5004 Homework 2

RONG Shuo

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Question: 1

1. Let $(V, \|\cdot\|)$ be a normed vector space.
   (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,

$$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

   (b) Let $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that

$$\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \|\boldsymbol{x}\|.$$

   (Hint: Use part (a).)

Answer (a):

We begin by recalling two fundamental properties of norms:

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \tag{1}$$
$$\|-\boldsymbol{x}\| = \|\boldsymbol{x}\| \tag{2}$$

For two vectors, $\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{y}$, according to (1), we get:

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\| \\ \|\boldsymbol{x}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\| \\ \|\boldsymbol{x}\| - \|\boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| \end{aligned}$$

Similarly, for two vectors, $\boldsymbol{y} - \boldsymbol{x}, \boldsymbol{x}$, we have:

$$\begin{aligned} \|\boldsymbol{y} - \boldsymbol{x} + \boldsymbol{x}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x}\| \\ \|\boldsymbol{y}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x}\| \\ \|\boldsymbol{y}\| - \|\boldsymbol{x}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\| \end{aligned}$$

We know (2), s.t.:

$$\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$$

Because of:

$$\begin{aligned} \|\boldsymbol{x}\| - \|\boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| \\ \|\boldsymbol{y}\| - \|\boldsymbol{x}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| \end{aligned}$$

We can conclude: For all $\boldsymbol{x}, \boldsymbol{y} \in V$,

$$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \tag{3}$$

Answer (b):

We know (3), and $0 \leq |x|, \forall x \in \mathbb{R}$, s.t.

$$0 \leq |\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \tag{4}$$

Since we know $\boldsymbol{x}_k \to \boldsymbol{x}$, we have:

$$\lim_{k \to \infty} \|\boldsymbol{x}_k - \boldsymbol{x}\| = 0 \tag{5}$$

According to (4), (5), we have:

$$\begin{aligned}
0 \leq |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq \|\boldsymbol{x}_k - \boldsymbol{x}\| \\
0 \leq \lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq \lim_{k \to \infty} \|\boldsymbol{x}_k - \boldsymbol{x}\| \\
0 \leq \lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq 0 \\
\lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &= 0 \\
\lim_{k \to \infty} \|\boldsymbol{x}_k\| &= \|\boldsymbol{x}\|
\end{aligned}$$

Question 2:

2. Let V be a vector space and $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ be a basis of V. If $\boldsymbol{u} = u_1\boldsymbol{a}_1 + \cdots + u_n\boldsymbol{a}_n$ and $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$ are two vectors in V, define

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1v_1 + \cdots + u_nv_n.$$

Show that this is an inner product on V.

Answer:

Positive Definite Property:

For any $\boldsymbol{u} \in V$, we know:

$$u_k^2 \geq 0, \forall u_k \in \mathbb{R} \tag{6}$$

s.t.

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = u_1^2 + \cdots + u_n^2 \geq 0$$

For any $\boldsymbol{u} = \boldsymbol{0}$, we have:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 + \cdots + 0 = 0$$

For any $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and (6), we have:

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{u} \rangle &= 0 \\
u_1^2 + \cdots + u_n^2 &= 0
\end{aligned}$$

Assume for the sake of contradiction that there exists at least one $u_j > 0$ for some $j \in \{1, 2, \cdots, n\}$.

