

18.06SC Final Exam

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- (a) Remembering that A is 3 by 4, find its row reduced echelon form R .
- (b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

You have enough information to find bases for one or more of these subspaces—find those bases.

- 2 (6+3+2=11 pts.)** (a) Find the inverse of a 3 by 3 upper triangular matrix U , with **nonzero** entries a, b, c, d, e, f . You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

- (b) Suppose the columns of U are eigenvectors of a matrix A . Show that A is also upper triangular.
- (c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^T$.

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows.

What can you say (*and explain why it is true*) about the comparison of

rank of A rank of the block matrix $\begin{bmatrix} A & B \end{bmatrix}$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r , what are the dimensions of these nullspaces?

Nullspace of A Nullspace of $\begin{bmatrix} A & A \end{bmatrix}$

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A ?

(b) Show that $A^T Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^T Ax = 0$ then obviously $x^T A^T Ax = 0$ and then (WHY?) $Ax = 0$.

(c) We now know that $A^T A$ is invertible. Explain why $B = (A^T A)^{-1} A^T$ is a one-sided inverse of A (which side of A ?). B is NOT a 2-sided inverse of A (*explain why not*).

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i .

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

- (a) Compute $x^T x$ and also $x^T A x$ in terms of the c 's and λ 's.
- (b) Looking at the ratio of $x^T A x$ in part (a) divided by $x^T x$ in part (a), what c 's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Conclusion: the ratio $x^T A x / x^T x$ is a maximum when x is _____.

- 6 (4+4+4=12 pts.)** (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u .
- (b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.
- (c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v .

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

- (b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?
- (c) Find the determinants of C and $C + I$ and $C + 2I$.

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

- (a) How do you find the best least squares solution \hat{x} to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for \hat{x} and also for $p = A\hat{x}$.
- (b) The projection p is in which fundamental subspace associated with A ? The error vector $e = b - p$ is in which fundamental subspace?
- (c) Find by any method the projection matrix P onto the column space of A :

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

- 9 (3+4+4=11 pts.)** This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

- (a) What are the determinants of A_2 and A_3 ?
- (b) The determinant of A_n is D_n . Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n :

$$(*) \quad D_n = a D_{n-1} + b D_{n-2}.$$

- (c) This equation (*) is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b , say how you would answer part (c) for any a and b) For 1 point, find D_5 .

18.06SC Final Exam Solutions

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- (a) Remembering that A is 3 by 4, find its row reduced echelon form R .
- (b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

You have enough information to find bases for one or more of these subspaces—find those bases.

Solution.

- (a) Each special solution tells us the solution to $Rx = 0$ when we set one free variable = 1 and the others = 0. Here, the third and fourth variables must be the two free variables,

and the other two are the pivots: $R = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now multiply out $Rx_1 = 0$ and $Rx_2 = 0$ to find the $*$'s: $R = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(The $$'s are just the negatives of the special solutions' pivot entries.)*

- (b) We know the nullspace $N(A)$ has $n - r = 4 - 2 = 2$ dimensions: the special solutions x_1, x_2 form a basis.

The row space $C(A^T)$ has $r = 2$ dimensions. It's orthogonal to $N(A)$, so just pick two linearly-independent vectors orthogonal to x_1 and x_2 to form a basis: for example,

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

(Or: $C(A^T) = C(R^T)$ is just the row space of R , so the first two rows are a basis. Same thing!)

The column space $C(A)$ has $r = 2$ dimensions (same as $C(A^T)$). We can't write down a basis because we don't know what A is, but we can say that the first two columns of A are a basis.

The left nullspace $N(A^T)$ has $m - r = 1$ dimension; it's orthogonal to $C(A)$, so any vector orthogonal to the first two columns of A (whatever they are) will be a basis.

- 2 (6+3+2=11 pts.)** (a) Find the inverse of a 3 by 3 upper triangular matrix U , with **nonzero** entries a, b, c, d, e, f . You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

- (b) Suppose the columns of U are eigenvectors of a matrix A . Show that A is also upper triangular.
- (c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^T$.

Solution.

- (a) *By elimination:* (We keep track of the elimination matrix E on one side, and the product EU on the other. When $EU = I$, then $E = U^{-1}$.)

$$\begin{aligned} \begin{bmatrix} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{bmatrix} &\rightsquigarrow \begin{bmatrix} 1 & b/a & c/a & 1/a & 0 & 0 \\ 0 & 1 & e/d & 0 & 1/d & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1/a & -b/ad & (be - cd)/adf \\ 0 & 1 & 0 & 0 & 1/d & -e/df \\ 0 & 0 & 1 & 0 & 0 & 1/f \end{bmatrix} = \begin{bmatrix} I & U^{-1} \end{bmatrix} \end{aligned}$$

By cofactors: (Take the minor, then “checkerboard” the signs to get the cofactor matrix, then transpose and divide by $\det(U) = adf$.)

$$\begin{aligned} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} &\rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ bf & af & 0 \\ be - cd & ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & 0 & 0 \\ -bf & af & 0 \\ be - cd & -ae & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} df & -bf & be - cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix} \rightsquigarrow \\ \begin{bmatrix} 1/a & -b/ad & (be - cd)/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{bmatrix} &= U^{-1} \end{aligned}$$

- (b) We have a complete set of eigenvectors for A , so we can diagonalize: $A = U\Lambda U^{-1}$. We know U is upper-triangular, and so is the diagonal matrix Λ , and we’ve just shown that U^{-1} is upper-triangular too. So their product A is also upper-triangular.
- (c) The columns aren’t orthogonal! (For example, the product $u_1^T u_2$ of the first two columns is $ab + 0d + 0 \cdot 0 = ab$, which is nonzero because we’re assuming all the entries are nonzero.)

