5004 Homework2

RONG Shuo

October 15, 2024

Question: 1

- 1. Let $(V, \|\cdot\|)$ be a normed vector space.
 - (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,

$$|||x|| - ||y||| \le ||x - y||.$$

(b) Let $\{x_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $x\in V$. Prove that

$$\lim_{k\to\infty}\|\boldsymbol{x}_k\|=\|\boldsymbol{x}\|.$$

(Hint: Use part (a).)

Answer (a):

We know for every norm,

$$||x + y|| \le ||x|| + ||y||$$
 (1)

$$||-\boldsymbol{x}|| = ||\boldsymbol{x}|| \tag{2}$$

For two vectors, x - y, y, according to (1), we get:

$$\|x - y + y\| \le \|x - y\| + \|y\|$$
 $\|x\| \le \|x - y\| + \|y\|$
 $\|x\| - \|y\| \le \|x - y\|$

Similarly, for two vectors, y - x, we have:

$$\|y-x+x\| \le \|y-x\| + \|x\| \ \|y\| \le \|y-x\| + \|x\| \ \|y\| - \|x\| \le \|y-x\|$$

We know (2), s.t.:

$$\|y\| - \|x\| \le \|x - y\|$$

Because of:

$$\|x\| - \|y\| \le \|x - y\|$$

 $\|y\| - \|x\| \le \|x - y\|$

We can conclude: For all $x, y \in V$,

$$|||\boldsymbol{x}|| - ||\boldsymbol{y}||| \le ||\boldsymbol{x} - \boldsymbol{y}|| \tag{3}$$

Answer (b):

We know (3), and $0 \le |x|, \forall x \in \mathbb{R}$, s.t.

$$0 \le |||x|| - ||y||| \le ||x - y|| \tag{4}$$

Since we know $\boldsymbol{x}_k \to \boldsymbol{x}$, we have:

$$\lim_{k \to \infty} \|\boldsymbol{x}_k - \boldsymbol{x}\| = 0 \tag{5}$$

According to (4), (5), we have:

$$0 \le |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| \le \|\boldsymbol{x}_k - \boldsymbol{x}\|$$

$$0 \le \lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| \le \lim_{k \to \infty} \|\boldsymbol{x}_k - \boldsymbol{x}\|$$

$$0 \le \lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| \le 0$$

$$\lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|\| = 0$$

$$\lim_{k \to \infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|\| = \|\boldsymbol{x}\|$$

Question 2:

2. Let V be a vector space and $\{a_1, a_2, \dots, a_n\}$ be a basis of V. If $\mathbf{u} = u_1 \mathbf{a}_1 + \dots + u_n \mathbf{a}_n$ and $\mathbf{v} = v_1 \mathbf{a}_1 + \dots + v_n \mathbf{a}_n$ are two vectors in V, define

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \dots + u_n v_n.$$

Show that this is an inner product on V.

Answer:

Positive Definite Property:

For any $\boldsymbol{u} \in V$, we know:

$$u_k^2 \ge 0, \forall u_k \in \mathbb{R} \tag{6}$$

s.t.

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = u_1^2 + \dots + u_n^2 \ge 0$$

For any u = 0, we have:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 + \dots + 0 = 0$$

For any $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and (6), we have:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$$
$$u_1^2 + \dots + u_n^2 = 0$$

Assume for the sake of contradiction that there exists at least one $u_j > 0$ for some $j \in \{1, 2, \dots, n\}$.

Since $u_j > 0$, we can express it as:

$$u_{j} = c \qquad \text{where } c > 0$$

$$u_{1} + u_{2} + \dots + u_{n} = u_{1} + u_{2} + \dots + u_{j-1} + c + u_{j+1} + \dots + u_{n}.$$

$$u_{1} + u_{2} + \dots + u_{j-1} + c + u_{j+1} + \dots + u_{n} \geq c,$$
 Since $u_{k} \geq 0, \forall k$

This contradicts the initial condition. $u_1 + u_2 + \cdots + u_n = 0$. In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \ge 0, \forall \boldsymbol{u} \in V.$$

 $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \iff \boldsymbol{u} = 0.$

Symmetric:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \dots + u_n v_n$$

$$u_1 v_1 + \dots + u_n v_n = v_1 u_1 + \dots + v_n u_n$$

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = v_1 u_1 + \dots + v_n u_n$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

Linearity:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i$$

$$\sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i = \sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i)$$

$$\sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i) = \alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i$$

$$\alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

In conclusion:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

Question 3:

3. Let V be a vector space with a norm $\|\cdot\|$ that satisfies the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \forall x, y \in V.$$

Note that we don; thave an inner product on V so far. For any $x, y \in V$, define

$$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2} (\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

- (a) Prove f(x, x) > 0 for any $x \in V$, and f(x, x) = 0 if and only if x = 0.
 - (b) Prove $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$
 - (c) Prove $f(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$
 - (d) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in V$
 - (e) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$
- (c)(d)(e) together with some other technique can show that $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$. Therefore, we can finally prove f defines an inner product. This question showed that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be an induced by an inner product.

