 3.3. We localize on X and consider maps $X \to \mathbb{A}^1_S$; denoting $S(x)$ the strict henselization of $\mathbb{A}^1_S$ at the generic point of the special fiber, we apply the induction hypothesis to $X_1/S(x)$ ($X_1 = X \times_{\mathbb{A}^1} S(x)$) and to $S'(x)$. Taking invariants by a pro-p-group, we find that Δ is trivial on the geometric generic fiber of $X_{s'} \to \mathbb{A}^1_s$. This being true, locally on X and for any projection, we have (A).

3.9

For (B), the properness of X over S ensures that

$$\mathsf{R}\Gamma(X'_{s'}, \mathsf{R}\Psi_{\eta'}(\mathscr{F}')) = \mathsf{R}\Gamma(X'_{\bar{\eta}}, \mathscr{F}')$$

and

$$\mathsf{R}\Gamma(X_s, \mathsf{R}\Psi_\eta(\mathscr{F})) = \mathsf{R}\Gamma(X_{\bar{\eta}}, \mathscr{F}).$$

We have $\mathsf{R}\Gamma(X_s, \mathsf{R}\Psi_\eta(\mathscr{F})) \xrightarrow{\sim} \mathsf{R}\Gamma(X'_{s'}, g^*\mathsf{R}\Psi_\eta(\mathscr{F}))$ (invariance by change of separably closed base field); $\mathsf{R}\Gamma(X'_{s'}, \Delta)$ is therefore the mapping cylinder of $\mathsf{R}\Gamma(X_{\bar{\eta}}, \mathscr{F}) \to \mathsf{R}\Gamma(X'_{\bar{\eta}}, \mathscr{F}')$, an isomorphism by the same invariance theorem.

3.10

Let us prove 1.1. We may assume S is connected, hence integral. If $\dim S = 0$, (S is the spectrum of a field); we apply 1.9, 1.10. If S is of dimension 1, the theorem 1.9 assures that the $\mathsf{R}^i f_* \mathscr{F}$ are constructible over the complement of a finite set T of points of S. It remains to see that their restrictions to the Y_t ($t \in T$) are constructible. It suffices to see it after localizing at t: we may assume S is a strict local line. The essential case is the following.

Lemme 3.11. *Let X be of finite type over S, $j : X_\eta \hookrightarrow X$, $i : X_s \hookrightarrow X$ and $\mathscr{F}$ constructible over X_η. Then, $i^*\mathsf{R}j_*\mathscr{F}$ is constructible.*

Let $I = \mathrm{Gal}(\bar\eta/\eta)$ be the inertia group. We know it is an extension of $\widehat{\mathbb{Z}}_{p'}(1)$ by a pro-p-group P (p is the characteristic of the residue field). The Hochschild-Serre spectral sequence gives

$$E_2^{pq} = \mathscr{H}^p(I, \mathsf{R}^q\Psi_\eta(\mathscr{F})) \Rightarrow i^*\mathsf{R}^{p+q}j_*\mathscr{F}.$$

Let $R_t^q = \mathsf{R}^q\Psi_\eta(\mathscr{F})^P$ (t for moderate). If σ is a generator of $\widehat{\mathbb{Z}}_{p'}(1)$, we have $E_2^{0q} = \ker(\sigma - 1, R_t^q)$, $E_2^{1q} = \mathrm{coker}(\sigma - 1, R_t^q)$ and $E_2^{pq} = 0$ for $p \neq 0, 1$. These sheaves are constructible, since vanishing cycles sheaves are, and 3.11 follows.

3.12

Let us treat the general case, S still being a strict henselian line. If $X = X_\eta$, f is the composition $X \to Y_\eta \to Y$ and we apply 1.9 and 3.11. In the general case, let $j : X_\eta \hookrightarrow X$ and let Δ be the mapping cylinder of $\mathscr{F} \to \mathsf{R}j_*j^*\mathscr{F}$. Its cohomology sheaves are supported in X_s; according to 1.9 applied to $X_s \to Y_{s'}$, $\mathsf{R}f_*\Delta$ is therefore constructible. The complex $\mathsf{R}f_*\mathsf{R}j_*j^*\mathscr{F} = \mathsf{R}(fj)_*j^*\mathscr{F}$ is also constructible, and we conclude by the triangle

$$\longrightarrow \mathsf{R}f_*\mathscr{F} \longrightarrow \mathsf{R}f_*\mathsf{R}j_*j^*\mathscr{F} \longrightarrow \mathsf{R}f_*\Delta \longrightarrow .$$

