

Cramer's rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used.

Formula for A^{-1}

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Can we get a formula for the inverse of a 3 by 3 or n by n matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A} C^T$$

where C is the matrix of cofactors – please notice the transpose! Cofactors of row one of A go into column 1 of A^{-1} , and then we divide by the determinant.

The determinant of A involves products with n terms and the cofactor matrix involves products of $n - 1$ terms. A and $\frac{1}{\det A} C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

To more formally verify the formula, we'll check that $AC^T = (\det A)I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^n a_{1j} C_{j1} = \det A.$$

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of AC^T .

To finish proving that $AC^T = (\det A)I$, we just need to check that the off-diagonal entries of AC^T are zero. In the two by two case, multiplying the entries in row 1 of A by the entries in column 2 of C^T gives $a(-b) + b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of A and the last column of C^T equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A} C^T$.

This formula helps us answer questions about how the inverse changes when the matrix changes.

Cramer's Rule for $\mathbf{x} = A^{-1}\mathbf{b}$

We know that if $A\mathbf{x} = \mathbf{b}$ and A is nonsingular, then $\mathbf{x} = A^{-1}\mathbf{b}$. Applying the formula $A^{-1} = C^T / \det A$ gives us:

$$\mathbf{x} = \frac{1}{\det A} C^T \mathbf{b}.$$

Cramer's rule gives us another way of looking at this equation. To derive this rule we break \mathbf{x} down into its components. Because the i 'th component of $C^T \mathbf{b}$ is a sum of cofactors times some number, it is the determinant of some matrix B_j .

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the matrix created by starting with A and then replacing column j with \mathbf{b} , so:

$$\begin{aligned} B_1 &= \begin{bmatrix} & \text{last } n-1 \\ \mathbf{b} & \text{columns} \\ & \text{of } A \end{bmatrix} \quad \text{and} \\ B_n &= \begin{bmatrix} \text{first } n-1 & \\ \text{columns} & \mathbf{b} \\ \text{of } A & \end{bmatrix}. \end{aligned}$$

This agrees with our formula $x_1 = \frac{\det B_1}{\det A}$. When taking the determinant of B_1 we get a sum whose first term is b_1 times the cofactor C_{11} of A .

Computing inverses using Cramer's rule is usually less efficient than using elimination.

$|\det A| = \text{volume of box}$

Claim: $|\det A|$ is the volume of the box (*parallelepiped*) whose edges are the column vectors of A . (We could equally well use the row vectors, forming a different box with the same volume.)

If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^T Q = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

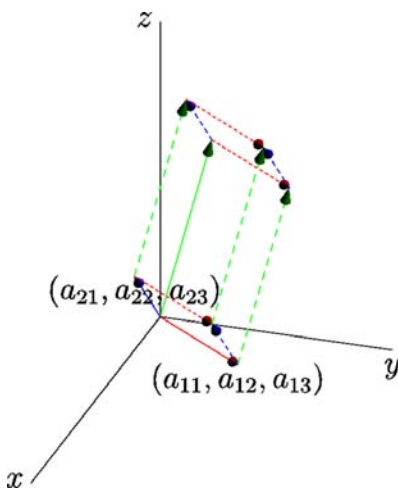


Figure 1: The box whose edges are the column vectors of A .

If we double the length of one column of A , we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Figure 2 illustrates why this should be true.

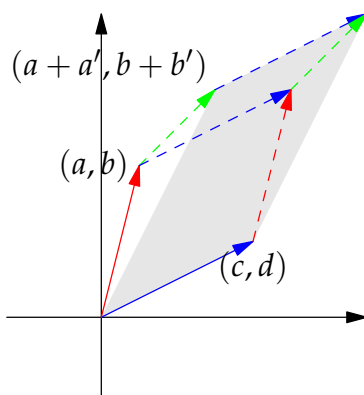


Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.

Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$. The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is:

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Exercises on Cramer's rule, inverse matrix, and volume

Problem 20.1: (5.3 #8. *Introduction to Linear Algebra*: Strang) Suppose

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Find its cofactor matrix C and multiply AC^T to find $\det(A)$.

$$C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \text{ and } AC^T = \text{_____}.$$

If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

Problem 20.2: (5.3 #28.) Spherical coordinates ρ, ϕ, θ satisfy

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta \text{ and } z = \rho \cos \phi.$$

Find the three by three matrix of partial derivatives:

$$\begin{bmatrix} \partial x / \partial \rho & \partial x / \partial \phi & \partial x / \partial \theta \\ \partial y / \partial \rho & \partial y / \partial \phi & \partial y / \partial \theta \\ \partial z / \partial \rho & \partial z / \partial \phi & \partial z / \partial \theta \end{bmatrix}.$$

Simplify its determinant to $J = \rho^2 \sin \phi$. In spherical coordinates,

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

is the volume of an infinitesimal "coordinate box."

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If you change $a_{1,3} = 4$ to 100, why is $\det(A)$ unchanged?

Solution: We fill in the cofactor matrix C and then multiply to obtain AC^T :

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$$

and

$$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I.$$

Since $AC^T = \det(A)I$, we have $\det(A) = 3$. If 4 is changed to 100, $\det(A)$ is unchanged because the cofactor of that entry is 0, and thus its value does not contribute to the determinant.

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Solution: The rows are formed by the partials of x, y , and z with respect to ρ, ϕ , and θ :

$$\begin{bmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{bmatrix}.$$

Expanding its determinant J along the bottom row, we get:

$$\begin{aligned} J &= \cos \phi \begin{bmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{bmatrix} \\ &\quad - (-\rho \sin \phi) \begin{bmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{bmatrix} + 0 \\ &= \cos \phi (\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \cos \phi \sin \phi \sin^2 \theta) \\ &\quad + \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \\ &= \cos \phi (\rho^2 \cos \phi \sin \phi (\cos^2 \theta + \sin^2 \theta)) + \rho \sin \phi (\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)) \\ &= \cos \phi (\rho^2 \cos \phi \sin \phi) + \rho^2 \sin^3 \phi \\ &= \rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ J &= \rho^2 \sin \phi. \end{aligned}$$