

5004 Homework 2

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Question: 1

1. Let $(V, \|\cdot\|)$ be a normed vector space.

(a) Prove that, for all $\mathbf{x}, \mathbf{y} \in V$,

$$||\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|.$$

(b) Let $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$ be a convergent sequence in V with limit $\mathbf{x} \in V$. Prove that

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| = \|\mathbf{x}\|.$$

(Hint: Use part (a).)

Answer (a):

We begin by recalling two fundamental properties of norms:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \tag{1}$$

$$\|-\mathbf{x}\| = \|\mathbf{x}\| \tag{2}$$

For two vectors, $\mathbf{x} - \mathbf{y}, \mathbf{y}$, according to (1), we get:

$$\|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$$

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$$

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

Similarly, for two vectors, $\mathbf{y} - \mathbf{x}, \mathbf{x}$, we have:

$$\|\mathbf{y} - \mathbf{x} + \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|$$

$$\|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|$$

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\|$$

We know (2), s.t.:

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

Because of:

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

We can conclude: For all $\mathbf{x}, \mathbf{y} \in V$,

$$||\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\| \tag{3}$$

Answer (b):

We know (3), and $0 \leq |x|, \forall x \in \mathbb{R}$, s.t.

$$0 \leq ||\mathbf{x}| - |\mathbf{y}|| \leq \|\mathbf{x} - \mathbf{y}\| \quad (4)$$

Since we know $\mathbf{x}_k \rightarrow \mathbf{x}$, we have:

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}\| = 0 \quad (5)$$

According to (4), (5), we have:

$$\begin{aligned} 0 &\leq ||\mathbf{x}_k| - |\mathbf{x}|| \leq \|\mathbf{x}_k - \mathbf{x}\| \\ 0 &\leq \lim_{k \rightarrow \infty} ||\mathbf{x}_k| - |\mathbf{x}|| \leq \lim_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}\| \\ 0 &\leq \lim_{k \rightarrow \infty} ||\mathbf{x}_k| - |\mathbf{x}|| \leq 0 \\ \lim_{k \rightarrow \infty} ||\mathbf{x}_k| - |\mathbf{x}|| &= 0 \\ \lim_{k \rightarrow \infty} \|\mathbf{x}_k\| &= \|\mathbf{x}\| \end{aligned}$$

Question 2:

2. Let V be a vector space and $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ be a basis of V . If $\mathbf{u} = u_1\mathbf{a}_1 + \dots + u_n\mathbf{a}_n$ and $\mathbf{v} = v_1\mathbf{a}_1 + \dots + v_n\mathbf{a}_n$ are two vectors in V , define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \dots + u_nv_n.$$

Show that this is an inner product on V .

Answer:

Positive Definite Property:

For any $\mathbf{u} \in V$, we know:

$$u_k^2 \geq 0, \forall u_k \in \mathbb{R} \quad (6)$$

s.t.

$$\langle \mathbf{u}, \mathbf{u} \rangle = u_1^2 + \dots + u_n^2 \geq 0$$

For any $\mathbf{u} = \mathbf{0}$, we have:

$$\langle \mathbf{u}, \mathbf{u} \rangle = 0 + \dots + 0 = 0$$

For any $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and (6), we have:

$$\begin{aligned} \langle \mathbf{u}, \mathbf{u} \rangle &= 0 \\ u_1^2 + \dots + u_n^2 &= 0 \end{aligned}$$

Assume for the sake of contradiction that there exists at least one $u_j \neq 0$ for some $j \in \{1, 2, \dots, n\}$.

Since $u_j^2 > 0$, we can express it as:

$$\begin{aligned} u_j^2 &= c && \text{where } c > 0 \\ u_1^2 + u_2^2 + \cdots + u_n^2 &= u_1^2 + u_2^2 + \cdots + u_{j-1}^2 + c + u_{j+1}^2 + \cdots + u_n^2. \\ u_1^2 + u_2^2 + \cdots + u_{j-1}^2 + c + u_{j+1}^2 + \cdots + u_n^2 &\geq c, \end{aligned}$$

Since $u_k^2 \geq 0, \forall k$

This contradicts the initial condition. $u_1^2 + u_2^2 + \cdots + u_n^2 = 0$.