- 3 (3+3+5=11 pts.)** (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } A \qquad \text{rank of the block matrix } \begin{bmatrix} A & B \end{bmatrix}$$

- (b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

- (c) If A is m by n of rank r , what are the dimensions of these nullspaces?

$$\text{Nullspace of } A \qquad \text{Nullspace of } \begin{bmatrix} A & A \end{bmatrix}$$

Solution.

- (a) All you can say is that $\text{rank } A \leq \text{rank } [A \ B]$. (A can have any number r of pivot columns, and these will all be pivot columns for $[A \ B]$; but there could be more pivot columns among the columns of B .)
- (b) Now $\text{rank } A = \text{rank } [A \ A^2]$. (Every column of A^2 is a linear combination of columns of A . For instance, if we call A 's first column a_1 , then Aa_1 is the first column of A^2 . So there are no new pivot columns in the A^2 part of $[A \ A^2]$.)
- (c) The nullspace $N(A)$ has dimension $n - r$, as always. Since $[A \ A]$ only has r pivot columns — the n columns we added are all duplicates — $[A \ A]$ is an m -by- $2n$ matrix of rank r , and its nullspace $N([A \ A])$ has dimension $2n - r$.

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

- (a) What can you say about the columns of A ?
- (b) Show that $A^T Ax$ is also never zero (except when $x = 0$) by explaining this key step:
If $A^T Ax = 0$ then obviously $x^T A^T Ax = 0$ and then (WHY?) $Ax = 0$.
- (c) We now know that $A^T A$ is invertible. Explain why $B = (A^T A)^{-1} A^T$ is a one-sided inverse of A (which side of A ?). B is NOT a 2-sided inverse of A (*explain why not*).

Solution.

- (a) $N(A) = 0$ so A has full column rank $r = n = 3$: the columns are linearly independent.
- (b) $x^T A^T Ax = (Ax)^T Ax$ is the squared length of Ax . The only way it can be zero is if Ax has zero length (meaning $Ax = 0$).
- (c) B is a left inverse of A , since $BA = (A^T A)^{-1} A^T A = I$ is the (3-by-3) identity matrix. B is not a right inverse of A , because AB is a 5-by-5 matrix but can only have rank 3. (In fact, $BA = A(A^T A)^{-1} A^T$ is the projection onto the (3-dimensional) column space of A .)

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i .

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

- (a) Compute $x^T x$ and also $x^T A x$ in terms of the c 's and λ 's.
- (b) Looking at the ratio of $x^T A x$ in part (a) divided by $x^T x$ in part (a), what c 's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Conclusion: the ratio $x^T A x / x^T x$ is a maximum when x is _____.

Solution.

(a)

$$\begin{aligned} x^T x &= (c_1 q_1^T + c_2 q_2^T + c_3 q_3^T)(c_1 q_1 + c_2 q_2 + c_3 q_3) \\ &= c_1^2 q_1^T q_1 + c_1 c_2 q_1^T q_2 + \dots + c_3 c_2 q_3^T q_2 + c_3^2 q_3^T q_3 \\ &= c_1^2 + c_2^2 + c_3^2. \end{aligned}$$

$$\begin{aligned} x^T A x &= (c_1 q_1^T + c_2 q_2^T + c_3 q_3^T)(c_1 A q_1 + c_2 A q_2 + c_3 A q_3) \\ &= (c_1 q_1^T + c_2 q_2^T + c_3 q_3^T)(c_1 \lambda_1 q_1 + c_2 \lambda_2 q_2 + c_3 \lambda_3 q_3) \\ &= c_1^2 \lambda_1 q_1^T q_1 + c_1 c_2 \lambda_2 q_1^T q_2 + \dots + c_3 c_2 \lambda_2 q_3^T q_2 + c_3^2 \lambda_3 q_3^T q_3 \\ &= c_1^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3. \end{aligned}$$

- (b) We maximize $(c_1^2 \lambda_1 + c_2^2 \lambda_2 + c_3^2 \lambda_3) / (c_1^2 + c_2^2 + c_3^2)$ when $c_1 = c_2 = 0$, so $x = c_3 q_3$ is a multiple of the eigenvector q_3 with the largest eigenvalue λ_3 .

(Also notice that the maximum value of this "Rayleigh quotient" $x^T A x / x^T x$ is the largest eigenvalue itself. This is another way of finding eigenvectors: maximize $x^T A x / x^T x$ numerically.)

- 6 (4+4+4=12 pts.)** (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u .
- (b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.
- (c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v .

