5004 Homework 2

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Question 1:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{, and } T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$$

.

Answer

We know that

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

So in conclusion,

$$2T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-3 \\ 6+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Question 2:

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.
(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^n$.
(b) $f : R^n \to \mathbb{R}^n$ is defined by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{a}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$.

Answer

Question 3:

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^2y, x - y)$ and $h = f \circ g = f(g(x, y))$. Find $\frac{\partial h}{\partial x}|_{x=1,y=2}$ if $\frac{\partial f}{\partial x}|_{x=2,y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2,y=-1} = -2$. (Hint: use the chain rule)

0.1 Answer :

Question 4:

Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on $\mathbb{R}$, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t-s) f_2(s) ds$$

Let a, a_1, a_2 be real number.

(i) Prove the following identity:

$$f_1(t-a) * f_2(t) = f_1(t) * f_2(t-a) = f(t-a).$$

(ii) Prove the following identity:

$$f_1(t-a_1) * f_2(t-a_2) = f(t-a_1-a_2).$$

Question 5:

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{V_1}$, and $\langle \cdot, \cdot \rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \to V_2$ be a bounded linear operator.
(a) Let $S : V_2 \to V_1$ be an operator satisfying $\langle T\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} = \langle \boldsymbol{x}, S\boldsymbol{y} \rangle_{V_1}$ for any $\boldsymbol{x} \in V_1$ and $\boldsymbol{y} \in V_2$. Prove that S is a bounded linear operator. (COnsequently, S is the adjoint of T, i.e., $S = T^*$)
(b) Prove that $(T^*)^* = T$.
(c) Prove that $\|T\| = \|T^*\|$.

Answer

Question 6:

Consider the vector space ℓ_∞ equipped with the norm $\| \cdot \|_\infty$. Define the operator $T : \ell_\infty \to \ell_\infty$ by $T(\{x_n\}_{n \in N}) = \{y_n\}_{n \in N}$ where $y_n = x_{n+1}$.
(a) Prove that T is a linear operator.
(b) Prove that T is a bounded operator.
(c) Prove that $\|T\| = 1$.

Answer