Orthogonal vectors and subspaces

In this lecture we learn what it means for vectors, bases and subspaces to be *orthogonal*. The symbol for this is $\perp$.

The "big picture" of this course is that the row space of a matrix' is orthogonal to its nullspace, and its column space is orthogonal to its left nullspace.

row space	column space
dimension r	dimension r
$\perp$	$\perp$
nullspace	left nullspace $N(A^T)$
dimension $n-r$	dimension $m-r$

Orthogonal vectors

Orthogonal is just another word for *perpendicular*. Two vectors are *orthogonal* if the angle between them is 90 degrees. If two vectors are orthogonal, they form a right triangle whose hypotenuse is the sum of the vectors. Thus, we can use the Pythagorean theorem to prove that *the dot product* $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$ *is zero* exactly when $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. (The length squared $||\mathbf{x}||^2$ equals $\mathbf{x}^T\mathbf{x}$.)

Note that all vectors are orthogonal to the zero vector.

Orthogonal subspaces

Subspace S is *orthogonal* to subspace T means: every vector in S is orthogonal to every vector in T. The blackboard is not orthogonal to the floor; two vectors in the line where the blackboard meets the floor aren't orthogonal to each other.

In the plane, the space containing only the zero vector and any line through the origin are orthogonal subspaces. A line through the origin and the whole plane are never orthogonal subspaces. Two lines through the origin are orthogonal subspaces if they meet at right angles.

Nullspace is perpendicular to row space

The row space of a matrix is orthogonal to the nullspace, because $A\mathbf{x} = \mathbf{0}$ means the dot product of $\mathbf{x}$ with each row of A is 0. But then the product of $\mathbf{x}$ with any combination of rows of A must be 0.

The column space is orthogonal to the left nullspace of A because the row space of A^T is perpendicular to the nullspace of A^T.

In some sense, the row space and the nullspace of a matrix subdivide $\mathbb{R}^n$ into two perpendicular subspaces. For $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$, the row space has dimension 1 and basis $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and the nullspace has dimension 2 and is the plane through the origin perpendicular to the vector $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

Not only is the nullspace orthogonal to the row space, their dimensions add up to the dimension of the whole space. We say that the nullspace and the row space are *orthogonal complements* in $\mathbb{R}^n$. The nullspace contains all the vectors that are perpendicular to the row space, and vice versa.

We could say that this is part two of the fundamental theorem of linear algebra. Part one gives the dimensions of the four subspaces, part two says those subspaces come in orthogonal pairs, and part three will be about orthogonal bases for these subspaces.

$N(A^TA) = N(A)$

Due to measurement error, $A\mathbf{x} = \mathbf{b}$ is often unsolvable if $m > n$. Our next challenge is to find the best possible solution in this case. The matrix A^TA plays a key role in this effort: the central equation is $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

We know that A^TA is square ($n \times n$) and symmetric. When is it invertible?

Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$. Then:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

is invertible. A^TA is not always invertible. In fact:

$$\begin{aligned} N(A^TA) &= N(A) \\ \text{rank of } A^TA &= \text{rank of } A. \end{aligned}$$

We conclude that A^TA is invertible exactly when A has independent columns.

Exercises on orthogonal vectors and subspaces

Problem 16.1: (4.1 #7. *Introduction to Linear Algebra:* Strang) For every system of m equations with no solution, there are numbers $y_1, ..., y_m$ that multiply the equations so they add up to $0 = 1$. This is called *Fredholm's Alternative*:

> Exactly one of these problems has a solution:
> $A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$, $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2 and y_3 chosen so that the equations add up to $0 = 1$.

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}$, $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

Exercises on orthogonal vectors and subspaces

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Solution: Let $y_1 = 1$, $y_2 = 1$ and $y_3 = -1$. Then the left-hand side of the sum of the equations is:

$$(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = x_1 - x_2 + x_2 - x_3 + x_3 - x_1 = 0,$$

and the right-hand side verifies that $\mathbf{y}^T\mathbf{b} = 1$:

$$1 + 1 - 1 = 1.$$

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}$, $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

Solution:

a) In order for $\mathbf{r}$ and $\mathbf{n}$ to be bases for $N(A)$ and $C(A^T)$, we must have

$$\mathbf{r} \cdot \mathbf{n} = 0,$$

as the row space and null space must be orthogonal. Similarly, in order for $\mathbf{c}$ and $\mathbf{l}$ to form bases for $C(A)$ and $N(A^T)$ we need

$$\mathbf{c} \cdot \mathbf{l} = 0,$$

as the column space and the left nullspace are orthogonal. In addition, we need:

$$\dim N(A) + \dim C(A^T) = n \quad \text{and} \quad \dim N(A^T) + \dim C(A) = m;$$

however, in this case $n = m = 1$, and as the four vectors we are given are nonzero both of these equations reduce to $1 + 1 = 2$, which is automatically satisfied.

b) One possible such matrix is $A = \mathbf{c}\mathbf{r}^T$.

Note that each column of A will be a multiple of $\mathbf{c}$, so it will have the desired column space. On the other hand, each row of A will be a multiple of $\mathbf{r}$, so A will have the desired row space. The nullspaces don't need to be checked, as any matrix with the correct row and column space will have the desired nullspaces (as the nullspaces are just the orthogonal complements of the row and column spaces).

Projections onto subspaces

Projections

If we have a vector $\mathbf{b}$ and a line determined by a vector $\mathbf{a}$, how do we find the point on the line that is closest to $\mathbf{b}$?

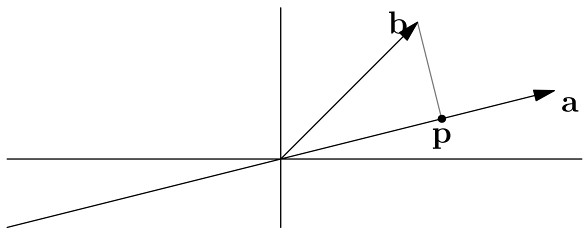

Figure 1: The point closest to $\mathbf{b}$ on the line determined by $\mathbf{a}$.

We can see from Figure 1 that this closest point $\mathbf{p}$ is at the intersection formed by a line through $\mathbf{b}$ that is orthogonal to $\mathbf{a}$. If we think of $\mathbf{p}$ as an approximation of $\mathbf{b}$, then the length of $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is the error in that approximation.

We could try to find $\mathbf{p}$ using trigonometry or calculus, but it's easier to use linear algebra. Since $\mathbf{p}$ lies on the line through $\mathbf{a}$, we know $\mathbf{p} = x\mathbf{a}$ for some number x. We also know that $\mathbf{a}$ is perpendicular to $\mathbf{e} = \mathbf{b} - \mathbf{xa}$:

$$\begin{aligned} \mathbf{a}^T(\mathbf{b} - x\mathbf{a}) &= 0 \\ x\mathbf{a}^T\mathbf{a} &= \mathbf{a}^T\mathbf{b} \\ x &= \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}, \end{aligned}$$

and $\mathbf{p} = \mathbf{a}x = \mathbf{a}\dfrac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$. Doubling $\mathbf{b}$ doubles $\mathbf{p}$. Doubling $\mathbf{a}$ does not affect $\mathbf{p}$.

Projection matrix

We'd like to write this projection in terms of a *projection matrix* P: $\mathbf{p} = P\mathbf{b}$.

$$\mathbf{p} = \mathbf{xa} = \frac{\mathbf{a}\mathbf{a}^T\mathbf{a}}{\mathbf{a}^T\mathbf{a}},$$

so the matrix is:

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

Note that $\mathbf{a}\mathbf{a}^T$ is a three by three matrix, not a number; matrix multiplication is not commutative.

The column space of P is spanned by $\mathbf{a}$ because for any $\mathbf{b}$, $P\mathbf{b}$ lies on the line determined by $\mathbf{a}$. The rank of P is 1. P is symmetric. $P^2\mathbf{b} = P\mathbf{b}$ because the projection of a vector already on the line through $\mathbf{a}$ is just that vector. In general, projection matrices have the properties:

$$P^T = P \qquad \text{and} \qquad P^2 = P.$$

Why project?

As we know, the equation $A\mathbf{x} = \mathbf{b}$ may have no solution. The vector $A\mathbf{x}$ is always in the column space of A, and $\mathbf{b}$ is unlikely to be in the column space. So, we project $\mathbf{b}$ onto a vector $\mathbf{p}$ in the column space of A and solve $A\hat{\mathbf{x}} = \mathbf{p}$.

Projection in higher dimensions

In $\mathbb{R}^3$, how do we project a vector $\mathbf{b}$ onto the closest point $\mathbf{p}$ in a plane?

If $\mathbf{a}_1$ and $\mathbf{a}_2$ form a basis for the plane, then that plane is the column space of the matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$.

We know that $\mathbf{p} = \hat{x}_1\mathbf{a}_1 + \hat{x}_2\mathbf{a}_2 = A\hat{\mathbf{x}}$. We want to find $\hat{\mathbf{x}}$. There are many ways to show that $\mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - A\hat{\mathbf{x}}$ is orthogonal to the plane we're projecting onto, after which we can use the fact that $\mathbf{e}$ is perpendicular to $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \quad \text{and} \quad \mathbf{a}_2^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0.$$

In matrix form, $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$. When we were projecting onto a line, A only had one column and so this equation looked like: $a^T(\mathbf{b} - x\mathbf{a}) = \mathbf{0}$.

Note that $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is in the nullspace of A^T and so is in the left nullspace of A. We know that everything in the left nullspace of A is perpendicular to the column space of A, so this is another confirmation that our calculations are correct.

We can rewrite the equation $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$ as:

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}.$$

When projecting onto a line, A^TA was just a number; now it is a square matrix. So instead of dividing by $\mathbf{a}^T\mathbf{a}$ we now have to multiply by $(A^TA)^{-1}$

In n dimensions,

$$\begin{aligned} \hat{\mathbf{x}} &= (A^TA)^{-1}A^T\mathbf{b} \\ \mathbf{p} = A\hat{\mathbf{x}} &= A(A^TA)^{-1}A^T\mathbf{b} \\ P &= A(A^TA)^{-1}A^T. \end{aligned}$$

It's tempting to try to simplify these expressions, but if A isn't a square matrix we can't say that $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$. If A does happen to be a square, invertible matrix then its column space is the whole space and contains $\mathbf{b}$. In this case P is the identity, as we find when we simplify. It is still true that:

$$P^T = P \qquad \text{and} \qquad P^2 = P.$$

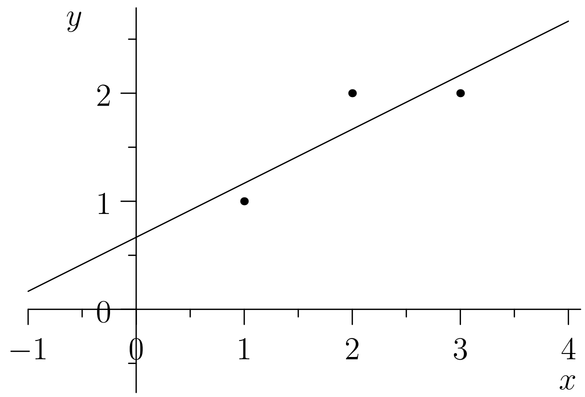

Figure 2: Three points and a line close to them.

Least Squares

Suppose we're given a collection of data points (t, b):

$$\{(1,1),(2,2),(3,2)\}$$

and we want to find the closest line $b = C + Dt$ to that collection. If the line went through all three points, we'd have:

$$\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2, \end{aligned}$$

which is equivalent to:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{\mathbf{b}}.$$

In our example the line does not go through all three points, so this equation is not solvable. Instead we'll solve:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

Exercises on projections onto subspaces

Problem 15.1: (4.2 #13. *Introduction to Linear Algebra:* Strang) Suppose A is the four by four identity matrix with its last column removed; A is four by three. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Problem 15.2: (4.2 #17.) If $P^2 = P$, show that $(I - P)^2 = I - P$. For the matrices A and P from the previous question, P projects onto the column space of A and $I - P$ projects onto the ________.

