

Singular value decomposition

The *singular value decomposition* of a matrix is usually referred to as the *SVD*. This is the final and best factorization of a matrix:

$$A = U\Sigma V^T$$

where U is orthogonal, Σ is diagonal, and V is orthogonal.

In the decomposition $A = U\Sigma V^T$, A can be *any* matrix. We know that if A is symmetric positive definite its eigenvectors are orthogonal and we can write $A = Q\Lambda Q^T$. This is a special case of a SVD, with $U = V = Q$. For more general A , the SVD requires two different matrices U and V .

We've also learned how to write $A = S\Lambda S^{-1}$, where S is the matrix of n distinct eigenvectors of A . However, S may not be orthogonal; the matrices U and V in the SVD will be.

How it works

We can think of A as a linear transformation taking a vector \mathbf{v}_1 in its row space to a vector $\mathbf{u}_1 = A\mathbf{v}_1$ in its column space. The SVD arises from finding an orthogonal basis for the row space that gets transformed into an orthogonal basis for the column space: $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$.

It's not hard to find an orthogonal basis for the row space – the Gram-Schmidt process gives us one right away. But in general, there's no reason to expect A to transform that basis to another orthogonal basis.

You may be wondering about the vectors in the nullspaces of A and A^T . These are no problem – zeros on the diagonal of Σ will take care of them.

Matrix language

The heart of the problem is to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ for the row space of A for which

$$\begin{aligned} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix} &= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \cdots & \sigma_r\mathbf{u}_r \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}, \end{aligned}$$

with $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ an orthonormal basis for the column space of A . Once we add in the nullspaces, this equation will become $AV = U\Sigma$. (We can complete the orthonormal bases $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{u}_1, \dots, \mathbf{u}_r$ to orthonormal bases for the entire space any way we want. Since $\mathbf{v}_{r+1}, \dots, \mathbf{v}_n$ will be in the nullspace of A , the diagonal entries $\sigma_{r+1}, \dots, \sigma_n$ will be 0.)

The columns of U and V are bases for the row and column spaces, respectively. Usually $U \neq V$, but if A is positive definite we can use the *same* basis for its row and column space!

Calculation

Suppose A is the invertible matrix $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$. We want to find vectors \mathbf{v}_1 and \mathbf{v}_2 in the row space \mathbb{R}^2 , \mathbf{u}_1 and \mathbf{u}_2 in the column space \mathbb{R}^2 , and positive numbers σ_1 and σ_2 so that the \mathbf{v}_i are orthonormal, the \mathbf{u}_i are orthonormal, and the σ_i are the scaling factors for which $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$.

This is a big step toward finding orthonormal matrices V and U and a diagonal matrix Σ for which:

$$AV = U\Sigma.$$

Since V is orthogonal, we can multiply both sides by $V^{-1} = V^T$ to get:

$$A = U\Sigma V^T.$$

Rather than solving for U , V and Σ simultaneously, we multiply both sides by $A^T = V\Sigma^T U^T$ to get:

$$\begin{aligned} A^T A &= V\Sigma U^{-1} U\Sigma V^T \\ &= V\Sigma^2 V^T \\ &= V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} V^T. \end{aligned}$$

This is in the form $Q\Lambda Q^T$; we can now find V by diagonalizing the symmetric positive definite (or semidefinite) matrix $A^T A$. The columns of V are eigenvectors of $A^T A$ and the eigenvalues of $A^T A$ are the values σ_i^2 . (We choose σ_i to be the positive square root of λ_i .)

To find U , we do the same thing with AA^T .

SVD example

We return to our matrix $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$. We start by computing

$$\begin{aligned} A^T A &= \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}. \end{aligned}$$

The eigenvectors of this matrix will give us the vectors \mathbf{v}_i , and the eigenvalues will give us the numbers σ_i .

Two orthogonal eigenvectors of $A^T A$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. To get an orthonormal basis, let $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. These have eigenvalues $\sigma_1^2 = 32$ and $\sigma_2^2 = 18$. We now have:

$$\begin{array}{c} A \\ \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \end{array} = \begin{array}{c} U \\ \begin{bmatrix} \\ \end{bmatrix} \end{array} \begin{array}{c} \Sigma \\ \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} V^T \\ \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{array}.$$

We could solve this for U , but for practice we'll find U by finding orthonormal eigenvectors \mathbf{u}_1 and \mathbf{u}_2 for $AA^T = U\Sigma^2U^T$.

$$\begin{aligned} AA^T &= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}. \end{aligned}$$

Luckily, AA^T happens to be diagonal. It's tempting to let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as Professor Strang did in the lecture, but because $A\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3\sqrt{2} \end{bmatrix}$ we instead have $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that this also gives us a chance to double check our calculation of σ_1 and σ_2 .

Thus, the SVD of A is:

$$\begin{array}{c} A \\ \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \end{array} = \begin{array}{c} U \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{array} \begin{array}{c} \Sigma \\ \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} V^T \\ \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{array}.$$

Example with a nullspace

Now let $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$. This has a one dimensional nullspace and one dimensional row and column spaces.

The row space of A consists of the multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. The column space of A is made up of multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$. The nullspace and left nullspace are perpendicular to the row and column spaces, respectively.

Unit basis vectors of the row and column spaces are $\mathbf{v}_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix}$ and $\mathbf{u}_1 =$

$\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. To compute σ_1 we find the nonzero eigenvalue of $A^T A$.

$$\begin{aligned} A^T A &= \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}. \end{aligned}$$

Because this is a rank 1 matrix, one eigenvalue must be 0. The other must equal the trace, so $\sigma_1^2 = 125$. After finding unit vectors perpendicular to \mathbf{u}_1 and \mathbf{v}_1 (basis vectors for the left nullspace and nullspace, respectively) we see that the SVD of A is:

$$\begin{array}{ccc} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} & = & \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} \\ A & & U \qquad \qquad \Sigma \qquad \qquad V^T \end{array}$$

The singular value decomposition combines topics in linear algebra ranging from positive definite matrices to the four fundamental subspaces.

- $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ is an orthonormal basis for the row space.
- $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ is an orthonormal basis for the column space.
- $\mathbf{v}_{r+1}, \dots, \mathbf{v}_n$ is an orthonormal basis for the nullspace.
- $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ is an orthonormal basis for the left nullspace.

These are the “right” bases to use, because $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$.

Exercises on singular value decomposition

Problem 29.1: (Based on 6.7 #4. *Introduction to Linear Algebra*: Strang) Verify that if we compute the singular value decomposition $A = U\Sigma V^T$ of the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}.$$

Problem 29.2: (6.7 #11.) Suppose A has orthogonal columns $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ of lengths $\sigma_1, \sigma_2, \dots, \sigma_n$. Calculate $A^T A$. What are U, Σ , and V in the SVD?

Exercises on singular value decomposition

Problem 29.1: (Based on 6.7 #4. *Introduction to Linear Algebra*: Strang)
Verify that if we compute the singular value decomposition $A = U\Sigma V^T$ of the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{bmatrix}.$$

Solution:

$$A^T A = A A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of this matrix are the roots of $x^2 - 3x + 1$, which are $\frac{3 \pm \sqrt{5}}{2}$. Thus we have:

$$\sigma_1^2 = \frac{3 + \sqrt{5}}{2} \quad \text{and} \quad \sigma_2^2 = \frac{3 - \sqrt{5}}{2}.$$

To check that $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, we will square the entries of the matrix Σ given above.

$$\left(\frac{1 + \sqrt{5}}{2} \right)^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{3 + \sqrt{5}}{2}. \checkmark$$

$$\left(\frac{\sqrt{5} - 1}{2} \right)^2 = \frac{5 - 2\sqrt{5} + 1}{4} = \frac{3 - \sqrt{5}}{2}. \checkmark$$

Problem 29.2: (6.7 #11.) Suppose A has orthogonal columns $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ of lengths $\sigma_1, \sigma_2, \dots, \sigma_n$. Calculate $A^T A$. What are U, Σ , and V in the SVD?

Solution: Since the columns of A are orthogonal, $A^T A$ is a diagonal matrix with entries $\sigma_1^2, \dots, \sigma_n^2$. Since $A^T A = V \Sigma^2 V^T$, we find that Σ^2 is the matrix with diagonal entries $\sigma_1^2, \dots, \sigma_n^2$ and thus that Σ is the matrix with diagonal entries $\sigma_1, \dots, \sigma_n$.

Referring again to the equation $A^T A = V \Sigma^2 V^T$, we conclude also that $V = I$.

The equation $A = U \Sigma V^T$ then tells us that U must be the matrix whose columns are $\frac{1}{\sigma_i} \mathbf{w}_i$.