5004 Homework 2

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Question 1:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{, and } T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$$

.

Answer

We know that

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

So in conclusion,

$$2T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 - 3 \\ 6 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Question 2:

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.
(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^n$.
(b) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{a}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$.

Answer

Assume that:

$$\boldsymbol{A}_i = [A_{ij}]_{j=1}^n, \boldsymbol{A}_i \in \mathbb{R}^n$$
$$b_i = \boldsymbol{b}_i, b_i \in \mathbb{R}$$
$$f_i(\boldsymbol{x}) = \langle \boldsymbol{A}_i, \boldsymbol{x} \rangle - b_i$$

We know that:

$$f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$$

$$f(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix}$$

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$$Df(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{A}_1^T \\ \vdots \\ \boldsymbol{A}_n^T \end{bmatrix} = \boldsymbol{A}$$

We know that Jacobian matrix is the differentiation of $f : \mathbb{R}^n \to \mathbb{R}^m$.
So In conclusion, the Jacobian matrix is $\boldsymbol{A}$.

(b) Assume that:

$$\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T = [x_i x_j]_{i=1,j=1}^{n,n}, \boldsymbol{X} \in \mathbb{R}^{n \times n}$$
$$\boldsymbol{X}_i = [x_i x_j]_{j=1}^{n}, \boldsymbol{X}_i \in \mathbb{R}^n$$
$$f_i(\boldsymbol{x}) = \langle \boldsymbol{X}_i, \boldsymbol{a} \rangle$$

if $i = k$

$$\frac{\partial f_i}{x_k} = \sum_{j=1}^{n} x_j a_j + x_k a_k$$

if $i \neq k$

$$\frac{\partial f_i}{\partial x_k} = x_i a_k$$

$$f(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix} = \begin{bmatrix} x_1 a_1 & x_1 a_2 & \cdots & x_1 a_n \\ \vdots & \vdots & \cdots & \vdots \\ x_n a_1 & x_n a_2 & \cdots & x_n a_n \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n} x_j a_j & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{n} x_j a_j \end{bmatrix} = \boldsymbol{x}\boldsymbol{a}^T + (\boldsymbol{x}^T\boldsymbol{a})\boldsymbol{I}$$

Question 3:

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^2 y, x - y)$ and $h = f \circ g = f(g(x, y))$. Find $\frac{\partial h}{\partial x}|_{x=2, y=-1}$ if $\frac{\partial f}{\partial x}|_{x=2, y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2, y=-1} = -2$. (Hint: use the chain rule)

Answer :

We know that:

$$g_1(x, y) = x^2 y$$
$$g_2(x, y) = x - y$$
$$h(x, y) = f(g_1(x, y), g_2(x, y))$$
$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial x}$$
$$\frac{\partial h}{\partial x}\Big|_{x=1, y=2} = 3\frac{\partial g_1}{\partial x} - 2\frac{\partial g_2}{\partial x} = 3 * (2xy)|_{x=2, y=-1} - 2 * 1 = -14$$

Question 4:

Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on $\mathbb{R}$, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t - s) f_2(s) ds$$

Let a, a_1, a_2 be real number.

(i) Prove the following identity:
$$f_1(t - a) * f_2(t) = f_1(t) * f_2(t - a) = f(t - a).$$

(ii) Prove the following identity:
$$f_1(t - a_1) * f_2(t - a_2) = f(t - a_1 - a_2).$$

Answer :

(i) Prove $f_1(t - a) * f_2(t) = f(t - a)$:

$$f(t - a) = \int_{-\infty}^{+\infty} f_1(t - s - a) f_2(s) ds$$
$$f_1'(t) = f_1(t - a)$$
$$f(t - a) = \int_{-\infty}^{+\infty} f_1'(t - s) f_2(s) ds = f_1'(t) * f_2(t) = f_1(t - a) * f_2(t)$$

Prove $f_1(t) * f_2(t - a) = f(t - a)$:

$$f(t - a) = \int_{-\infty}^{+\infty} f_1(t - s - a) f_2(s) ds$$
$$f_2'(t) = f_2(t - a)$$
$$f(t - a) = \int_{-\infty}^{+\infty} f_1(t - (s + a)) f_2(s) d(s + a) = \int_{-\infty}^{+\infty} f_1(t - (s + a)) f_2'(s + a) d(s + a)$$
$$= \int_{-\infty}^{+\infty} f_1(t - u) f_2'(u) du$$
$$= f_1(t) * f_2'(t)$$
$$= f_1() * f_2(t - a)$$

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(ii)

$$f_1(t - a_1) * f_2(t - a_2) = \int_{-\infty}^{+\infty} f_1(t - a_1 - s) f_2(s - a_2) ds$$

$$= \int_{-\infty}^{+\infty} f_1(t - a_1 - s) f_2(s - a_2) d(s - a_2)$$

$$= \int_{-\infty}^{+\infty} f_1(t - a_1 - a_2 - (s - a_2)) f_2(s - a_2) d(s - a_2)$$

$$= \int_{-\infty}^{+\infty} f_1(t - (a_1 + a_2) - u) f_2(u) du$$

$$= f(t - a_1 - a_2).$$

Question 5:

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{V_1}$, and $\langle \cdot, \cdot \rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \to V_2$ be a bounded linear operator.
(a) Let $S : V_2 \to V_1$ be an operator satisfying $\langle Tx, y \rangle_{V_2} = \langle x, Sy \rangle_{V_1}$ for any $x \in V_1$ and $y \in V_2$. Prove that S is a bounded linear operator. (Consequently, S is the adjoint of T, i.e., $S = T^*$)
(b) Prove that $(T^*)^* = T$.
(c) Prove that $\|T\| = \|T^*\|$.

