Exam 1 review

This lecture is a review for the exam. The majority of the exam is on what we've learned about rectangular matrices.

Sample question 1

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-zero vectors in $\mathbb{R}^7$. They span a subspace of $\mathbb{R}^7$. What are the possible dimensions of that vector space?

The answer is 1, 2 or 3. The dimension can't be higher because a basis for this subspace has at most three vectors. It can't be 0 because the vectors are non-zero.

Sample question 2

Suppose a 5 by 3 matrix R in reduced row echelon form has $r = 3$ pivots.

1. What's the nullspace of R?

   Since the rank is 3 and there are 3 columns, there is no combination of the columns that equals $\mathbf{0}$ except the trivial one. $N(R) = \{\mathbf{0}\}$.

2. Let B be the 10 by 3 matrix $\begin{bmatrix} R \\ 2R \end{bmatrix}$. What's the reduced row echelon form of B?

   Answer: $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

3. What is the rank of B?

   Answer: 3.

4. What is the reduced row echelon form of $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$?

   When we perform row reduction we get:

   $$\begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

   Then we might need to move some zero rows to the bottom of the matrix.

5. What is the rank of C?

   Answer: 6.

6. What is the dimension of the nullspace of C^T?

   $m = 10$ and $r = 6$ so $\dim N(C^T) = 10 - 6 = 4$.

Sample question 3

Suppose we know that $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and that:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a complete solution.

Note that in this problem we don't know what A is.

1. What is the shape of the matrix A?

   Answer: 3 by 3, because $\mathbf{x}$ and $\mathbf{b}$ both have three components.

2. What's the dimension of the row space of A?

   From the complete solution we can see that the dimension of the nullspace of A is 2, so the rank of A must be $3 - 2 = 1$.

3. What is A?

   Because the second and third components of the particular solution $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ are zero, we see that the first column vector of A must be $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   Knowing that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in the nullspace tells us that the third column of A must be $\mathbf{0}$. The fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace tells us that the second column must be the negative of the first. So,

   $$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

   If we had time, we could check that this A times $\mathbf{x}$ equals $\mathbf{b}$.

4. For what vectors b does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

   This equation has a solution exactly when $\mathbf{b}$ is in the column space of A, so when $\mathbf{b}$ is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. This makes sense; we know that the rank of A is 1 and the nullspace is large.

   In contrast, we might have had $r = m$ or $r = n$.

Sample question 4

Suppose:

$$B = CD = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Try to answer the questions below without performing this matrix multiplication CD.

1. Give a basis for the nullspace of B.

   The matrix B is 3 by 4, so $N(B) \subseteq \mathbb{R}^4$. Because $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible, the nullspace of B is the same as the nullspace of $D = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Matrix D is in reduced form, so its special solutions form a basis for $N(D) = N(B)$:

   $$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   These vectors are independent, and if time permits we can multiply to check that they are in $N(B)$.

2. Find the complete solution to $B\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   We can now describe any vector in the nullspace, so all we need to do is find a particular solution. There are many possible particular solutions; the simplest one is given below.

   One way to solve this is to notice that $C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and then find a vector $\mathbf{x}$ for which $D\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Another approach is to notice that the first column of $B = CD$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. In either case, we get the complete solution:

   $$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Again, we can check our work by multiplying.

Short questions

There may not be true/false questions on the exam, but it's a good idea to review these:

1. Given a square matrix A whose nullspace is just $\{\mathbf{0}\}$, what is the nullspace of A^T?

   $N(A^T)$ is also $\{\mathbf{0}\}$ because A is square.

2. Do the invertible matrices form a subspace of the vector space of 5 by 5 matrices?

   No. The sum of two invertible matrices may not be invertible. Also, 0 is not invertible, so is not in the collection of invertible matrices.

3. True or false: If $B^2 = 0$, then it must be true that $B = 0$.

   False. We could have $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. True or false: A system $A\mathbf{x} = \mathbf{b}$ of n equations with n unknowns is solvable for every right hand side $\mathbf{b}$ if the columns of A are independent.

   True. A is invertible, and $\mathbf{x} = A^{-1}\mathbf{b}$ is a (unique) solution.

5. True or false: If $m = n$ then the row space equals the column space.

   False. The dimensions are equal, but the spaces are not. A good example to look at is $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. True or false: The matrices A and $-A$ share the same four spaces.

   True, because whenever a vector $\mathbf{v}$ is in a space, so is $-\mathbf{v}$.

7. True or false: If A and B have the same four subspaces, then A is a multiple of B.

   A good way to approach this question is to first try to convince yourself that it isn't true – look for a counterexample. If A is 3 by 3 and invertible, then its row and column space are both $\mathbb{R}^3$ and its nullspaces are $\{\mathbf{0}\}$. If B is any other invertible 3 by 3 matrix it will have the same four subspaces, and it may not be a multiple of A. So we answer "false".

   It's good to ask how we could truthfully complete the statement "If A and B have the same four subspaces, then ..."

8. If we exchange two rows of A, which subspaces stay the same?

   The row space and the nullspace stay the same.

9. Why can't a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be in the nullspace of A and also be a row of A?

   Because if $\mathbf{v}$ is the n^{th} row of A, the n^{th} component of the vector $A\mathbf{v}$ would be 14, not 0. The vector $\mathbf{v}$ could not be a solution to $A\mathbf{v} = \mathbf{0}$.

   In fact, we will learn that the row space is perpendicular to the nullspace.