5004 Homework 2

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Question 1:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ , and } T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$$

.

Answer

We know that

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

So in conclusion,

$$2T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-3 \\ 6+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Question 2:

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.
(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^n$.
(b) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{a}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$.

Answer

Assume that:

$$\begin{aligned} \boldsymbol{A}_i &= [A_{ij}]_{j=1}^n, \boldsymbol{A}_i \in \mathbb{R}^n \\ b_i &= \boldsymbol{b}_i, b_i \in \mathbb{R} \\ f_i(\boldsymbol{x}) &= \langle \boldsymbol{A}_i, \boldsymbol{x} \rangle - b_i \end{aligned}$$

We know that:

$$
\begin{aligned}
f(\boldsymbol{x}) &= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\
f(\boldsymbol{x}) &= \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix} \\
Df(\boldsymbol{x}) &= \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix} \\
Df(\boldsymbol{x}) &= \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix} \\
Df(\boldsymbol{x}) &= \begin{bmatrix} \boldsymbol{A}_1^T \\ \vdots \\ \boldsymbol{A}_n^T \end{bmatrix} = \boldsymbol{A}
\end{aligned}
$$

We know that Jacobian matrix is the differentiation of $f : \mathbb{R}^n \to \mathbb{R}^m$.

So In conclusion, the Jacobian matrix is $\boldsymbol{A}$.

(b) Assume that:

$$
\begin{aligned}
\boldsymbol{X} &= \boldsymbol{x}\boldsymbol{x}^T = [x_i x_j]_{i=1,j=1}^{n,n}, \boldsymbol{X} \in \mathbb{R}^{n\times n} \\
\boldsymbol{X}_i &= [x_i x_j]_{j=1}^{n}, \boldsymbol{X}_i \in \mathbb{R}^n \\
f_i(\boldsymbol{x}) &= \langle \boldsymbol{X}_i, \boldsymbol{a} \rangle
\end{aligned}
$$

if $i = k$

$$
\frac{\partial f_i}{x_k} = \sum_{j=1}^{n} x_j a_j + x_k a_k
$$

if $i \neq k$

$$
\frac{\partial f_i}{\partial x_k} = x_i a_k
$$

$$
\begin{aligned}
f(\boldsymbol{x}) &= \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix} \\
Df(\boldsymbol{x}) &= \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix} = \begin{bmatrix} x_1 a_1 & x_1 a_2 & \cdots & x_1 a_n \\ \vdots & \vdots & \dots & \vdots \\ x_n a_1 & x_n a_2 & \cdots & x_n a_n \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n x_j a_j & 0 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^n x_j a_j \end{bmatrix} = \boldsymbol{x}\boldsymbol{a}^T + (\boldsymbol{x}^T\boldsymbol{a})\boldsymbol{I}
\end{aligned}
$$

Question 3:

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^2 y, x - y)$ and $h = f \circ g = f(g(x, y))$. Find $\frac{\partial h}{\partial x}|_{x=2,y=-1}$ if $\frac{\partial f}{\partial x}|_{x=2,y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2,y=-1} = -2$. (Hint: use the chain rule)

Answer :

We know that:

$$\begin{aligned}
g_1(x,y) &= x^2y \\
g_2(x,y) &= x-y \\
h(x,y) &= f(g_1(x,y), g_2(x,y)) \\
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial x} \\
\frac{\partial h}{\partial x}|_{x=1,y=2} &= 3\frac{\partial g_1}{\partial x} - 2\frac{\partial g_2}{\partial x} = 3*(2xy)|_{x=2,y=-1} - 2*1 = -14
\end{aligned}$$

Question 4:

Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on $\mathbb{R}$, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t-s)f_2(s)ds$$

Let a, a_1, a_2 be real number.

(i) Prove the following identity:

$$f_1(t-a) * f_2(t) = f_1(t) * f_2(t-a) = f(t-a).$$

(ii) Prove the following identity:

$$f_1(t-a_1) * f_2(t-a_2) = f(t-a_1-a_2).$$

Answer :

(i) Prove $f_1(t-a) * f_2(t) = f(t-a)$:

$$\begin{aligned}
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds \\
f_1'(t) &= f_1(t-a) \\
f(t-a) &= \int_{-\infty}^{+\infty} f_1'(t-s)f_2(s)ds = f_1'(t) * f_2(t) = f_1(t-a) * f_2(t)
\end{aligned}$$

Prove $f_1(t) * f_2(t-a) = f(t-a)$:

$$\begin{aligned}
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds \\
f_2'(t) &= f_2(t-a) \\
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-(s+a))f_2(s)d(s+a) = \int_{-\infty}^{+\infty} f_1(t-(s+a))f_2'(s+a)d(s+a) \\
&= \int_{-\infty}^{+\infty} f_1(t-u)f_2'(u)du \\
&= f_1(t) * f_2'(t) \\
&= f_1() * f_2(t-a)
\end{aligned}$$