Exercises on projections onto subspaces

Problem 15.1: (4.2 #13. *Introduction to Linear Algebra:* Strang) Suppose A is the four by four identity matrix with its last column removed; A is four by three. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution: P will be four by four since we are projecting a 4-dimensional vector to another 4-dimensional vector. We will have:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This can be seen by observing that the column space of A is the $wxy-$space, so we just need to subtract the z coordinate from the 4-dimensional vector (w, x, y, z) we're projecting. The projection of $\mathbf{b}$ is therefore:

$$\mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem 15.2: (4.2 #17.) If $P^2 = P$, show that $(I - P)^2 = I - P$. For the matrices A and P from the previous question, P projects onto the column space of A and $I - P$ projects onto the __________.

Solution:

$$(I - P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

Using the matrices A and P from the previous question,

$$I - P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

projects onto the **left nullspace** of A.

Projection matrices and least squares

Projections

Last lecture, we learned that $P = A(A^TA)^{-1}A^T$ is the matrix that projects a vector $\mathbf{b}$ onto the space spanned by the columns of A. If $\mathbf{b}$ is perpendicular to the column space, then it's in the left nullspace $N(A^T)$ of A and $P\mathbf{b} = \mathbf{0}$. If $\mathbf{b}$ is in the column space then $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$, and $P\mathbf{b} = \mathbf{b}$.

A typical vector will have a component $\mathbf{p}$ in the column space and a component $\mathbf{e}$ perpendicular to the column space (in the left nullspace); its projection is just the component in the column space.

The matrix projecting $\mathbf{b}$ onto $N(A^T)$ is $I - P$:

$$\begin{aligned} \mathbf{e} &= \mathbf{b} - \mathbf{p} \\ \mathbf{e} &= (I - P)\mathbf{b}. \end{aligned}$$

Naturally, $I - P$ has all the properties of a projection matrix.

Least squares

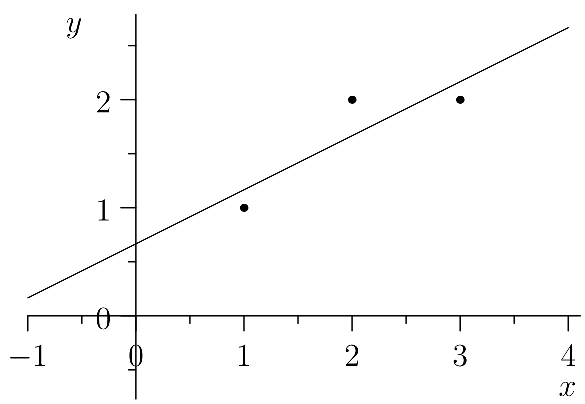

Figure 1: Three points and a line close to them.

We want to find the closest line $b = C + Dt$ to the points $(1, 1)$, $(2, 2)$, and $(3, 2)$. The process we're going to use is called *linear regression*; this technique is most useful if none of the data points are *outliers*.

By "closest" line we mean one that minimizes the error represented by the distance from the points to the line. We measure that error by adding up the squares of these distances. In other words, we want to minimize $||A\mathbf{x} - \mathbf{b}||^2 = ||\mathbf{e}||^2$.

If the line went through all three points, we'd have:

$$\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2, \end{aligned}$$

but this system is unsolvable. It's equivalent to $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} C \\ D \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

There are two ways of viewing this. In the space of the line we're trying to find, e_1, e_2 and e_3 are the vertical distances from the data points to the line. The components p_1, p_2 and p_3 are the values of $C + Dt$ near each data point; $\mathbf{p} \approx \mathbf{b}$.

In the other view we have a vector $\mathbf{b}$ in $\mathbb{R}^3$, its projection $\mathbf{p}$ onto the column space of A, and its projection $\mathbf{e}$ onto $N(A^T)$.

We will now find $\hat{\mathbf{x}} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$ and $\mathbf{p}$. We know:

$$\begin{aligned} A^T A\hat{\mathbf{x}} &= A^T\mathbf{b} \\ \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} &= \begin{bmatrix} 5 \\ 11 \end{bmatrix}. \end{aligned}$$

From this we get the *normal equations*:

$$\begin{aligned} 3\hat{C} + 6\hat{D} &= 5 \\ 6\hat{C} + 14\hat{D} &= 11. \end{aligned}$$

We solve these to find $\hat{D} = 1/2$ and $\hat{C} = 2/3$.

We could also have used calculus to find the minimum of the following function of two variables:

$$e_1^2 + e_2^2 + e_3^2 = (C + D - 1)^2 + (C + 2D - 2)^2 + (C + 3D - 2)^2.$$

Either way, we end up solving a system of linear equations to find that the closest line to our points is $b = \frac{2}{3} + \frac{1}{2}t$.

This gives us:

i	p_i	e_i
1	$7/6$	$-1/6$
2	$5/3$	$1/3$
3	$13/6$	$-1/6$

or $\mathbf{p} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}$ and $\mathbf{e} = \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$. Note that $\mathbf{p}$ and $\mathbf{e}$ are orthogonal, and also that $\mathbf{e}$ is perpendicular to the columns of A.

The matrix A^TA

We've been assuming that the matrix A^TA is invertible. Is this justified?

If A has independent columns, then A^TA is invertible.

To prove this we assume that $A^TA\mathbf{x} = \mathbf{0}$, then show that it must be true that $\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} A^TA\mathbf{x} &= \mathbf{0} \\ \mathbf{x}^TA^TA\mathbf{x} &= \mathbf{x}^T\mathbf{0} \\ (A\mathbf{x})^T(A\mathbf{x}) &= \mathbf{0} \\ A\mathbf{x} &= \mathbf{0}. \end{aligned}$$

Since A has independent columns, $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$.

As long as the columns of A are independent, we can use linear regression to find approximate solutions to unsolvable systems of linear equations. The columns of A are guaranteed to be independent if they are *orthonormal*, i.e. if they are perpendicular unit vectors like $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, or like $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

Exercises on projection matrices and least squares

Problem 16.1: (4.3 #17. *Introduction to Linear Algebra:* Strang) Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.

Problem 16.2: (4.3 #18.) Find the projection $\mathbf{p} = A\hat{\mathbf{x}}$ in the previous problem. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$. Why is $P\mathbf{e} = \mathbf{0}$?

Problem 16.3: (4.3 #19.) Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in the previous problem. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to __________ so the projection is $\mathbf{p} = \mathbf{0}$.

Problem 16.4: (4.3 #20.) Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and $\mathbf{e}$. The error is $\mathbf{e} = \mathbf{0}$ because this $\mathbf{b}$ is __________.

Problem 16.5: (4.3 #21.) Which of the four subspaces contains the error vector $\mathbf{e}$? Which contains $\mathbf{p}$? Which contains $\hat{\mathbf{x}}$? What is the nullspace of A?

Problem 16.6: (4.3 #22.) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Exercises on projection matrices and least squares

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Solution: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$.

The solution $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

Problem 16.2: (4.3 #18.) Find the projection $\mathbf{p} = A\hat{\mathbf{x}}$ in the previous problem. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$. Why is $P\mathbf{e} = \mathbf{0}$?

Solution: $\mathbf{p} = A\hat{\mathbf{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\mathbf{b} - \mathbf{p} = (2, -6, 4)$. This error $\mathbf{e}$ has $P\mathbf{e} = P\mathbf{b} - P\mathbf{p} = \mathbf{p} - \mathbf{p} = \mathbf{0}$.

Problem 16.3: (4.3 #19.) Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in the previous problem. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to __________ so the projection is $\mathbf{p} = \mathbf{0}$.

Solution: If $\mathbf{b}$ = error $\mathbf{e}$ then $\mathbf{b}$ is perpendicular to the column space of A. Projection $\mathbf{p} = \mathbf{0}$.

Problem 16.4: (4.3 #20.) Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and $\mathbf{e}$. The error is $\mathbf{e} = \mathbf{0}$ because this $\mathbf{b}$ is __________.

Solution: If $\mathbf{b} = A\hat{\mathbf{x}} = (5, 13, 17)$ then $\hat{\mathbf{x}} = (9, 4)$ and $\mathbf{e} = \mathbf{0}$ since $\mathbf{b}$ is *in the column space of* A.

Problem 16.5: (4.3 #21.) Which of the four subspaces contains the error vector $\mathbf{e}$? Which contains $\mathbf{p}$? Which contains $\hat{\mathbf{x}}$? What is the nullspace of A?

Solution: $\mathbf{e}$ is in $\mathbf{N}(A^T)$; $\mathbf{p}$ is in $\mathbf{C}(A)$; $\hat{\mathbf{x}}$ is in $\mathbf{C}(A^T)$; $\mathbf{N}(A) = \{\mathbf{0}\}$ = zero vector only.

Problem 16.6: (4.3 #22.) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Solution: The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.
Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^T A$

Orthogonal matrices and Gram-Schmidt

In this lecture we finish introducing orthogonality. Using an orthonormal basis or a matrix with orthonormal columns makes calculations much easier. The Gram-Schmidt process starts with any basis and produces an orthonormal basis that spans the same space as the original basis.

Orthonormal vectors

The vectors $\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_n$ are *orthonormal* if:

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

In other words, they all have (normal) length 1 and are perpendicular (ortho) to each other. Orthonormal vectors are always independent.

Orthonormal matrix

If the columns of $Q = \begin{bmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{bmatrix}$ are orthonormal, then $Q^T Q = I$ is the identity.

Matrices with orthonormal columns are a new class of important matrices to add to those on our list: triangular, diagonal, permutation, symmetric, reduced row echelon, and projection matrices. We'll call them "orthonormal matrices".

A square orthonormal matrix Q is called an *orthogonal matrix*. If Q is square, then $Q^T Q = I$ tells us that $Q^T = Q^{-1}$.

For example, if $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ then $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Both Q and Q^T are orthogonal matrices, and their product is the identity.

The matrix $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is not, but we can adjust that matrix to get the orthogonal matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. We can use the same tactic to find some larger orthogonal matrices called *Hadamard matrices*:

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

An example of a rectangular matrix with orthonormal columns is:

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}.$$

We can extend this to a (square) orthogonal matrix:

$$\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}.$$

These examples are particularly nice because they don't include complicated square roots.

Orthonormal columns are good

Suppose Q has orthonormal columns. The matrix that projects onto the column space of Q is:

$$P = Q^T(Q^TQ)^{-1}Q^T.$$

If the columns of Q are orthonormal, then $Q^TQ = I$ and $P = QQ^T$. If Q is square, then $P = I$ because the columns of Q span the entire space.

Many equations become trivial when using a matrix with orthonormal columns. If our basis is orthonormal, the projection component $\hat{x}_i$ is just $\mathbf{q}_i^T\mathbf{b}$ because $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ becomes $\hat{\mathbf{x}} = Q^T\mathbf{b}$.

Gram-Schmidt

With elimination, our goal was "make the matrix triangular". Now our goal is "make the matrix orthonormal".

We start with two independent vectors $\mathbf{a}$ and $\mathbf{b}$ and want to find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ that span the same plane. We start by finding orthogonal vectors $\mathbf{A}$ and $\mathbf{B}$ that span the same space as $\mathbf{a}$ and $\mathbf{b}$. Then the unit vectors $\mathbf{q}_1 = \frac{\mathbf{A}}{||\mathbf{A}||}$ and $\mathbf{q}_2 = \frac{\mathbf{B}}{||\mathbf{B}||}$ form the desired orthonormal basis.

Let $\mathbf{A} = \mathbf{a}$. We get a vector orthogonal to $\mathbf{A}$ in the space spanned by $\mathbf{a}$ and $\mathbf{b}$ by projecting $\mathbf{b}$ onto $\mathbf{a}$ and letting $\mathbf{B} = \mathbf{b} - \mathbf{p}$. ($\mathbf{B}$ is what we previously called $\mathbf{e}$.)