Answer

(a)We know

$$\|Ax\|_{V_2} \le \|A\| \|x\|_{V_1} \tag{1}$$

We know, by using Cauchy–Schwarz inequality and (1):

$$\langle Tx, y \rangle_{V_2} \le \|Tx\|_{V_2} \|y\|_{V_2} \le \|T\| \|x\|_{V_1} \|y\|_{V_2}$$

We choose x as Sy

$$\langle TSy, y \rangle_{V_2} = \langle Sy, Sy \rangle_{V1} = \|Sy\|_{V_1}^2 \le \|TSy\|_{V_2} \|y\|_{V_2} \le \|T\| \|Sy\|_{V_1} \|y\|_{V_2}$$
$$\|Sy\|_{V_1}^2 \le \|T\| \|Sy\|_{V_1} \|y\|_{V_2}$$
$$\|Sy\|_{V_1} \le \|T\| \|y\|_{V_2} \le \infty$$

Thus, S is a bounded linear operator.

(b) We know:

$$\langle Tx, y \rangle_{V_2} = \langle x, T^* y \rangle_{V_1}$$

$$\langle x, T^* y \rangle_{V_1} = \langle (T^*)^* x, y \rangle_{V_2}$$
$$\langle Tx, y \rangle_{V_2} = \langle (T^*)^* x, y \rangle_{V_2}$$
$$T = (T^*)^*$$

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(c) Proof of

$$\|\boldsymbol{T}\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{Ty}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{Ty} \rangle$$

By using Cauchy Schwarz inequality

$$\langle \boldsymbol{x}, \boldsymbol{Ty} \rangle \leq \|\boldsymbol{x}\| \|\boldsymbol{Ty}\|$$
$$\|\boldsymbol{x}\| = 1$$
$$\langle \boldsymbol{x}, \boldsymbol{Ty} \rangle \leq \|\boldsymbol{Ty}\|$$
$$\|\boldsymbol{Ty}\| = \sup_{\|x\|=1} \langle \boldsymbol{x}, \boldsymbol{Ty} \rangle$$
$$\|\boldsymbol{T}\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{Ty}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{Ty} \rangle$$

Then

$$\|\boldsymbol{T}^*\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}^*\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y} \rangle$$
$$= \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{Tx}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \sup_{\|\boldsymbol{y}\|=1} \langle \boldsymbol{Tx}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \|\boldsymbol{Tx}\| = \|\boldsymbol{T}\|$$

In conclusion, $\|\boldsymbol{T}^*\| = \|\boldsymbol{T}\|$

Question 6:

Consider the vector space ℓ_∞ equipped with the norm $\|\cdot\|_\infty$. Define the operator $T : \ell_\infty \to \ell_\infty$ by $T(\{x_n\}_{n\in N}) = \{y_n\}_{n\in N}$ where $y_n = x_{n+1}$.
(a) Prove that T is a linear operator.
(b) Prove that T is a bounded operator.
(c) Prove that $\|T\| = 1$.

Answer

(a) Additivity: Proof:
For any $\{x_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \in \ell_\infty$

$T(\{x_n\}_{n\in\mathbb{N}} + \{z_n\}_{n\in\mathbb{N}}) = T(\{x_n + z_n\}_{n\in\mathbb{N}}) =$
$\{(x_n + z_n)_{n+1}\}_{n\in\mathbb{N}} = \{x_{n+1} + z_{n+1}\}_{n\in\mathbb{N}} = \{x_{n+1}\}_{n\in\mathbb{N}} + \{z_{n+1}\}_{n\in\mathbb{N}} = T(\{x_n\}_{n\in\mathbb{N}}) + T(\{z_n\}_{n\in\mathbb{N}})$

Homogeneity Proof:

$$T(\alpha\{x_n\}_{n\in\mathbb{N}}) = T(\{\alpha x_n\}_{n\in\mathbb{N}}) = \{\alpha x_{n+1}\}_{n\in\mathbb{N}} = \alpha\{x_{n+1}\}_{n\in\mathbb{N}} = \alpha T(\{x_n\}_{n\in\mathbb{N}})$$

(b) To prove T is bounded operator, we can prove, for a constance C, that :

$$\|T(\{x_n\}_{n\in\mathbb{N}})\|_\infty \leq C\|\{x_n\}_{n\in\mathbb{N}}\|_\infty, \forall \{x_n\}_{n\in\mathbb{N}} \in \ell_\infty$$
$$\|T(\{x_n\}_{n\in\mathbb{N}})\|_\infty = \sup_{n\in\mathbb{N}} |y_n| = \sup_{n\in\mathbb{N}} |x_{n+1}| \leq \sup_{n\in\mathbb{N}} |x_n| = \|\{x_n\}_{n\in\mathbb{N}}\|_\infty$$
$$\|T(\{x_n\})\|_\infty \leq \|\{x_n\}\|_\infty$$

In conclusion, T is a bounded operator.

(c) From (b), we know that $\|T(\{x_n\})\|_\infty \leq \|\{x_n\}\|_\infty$, s.t. $\|T\| \leq 1$.
Consider the sequence $\{x_n\}_{n\in\mathbb{N}} = \{1, 1, \cdots\} = \{x_{n+1}\}_{n\in\mathbb{N}}$ By definition, we know:

$$\|T\| \geq \frac{\|T(\{x_n\})\|_\infty}{\|\{x_n\}\|_\infty} = 1$$
$$1 \leq \|T\| \leq 1$$

Thus, we conclude that $\|T\| = 1$