(ii)

$$\begin{aligned}
f_1(t-a_1) * f_2(t-a_2) &= \int_{-\infty}^{+\infty} f_1(t-a_1-s) f_2(s-a_2) ds \\
&= \int_{-\infty}^{+\infty} f_1(t-a_1-s) f_2(s-a_2) d(s-a_2) \\
&= \int_{-\infty}^{+\infty} f_1(t-a_1-a_2-(s-a_2)) f_2(s-a_2) d(s-a_2) \\
&= \int_{-\infty}^{+\infty} f_1(t-(a_1+a_2)-u) f_2(u) du \\
&= f(t-a_1-a_2).
\end{aligned}$$

Question 5:

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle\cdot,\cdot\rangle_{V_1}$, and $\langle\cdot,\cdot\rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \to V_2$ be a bounded linear operator.
(a) Let $S : V_2 \to V_1$ be an operator satisfying $\langle T\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} = \langle \boldsymbol{x}, S\boldsymbol{y}\rangle_{V_1}$ for any $\boldsymbol{x} \in V_1$ and $\boldsymbol{y} \in V_2$. Prove that S is a bounded linear operator. (Consequently, S is the adjoint of T, i.e., $S = T^*$)
(b) Prove that $(T^*)^* = T$.
(c) Prove that $\|T\| = \|T^*\|$.

Answer

We know

$$\|\boldsymbol{A}\boldsymbol{x}\|_{V_2} \leq \|\boldsymbol{A}\|\|\boldsymbol{x}\|_{V_1} \tag{1}$$

We know, by using Cauchy–Schwarz inequality and (1):

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} \leq \|\boldsymbol{T}\boldsymbol{x}\|_{V_2}\|\boldsymbol{y}\|_{V_2} \leq \|\boldsymbol{T}\|\|\boldsymbol{x}\|_{V_1}\|\boldsymbol{y}\|_{V_2}$$

We choose $\boldsymbol{x}$ as $\boldsymbol{S}\boldsymbol{y}$

$$\begin{aligned}
\langle \boldsymbol{T}\boldsymbol{S}\boldsymbol{y}, \boldsymbol{y}\rangle_{V_2} = \langle \boldsymbol{S}\boldsymbol{y}, \boldsymbol{S}\boldsymbol{y}\rangle_{V1} = \|\boldsymbol{S}\boldsymbol{y}\|_{V_1}^2 &\leq \|\boldsymbol{T}\boldsymbol{S}\boldsymbol{y}\|_{V_2}\|\boldsymbol{y}\|_{V_2} \leq \|\boldsymbol{T}\|\|\boldsymbol{S}\boldsymbol{y}\|_{V_1}\|\boldsymbol{y}\|_{V_2} \\
\|\boldsymbol{S}\boldsymbol{y}\|_{V_1}^2 &\leq \|\boldsymbol{T}\|\|\boldsymbol{S}\boldsymbol{y}\|_{V_1}\|\boldsymbol{y}\|_{V_2} \\
\|\boldsymbol{S}\boldsymbol{y}\|_{V_1} &\leq \|\boldsymbol{T}\|\|\boldsymbol{y}\|_{V_2} \leq \infty
\end{aligned}$$

Thus, S is a bounded linear operator.

(b) We know:

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} = \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y}\rangle_{V_1}$$

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y}\rangle_{V_1} &= \langle (\boldsymbol{T}^*)^*\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} \\
\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} &= \langle (\boldsymbol{T}^*)^*\boldsymbol{x}, \boldsymbol{y}\rangle_{V_2} \\
\boldsymbol{T} &= (\boldsymbol{T}^*)^*
\end{aligned}$$

(c) Proof of

$$\|\boldsymbol{T}\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y}\rangle$$

By using Cauchy Schwarz inequality

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle &\leq \|\boldsymbol{x}\| \|\boldsymbol{T}\boldsymbol{y}\| \\
\|\boldsymbol{x}\| &= 1 \\
\langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle &\leq \|\boldsymbol{T}\boldsymbol{y}\| \\
\|\boldsymbol{T}\boldsymbol{y}\| &= \sup_{\|x\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle \\
\|\boldsymbol{T}\| &= \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle
\end{aligned}$$

Then

$$\begin{aligned}
\|\boldsymbol{T}^*\| &= \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}^*\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y} \rangle \\
&= \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \sup_{\|\boldsymbol{y}\|=1} \langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \|\boldsymbol{T}\boldsymbol{x}\| = \|\boldsymbol{T}\|
\end{aligned}$$

In conclusion, $\|\boldsymbol{T}^*\| = \|\boldsymbol{T}\|$

Question 6:

Consider the vector space ℓ_∞ equipped with the norm $\|\cdot\|_\infty$. Define the operator $T : \ell_\infty \to \ell_\infty$ by $T(\{x_n\}_{n\in N}) = \{y_n\}_{n\in N}$ where $y_n = x_{n+1}$.
(a) Prove that T is a linear operator.
(b) Prove that T is a bounded operator.
(c) Prove that $\|T\| = 1$.

Answer