18.06SC Unit 3 Exam

1 (34 pts.)

(a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive ? zero ? negative ? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A ?

3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \,.$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1} b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \,.$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

18.06SC Unit 3 Exam Solutions

1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

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Solution.

(a) If $\sigma = I$ then $A = UV^{\mathrm{T}}$ = product of orthogonal matrices = orthogonal matrix.

2nd proof: All $\sigma_i = 1$ implies $A^{\mathrm{T}}A = I$. So A is orthogonal.

(A is *never* singular, and it won't always be symmetric — take $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $V = I$, for example. This also shows it can't be diagonal, or positive definite or semidefinite.)

(b) $B = H\Lambda H^{-1}$ with $\Lambda = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$

$(B+I)^{-1} = H(\Lambda+I)^{-1}H^{-1}$ with (same eigenvectors) $(\Lambda+I)^{-1} = \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/3 & \\ & & & 1/4 \end{bmatrix}$

(c) B is singular, symmetric, positive semidefinite.

C is symmetric positive definite.

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

Solution.

(a) The eigenvalues are $-1, 0, 1$ since A is triangular.

$$\lambda = -1 \text{ has } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda = 0 \text{ has } x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 1 \text{ has } x = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}.$$

Those vectors x are the columns of S (upper triangular!).

(b) $A = \bullet \Lambda S^{-1}$ and $A^{1001} = S\Lambda^{1001}S^{-1}$. Notice $\Lambda^{1001} = \Lambda$, $A^{1000} \neq I$ (A is singular) $(0^{1000} = 0 \neq 1)$.

e^{At} has e^{-1t}, $e^{0t} = 1$, e^{t} on its diagonal. *Proof using series:*

$\sum_0^\infty (At)^n/n!$ has triangular matrices so the diagonal has $\sum(-t)^n/n! = e^{-t}$, $\sum 0^n/n! = 1$, $\sum t^n/n! = e^t$.

Proof using $S\Lambda S^{-1}$*:*

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{bmatrix} \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) $A^{\mathrm{T}}A$ has 2 positive eigenvalues (it has rank 2, its eigenvalues can never be negative).

One eigenvalue is zero because $A^{\mathrm{T}}A$ is singular. And $3 - 2 = 1$.

(Or: $A^{\mathrm{T}}A$ is symmetric, so the eigenvalues have the same signs as the pivots.

Do elimination: the pivots are 1, 0, and $42 - 16 = 26$.)

3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n .$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1}b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n .$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

Solution.

(a) A^{-1} has eigenvalues $\dfrac{1}{\lambda_j}$ with the same eigenvectors

$$Aq_j = \lambda_j q_j \longrightarrow q_j = \lambda_j A^{-1} q_j \longrightarrow A^{-1} q_j = \frac{1}{\lambda_j}\, q_j\,.$$

(b) Multiply $b = c_1 q_1 + \cdots + c_n q_n$ by q_1^{T}.

Orthogonality gives $q_1^{\mathrm{T}} b = c_1 q_1^{\mathrm{T}} q_1$ so $c_1 = \dfrac{q_1^{\mathrm{T}} b}{q_1^{\mathrm{T}} q_1} = q_1^{\mathrm{T}} b.$

(c) Multiplying b by A^{-1} will multiply each q_i by $\dfrac{1}{\lambda_i}$ (part (a)). So c_i becomes

$$d_1 = \frac{c_1}{\lambda_1} \quad \left(= \frac{q_1^{\mathrm{T}} b}{\lambda_1 q_1^{\mathrm{T}} q_1} \text{ or } \frac{q_1^{\mathrm{T}} b}{\lambda_1} \right).$$