Since $u_j > 0$, we can express it as:

$$u_j = c \qquad \text{where } c > 0$$
$$u_1 + u_2 + \cdots + u_n = u_1 + u_2 + \cdots + u_{j-1} + c + u_{j+1} + \cdots + u_n.$$
$$u_1 + u_2 + \cdots + u_{j-1} + c + u_{j+1} + \cdots + u_n \geq c, \qquad \text{Since } u_k \geq 0, \forall k$$

This contradicts the initial condition. $u_1 + u_2 + \cdots + u_n = 0$.
In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0, \forall \boldsymbol{u} \in V.$$
$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \iff \boldsymbol{u} = 0.$$

Symmetric:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$
$$u_1 v_1 + \cdots + u_n v_n = v_1 u_1 + \cdots + v_n u_n$$
$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = v_1 u_1 + \cdots + v_n u_n$$
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

Linearity:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i$$
$$\sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i = \sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i)$$
$$\sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i) = \alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i$$
$$\alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

In conclusion:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

Question 3:

3. Let V be a vector space with a norm $\| \cdot \|$ that satisfies the parallelogram identity

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2, \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}.$$

Note that we don;t have an inner product on V so far. For any $\boldsymbol{x}, \boldsymbol{y} \in V$, define

$$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

(a) Prove $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$.

(b) Prove $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$

(c) Prove $f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$

(d) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in V$

(e) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$

(c)(d)(e) together with some other technique can show that $f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}, \boldsymbol{z}) = \alpha f(\boldsymbol{x}, \boldsymbol{z}) + \beta f(\boldsymbol{y}, \boldsymbol{z})$. Therefore, we can finally prove f defines an inner product. This question showed that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be an induced by an inner product.

Answer

(a)

$$\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(4\|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(2\|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \|\boldsymbol{x}\|^2 \\
\|\boldsymbol{x}\|^2 &\geq 0
\end{aligned}$$

We know that:

$$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0} \tag{7}$$

s.t.

$$\|\boldsymbol{x}\|^2 = 0 \iff \boldsymbol{x} = \boldsymbol{0}$$

In conclusion, $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$.

(b) We know that:

$$\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{y}+\boldsymbol{x}\| \tag{8}$$

$$\begin{aligned}
f(\boldsymbol{y}, \boldsymbol{x}) &= \frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2) \\
\frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{y}) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)
\end{aligned}$$

$$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x}) \tag{9}$$

(c)

$$\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2+\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2=2\|\boldsymbol{x}+\boldsymbol{y}\|^2+2\|\boldsymbol{z}\|^2$$

$$f(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{x}+\boldsymbol{y}\|^2-\|\boldsymbol{z}\|^2)$$

$$f(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})=\frac{1}{4}(\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2) \tag{i}$$

$$\begin{aligned}&f(\boldsymbol{x},\boldsymbol{z})+f(\boldsymbol{y},\boldsymbol{z})\\&=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{z}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{z}\|^2)+\frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{y}\|^2-\|\boldsymbol{z}\|^2)\\&=\frac{1}{4}(\|\boldsymbol{x}+\boldsymbol{z}\|^2-\|\boldsymbol{x}-\boldsymbol{z}\|^2)+\frac{1}{4}(\|\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{y}-\boldsymbol{z}\|^2)\end{aligned} \tag{ii}$$

$$\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2=2\|\boldsymbol{x}+\boldsymbol{z}\|^2+2\|\boldsymbol{y}\|^2-\|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z}\|^2$$
$$\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2=2\|\boldsymbol{y}+\boldsymbol{z}\|^2+2\|\boldsymbol{x}\|^2-\|-\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2$$
$$\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2=\|\boldsymbol{x}+\boldsymbol{z}\|^2+\|\boldsymbol{y}+\boldsymbol{z}\|^2+\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{z}\|^2+\|-\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2)$$

$$\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2=2\|\boldsymbol{x}-\boldsymbol{z}\|^2+2\|\boldsymbol{y}\|^2-\|\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z}\|^2$$
$$\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2=2\|\boldsymbol{y}-\boldsymbol{z}\|^2+2\|\boldsymbol{x}\|^2-\|-\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2$$
$$\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2=\|\boldsymbol{x}-\boldsymbol{z}\|^2+\|\boldsymbol{y}-\boldsymbol{z}\|^2+\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z}\|^2+\|-x+\boldsymbol{y}-\boldsymbol{z}\|^2)$$