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A}.$$

If we multiply both sides of this equation by $\mathbf{A}^T$, we see that $\mathbf{A}^T\mathbf{B} = 0$.

What if we had started with three independent vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$? Then we'd find a vector $\mathbf{C}$ orthogonal to both $\mathbf{A}$ and $\mathbf{B}$ by subtracting from $\mathbf{c}$ its components in the $\mathbf{A}$ and $\mathbf{B}$ directions:

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}^T\mathbf{c}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} - \frac{\mathbf{B}^T\mathbf{c}}{\mathbf{B}^T\mathbf{B}}\mathbf{B}.$$

For example, suppose $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Then $\mathbf{A} = \mathbf{a}$ and:

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Normalizing, we get:

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}.$$

The column space of Q is the plane spanned by $\mathbf{a}$ and $\mathbf{b}$.

When we studied elimination, we wrote the process in terms of matrices and found $A = LU$. A similar equation $A = QR$ relates our starting matrix A to the result Q of the Gram-Schmidt process. Where L was lower triangular, R is upper triangular.

Suppose $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Then:

$$\underset{A}{\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}} \underset{R}{\begin{bmatrix} \mathbf{a}_1^T\mathbf{q}_1 & \mathbf{a}_2^T\mathbf{q}_1 \\ \mathbf{a}_1^T\mathbf{q}_2 & \mathbf{a}_2^T\mathbf{q}_2 \end{bmatrix}}.$$

If R is upper triangular, then it should be true that $\mathbf{a}_1^T\mathbf{q}_2 = 0$. This must be true because we chose $\mathbf{q}_1$ to be a unit vector in the direction of $\mathbf{a}_1$. All the later $\mathbf{q}_i$ were chosen to be perpendicular to the earlier ones.

Notice that $R = Q^T A$. This makes sense; $Q^T Q = I$.

Exercises on orthogonal matrices and Gram-Schmidt

Problem 17.1: (4.4 #10.b *Introduction to Linear Algebra:* Strang)

Orthonormal vectors are automatically linearly independent.

Matrix Proof: Show that $Q\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

Problem 17.2: (4.4 #18) Given the vectors **a**, **b** and **c** listed below, use the Gram-Schmidt process to find orthogonal vectors **A**, **B**, and **C** that span the same space.

$$\mathbf{a} = (1, -1, 0, 0), \mathbf{b} = (0, 1, -1, 0), \mathbf{c} = (0, 0, 1, -1).$$

Show that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are bases for the space of vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

Exercises on orthogonal matrices and Gram-Schmidt

Problem 17.1: (4.4 #10.b *Introduction to Linear Algebra:* Strang)
Orthonormal vectors are automatically linearly independent.
Matrix Proof: Show that $Q\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

Solution: By definition, Q is a matrix whose columns are orthonormal, and so we know that $Q^TQ = I$ (where Q may be rectangular). Then:

$$Q\mathbf{x} = \mathbf{0} \Longrightarrow Q^TQ\mathbf{x} = Q^T\mathbf{0} \Longrightarrow I\mathbf{x} = \mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}.$$

Thus the nullspace of Q is the zero vector, and so the columns of Q are linearly independent. There are no non-zero linear combinations of the columns that equal the zero vector. Thus, orthonormal vectors are automatically linearly independent.

Problem 17.2: (4.4 #18) Given the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ listed below, use the Gram-Schmidt process to find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ that span the same space.

$$\mathbf{a} = (1,-1,0,0), \mathbf{b} = (0,1,-1,0), \mathbf{c} = (0,0,1,-1).$$

Show that $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ and $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ are bases for the space of vectors perpendicular to $\mathbf{d} = (1,1,1,1)$.

Solution: We apply Gram-Schmidt to $\mathbf{a},\mathbf{b},\mathbf{c}$. First, we set

$$\mathbf{A} = \mathbf{a} = (\mathbf{1},-\mathbf{1},\mathbf{0},\mathbf{0}).$$

Next we find $\mathbf{B}$:

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A^Tb}}{\mathbf{A^TA}}\mathbf{A} = (0,1,-1,0) + \frac{1}{2}(1,-1,0,0) = \left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1},\mathbf{0}\right).$$

And then we find $\mathbf{C}$:

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A^Tc}}{\mathbf{A^TA}}\mathbf{A} - \frac{\mathbf{B^Tc}}{\mathbf{B^TB}}\mathbf{B} = (0,0,1,-1) + \frac{2}{3}\left(\frac{1}{2},\frac{1}{2},-1,0\right) = \left(\frac{\mathbf{1}}{\mathbf{3}},\frac{\mathbf{1}}{\mathbf{3}},\frac{\mathbf{1}}{\mathbf{3}},-\mathbf{1}\right).$$

We know from the first problem that the elements of the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ are linearly independent, and each vector is orthogonal to (1,1,1,1). The space of vectors perpendicular to $\mathbf{d}$ is three dimensional (since the row space of $(1, 1, 1, 1)$ is one-dimensional, and the number of dimensions of the row space added to the number of dimensions of the nullspace add to 4). Therefore $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ forms a basis for the space of vectors perpendicular to $\mathbf{d}$.

Similarly, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for the space of vectors perpendicular to $\mathbf{d}$ because the vectors are linearly independent, orthogonal to (1,1,1,1), and because there are three of them.

Properties of determinants

Determinants

Now halfway through the course, we leave behind rectangular matrices and focus on square ones. Our next big topics are determinants and eigenvalues.

The *determinant* is a number associated with any square matrix; we'll write it as $\det A$ or $|A|$. The determinant encodes a lot of information about the matrix; the matrix is invertible exactly when the determinant is non-zero.

Properties

Rather than start with a big formula, we'll list the properties of the determinant. We already know that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; these properties will give us a formula for the determinant of square matrices of all sizes.

1. $\det I = 1$

2. If you exchange two rows of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.

3. (a) If we multiply one row of a matrix by t, the determinant is multiplied by t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

   (b) The determinant behaves like a linear function on the rows of the matrix:
   $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$. The determinant of a permutation matrix P is 1 or -1 depending on whether P exchanges an even or odd number of rows.

From these three properties we can deduce many others:

4. If two rows of a matrix are equal, its determinant is zero.

   This is because of property 2, the exchange rule. On the one hand, exchanging the two identical rows does not change the determinant. On the other hand, exchanging the two rows changes the sign of the determinant. Therefore the determinant must be 0.

5. If $i \neq j$, subtracting t times row i from row j doesn't change the determinant.

In two dimensions, this argument looks like:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c-ta & d-tb \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix} && \text{property 3(b)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t\begin{vmatrix} a & b \\ a & b \end{vmatrix} && \text{property 3(a)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} && \text{property 4.}
\end{aligned}$$

The proof for higher dimensional matrices is similar.

6. If A has a row that is all zeros, then $\det A = 0$.

   We get this from property 3 (a) by letting $t = 0$.

7. The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_1, d_2, ..., d_n$.

   Property 5 tells us that the determinant of the triangular matrix won't change if we use elimination to convert it to a diagonal matrix with the entries d_i on its diagonal. Then property 3 (a) tells us that the determinant of this diagonal matrix is the product $d_1 d_2 \cdots d_n$ times the determinant of the identity matrix. Property 1 completes the argument.

   Note that we cannot use elimination to get a diagonal matrix if one of the d_i is zero. In that case elimination will give us a row of zeros and property 6 gives us the conclusion we want.

8. $\det A = 0$ exactly when A is singular.

   If A is singular, then we can use elimination to get a row of zeros, and property 6 tells us that the determinant is zero.

   If A is not singular, then elimination produces a full set of pivots $d_1, d_2, ..., d_n$ and the determinant is $d_1 d_2 \cdots d_n \neq 0$ (with minus signs from row exchanges).

We now have a very practical formula for the determinant of a non-singular matrix. In fact, the way computers find the determinants of large matrices is to first perform elimination (keeping track of whether the number of row exchanges is odd or even) and then multiply the pivots:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}, \text{ if } a \neq 0, \text{ so}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a(d - \frac{c}{a}b) = ad - bc.$$

9. $\det AB = (\det A)(\det B)$

   This is very useful. Although the determinant of a sum does not equal the sum of the determinants, it is true that the determinant of a product equals the product of the determinants.

For example:

$$\det A^{-1} = \frac{1}{\det A},$$

because $A^{-1}A = 1$. (Note that if A is singular then A^{-1} does not exist and $\det A^{-1}$ is undefined.) Also, $\det A^2 = (\det A)^2$ and $\det 2A = 2^n \det A$ (applying property 3 to each row of the matrix). This reminds us of volume – if we double the length, width and height of a three dimensional box, we increase its volume by a multiple of $2^3 = 8$.

10. $\det A^T = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

This lets us translate properties (2, 3, 4, 5, 6) involving rows into statements about columns. For instance, if a column of a matrix is all zeros then the determinant of that matrix is zero.

To see why $|A^T| = |A|$, use elimination to write $A = LU$. The statement becomes $|U^T L^T| = |LU|$. Rule 9 then tells us $|U^T||L^T| = |L||U|$.

Matrix L is a lower triangular matrix with 1's on the diagonal, so rule 5 tells us that $|L| = |L^T| = 1$. Because U is upper triangular, rule 5 tells us that $|U| = |U^T|$. Therefore $|U^T||L^T| = |L||U|$ and $|A^T| = |A|$.

We have one loose end to worry about. Rule 2 told us that a row exchange changes the sign of the determinant. If it's possible to do seven row exchanges and get the same matrix you would by doing ten row exchanges, then we could prove that the determinant equals its negative. To complete the proof that the determinant is well defined by properties 1, 2 and 3 we'd need to show that the result of an odd number of row exchanges (odd permutation) can never be the same as the result of an even number of row exchanges (even permutation).

Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Problem 18.2: (5.1 #18.) Use row operations and the properties of the determinant to calculate the three by three "Vandermonde determinant":

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Solution: If the entries of every row of A sum to zero, then $A\mathbf{x} = 0$ when $\mathbf{x} = (1, \ldots, 1)$ since each component of $A\mathbf{x}$ is the sum of the entries in a row of A. Since A has a non-zero nullspace, it is not invertible and $\det A = 0$.

If the entries of every row of A sum to one, then the entries in every row of $A - I$ sum to zero. Hence $A - I$ has a non-zero nullspace and $\det(A - I) = 0$.

If $\det(A - I) = 0$ it is **not** necessarily true that $\det A = 1$. For example, the rows of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sum to one but $\det A = -1$.

Problem 18.2: (5.1 #18.) Use row operations and the properties of the determinant to calculate the three by three "Vandermonde determinant":

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b - a)(c - a)(c - b).$$

Solution: Using row operations and properties of the determinant, we have:

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c-a & c^2-a^2 \end{bmatrix}$$

$$= (b-a)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}$$

$$= (b-a)(c-a)(c-b)\det\begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(c-b)\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(c-b).\checkmark$$

Determinant formulas and cofactors

Now that we know the properties of the determinant, it's time to learn some (rather messy) formulas for computing it.

Formula for the determinant

We know that the determinant has the following three properties:

1. $\det I = 1$
2. Exchanging rows reverses the sign of the determinant.
3. The determinant is linear in each row separately.

Last class we listed seven consequences of these properties. We can use these ten properties to find a formula for the determinant of a 2 by 2 matrix:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
&= 0 + ad + (-cb) + 0 \\
&= ad - bc.
\end{aligned}$$

By applying property 3 to separate the individual entries of each row we could get a formula for any other square matrix. However, for a 3 by 3 matrix we'll have to add the determinants of twenty seven different matrices! Many of those determinants are zero. The non-zero pieces are:

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} - a_{12}a_{21}a_{33} \\
&\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.
\end{aligned}$$

Each of the non-zero pieces has one entry from each row in each column, as in a permutation matrix. Since the determinant of a permutation matrix is either 1 or -1, we can again use property 3 to find the determinants of each of these summands and obtain our formula.

