

## Differential equations and $e^{At}$

The system of equations below describes how the values of variables  $u_1$  and  $u_2$  affect each other over time:

$$\begin{aligned}\frac{du_1}{dt} &= -u_1 + 2u_2 \\ \frac{du_2}{dt} &= u_1 - 2u_2.\end{aligned}$$

Just as we applied linear algebra to solve a difference equation, we can use it to solve this differential equation. For example, the initial condition  $u_1 = 1$ ,  $u_2 = 0$  can be written  $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

### Differential equations $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

By looking at the equations above, we might guess that over time  $u_1$  will decrease. We can get the same sort of information more safely by looking at the eigenvalues of the matrix  $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$  of our system  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ . Because  $A$  is singular and its trace is  $-3$  we know that its eigenvalues are  $\lambda_1 = 0$  and  $\lambda_2 = -3$ . The solution will turn out to include  $e^{-3t}$  and  $e^{0t}$ . As  $t$  increases,  $e^{-3t}$  vanishes and  $e^{0t} = 1$  remains constant. Eigenvalues equal to zero have eigenvectors that are *steady state* solutions.

$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is an eigenvector for which  $A\mathbf{x}_1 = 0\mathbf{x}_1$ . To find an eigenvector corresponding to  $\lambda_2 = -3$  we solve  $(A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$ :

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = \mathbf{0} \quad \text{so} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we can check that  $A\mathbf{x}_2 = -3\mathbf{x}_2$ . The general solution to this system of differential equations will be:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2.$$

Is  $e^{\lambda_1 t} \mathbf{x}_1$  really a solution to  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ ? To find out, plug in  $\mathbf{u} = e^{\lambda_1 t} \mathbf{x}_1$ :

$$\frac{d\mathbf{u}}{dt} = \lambda_1 e^{\lambda_1 t} \mathbf{x}_1,$$

which agrees with:

$$A\mathbf{u} = e^{\lambda_1 t} A\mathbf{x}_1 = \lambda_1 e^{\lambda_1 t} \mathbf{x}_1.$$

The two “pure” terms  $e^{\lambda_1 t} \mathbf{x}_1$  and  $e^{\lambda_2 t} \mathbf{x}_2$  are analogous to the terms  $\lambda_i^k \mathbf{x}_i$  we saw in the solution  $c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 + \cdots + c_n \lambda_n^k \mathbf{x}_n$  to the difference equation  $\mathbf{u}_{k+1} = A\mathbf{u}_k$ .

Plugging in the values of the eigenvectors, we get:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We know  $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so at  $t = 0$ :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$c_1 = c_2 = 1/3 \text{ and } \mathbf{u}(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This tells us that the system starts with  $u_1 = 1$  and  $u_2 = 0$  but that as  $t$  approaches infinity,  $u_1$  decays to  $2/3$  and  $u_2$  increases to  $1/3$ . This might describe stuff moving from  $u_1$  to  $u_2$ .

$$\text{The steady state of this system is } \mathbf{u}(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}.$$

## Stability

Not all systems have a steady state. The eigenvalues of  $A$  will tell us what sort of solutions to expect:

1. Stability:  $\mathbf{u}(t) \rightarrow 0$  when  $\text{Re}(\lambda) < 0$ .
2. Steady state: One eigenvalue is 0 and all other eigenvalues have negative real part.
3. Blow up: if  $\text{Re}(\lambda) > 0$  for any eigenvalue  $\lambda$ .

If a two by two matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  has two eigenvalues with negative real part, its trace  $a + d$  is negative. The converse is not true:  $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$  has negative trace but one of its eigenvalues is 1 and  $e^{1t}$  blows up. If  $A$  has a positive determinant and negative trace then the corresponding solutions must be stable.

## Applying S

The final step of our solution to the system  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$  was to solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

or  $S\mathbf{c} = \mathbf{u}(0)$ , where  $S$  is the eigenvector matrix. The components of  $\mathbf{c}$  determine the contribution from each pure exponential solution, based on the initial conditions of the system.