Answer

(a)

$$f(\boldsymbol{x}, \boldsymbol{x}) = \frac{1}{2} (\|\boldsymbol{x} + \boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2)$$

$$f(\boldsymbol{x}, \boldsymbol{x}) = \frac{1}{2} (4\|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2)$$

$$f(\boldsymbol{x}, \boldsymbol{x}) = \frac{1}{2} (2\|\boldsymbol{x}\|^2)$$

$$f(\boldsymbol{x}, \boldsymbol{x}) = \|\boldsymbol{x}\|^2$$

$$\|\boldsymbol{x}\|^2 \ge 0$$

We know that:

$$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0} \tag{7}$$

s.t.

$$\|\boldsymbol{x}\|^2 = 0 \iff \boldsymbol{x} = \boldsymbol{0}$$

In conclusion, $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$.

(b) We know that:

$$\|\boldsymbol{x} + \boldsymbol{y}\| = \|\boldsymbol{y} + \boldsymbol{x}\|$$

$$f(\boldsymbol{y}, \boldsymbol{x}) = \frac{1}{2}(\|\boldsymbol{y} + \boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2)$$

$$\frac{1}{2}(\|\boldsymbol{y} + \boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2) = \frac{1}{2}(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

$$f(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$
(8)

$$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x}) \tag{9}$$

(c)

$$\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^{2} + \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^{2} = 2\|\mathbf{x} + \mathbf{y}\|^{2} + 2\|\mathbf{z}\|^{2}$$

$$f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \frac{1}{2}(\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^{2} - \|\mathbf{x} + \mathbf{y}\|^{2} - \|\mathbf{z}\|^{2})$$

$$f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \frac{1}{4}(\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^{2} - \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^{2})$$
(i)

$$f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$$

$$= \frac{1}{2} (\|\boldsymbol{x} + \boldsymbol{z}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{z}\|^2) + \frac{1}{2} (\|\boldsymbol{y} + \boldsymbol{z}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{z}\|^2)$$

$$= \frac{1}{4} (\|\boldsymbol{x} + \boldsymbol{z}\|^2 - \|\boldsymbol{x} - \boldsymbol{z}\|^2) + \frac{1}{4} (\|\boldsymbol{y} + \boldsymbol{z}\|^2 - \|\boldsymbol{y} - \boldsymbol{z}\|^2)$$
(ii)

$$\begin{aligned} \|\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\|^2 &= 2\|\boldsymbol{x} + \boldsymbol{z}\|^2 + 2\|\boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{z}\|^2 \\ \|\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\|^2 &= 2\|\boldsymbol{y} + \boldsymbol{z}\|^2 + 2\|\boldsymbol{x}\|^2 - \|-\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\|^2 \\ \|\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\|^2 &= \|\boldsymbol{x} + \boldsymbol{z}\|^2 + \|\boldsymbol{y} + \boldsymbol{z}\|^2 + \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - \frac{1}{2}(\|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{z}\|^2 + \|-\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\|^2) \end{aligned}$$

$$||x + y - z||^{2} = 2||x - z||^{2} + 2||y||^{2} - ||x - y - z||^{2}$$

$$||x + y - z||^{2} = 2||y - z||^{2} + 2||x||^{2} - ||-x + y - z||^{2}$$

$$||x + y - z||^{2} = ||x - z||^{2} + ||y - z||^{2} + ||x||^{2} + ||y||^{2} - \frac{1}{2}(||x - y - z||^{2} + ||-x + y - z||^{2})$$

$$||x + y + z||^{2} - ||x + y - z||^{2} = ||x + z||^{2} + ||y + z||^{2} - (||x - z||^{2} + ||y - z||^{2})$$

$$= (||x + z||^{2} - ||x - z||^{2}) + (||y + z||^{2} - ||y - z||^{2})$$
(iii)

We know (i),(ii),(iii) s.t.

$$f(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$$

In conclusion, $f(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$ (d) We know(2),

$$\|x + y\|^2 = -\|x - y\|^2 + 2\|x\|^2 + 2\|y\|^2, \forall x, y \in V.$$

s.t.

$$f(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2} (\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

$$= \frac{1}{2} (-\|\boldsymbol{x} - \boldsymbol{y}\|^2 + 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

$$= \frac{1}{2} (-\|\boldsymbol{x} - \boldsymbol{y}\|^2 + \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)$$

$$= -\frac{1}{2} (\|\boldsymbol{x} - \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

$$= -\frac{1}{2} (\|\boldsymbol{x} - \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|-\boldsymbol{y}\|^2)$$

$$= -f(\boldsymbol{x}, -\boldsymbol{y})$$

$$f(\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, -\boldsymbol{y})$$

we know (9), s.t.

$$f(\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, -\boldsymbol{y}) \equiv f(\boldsymbol{y}, \boldsymbol{x}) = -f(-\boldsymbol{y}, \boldsymbol{x})$$

$$f(-\boldsymbol{y}, \boldsymbol{x}) = -f(\boldsymbol{y}, \boldsymbol{x}) \equiv f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$$

In conclusion $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ (e) We know:

$$\|x + y\| \le \|x\| + \|y\|$$

s.t.

$$||x + y||^{2} \le ||x||^{2} + ||y||^{2} + 2||x||||y||$$

$$||x + y||^{2} - ||x||^{2} - ||y||^{2} \le 2||x||||y||$$

$$f(x, y) \le ||x||||y||$$

$$f(x, y)^{2} \le ||x||^{2}||y||^{2}$$

$$f(x, y)^{2} \le f(x, x)f(y, y)$$

In conclusion, $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$