In conclusion:

$$\begin{aligned} \langle \mathbf{u}, \mathbf{u} \rangle &\geq 0, \forall \mathbf{u} \in V. \\ \langle \mathbf{u}, \mathbf{u} \rangle &= 0 \iff \mathbf{u} = 0. \end{aligned}$$



Symmetric:

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= u_1 v_1 + \cdots + u_n v_n \\ u_1 v_1 + \cdots + u_n v_n &= v_1 u_1 + \cdots + v_n u_n \\ \langle \mathbf{v}, \mathbf{u} \rangle &= v_1 u_1 + \cdots + v_n u_n \\ \langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{v}, \mathbf{u} \rangle \end{aligned}$$



In conclusion:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

Linearity:

$$\begin{aligned} \langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle &= \sum_{i=1}^n (\alpha u_i + \beta v_i) w_i \\ \sum_{i=1}^n (\alpha u_i + \beta v_i) w_i &= \sum_{i=1}^n (\alpha u_i w_i) + (\beta v_i w_i) \\ \sum_{i=1}^n (\alpha u_i w_i) + (\beta v_i w_i) &= \alpha \sum_{i=1}^n u_i w_i + \beta \sum_{i=1}^n v_i w_i \\ \alpha \sum_{i=1}^n u_i w_i + \beta \sum_{i=1}^n v_i w_i &= \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$



In conclusion:

$$\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$$

Question 3:

3. Let V be a vector space with a norm $\| \cdot \|$ that satisfies the parallelogram identity

$$\| \mathbf{x} + \mathbf{y} \|^2 + \| \mathbf{x} - \mathbf{y} \|^2 = 2\| \mathbf{x} \|^2 + 2\| \mathbf{y} \|^2, \forall \mathbf{x}, \mathbf{y} \in V.$$

Note that we don't have an inner product on V so far. For any $\mathbf{x}, \mathbf{y} \in V$, define

$$f(\mathbf{x}, \mathbf{y}) := \frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)$$

(a) Prove $f(\mathbf{x}, \mathbf{x}) \geq 0$ for any $\mathbf{x} \in V$, and $f(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(b) Prove $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$

(c) Prove $f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

(d) Prove $f(-\mathbf{x}, \mathbf{y}) = -f(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in V$

(e) Prove $(f(\mathbf{x}, \mathbf{y}))^2 \leq f(\mathbf{x}, \mathbf{x})f(\mathbf{y}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$

(c)(d)(e) together with some other technique can show that $f(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha f(\mathbf{x}, \mathbf{z}) + \beta f(\mathbf{y}, \mathbf{z})$. Therefore, we can finally prove f defines an inner product. This question showed that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be induced by an inner product.

Answer

(a)

$$f(\mathbf{x}, \mathbf{x}) = \frac{1}{2}(\|\mathbf{x} + \mathbf{x}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2)$$

$$f(\mathbf{x}, \mathbf{x}) = \frac{1}{2}(4\|\mathbf{x}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{x}\|^2)$$

$$f(\mathbf{x}, \mathbf{x}) = \frac{1}{2}(2\|\mathbf{x}\|^2)$$

$$f(\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2$$

$$\|\mathbf{x}\|^2 \geq 0$$

We know that:

$$\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0} \quad (7)$$

s.t.

$$\|\mathbf{x}\|^2 = 0 \iff \mathbf{x} = \mathbf{0}$$

In conclusion, $f(\mathbf{x}, \mathbf{x}) \geq 0$ for any $\mathbf{x} \in V$, and $f(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(b) We know that:

$$\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{y} + \mathbf{x}\| \quad (8)$$

$$f(\mathbf{y}, \mathbf{x}) = \frac{1}{2}(\|\mathbf{y} + \mathbf{x}\|^2 - \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2)$$

$$\frac{1}{2}(\|\mathbf{y} + \mathbf{x}\|^2 - \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2) = \frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)$$

$$f(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2)$$

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x}) \quad (9)$$

(c)

$$\begin{aligned}\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 + \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2 &= 2\|\mathbf{x} + \mathbf{y}\|^2 + 2\|\mathbf{z}\|^2 \\ f(\mathbf{x} + \mathbf{y}, \mathbf{z}) &= \frac{1}{2}(\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{z}\|^2) \\ f(\mathbf{x} + \mathbf{y}, \mathbf{z}) &= \frac{1}{4}(\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2)\end{aligned}\tag{i}$$

