Question 1:

Determine whether each of the following scalar-valued functions of n-vectors is linear. If it is a linear function, give its inner product representation, ie., an n-vector $\boldsymbol{a}$ for which $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$ for all $\boldsymbol{x}$. If it is not linear, give specific $\boldsymbol{x}, \boldsymbol{y}$, α and β such that

$$f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) \neq \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}).$$

(a) The spread of values of the vector, defined as $f(\boldsymbol{x}) = max_k x_k - min_k x_k$.
(b) The difference of the last element and the first, $f(\boldsymbol{x}) = x_n - x_1$.

Answer :

(a)
Take $\boldsymbol{x} = (1, 2, 3)$ and $\alpha = 1, \beta = 1$ for example:

$$\begin{aligned}
f(\boldsymbol{x}) &= 3 - 1 = 2 \\
f(-\boldsymbol{x}) &= -1 + 3 = 2 \\
f(\boldsymbol{0}) &= 0 - 0 = 0 \\
f(\boldsymbol{x} + (-\boldsymbol{x})) &= f(\boldsymbol{0}) = 0 \\
f(\boldsymbol{x}) + f(-\boldsymbol{x}) &= 2 + 2 = 4 \\
f(\boldsymbol{x} + (-\boldsymbol{x})) &\neq f(\boldsymbol{x}) + f(-\boldsymbol{x})
\end{aligned}$$

In conclusion, $f(\boldsymbol{x}) = max_k x_k - min_k x_k$ is not a linear function.

(b)
We know:

$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} = (\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n)$$

$$\begin{aligned}
f(\boldsymbol{x}) &= x_n - x_1 \\
f(\boldsymbol{y}) &= y_n - y_1
\end{aligned}$$

$$\begin{aligned}
f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) &= \alpha x_n + \beta y_n - (\alpha x_1 + \beta y_1) \\
\alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}) &= \alpha(x_n - x_1) + \beta(y_n - y_1) \\
&= \alpha x_n + \beta y_n - (\alpha x_1 + \beta y_1) \\
f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) &= \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}).
\end{aligned}$$

Let's denote $\boldsymbol{e}_i$ as the vector in $\mathbb{R}^n$ where the i-th entry is equal to 1, and all other entries are equal to 0.

$$f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x} = (\boldsymbol{e}_n - \boldsymbol{e}_1)^T\boldsymbol{x}$$

In conclusion, $f(\boldsymbol{x}) = x_n - x_1$ is a linear function.