One way to remember this formula is that the positive terms are products of entries going down and to the right in our original matrix, and the negative terms are products going down and to the left. This rule of thumb doesn't work for matrices larger than 3 by 3.

The number of parts with non-zero determinants was 2 in the 2 by 2 case, 6 in the 3 by 3 case, and will be $24 = 4!$ in the 4 by 4 case. This is because there are n ways to choose an element from the first row (i.e. a value for α), after which there are only $n-1$ ways to choose an element from the second row that avoids a zero determinant. Then there are $n-2$ choices from the third row, $n-3$ from the fourth, and so on.

The big formula for computing the determinant of any square matrix is:

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \ldots a_{n\omega}$$

where $(\alpha, \beta, \gamma, \ldots \omega)$ is some permutation of $(1, 2, 3, \ldots, n)$. If we test this on the identity matrix, we find that all the terms are zero except the one corresponding to the trivial permutation $\alpha = 1$, $\beta = 2$, ..., $\omega = n$. This agrees with the first property: $\det I = 1$. It's possible to check all the other properties as well, but we won't do that here.

Applying the method of elimination and multiplying the diagonal entries of the result (the pivots) is another good way to find the determinant of a matrix.

Example

In a matrix with many zero entries, many terms in the formula are zero. We can compute the determinant of:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

by choosing a non-zero entry from each row and column, multiplying those entries, giving the product the appropriate sign, then adding the results.

The permutation corresponding to the diagonal running from a_{14} to a_{41} is $(4, 3, 2, 1)$. This contributes 1 to the determinant of the matrix; the contribution is positive because it takes two row exchanges to convert the permutation $(4, 3, 2, 1)$ to the identity $(1, 2, 3, 4)$.

Another non-zero term of $\sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4\omega}$ comes from the permutation $(3, 2, 1, 4)$. This contributes -1 to the sum, because one exchange (of the first and third rows) leads to the identity.

These are the only two non-zero terms in the sum, so the determinant is 0. We can confirm this by noting that row 1 minus row 2 plus row 3 minus row 4 equals zero.

Cofactor formula

The cofactor formula rewrites the big formula for the determinant of an n by n matrix in terms of the determinants of smaller matrices.

In the 3×3 case, the formula looks like:

$$\begin{aligned}
\det A &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}
\end{aligned}$$

This comes from grouping all the multiples of a_{ij} in the big formula. Each element is multiplied by the *cofactors* in the parentheses following it. Note that each cofactor is (plus or minus) the determinant of a two by two matrix. That determinant is made up of products of elements in the rows and columns NOT containing a_{1j}.

In general, the cofactor C_{ij} of a_{ij} can be found by looking at all the terms in the big formula that contain a_{ij}. C_{ij} equals $(-1)^{i+j}$ times the determinant of the $n-1$ by $n-1$ square matrix obtained by removing row i and column j. (C_{ij} is positive if $i+j$ is even and negative if $i+j$ is odd.)

For $n \times n$ matrices, the cofactor formula is:

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.}$$

Applying this to a 2×2 matrix gives us:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c).$$

Tridiagonal matrix

A *tridiagonal matrix* is one for which the only non-zero entries lie on or adjacent to the diagonal. For example, the 4×4 tridiagonal matrix of 1's is:

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

What is the determinant of an $n \times n$ tridiagonal matrix of 1's?

$$|A_1| = 1, |A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, |A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$|A_4| = 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = |A_3| - 1|A_2| = -1$$

In fact, $|A_n| = |A_{n-1}| - |A_{n-2}|$. We get a sequence which repeats every six terms:

$$|A_1| = 1, |A_2| = 0, |A_3| = -1, |A_4| = -1, |A_5| = 0, |A_6| = 1, |A_7| = 1.$$

Exercises on determinant formulas and cofactors

Problem 19.1: Compute the determinant of:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Which method of computing the determinant do you prefer for this problem, and why?

Problem 19.2: (5.2 #33. *Introduction to Linear Algebra:* Strang) The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain)}.$$

Exercises on determinant formulas and cofactors

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Which method of computing the determinant do you prefer for this problem, and why?

Solution: The preferred method is that of using **cofactors.** We apply the Big Formula:

$$\det A = \sum_{P=(\alpha,\beta,\ldots,\omega)} (\det P) a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

to A:

$$\det A = 0 \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

This is quicker than row exchange:

$$\det A = \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= -1.$$

Problem 19.2: (5.2 #33. *Introduction to Linear Algebra:* Strang) The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

Solution: The difference in the n, n entry (in the example, the difference between 19 and 20) multiplies its cofactor, the determinant of the $n-1$ by $n-1$ symmetric Pascal matrix. In our example this matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

We're told that this matrix has determinant 1. Since the n, n entry multiplies its cofactor positively, the overall determinant drops by 1 to become 0.

Cramer's rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used.

Formula for A^{-1}

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Can we get a formula for the inverse of a 3 by 3 or n by n matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A}C^T$$

where C is the matrix of cofactors – please notice the transpose! Cofactors of row one of A go into column 1 of A^{-1}, and then we divide by the determinant.

The determinant of A involves products with n terms and the cofactor matrix involves products of $n-1$ terms. A and $\frac{1}{\det A}C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

To more formally verify the formula, we'll check that $AC^T = (\det A)I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j}C_{j1} = \det A.$$

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of AC^T.

To finish proving that $AC^T = (\det A)I$, we just need to check that the off-diagonal entries of AC^T are zero. In the two by two case, multiplying the entries in row 1 of A by the entries in column 2 of C^T gives $a(-b)+b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of A and the last column of C^T equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A}C^T$.

This formula helps us answer questions about how the inverse changes when the matrix changes.

Cramer's Rule for $\mathbf{x} = A^{-1}\mathbf{b}$

We know that if $A\mathbf{x} = \mathbf{b}$ and A is nonsingular, then $\mathbf{x} = A^{-1}\mathbf{b}$. Applying the formula $A^{-1} = C^T/\det A$ gives us:

$$\mathbf{x} = \frac{1}{\det A} C^T \mathbf{b}.$$

Cramer's rule gives us another way of looking at this equation. To derive this rule we break $\mathbf{x}$ down into its components. Because the i'th component of $C^T\mathbf{b}$ is a sum of cofactors times some number, it is the determinant of some matrix B_j.

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the matrix created by starting with A and then replacing column j with $\mathbf{b}$, so:

$$\begin{aligned} B_1 &= \begin{bmatrix} \mathbf{b} & \begin{array}{c}\text{last n-1}\\ \text{columns}\\ \text{of } A\end{array} \end{bmatrix} \qquad \text{and} \\ B_n &= \begin{bmatrix} \begin{array}{c}\text{first n-1}\\ \text{columns}\\ \text{of } A\end{array} & \mathbf{b} \end{bmatrix}. \end{aligned}$$

This agrees with our formula $x_1 = \frac{\det B_1}{\det A}$. When taking the determinant of B_1 we get a sum whose first term is b_1 times the cofactor C_{11} of A.

Computing inverses using Cramer's rule is usually less efficient than using elimination.

$|\det A|$ = volume of box

Claim: $|\det A|$ is the volume of the box (*parallelepiped*) whose edges are the column vectors of A. (We could equally well use the row vectors, forming a different box with the same volume.)

If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^TQ = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

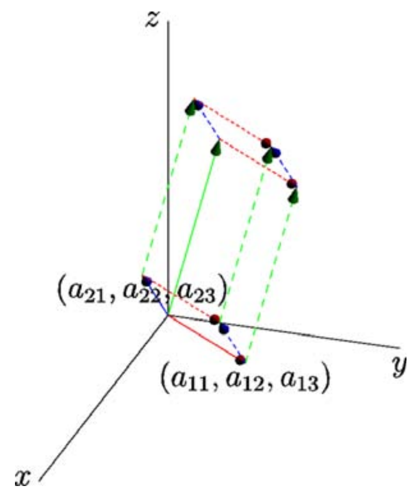

Figure 1: The box whose edges are the column vectors of A.

If we double the length of one column of A, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Figure 2 illustrates why this should be true.

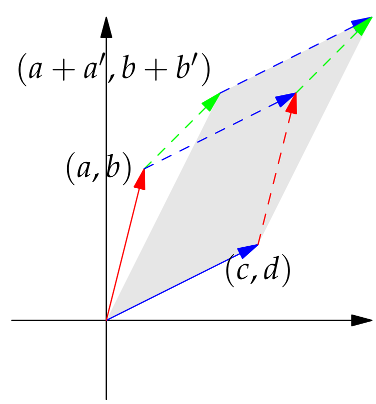

Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.

Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$. The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Exercises on Cramer's rule, inverse matrix, and volume

Problem 20.1: (5.3 #8. *Introduction to Linear Algebra:* Strang) Suppose

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Find its cofactor matrix C and multiply AC^T to find $\det(A)$.

$$C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ and } AC^T = \underline{\qquad}.$$

If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

Problem 20.2: (5.3 #28.) Spherical coordinates ρ, ϕ, θ satisfy

$$x = \rho \sin\phi \cos\theta, \; y = \rho \sin\phi \sin\theta \text{ and } z = \rho\cos\phi.$$

Find the three by three matrix of partial derivatives:

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}.$$

Simplify its determinant to $J = \rho^2 \sin\phi$. In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

is the volume of an infinitesimal "coordinate box."

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$$C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ and } AC^T = \underline{\qquad}.$$

If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

Solution: We fill in the cofactor matrix C and then multiply to obtain AC^T:

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

and

$$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I.$$

Since $AC^T = \det(A)I$, we have $\det(A) = 3$. If 4 is changed to 100, $\det(A)$ is unchanged because the cofactor of that entry is 0, and thus its value does not contribute to the determinant.

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Find the three by three matrix of partial derivatives:

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}.$$

Simplify its determinant to $J = \rho^2 \sin\phi$. In spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

is the volume of an infinitesimal "coordinate box."

Solution: The rows are formed by the partials of x, y, and z with respect to ρ, ϕ, and θ:

$$\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Expanding its determinant J along the bottom row, we get:

$$\begin{aligned}
J &= \cos\phi \begin{bmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} \\
&\quad -(-\rho\sin\phi) \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix} + 0 \\
&= \cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta) \\
&\quad +\rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\
&= \cos\phi(\rho^2\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta)) + \rho\sin\phi(\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)) \\
&= \cos\phi(\rho^2\cos\phi\sin\phi) + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
J &= \rho^2\sin\phi.
\end{aligned}$$

Eigenvalues and eigenvectors

The subject of eigenvalues and eigenvectors will take up most of the rest of the course. We will again be working with square matrices. Eigenvalues are special numbers associated with a matrix and eigenvectors are special vectors.

Eigenvectors and eigenvalues

A matrix A *acts on* vectors $\mathbf{x}$ like a function does, with input $\mathbf{x}$ and output $A\mathbf{x}$. *Eigenvectors* are vectors for which $A\mathbf{x}$ is parallel to $\mathbf{x}$. In other words:

$$A\mathbf{x} = \lambda\mathbf{x}.$$

In this equation, $\mathbf{x}$ is an eigenvector of A and λ is an *eigenvalue* of A.

Eigenvalue 0

If the eigenvalue λ equals 0 then $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. Vectors with eigenvalue 0 make up the nullspace of A; if A is singular, then $\lambda = 0$ is an eigenvalue of A.

Examples

Suppose P is the matrix of a projection onto a plane. For any $\mathbf{x}$ in the plane $P\mathbf{x} = \mathbf{x}$, so $\mathbf{x}$ is an eigenvector with eigenvalue 1. A vector $\mathbf{x}$ perpendicular to the plane has $P\mathbf{x} = \mathbf{0}$, so this is an eigenvector with eigenvalue $\lambda = 0$. The eigenvectors of P span the whole space (but this is not true for every matrix).

The matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has an eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 1 and another eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue -1. These eigenvectors span the space. They are perpendicular because $B = B^T$ (as we will prove).

$\det(A - \lambda I) = 0$

An n by n matrix will have n eigenvalues, and their sum will be the sum of the diagonal entries of the matrix: $a_{11} + a_{22} + \cdots + a_{nn}$. This sum is the *trace* of the matrix. For a two by two matrix, if we know one eigenvalue we can use this fact to find the second.

Can we solve $A\mathbf{x} = \lambda\mathbf{x}$ for the eigenvalues and eigenvectors of A? Both λ and $\mathbf{x}$ are unknown; we need to be clever to solve this problem:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$

In order for λ to be an eigenvector, $A - \lambda I$ must be singular. In other words, $\det(A - \lambda I) = 0$. We can solve this *characteristic equation* for λ to get n solutions.

If we're lucky, the solutions are distinct. If not, we have one or more *repeated eigenvalues*.

Once we've found an eigenvalue λ, we can use elimination to find the nullspace of $A - \lambda I$. The vectors in that nullspace are eigenvectors of A with eigenvalue λ.

Calculating eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)^2 - 1 \\ &= \lambda^2 - 6\lambda + 8. \end{aligned}$$

Note that the coefficient 6 is the trace (sum of diagonal entries) and 8 is the determinant of A. In general, the eigenvalues of a two by two matrix are the solutions to:

$$\lambda^2 - \operatorname{trace}(A) \cdot \lambda + \det A = 0.$$

Just as the trace is the sum of the eigenvalues of a matrix, the product of the eigenvalues of any matrix equals its determinant.

For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. We find the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 4$ in the nullspace of $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

x_2 will be in the nullspace of $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The nullspace is an entire line; x_2 could be any vector on that line. A natural choice is $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that these eigenvectors are the same as those of $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Adding $3I$ to the matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ added 3 to each of its eigenvalues and did not change its eigenvectors, because $A\mathbf{x} = (B + 3I)\mathbf{x} = \lambda\mathbf{x} + 3\mathbf{x} = (\lambda + 3)\mathbf{x}$.

A caution

Similarly, if $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{x} = \alpha\mathbf{x}$, $(A + B)\mathbf{x} = (\lambda + \alpha)\mathbf{x}$. It would be nice if the eigenvalues of a matrix sum were always the sums of the eigenvalues, but this is only true if A and B have the same eigenvectors. The eigenvalues of the product AB aren't usually equal to the products $\lambda(A)\lambda(b)$, either.

Complex eigenvalues

The matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector in the plane by 90°. It has trace $0 = \lambda_1 + \lambda_2$ and determinant $1 = \lambda_1 \cdot \lambda_2$. Its only real eigenvector is the zero vector; any other vector's direction changes when it is multiplied by Q. How will this affect our eigenvalue calculation?

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 + 1. \end{aligned}$$

$\det(A - \lambda I) = 0$ has solutions $\lambda_1 = i$ and $\lambda_2 = -i$. If a matrix has a complex eigenvalue $a + bi$ then the *complex conjugate* $a - bi$ is also an eigenvalue of that matrix.

Symmetric matrices have real eigenvalues. For *antisymmetric* matrices like Q, for which $A^T = -A$, all eigenvalues are imaginary ($\lambda = bi$).

Triangular matrices and repeated eigenvalues

For triangular matrices such as $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, the eigenvalues are exactly the entries on the diagonal. In this case, the eigenvalues are 3 and 3:

$$\begin{aligned} \det(A - \lambda \det I) &= \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)(3-\lambda) \quad \Big(= (a_{11} - \lambda)(a_{22} - \lambda) \Big) \\ &= 0, \end{aligned}$$

so $\lambda_1 = 3$ and $\lambda_2 = 3$. To find the eigenvectors, solve:

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

to get $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. There is no independent eigenvector $\mathbf{x}_2$.

Exercises on eigenvalues and eigenvectors

Problem 21.1: (6.1 #19. *Introduction to Linear Algebra:* Strang) A three by three matrix B is known to have eigenvalues 0, 1 and 2. This information is enough to find three of these (give the answers where possible):

a) The rank of B

b) The determinant of $B^T B$

c) The eigenvalues of $B^T B$

d) The eigenvalues of $(B^2 + I)^{-1}$

Problem 21.2: (6.1 #29.) Find the eigenvalues of A, B, and C when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Exercises on eigenvalues and eigenvectors

Problem 21.1: (6.1 #19. *Introduction to Linear Algebra:* Strang) A three by three matrix B is known to have eigenvalues 0, 1 and 2. This information is enough to find three of these (give the answers where possible):

a) The rank of B

b) The determinant of $B^T B$

c) The eigenvalues of $B^T B$

d) The eigenvalues of $(B^2 + I)^{-1}$

Solution:

a) B has 0 as an eigenvalue and is therefore singular (not invertible). Since B is a three by three matrix, this means that its rank can be at most 2. Since B has two distinct nonzero eigenvalues, its rank is exactly 2.

b) Since B is singular, $\det(B) = 0$. Thus $\det(B^T B) = \det(B^T)\det(B) = 0$.

c) There is not enough information to find the eigenvalues of $B^T B$. For example:

$$\text{If } B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 4 \end{bmatrix}.$$

$$\text{If } B = \begin{bmatrix} 0 & 1 & \\ & 1 & \\ & & 2 \end{bmatrix} \text{ then } B^T B = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}.$$

d) If $p(t)$ is a polynomial and if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then

$$p(A)\mathbf{x} = p(\lambda)\mathbf{x}.$$

We also know that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $\frac{1}{0^2+1}$, $\frac{1}{1^2+1}$ and $\frac{1}{2^2+1}$, or **1, 1/2** and **1/5**.

Problem 21.2: (6.1 #29.) Find the eigenvalues of A, B, and C when

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Solution: Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 1,4, and 6. For B we have:

$$\begin{aligned} \det(B - \lambda I) &= (-\lambda)(2-\lambda)(-\lambda) - 3(2-\lambda) \\ &= (\lambda^2 - 3)(2-\lambda). \end{aligned}$$

Hence the eigenvalues of B are $\pm\sqrt{3}$ and 2. Finally, for C we have:

$$\begin{aligned} \det(C - \lambda I) &= (2-\lambda)[(2-\lambda)^2 - 4] - 2[2(2-\lambda) - 4] + 2[4 - 2(2-\lambda)] \\ &= \lambda^3 - 6\lambda^2 = \lambda^2(\lambda - 6). \end{aligned}$$

The eigenvalues of C are 6, 0, and 0.

We can quickly check our answers by computing the determinants of A and B and by noting that C is singular.

Diagonalization and powers of A

We know how to find eigenvalues and eigenvectors. In this lecture we learn to *diagonalize* any matrix that has n independent eigenvectors and see how diagonalization simplifies calculations. The lecture concludes by using eigenvalues and eigenvectors to solve *difference equations*.

Diagonalizing a matrix $S^{-1}AS = \Lambda$

If A has n linearly independent eigenvectors, we can put those vectors in the columns of a (square, invertible) matrix S. Then

$$\begin{aligned} AS &= A\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix} \\ &= S\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = S\Lambda. \end{aligned}$$

Note that Λ is a diagonal matrix whose non-zero entries are the eigenvalues of A. Because the columns of S are independent, S^{-1} exists and we can multiply both sides of $AS = S\Lambda$ by S^{-1}:

$$S^{-1}AS = \Lambda.$$

Equivalently, $A = S\Lambda S^{-1}$.

Powers of A

What are the eigenvalues and eigenvectors of A^2?

$$\begin{aligned} \text{If } A\mathbf{x} &= \lambda\mathbf{x}, \\ \text{then } A^2\mathbf{x} &= \lambda A\mathbf{x} = \lambda^2\mathbf{x}. \end{aligned}$$

The eigenvalues of A^2 are the squares of the eigenvalues of A. The eigenvectors of A^2 are the same as the eigenvectors of A. If we write $A = S\Lambda S^{-1}$ then:

$$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}.$$

Similarly, $A^k = S\Lambda^k S^{-1}$ tells us that raising the eigenvalues of A to the kth power gives us the eigenvalues of A^k, and that the eigenvectors of A^k are the same as those of A.

Theorem: If A has n independent eigenvectors with eigenvalues λ_i, then $A^k \to 0$ as $k \to \infty$ if and only if all $|\lambda_i| < 1$.

A is guaranteed to have n independent eigenvectors (and be *diagonalizable*) if all its eigenvalues are different. Most matrices do have distinct eigenvalues.

Repeated eigenvalues

If A has repeated eigenvalues, it may or may not have n independent eigenvectors. For example, the eigenvalues of the identity matrix are all 1, but that matrix still has n independent eigenvectors.

If A is the triangular matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ its eigenvalues are 2 and 2. Its eigenvectors are in the nullspace of $A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is spanned by $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This particular A does not have two independent eigenvectors.

Difference equations $\mathbf{u}_{k+1} = A\mathbf{u}_k$

Start with a given vector $\mathbf{u}_0$. We can create a sequence of vectors in which each new vector is A times the previous vector: $\mathbf{u}_{k+1} = A\mathbf{u}_k$. $\mathbf{u}_{k+1} = A\mathbf{u}_k$ is a *first order difference equation*, and $\mathbf{u}_k = A^k\mathbf{u}_0$ is a solution to this system.

We get a more satisfying solution if we write $\mathbf{u}_0$ as a combination of eigenvectors of A:

$$\mathbf{u}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = S\mathbf{c}.$$

Then:

$$A\mathbf{u}_0 = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_n\lambda_n\mathbf{x}_n$$

and:

$$\mathbf{u}_k = A^k\mathbf{u}_0 = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + \cdots + c_n\lambda_n^k\mathbf{x}_n = \Lambda^k S\mathbf{c}.$$

Fibonacci sequence

The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, 13, ...$ In general, $F_{k+2} = F_{k+1} + F_k$. If we could understand this in terms of matrices, the eigenvalues of the matrices would tell us how fast the numbers in the sequence are increasing.

$\mathbf{u}_{k+1} = A\mathbf{u}_k$ was a first order system. $F_{k+2} = F_{k+1} + F_k$ is a second order scalar equation, but we can convert it to first order linear system by using a clever trick. If $\mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, then:

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \qquad (1) \\ F_{k+1} &= F_{k+1}. \qquad (2) \end{aligned}$$

is equivalent to the first order system $\mathbf{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k$.

What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$? Because A is symmetric, its eigenvalues will be real and its eigenvectors will be orthogonal.

Because A is a two by two matrix we know its eigenvalues sum to 1 (the trace) and their product is -1 (the determinant).

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

Setting this to zero we find $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$; i.e. $\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -.618$. The growth rate of the F_k is controlled by λ_1, the only eigenvalue with absolute value greater than 1. This tells us that for large k, $F_k \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^k$ for some constant c_1. (Remember $\mathbf{u}_k = A^k \mathbf{u}_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$, and here λ_2^k goes to zero since $|\lambda_2| < 1$.)

To find the eigenvectors of A note that:

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \mathbf{x}$$

equals $\mathbf{0}$ when $\mathbf{x} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$, so $\mathbf{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

Finally, $\mathbf{u}_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$ tells us that $c_1 = -c_2 = \frac{1}{\sqrt{5}}$. Because $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \mathbf{u}_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$, we get:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^k.$$

Using eigenvalues and eigenvectors, we have found a *closed form expression* for the Fibonacci numbers.