4 Local biduality

4.1

Let S be a regular dimension 0 or 1 and let $K_S = A$ (a constant sheaf of value A). For $K \in \mathrm{Ob}\,\mathsf{D}^b_{\mathrm{ctf}}(S, A)$, we define $DK = \mathsf{R}\mathscr{H}om(K, K_S) \in \mathrm{Ob}\,\mathsf{D}^+(S, A^\circ)$ (a complex of A-module sheaves on the right). We still have $DK \in \mathrm{Ob}\,\mathsf{D}^b_{\mathrm{ctf}}(S, A^0)$, and an explicit calculation, possible because S is dimension 1 shows $K \xrightarrow{\sim} DDK$ (for $A = \mathbb{Z}/n$, Duality 1.4).

4.2

We will assume that A is commutative, probably an unnecessary hypothesis. For $a : X \to S$ of finite type over S, we define $K_X = \mathsf{R}a^!K_S$. For $K \in \mathrm{Ob}\,\mathsf{D}^b_{\mathrm{ctf}}(S, A)$, we define $DK = \mathsf{R}\mathscr{H}om(K, K_X) \in \mathrm{Ob}\,\mathsf{D}^b_{\mathrm{ctf}}(X, A)$. Our main result is the following

Théorème 4.3. *We have $K \xrightarrow{\sim} DDK$ ($K \in \mathrm{Ob}\,\mathsf{D}^b_{\mathrm{ctf}}(X, A)$).*

Lemme 4.4. *If a is proper, we have $\mathsf{R}a_*K \xrightarrow{\sim} \mathsf{R}a_*DDK$.*

The Poincaré duality

$$\mathsf{R}a_!\,\mathsf{R}\mathscr{H}om(K, \mathsf{R}a^!K_S) = \mathsf{R}\mathscr{H}om(\mathsf{R}a_!K, K_S)$$

provides us with an isomorphism $\mathsf{R}a_*D = D\mathsf{R}a_! = D\mathsf{R}a_*$. The lemma follows from the commutativity of the diagram (Dualité, 1.2)

$$\begin{tikzcd} \mathsf{R}a_*K \arrow[r] \arrow[d, "\wr"] & \mathsf{R}a_*DDK \arrow[d, "\wr"] \\ DD\mathsf{R}a_*K \arrow[r, "\sim"] & D\mathsf{R}a_*DK. \end{tikzcd}$$

4.5

It is seen by localization that it is enough to treat the cases where S is the spectrum of a field, or a line. For S the spectrum of a field, we proceed by induction on $\dim X$. The problem being local, we can assume X proper. Furthermore, the induction hypothesis assures, by the usual arguments, that the mapping cylinder Δ of $K \to DDK$ has cohomology sheaves in skyscraper. The lemma then assures that $\mathsf{R}a_*\Delta = 0$, thus that $\Delta = 0$.

4.6

For S a line, we already know that 4.3 holds on the generic fiber (4.5). We can also assume X proper over S and we proceed by induction on $\dim X_s$, X proper over S. The mapping cylinder δ still has cohomology sheaves in skyscraper, on X_s, and we conclude as above.

4.7

Suppose that $A = \mathbb{Z}/m$. We then have a variant of the previous theory, by taking any K in $\mathsf{D}_c^b(X, A)$. The complex $\mathsf{R}a^!\mathbb{Z}/m$ is of finite injective dimension [2, XVIII 3.1.7]. This assures that DK is still in $\mathsf{D}_c^b(X, A)$. To prove the local duality on S, we use that $\mathbb{Z}/m$ is the dualizing A-module. Then, we proceed as above.

5 Appendix, by L. Illusie

In this appendix, which is a revision of parts of [20, II], and a letter of P. Deligne to the author, we prove 2.14 as well as various generalizations and variants of the specialization theorems [2, XVI 2.1] and (Cohomologie étale 5.1.7).

The schemes and morphisms considered will be assumed quasi-compact and quasi-separated. If X is a scheme and x a geometric point of X, we will denote $X(x)$ the strict localization of X at x.

