Solving $A\mathbf{x} = \mathbf{b}$: row reduced form R

When does $A\mathbf{x} = \mathbf{b}$ have solutions $\mathbf{x}$, and how can we describe those solutions?

Solvability conditions on b

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

The third row of A is the sum of its first and second rows, so we know that if $A\mathbf{x} = \mathbf{b}$ the third component of $\mathbf{b}$ equals the sum of its first and second components. If $\mathbf{b}$ does not satisfy $b_3 = b_1 + b_2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of $\mathbf{b}$ must equal zero.

One way to find out whether $A\mathbf{x} = \mathbf{b}$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in $\mathbf{b}$. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.$$

If $A\mathbf{x} = \mathbf{b}$ has a solution, then $b_3 - b_2 - b_1 = 0$. For example, we could choose $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $A\mathbf{x} = \mathbf{b}$ is solvable exactly when $\mathbf{b}$ is in the column space $C(A)$. We have these two conditions on $\mathbf{b}$; in fact they are equivalent.

Complete solution

In order to find all solutions to $A\mathbf{x} = \mathbf{b}$ we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ is to set all free variables to zero, then solve for the pivot variables.

For our example matrix A, we let $x_2 = x_4 = 0$ to get the system of equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our particular solution is:

$$\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}.$$

Combined with the nullspace

The general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is a generic vector in the nullspace. To see this, we add $A\mathbf{x}_p = \mathbf{b}$ to $A\mathbf{x}_n = \mathbf{0}$ and get $A\left(\mathbf{x}_p + \mathbf{x}_n\right) = \mathbf{b}$ for every vector $\mathbf{x}_n$ in the nullspace.

Last lecture we learned that the nullspace of A is the collection of all combinations of the special solutions $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are real numbers.

The nullspace of A is a two dimensional subspace of $\mathbb{R}^4$, and the solutions to the equation $A\mathbf{x} = \mathbf{b}$ form a plane parallel to that through $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of pivots of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then from the previous lecture we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$. For any vector $\mathbf{b}$ in $\mathbb{R}^m$ that's not a linear combination of the columns of A, there is no solution to $A\mathbf{x} = \mathbf{b}$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of $\mathbf{b}$ to satisfy. The equation $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$. There are $n - r = n - m$ free variables, so there are $n - m$ special solutions to $A\mathbf{x} = \mathbf{0}$.

Full row and column rank

If $r = m = n$ is the number of pivots of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Summary

If R is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many

Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Problem 8.2: (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \;\; 0]$ and $[U \;\; \mathbf{c}]$ to $[R \;\; 0]$ and $[R \;\; \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

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Solution:

a) The coefficient of $\mathbf{x}_p$ must be one.

b) If $\mathbf{x}_n \in \mathbf{N}(A)$ is in the nullspace of A and $\mathbf{x}_p$ is one particular solution, then $\mathbf{x}_p + \mathbf{x}_n$ is also a particular solution.

c) There's always $\mathbf{x}_n = 0$.

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Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Solution: First we transform $[U \;\; 0]$ into $[R \;\; 0]$:

$$[U \;\; 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [R \;\; 0].$$

We now solve $R\mathbf{x} = \mathbf{0}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$

where we used the free variable $x_2 = -1$. ($c\mathbf{x}$ is a solution for all c.)

We check that this is a correct solution by plugging it into $U\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

Next, we transform $[U \;\; \mathbf{c}]$ into $[R \;\; \mathbf{d}]$:

$$[U \;\; \mathbf{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [R \;\; \mathbf{d}].$$

We now solve $R\mathbf{x} = \mathbf{d}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix},$$

where we used the free variable $x_2 = 1$.

Finally, we check that this is the correct solution by plugging it into the equation $U\mathbf{x} = \mathbf{c}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Solution: **Yes.** In order to check that $A = C$ as matrices, it is enough to check that $A\mathbf{y} = C\mathbf{y}$ for all vectors $\mathbf{y}$ of the correct size (or just for the standard basis vectors, since multiplication by them "picks out the columns"). So let $\mathbf{y}$ be any vector of the correct size, and set $\mathbf{b} = A\mathbf{y}$. Then $\mathbf{y}$ is certainly a solution to $A\mathbf{x} = \mathbf{b}$, and so by our hypothesis must also be a solution to $C\mathbf{x} = \mathbf{b}$; in other words, $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$.