Properties of determinants

Determinants

Now halfway through the course, we leave behind rectangular matrices and focus on square ones. Our next big topics are determinants and eigenvalues.

The *determinant* is a number associated with any square matrix; we'll write it as $\det A$ or $|A|$. The determinant encodes a lot of information about the matrix; the matrix is invertible exactly when the determinant is non-zero.

Properties

Rather than start with a big formula, we'll list the properties of the determinant. We already know that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; these properties will give us a formula for the determinant of square matrices of all sizes.

1. $\det I = 1$
2. If you exchange two rows of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.
3. (a) If we multiply one row of a matrix by t, the determinant is multiplied by t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

   (b) The determinant behaves like a linear function on the rows of the matrix:
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$. The determinant of a permutation matrix P is 1 or -1 depending on whether P exchanges an even or odd number of rows.

From these three properties we can deduce many others:

4. If two rows of a matrix are equal, its determinant is zero.

   This is because of property 2, the exchange rule. On the one hand, exchanging the two identical rows does not change the determinant. On the other hand, exchanging the two rows changes the sign of the determinant. Therefore the determinant must be 0.

5. If $i \neq j$, subtracting t times row i from row j doesn't change the determinant.

In two dimensions, this argument looks like:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c-ta & d-tb \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix} && \text{property 3(b)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t\begin{vmatrix} a & b \\ a & b \end{vmatrix} && \text{property 3(a)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} && \text{property 4.}
\end{aligned}$$

The proof for higher dimensional matrices is similar.

6. If A has a row that is all zeros, then $\det A = 0$.

   We get this from property 3 (a) by letting $t = 0$.

7. The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_1, d_2, ..., d_n$.

   Property 5 tells us that the determinant of the triangular matrix won't change if we use elimination to convert it to a diagonal matrix with the entries d_i on its diagonal. Then property 3 (a) tells us that the determinant of this diagonal matrix is the product $d_1 d_2 \cdots d_n$ times the determinant of the identity matrix. Property 1 completes the argument.

   Note that we cannot use elimination to get a diagonal matrix if one of the d_i is zero. In that case elimination will give us a row of zeros and property 6 gives us the conclusion we want.

8. $\det A = 0$ exactly when A is singular.

   If A is singular, then we can use elimination to get a row of zeros, and property 6 tells us that the determinant is zero.

   If A is not singular, then elimination produces a full set of pivots $d_1, d_2, ..., d_n$ and the determinant is $d_1 d_2 \cdots d_n \neq 0$ (with minus signs from row exchanges).

We now have a very practical formula for the determinant of a non-singular matrix. In fact, the way computers find the determinants of large matrices is to first perform elimination (keeping track of whether the number of row exchanges is odd or even) and then multiply the pivots:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}, \text{ if } a \neq 0, \text{ so}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a(d - \frac{c}{a}b) = ad - bc.$$

9. $\det AB = (\det A)(\det B)$

   This is very useful. Although the determinant of a sum does not equal the sum of the determinants, it is true that the determinant of a product equals the product of the determinants.

For example:

$$\det A^{-1} = \frac{1}{\det A},$$

because $A^{-1}A = 1$. (Note that if A is singular then A^{-1} does not exist and $\det A^{-1}$ is undefined.) Also, $\det A^2 = (\det A)^2$ and $\det 2A = 2^n \det A$ (applying property 3 to each row of the matrix). This reminds us of volume – if we double the length, width and height of a three dimensional box, we increase its volume by a multiple of $2^3 = 8$.

10. $\det A^T = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

This lets us translate properties (2, 3, 4, 5, 6) involving rows into statements about columns. For instance, if a column of a matrix is all zeros then the determinant of that matrix is zero.

To see why $|A^T| = |A|$, use elimination to write $A = LU$. The statement becomes $|U^T L^T| = |LU|$. Rule 9 then tells us $|U^T||L^T| = |L||U|$.

Matrix L is a lower triangular matrix with 1's on the diagonal, so rule 5 tells us that $|L| = |L^T| = 1$. Because U is upper triangular, rule 5 tells us that $|U| = |U^T|$. Therefore $|U^T||L^T| = |L||U|$ and $|A^T| = |A|$.

We have one loose end to worry about. Rule 2 told us that a row exchange changes the sign of the determinant. If it's possible to do seven row exchanges and get the same matrix you would by doing ten row exchanges, then we could prove that the determinant equals its negative. To complete the proof that the determinant is well defined by properties 1, 2 and 3 we'd need to show that the result of an odd number of row exchanges (odd permutation) can never be the same as the result of an even number of row exchanges (even permutation).

Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Problem 18.2: (5.1 #18.) Use row operations and the properties of the determinant to calculate the three by three "Vandermonde determinant":

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Exercises on properties of determinants

Problem 18.1: (5.1 #10. *Introduction to Linear Algebra:* Strang) If the entries in every row of a square matrix A add to zero, solve $A\mathbf{x} = \mathbf{0}$ to prove that $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean that $\det A = 1$?

Solution: If the entries of every row of A sum to zero, then $A\mathbf{x} = 0$ when $\mathbf{x} = (1, \ldots, 1)$ since each component of $A\mathbf{x}$ is the sum of the entries in a row of A. Since A has a non-zero nullspace, it is not invertible and $\det A = 0$.

If the entries of every row of A sum to one, then the entries in every row of $A - I$ sum to zero. Hence $A - I$ has a non-zero nullspace and $\det(A - I) = 0$.

If $\det(A - I) = 0$ it is **not** necessarily true that $\det A = 1$. For example, the rows of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sum to one but $\det A = -1$.

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$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Solution: Using row operations and properties of the determinant, we have:

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$$= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c-a & c^2-a^2 \end{bmatrix}$$

$$= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}$$

$$= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(c-b). \checkmark$$