$$\begin{aligned}&\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}\|^2=\|\boldsymbol{x}+\boldsymbol{z}\|^2+\|\boldsymbol{y}+\boldsymbol{z}\|^2-(\|\boldsymbol{x}-\boldsymbol{z}\|^2+\|\boldsymbol{y}-\boldsymbol{z}\|^2)\\&=(\|\boldsymbol{x}+\boldsymbol{z}\|^2-\|\boldsymbol{x}-\boldsymbol{z}\|^2)+(\|\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{y}-\boldsymbol{z}\|^2)\end{aligned} \tag{iii}$$

We know (i),(ii),(iii) s.t.

$$f(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})=f(\boldsymbol{x},\boldsymbol{z})+f(\boldsymbol{y},\boldsymbol{z})$$

In conclusion, $f(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})=f(\boldsymbol{x},\boldsymbol{z})+f(\boldsymbol{y},\boldsymbol{z})$ for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in V$

(d) We know(2),

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2=-\|\boldsymbol{x}-\boldsymbol{y}\|^2+2\|\boldsymbol{x}\|^2+2\|\boldsymbol{y}\|^2,\forall\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}.$$

s.t.

$$\begin{aligned}f(\boldsymbol{x},\boldsymbol{y})&=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\&=\frac{1}{2}(-\|\boldsymbol{x}-\boldsymbol{y}\|^2+2\|\boldsymbol{x}\|^2+2\|\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\&=\frac{1}{2}(-\|\boldsymbol{x}-\boldsymbol{y}\|^2+\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2)\\&=-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\&=-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|-\boldsymbol{y}\|^2)\\&=-f(\boldsymbol{x},-\boldsymbol{y})\end{aligned}$$

$$f(\boldsymbol{x},\boldsymbol{y})=-f(\boldsymbol{x},-\boldsymbol{y})$$

we know (9), s.t.

$$f(\boldsymbol{x},\boldsymbol{y}) = -f(\boldsymbol{x},-\boldsymbol{y}) \equiv f(\boldsymbol{y},\boldsymbol{x}) = -f(-\boldsymbol{y},\boldsymbol{x})$$
$$f(-\boldsymbol{y},\boldsymbol{x}) = -f(\boldsymbol{y},\boldsymbol{x}) \equiv f(-\boldsymbol{x},\boldsymbol{y}) = -f(\boldsymbol{x},\boldsymbol{y})$$

In conclusion $f(-\boldsymbol{x},\boldsymbol{y}) = -f(\boldsymbol{x},\boldsymbol{y})$
(e) We know:

$$\|\boldsymbol{x}+\boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$$

s.t.

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 &\le \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| \\
\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 &\le 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| \\
f(\boldsymbol{x},\boldsymbol{y}) &\le \|\boldsymbol{x}\|\|\boldsymbol{y}\| \\
f(\boldsymbol{x},\boldsymbol{y})^2 &\le \|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2 \\
f(\boldsymbol{x},\boldsymbol{y})^2 &\le f(\boldsymbol{x},\boldsymbol{x})f(\boldsymbol{y},\boldsymbol{y})
\end{aligned}$$

In conclusion, $(f(\boldsymbol{x},\boldsymbol{y}))^2 \le f(\boldsymbol{x},\boldsymbol{x})f(\boldsymbol{y},\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y} \in V$

Question 4:

Consider the kernel $K(\boldsymbol{x},\boldsymbol{y}) = e^{\boldsymbol{x}^T\boldsymbol{y}}$ for $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature space H (a Hilbert space) and the feature map $\phi : \mathbb{R}^2 \to H$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = K(\boldsymbol{x},\boldsymbol{y})$
What is the inner product and the induced norm on H?
H might be infinite dimensional, and consider the Taylor's expansion $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$.