We fix a base scheme S and a sheaf of rings A on S (not necessarily commutative or Noetherian). If X is an S-scheme, we will write $\mathsf{D}(X)$ for $\mathsf{D}(X, A_X)$.

5.1 Cohomological purity

Proposition 5.1.1. *Let $f : X \to Y$ be an S-morphism and $E \in \operatorname{Ob} \mathsf{D}^+(X)$. The following conditions are equivalent:*

(i) The formation of $\mathsf{R}f_*E$ commutes with any finite base change $S' \to S$.

(ii) The formation of $\mathsf{R}f_*E$ commutes with any quasi-finite base change (or projective limit of quasi-finite morphisms) $S' \to S$.

(iii) For any specialization arrow $i : t \to S(s)$ (Etale Cohomology, 5.1.2), if we denote by $\widetilde{S}$ the normalized in $k(\bar{t})$ of the integral scheme closure of $i(t)$ in $S(s)$ (according to [19, IV, 6.15.5, 18.6.11, 18.8.16]) $\bar{S}$ is thus local integral, of radical point over s), $\bar{f} : \bar{X} \to \bar{Y}$ the morphism deduced from f by the base change $\bar{S} \to S$, $f_s : X_s \to Y_s$ the fiber of f at s, then the base change arrow

$$(\mathsf{R}\bar{f}_*(E|\bar{X}))|Y_s \to \mathsf{R}f_{s*}(E|X_s)$$

is an isomorphism.

The equivalence of (i) and (ii) follows from the Zariski Main Theorem, and it is clear that (ii) implies (iii). We prove that (iii) implies (ii). For any $S' \to S$, projective limit of quasi-finite morphisms, and any geometric point s of S', we denote by $C(S', s)$ the cone of the base change arrow

$$(\mathsf{R}f'_*(E|X'))|Y_s \to \mathsf{R}f_{s*}(E|X_s),$$

where $f' : X' \to Y'$ is the morphism deduced from f by the base change $S' \to S$ and $f_s : X_s \to Y_s$ the fiber of f at s. It is enough to prove that, for any (S', s) as above, $C(S', s)$ is acyclic. We will show, by recurrence on N, that $\mathrm{H}^i\, C(S', s) = 0$ for $i \leqslant N$. By the Main Theorem, passage to the limit, and cleaning of radical morphisms, we can assume that S and S' are strictly local, S' integral, of closed point s, $S' \to S$ finite. Let t be a generic geometric point of S', and $\bar{S}$ the normalized of S' in $k(t)$. Let $\bar{S}_\bullet$ be the coskeleton simplicial scheme of $S' \to S'$ ($\bar{S}_0 = \bar{S}, \ldots \bar{S}_n = (\bar{S}/S')^{n+1}, \ldots$), $X_\bullet = X \times_S \bar{S}_\bullet$. $g_\bullet : \bar{X}_\bullet \to X'$ the canonical projection. We set $G|X' = G'$. The bicomplex $g_{\bullet *} g_\bullet^* G'$, of columns the $g_{n*} g_n^* G'$, is a resolution of G' [2, VIII 8]. We can thus represent $C(S', s)$ by a bicomplex whose columns identify with the $C(\bar{S}_n, s)$. According to (iii), $C(\bar{S}_0, s) = C(\bar{S}, s)$ is acyclic. On the other hand, by $n > 0$, we have $\mathrm{H}^i(\bar{S}_n, s) = 0$ for $i \leqslant N$ by the recurrence hypothesis. We conclude that $\mathrm{H}^i\, C(S', s) = 0$ for $i \leqslant N + 1$, which ends the proof.

5.1.2 Remarque

Let (f, E) be as in condition (i) of 5.1.1.

a) If $E \in \operatorname{Ob} \mathsf{D}^b(X)$, and f_* has finite cohomological dimension, then, for all $F \in \operatorname{Ob} \mathsf{D}^-(S, A^\circ)$, the canonical arrow

$$\mathsf{R}f_*(E) \overset{\mathsf{L}}{\otimes}_A q^* F \to \mathsf{R}f_*(E \overset{\mathsf{L}}{\otimes}_A p^* F), \qquad (*)$$

where $p : X \to S$, $q : Y \to S$ are the projections, is an isomorphism. The same conclusion holds with $E \in \operatorname{Ob} \mathsf{D}^-(X)$.

