Column space and nullspace

In this lecture we continue to study subspaces, particularly the column space and nullspace of a matrix.

Review of subspaces

A vector space is a collection of vectors which is closed under linear combinations. In other words, for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in the space and any two real numbers c and d, the vector $c\mathbf{v} + d\mathbf{w}$ is also in the vector space. A subspace is a vector space contained inside a vector space.

A plane P containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and a line L containing $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are both subspaces of $\mathbb{R}^3$. The union $P \cup L$ of those two subspaces is generally not a subspace, because the sum of a vector in P and a vector in L is probably not contained in $P \cup L$. The intersection $S \cap T$ of two subspaces S and T is a subspace. To prove this, use the fact that both S and T are closed under linear combinations to show that their intersection is closed under linear combinations.

Column space of A

The *column space* of a matrix A is the vector space made up of all linear combinations of the columns of A.

Solving $A\mathbf{x} = \mathbf{b}$

Given a matrix A, for what vectors $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}.$$

Then $A\mathbf{x} = \mathbf{b}$ does not have a solution for every choice of $\mathbf{b}$ because solving $A\mathbf{x} = \mathbf{b}$ is equivalent to solving four linear equations in three unknowns. If there is a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ must be a linear combination of the columns of A. Only three columns cannot fill the entire four dimensional vector space – some vectors $\mathbf{b}$ cannot be expressed as linear combinations of columns of A.

Big question: what $\mathbf{b}$'s allow $A\mathbf{x} = \mathbf{b}$ to be solved?

A useful approach is to choose $\mathbf{x}$ and find the vector $\mathbf{b} = A\mathbf{x}$ corresponding to that solution. The components of $\mathbf{x}$ are just the coefficients in a linear combination of columns of A.

The system of linear equations $A\mathbf{x} = \mathbf{b}$ is *solvable* exactly when $\mathbf{b}$ is a vector in the *column space* of A.

For our example matrix A, what can we say about the column space of A? Are the columns of A *independent*? In other words, does each column contribute something new to the subspace?

The third column of A is the sum of the first two columns, so does not add anything to the subspace. The column space of our matrix A is a two dimensional subspace of $\mathbb{R}^4$.

Nullspace of A

The *nullspace* of a matrix A is the collection of all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $A\mathbf{x} = 0$.

The column space of the matrix in our example was a subspace of $\mathbb{R}^4$. The nullspace of A is a subspace of $\mathbb{R}^3$. To see that it's a vector space, check that any sum or multiple of solutions to $A\mathbf{x} = \mathbf{0}$ is also a solution: $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + \mathrm{A}\mathbf{x}_2 = \mathbf{0} + \mathbf{0}$ and $A(c\mathbf{x}) = cA\mathbf{x} = c(\mathbf{0})$.

In the example:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

the nullspace $N(A)$ consists of all multiples of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$; column 1 plus column 2 minus column 3 equals the zero vector. This nullspace is a line in $\mathbb{R}^3$.

Other values of b

The solutions to the equation:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

do not form a subspace. The zero vector is not a solution to this equation. The set of solutions forms a line in $\mathbb{R}^3$ that passes through the points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ but not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S}+\mathbf{T}$ contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S}+\mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S}+\mathbf{T}$ and $\mathbf{S}\cup\mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S}\cup\mathbf{T}$ *is* $\mathbf{S}+\mathbf{T}$.

Problem 6.2: (3.2 #18.) The plane $x-3y-z=12$ is parallel to the plane $x-3y-x=0$. One particular point on this plane is $(12,0,0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S} + \mathbf{T}$ contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

Solution:

a) Let $\mathbf{s}, \mathbf{s}'$ be vectors in $\mathbf{S}$, let $\mathbf{t}, \mathbf{t}'$ be vectors in $\mathbf{T}$, and let c be a scalar. Then

$$(\mathbf{s} + \mathbf{t}) + (\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{s}') + (\mathbf{t} + \mathbf{t}') \text{ and } c(\mathbf{s} + \mathbf{t}) = c\mathbf{s} + c\mathbf{t}.$$

Thus $\mathbf{S} + \mathbf{T}$ is closed under addition and scalar multiplication; in other words, it satisfies the two requirements for a vector space.

b) If $\mathbf{S}$ and $\mathbf{T}$ are distinct lines, then $\mathbf{S} + \mathbf{T}$ is a plane, whereas $\mathbf{S} \cup \mathbf{T}$ is only the two lines. The span of $\mathbf{S} \cup \mathbf{T}$ is the set of all combinations of vectors in this union of two lines. In particular, it contains all sums $\mathbf{s} + \mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$, and these sums form $\mathbf{S} + \mathbf{T}$.

Since $\mathbf{S} + \mathbf{T}$ contains both $\mathbf{S}$ and $\mathbf{T}$, it contains $\mathbf{S} \cup \mathbf{T}$. Further, $\mathbf{S} + \mathbf{T}$ is a vector space. So it contains all combinations of vectors in itself; in particular, it contains the span of $\mathbf{S} \cup \mathbf{T}$. Thus the span of $\mathbf{S} \cup \mathbf{T}$ is $\mathbf{S} + \mathbf{T}$.

Problem 6.2: (3.2 #18.) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - x = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution: The equation $x = 12 + 3y + z$ says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(= \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \boxed{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \boxed{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \boxed{1} \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution: $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$ contains all vectors that are in both nullspaces:

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix} = 0$$

if and only if $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.