Answer :

$$\begin{aligned}
e^{\boldsymbol{x}^T\boldsymbol{y}} &= 1 + \frac{\sum_{i=1}^2 x_i y_i}{1!} + \frac{(\sum_{i=1}^2 x_i y_i)^2}{2!} + \frac{(\sum_{i=1}^2 x_i y_i)^3}{3!} + \cdots \\
&= \sum_{j=0}^{\infty} \frac{(\sum_{i=1}^2 x_i y_i)^j}{j!} = \sum_{j=0}^{\infty} \frac{(x_1y_1 + x_2y_2)^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{\sum_{i=0}^{j} \binom{j}{i} x_1^i y_1^i x_2^{j-i} y_2^{j-i}}{j!} \\
&= \sum_{j=0}^{\infty} \frac{\sum_{i=0}^{j} \binom{j}{i} x_1^i x_2^{j-i} y_1^i y_2^{j-i}}{j!}
\end{aligned}$$

We can see that:

$$\phi(\boldsymbol{x}) = \left(\frac{\binom{0}{0}x_1^0x_2^0}{0!}, \frac{\binom{1}{0}x_1^0x_2^1}{1!}, \frac{\binom{1}{1}x_1^1x_2^0}{1!}, \frac{\binom{2}{0}x_1^0x_2^2}{2!}, \frac{\binom{2}{1}x_1^1x_2^1}{2!}, \frac{\binom{2}{2}x_1^2x_2^0}{2!}, \frac{\binom{3}{0}x_1^0x_2^3}{3!}, \cdots \right)$$

$$\phi(\boldsymbol{x}) = \bigcup_{n=0}^{N} \left(\frac{\binom{n}{k} x_1^k x_2^{n-k}}{n!} : k = 0, 1, \cdots, n \right)$$

It is obvious that H with a standard inner product$\langle\cdot\rangle$:

$$\phi(\boldsymbol{x})^T\phi(\boldsymbol{y}) = \langle\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle = e^{\boldsymbol{x}^T\boldsymbol{y}} = K(\boldsymbol{x}, \boldsymbol{y})$$

We know the induced norm:

$$\begin{aligned} \|\phi(\boldsymbol{x})\|^2 = \langle\phi(\boldsymbol{x}), \phi(\boldsymbol{x})\rangle = e^{\boldsymbol{x}^T\boldsymbol{x}} \\ \|\phi(\boldsymbol{x})\| = e^{\frac{\boldsymbol{x}^T\boldsymbol{x}}{2}} \end{aligned}$$

Question 5:

Let $X \in \mathbb{R}^2$ be a two-dimensional input space, and consider the feature map: $\phi : X \to \mathbb{R}^3$ defined by

$$\phi(\boldsymbol{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2),$$

where $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. We are given the function K: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$K(\boldsymbol{x}, \boldsymbol{y}) = \langle\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^3$. Prove that K is a kernel function.

Answer:

$$\begin{aligned} K(\boldsymbol{x}, \boldsymbol{y}) &= \langle\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle = \phi(\boldsymbol{x})^T\phi(\boldsymbol{y}) \\ K(\boldsymbol{x}, \boldsymbol{y}) &= x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2 \\ &= (x_1y_1 + x_2y_2)^2 \\ &= (\boldsymbol{x}^T\boldsymbol{y})^2 \end{aligned}$$

We know polynomial kernel:

$$K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T\boldsymbol{y} + c)^d$$

with $c = 0$ and $d = 2$.

Polynomial kernels are known to be valid kernel functions. They satisfy the necessary properties:

1. Symmetry: $K(\boldsymbol{x}, \boldsymbol{y}) = K(\boldsymbol{y}, \boldsymbol{x})$ 2. Positive semi-definiteness: For any finite set of points $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$, the Gram matrix $K_{ij} = K(\boldsymbol{x}_i, \boldsymbol{x}_j)$ is positive semi-definite.

Therefore, $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T\boldsymbol{y})^2$ is indeed a valid kernel function.