Indeed, by devissage, it is enough to check the assertion for F of the form $i_! A_{S'}$, with $i : S' \to S$ quasi-finite. By the Main Theorem, we may assume i is finite, the conclusion then amounts to the commutation of $\mathsf{R}f_*(E)$ with base change by i.

b) We assume A is Noetherian, constructible. Then, for all $F \in \operatorname{Ob} \mathsf{D}^b(S, A^\circ)$, constructible and of finite tor-dimension, the arrow $(*)$ above is an isomorphism.

Indeed, by devissage, we may assume F is of the form $i_! M$, with $i : S' \to S$ locally closed, A locally constant on S', M constructible, of finite tor-dimension, and locally constant cohomology on S'. Changing base from S to S', we are thus reduced to the case where F is perfect [5, I], and finally, by localization and devissage, to the case where $F = A$, for which the conclusion is trivial.

5.1.3

Let $f : X \to Y$ be an S-morphism, and $E \in \operatorname{Ob} \mathsf{D}^+(X)$. We will say that (f, E) is *cohomologically proper* relative to S if the formation of $\mathsf{R}f_* E$ commutes with any base change $S' \to S$. It is equivalent to saying that, after any base change $S' \to S$, the formation of $\mathsf{R}f'_* E'$ (where $f' = f \times_S S'$, E' =inverse image of E on $X \times_S S'$) commutes with any finite base change $S'' \to S'$ (cf. [2, XII 6.1]). Here are some examples of cohomological properness (for a study of the analogous notion for sheaves of sets or non-commutative groups, the reader is referred to [22, XIII]).

1. We suppose A has torsion and f is proper. Then, for all $E \in \operatorname{Ob} \mathsf{D}^+(X)$, (f, E) is cohomologically proper relative to S (proper base change theorem, [2, XII 5.1]).

2. We suppose S is finite, X and Y are of finite type over S, and A is annihilated by an invertible integer on S. Then, for all $E \in \operatorname{Ob} \mathsf{D}^+(X)$, (f, E) is cohomologically proper relative to S (1.9).

3. We suppose A is constant, noetherian, annihilated by an integer n invertible on S, $f : X = Y \smallsetminus D \to Y$ is the inclusion of the complement of a divisor D on Y, with normal crossings relative to S [22, XIII 2.1]. Let E be a sheaf of A-modules on X satisfying one of the following conditions:

   (i) E is locally constant and constructible, and mildly ramified along D;

   (ii) E is locally for the étale topology on Y and on S the inverse image of a sheaf of A-modules on S.

   Then (f, E) is cohomologically proper relative to S, and $\mathsf{R}f_*E$ is constructible (compare with [22, XIII 2.4]).

We sketch the proof in case (i). The question is local in a neighborhood of a point y of Y. We can assume D is a sum of smooth divisors, $D = \sum_{i=1}^{m} D_i$. By the Abhyankar relative lemma [22, XIII 5.5], we can, in a neighborhood of y, trivialize E by a finite covering $\widetilde{Y} \to Y$ of the form $\widetilde{Y} = Y[T_1, \dots, T_r]/(T_1^{n_1} - t_1, \dots, T_r^{n_r} - t_r)$ where the t_i are local equations of the smooth divisors passing through y, and the n_i are integers prime to the characteristic of $k(y)$. The inverse image $\widetilde{D}$ of D in Y is still with normal crossings relative to S, and if $g : \widetilde{X} = \widetilde{Y} \smallsetminus \widetilde{D} \to X$ is the projection, g^*E is constant. As E injects into g_*g^*E and the quotient is mildly ramified, a simple unscrewing reduces to the case where E is constant. We denote by $p : X \to S$, $q : Y \to S$ the projections, and, for $1 \leqslant i \leqslant m$, $f_i : Y \smallsetminus D_i \to D$ the canonical inclusion. From the relative purity theorem [2, XVI 3.7] we deduce easily, by induction on m, that, for all locally constant A-module M on S, the canonical map

$$\mathsf{R}f_*(\mathbb{Z}/n) \overset{\mathsf{L}}{\otimes} q^*M \to \mathsf{R}f_*(p^*M) \tag{5.1.3.1}$$