Solution.

- (a) You could just write down $w = 0u + 0v = 0$ — that's perpendicular to everything! But a more useful choice is to subtract off just enough u so that $w = v - cu$ is perpendicular to u . That means $0 = w^T u = v^T u - cu^T u$, so $c = (v^T u)/(u^T u)$ and

$$w = v - \left(\frac{v^T u}{u^T u}\right)u.$$

- (b) We already know u and w are orthogonal; just normalize them! Take $q_1 = u/|u|$ and $q_2 = w/|w|$. Then solve for the columns r_1, r_2 of R : $Qr_1 = u$ so $r_1 = \begin{bmatrix} |u| \\ 0 \end{bmatrix}$, and $Qr_2 = v$ so $r_2 = \begin{bmatrix} c|u| \\ |w| \end{bmatrix}$. (Where $c = (v^T u)/(u^T u)$ as before.)

Then $Q = [q_1 \ q_2]$ and $R = [r_1 \ r_2]$.

- (c) $P = A(A^T A)^{-1} A^T = (QR)(R^T Q^T QR)^{-1} (R^T Q^T) = (QR)(R^T Q^T) = \underline{QQ^T}$.

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

- (b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?
- (c) Find the determinants of C and $C + I$ and $C + 2I$.

Solution.

- (a) Take the determinant of $C - \lambda I$ (I expanded by cofactors): $\lambda^4 - 1 = 0$. The roots of this “characteristic equation” are the eigenvalues: $+1, -1, i, -i$.

The eigenvalues of C^2 are just $\lambda^2 = \pm 1$ (two of each).

(Here’s a “guessing” approach. Since $C^4 = I$, all the eigenvalues λ^4 of C^4 are 1: so $\lambda = 1, -1, i, -i$ are the only possibilities. Just check to see which ones work. Then the eigenvalues of C^2 must be ± 1 .)

- (b) For any permutation matrix, $C^{-1} = C^T$: so

$$C^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $(C^2)^{-1} = C^2$ is itself.

- (c) The determinant of C is the product of its eigenvalues: $1(-1)i(-i) = \underline{-1}$.

Add 1 to every eigenvalue to get the eigenvalues of $C + I$ (if $C = SAS^{-1}$, then $C + I = S(\Lambda + I)S^{-1}$): $2(0)(1+i)(1-i) = \underline{0}$.

(Or let $\lambda = -1$ in the characteristic equation $\det(C - \lambda I)$.)

Add 2 to get the eigenvalues of $C + 2I$ (or let $\lambda = -2$): $3(1)(2+i)(2-i) = \underline{15}$.

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

- (a) How do you find the best least squares solution \hat{x} to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for \hat{x} and also for $p = A\hat{x}$.
- (b) The projection p is in which fundamental subspace associated with A ? The error vector $e = b - p$ is in which fundamental subspace?
- (c) Find by any method the projection matrix P onto the column space of A :

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

Solution.

(a)

$$Ax = b$$

$$\text{Least-squares "solution": } A^T A \hat{x} = A^T b$$

$$A^T A \text{ is invertible: } \hat{x} = (A^T A)^{-1} A^T b$$

$$\text{and } p = A\hat{x} \text{ is: } A\hat{x} = A(A^T A)^{-1} A^T b$$

- (b) $p = A\hat{x}$ is a linear combination of columns of A , so it's in the column space $C(A)$. The error $e = b - p$ is orthogonal to this space, so it's in the left nullspace $N(A^T)$.

- (c) I used $P = A(A^T A)^{-1} A^T$. Since $A^T A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, its inverse is $\begin{bmatrix} 1/10 & 0 \\ 0 & 1/10 \end{bmatrix} = \frac{1}{10}I$,

and

$$P = \frac{1}{10} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

- 9 (3+4+4=11 pts.)** This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

- (a) What are the determinants of A_2 and A_3 ?
- (b) The determinant of A_n is D_n . Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n :

$$(*) \quad D_n = a D_{n-1} + b D_{n-2}.$$

- (c) This equation (*) is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

>From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b , say how you would answer part (c) for any a and b) For 1 point, find D_5 .

Solution.

(a) $\det(A_2) = 3 \cdot 3 - 1 \cdot 2 = 7$ and $\det(A_3) = 3 \det(A_2) - 2 \cdot 1 \cdot 3 = 15$.

(b) $D_n = 3D_{n-1} + (-2)D_{n-2}$. (*Show your work.*)

(c) The trace of that matrix A is $a = 3$, and the determinant is $-b = 2$. So the characteristic equation of A is $\lambda^2 - a\lambda - b = 0$, which has roots (the eigenvalues of A)

$$\lambda_{\pm} = \frac{a \pm \sqrt{a^2 - 4(-b)}}{2} = \frac{3 \pm 1}{2} = 1 \text{ or } 2.$$

D_n grows at the same rate as the largest eigenvalue of A^n , $\lambda_+^n = 2^n$.

The final point: $D_5 = 3D_4 + 2D_3 = 3(3D_3 + 2D_2) + 2D_3 = 11D_3 + 6D_2 = 207$.