Summary: When a sequence evolves over time according to the rules of a first order system, the eigenvalues of the matrix of that system determine the long term behavior of the series. To get an exact formula for the series we find the eigenvectors of the matrix and then solve for the coefficients $c_1, c_2, ...$

Exercises on diagonalization and powers of A

Problem 22.1: (6.2 #6. *Introduction to Linear Algebra:* Strang) Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

Problem 22.2: (6.2 #16.) Find Λ and S to diagonalize A :

$$A = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}.$$

What is the limit of Λ^k as $k \to \infty$? What is the limit matrix of $S\Lambda^k S^{-1}$? In the columns of this matrix you see the __________.

Exercises on diagonalization and powers of A

Problem 22.1: (6.2 #6. *Introduction to Linear Algebra:* Strang) Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

Solution: To find the eigenvectors of A, we first find the eigenvalues:

$$\det \begin{bmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = 0 \Longrightarrow (4-\lambda)(2-\lambda) = 0.$$

Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$. Using these values, we find the eigenvectors by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = 2z,$$

thus any multiple of **(2,1)** is an eigenvector for λ_1.

$$(A - \lambda_2 I)\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = 0, z = \text{free variable},$$

thus any multiple of **(0,1)** is an eigenvector for λ_2. Therefore the columns of the matrices S that diagonalize A are nonzero multiples of (2,1) and (1,0). They can appear in either order.

Finally, because $A^{-1} = S\Lambda^{-1}S^{-1}$ the same matrices S will diagonalize A^{-1}.

Problem 22.2: (6.2 #16.) Find Λ and S to diagonalize A :

$$A = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}.$$

What is the limit of Λ^k as $k \to \infty$? What is the limit matrix of $S\Lambda^k S^{-1}$? In the columns of this matrix you see the __________.

Solution: Since each of the columns of A sums to one, A is a Markov matrix and definitely has eigenvalue $\lambda_1 = 1$. The trace of A is .7, so the other eigenvalue is $\lambda_2 = .7 - 1 = -.3$. To find S we need to find the corresponding eigenvectors:

$$(A - \lambda_1 I)\mathbf{x_1} = \begin{bmatrix} -.4 & .9 \\ .4 & -.9 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \mathbf{x_1} = (9,4).$$

$$(A - \lambda_2 I)\mathbf{x_2} = \begin{bmatrix} .9 & .9 \\ .4 & .4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow y = -z \Longrightarrow \mathbf{x_2} = (1,-1).$$

Putting these together, we have:

$$S = \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & \\ & -.3 \end{bmatrix}. \text{ As } k \to \infty,\ \ \Lambda^k \to \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}.$$

So

$$S\Lambda^\infty S^{-1} \to \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \left(\frac{1}{13}\right) \begin{bmatrix} 1 & 1 \\ 4 & -9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 9 \\ 4 & 4 \end{bmatrix}.$$

In the columns of this matrix you see the **steady state vector.**

Differential equations and e^{At}

The system of equations below describes how the values of variables u_1 and u_2 affect each other over time:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2.$$

Just as we applied linear algebra to solve a difference equation, we can use it to solve this differential equation. For example, the initial condition $u_1 = 1$, $u_2 = 0$ can be written $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Differential equations $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

By looking at the equations above, we might guess that over time u_1 will decrease. We can get the same sort of information more safely by looking at the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ of our system $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u}$. Because A is singular and its trace is -3 we know that its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -3$. The solution will turn out to include e^{-3t} and e^{0t}. As t increases, e^{-3t} vanishes and $e^{0t} = 1$ remains constant. Eigenvalues equal to zero have eigenvectors that are *steady state* solutions.

$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for which $A\mathbf{x}_1 = 0\mathbf{x}_1$. To find an eigenvector corresponding to $\lambda_2 = -3$ we solve $(A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = 0 \quad \text{so} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we can check that $A\mathbf{x}_2 = -3\mathbf{x}_2$. The general solution to this system of differential equations will be:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2.$$

Is $e^{\lambda_1 t}\mathbf{x}_1$ really a solution to $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$? To find out, plug in $\mathbf{u} = e^{\lambda_1 t}\mathbf{x}_1$:

$$\frac{d\mathbf{u}}{dt} = \lambda_1 e^{\lambda_1 t}\mathbf{x}_1,$$

which agrees with:

$$A\mathbf{u} = e^{\lambda_1 t}A\mathbf{x}_1 = \lambda_1 e^{\lambda_1 t}\mathbf{x}_1.$$

The two "pure" terms $e^{\lambda_1 t}\mathbf{x}_1$ and $e^{\lambda_2 t}\mathbf{x}_2$ are analogous to the terms $\lambda_i^k\mathbf{x}_i$ we saw in the solution $c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + \cdots + c_n\lambda_n^k\mathbf{x}_n$ to the difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

Plugging in the values of the eigenvectors, we get:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We know $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so at $t = 0$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$c_1 = c_2 = 1/3$ and $\mathbf{u}(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

This tells us that the system starts with $u_1 = 1$ and $u_2 = 0$ but that as t approaches infinity, u_1 decays to $2/3$ and u_2 increases to $1/3$. This might describe stuff moving from u_1 to u_2.

The steady state of this system is $\mathbf{u}(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

Stability

Not all systems have a steady state. The eigenvalues of A will tell us what sort of solutions to expect:

1. Stability: $\mathbf{u}(t) \to 0$ when $\text{Re}(\lambda) < 0$.
2. Steady state: One eigenvalue is 0 and all other eigenvalues have negative real part.
3. Blow up: if $\text{Re}(\lambda) > 0$ for any eigenvalue λ.

If a two by two matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues with negative real part, its trace $a + d$ is negative. The converse is not true: $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ has negative trace but one of its eigenvalues is 1 and e^{1t} blows up. If A has a positive determinant and negative trace then the corresponding solutions must be stable.

Applying S

The final step of our solution to the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ was to solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

or $S\mathbf{c} = \mathbf{u}(0)$, where S is the eigenvector matrix. The components of $\mathbf{c}$ determine the contribution from each pure exponential solution, based on the initial conditions of the system.

In the equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$, the matrix A couples the pure solutions. We set $\mathbf{u} = S\mathbf{v}$, where S is the matrix of eigenvectors of A, to get:

$$S\frac{d\mathbf{v}}{dt} = AS\mathbf{v}$$

or:

$$\frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v} = \Lambda\mathbf{v}.$$

This diagonalizes the system: $\frac{dv_i}{dt} = \lambda_i v_i$. The general solution is then:

$$\begin{aligned} \mathbf{v}(t) &= e^{\Lambda t}\mathbf{v}(0), \qquad \text{and} \\ \mathbf{u}(t) &= Se^{\Lambda t}S^{-1}\mathbf{v}(0) = e^{At}\mathbf{u}(0). \end{aligned}$$

Matrix exponential e^{At}

What does e^{At} mean if A is a matrix? We know that for a real number x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

We can use the same formula to define e^{At}:

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots.$$

Similarly, if the eigenvalues of At are small, we can use the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to estimate $(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$.

We've said that $e^{At} = Se^{\Lambda t}S^{-1}$. If A has n independent eigenvectors we can prove this from the definition of e^{At} by using the formula $A = S\Lambda S^{-1}$:

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots \\ &= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \frac{S\Lambda^3 S^{-1}}{6}t^3 + \cdots \\ &= Se^{\Lambda t}S^{-1}. \end{aligned}$$

It's impractical to add up infinitely many matrices. Fortunately, there is an easier way to compute $e^{\Lambda t}$. Remember that:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

When we plug this in to our formula for e^{At} we find that:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}.$$

This is another way to see the relationship between the stability of $\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{v}(0)$ and the eigenvalues of A.

Second order

We can change the second order equation $y'' + by' + ky = 0$ into a two by two first order system using a method similar to the one we used to find a formula for the Fibonacci numbers. If $u = \begin{bmatrix} y' \\ y \end{bmatrix}$, then

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}.$$

We could use the methods we just learned to solve this system, and that would give us a solution to the second order scalar equation we started with.

If we start with a kth order equation we get a k by k matrix with coefficients of the equation in the first row and 1's on a diagonal below that; the rest of the entries are 0.

Exercises on differential equations and e^{At}

Problem 23.1: (6.3 #14.a *Introduction to Linear Algebra:* Strang) The matrix in this question is skew-symmetric $(A^T = -A)$:

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \quad \text{or} \quad \begin{aligned} u_1{}' &= cu_2 - bu_3 \\ u_2{}' &= au_3 - cu_1 \\ u_3{}' &= bu_1 - au_2. \end{aligned}$$

Find the derivative of $||\mathbf{u}(t)||^2$ using the definition:

$$||\mathbf{u}(t)||^2 = u_1{}^2 + u_2{}^2 + u_3{}^2.$$

What does this tell you about the rate of change of the length of $\mathbf{u}$? What does this tell you about the range of values of $\mathbf{u}(t)$?

Problem 23.2: (6.3 #24.) Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check your work by evaluating e^{At} and the derivative of e^{At} when $t = 0$.

Exercises on differential equations and e^{At}

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What does this tell you about the rate of change of the length of $\mathbf{u}$? What does this tell you about the range of values of $\mathbf{u}(t)$?

Solution:

$$\begin{aligned} \frac{d||\mathbf{u}(t)||^2}{dt} &= \frac{d(u_1{}^2 + u_2{}^2 + u_3{}^2)}{dt} \\ &= 2u_1u_1{}' + 2u_2u_2{}' + 2u_3u_3{}' \\ &= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) \\ &= 0. \end{aligned}$$

This means $||\mathbf{u}(t)||^2$ stays equal to $||\mathbf{u}(0)||^2$. Because $\mathbf{u}(t)$ never changes length, it is always on the circumference of a circle of radius $||\mathbf{u}(0)||$.

Problem 23.2: (6.3 #24.) Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check your work by evaluating e^{At} and the derivative of e^{At} when $t = 0$.

Solution: The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$, with corresponding eigenvectors $\mathbf{x_1} = (1,0)$ and $\mathbf{x_2} = (1,2)$. This gives us the following values for S, Λ, and S^{-1} :

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}.$$

We use these to find e^{At} :

$$Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} = e^{At}.$$

Check:

$$e^{At} = \begin{bmatrix} e^t & .5e^{3t} - .5e^t \\ 0 & e^{3t} \end{bmatrix} \text{ equals } I \text{ when } t = 0. \checkmark$$

$$\frac{de^{At}}{dt} = \begin{bmatrix} e^t & 1.5e^{3t} - .5e^t \\ 0 & 3e^{3t} \end{bmatrix}.$$

$$\left.\frac{de^{At}}{dt}\right|_{t=0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A. \checkmark$$

Markov matrices; Fourier series

In this lecture we look at Markov matrices and Fourier series – two applications of eigenvalues and projections.

Eigenvalues of A^T

The eigenvalues of A and the eigenvalues of A^T are the same:

$$(A - \lambda I)^T = A^T - \lambda I,$$

so property 10 of determinants tells us that $\det(A - \lambda I) = \det(A^T - \lambda I)$. If λ is an eigenvalue of A then $\det(A^T - \lambda I) = 0$ and λ is also an eigenvalue of A^T.

Markov matrices

A matrix like:

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

in which all entries are non-negative and each column adds to 1 is called a *Markov matrix*. These requirements come from Markov matrices' use in probability. Squaring or raising a Markov matrix to a power gives us another Markov matrix.

When dealing with systems of differential equations, eigenvectors with the eigenvalue 0 represented steady states. Here we're dealing with powers of matrices and get a steady state when $\lambda = 1$ is an eigenvalue.

The constraint that the columns add to 1 guarantees that 1 is an eigenvalue. All other eigenvalues will be less than 1. Remember that (if A has n independent eigenvectors) the solution to $\mathbf{u}_k = A^k\mathbf{u}_0$ is $\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + \cdots + c_n\lambda_n^k\mathbf{x}_n$. If $\lambda_1 = 1$ and all others eigenvalues are less than one the system approaches the steady state $c_1\mathbf{x}_1$. This is the $\mathbf{x}_1$ component of $\mathbf{u}_0$.