is an isomorphism, and on the other hand we have:

$$\begin{aligned} f_{i*}(\mathbb{Z}/n) &= \mathbb{Z}/n, \\ \mathsf{R}^1 f_{i*}(\mathbb{Z}/n) &= (\mathbb{Z}/n)(-1)_{D_i}, \end{aligned} \tag{5.1.3.2}$$

a canonical isomorphism being given by the fundamental class $\mathrm{cl}(D_i)$ (Cycle 2.1.4),

$$\begin{aligned} \mathsf{R}^q f_{i*}(\mathbb{Z}/n) &= 0 \text{ for } q > 1, \\ \mathsf{R}f_*(\mathbb{Z}/n) &= \overset{\mathsf{L}}{\bigotimes_i} \mathsf{R}f_{i*}(\mathbb{Z}/n) \text{ , i.e.:} \\ \mathsf{R}^1 f_*(\mathbb{Z}/n) &= \bigoplus_{i=1}^{m} (\mathbb{Z}/n)(-1)_{D_i}, \\ \textstyle\bigwedge^{\bullet} \mathsf{R}^1 f_*(\mathbb{Z}/n) &\xrightarrow{\sim} \mathsf{R}^{\bullet} f_*(\mathbb{Z}/n). \end{aligned}$$

The conclusion of 3, in case (i), therefore follows from (5.1.3.1) and (5.1.3.2). In case (ii), we reduce, by passage to the limit, to the case where $E = p^*M$, with M constructible, then, by devissage and change of base (using (i)), to the case M constant, already treated.

It follows from the previous proof that (5.1.3.1) is an isomorphism for all $M \in \mathrm{Ob}\,\mathsf{D}^+(S)$.

We compare the formulas (5.1.3.2) to the analogous formulas in entire cohomology in the case $S = \mathrm{Spec}(\mathbb{C})$ (see for example [12, 3.1]), which follow from the fact, locally for the classical topology, that the complement of D has the homotopy type of a torus.

By taking the limit, we deduce from (5.1.3.2) similar formulas in ℓ-adic cohomology, i.e. with $\mathbb{Z}/n$ replaced by $\mathbb{Z}_\ell$ (ℓ an invertible prime number on S). Suppose $q : Y \to S$ is proper and smooth. It then follows from the Weil conjectures [14, 15] that the Leray spectral sequence

$$E_2^{ij} = \mathsf{R}^i q_* \mathsf{R}^j f_* \mathbb{Z}_\ell \Rightarrow \mathsf{R}^\bullet p_* \mathbb{Z}_\ell$$

degenerates in E_3 modulo torsion [11, §6]. The same phenomenon occurs in integral cohomology in the case $S = \mathrm{Spec}(\mathbb{C})$, see (loc. cit.) and [12, 3.2.13].

5.2 Specialization and cospcialization

Let $f : X \to S$ and $E \in \mathrm{Ob}\,\mathsf{D}^+(X)$.

5.2.1

Suppose that the formation of $\mathsf{R}f_*E$ commutes with any finite base change. We then define a *specialization arrow*, denoted by

$$\mathrm{sp}(u) : \mathsf{R}\Gamma(X_s, E|X_s) \to \mathsf{R}\Gamma(X_t, E|X_t), \tag{5.2.1.1}$$

as follows. Let, as in 5.1.1(iii), $\bar{S}$ be the normalization in $k(t)$ of the closure of $u(t)$ in $S(s)$, and $\bar{f} : \bar{X} \to \bar{S}$ the morphism deduced from f by the base change $\bar{S} \to S$. By the trivial part (i)⇒(iii) of 5.1.1, the base change arrow

$$\mathsf{R}\Gamma(\bar{X}, E|\bar{X}) = \mathsf{R}\bar{f}_*(E|\bar{X})_s \to \mathsf{R}\Gamma(X_s, E|X_s) \tag{5.2.1.2}$$

is an isomorphism. We define (5.2.1.1) as the composition of the inverse of (5.2.1.2) and the restriction arrow

$$\mathsf{R}\Gamma(\bar{X}, E|\bar{X}) \to \mathsf{R}\Gamma(X_t, E|X_t). \tag{5.2.1.3}$$

