

## Solving $A\mathbf{x} = \mathbf{0}$ : pivot variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

### Computing the nullspace

The *nullspace* of a matrix  $A$  is made up of the vectors  $\mathbf{x}$  for which  $A\mathbf{x} = \mathbf{0}$ .

Suppose:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

(Note that the columns of this matrix  $A$  are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that  $A$  is not invertible. We don't need to use an augmented matrix because the right side (the vector  $\mathbf{b}$ ) is  $\mathbf{0}$  in this computation.

The row operations used in the method of elimination don't change the solution to  $A\mathbf{x} = \mathbf{b}$  so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

We don't find a pivot in the second column, so our next pivot is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The matrix  $U$  is in *echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated.

The *rank* of a matrix  $A$  equals the number of pivots it has. In this example, the rank of  $A$  (and of  $U$ ) is 2.

### Special solutions

Once we've found  $U$  we can use back-substitution to find the solutions  $\mathbf{x}$  to the equation  $U\mathbf{x} = \mathbf{0}$ . In our example, columns 1 and 3 are *pivot columns* containing pivots, and columns 2 and 4 are *free columns*. We can assign any value to  $x_2$  and  $x_4$ ; we call these *free variables*. Suppose  $x_2 = 1$  and  $x_4 = 0$ . Then:

$$2x_3 + 4x_4 = 0 \implies x_3 = 0$$

and:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \implies x_1 = -2.$$

So one solution is  $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  (because the second column is just twice the first column). Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has  $x_4 = 1$  and  $x_2 = 0$ . The nullspace of  $A$  is the collection of all linear combinations of these “special solution” vectors.

The rank  $r$  of  $A$  equals the number of pivot columns, so the number of free columns is  $n - r$ : the number of columns (variables) minus the number of pivot columns. This equals the number of special solution vectors and the dimension of the nullspace.

## Reduced row echelon form

By continuing to use the method of elimination we can convert  $U$  to a matrix  $R$  in *reduced row echelon form* (rref form), with pivots equal to 1 and zeros above and below the pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

By exchanging some columns,  $R$  can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of  $A$  are linearly dependent, the lower rows of the matrix  $R$  will be filled with zeros:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

(Here  $I$  is an  $r$  by  $r$  square matrix.)

If  $N$  is the *nullspace matrix*  $N = \begin{bmatrix} -F \\ I \end{bmatrix}$  then  $RN = 0$ . (Here  $I$  is an  $n - r$  by  $n - r$  square matrix and  $0$  is an  $m$  by  $n - r$  matrix.) The columns of  $N$  are the special solutions.

**Exercises on solving  $Ax = 0$ : pivot variables, special solutions**

**Problem 7.1:**

a) Find the row reduced form of:

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$$

b) What is the rank of this matrix?

c) Find any special solutions to the equation  $Ax = 0$ .

**Problem 7.2:** (3.3 #17.b *Introduction to Linear Algebra*: Strang) Find  $A_1$  and  $A_2$  so that  $\text{rank}(A_1 B) = 1$  and  $\text{rank}(A_2 B) = 0$  for  $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

### Exercises on solving $Ax = 0$ : pivot variables, special solutions

#### Problem 7.1:

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#### Solution:

a) To transform  $A$  into its reduced row form, we perform a series of row operations. Different operations are possible (same answer!). First, we multiply the first row by 2 and subtract it from the third row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix}.$$

We then multiply the second row by  $\frac{1}{4}$  to make the second pivot 1:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & -12 & -3 & -21 \end{bmatrix}.$$

Multiply the second row by 12 and add it to the third row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & -12 & -3 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, multiply the second row by 5 and subtract it from the first row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 23/4 & 1/4 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The matrix is of **rank 2** because it has 2 pivots.

c) The special solutions to  $A\mathbf{x} = \mathbf{0}$  are:

$$\begin{bmatrix} -23/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1/4 \\ -7/4 \\ 0 \\ 1 \end{bmatrix}$$

**Problem 7.2:** (3.3 #17.b *Introduction to Linear Algebra: Strang*) Find  $A_1$  and  $A_2$  so that  $\text{rank}(A_1 B) = 1$  and  $\text{rank}(A_2 B) = 0$  for  $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

**Solution:** Take  $A_1 = I_2$  and  $A_2 = 0_2$ .

A less trivial example is  $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ .