The geometry of linear equations

The fundamental problem of linear algebra is to solve n linear equations in n unknowns; for example:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

In this first lecture on linear algebra we view this problem in three ways.

The system above is two dimensional ($n = 2$). By adding a third variable z we could expand it to three dimensions.

Row Picture

Plot the points that satisfy each equation. The intersection of the plots (if they do intersect) represents the solution to the system of equations. Looking at Figure 1 we see that the solution to this system of equations is $x = 1, y = 2$.

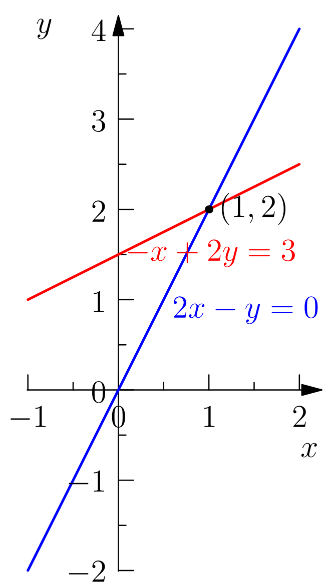

Figure 1: The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point $(1, 2)$.

We plug this solution in to the original system of equations to check our work:

$$\begin{aligned} 2 \cdot 1 - 2 &= 0 \\ -1 + 2 \cdot 2 &= 3. \end{aligned}$$

The solution to a three dimensional system of equations is the common point of intersection of three planes (if there is one).

Column Picture

In the column picture we rewrite the system of linear equations as a single equation by turning the coefficients in the columns of the system into vectors:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Given two vectors **c** and **d** and scalars x and y, the sum $x\mathbf{c} + y\mathbf{d}$ is called a *linear combination* of **c** and **d**. Linear combinations are important throughout this course.

Geometrically, we want to find numbers x and y so that x copies of vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ added to y copies of vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. As we see from Figure 2, $x = 1$ and $y = 2$, agreeing with the row picture in Figure 2.

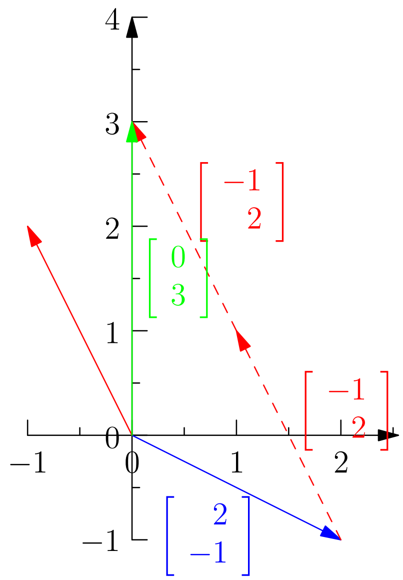

Figure 2: A linear combination of the column vectors equals the vector **b**.

In three dimensions, the column picture requires us to find a linear combination of three 3-dimensional vectors that equals the vector **b**.

Matrix Picture

We write the system of equations

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

as a single equation by using matrices and vectors:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is called the *coefficient matrix*. The vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector of unknowns. The values on the right hand side of the equations form the vector $\mathbf{b}$:

$$A\mathbf{x} = \mathbf{b}.$$

The three dimensional matrix picture is very like the two dimensional one, except that the vectors and matrices increase in size.

Matrix Multiplication

How do we multiply a matrix A by a vector $\mathbf{x}$?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \quad ?$$

One method is to think of the entries of $\mathbf{x}$ as the coefficients of a linear combination of the column vectors of the matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This technique shows that $A\mathbf{x}$ is a linear combination of the columns of A.

You may also calculate the product $A\mathbf{x}$ by taking the dot product of each row of A with the vector $\mathbf{x}$:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Linear Independence

In the column and matrix pictures, the right hand side of the equation is a vector $\mathbf{b}$. Given a matrix A, can we solve:

$$A\mathbf{x} = \mathbf{b}$$

for every possible vector $\mathbf{b}$? In other words, do the linear combinations of the column vectors fill the xy-plane (or space, in the three dimensional case)?

If the answer is "no", we say that A is a *singular matrix*. In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in two dimensions) or on a point, line or plane (in three dimensions). The combinations don't fill the whole space.

Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent)(dependent).

The three vectors lie in a ____________. The matrix W with those columns is *not invertible*.

Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

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Solution: We might observe that $\mathbf{w}_1 + \mathbf{w}_3 - 2\mathbf{w}_2 = 0$, or we might simultaneously solve the system of equations:

$$\begin{aligned} 1x_1 + 4x_2 + 7x_3 &= 0 \\ 2x_1 + 5x_2 + 8x_3 &= 0 \\ 3x_1 + 6x_2 + 9x_3 &= 0 \end{aligned}$$

Subtracting twice equation 1 from equation 2 gives us $-3x_2 - 6x_3 = 0$. Subtracting thrice equation 1 from equation 3 gives us $-6x_2 - 12x_3 = 0$, which is equivalent to the previous equation and so leads us to suspect that the vectors are dependent. At this point we might guess $x_2 = -2$ and $x_3 = 1$ which would lead us to the answer we observed above:

$$x_1 = 1,\ x_2 = -2,\ x_3 = 1 \text{ and } \mathbf{w}_1 - 2\mathbf{w}_2 + \mathbf{w}_3 = 0.$$

Those vectors are **dependent** because there is a combination of the vectors that gives the zero vector.

The three vectors lie in a **plane.**

Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Solution: $\begin{bmatrix} 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 \\ 6 + 0 + 3 \\ 12 - 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$.

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

Solution: The statement is true. In order to multiply two matrices, the number of columns of A must equal the number of rows of B. The product AB will have the same number of rows as the first matrix and the same number of columns as the second:

$$A(m \text{ by } n) \text{ times } B(n \text{ by } p) \text{ equals } AB(m \text{ by } p).$$

An overview of key ideas

This is an overview of linear algebra given at the start of a course on the mathematics of engineering.

Linear algebra progresses from vectors to matrices to subspaces.

Vectors

What do you do with vectors? Take combinations.

We can multiply vectors by scalars, add, and subtract. Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can form the *linear combination* $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$.

An example in $\mathbb{R}^3$ would be:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The collection of all multiples of $\mathbf{u}$ forms a line through the origin. The collection of all multiples of $\mathbf{v}$ forms another line. The collection of all combinations of $\mathbf{u}$ and $\mathbf{v}$ forms a plane. Taking *all combinations* of some vectors creates a *subspace*.

We could continue like this, or we can use a matrix to add in all multiples of $\mathbf{w}$.

Matrices

Create a matrix A with vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in its columns:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The product:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

equals the sum $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$. The product of a matrix and a vector is a combination of the columns of the matrix. (This particular matrix A is a *difference matrix* because the components of $A\mathbf{x}$ are differences of the components of that vector.)

When we say $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$ we're thinking about multiplying numbers by vectors; when we say $A\mathbf{x} = \mathbf{b}$ we're thinking about multiplying a matrix (whose columns are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$) by the numbers. The calculations are the same, but our perspective has changed.

For any input vector $\mathbf{x}$, the output of the operation "multiplication by A" is some vector $\mathbf{b}$:

$$A \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

A deeper question is to start with a vector $\mathbf{b}$ and ask "for what vectors $\mathbf{x}$ does $A\mathbf{x} = \mathbf{b}$?" In our example, this means solving three equations in three unknowns. Solving:

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is equivalent to solving:

$$\begin{aligned} x_1 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3. \end{aligned}$$

We see that $x_1 = b_1$ and so x_2 must equal $b_1 + b_2$. In vector form, the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}.$$

But this just says:

$$\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

or $\mathbf{x} = A^{-1}\mathbf{b}$. If the matrix A is invertible, we can multiply on both sides by A^{-1} to find the unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. We might say that A represents a transform $\mathbf{x} \to \mathbf{b}$ that has an inverse transform $\mathbf{b} \to \mathbf{x}$.

In particular, if $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The second example has the same columns $\mathbf{u}$ and $\mathbf{v}$ and replaces column vector $\mathbf{w}$:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then:

$$C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

and our system of three equations in three unknowns becomes circular.

Where before $A\mathbf{x} = \mathbf{0}$ implied $\mathbf{x} = \mathbf{0}$, there are non-zero vectors $\mathbf{x}$ for which $C\mathbf{x} = \mathbf{0}$. For any vector $\mathbf{x}$ with $x_1 = x_2 = x_3$, $C\mathbf{x} = \mathbf{0}$. This is a significant difference; we can't multiply both sides of $C\mathbf{x} = \mathbf{0}$ by an inverse to find a non-zero solution $\mathbf{x}$.

The system of equations encoded in $C\mathbf{x} = \mathbf{b}$ is:

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3. \end{aligned}$$

If we add these three equations together, we get:

$$0 = b_1 +_2 + b_3.$$

This tells us that $C\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ only when the components of $\mathbf{b}$ sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

Subspaces

Geometrically, the columns of C lie in the same plane (they are *dependent*; the columns of A are *independent*). There are many vectors in $\mathbb{R}^3$ which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of C and so correspond to values of $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$. The linear combinations of the columns of C form a two dimensional *subspace* of $\mathbb{R}^3$.

This plane of combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be described as "all vectors $C\mathbf{x}$". But we know that the vectors $\mathbf{b}$ for which $C\mathbf{x} = \mathbf{b}$ satisfy the condition $b_1 + b_2 + b_3 = 0$. So the plane of all combinations of $\mathbf{u}$ and $\mathbf{v}$ consists of all vectors whose components sum to 0.

If we take all combinations of:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the entire space $\mathbb{R}^3$; the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$. We say that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a *basis* for $\mathbb{R}^3$.

A *basis* for $\mathbb{R}^n$ is a collection of n independent vectors in $\mathbb{R}^n$. Equivalently, a basis is a collection of n vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

A *vector space* is a collection of vectors that is closed under linear combinations. A *subspace* is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace. A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

When you look at a matrix, try to see "what is it doing?"

Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix $A^T A$ that often appears when studying rectangular matrices may be invertible.

Elimination with matrices

Method of Elimination

Elimination is the technique most commonly used by computer software to solve systems of linear equations. It finds a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ whenever the matrix A is invertible. In the example used in class,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}.$$

The number 1 in the upper left corner of A is called the *first pivot*. We recopy the first row, then multiply the numbers in it by an appropriate value (in this case 3) and subtract those values from the numbers in the second row. The first number in the second row becomes 0. We have thus *eliminated* the 3 in row 2 column 1.

The next step is to perform another elimination to get a 0 in row 3 column 1; here this is already the case.

The *second pivot* is the value 2 which now appears in row 2 column 2. We find a multiplier (in this case 2) by which we multiply the second row to eliminate the 4 in row 3 column 2. The *third pivot* is then the 5 now in row 3 column 3.

We started with an invertible matrix A and ended with an *upper triangular* matrix U; the lower left portion of U is filled with zeros. Pivots 1, 2, 5 are on the diagonal of U.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

We repeat the multiplications and subtractions with the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$. For example, we multiply the 2 in the first position by 3 and subtract from 12 to get 6 in the second position. When calculating by hand we can do this efficiently by *augmenting* the matrix A, appending the vector $\mathbf{b}$ as a fourth or final column. The method of elimination transforms the equation $A\mathbf{x} = \mathbf{b}$ into a new equation $U\mathbf{x} = \mathbf{c}$. In the example above, $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$ comes from A and $\mathbf{c} = \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}$ comes from $\mathbf{b}$.

The equation $U\mathbf{x} = c$ is easy to solve by *back substitution*; in our example, $z = -2$, $y = 1$ and $x = 2$. This is also a solution to the original system $A\mathbf{x} = \mathbf{b}$.

The *determinant* of U is the product of the pivots. We will see this again.

Pivots may not be 0. If there is a zero in the pivot position, we must exchange that row with one below to get a non-zero value in the pivot position.

If there is a zero in the pivot position and no non-zero value below it, then the matrix A is not invertible. Elimination can not be used to find a unique solution to the system of equations – it doesn't exist.

Elimination Matrices

The product of a matrix (3x3) and a column vector (3x1) is a column vector (3x1) that is a linear combination of the columns of the matrix.

The product of a row (1x3) and a matrix (3x3) is a row (1x3) that is a linear combination of the rows of the matrix.

We can subtract 3 times row 1 of matrix A from row 2 of A by calculating the matrix product:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}.$$

The *elimination matrix* used to eliminate the entry in row m column n is denoted E_{mn}. The calculation above took us from A to $E_{21}A$. The three elimination steps leading to U were: $E_{32}(E_{31}(E_{21}A)) = U$, where $E_{31} = I$. Thus $E_{32}(E_{21}A) = U$.

Matrix multiplication is *associative*, so we can also write $(E_{32}E_{21})A = U$. The product $E_{32}E_{21}$ tells us how to get from A to U. The *inverse* of the matrix $E_{32}E_{21}$ tells us how to get from U to A.

If we solve $U\mathbf{x} = EA\mathbf{x} = E\mathbf{b}$, then it is also true that $A\mathbf{x} = \mathbf{b}$. This is why the method of elimination works: all steps can be reversed.

A *permutation matrix* exchanges two rows of a matrix; for example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The first and second rows of the matrix PA are the second and first rows of the matrix A. The matrix P is constructed by exchanging rows of the identity matrix.

To exchange the columns of a matrix, multiply on the right (as in AP) by a permutation matrix.

Note that matrix multiplication is not *commutative*: $PA \neq AP$.

Inverses

We have a matrix:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which subtracts 3 times row 1 from row 2. To “undo” this operation we must add 3 times row 1 to row 2 using the inverse matrix:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, $E_{21}^{-1}E_{21} = I$.

Exercises on elimination with matrices

Problem 2.1: In the two-by-two system of linear equations below, what multiple of the first equation should be subtracted from the second equation when using the method of elimination? Convert this system of equations to matrix form, apply elimination (what are the pivots?), and use back substitution to find a solution. Try to check your work before looking up the answer.

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 15y &= 12 \end{aligned}$$

Problem 2.2: (2.3 #29. *Introduction to Linear Algebra:* Strang) Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

Exercises on elimination with matrices

Problem 2.1: In the two-by-two system of linear equations below, what multiple of the first equation should be subtracted from the second equation when using the method of elimination? Convert this system of equations to matrix form, apply elimination (what are the pivots?), and use back substitution to find a solution. Try to check your work before looking up the answer.

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 15y &= 12 \end{aligned}$$

Solution: One subtracts **3** times the first equation from the second equation in order to eliminate the $6x$.

To convert to matrix form, use the general format $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 15y &= 12 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

We then apply elimination on matrix A. Using the first pivot (the number 2 in the upper left corner of A), we subtract three times the first row from the second row to get:

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \longrightarrow U = \begin{bmatrix} \mathbf{2} & 3 \\ 0 & \mathbf{6} \end{bmatrix}$$

where U is an upper triangular matrix with pivots 2 and 6. Doing the same to the right side $\mathbf{b} = (5, 12)$ gives a new equation of the form $U\mathbf{x} = \mathbf{c}$:

$$\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

To solve our new equation, we use back substitution:

$$6y = -3 \longrightarrow \boxed{y = -\tfrac{1}{2}}$$

and

$$2x + 3y = 5 \longrightarrow 2x + 3\left(-\frac{1}{2}\right) = 5 \longrightarrow 2x = 5 + \frac{3}{2} = \frac{13}{2} \longrightarrow \boxed{x = \tfrac{13}{4}}$$

We know that our solution fulfills the first equation; let's make sure that our values fulfill the second equation as a check on our work:

$$6x + 15y = 6\left(\frac{13}{4}\right) + 15\left(\frac{-1}{2}\right) = \frac{78 - 30}{4} = 12 \checkmark$$

Problem 2.2: (2.3 #29. *Introduction to Linear Algebra:* Strang) Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

Solution:

$$\text{The matrix is } E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

One can eliminate the second column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the third column with the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Multiplying these together, we get

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Since M reduces the Pascal matrix to I, M must be the inverse matrix!

Lecture 3: Multiplication and inverse matrices

Matrix Multiplication

We discuss four different ways of thinking about the product $AB = C$ of two matrices. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C is an $m \times p$ matrix. We use c_{ij} to denote the entry in row i and column j of matrix C.

Standard (row times column)

The standard way of describing a matrix product is to say that c_{ij} equals the dot product of row i of matrix A and column j of matrix B. In other words, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Columns

The product of matrix A and column j of matrix B equals column j of matrix C. This tells us that the columns of C are combinations of columns of A.

Rows

The product of row i of matrix A and matrix B equals row i of matrix C. So the rows of C are combinations of rows of B.

Column times row

A column of A is an $m \times 1$ vector and a row of B is a $1 \times p$ vector. Their product is a matrix:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}.$$

The columns of this matrix are multiples of the column of A and the rows are multiples of the row of B. If we think of the entries in these rows as the coordinates $(2, 12)$ or $(3, 18)$ or $(4, 24)$, all these points lie on the same line; similarly for the two column vectors. Later we'll see that this is equivalent to saying that the *row space* of this matrix is a single line, as is the *column space*.

The product of A and B is the sum of these "column times row" matrices:

$$AB = \sum_{k=1}^{n} \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kn} \end{bmatrix}.$$

Blocks

If we subdivide A and B into blocks that match properly, we can write the product $AB = C$ in terms of products of the blocks:

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

Here $C_1 = A_1B_1 + A_2B_3$.

Inverses

Square matrices

If A is a square matrix, the most important question you can ask about it is whether it has an inverse A^{-1}. If it does, then $A^{-1}A = I = AA^{-1}$ and we say that A is *invertible* or *nonsingular*.

If A is *singular* – i.e. A does not have an inverse – its determinant is zero and we can find some non-zero vector $\mathbf{x}$ for which $A\mathbf{x} = 0$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this example, three times the first column minus one times the second column equals the zero vector; the two column vectors lie on the same line.

Finding the inverse of a matrix is closely related to solving systems of linear equations:

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \quad \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} \quad = \quad \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

can be read as saying "A times column j of A^{-1} equals column j of the identity matrix". This is just a special form of the equation $A\mathbf{x} = \mathbf{b}$.

Gauss-Jordan Elimination

We can use the method of elimination to solve two or more linear equations at the same time. Just augment the matrix with the whole identity matrix I:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

(Once we have used Gauss' elimination method to convert the original matrix to upper triangular form, we go on to use Jordan's idea of eliminating entries in the upper right portion of the matrix.)

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

As in the last lecture, we can write the results of the elimination method as the product of a number of elimination matrices E_{ij} with the matrix A. Letting E be the product of all the E_{ij}, we write the result of this Gauss-Jordan elimination using block matrices: $E[\,A \mid I\,] = [\,I \mid E\,]$. But if $EA = I$, then $E = A^{-1}$.

Exercises on multiplication and inverse matrices

Problem 3.1: Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$$

Problem 3.2: (2.5 #24. *Introduction to Linear Algebra:* Strang) Use Gauss-Jordan elimination on $[U \;\; I]$ to find the upper triangular U^{-1} :

$$UU^{-1} = I \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercises on multiplication and inverse matrices

Problem 3.1: Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$$

Solution: We first add AB to AC :

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, \quad AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$$

$$\longrightarrow AB + AC = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix}.$$

We then compute $A(B+C)$:

$$B + C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 6 \end{bmatrix}$$

$$\longrightarrow A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix} = AB + AC.$$

Therefore, $AB + AC = A(B+C)$.

Problem 3.2: (2.5 #24. *Introduction to Linear Algebra:* Strang) Use Gauss-Jordan elimination on $[U \;\; I]$ to find the upper triangular U^{-1} :

$$UU^{-1} = I \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: Row reduce $[U \;\; I]$ to get $[I \;\; U^{-1}]$ as follows (here, R_i = row i)

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{array}{c} (R_1 = R_1 - aR_2) \\ (R_2 = R_2 - cR_2) \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} (R_1 = R_1 - (b-ac)R_3) \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I \quad L^{-1}]$$

Factorization into $A = LU$

One goal of today's lecture is to understand Gaussian elimination in terms of matrices; to find a matrix L such that $A = LU$. We start with some useful facts about matrix multiplication.

Inverse of a product

The inverse of a matrix product AB is $B^{-1}A^{-1}$.

Transpose of a product

We obtain the *transpose* of a matrix by exchanging its rows and columns. In other words, the entry in row i column j of A is the entry in row j column i of A^T.

The transpose of a matrix product AB is $B^T A^T$. For any invertible matrix A, the inverse of A^T is $\left(A^{-1}\right)^T$.

$A = LU$

We've seen how to use elimination to convert a suitable matrix A into an upper triangular matrix U. This leads to the factorization $A = LU$, which is very helpful in understanding the matrix A.

Recall that (when there are no row exchanges) we can describe the elimination of the entries of matrix A in terms of multiplication by a succession of elimination matrices E_{ij}, so that $A \to E_{21}A \to E_{31}E_{21}A \to \cdots \to U$. In the two by two case this looks like:

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}.$$

We can convert this to a factorization $A = LU$ by "canceling" the matrix E_{21}; multiply by its inverse to get $E_{21}^{-1}E_{21}A = E_{21}^{-1}U$.

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}.$$

The matrix U is upper triangular with pivots on the diagonal. The matrix L is *lower triangular* and has ones on the diagonal. Sometimes we will also want to factor out a diagonal matrix whose entries are the pivots:

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U'}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}.$$

In the three dimensional case, if $E_{32}E_{31}E_{21}A = U$ then $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$.

For example, suppose E_{31} is the identity matrix and E_{32} and E_{21} are as shown below:

$$\begin{array}{c}E_{32}\\ \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\end{array} \begin{array}{c}E_{21}\\ \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array} = \begin{array}{c}E\\ \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 10 & -5 & 1\end{bmatrix}\end{array}.$$

The 10 in the lower left corner arises because we subtracted twice the first row from the second row, then subtracted five times the new second row from the third.

The factorization $A = LU$ is preferable to the statement $EA = U$ because the combination of row subtractions does not have the effect on L that it did on E. Here $L = E^{-1} = E_{21}^{-1}E_{32}^{-1}$:

$$\begin{array}{c}E_{21}^{-1}\\ \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array} \begin{array}{c}E_{32}^{-1}\\ \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 5 & 1\end{bmatrix}\end{array} = \begin{array}{c}L\\ \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 5 & 1\end{bmatrix}\end{array}.$$

Notice the 0 in row three column one of $L = E^{-1}$, where E had a 10. If there are no row exchanges, the multipliers from the elimination matrices are copied directly into L.

How expensive is elimination?

Some applications require inverting very large matrices. This is done using a computer, of course. How hard will the computer have to work? How long will it take?

When using elimination to find the factorization $A = LU$ we just saw that we can build L as we go by keeping track of row subtractions. We have to remember L and (the matrix which will become) U; we don't have to store A or E_{ij} in the computer's memory.

How many operations does the computer perform during the elimination process for an $n \times n$ matrix? A typical operation is to multiply one row and then subtract it from another, which requires on the order of n operations. There are n rows, so the total number of operations used in eliminating entries in the first column is about n^2. The second row and column are shorter; that product costs about $(n-1)^2$ operations, and so on. The total number of operations needed to factor A into LU is on the order of n^3:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2 \approx \int_0^n x^2\,dx = \frac{1}{3}n^3.$$

While we're factoring A we're also operating on $\mathbf{b}$. That costs about n^2 operations, which is hardly worth counting compared to $\frac{1}{3}n^3$.

Row exchanges

What if there are row exchanges? In other words, what happens if there's a zero in a pivot position?

To swap two rows, we multiply on the left by a permutation matrix. For example,

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

swaps the first and second rows of a 3×3 matrix. The inverse of any permutation matrix P is $P^{-1} = P^T$.

There are $n!$ different ways to permute the rows of an $n \times n$ matrix (including the permutation that leaves all rows fixed) so there are $n!$ permutation matrices. These matrices form a *multiplicative group*.

Exercises on factorization into $A = LU$

Problem 4.1: What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Problem 4.2: (2.6 #13. *Introduction to Linear Algebra:* Strang) Compute L and U for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Exercises on factorization into $A = LU$

Problem 4.1: What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution: We will perform a series of row operations to transform the matrix A into an upper triangular matrix. First, we multiply the first row by 2 and then subtract it from the second row in order to make the first element of the second row 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Next, we multiply the first row by 2 (again) and subtract it from the third row in order to make the first element of the third row 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

Now, we multiply the second row by 3 and subtract it from the third row in order to make the second element of the third row 0:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

We take the three matrices we used to perform each operation and multiply them to get E:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} = E.$$

To check, we evaluate EA:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

To find E^{-1}, use the Gauss-Jordan elimination method (or just insert the multipliers 2, 2, 3 into E^{-1})

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 1 & -4 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = E^{-1}$$

We can check that this is in fact the inverse of E:

$$EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Finally, to factorize A into LU (where $L = E^{-1}$):

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 4.2: (2.6 #13. *Introduction to Linear Algebra:* Strang) Compute L and U for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Solution: Elimination subtracts row 1 from rows 2-4, then row 2 from rows 3-4, and finally row 3 from row 4; the result is U. All the multipliers ℓ_{ij} are equal to 1; so L is the lower triangular matrix with 1's on the diagonal and below it.

$$\mathbf{A} \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U, L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The pivots are the nonzero entries on the diagonal of U. So there are four pivots when these four conditions are satisfied: $a \neq 0, b \neq a, c \neq b$, and $d \neq c$.

Transposes, permutations, spaces R^n

In this lecture we introduce vector spaces and their subspaces.

Permutations

Multiplication by a permutation matrix P swaps the rows of a matrix; when applying the method of elimination we use permutation matrices to move zeros out of pivot positions. Our factorization $A = LU$ then becomes $PA = LU$, where P is a permutation matrix which reorders any number of rows of A. Recall that $P^{-1} = P^T$, i.e. that $P^T P = I$.

Transposes

When we take the transpose of a matrix, its rows become columns and its columns become rows. If we denote the entry in row i column j of matrix A by A_{ij}, then we can describe A^T by: $\left(A^T\right)_{ij} = A_{ji}$. For example:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}.$$

A matrix A is *symmetric* if $A^T = A$. Given any matrix R (not necessarily square) the product $R^T R$ is always symmetric, because $\left(R^T R\right)^T = R^T \left(R^T\right)^T = R^T R$. (Note that $\left(R^T\right)^T = R$.)

Vector spaces

We can add vectors and multiply them by numbers, which means we can discuss *linear combinations* of vectors. These combinations follow the rules of a *vector space*.

One such vector space is $\mathbb{R}^2$, the set of all vectors with exactly two real number components. We depict the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ by drawing an arrow from the origin to the point (a, b) which is a units to the right of the origin and b units above it, and we call $\mathbb{R}^2$ the "$x - y$ plane".

Another example of a space is $\mathbb{R}^n$, the set of (column) vectors with n real number components.

Closure

The collection of vectors with exactly two *positive* real valued components is *not* a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, -5, gives a vector that's not in the collection. We say that this collection of positive vectors is *closed* under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example, take any non-zero vector $\mathbf{v}$ in $\mathbb{R}^2$. Then the set of all vectors $c\mathbf{v}$, where c is a real number, forms a subspace of $\mathbb{R}^2$. This collection of vectors describes a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$ and is closed under addition.

A line in $\mathbb{R}^2$ that does not pass through the origin is *not* a subspace of $\mathbb{R}^2$. Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of $\mathbb{R}^2$ are:

1. all of $\mathbb{R}^2$,
2. any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and
3. the zero vector alone (Z).

The subspaces of $\mathbb{R}^3$ are:

1. all of $\mathbb{R}^3$,
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

Column space

Given a matrix A with columns in $\mathbb{R}^3$, these columns and all their linear combinations form a subspace of $\mathbb{R}^3$. This is the *column space* $C(A)$. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$, the column space of A is the plane through the origin in $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

Our next task will be to understand the equation $A\mathbf{x} = b$ in terms of subspaces and the column space of A.

Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

Solution:

a) Let P move the rows in a cycle: the first to the second, the second to the third, and the third to the first. So

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } P^3 = I.$$

b) Let $\widehat{P}$ be the block diagonal matrix with 1 and P on the diagonal; $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$. Since $P^3 = I$, also $\widehat{P}^3 = I$. So $\widehat{P}^4 = \widehat{P} \neq I$.

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric?

b) A is *skew-symmetric*? ($A^T = -A$)

Solution:

a) The most general form of a four by four symmetric matrix is:

$$A = \begin{bmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{bmatrix}.$$

Therefore **10** entries can be chosen independently.

b) The most general form of a four by four skew-symmetric matrix is:

$$A = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}.$$

Therefore **6** entries can be chosen independently.

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

Solution:

a) True: $A^T = A$ and $B^T = B$ lead to:

$$(A+B)^T = A^T + B^T = A + B, \text{ and } (cA)^T = cA.$$

b) True: $A^T = -A$ and $B^T = -B$ lead to:

$$(A+B)^T = A^T + B^T = -A - B = -(A+B), \text{ and } (cA)^T = -cA.$$

c) False: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$

Column space and nullspace

In this lecture we continue to study subspaces, particularly the column space and nullspace of a matrix.

Review of subspaces

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in the space and any two real numbers c and d, the vector $c\mathbf{v} + d\mathbf{w}$ is also in the vector space. A subspace is a vector space contained inside a vector space.

A plane P containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and a line L containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are both subspaces of $\mathbb{R}^3$. The union $P \cup L$ of those two subspaces is generally not a subspace, because the sum of a vector in P and a vector in L is probably not contained in $P \cup L$. The intersection $S \cap T$ of two subspaces S and T is a subspace. To prove this, use the fact that both S and T are closed under linear combinations to show that their intersection is closed under linear combinations.

Column space of A

The *column space* of a matrix A is the vector space made up of all linear combinations of the columns of A.

Solving $A\mathbf{x} = \mathbf{b}$

Given a matrix A, for what vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{b}$ does not have a solution for every choice of $\mathbf{b}$ because solving $A\mathbf{x} = \mathbf{b}$ is equivalent to solving four linear equations in three unknowns. If there is a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ must be a linear combination of the columns of A. Only three columns cannot fill the entire four dimensional vector space – some vectors $\mathbf{b}$ cannot be expressed as linear combinations of columns of A.

Big question: what $\mathbf{b}$'s allow $A\mathbf{x} = \mathbf{b}$ to be solved?

A useful approach is to choose $\mathbf{x}$ and find the vector $\mathbf{b} = A\mathbf{x}$ corresponding to that solution. The components of $\mathbf{x}$ are just the coefficients in a linear combination of columns of A.

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is *solvable* exactly when $\mathbf{b}$ is a vector in the *column space* of A.

For our example matrix A, what can we say about the column space of A? Are the columns of A *independent*? In other words, does each column contribute something new to the subspace?

The third column of A is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix A is a two dimensional subspace of $\mathbb{R}^4$.

Nullspace of A

The *nullspace* of a matrix A is the collection of all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $A\mathbf{x} = 0$.

The column space of the matrix in our example was a subspace of $\mathbb{R}^4$. The nullspace of A is a subspace of $\mathbb{R}^3$. To see that it's a vector space, check that any sum or multiple of solutions to $A\mathbf{x} = \mathbf{0}$ is also a solution: $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + \mathrm{A}\mathbf{x}_2 = \mathbf{0} + \mathbf{0}$ and $A(c\mathbf{x}) = cA\mathbf{x} = c(\mathbf{0})$.

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the nullspace $N(A)$ consists of all multiples of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$; column 1 plus column 2 minus column 3 equals the zero vector. This nullspace is a line in $\mathbb{R}^3$.

Other values of b

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in $\mathbb{R}^3$ that passes through the points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ but not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S} + \mathbf{T}$ contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

Problem 6.2: (3.2 #18.) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - x = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S}+\mathbf{T}$ contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S}+\mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S}+\mathbf{T}$ and $\mathbf{S}\cup\mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S}\cup\mathbf{T}$ *is* $\mathbf{S}+\mathbf{T}$.

Solution:

a) Let $\mathbf{s}, \mathbf{s}'$ be vectors in $\mathbf{S}$, let $\mathbf{t}, \mathbf{t}'$ be vectors in $\mathbf{T}$, and let c be a scalar. Then

$$(\mathbf{s}+\mathbf{t})+(\mathbf{s}'+\mathbf{t}') = (\mathbf{s}+\mathbf{s}')+(\mathbf{t}+\mathbf{t}') \text{ and } c(\mathbf{s}+\mathbf{t}) = c\mathbf{s}+c\mathbf{t}.$$

Thus $\mathbf{S}+\mathbf{T}$ is closed under addition and scalar multiplication; in other words, it satisfies the two requirements for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are distinct lines, then $\mathbf{S}+\mathbf{T}$ is a plane, whereas $\mathbf{S}\cup\mathbf{T}$ is only the two lines. The span of $\mathbf{S}\cup\mathbf{T}$ is the set of all combinations of vectors in this union of two lines. In particular, it contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$, and these sums form $\mathbf{S}+\mathbf{T}$.

Since $\mathbf{S}+\mathbf{T}$ contains both $\mathbf{S}$ and $\mathbf{T}$, it contains $\mathbf{S}\cup\mathbf{T}$. Further, $\mathbf{S}+\mathbf{T}$ is a vector space. So it contains all combinations of vectors in itself; in particular, it contains the span of $\mathbf{S}\cup\mathbf{T}$. Thus the span of $\mathbf{S}\cup\mathbf{T}$ is $\mathbf{S}+\mathbf{T}$.

Problem 6.2: (3.2 #18.) The plane $x-3y-z=12$ is parallel to the plane $x-3y-x=0$. One particular point on this plane is $(12,0,0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution: The equation $x = 12 + 3y + z$ says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(= \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \boxed{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \boxed{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \boxed{1} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution: $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$ contains all vectors that are in both nullspaces:

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix} = 0$$

if and only if $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

Solving $A\mathbf{x} = \mathbf{0}$: pivot variables, special solutions

We have a definition for the column space and the nullspace of a matrix, but how do we compute these subspaces?

Computing the nullspace

The *nullspace* of a matrix A is made up of the vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$.

Suppose:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

(Note that the columns of this matrix A are not independent.) Our algorithm for computing the nullspace of this matrix uses the method of elimination, despite the fact that A is not invertible. We don't need to use an augmented matrix because the right side (the vector $\mathbf{b}$) is $\mathbf{0}$ in this computation.

The row operations used in the method of elimination don't change the solution to $A\mathbf{x} = \mathbf{b}$ so they don't change the nullspace. (They do affect the column space.)

The first step of elimination gives us:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

We don't find a pivot in the second column, so our next pivot is the 2 in the third column of the second row:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The matrix U is in *echelon* (staircase) form. The third row is zero because row 3 was a linear combination of rows 1 and 2; it was eliminated.

The *rank* of a matrix A equals the number of pivots it has. In this example, the rank of A (and of U) is 2.

Special solutions

Once we've found U we can use back-substitution to find the solutions $\mathbf{x}$ to the equation $U\mathbf{x} = \mathbf{0}$. In our example, columns 1 and 3 are *pivot columns* containing pivots, and columns 2 and 4 are *free columns*. We can assign any value to x_2 and x_4; we call these *free variables*. Suppose $x_2 = 1$ and $x_4 = 0$. Then:

$$2x_3 + 4x_4 = 0 \quad \Longrightarrow \quad x_3 = 0$$

and:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \quad \Longrightarrow \quad x_1 = -2.$$

So one solution is $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (because the second column is just twice the first column). Any multiple of this vector is in the nullspace.

Letting a different free variable equal 1 and setting the other free variables equal to zero gives us other vectors in the nullspace. For example:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

has $x_4 = 1$ and $x_2 = 0$. The nullspace of A is the collection of all linear combinations of these "special solution" vectors.

The rank r of A equals the number of pivot columns, so the number of free columns is $n - r$: the number of columns (variables) minus the number of pivot columns. This equals the number of special solution vectors and the dimension of the nullspace.

Reduced row echelon form

By continuing to use the method of elimination we can convert U to a matrix R in *reduced row echelon form* (rref form), with pivots equal to 1 and zeros above and below the pivots.

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

By exchanging some columns, R can be rewritten with a copy of the identity matrix in the upper left corner, possibly followed by some free columns on the right. If some rows of A are linearly dependent, the lower rows of the matrix R will be filled with zeros:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

(Here I is an r by r square matrix.)

If N is the *nullspace matrix* $N = \begin{bmatrix} -F \\ I \end{bmatrix}$ then $RN = 0$. (Here I is an $n - r$ by $n - r$ square matrix and 0 is an m by $n - r$ matrix.) The columns of N are the special solutions.

Exercises on solving $A\mathbf{x} = \mathbf{0}$: pivot variables, special solutions

Problem 7.1:

a) Find the row reduced form of:

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$$

b) What is the rank of this matrix?

c) Find any special solutions to the equation $A\mathbf{x} = \mathbf{0}$.

Problem 7.2: (3.3 #17.b *Introduction to Linear Algebra:* Strang) Find A_1 and A_2 so that rank(A_1B) = 1 and rank(A_2B) = 0 for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Exercises on solving $A\mathbf{x} = \mathbf{0}$: pivot variables, special solutions

Problem 7.1:

a) Find the row reduced form of:

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$$

b) What is the rank of this matrix?

c) Find any special solutions to the equation $A\mathbf{x} = \mathbf{0}$.

Solution:

a) To transform A into its reduced row form, we perform a series of row operations. Different operations are possible (same answer!). First, we multiply the first row by 2 and subtract it from the third row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix}.$$

We then multiply the second row by $\frac{1}{4}$ to make the second pivot 1:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 0 & -12 & -3 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & -12 & -3 & -21 \end{bmatrix}.$$

Multiply the second row by 12 and add it to the third row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & -12 & -3 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, multiply the second row by 5 and subtract it from the first row:

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 23/4 & 1/4 \\ 0 & 1 & 1/4 & 7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The matrix is of **rank 2** because it has 2 pivots.

c) The special solutions to $A\mathbf{x} = \mathbf{0}$ are:

$$\begin{bmatrix} -23/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1/4 \\ -7/4 \\ 0 \\ 1 \end{bmatrix}$$

Problem 7.2: (3.3 #17.b *Introduction to Linear Algebra:* Strang) Find A_1 and A_2 so that $\text{rank}(A_1B) = 1$ and $\text{rank}(A_2B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: Take $A_1 = I_2$ and $A_2 = 0_2$.

A less trivial example is $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solving $A\mathbf{x} = \mathbf{b}$: row reduced form R

When does $A\mathbf{x} = \mathbf{b}$ have solutions $\mathbf{x}$, and how can we describe those solutions?

Solvability conditions on b

We again use the example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}.$$

The third row of A is the sum of its first and second rows, so we know that if $A\mathbf{x} = \mathbf{b}$ the third component of $\mathbf{b}$ equals the sum of its first and second components. If $\mathbf{b}$ does not satisfy $b_3 = b_1 + b_2$ the system has no solution. If a combination of the rows of A gives the zero row, then the same combination of the entries of $\mathbf{b}$ must equal zero.

One way to find out whether $A\mathbf{x} = \mathbf{b}$ is solvable is to use elimination on the augmented matrix. If a row of A is completely eliminated, so is the corresponding entry in $\mathbf{b}$. In our example, row 3 of A is completely eliminated:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.$$

If $A\mathbf{x} = \mathbf{b}$ has a solution, then $b_3 - b_2 - b_1 = 0$. For example, we could choose $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$.

From an earlier lecture, we know that $A\mathbf{x} = \mathbf{b}$ is solvable exactly when $\mathbf{b}$ is in the column space $C(A)$. We have these two conditions on $\mathbf{b}$; in fact they are equivalent.

Complete solution

In order to find all solutions to $A\mathbf{x} = \mathbf{b}$ we first check that the equation is solvable, then find a particular solution. We get the complete solution of the equation by adding the particular solution to all the vectors in the nullspace.

A particular solution

One way to find a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ is to set all free variables to zero, then solve for the pivot variables.

For our example matrix A, we let $x_2 = x_4 = 0$ to get the system of equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

which has the solution $x_3 = 3/2$, $x_1 = -2$. Our particular solution is:

$$\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}.$$

Combined with the nullspace

The general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_{\text{complete}} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is a generic vector in the nullspace. To see this, we add $A\mathbf{x}_p = \mathbf{b}$ to $A\mathbf{x}_n = \mathbf{0}$ and get $A\left(\mathbf{x}_p + \mathbf{x}_n\right) = \mathbf{b}$ for every vector $\mathbf{x}_n$ in the nullspace.

Last lecture we learned that the nullspace of A is the collection of all combinations of the special solutions $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. So the complete solution to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ is:

$$\mathbf{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are real numbers.

The nullspace of A is a two dimensional subspace of $\mathbb{R}^4$, and the solutions to the equation $A\mathbf{x} = \mathbf{b}$ form a plane parallel to that through $x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$.

Rank

The rank of a matrix equals the number of pivots of that matrix. If A is an m by n matrix of rank r, we know $r \leq m$ and $r \leq n$.

Full column rank

If $r = n$, then from the previous lecture we know that the nullspace has dimension $n - r = 0$ and contains only the zero vector. There are no free variables or special solutions.

If $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique; there is either 0 or 1 solution. Examples like this, in which the columns are independent, are common in applications.

We know $r \leq m$, so if $r = n$ the number of columns of the matrix is less than or equal to the number of rows. The row reduced echelon form of the matrix will look like $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$. For any vector $\mathbf{b}$ in $\mathbb{R}^m$ that's not a linear combination of the columns of A, there is no solution to $A\mathbf{x} = \mathbf{b}$.

Full row rank

If $r = m$, then the reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$ has no rows of zeros and so there are no requirements for the entries of $\mathbf{b}$ to satisfy. The equation $A\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$. There are $n - r = n - m$ free variables, so there are $n - m$ special solutions to $A\mathbf{x} = \mathbf{0}$.

Full row and column rank

If $r = m = n$ is the number of pivots of A, then A is an invertible square matrix and R is the identity matrix. The nullspace has dimension zero, and $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Summary

If R is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many

Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Problem 8.2: (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \;\; 0]$ and $[U \;\; \mathbf{c}]$ to $[R \;\; 0]$ and $[R \;\; \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Solution:

a) The coefficient of $\mathbf{x}_p$ must be one.

b) If $\mathbf{x}_n \in \mathbf{N}(A)$ is in the nullspace of A and $\mathbf{x}_p$ is one particular solution, then $\mathbf{x}_p + \mathbf{x}_n$ is also a particular solution.

c) There's always $\mathbf{x}_n = 0$.

Problem 8.2: (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \;\; 0]$ and $[U \;\; \mathbf{c}]$ to $[R \;\; 0]$ and $[R \;\; \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Solution: First we transform $[U \;\; 0]$ into $[R \;\; 0]$:

$$[U \;\; 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [R \;\; 0].$$

We now solve $R\mathbf{x} = \mathbf{0}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$

where we used the free variable $x_2 = -1$. ($c\mathbf{x}$ is a solution for all c.)

We check that this is a correct solution by plugging it into $U\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

Next, we transform $[U\ \ \mathbf{c}]$ into $[R\ \ \mathbf{d}]$:

$$[U\ \ \mathbf{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [R\ \ \mathbf{d}].$$

We now solve $R\mathbf{x} = \mathbf{d}$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix},$$

where we used the free variable $x_2 = 1$.

Finally, we check that this is the correct solution by plugging it into the equation $U\mathbf{x} = \mathbf{c}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Solution: **Yes.** In order to check that $A = C$ as matrices, it is enough to check that $A\mathbf{y} = C\mathbf{y}$ for all vectors $\mathbf{y}$ of the correct size (or just for the standard basis vectors, since multiplication by them "picks out the columns"). So let $\mathbf{y}$ be any vector of the correct size, and set $\mathbf{b} = A\mathbf{y}$. Then $\mathbf{y}$ is certainly a solution to $A\mathbf{x} = \mathbf{b}$, and so by our hypothesis must also be a solution to $C\mathbf{x} = \mathbf{b}$; in other words, $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$.

Independence, basis, and dimension

What does it mean for vectors to be independent? How does the idea of independence help us describe subspaces like the nullspace?

Linear independence

Suppose A is an m by n matrix with $m < n$ (so $A\mathbf{x} = \mathbf{b}$ has more unknowns than equations). A has at least one free variable, so there are nonzero solutions to $A\mathbf{x} = \mathbf{0}$. A combination of the columns is zero, so the columns of this A are *dependent*.

We say vectors $\mathbf{x}_1, \mathbf{x}_2, ...\mathbf{x}_n$ are *linearly independent* (or just *independent*) if $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ only when $c_1, c_2, ..., c_n$ are all 0. When those vectors are the columns of A, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Two vectors are independent if they do not lie on the same line. Three vectors are independent if they do not lie in the same plane. Thinking of $A\mathbf{x}$ as a linear combination of the column vectors of A, we see that the column vectors of A are independent exactly when the nullspace of A contains only the zero vector.

If the columns of A are independent then all columns are pivot columns, the rank of A is n, and there are no free variables. If the columns of A are dependent then the rank of A is less than n and there are free variables.

Spanning a space

Vectors $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_k$ *span* a space when the space consists of all combinations of those vectors. For example, the column vectors of A span the column space of A.

If vectors $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_k$ span a space S, then S is the smallest space containing those vectors.

Basis and dimension

A *basis* for a vector space is a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_d$ with two properties:

- $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_d$ are independent
- $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_d$ span the vector space.

The basis of a space tells us everything we need to know about that space.

Example: $\mathbb{R}^3$

One basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. These are independent because:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is only possible when $c_1 = c_2 = c_3 = 0$. These vectors span $\mathbb{R}^3$.

As discussed at the start of Lecture 10, the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ do not form a basis for $\mathbb{R}^3$ because these are the column vectors of a matrix that has two identical rows. The three vectors are not linearly independent.

In general, n vectors in $\mathbb{R}^n$ form a basis if they are the column vectors of an invertible matrix.

Basis for a subspace

The vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ span a plane in $\mathbb{R}^3$ but they cannot form a basis for $\mathbb{R}^3$. Given a space, every basis for that space has the same number of vectors; that number is the *dimension* of the space. So there are exactly n vectors in every basis for $\mathbb{R}^n$.

Bases of a column space and nullspace

Suppose:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}.$$

By definition, the four column vectors of A span the column space of A. The third and fourth column vectors are dependent on the first and second, and the first two columns are independent. Therefore, the first two column vectors are the pivot columns. They form a basis for the column space $C(A)$. The matrix has rank 2. In fact, for any matrix A we can say:

$$\text{rank}(A) = \text{number of pivot columns of } A = \text{dimension of } C(A).$$

(Note that matrices have a rank but not a dimension. Subspaces have a dimension but not a rank.)

The column vectors of this A are not independent, so the nullspace $N(A)$ contains more than just the zero vector. Because the third column is the sum of the first two, we know that the vector $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace. Similarly, $\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is also in $N(A)$. These are the two special solutions to $A\mathbf{x} = \mathbf{0}$. We'll see that:

$$\text{dimension of } N(A) = \text{number of free variables} = n - r,$$

so we know that the dimension of $N(A)$ is $4 - 2 = 2$. These two special solutions form a basis for the nullspace.

Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra:* Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra:* Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution: Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, the vectors $\mathbf{v}_4, \mathbf{v}_5$, and $\mathbf{v}_6$ are dependent on the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. To determine the relationship between the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ we apply row reduction to the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are independent. Therefore the largest number of independent vectors among the given six vectors is **three.** This will be the rank of the 4 by 6 matrix of $\mathbf{v}$'s.

Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

Solution: This plane is the nullspace of the matrix $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ and also

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The special solutions to $A\mathbf{x} = \mathbf{0}$ are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

These form a basis for the nullspace of A and thus for the plane.

The intersection of this plane with the xy plane contains $\mathbf{v}_1$ and does not contain $\mathbf{v}_2$; the intersection must be a line. Since $\mathbf{v_1}$ lies on this line it also provides a basis for it.

Finally, we can use "inspection" or the cross product to find the vector

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix},$$

which is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. It is therefore perpendicular to the plane. Since the space of vectors perpendicular to a plane in $\mathbb{R}^3$ is one-dimensional, $\mathbf{v}_3$ serves as a basis for that space.

The four fundamental subspaces

In this lecture we discuss the four fundamental spaces associated with a matrix and the relations between them.

Four subspaces

Any m by n matrix A determines four subspaces (possibly containing only the zero vector):

Column space, $C(A)$

$C(A)$ consists of all combinations of the columns of A and is a vector space in $\mathbb{R}^m$.

Nullspace, $N(A)$

This consists of all solutions $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{0}$ and lies in $\mathbb{R}^n$.

Row space, $C(A^T)$

The combinations of the row vectors of A form a subspace of R^n. We equate this with $C(A^T)$, the column space of the transpose of A.

Left nullspace, $N(A^T)$

We call the nullspace of A^T the *left nullspace* of A. This is a subspace of $\mathbb{R}^m$.

Basis and Dimension

Column space

The r pivot columns form a basis for $C(A)$

$$\dim C(A) = r.$$

Nullspace

The special solutions to $A\mathbf{x} = \mathbf{0}$ correspond to free variables and form a basis for $N(A)$. An m by n matrix has $n - r$ free variables:

$$\dim N(A) = n - r.$$

Row space

We could perform row reduction on A^T, but instead we make use of R, the row reduced echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

Although the column spaces of A and R are different, the row space of R is the same as the row space of A. The rows of R are combinations of the rows of A, and because reduction is reversible the rows of A are combinations of the rows of R.

The first r rows of R are the "echelon" basis for the row space of A:

$$\dim C(A^T) = r.$$

Left nullspace

The matrix A^T has m columns. We just saw that r is the rank of A^T, so the number of free columns of A^T must be $m - r$:

$$\dim N(A^T) = m - r.$$

The left nullspace is the collection of vectors y for which $A^T y = 0$. Equivalently, $y^T A = 0$; here y and 0 are row vectors. We say "left nullspace" because y^T is on the left of A in this equation.

To find a basis for the left nullspace we reduce an augmented version of A:

$$\begin{bmatrix} A_{m \times n} & I_{m \times n} \end{bmatrix} \longrightarrow \begin{bmatrix} R_{m \times n} & E_{m \times n} \end{bmatrix}.$$

From this we get the matrix E for which $EA = R$. (If A is a square, invertible matrix then $E = A^{-1}$.) In our example,

$$EA = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

The bottom $m - r$ rows of E describe linear dependencies of rows of A, because the bottom $m - r$ rows of R are zero. Here $m - r = 1$ (one zero row in R).

The bottom $m - r$ rows of E satisfy the equation $\mathbf{y}^T A = \mathbf{0}$ and form a basis for the left nullspace of A.

New vector space

The collection of all 3×3 matrices forms a vector space; call it M. We can add matrices and multiply them by scalars and there's a zero matrix (additive identity). If we ignore the fact that we can multiply matrices by each other, they behave just like vectors.

Some subspaces of M include:

- all upper triangular matrices
- all symmetric matrices
- D, all diagonal matrices

D is the intersection of the first two spaces. Its dimension is 3; one basis for D is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra:* Strang) A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

a) What are all the inequalities ($<$ or $\leq$) that must be true between $m, n,$ and r?

b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the __________ contains only the zero vector.

Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra:* Strang) A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

Solution:

a) The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$ and $r \leq n$. The second statement tells us that the column space is not all of $\mathbb{R}^n$, so $r < m$.

b) These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it).

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the __________ contains only the zero vector.

Solution: It is solvable when $\mathbf{d}$ is in the row space, which consists of all vectors $A^T\mathbf{y}$. The solution $\mathbf{y}$ is unique when the **left nullspace** contains only the zero vector.

Matrix spaces; rank 1; small world graphs

We've talked a lot about $\mathbb{R}^n$, but we can think about vector spaces made up of any sort of "vectors" that allow addition and scalar multiplication.

New vector spaces

3 by 3 matrices

We were looking at the space M of all 3 by 3 matrices. We identified some subspaces; the symmetric 3 by 3 matrices S, the upper triangular 3 by 3 matrices U, and the intersection D of these two spaces – the space of diagonal 3 by 3 matrices.

The dimension of M is 9; we must choose 9 numbers to specify an element of M. The space M is very similar to $\mathbb{R}^9$. A good choice of basis is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The subspace of symmetric matrices S has dimension 6. When choosing an element of S we pick three numbers on the diagonal and three in the upper right, which tell us what must appear in the lower left of the matrix. One basis for S is the collection:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The dimension of U is again 6; we have the same amount of freedom in selecting the entries of an upper triangular matrix as we did in choosing a symmetric matrix. A basis for U is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This happens to be a subset of the basis we chose for M, but there is no basis for S that is a subset of the basis we chose for M.

The subspace $D = S \cap U$ of diagonal 3 by 3 matrices has dimension 3. Because of the way we chose bases for U and S, a good basis for D is the intersection of those bases.

Is $S \cup U$, the set of 3 by 3 matrices which are either symmetric or upper triangular, a subspace of M? No. This is like taking two lines in $\mathbb{R}^2$ and asking if together they form a subspace; we have to fill in between them. If we take all possible sums of elements of S and elements of U we get what we call the *sum* $S + U$. This is a subspace of M. In fact, $S + U = M$. For unions and sums, dimensions follow this rule:

$$\dim S + \dim U = \dim S \cup U + \dim S \cap U.$$

Differential equations

Another example of a vector space that's not $\mathbb{R}^n$ appears in differential equations.

We can think of the solutions y to $\frac{d^2y}{dx^2} + y = 0$ as the elements of a nullspace. Some solutions are:

$$y = \cos x, \quad y = \sin x, \quad \text{and} \quad y = e^{ix}.$$

The complete solution is:

$$y = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 can be any complex numbers. This solution space is a two dimensional vector space with basis vectors $\cos x$ and $\sin x$. (Even though these don't "look like" vectors, we can build a vector space from them because they can be added and multiplied by a constant.)

Rank 4 matrices

Now let M be the space of 5×17 matrices. The subset of M containing all rank 4 matrices is not a subspace, even if we include the zero matrix, because the sum of two rank 4 matrices may not have rank 4.

In $\mathbb{R}^4$, the set of all vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ for which $v_1 + v_2 + v_3 + v_4 = 0$ is a subspace. It contains the zero vector and is closed under addition and scalar multiplication. It is the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Because A has rank 1, the dimension of this nullspace is $n - r = 3$. The subspace has the basis of special solutions:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The column space of A is $\mathbb{R}^1$. The left nullspace contains only the zero vector, has dimension zero, and its basis is the empty set. The row space of A also has dimension 1.

Rank one matrices

The rank of a matrix is the dimension of its column (or row) space. The matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

has rank 1 because each of its columns is a multiple of the first column.

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}.$$

Every rank 1 matrix A can be written $A = \mathbf{U}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are column vectors. We'll use rank 1 matrices as building blocks for more complex matrices.

Small world graphs

In this class, a *graph* G is a collection of nodes joined by edges:

$$G = \{\text{nodes}, \text{edges}\}.$$

A typical graph appears in Figure 1. Another example of a graph is one in

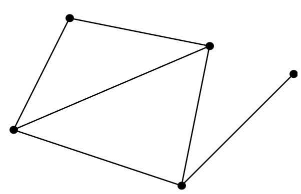


Figure 1: A graph with 5 nodes and 6 edges.

which each node is a person. Two nodes are connected by an edge if the people are friends. We can ask how close two people are to each other in the graph – what's the smallest number of friend to friend connections joining them? The question "what's the farthest distance between two people in the graph?" lies behind phrases like "six degrees of separation" and "it's a small world".

Another graph is the world wide web: its nodes are web sites and its edges are links.

We'll describe graphs in terms of matrices, which will make it easy to answer questions about distances between nodes.

Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Problem 11.2: (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix X in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices X lead to $AX = 0$?

b) Which matrices have the form AX for some matrix X?

c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of **M**? Why do the dimensions add to $(n - r) + r = 9$?

Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Solution: The other five permutation matrices are:

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \text{ and } P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Since $P_{21} + P_{31} + P_{32}$ is the all ones matrix and $P_{32}P_{21} + P_{21}P_{32}$ is the matrix with zeros on the diagonal and ones elsewhere,

$$I = P_{21} + P_{31} + P_{32} - P_{32}P_{21} - P_{21}P_{32}.$$

For the second part, setting $c_1P_1 + \cdots + c_5P_5$ equal to zero gives:

$$\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0.$$

So $c_1 = c_2 = c_3 = 0$ along the diagonal, and $c_4 = c_5 = 0$ from the off-diagonal entries.

Problem 11.2: (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix X in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices X lead to $AX = 0$?

b) Which matrices have the form AX for some matrix X?

c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n-r)+r=9$?

Solution:

a) We can use row reduction or some other method to see that the rows of A are dependent and that A has rank 2. Its nullspace has the basis:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$AX = 0$ precisely when the columns of X are in the nullspace of A, i.e. when they are multiples of the basis of $N(A)$. Therefore, if $AX = 0$ then X must have the form:

$$X = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}.$$

b) On the other hand, the columns of any matrix of the form AX are linear combinations of the columns of A. That is, they are vectors whose components all sum to 0, so a matrix has the form AX if and only if all of its columns individually sum to 0:

$$AX = B \text{ if and only if } B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}.$$

c) The dimension of the "nullspace" is 3, while the dimension of the "column space" is 6. These add up to 9, which is the dimension of the space of "inputs" $\mathbf{M}$.

Graphs, networks, incidence matrices

When we use linear algebra to understand physical systems, we often find more structure in the matrices and vectors than appears in the examples we make up in class. There are many applications of linear algebra; for example, chemists might use row reduction to get a clearer picture of what elements go into a complicated reaction. In this lecture we explore the linear algebra associated with electrical networks.

Graphs and networks

A *graph* is a collection of nodes joined by edges; Figure 1 shows one small graph.

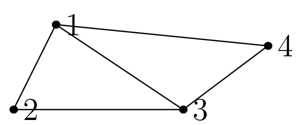


Figure 1: A graph with $n = 4$ nodes and $m = 5$ edges.

We put an arrow on each edge to indicate the positive direction for currents running through the graph.

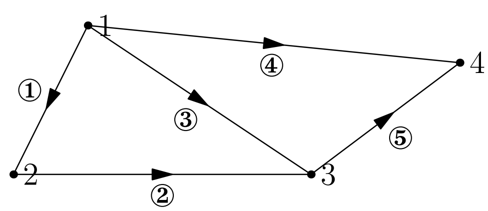


Figure 2: The graph of Figure 1 with a direction on each edge.

Incidence matrices

The *incidence matrix* of this directed graph has one column for each node of the graph and one row for each edge of the graph:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

If an edge runs from node a to node b, the row corresponding to that edge has -1 in column a and 1 in column b; all other entries in that row are 0. If we were studying a larger graph we would get a larger matrix but it would be *sparse*; most of the entries in that matrix would be 0. This is one of the ways matrices arising from applications might have extra structure.

Note that nodes 1, 2 and 3 and edges ①, ② and ③ form a loop. The matrix describing just those nodes and edges looks like:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Note that the third row is the sum of the first two rows; loops in the graph correspond to linearly dependent rows of the matrix.

To find the nullspace of A, we solve $A\mathbf{x} = \mathbf{0}$:

$$A\mathbf{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

If the components x_i of the vector $\mathbf{x}$ describe the electrical potential at the nodes i of the graph, then $A\mathbf{x}$ is a vector describing the *difference* in potential across each edge of the graph. We see $A\mathbf{x} = \mathbf{0}$ when $x_1 = x_2 = x_3 = x_4$, so the nullspace has dimension 1. In terms of an electrical network, the potential difference is zero on each edge if each node has the same potential. We can't tell what that potential is by observing the flow of electricity through the network, but if one node of the network is grounded then its potential is zero. From that we can determine the potential of all other nodes of the graph.

The matrix has 4 columns and a 1 dimensional nullspace, so its rank is 3. The first, second and fourth columns are its pivot columns; these edges connect all the nodes of the graph without forming a loop – a graph with no loops is called a *tree*.

The left nullspace of A consists of the solutions $\mathbf{y}$ to the equation: $A^T\mathbf{y} = \mathbf{0}$. Since A^T has 5 columns and rank 3 we know that the dimension of $N(A^T)$ is $m - r = 2$. Note that 2 is the number of loops in the graph and m is the number of edges. The rank r is $n - 1$, one less than the number of nodes. This gives us # loops = # edges − (# nodes − 1), or:

$$\text{number of nodes} - \text{number of edges} + \text{number of loops} = 1.$$

This is Euler's formula for connected graphs.

Kirchhoff's law

In our example of an electrical network, we started with the potentials x_i of the nodes. The matrix A then told us something about potential differences. An engineer could create a matrix C using Ohm's law and information about

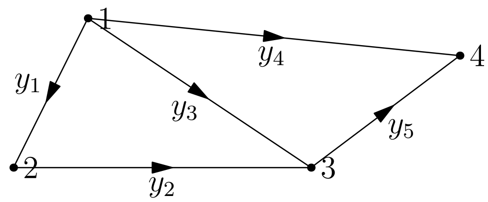


Figure 3: The currents in our graph.

the conductance of the edges and use that matrix to determine the current y_i on each edge. Kirchhoff's Current Law then says that $A^T\mathbf{y} = \mathbf{0}$, where $\mathbf{y}$ is the vector with components y_1, y_2, y_3, y_4, y_5. Vectors in the nullspace of A^T correspond to collections of currents that satisfy Kirchhoff's law.

$$
\begin{array}{ccc}
\begin{array}{c}\mathbf{x} = x_1, x_2, x_3, x_4 \\ \text{potentials at nodes}\end{array} & & \begin{array}{c}A^T\mathbf{y} = 0 \\ \text{Kirchhoff's Current Law}\end{array} \\[2em]
\mathbf{e} = A\mathbf{x} \downarrow & & \uparrow A^T\mathbf{y} \\[2em]
\begin{array}{c}x_2 - x_1, \text{etc.} \\ \text{potential differences}\end{array} & \begin{array}{c}\mathbf{y} = C\mathbf{e} \\ \longrightarrow \\ \text{Ohm's Law}\end{array} & \begin{array}{c}y_1, y_2, y_3, y_4, y_5 \\ \text{currents on edges}\end{array}
\end{array}
$$

Written out, $A^T\mathbf{y} = \mathbf{0}$ looks like:

$$
\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Multiplying the first row by the column vector $\mathbf{y}$ we get $-y_1 - y_3 - y_4 = 0$. This tells us that the total current flowing out of node 1 is zero – it's a balance equation, or a conservation law. Multiplying the second row by $\mathbf{y}$ tells us $y_1 - y_2 = 0$; the current coming into node 2 is balanced with the current going out. Multiplying the bottom rows, we get $y_2 + y_3 - y_5 = 0$ and $y_4 + y_5 = 0$.

We could use the method of elimination on A^T to find its column space, but we already know the rank. To get a basis for $N(A^T)$ we just need to find two independent vectors in this space. Looking at the equations $y_1 - y_2 = 0$ we might guess $y_1 = y_2 = 1$. Then we could use the conservation laws for node 3 to guess $y_3 = -1$ and $y_5 = 0$. We satisfy the conservation conditions on node 4 with $y_4 = 0$, giving us a basis vector $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. This vector represents one unit of current flowing around the loop joining nodes 1, 2 and 3; a multiple of this vector represents a different amount of current around the same loop.

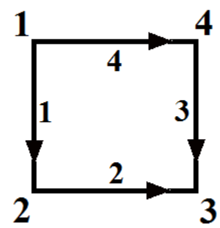
We find a second basis vector for $N(A^T)$ by looking at the loop formed by nodes 1, 3 and 4: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ that represents a current around the outer loop is also in the nullspace, but it is the sum of the first two vectors we found.

We've almost completely covered the mathematics of simple circuits. More complex circuits might have batteries in the edges, or current sources between nodes. Adding current sources changes the $A^T\mathbf{y} = \mathbf{0}$ in Kirchhoff's current law to $A^T\mathbf{y} = \mathbf{f}$. Combining the equations $\mathbf{e} = A\mathbf{x}$, $\mathbf{y} = C\mathbf{e}$ and $A^T\mathbf{y} = \mathbf{f}$ gives us:

$$A^T C A\mathbf{x} = \mathbf{f}.$$

Exercises on graphs, networks, and incidence matrices

Problem 12.1: (8.2 #1. *Introduction to Linear Algebra:* Strang) Write down the four by four incidence matrix A for the square graph, shown below. (Hint: the first row has -1 in column 1 and +1 in column 2.) What vectors (x_1, x_2, x_3, x_4) are in the nullspace of A? How do you know that (1,0,0,0) is not in the row space of A?



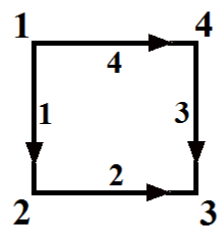
Problem 12.2: (8.2 #7.) Continuing with the network from problem one, suppose the conductance matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiply matrices to find $A^T CA$. For $\mathbf{f} = (1, 0, -1, 0)$, find a solution to $A^T CA\mathbf{x} = \mathbf{f}$. Write the potentials $\mathbf{x}$ and currents $\mathbf{y} = -CA\mathbf{x}$ on the square graph (see above) for this current source $\mathbf{f}$ going into node 1 and out from node 3.

Exercises on graphs, networks, and incidence matrices

Problem 12.1: (8.2 #1. *Introduction to Linear Algebra:* Strang) Write down the four by four incidence matrix A for the square graph, shown below. (Hint: the first row has -1 in column 1 and +1 in column 2.) What vectors (x_1, x_2, x_3, x_4) are in the nullspace of A? How do you know that (1,0,0,0) is not in the row space of A?



Solution: The incidence matrix A is written as:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

To find the vectors in the nullspace, we solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_4 \\ x_4 - x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $x_1 = x_2 = x_3 = x_4$. Therefore, the nullspace consists of vectors of the form (a, a, a, a).

Finally, (1,0,0,0) is not in the row space of A because it is not orthogonal to the nullspace.

Problem 12.2: (8.2 #7.) Continuing with the network from problem one, suppose the conductance matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiply matrices to find $A^T CA$. For $\mathbf{f} = (1, 0, -1, 0)$, find a solution to $A^T CA\mathbf{x} = \mathbf{f}$. Write the potentials $\mathbf{x}$ and currents $\mathbf{y} = -CA\mathbf{x}$ on the square graph (see above) for this current source $\mathbf{f}$ going into node 1 and out from node 3.

Solution: From the previous question, we know that the incidence matrix is:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Multiply to obtain $A^T CA$:

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}.$$

Solving the equation $A^T CA\mathbf{x} = \mathbf{f}$ by performing row reduction on the augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

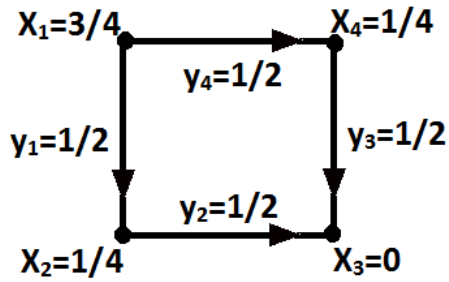
and choosing $x_3 = 0$ to represent a grounded node gives:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \mathbf{3/4} \\ \mathbf{1/4} \\ \mathbf{0} \\ \mathbf{1/4} \end{bmatrix}.$$

We know $\mathbf{y} = -CA\mathbf{x}$, so

$$\mathbf{y} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} \mathbf{1/2} \\ \mathbf{1/2} \\ \mathbf{1/2} \\ \mathbf{1/2} \end{bmatrix}.$$

We draw these values on the square graph:


X1=3/4
X4=1/4
y4=1/2
y1=1/2
y3=1/2
y2=1/2
X2=1/4
X3=0

Exam 1 review

This lecture is a review for the exam. The majority of the exam is on what we've learned about rectangular matrices.

Sample question 1

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-zero vectors in $\mathbb{R}^7$. They span a subspace of $\mathbb{R}^7$. What are the possible dimensions of that vector space?

The answer is 1, 2 or 3. The dimension can't be higher because a basis for this subspace has at most three vectors. It can't be 0 because the vectors are non-zero.

Sample question 2

Suppose a 5 by 3 matrix R in reduced row echelon form has $r = 3$ pivots.

1. What's the nullspace of R?

   Since the rank is 3 and there are 3 columns, there is no combination of the columns that equals $\mathbf{0}$ except the trivial one. $N(R) = \{\mathbf{0}\}$.

2. Let B be the 10 by 3 matrix $\begin{bmatrix} R \\ 2R \end{bmatrix}$. What's the reduced row echelon form of B?

   Answer: $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

3. What is the rank of B?

   Answer: 3.

4. What is the reduced row echelon form of $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$?

   When we perform row reduction we get:

$$\begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

   Then we might need to move some zero rows to the bottom of the matrix.

5. What is the rank of C?

   Answer: 6.

6. What is the dimension of the nullspace of C^T?

   $m = 10$ and $r = 6$ so $\dim N(C^T) = 10 - 6 = 4$.

Sample question 3

Suppose we know that $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and that:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a complete solution.

Note that in this problem we don't know what A is.

1. What is the shape of the matrix A?

   Answer: 3 by 3, because $\mathbf{x}$ and $\mathbf{b}$ both have three components.

2. What's the dimension of the row space of A?

   From the complete solution we can see that the dimension of the nullspace of A is 2, so the rank of A must be $3 - 2 = 1$.

3. What is A?

   Because the second and third components of the particular solution $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ are zero, we see that the first column vector of A must be $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   Knowing that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in the nullspace tells us that the third column of A must be $\mathbf{0}$. The fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace tells us that the second column must be the negative of the first. So,

   $$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

   If we had time, we could check that this A times $\mathbf{x}$ equals $\mathbf{b}$.

4. For what vectors b does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

   This equation has a solution exactly when $\mathbf{b}$ is in the column space of A, so when $\mathbf{b}$ is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. This makes sense; we know that the rank of A is 1 and the nullspace is large.

   In contrast, we might have had $r = m$ or $r = n$.

Sample question 4

Suppose:

$$B = CD = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Try to answer the questions below without performing this matrix multiplication CD.

1. Give a basis for the nullspace of B.

   The matrix B is 3 by 4, so $N(B) \subseteq \mathbb{R}^4$. Because $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible, the nullspace of B is the same as the nullspace of $D = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Matrix D is in reduced form, so its special solutions form a basis for $N(D) = N(B)$:

   $$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   These vectors are independent, and if time permits we can multiply to check that they are in $N(B)$.

2. Find the complete solution to $B\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   We can now describe any vector in the nullspace, so all we need to do is find a particular solution. There are many possible particular solutions; the simplest one is given below.

   One way to solve this is to notice that $C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and then find a vector $\mathbf{x}$ for which $D\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Another approach is to notice that the first column of $B = CD$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. In either case, we get the complete solution:

   $$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Again, we can check our work by multiplying.

Short questions

There may not be true/false questions on the exam, but it's a good idea to review these:

1. Given a square matrix A whose nullspace is just $\{\mathbf{0}\}$, what is the nullspace of A^T?

   $N(A^T)$ is also $\{\mathbf{0}\}$ because A is square.

2. Do the invertible matrices form a subspace of the vector space of 5 by 5 matrices?

   No. The sum of two invertible matrices may not be invertible. Also, 0 is not invertible, so is not in the collection of invertible matrices.

3. True or false: If $B^2 = 0$, then it must be true that $B = 0$.

   False. We could have $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. True or false: A system $A\mathbf{x} = \mathbf{b}$ of n equations with n unknowns is solvable for every right hand side $\mathbf{b}$ if the columns of A are independent.

   True. A is invertible, and $\mathbf{x} = A^{-1}\mathbf{b}$ is a (unique) solution.

5. True or false: If $m = n$ then the row space equals the column space.

   False. The dimensions are equal, but the spaces are not. A good example to look at is $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. True or false: The matrices A and $-A$ share the same four spaces.

   True, because whenever a vector $\mathbf{v}$ is in a space, so is $-\mathbf{v}$.

7. True or false: If A and B have the same four subspaces, then A is a multiple of B.

   A good way to approach this question is to first try to convince yourself that it isn't true – look for a counterexample. If A is 3 by 3 and invertible, then its row and column space are both $\mathbb{R}^3$ and its nullspaces are $\{\mathbf{0}\}$. If B is any other invertible 3 by 3 matrix it will have the same four subspaces, and it may not be a multiple of A. So we answer "false".

   It's good to ask how we could truthfully complete the statement "If A and B have the same four subspaces, then ..."

8. If we exchange two rows of A, which subspaces stay the same?

   The row space and the nullspace stay the same.

9. Why can't a vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be in the nullspace of A and also be a row of A?

   Because if $\mathbf{v}$ is the n^{th} row of A, the n^{th} component of the vector $A\mathbf{v}$ would be 14, not 0. The vector $\mathbf{v}$ could not be a solution to $A\mathbf{v} = \mathbf{0}$.

   In fact, we will learn that the row space is perpendicular to the nullspace.

18.06SC Unit 1 Exam

1 (24 pts.) This question is about an m by n matrix A for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about m and n and the rank r of A.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix A that fits the description in part (a).

2 (24 pts.) The 3 by 3 matrix A reduces to the identity matrix I by the following three row operations (in order):

E_{21} :	Subtract 4 (row 1) from row 2.
E_{31} :	Subtract 3 (row 1) from row 3.
E_{23} :	Subtract row 3 from row 2.

(a) Write the inverse matrix A^{-1} in terms of the E's. **Then compute $\boldsymbol{A^{-1}}$.**

(b) What is the original matrix A?

(c) What is the lower triangular factor L in $A = LU$?

3 (28 pts.) This 3 by 4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each* c find a basis for the column space of A.

(b) *For each* c find a basis for the nullspace of A.

(c) *For each* c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

4 (24 pts.) (a) If A is a 3 by 5 matrix, what information do you have about the nullspace of A?

(b) Suppose row operations on A lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of A.

(c) In the vector space M of all 3 by 3 matrices (you could call this a matrix space), what subspace S is spanned by all possible row reduced echelon forms R?

18.06SC Unit 1 Exam Solutions

1 (24 pts.) This question is about an m by n matrix A for which

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Give all possible information about m and n and the rank r of A.

(b) Find all solutions to $Ax = 0$ and **explain your answer**.

(c) Write down an example of a matrix A that fits the description in part (a).

Solution.

(a) $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has *one* solution $\Longrightarrow N(A) = \{0\}$ so $r = n$. (Also, $m = 3$ since $Ax \in \mathbb{R}^3$.)

$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solution $\Longrightarrow C(A) \neq \mathbb{R}^3$, so $r < m$.

There are two possibilities: $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

(b) Since $N(A) = \{0\}$ (because $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has **1** solution), there is a unique solution to $Ax = 0$, which is clearly $x = 0$. (Can be either $x = \begin{bmatrix} 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ depending on if $n = 1$ or $n = 2$.)

(c) A could be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (many more possibilities).

2 (24 pts.) The 3 by 3 matrix A reduces to the identity matrix I by the following three row operations (in order):

E_{21}: Subtract 4 (row 1) from row 2.

E_{31}: Subtract 3 (row 1) from row 3.

E_{23}: Subtract row 3 from row 2.

(a) Write the inverse matrix A^{-1} in terms of the E's. **Then compute A^{-1}.**

(b) What is the original matrix A?

(c) What is the lower triangular factor L in $A = LU$?

Solution.

(a) Apply the three operations to I, i.e. $A^{-1} = E_{23}E_{31}E_{21}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = A^{-1}$$

(b) Apply the inverse operations in reverse order to I, i.e. $A = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} = A$$

$$\text{Check} \quad \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $$L = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

3 (28 pts.) This 3 by 4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) *For each* c find a basis for the column space of A.

(b) *For each* c find a basis for the nullspace of A.

(c) *For each* c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

Solution.

(a) Elimination gives $\begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ so there are two cases:

If $c \neq 3$, $c-3$ is a pivot and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & \boxed{c-3} & -4 & -4 \\ 0 & 0 & \boxed{2} & 2 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

so a basis for $C(A)$ is the first three columns of A: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

If $c = 3$, $c - 3 = 0$ and $U = \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & 0 & \boxed{-4} & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

so take the first and third columns of A as a basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(b) If $c \neq 3$, the special solutions give $N(A) = \left\{ x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

If $c = 3$, the special solutions give $N(A) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(c) By inspection, $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is one particular solution (other correct answers)

for $c \neq 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

for $c = 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

4 (24 pts.) (a) If A is a 3 by 5 matrix, what information do you have about the nullspace of A?

(b) Suppose row operations on A lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of A.

(c) In the vector space M of all 3 by 3 matrices (you could call this a matrix space), what subspace S is spanned by all possible row reduced echelon forms R?

Solution.

(a) $N(A)$ has dimension *at least* 2 (and at most 5).

(b) (**7pts**) Columns 1, 4, 5 of A form a basis for $C(A)$.

($\approx$ **1pt**) Column 2 is $4 \times$ (Column 1); Column 3 is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) $A = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \right\}$, the set of upper triangular matrices.

(A basis of six echelon forms is

$$\left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & 1 \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & & 1 \\ & & \\ & & \end{bmatrix} \right\}.)$$