5.2.2

Suppose that f is locally acyclic relative to E (2.12). For any specialization $u : t \to S(s)$ of geometric points of S, we then define an arrow, called the *specialization arrow*,

$$\mathrm{cosp}(u) : \mathsf{R}\Gamma(X_t, E|X_t) \to \mathsf{R}\Gamma(X_s, E|X_s), \tag{5.2.2.1}$$

defined as follows. Let $g : X_t \to X$, $g' : \bar{X} \to X$ be the canonical arrows. It follows from the local acyclicity hypothesis that the canonical arrow

$$\mathsf{R}g'_*(E|\bar{X}) \to \mathsf{R}g_*(E|X_t) \tag{5.2.2.2}$$

is an isomorphism. Indeed, we can assume that S is strictly local, integral, with closed point s, t being a generic geometric point. We compute the fibre of (5.2.2.2) at a geometric point x in X, with image y in S. Replacing S by the strict localization of S at y, we can assume that $y = s$. As $\bar{S} \to S$ is the projective limit of finite morphisms, and radical at s, we find that $\mathsf{R}g'_*(E|\bar{X})_x = E_x$ and that the fibre of (5.2.2.2) at x is identified with the restriction $E_x = \mathsf{R}\Gamma(X(x), E) \to \mathsf{R}\Gamma(X(x)_t, E|X(x)_t)$, which is an isomorphism by hypothesis. Applying $\mathsf{R}\Gamma(X, -)$ to (5.2.2.2), we deduce that the restriction arrow (5.2.1.3) is an isomorphism. We define (5.2.2.1) as the composition of the inverse of (5.2.1.3) and the arrow of change of base (5.2.1.2).

5.2.3

If $s'' \xrightarrow{u'} s' \xrightarrow{u} s$ are the specialization arrows of geometric points of S, we verify that we have

$$\operatorname{sp}(u')\operatorname{sp}(u'') = \operatorname{sp}(uu'),$$
$$\operatorname{cosp}(u)\operatorname{cosp}(u') = \operatorname{cosp}(uu'),$$

whenever the two sides are defined.

5.2.4

Suppose that the formation of $\mathsf{R}f_*E$ commutes with any finite base change, and that f is locally acyclic relative to E. It follows from the definitions that, for any specialization $u : t \to S(s)$ of geometric points of S; the arrows $\operatorname{sp}(u)$ and $\operatorname{cosp}(u)$ are isomorphisms, inverses of each other.

The previous hypotheses are verified in particular when A is Noetherian, annihilated by an integer n invertible on S, f is proper and smooth, and E has locally constant cohomology (cohomological purity of proper morphisms (5.1.3), and local acyclicity of smooth morphisms [2, XV]). We recover the "specialization theorem" [2, XVI 2.1].

On the other hand, 5.1.1 has the following consequence:

Proposition 5.2.5. *Suppose that f is locally acyclic relative to E, and that, for any specialization $u : t \to S(s)$, the cospcialization arrow $\operatorname{cosp}(u)$ is an isomorphism. Then the formation of $\mathsf{R}f_*E$ commutes with any finite base change.*

Taking into account 5.2.4, 5.2.5 generalizes (Etale Cohomology 5.1.7).

Corollaire 5.2.6. *Suppose that f is locally acyclic relative to E. Let x be a geometric point of X, with image s in S, let $f_{(x)} : X(x) \to S(s)$ be the morphism induced by f. Then the formation of $\mathsf{R}{f_{(x)}}_*(E|X(x))$ commutes with any base change $S' \to S(s)$.*

It is enough, according to 5.2.5, to verify that the cospcialization arrows for $(f_{(x)}, E|X(x))$ are isomorphisms. By transitivity of cospcialization arrows (5.2.3), it is enough to show that, if $u : t \to S(s)$ is a specialization, then $\operatorname{cosp}(u) : \mathsf{R}\Gamma(X(x)_t, E|X(x)_t) \to \mathsf{R}\Gamma(X(x)_s, E|X(x)_s)$ is an isomorphism. But, as s is closed in $S(s)$, $X(x)_s$ is strictly local at the closed point x, $\mathsf{R}\Gamma(X(x)_s, E|X(x)_s) = E_x$, and $\operatorname{cosp}(u)$ is an isomorphism, inverse of the restriction isomorphism $E_x \to \mathsf{R}\Gamma(X(x)_t, E|X(x)_t)$.