$$\begin{aligned}f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z}) &= \frac{1}{2}(\|\mathbf{x} + \mathbf{z}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{z}\|^2) + \frac{1}{2}(\|\mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{y}\|^2 - \|\mathbf{z}\|^2) \\ &= \frac{1}{4}(\|\mathbf{x} + \mathbf{z}\|^2 - \|\mathbf{x} - \mathbf{z}\|^2) + \frac{1}{4}(\|\mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{y} - \mathbf{z}\|^2)\end{aligned}\tag{ii}$$

$$\begin{aligned}\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 &= 2\|\mathbf{x} + \mathbf{z}\|^2 + 2\|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y} + \mathbf{z}\|^2 \\ \|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 &= 2\|\mathbf{y} + \mathbf{z}\|^2 + 2\|\mathbf{x}\|^2 - \|\mathbf{x} - \mathbf{y} + \mathbf{z}\|^2 \\ \|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 &= \|\mathbf{x} + \mathbf{z}\|^2 + \|\mathbf{y} + \mathbf{z}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \frac{1}{2}(\|\mathbf{x} - \mathbf{y} + \mathbf{z}\|^2 + \|\mathbf{x} - \mathbf{y} - \mathbf{z}\|^2) \\ \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2 &= 2\|\mathbf{x} - \mathbf{z}\|^2 + 2\|\mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y} - \mathbf{z}\|^2 \\ \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2 &= 2\|\mathbf{y} - \mathbf{z}\|^2 + 2\|\mathbf{x}\|^2 - \|\mathbf{x} - \mathbf{y} - \mathbf{z}\|^2 \\ \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2 &= \|\mathbf{x} - \mathbf{z}\|^2 + \|\mathbf{y} - \mathbf{z}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \frac{1}{2}(\|\mathbf{x} - \mathbf{y} - \mathbf{z}\|^2 + \|\mathbf{x} - \mathbf{y} + \mathbf{z}\|^2) \\ \|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{x} + \mathbf{y} - \mathbf{z}\|^2 &= \|\mathbf{x} + \mathbf{z}\|^2 + \|\mathbf{y} + \mathbf{z}\|^2 - (\|\mathbf{x} - \mathbf{z}\|^2 + \|\mathbf{y} - \mathbf{z}\|^2) \\ &= (\|\mathbf{x} + \mathbf{z}\|^2 - \|\mathbf{x} - \mathbf{z}\|^2) + (\|\mathbf{y} + \mathbf{z}\|^2 - \|\mathbf{y} - \mathbf{z}\|^2)\end{aligned}\tag{iii}$$

We know (i),(ii),(iii) s.t.

$$f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z})$$

In conclusion, $f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

(d) We know(2),

$$\|\mathbf{x} + \mathbf{y}\|^2 = -\|\mathbf{x} - \mathbf{y}\|^2 + 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2, \forall \mathbf{x}, \mathbf{y} \in V.$$

s.t.

$$\begin{aligned}f(\mathbf{x}, \mathbf{y}) &= \frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2) \\ &= \frac{1}{2}(-\|\mathbf{x} - \mathbf{y}\|^2 + 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2) \\ &= \frac{1}{2}(-\|\mathbf{x} - \mathbf{y}\|^2 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) \\ &= -\frac{1}{2}(\|\mathbf{x} - \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2) \\ &= -\frac{1}{2}(\|\mathbf{x} - \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2) \\ &= -f(\mathbf{x}, -\mathbf{y}) \\ f(\mathbf{x}, \mathbf{y}) &= -f(\mathbf{x}, -\mathbf{y})\end{aligned}$$

we know (9), s.t.

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}) &= -f(\mathbf{x}, -\mathbf{y}) \equiv f(\mathbf{y}, \mathbf{x}) = -f(-\mathbf{y}, \mathbf{x}) \\ f(-\mathbf{y}, \mathbf{x}) &= -f(\mathbf{y}, \mathbf{x}) \equiv f(-\mathbf{x}, \mathbf{y}) = -f(\mathbf{x}, \mathbf{y}) \end{aligned}$$

In conclusion $f(-\mathbf{x}, \mathbf{y}) = -f(\mathbf{x}, \mathbf{y})$

(e) We know:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

s.t.