In the equation  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ , the matrix  $A$  couples the pure solutions. We set  $\mathbf{u} = S\mathbf{v}$ , where  $S$  is the matrix of eigenvectors of  $A$ , to get:

$$S \frac{d\mathbf{v}}{dt} = AS\mathbf{v}$$

or:

$$\frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v} = \Lambda\mathbf{v}.$$

This diagonalizes the system:  $\frac{dv_i}{dt} = \lambda_i v_i$ . The general solution is then:

$$\begin{aligned}\mathbf{v}(t) &= e^{\Lambda t} \mathbf{v}(0), \quad \text{and} \\ \mathbf{u}(t) &= S e^{\Lambda t} S^{-1} \mathbf{v}(0) = e^{At} \mathbf{u}(0).\end{aligned}$$

### Matrix exponential $e^{At}$

What does  $e^{At}$  mean if  $A$  is a matrix? We know that for a real number  $x$ ,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

We can use the same formula to define  $e^{At}$ :

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots$$

Similarly, if the eigenvalues of  $At$  are small, we can use the geometric series  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  to estimate  $(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \dots$ .

We've said that  $e^{At} = S e^{\Lambda t} S^{-1}$ . If  $A$  has  $n$  independent eigenvectors we can prove this from the definition of  $e^{At}$  by using the formula  $A = S\Lambda S^{-1}$ :

$$\begin{aligned}e^{At} &= I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots \\ &= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \frac{S\Lambda^3 S^{-1}}{6}t^3 + \dots \\ &= S e^{\Lambda t} S^{-1}.\end{aligned}$$

It's impractical to add up infinitely many matrices. Fortunately, there is an easier way to compute  $e^{\Lambda t}$ . Remember that:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}.$$

When we plug this in to our formula for  $e^{At}$  we find that:

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}.$$

This is another way to see the relationship between the stability of  $\mathbf{u}(t) = Se^{At}S^{-1}\mathbf{v}(0)$  and the eigenvalues of  $A$ .

## Second order

We can change the second order equation  $y'' + by' + ky = 0$  into a two by two first order system using a method similar to the one we used to find a formula for the Fibonacci numbers. If  $u = \begin{bmatrix} y' \\ y \end{bmatrix}$ , then

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}.$$

We could use the methods we just learned to solve this system, and that would give us a solution to the second order scalar equation we started with.

If we start with a  $k$ th order equation we get a  $k$  by  $k$  matrix with coefficients of the equation in the first row and 1's on a diagonal below that; the rest of the entries are 0.

### Exercises on differential equations and $e^{At}$

**Problem 23.1:** (6.3 #14.a *Introduction to Linear Algebra*: Strang) The matrix in this question is skew-symmetric ( $A^T = -A$ ) :

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \quad \text{or} \quad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2. \end{aligned}$$

Find the derivative of  $\|\mathbf{u}(t)\|^2$  using the definition:

$$\|\mathbf{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2.$$

What does this tell you about the rate of change of the length of  $\mathbf{u}$ ? What does this tell you about the range of values of  $\mathbf{u}(t)$ ?

**Problem 23.2:** (6.3 #24.) Write  $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$  as  $S\Lambda S^{-1}$ . Multiply  $Se^{\Lambda t}S^{-1}$  to find the matrix exponential  $e^{At}$ . Check your work by evaluating  $e^{At}$  and the derivative of  $e^{At}$  when  $t = 0$ .

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**Solution:**

$$\begin{aligned} \frac{d\|\mathbf{u}(t)\|^2}{dt} &= \frac{d(u_1^2 + u_2^2 + u_3^2)}{dt} \\ &= 2u_1u_1' + 2u_2u_2' + 2u_3u_3' \\ &= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) \\ &= 0. \end{aligned}$$

This means  $\|\mathbf{u}(t)\|^2$  stays equal to  $\|\mathbf{u}(0)\|^2$ . Because  $\mathbf{u}(t)$  never changes length, it is always on the circumference of a circle of radius  $\|\mathbf{u}(0)\|$ .

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**Solution:** The eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = 3$ , with corresponding eigenvectors  $\mathbf{x}_1 = (1, 0)$  and  $\mathbf{x}_2 = (1, 2)$ . This gives us the following values for  $S$ ,  $\Lambda$ , and  $S^{-1}$ :

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}.$$

We use these to find  $e^{At}$  :

$$Se^{At}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} = e^{At}.$$

Check:

$$e^{At} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} \text{ equals } I \text{ when } t = 0. \checkmark$$

$$\frac{de^{At}}{dt} = \begin{bmatrix} e^t & 1.5e^{3t} - .5e^t \\ 0 & 3e^{3t} \end{bmatrix}.$$

$$\left. \frac{de^{At}}{dt} \right|_{t=0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A. \checkmark$$