

The four fundamental subspaces

In this lecture we discuss the four fundamental spaces associated with a matrix and the relations between them.

Four subspaces

Any m by n matrix A determines four subspaces (possibly containing only the zero vector):

Column space, $C(A)$

$C(A)$ consists of all combinations of the columns of A and is a vector space in \mathbb{R}^m .

Nullspace, $N(A)$

This consists of all solutions \mathbf{x} of the equation $A\mathbf{x} = \mathbf{0}$ and lies in \mathbb{R}^n .

Row space, $C(A^T)$

The combinations of the row vectors of A form a subspace of \mathbb{R}^n . We equate this with $C(A^T)$, the column space of the transpose of A .

Left nullspace, $N(A^T)$

We call the nullspace of A^T the *left nullspace* of A . This is a subspace of \mathbb{R}^m .

Basis and Dimension

Column space

The r pivot columns form a basis for $C(A)$

$$\dim C(A) = r.$$

Nullspace

The special solutions to $A\mathbf{x} = \mathbf{0}$ correspond to free variables and form a basis for $N(A)$. An m by n matrix has $n - r$ free variables:

$$\dim N(A) = n - r.$$

Row space

We could perform row reduction on A^T , but instead we make use of R , the row reduced echelon form of A .

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

Although the column spaces of A and R are different, the row space of R is the same as the row space of A . The rows of R are combinations of the rows of A , and because reduction is reversible the rows of A are combinations of the rows of R .

The first r rows of R are the "echelon" basis for the row space of A :

$$\dim C(A^T) = r.$$

Left nullspace

The matrix A^T has m columns. We just saw that r is the rank of A^T , so the number of free columns of A^T must be $m - r$:

$$\dim N(A^T) = m - r.$$

The left nullspace is the collection of vectors y for which $A^T y = 0$. Equivalently, $y^T A = 0$; here y and 0 are row vectors. We say "left nullspace" because y^T is on the left of A in this equation.

To find a basis for the left nullspace we reduce an augmented version of A :

$$\begin{bmatrix} A_{m \times n} & I_{m \times n} \end{bmatrix} \longrightarrow \begin{bmatrix} R_{m \times n} & E_{m \times n} \end{bmatrix}.$$

From this we get the matrix E for which $EA = R$. (If A is a square, invertible matrix then $E = A^{-1}$.) In our example,

$$EA = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

The bottom $m - r$ rows of E describe linear dependencies of rows of A , because the bottom $m - r$ rows of R are zero. Here $m - r = 1$ (one zero row in R).

The bottom $m - r$ rows of E satisfy the equation $y^T A = \mathbf{0}$ and form a basis for the left nullspace of A .

New vector space

The collection of all 3×3 matrices forms a vector space; call it M . We can add matrices and multiply them by scalars and there's a zero matrix (additive identity). If we ignore the fact that we can multiply matrices by each other, they behave just like vectors.

Some subspaces of M include:

- all upper triangular matrices
- all symmetric matrices
- D , all diagonal matrices

D is the intersection of the first two spaces. Its dimension is 3; one basis for D is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra*: Strang) A is an m by n matrix of rank r . Suppose there are right sides \mathbf{b} for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

- a) What are all the inequalities ($<$ or \leq) that must be true between m, n , and r ?
- b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when \mathbf{d} is in which of the four subspaces? The solution \mathbf{y} is unique when the _____ contains only the zero vector.

Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra*: Strang) A is an m by n matrix of rank r . Suppose there are right sides \mathbf{b} for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

- a) What are all the inequalities ($<$ or \leq) that must be true between m, n , and r ?
- b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

Solution:

- a) The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$ and $r \leq n$. The second statement tells us that the column space is not all of \mathbb{R}^n , so $r < n$.
- b) These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it).

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when \mathbf{d} is in which of the four subspaces? The solution \mathbf{y} is unique when the _____ contains only the zero vector.

Solution: It is solvable when \mathbf{d} is in the row space, which consists of all vectors $A^T\mathbf{y}$. The solution \mathbf{y} is unique when the **left nullspace** contains only the zero vector.