Corollaire 5.2.7. *Let $f : X \to Y$ be an S-morphism, $p : X \to S$, $q : Y \to S$ the projections, and $E \in \operatorname{Ob}\mathsf{D}^+(X)$. We assume A is locally constant, f is locally acyclic rel. to E, and q is locally acyclic at any geometric point y of Y rel. to any locally constant sheaf at a neighborhood of y. Then p is locally acyclic rel. to E.*

Let x be a geometric point of x, with images y and s in Y and S, $t \to S(s)$ a specialization, $f_{(x)} : X(x) \to Y(y)$ the morphism induced by f. We need to show that the restriction

$$E_x \to \mathsf{R}\Gamma(X(x)_t, E|X(x)_t) \tag{$*$}$$

is an isomorphism. We can assume A is constant. According to 5.2.4 and 5.2.5, for any specialization $z \to Y(y)$, the specialization arrow

$$E_x = \mathsf{R}f_{(x)_*}(E|X(x))_y \to \mathsf{R}f_{(x)_*}(E|X(x)|_z = \mathsf{R}\Gamma(X(x)_z, E|X(x)_z)$$

is an isomorphism, i.e. $\mathsf{R}f_{(x)_*}(E|X(x))$ is constant with value E_x. The acyclicity hypothesis on q therefore implies that the restriction

$$E_x \to \mathsf{R}\Gamma(Y(y)_y, (E_x)|Y(y)_t) = \mathsf{R}\Gamma\left(Y(y)_t, \mathsf{R}f_{(x)_*}(E|X(x))|Y(y)_t\right)$$

is an isomorphism. But this arrow is identified with $(*)$, which gives the conclusion.

The hypothesis of 5.2.7 on q is verified in particular when q is smooth, and when A is killed by an integer n that is invertible on S [2, XV]. On the other hand, if the hypotheses of 5.2.7 are verified after any base change $S' \to S$, then we conclude that p is universally locally acyclic relative to E. From these remarks it follows in particular 2.14.

Corollaire 5.2.8. *Let $f : X \to S$ and $E \in \operatorname{Ob}\mathsf{D}^+(X)$. We suppose that f is universally locally acyclic relative to E, that the formation of $\mathsf{R}f_*E$ commutes to any finite base change, and that for any morphism $s' \to s$, where s is a geometric point of S and s' is the spectrum of a separably closed field, the base change arrow $\mathsf{R}\Gamma(X_s, E|X_s) \to \mathsf{R}\Gamma(X_{s'}, E|X_{s'})$ is an isomorphism. Then (f, E) is cohomologically proper (5.1.3).*

Let $f' : X' \to S'$ be the morphism deduced from f by a base change $S' \to S$, we need to show that the formation of $\mathsf{R}f'_*(E|X')$ commutes to any finite base change. According to 5.2.5, it is enough to show that, for any specialization $u' : t' \to S'(s')$, the cospacialization arrow $\operatorname{cosp}(u')$: $\mathsf{R}\Gamma(X'_{t'}, E|X'_{t'}) \to \mathsf{R}\Gamma(X'_{s'}, E|X'_{s'})$ is an isomorphism. Let s (resp. t) be the geometric point image of s (resp. t) in S, $u : t \to S(s)$ the specialization defined by u'. It is easily seen that we have a commutative square

$$\begin{array}{ccc}
\mathsf{R}\Gamma(X_t, E|X_t) & \xrightarrow{\operatorname{cosp}(u)} & \mathsf{R}\Gamma(X_s, E|X_s) \\
\downarrow & & \downarrow \\
\mathsf{R}\Gamma(X'_{t'}, E|X'_{t'}) & \xrightarrow{\operatorname{cosp}(u')} & \mathsf{R}\Gamma(X'_{s'}, E|X'_{s'})
\end{array}$$

where the vertical arrows are the base change arrows. By hypothesis, these are isomorphisms. On the other hand (5.2.4) $\operatorname{cosp}(u)$ is an isomorphism, so $\operatorname{cosp}(u')$ is an isomorphism, which finishes the proof.

Here is an example of the application of 5.2.8. Suppose A is annulled by an integer n that is invertible on S, f is locally of finite type and universally locally acyclic. Let x be a geometric point of X, with image s in S, $f_{(x)} : X(x) \to S(s)$ the morphism induced by f. It follows from 5.2.6 and 1.9 (by passage to the limit) that the hypotheses of 5.2.8 are verified by the couple $(f_{(x)}, E|X(x))$, which is therefore cohomologically proper.

5.2.9

Suppose A is constant. Let $f : X \to S$, and $E \in \operatorname{Ob} \mathsf{D}^+(X)$ the complex of finite tor-dimension. We will say that f is *strongly locally acyclic* with respect to E if, for every geometric point x of X, with image s in S, every specialization $t \to S(s)$, and every A°-module M, the restriction map

$$E_x \overset{\mathsf{L}}{\otimes}_A M \to \mathsf{R}\Gamma(X(x)_t, (E|X(x)_t) \overset{\mathsf{L}}{\otimes}_A M)$$

is an isomorphism. I do not know if "locally acyclic" implies "strongly locally acyclic."

Proposition 5.2.10. *Under the hypotheses of 5.2.9, suppose A is Noetherian of torsion, S is Noetherian, and f is strongly locally acyclic with respect to E. Then, for every cartesian square diagram*

$$\begin{tikzcd} X & X' \arrow[l, "j"'] \arrow[d, "f'"] & X'' \arrow[l, "j'"'] \arrow[d, "f''"] \\ S \arrow[from=u, "f"] & S' \arrow[l, "i"'] & S'', \arrow[l, "i'"'] \end{tikzcd}$$

with i quasi-finite and i' open immersion, and every $F \in \operatorname{Ob} \mathsf{D}^+(S'', A^\circ)$, the canonical map

$$j^* E \overset{\mathsf{L}}{\otimes} f'^* \mathsf{R}i'_* F \to \mathsf{R}j'_*((jj')^* E \overset{\mathsf{L}}{\otimes} f''^* F) \tag{$*$}$$

is an isomorphism.

We will compare this statement with [2, XV 1.17].

Let us prove 5.2.10. We easily reduce to the case where $S = S'$, strictly local, and F is a constructible sheaf concentrated in degree 0. We then solve F to the right by finite sums of sheaves of the form $u_* M$, where $u : t \to S$ is a geometric point and M is an A°-module. The hypothesis implies that $(*)$ is an isomorphism for $F = u_* M$, from which the result follows.

Appendix A

Erratum for SGA 4, volume 3

XIV p.18 1.14 (XIX 6) instead of (XX 6).

XVI 2.2 We must assume F is locally constant!

XVI 5.2 The given proof is incomplete. After stating the induction hypothesis, we must first reduce to the case where F is constant (F becomes constant on a Galois étale covering $\pi : X' \to X$, with Galois group G, and we invoke the Hochschild-Serre spectral sequence $E_2^{pq} = \mathrm{H}^p(G, \mathrm{H}^q(X', \pi^* F)) \Rightarrow \mathrm{H}^{p+q}(X, F)$). The following arguments are then correct.

XVII 1.1.8 The sign (1.1.8.1) is incorrect when F is contravariant in certain variables. We must read:

$$\rho^{\underline{k}} = (-1)^{A(\underline{k})} : \text{ automorphism of } F \circ (G_j) \left(K_i^{k_i \varepsilon(i) \varepsilon(\psi(i))} \right) \tag{1.1.8.1}$$

with

$$A(\underline{k}) = \sum_{\varepsilon(i) = \varepsilon(\psi(i)) = -} k_i + \sum_{\varepsilon(j) = -} \sum_{\substack{\psi(a) = \psi(b) = j \\ a < b}} k_a k_b$$

XVII 2.1.3 The demonstration contains blatant errors. The 3rd line (p. 34 1.9) must be deleted and the 6th and 7th (p. 34 1.12, 13) replaced with:

The arrows in the diagram (2.1.3.2) induce applications

$$\hom_y(F, F) \to \hom_{fg'}({g'}^* F, f_* F) = \hom_{x'}(f'_* {g'}^* F, g^* f_* F),$$

XVIII 2.14.4 Read XVII 6.2.7.2 instead of XVII 6.2.4.

XVIII p. 99 1-1 Read u instead of U and 3.1.16.1 instead of 3.1.11.1.

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