5004 Homework2

RONG Shuo

October 15, 2024

Question: 1

1. Let $(V, \|\cdot\|)$ be a normed vector space.
   (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,

$$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

   (b) Let $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that

$$\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \|\boldsymbol{x}\|.$$

   (Hint: Use part (a).)

Answer (a):

We know for every norm,

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \tag{1}$$
$$\|-\boldsymbol{x}\| = \|\boldsymbol{x}\| \tag{2}$$

For two vectors, $\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{y}$, according to (1), we get:

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\| \\ \|\boldsymbol{x}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\| \\ \|\boldsymbol{x}\| - \|\boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\|\end{aligned}$$

Similarly, for two vectors, $\boldsymbol{y} - \boldsymbol{x}, \boldsymbol{x}$, we have:

$$\begin{aligned}\|\boldsymbol{y} - \boldsymbol{x} + \boldsymbol{x}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x}\| \\ \|\boldsymbol{y}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x}\| \\ \|\boldsymbol{y}\| - \|\boldsymbol{x}\| &\leq \|\boldsymbol{y} - \boldsymbol{x}\|\end{aligned}$$

We know (2), s.t.:

$$\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$$

Because of:

$$\begin{aligned}\|\boldsymbol{x}\| - \|\boldsymbol{y}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\| \\ \|\boldsymbol{y}\| - \|\boldsymbol{x}\| &\leq \|\boldsymbol{x} - \boldsymbol{y}\|\end{aligned}$$

We can conclude: For all $\boldsymbol{x}, \boldsymbol{y} \in V$,

$$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \tag{3}$$

Answer (b):

We know (3), and $0 \leq |x|, \forall x \in \mathbb{R}$, s.t.

$$0 \leq |\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \tag{4}$$

Since we know $\boldsymbol{x}_k \to \boldsymbol{x}$, we have:

$$\lim_{k\to\infty} \|\boldsymbol{x}_k - \boldsymbol{x}\| = 0 \tag{5}$$

According to (4), (5), we have:

$$\begin{aligned}
0 \leq |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq \|\boldsymbol{x}_k - \boldsymbol{x}\| \\
0 \leq \lim_{k\to\infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq \lim_{k\to\infty} \|\boldsymbol{x}_k - \boldsymbol{x}\| \\
0 \leq \lim_{k\to\infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &\leq 0 \\
\lim_{k\to\infty} |\|\boldsymbol{x}_k\| - \|\boldsymbol{x}\|| &= 0 \\
\lim_{k\to\infty} \|\boldsymbol{x}_k\| &= \|\boldsymbol{x}\|
\end{aligned}$$

Question 2:

2. Let V be a vector space and $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ be a basis of V. If $\boldsymbol{u} = u_1\boldsymbol{a}_1 + \cdots + u_n\boldsymbol{a}_n$ and $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$ are two vectors in V, define

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_n v_n.$$

Show that this is an inner product on V.

Answer:

Positive Definite Property:

For any $\boldsymbol{u} \in V$, we know:

$$u_k^2 \geq 0, \forall u_k \in \mathbb{R} \tag{6}$$

s.t.

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = u_1^2 + \cdots + u_n^2 \geq 0$$

For any $\boldsymbol{u} = \mathbf{0}$, we have:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 + \cdots + 0 = 0$$

For any $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and (6), we have:

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{u} \rangle &= 0 \\
u_1^2 + \cdots + u_n^2 &= 0
\end{aligned}$$

Assume for the sake of contradiction that there exists at least one $u_j > 0$ for some $j \in \{1, 2, \cdots, n\}$.

Since $u_j > 0$, we can express it as:

$$u_j = c \quad \text{where } c > 0$$
$$u_1 + u_2 + \cdots + u_n = u_1 + u_2 + \cdots + u_{j-1} + c + u_{j+1} + \cdots + u_n.$$
$$u_1 + u_2 + \cdots + u_{j-1} + c + u_{j+1} + \cdots + u_n \geq c, \quad \text{Since } u_k \geq 0, \forall k$$

This contradicts the initial condition. $u_1 + u_2 + \cdots + u_n = 0$.