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| \\ \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 &\leq 2\|\mathbf{x}\|\|\mathbf{y}\| \\ f(\mathbf{x}, \mathbf{y}) &\leq \|\mathbf{x}\|\|\mathbf{y}\| \\ f(\mathbf{x}, \mathbf{y})^2 &\leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \\ f(\mathbf{x}, \mathbf{y})^2 &\leq f(\mathbf{x}, \mathbf{x})f(\mathbf{y}, \mathbf{y}) \end{aligned}$$



In conclusion, $(f(\mathbf{x}, \mathbf{y}))^2 \leq f(\mathbf{x}, \mathbf{x})f(\mathbf{y}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$

Question 4:

Consider the kernel $K(\mathbf{x}, \mathbf{y}) = e^{\mathbf{x}^T \mathbf{y}}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Find an explicit feature space H (a Hilbert space) and the feature map $\phi: \mathbb{R}^2 \rightarrow H$ satisfying $\langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle = K(\mathbf{x}, \mathbf{y})$

What is the inner product and the induced norm on H ?

H might be infinite dimensional, and consider the Taylor's expansion $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$

Answer :

$$e^{\mathbf{x}^T \mathbf{y}} = \sum_{n=0}^{\infty} \frac{(\mathbf{x}^T \mathbf{y})^n}{n!}$$

And we know, the feature map of kernel function $K(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{y}$ is $\phi(\mathbf{x}) = \mathbf{x}$

For $K(\mathbf{x}, \mathbf{y}) = K_1(\mathbf{x}, \mathbf{y}) \cdot K_2(\mathbf{x}, \mathbf{y})$, the feature map is $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) \otimes \phi_2(\mathbf{x})$

As a result, we can define the feature map as

$$\begin{aligned} \phi(\mathbf{x}) &= \left(1, \mathbf{x}, \frac{1}{\sqrt{2!}} \mathbf{x} \otimes \mathbf{x}, \frac{1}{\sqrt{3!}} \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}, \dots \right) \\ &= (1, x_1, x_2, \frac{1}{\sqrt{2!}} x_1 x_1, \dots) \end{aligned}$$



You need to express the k -th term of $\phi(\mathbf{x})$.

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{x} \otimes \mathbf{x} = (x_1 x_1, x_1 x_2, x_2 x_1, x_2 x_2) \dots$

It is obvious that H with a standard inner product $\langle \cdot \rangle$:

$$\phi(\mathbf{x})^T \phi(\mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle = e^{\mathbf{x}^T \mathbf{y}} = K(\mathbf{x}, \mathbf{y})$$

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We know the induced norm:

$$\|\phi(\mathbf{x})\|^2 = \langle \phi(\mathbf{x}), \phi(\mathbf{x}) \rangle = e^{\mathbf{x}^T \mathbf{x}}$$

$$\|\phi(\mathbf{x})\| = e^{\frac{\mathbf{x}^T \mathbf{x}}{2}}$$



Question 5:

Let $X \in \mathbb{R}^2$ be a two-dimensional input space, and consider the feature map: $\phi : X \rightarrow \mathbb{R}^3$ defined by

$$\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2),$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. We are given the function $K: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$K(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^3 . Prove that K is a kernel function.

Answer:

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) &= \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle = \phi(\mathbf{x})^T \phi(\mathbf{y}) \\ K(\mathbf{x}, \mathbf{y}) &= x_1^2 y_1^2 + x_2^2 y_2^2 + 2x_1 x_2 y_1 y_2 \\ &= (x_1 y_1 + x_2 y_2)^2 \\ &= (\mathbf{x}^T \mathbf{y})^2 \end{aligned}$$

We know polynomial kernel:

$$K(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + c)^d$$

with $c = 0$ and $d = 2$.

Polynomial kernels are known to be valid kernel functions. They satisfy the necessary properties:

1. Symmetry: $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ 2. Positive semi-definiteness: For any finite set of points $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, the Gram matrix $K_{ij} = K(\mathbf{x}_i, \mathbf{x}_j)$ is positive semi-definite.

Therefore, $K(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y})^2$ is indeed a valid kernel function.

You need to formally prove that K is PSD, like $\mathbf{C}^T \mathbf{K} \mathbf{C} \geq 0$

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