Why does the fact that the columns sum to 1 guarantee that 1 is an eigenvalue? If 1 is an eigenvalue of A, then:

$$A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

should be singular. Since we've subtracted 1 from each diagonal entry, the sum of the entries in each column of $A - I$ is zero. But then the sum of the rows of $A - I$ must be the zero row, and so $A - I$ is singular. The eigenvector $\mathbf{x}_1$ is in the nullspace of $A - I$ and has eigenvalue 1. It's not very hard to find $\mathbf{x}_1 = \begin{bmatrix} .6 \\ 33 \\ .7 \end{bmatrix}$.

We're studying the equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$ where A is a Markov matrix. For example u_1 might be the population of (number of people in) Massachusetts and u_2 might be the population of California. A might describe what fraction of the population moves from state to state, or the probability of a single person moving. We can't have negative numbers of people, so the entries of A will always be positive. We want to account for all the people in our model, so the columns of A add to $1 = 100\%$.

For example:

$$\begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_{t=k}$$

assumes that there's a 90% chance that a person in California will stay in California and only a 10% chance that she or he will move, while there's a 20% percent chance that a Massachusetts resident will move to California. If our initial conditions are $\begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$, then after one move $\mathbf{u}_1 = A\mathbf{u}_0$ is:

$$\begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_1 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}.$$

For the next few values of k, the Massachusetts population will decrease and the California population will increase while the total population remains constant at 1000.

To understand the long term behavior of this system we'll need the eigenvectors and eigenvalues of $\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. We know that one eigenvalue is $\lambda_1 = 1$. Because the trace $.9 + .8 = 1.7$ is the sum of the eigenvalues, we see that $\lambda_2 = .7$.

Next we calculate the eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \mathbf{x}_1 = \mathbf{0},$$

so we choose $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The eigenvalue 1 corresponds to the steady state solution, and $\lambda_2 = .7 < 1$, so the system approaches a limit in which $2/3$ of 1000 people live in California and $1/3$ of 1000 people are in Massachusetts. This will be the limit from any starting vector $\mathbf{u}_0$.

To know how the population is distributed after a finite number of steps we look for an eigenvector corresponding to $\lambda_2 = .7$:

$$A - \lambda_2 I = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \mathbf{x}_1 = \mathbf{0},$$

so let $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

From what we learned about difference equations we know that:

$$\mathbf{u}_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

When $k = 0$ we have:

$$\mathbf{u}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so $c_1 = \frac{1000}{3}$ and $c_2 = \frac{2000}{3}$.

In some applications Markov matrices are defined differently – their rows add to 1 rather than their columns. In this case, the calculations are the transpose of everything we've done here.

Fourier series and projections

Expansion with an orthonormal basis

If we have an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n$ then we can write any vector $\mathbf{v}$ as $\mathbf{v} = x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + \cdots + x_n\mathbf{q}_n$, where:

$$\mathbf{q}_i^T\mathbf{v} = x_1\mathbf{q}_i^T\mathbf{q}_1 + x_2\mathbf{q}_i^T\mathbf{q}_2 + \cdots + x_n\mathbf{q}_i^T\mathbf{q}_n = x_i.$$

Since $\mathbf{q}_i^T\mathbf{q}_j = 0$ unless $i = j$, this equation gives $x_i = \mathbf{q}_i^T\mathbf{v}$.

In terms of matrices, $\begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{v}$, or $Q\mathbf{x} = \mathbf{v}$. So $\mathbf{x} = Q^{-1}\mathbf{v}$. Because the q_i form an orthonormal basis, $Q^{-1} = Q^T$ and $\mathbf{x} = Q^T\mathbf{v}$. This is another way to see that $x_i = \mathbf{q}_i^T\mathbf{v}$.

Fourier series

The key idea above was that the basis of vectors $\mathbf{q}_i$ was orthonormal. Fourier series are built on this idea. We can describe a function $f(x)$ in terms of trigonometric functions:

$$f(x) = a_0 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \cdots.$$

This *Fourier series* is an infinite sum and the previous example was finite, but the two are related by the fact that the cosines and sines in the Fourier series are orthogonal.

We're now working in an infinite dimensional vector space. The vectors in this space are functions and the (orthogonal) basis vectors are 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ...

What does "orthogonal" mean in this context? How do we compute a dot product or *inner product* in this vector space? For vectors in $\mathbb{R}^n$ the inner product is $\mathbf{v}^T\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$. Functions are described by a continuum of values $f(x)$ rather than by a discrete collection of components v_i. The best parallel to the vector dot product is:

$$f^T g = \int_0^{2\pi} f(x)g(x)\,dx.$$

We integrate from 0 to 2π because Fourier series are periodic:

$$f(x) = f(x + 2\pi).$$

The inner product of two basis vectors is zero, as desired. For example,

$$\int_0^{2\pi} \sin x \cos x\,dx = \frac{1}{2}(\sin x)^2\bigg|_0^{2\pi} = 0.$$

How do we find a_0, a_1, etc. to find the coordinates or *Fourier coefficients* of a function in this space? The constant term a_0 is the average value of the function. Because we're working with an orthonormal basis, we can use the inner product to find the coefficients a_i.

$$\begin{aligned}
\int_0^{2\pi} f(x)\cos x\,dx &= \int_0^{2\pi} (a_0 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + \cdots)\cos x\,dx \\
&= 0 + \int_0^{2\pi} a_1 \cos^2 x\,dx + 0 + 0 + \cdots \\
&= a_1\pi.
\end{aligned}$$

We conclude that $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos x\,dx$. We can use the same technique to find any of the values a_i.

Exercises on Markov matrices; Fourier series

Problem 24.1: (6.4 #7. *Introduction to Linear Algebra:* Strang)

a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

b) How do you know it must have a negative pivot?

c) How do you know it can't have two negative eigenvalues?

Problem 24.2: (6.4 #23.) Which of these classes of matrices do A and B belong to: invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, or $Q\Lambda Q^T$?

Problem 24.3: (8.3 #11.) Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\mathbf{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \text{---} & \text{---} & \text{---} \end{bmatrix}.$$

Exercises on Markov matrices; Fourier series

Problem 24.1: (6.4 #7. *Introduction to Linear Algebra:* Strang)

a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

b) How do you know it must have a negative pivot?

c) How do you know it can't have two negative eigenvalues?

Solution:

a) The eigenvalues of that matrix are $1 \pm b$. If $b > 1$ or $b < -1$ the matrix has a negative eigenvalue.

b) The pivots have the same signs as the eigenvalues. If the matrix has a negative eigenvalue, then it must have a negative pivot.

c) To obtain one negative eigenvalue, we choose either $b > 1$ or $b < -1$ (as stated in part (a)). If we choose $b > 1$, then $\lambda_1 = 1 + b$ will be positive while $\lambda_2 = 1 - b$ will be negative. Alternatively, if we choose $b < -1$, then $\lambda_1 = 1 + b$ will be negative while $\lambda_2 = 1 - b$ will be positive. Therefore this matrix cannot have two negative eigenvalues.

Problem 24.2: (6.4 #23.) Which of these classes of matrices do A and B belong to: invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, or $Q\Lambda Q^T$?

Solution:

a) For A :

$\det A = -1 \neq 0$.	A is **invertible**.
$AA^T = I$.	A is **orthogonal**.
$A^2 = I \neq A$.	A is **not a projection**.
A has one 1 in each row and column with 0's elsewhere.	A is **a permutation**.
$A = A^T$, so A is symmetric.	A is **diagonalizable**.
Each column of A sums to one.	A is **Markov**.

$A = LU$ is not possible because $A_{11} = 0$. QR is possible because A has independent columns, $S\Lambda S^{-1}$ is possible because it is diagonalizable, and $Q\Lambda Q^T$ is possible because it is symmetric.

b) For B :

$\det B = 0$.	B is **not invertible**.
$BB^T \neq I$.	B is **not orthogonal**.
$B^2 = B$.	B is **a projection**.
B does not have one 1 in each row and each column, with 0's elsewhere.	B is **not a permutation**.
$B = B^T$ so B is symmetric.	B is **diagonalizable**.
Each column of B sums to one.	B is **Markov**.

$B = LU$ is possible but U only contains one nonzero pivot. QR is impossible because B has dependent columns, $S\Lambda S^{-1}$ is possible because it is diagonalizable, and $Q\Lambda Q^T$ is possible because it is symmetric.

Problem 24.3: (8.3 #11.) Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\mathbf{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \text{---} & \text{---} & \text{---} \end{bmatrix}.$$

Solution: Matrix A becomes:

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \mathbf{.2} & \mathbf{.3} & \mathbf{.5} \end{bmatrix},$$

with steady state vector (1,1,1). When A is a *symmetric* Markov matrix, the elements of each row sum to one. The elements of each row of $A - I$ then sum to zero. Since the steady state vector $\mathbf{x}$ is the eigenvector associated with eigenvalue $\lambda = 1$, we solve $(A - \lambda I)\mathbf{x} = (A - I)\mathbf{x} = \mathbf{0}$ to get $\mathbf{x} = (1, \ldots, 1)$.

Exam 2 Review

Material covered by the exam

- Orthogonal matrices $Q = \begin{bmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{bmatrix}$. $Q^TQ = I$.

  Projections – Least Squares "best fit" solution to $A\mathbf{x} = \mathbf{b}$.

  Gram-Schmidt process for getting an orthonormal basis from any basis.

- $\det A$

  Properties 1-3 that define the determinant.

  Big formula for the determinant with $n!$ terms, each with $+$ or $-$.

  Cofactors formula, leading to a formula for A^{-1}.

- Eigenvalues $A\mathbf{x} = \lambda\mathbf{x}$.

  $\det(A - \lambda I) = 0$.

  Diagonalization: If A has n independent eigenvectors, then $S^{-1}AS = \Lambda$ (this is $A\mathbf{x} = \lambda\mathbf{x}$ for all n eigenvectors at once).

  Powers of A: $A^k = (S\Lambda S^{-1})^k = S\Lambda^k S^{-1}$.

Sample questions

1. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

   a) Find the projection matrix P that projects onto $\mathbf{a}$.
   To answer this, we just use the formula for P. Ordinarily $P = A(A^TA)^{-1}A^T$, but here A is a column vector so:

   $$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}.$$

   b) What is the rank of P?
   P has rank 1 because each of its columns is some multiple of its second column, or because it projects onto a one dimensional subspace.

   c) What is the column space of P?
   The line determined by $\mathbf{a}$.

   d) What are the eigenvalues of P?
   Since P has rank 1 we know it has a repeated eigenvalue of 0. We can use its trace or the fact that it's a projection matrix to find that P has the eigenvalue 1.
   The eigenvalues of P are 0, 0 and 1.

e) Find an eigenvector of P that has eigenvalue 1.

Eigenvector $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ has eigenvalue one. Because P is a projection matrix, any vector in the space it's projecting onto will be an eigenvector with eigenvalue 1.

f) Suppose $\mathbf{u}_{k+1} = P\mathbf{u}_k$ with initial condition $\mathbf{u}_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$. Find $\mathbf{u}_k$.

We're repeatedly projecting a vector onto a line:

$$\mathbf{u}_1 = P\mathbf{u}_0 = \mathbf{a}\frac{\mathbf{a}^T\mathbf{u}_0}{\mathbf{a}^T\mathbf{a}} = \mathbf{a}\frac{27}{9} = 3\mathbf{a} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}.$$

$\mathbf{u}_2$ is the projection of $\mathbf{u}_1$ onto the line determined by $\mathbf{a}$. But $\mathbf{u}_1$ already lies on the line through $\mathbf{a}$. In fact, $\mathbf{u}_k = P^k\mathbf{u}_0 = P\mathbf{u}_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$.

g) The exam might have a difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$ in which A is not a projection matrix with $P^k = P$. In that case we would find its eigenvalues and eigenvectors to calculate $\mathbf{u}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$. Then $\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + c_3\lambda_3^k\mathbf{x}_3$. (For the projection matrix P above, two eigenvalues are 0 and the third is 1, so two terms vanish and for the third $\lambda^k = 1$. Then $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \cdots$.)