In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0, \forall \boldsymbol{u} \in V.$$
$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \iff \boldsymbol{u} = 0.$$

Symmetric:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$
$$u_1 v_1 + \cdots + u_n v_n = v_1 u_1 + \cdots + v_n u_n$$
$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle = v_1 u_1 + \cdots + v_n u_n$$
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

In conclusion:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$$

Linearity:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i$$
$$\sum_{i=1}^{n} (\alpha u_i + \beta v_i) w_i = \sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i)$$
$$\sum_{i=1}^{n} (\alpha u_i w_i) + (\beta v_i w_i) = \alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i$$
$$\alpha \sum_{i=1}^{n} u_i w_i + \beta \sum_{i=1}^{n} v_i w_i = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

In conclusion:

$$\langle \alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

Question 3:

3. Let V be a vector space with a norm $\|\cdot\|$ that satisfies the parallelogram identity

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2, \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}.$$

Note that we don;t have an inner product on V so far. For any $\boldsymbol{x}, \boldsymbol{y} \in V$, define

$$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)$$

(a) Prove $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$.
(b) Prove $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$
(c) Prove $f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$
(d) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in V$
(e) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$
(c)(d)(e) together with some other technique can show that $f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}, \boldsymbol{z}) = \alpha f(\boldsymbol{x}, \boldsymbol{z}) + \beta f(\boldsymbol{y}, \boldsymbol{z})$.

Answer

(a)

$$\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(4\|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \frac{1}{2}(2\|\boldsymbol{x}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{x}) &= \|\boldsymbol{x}\|^2 \\
\|\boldsymbol{x}\|^2 &\geq 0
\end{aligned}$$

We know that:

$$\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbf{0} \tag{7}$$

s.t.

$$\|\boldsymbol{x}\|^2 = 0 \iff \boldsymbol{x} = \mathbf{0}$$

In conclusion, $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for any $\boldsymbol{x} \in V$, and $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$.
(b) We know that:

$$\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{y}+\boldsymbol{x}\| \tag{8}$$

$$\begin{aligned}
f(\boldsymbol{y}, \boldsymbol{x}) &= \frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2) \\
\frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2) \\
f(\boldsymbol{x}, \boldsymbol{y}) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2)
\end{aligned}$$

$$f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x}) \tag{9}$$

(c)

$$\begin{aligned}
f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2 - \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{z}\|^2) \\
\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2 - \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{z}\|^2) &= \frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\|^2 - \|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{z}\|^2)
\end{aligned}$$

$$
\begin{aligned}
&f(\boldsymbol{x},\boldsymbol{z})+f(\boldsymbol{y},\boldsymbol{z})\\
&=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{z}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{z}\|^2)+\frac{1}{2}(\|\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{y}\|^2-\|\boldsymbol{z}\|^2)\\
&=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{z}\|^2+\|\boldsymbol{y}+\boldsymbol{z}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2-2\|\boldsymbol{z}\|^2)
\end{aligned}
$$

(d) We know(2),

$$
\|\boldsymbol{x}+\boldsymbol{y}\|^2=-\|\boldsymbol{x}-\boldsymbol{y}\|^2+2\|\boldsymbol{x}\|^2+2\|\boldsymbol{y}\|^2,\forall\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V}. \tag{10}
$$

s.t.

$$
\begin{aligned}
f(\boldsymbol{x},\boldsymbol{y})&=\frac{1}{2}(\|\boldsymbol{x}+\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\
&=\frac{1}{2}(-\|\boldsymbol{x}-\boldsymbol{y}\|^2+2\|\boldsymbol{x}\|^2+2\|\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\
&=\frac{1}{2}(-\|\boldsymbol{x}-\boldsymbol{y}\|^2+\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2)\\
&=-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|\boldsymbol{y}\|^2)\\
&=-\frac{1}{2}(\|\boldsymbol{x}-\boldsymbol{y}\|^2-\|\boldsymbol{x}\|^2-\|-\boldsymbol{y}\|^2)\\
&=-f(\boldsymbol{x},-\boldsymbol{y})
\end{aligned}
$$

$$
f(\boldsymbol{x},\boldsymbol{y})=-f(\boldsymbol{x},-\boldsymbol{y})
$$

we know (9), s.t.

$$
\begin{aligned}
f(\boldsymbol{x},\boldsymbol{y})=-f(\boldsymbol{x},-\boldsymbol{y})\equiv f(\boldsymbol{y},\boldsymbol{x})=-f(-\boldsymbol{y},\boldsymbol{x})\\
f(-\boldsymbol{y},\boldsymbol{x})=-f(\boldsymbol{y},\boldsymbol{x})\equiv f(-\boldsymbol{x},\boldsymbol{y})=-f(\boldsymbol{x},\boldsymbol{y})
\end{aligned}
$$

In conclusion $f(-\boldsymbol{x},\boldsymbol{y})=-f(\boldsymbol{x},\boldsymbol{y})$