Transposes, permutations, spaces R^n

In this lecture we introduce vector spaces and their subspaces.

Permutations

Multiplication by a permutation matrix P swaps the rows of a matrix; when applying the method of elimination we use permutation matrices to move zeros out of pivot positions. Our factorization $A = LU$ then becomes $PA = LU$, where P is a permutation matrix which reorders any number of rows of A. Recall that $P^{-1} = P^T$, i.e. that $P^T P = I$.

Transposes

When we take the transpose of a matrix, its rows become columns and its columns become rows. If we denote the entry in row i column j of matrix A by A_{ij}, then we can describe A^T by: $\left(A^T\right)_{ij} = A_{ji}$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}.$$

A matrix A is *symmetric* if $A^T = A$. Given any matrix R (not necessarily square) the product $R^T R$ is always symmetric, because $\left(R^T R\right)^T = R^T \left(R^T\right)^T = R^T R$. (Note that $\left(R^T\right)^T = R$.)

Vector spaces

We can add vectors and multiply them by numbers, which means we can discuss *linear combinations* of vectors. These combinations follow the rules of a *vector space*.

One such vector space is $\mathbb{R}^2$, the set of all vectors with exactly two real number components. We depict the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ by drawing an arrow from the origin to the point (a, b) which is a units to the right of the origin and b units above it, and we call $\mathbb{R}^2$ the "$x - y$ plane".

Another example of a space is $\mathbb{R}^n$, the set of (column) vectors with n real number components.

Closure

The collection of vectors with exactly two *positive* real valued components is *not* a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, -5, gives a vector that's not in the collection. We say that this collection of positive vectors is *closed* under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example, take any non-zero vector $\mathbf{v}$ in $\mathbb{R}^2$. Then the set of all vectors $c\mathbf{v}$, where c is a real number, forms a subspace of $\mathbb{R}^2$. This collection of vectors describes a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$ and is closed under addition.

A line in $\mathbb{R}^2$ that does not pass through the origin is *not* a subspace of $\mathbb{R}^2$. Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of $\mathbb{R}^2$ are:

1. all of $\mathbb{R}^2$,
2. any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and
3. the zero vector alone (Z).

The subspaces of $\mathbb{R}^3$ are:

1. all of $\mathbb{R}^3$,
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

Column space

Given a matrix A with columns in $\mathbb{R}^3$, these columns and all their linear combinations form a subspace of $\mathbb{R}^3$. This is the *column space* $C(A)$. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$, the column space of A is the plane through the origin in $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

Our next task will be to understand the equation $A\mathbf{x} = b$ in terms of subspaces and the column space of A.

Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

Solution:

a) Let P move the rows in a cycle: the first to the second, the second to the third, and the third to the first. So

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } P^3 = I.$$

b) Let $\widehat{P}$ be the block diagonal matrix with 1 and P on the diagonal; $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$. Since $P^3 = I$, also $\widehat{P}^3 = I$. So $\widehat{P}^4 = \widehat{P} \neq I$.

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

Solution:

a) The most general form of a four by four symmetric matrix is:

$$A = \begin{bmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{bmatrix}.$$

Therefore **10** entries can be chosen independently.

b) The most general form of a four by four skew-symmetric matrix is:

$$A = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}.$$

Therefore **6** entries can be chosen independently.

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

Solution:

a) True: $A^T = A$ and $B^T = B$ lead to:

$$(A+B)^T = A^T + B^T = A + B, \text{ and } (cA)^T = cA.$$

b) True: $A^T = -A$ and $B^T = -B$ lead to:

$$(A+B)^T = A^T + B^T = -A - B = -(A+B), \text{ and } (cA)^T = -cA.$$

c) False: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$