Eigenvalues and eigenvectors

The subject of eigenvalues and eigenvectors will take up most of the rest of the course. We will again be working with square matrices. Eigenvalues are special numbers associated with a matrix and eigenvectors are special vectors.

Eigenvectors and eigenvalues

A matrix A *acts on* vectors $\mathbf{x}$ like a function does, with input $\mathbf{x}$ and output $A\mathbf{x}$. *Eigenvectors* are vectors for which $A\mathbf{x}$ is parallel to $\mathbf{x}$. In other words:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

In this equation, $\mathbf{x}$ is an eigenvector of A and λ is an *eigenvalue* of A.

Eigenvalue 0

If the eigenvalue λ equals 0 then $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. Vectors with eigenvalue 0 make up the nullspace of A; if A is singular, then $\lambda = 0$ is an eigenvalue of A.

Examples

Suppose P is the matrix of a projection onto a plane. For any $\mathbf{x}$ in the plane $P\mathbf{x} = \mathbf{x}$, so $\mathbf{x}$ is an eigenvector with eigenvalue 1. A vector $\mathbf{x}$ perpendicular to the plane has $P\mathbf{x} = \mathbf{0}$, so this is an eigenvector with eigenvalue $\lambda = 0$. The eigenvectors of P span the whole space (but this is not true for every matrix).

The matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has an eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 1 and another eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue -1. These eigenvectors span the space. They are perpendicular because $B = B^T$ (as we will prove).

$\det(A - \lambda I) = 0$

An n by n matrix will have n eigenvalues, and their sum will be the sum of the diagonal entries of the matrix: $a_{11} + a_{22} + \cdots + a_{nn}$. This sum is the *trace* of the matrix. For a two by two matrix, if we know one eigenvalue we can use this fact to find the second.

Can we solve $A\mathbf{x} = \lambda\mathbf{x}$ for the eigenvalues and eigenvectors of A? Both λ and $\mathbf{x}$ are unknown; we need to be clever to solve this problem:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

In order for λ to be an eigenvector, $A - \lambda I$ must be singular. In other words, $\det(A - \lambda I) = 0$. We can solve this *characteristic equation* for λ to get n solutions.

If we're lucky, the solutions are distinct. If not, we have one or more *repeated eigenvalues*.

Once we've found an eigenvalue λ, we can use elimination to find the nullspace of $A - \lambda I$. The vectors in that nullspace are eigenvectors of A with eigenvalue λ.

Calculating eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)^2 - 1 \\ &= \lambda^2 - 6\lambda + 8. \end{aligned}$$

Note that the coefficient 6 is the trace (sum of diagonal entries) and 8 is the determinant of A. In general, the eigenvalues of a two by two matrix are the solutions to:

$$\lambda^2 - \operatorname{trace}(A) \cdot \lambda + \det A = 0.$$

Just as the trace is the sum of the eigenvalues of a matrix, the product of the eigenvalues of any matrix equals its determinant.

For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. We find the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 4$ in the nullspace of $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

x_2 will be in the nullspace of $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The nullspace is an entire line; x_2 could be any vector on that line. A natural choice is $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that these eigenvectors are the same as those of $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Adding $3I$ to the matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ added 3 to each of its eigenvalues and did not change its eigenvectors, because $A\mathbf{x} = (B + 3I)\mathbf{x} = \lambda\mathbf{x} + 3\mathbf{x} = (\lambda + 3)\mathbf{x}$.

A caution

Similarly, if $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{x} = \alpha\mathbf{x}$, $(A + B)\mathbf{x} = (\lambda + \alpha)\mathbf{x}$. It would be nice if the eigenvalues of a matrix sum were always the sums of the eigenvalues, but this is only true if A and B have the same eigenvectors. The eigenvalues of the product AB aren't usually equal to the products $\lambda(A)\lambda(b)$, either.

Complex eigenvalues

The matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector in the plane by 90°. It has trace $0 = \lambda_1 + \lambda_2$ and determinant $1 = \lambda_1 \cdot \lambda_2$. Its only real eigenvector is the zero vector; any other vector's direction changes when it is multiplied by Q. How will this affect our eigenvalue calculation?

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 + 1. \end{aligned}$$

$\det(A - \lambda I) = 0$ has solutions $\lambda_1 = i$ and $\lambda_2 = -i$. If a matrix has a complex eigenvalue $a + bi$ then the *complex conjugate* $a - bi$ is also an eigenvalue of that matrix.

Symmetric matrices have real eigenvalues. For *antisymmetric* matrices like Q, for which $A^T = -A$, all eigenvalues are imaginary ($\lambda = bi$).

Triangular matrices and repeated eigenvalues

For triangular matrices such as $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, the eigenvalues are exactly the entries on the diagonal. In this case, the eigenvalues are 3 and 3:

$$\begin{aligned} \det(A - \lambda \det I) &= \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)(3-\lambda) \quad \Big(= (a_{11} - \lambda)(a_{22} - \lambda) \Big) \\ &= 0, \end{aligned}$$

so $\lambda_1 = 3$ and $\lambda_2 = 3$. To find the eigenvectors, solve:

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

to get $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. There is no independent eigenvector $\mathbf{x}_2$.

Exercises on eigenvalues and eigenvectors

Problem 21.1: (6.1 #19. *Introduction to Linear Algebra:* Strang) A three by three matrix B is known to have eigenvalues 0, 1 and 2. This information is enough to find three of these (give the answers where possible):

a) The rank of B

b) The determinant of $B^T B$

c) The eigenvalues of $B^T B$

d) The eigenvalues of $(B^2 + I)^{-1}$

Problem 21.2: (6.1 #29.) Find the eigenvalues of A, B, and C when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Exercises on eigenvalues and eigenvectors

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Solution:

a) B has 0 as an eigenvalue and is therefore singular (not invertible). Since B is a three by three matrix, this means that its rank can be at most 2. Since B has two distinct nonzero eigenvalues, its rank is exactly 2.

b) Since B is singular, $\det(B) = 0$. Thus $\det(B^T B) = \det(B^T)\det(B) = 0$.

c) There is not enough information to find the eigenvalues of $B^T B$. For example:

$$\text{If } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}.$$

$$\text{If } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}.$$

d) If $p(t)$ is a polynomial and if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then

$$p(A)\mathbf{x} = p(\lambda)\mathbf{x}.$$

We also know that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $\frac{1}{0^2+1}$, $\frac{1}{1^2+1}$ and $\frac{1}{2^2+1}$, or **1, 1/2** and **1/5**.

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Solution: Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 1,4, and 6. For B we have:

$$\begin{aligned} \det(B - \lambda I) &= (-\lambda)(2-\lambda)(-\lambda) - 3(2-\lambda) \\ &= (\lambda^2 - 3)(2-\lambda). \end{aligned}$$

Hence the eigenvalues of B are $\pm\sqrt{3}$ and 2. Finally, for C we have:

$$\begin{aligned} \det(C - \lambda I) &= (2-\lambda)[(2-\lambda)^2 - 4] - 2[2(2-\lambda) - 4] + 2[4 - 2(2-\lambda)] \\ &= \lambda^3 - 6\lambda^2 = \lambda^2(\lambda - 6). \end{aligned}$$

The eigenvalues of C are 6, 0, and 0.

We can quickly check our answers by computing the determinants of A and B and by noting that C is singular.