2. We're given the following data points:

t	y
1	4
2	5
3	8

a) Find the straight line through the origin that best fits these points.
The equation of this line will be $y = Dt$. There's only one unknown, D. We would like a solution to the 3 equations:

$$\begin{aligned} 1 \cdot D &= 4 \\ 2 \cdot D &= 5 \\ 3 \cdot D &= 8. \end{aligned}$$

or $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$, if we put this in the form $A\mathbf{x} = \mathbf{b}$. To find the best value for D we solve the equation:

$$A^TA\hat{D} = A^T\mathbf{b}$$

$$14\hat{D} = 38$$

$$\hat{D} = \frac{38}{14} = \frac{19}{7}.$$

We conclude that the best fit line through the origin is $y = \frac{19}{7}t$. We can roughly check our answer by noting that the line $y = 3t$ runs fairly close to the data points.

b) What vector did we just project onto what line?

There are two ways to think about least squares problems. The first is to think about the best fit line in the ty-plane. The other way is to think in terms of projections – we're projecting $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$ onto the column space of A (the line through $(1, 2, 3)$) to get as close as possible to a solution to $A\mathbf{x} = \mathbf{b}$.

3. The vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ determine a plane. Find two orthogonal vectors in the plane.

To answer this question we use the Gram-Shmidt process. We start with $\mathbf{a}_1$ and find a second vector $\mathbf{B}$ perpendicular to $\mathbf{a}_1$ by subtracting the component of $\mathbf{a}_2$ that lies in the $\mathbf{a}_1$ direction.

$$
\begin{aligned}
\mathbf{B} &= \mathbf{a}_2 - \frac{\mathbf{a}_1^T\mathbf{a}_2}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}_1 \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/7 \\ 6/7 \\ 9/7 \end{bmatrix} \\
&= \begin{bmatrix} 4/7 \\ 1/7 \\ -2/7 \end{bmatrix}.
\end{aligned}
$$

Because the dot product of $\mathbf{a}_1$ and $\mathbf{B}$ is zero, we know this answer is correct. These are our orthogonal vectors.

4. We're given a 4 by 4 matrix A with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

a) What conditions must the λ_i satisfy for the matrix to be invertible?

A is invertible if and only if none of the λ_i are 0.

If one of the λ_i is zero, then there is a non-zero vector in the nullspace of A and A is not invertible.

b) What is $\det A^{-1}$?

The eigenvalues of A^{-1} are the inverses of the eigenvalues of A, so $\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_4}\right)$.

c) What is trace$(A+I)$?
We know the trace of A is the sum of the eigenvalues of A, so trace$(A+I) = (\lambda_1+1)+(\lambda_2+1)+(\lambda_3+1)+(\lambda_4+1) = \lambda_1+\lambda_2+\lambda_3+\lambda_4+4$.

5. Remember the family of *tridiagonal matrices*; for example:

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $D_n = \det A_n$.

a) Use cofactors to show that $D_n = aD_{n-1} + bD_{n-2}$ and find values for a and b.
Using the cofactor formula we find that the determinant of A_4 is:

$$\begin{aligned} D_4 &= 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 0 - 0 \\ &= 1D_3 - 1\cdot 1D_2 \\ &= D_3 - D_2. \end{aligned}$$

In general, $D_n = D_{n-1} - D_{n-2}$. The answer is $a = 1$ and $b = -1$.

b) In part (a) you found a recurrence relation $D_n = aD_{n-1} + bD_{n-2}$. Find a way to predict the value of D_n for any n. Note: if your computations are all correct it should be true that $\lambda_1^6 = \lambda_2^6 = 1$.
We can quickly compute $D_1 = 1$ and $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.
We set up the system for $D_n = D_{n-1} - D_{n-2}$:

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

to get an equation of the form $\mathbf{u}_k = A\mathbf{u}_{k-1}$.
To find the eigenvalues λ_i, solve $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0.$$

The quadratic formula gives us $\lambda = \dfrac{1 \pm \sqrt{1-4}}{2}$, so:

$$\lambda_1 = \frac{1+\sqrt{3}i}{2} = e^{i\pi/3} \text{ and } \lambda_2 = \frac{1-\sqrt{3}i}{2} = e^{-i\pi/3}.$$

The magnitude of these complex numbers is 1. This tells us that the system is stable. The fact that $\lambda_1^6 = \lambda_2^6$ tells us that $A^6 = I$ and so the sequence of vectors $\mathbf{u}_k = A^k \mathbf{u}_{k-1}$ will repeat every 6 steps.

To finish answering the problem, we can use the recurrence relation $D_n = D_{n-1} - D_{n-2}$ starting from $D_1 = 1$ and $D_2 = 0$ to find $D_3 = -1$, $D_4 = -1$, $D_5 = 0$ and $D_6 = 1$. The sequence will then repeat, with $D_7 = D_1 = 1$, $D_8 = D_2 = 0$ and so on. If $n = 6j + k$ for positive integers j and k, then $D_n = D_k$.

6. Consider the following family of symmetric matrices:

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \ldots$$

a) Find the projection matrix P onto the column space of A_3.

We know that A_3 is singular because column 3 is a multiple of column 1, so P is a projection matrix onto a plane. Columns 1 and 2 form a basis for the column space of A_3, so we could use the formula $P = A(A^TA)^{-1}A^T$ with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$ to find:

$$P = \begin{bmatrix} 1/5 & 0 & 2/5 \\ 0 & 1 & 0 \\ 2/5 & 0 & 4/5 \end{bmatrix}.$$

However, there may be a quicker way to solve this problem.

To check our work we multiply P by the column vectors of A to see that $PA = A$.

b) What are the eigenvalues and eigenvectors of A_3?

$$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = -\lambda^3 + 5\lambda.$$

Setting $|A_3 - \lambda I| = 0$ gives us $\lambda(-\lambda^2 + 5) = 0$, so $\lambda_1 = 0$, $\lambda_2 = \sqrt{5}$, $\lambda_3 = -\sqrt{5}$.

We check that the trace of A_3 equals the sum of its eigenvalues.

Next we solve $(A_3 - \lambda I)\mathbf{x} = \mathbf{0}$ to find our eigenvectors. A good strategy for doing this is to choose one component of $\mathbf{x}$ to set equal to 1, then determine what the other components of $\mathbf{x}$ must be for the product to equal the zero vector.

$(A_3 - 0I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

$(A_3 - \sqrt{5}I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_2 = \begin{bmatrix} 1 \\ \sqrt{5} \\ 2 \end{bmatrix}$.

$(A_3 + \sqrt{5}I)\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \\ 2 \end{bmatrix}$.

If time permits, we can check this answer by multiplying each eigenvector by A_3.

c) (This is not difficult.) What is the projection matrix onto the column space of A_4?

How could this not be difficult? If A_4 is invertible, then its column space is $\mathbb{R}^4$ and the answer is $P = I$.

To confirm that A_4 is invertible, we can check that its determinant is non-zero. This is not difficult if we use cofactors:

$$\det A_4 = (-1)\left(1 \cdot \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}\right) = 9.$$

Because A_4 is invertible, the projection matrix onto its column space is I.

d) Bonus question: Prove or disprove that A_n is singular if n is odd and invertible if n is even.

18.06SC Unit 2 Exam

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $\boldsymbol{P_Q = P_A}$ and why ? What is $\boldsymbol{P_Q}$ times $\boldsymbol{Q}$? What is* det $\boldsymbol{P_Q}$ *?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4 ? (P_A and P_Q from above)

$$\textbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \textbf{2.}\ \frac{a_4 - \frac{a_4^{\mathrm{T}} a_1}{a_1^{\mathrm{T}} a_1} a_1 - \frac{a_4^{\mathrm{T}} a_2}{a_2^{\mathrm{T}} a_2} a_2 - \frac{a_4^{\mathrm{T}} a_3}{a_3^{\mathrm{T}} a_3} a_3}{\|\text{ norm of that vector }\|} \qquad \textbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

18.06SC Unit 2 Exam Solutions

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

Solution.

(a) The determinant of any square matrix with orthonormal columns (“orthogonal matrix”) is ± 1.

(b) Here are two ways you could do this:

(1) The determinant is *linear in each column*:

$$\begin{aligned} \det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} &= \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} \\ &= \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_3 + q_1 \end{bmatrix} \\ &= \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} \end{aligned}$$

Both of these determinants are equal (see (c)), so the total determinant is ± 2.

(2) You could also *use row reduction*. Here's what happens:

$$\begin{aligned}
\det\begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix} &= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3+q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & 2q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}
\end{aligned}$$

Again, whatever $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is, this determinant will be twice that, or ± 2.

(c) The second matrix is an *even* permutation of the columns of the first matrix (swap q_1/q_2 then swap q_2/q_3), so it has the *same* determinant as the first matrix. Whether the first matrix has determinant $+1$ or -1, the product will be $+1$.

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

Solution.

(a) If the line went exactly through the 21 points, then the 21 equations

$$\begin{bmatrix} 1 & -10 \\ 1 & -9 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

would be exactly solvable. Since we can't solve this equation $Ax = b$ exactly, we look for a least-squares solution $A^{\mathrm{T}}A\hat{x} = A^{\mathrm{T}}b$.

$$\begin{bmatrix} 21 & 0 \\ 0 & 770 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So the line of best fit is the horizontal line $\widehat{C} = \frac{1}{21}$, $\widehat{D} = 0$.

(b) We are projecting b onto the column space of A above (basis: $\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -10 & \ldots & 10 \end{bmatrix}^{\mathrm{T}}$). There are lots of vectors perpendicular to this subspace; one is the error vector $e = b - P_A b = \frac{1}{21}\begin{bmatrix} (\text{ten } -1\text{'s}) & 20 & (\text{ten } -1\text{'s}) \end{bmatrix}^{\mathrm{T}}$.

3 **(9 + 12 + 9 pts.)** The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $P_Q = P_A$ and why ? What is P_Q times Q ? What is $\det P_Q$?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

$$\textbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \textbf{2.}\ \frac{a_4 - \frac{a_4^{\mathrm{T}} a_1}{a_1^{\mathrm{T}} a_1} a_1 - \frac{a_4^{\mathrm{T}} a_2}{a_2^{\mathrm{T}} a_2} a_2 - \frac{a_4^{\mathrm{T}} a_3}{a_3^{\mathrm{T}} a_3} a_3}{\|\text{ norm of that vector }\|} \qquad \textbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

Solution.

(a) $P_A = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and $P_Q = Q(Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}} = QQ^{\mathrm{T}}$.

(b) $P_A = P_Q$ because both projections project onto the same subspace. *(Some people did this the hard way, by substituting $A = QR$ into the projection formula and simplifying. That also works.)* The determinant is zero, because P_Q is singular (like all non-identity projections): all vectors orthogonal to the column space of Q are projected to 0.

(c) Answer: choice 3. (Choice 2 is tempting, and would be correct if the a_i were replaced by the q_i. But the a_i are not orthogonal!)

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \\ \times & & \text{any numbers} & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

Solution.

(a) Every term in the big formula for $\det(A)$ takes one entry from each row and column, so we can choose at most two $\times$'s and the determinant has degree 2.

(b) You can find this by cofactor expansion; here's another way:

$$\det(A) = \times \det \begin{bmatrix} 1 & \times & \times & \times \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \times \det \begin{bmatrix} 1-3\times & \times & \times & \times \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \times(1-3\times) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \times(1-3\times).$$

This is zero when $\times = 0$ or $\times = \frac{1}{3}$.