5004 Homework 2

RONG Shuo

November 28, 2024

Question 1:

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ , and } T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$$

.

Answer

We know that

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

So in conclusion,

$$2T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-3 \\ 6+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Question 2:

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.
(a) $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^n$.
(b) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^T\boldsymbol{a}$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$.

Answer

Assume that:

$$\begin{aligned} \boldsymbol{A}_i &= [A_{ij}]_{j=1}^n, \boldsymbol{A}_i \in \mathbb{R}^n \\ b_i &= \boldsymbol{b}_i, b_i \in \mathbb{R} \\ f_i(\boldsymbol{x}) &= \langle \boldsymbol{A}_i, \boldsymbol{x} \rangle - b_i \end{aligned}$$

We know that:

$$f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$$

$$f(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{A}_1^T \\ \vdots \\ \boldsymbol{A}_n^T \end{bmatrix} = \boldsymbol{A}$$

We know that Jacobian matrix is the differentiation of $f : \mathbb{R}^n \to \mathbb{R}^m$.
So In conclusion, the Jacobian matrix is $\boldsymbol{A}$.

(b) Assume that:

$$\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T = [x_i x_j]_{i=1,j=1}^{n,n}, \boldsymbol{X} \in \mathbb{R}^{n\times n}$$
$$\boldsymbol{X}_i = [x_i x_j]_{j=1}^{n}, \boldsymbol{X}_i \in \mathbb{R}^n$$
$$f_i(\boldsymbol{x}) = \langle \boldsymbol{X}_i, \boldsymbol{a} \rangle$$

if $i = k$

$$\frac{\partial f_i}{x_k} = \sum_{j=1}^{n} x_j a_j + x_k a_k$$

if $i \neq k$

$$\frac{\partial f_i}{\partial x_k} = x_i a_k$$

$$f(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}$$

$$Df(\boldsymbol{x}) = \begin{bmatrix} \nabla f_1(\boldsymbol{x})^T \\ \vdots \\ \nabla f_n(\boldsymbol{x})^T \end{bmatrix} = \begin{bmatrix} x_1 a_1 & x_1 a_2 & \cdots & x_1 a_n \\ \vdots & \vdots & \ldots & \vdots \\ x_n a_1 & x_n a_2 & \cdots & x_n a_n \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n x_j a_j & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^n x_j a_j \end{bmatrix} = \boldsymbol{x}\boldsymbol{a}^T + (\boldsymbol{x}^T\boldsymbol{a})\boldsymbol{I}$$

Question 3:

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (x^2 y, x - y)$ and $h = f \circ g = f(g(x,y))$. Find $\frac{\partial h}{\partial x}|_{x=1,y=2}$ if $\frac{\partial f}{\partial x}|_{x=2,y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2,y=-1} = -2$. (Hint: use the chain rule)

Answer :

We know that:

$$
\begin{aligned}
g_1(x,y) &= x^2y \\
g_2(x,y) &= x-y \\
h(x,y) &= f(g_1(x,y), g_2(x,y)) \\
\frac{\partial h}{\partial x}|_{x=1,y=2} &= \frac{\partial f}{\partial x}|_{x=2,y=-1}\frac{\partial g_1}{\partial x}|_{x=1,y=2} + \frac{\partial f}{\partial y}|_{x=2,y=-1}\frac{\partial g_2}{\partial x}|_{x=1,y=2} \\
\frac{\partial h}{\partial x}|_{x=1,y=2} &= 3\frac{\partial g_1}{\partial x} - 2\frac{\partial g_2}{\partial x} = 3*(2xy)|_{x=1,y=2} - 2*1 = 10
\end{aligned}
$$

Question 4:

Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on $\mathbb{R}$, i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t-s)f_2(s)ds$$

Let a, a_1, a_2 be real number.

(i) Prove the following identity:

$$f_1(t-a) * f_2(t) = f_1(t) * f_2(t-a) = f(t-a).$$

(ii) Prove the following identity:

$$f_1(t-a_1) * f_2(t-a_2) = f(t-a_1-a_2).$$

Answer :

(i) Prove $f_1(t-a) * f_2(t) = f(t-a)$:

$$
\begin{aligned}
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds \\
\text{Assume } f_1'(t) &= f_1(t-a) \\
f(t-a) &= \int_{-\infty}^{+\infty} f_1'(t-s)f_2(s)ds = f_1'(t) * f_2(t) = f_1(t-a) * f_2(t)
\end{aligned}
$$

Prove $f_1(t) * f_2(t-a) = f(t-a)$:

$$
\begin{aligned}
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds \\
\text{Assume } f_2'(t) &= f_2(t-a) \\
f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-(s+a))f_2(s)d(s+a) = \int_{-\infty}^{+\infty} f_1(t-(s+a))f_2'(s+a)d(s+a) \\
&= \int_{-\infty}^{+\infty} f_1(t-u)f_2'(u)du \\
&= f_1(t) * f_2'(t) \\
&= f_1(t) * f_2(t-a)
\end{aligned}
$$

(ii)

$$\begin{aligned} f_1(t-a_1) * f_2(t-a_2) &= \int_{-\infty}^{+\infty} f_1(t-a_1-s) f_2(s-a_2) ds \\ &= \int_{-\infty}^{+\infty} f_1(t-a_1-s) f_2(s-a_2) d(s-a_2) \\ &= \int_{-\infty}^{+\infty} f_1(t-a_1-a_2-(s-a_2)) f_2(s-a_2) d(s-a_2) \\ &= \int_{-\infty}^{+\infty} f_1(t-(a_1+a_2)-u) f_2(u) du \\ &= f(t-a_1-a_2). \end{aligned}$$

Question 5:

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{V_1}$, and $\langle \cdot, \cdot \rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \to V_2$ be a bounded linear operator.
(a) Let $S : V_2 \to V_1$ be an operator satisfying $\langle T\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} = \langle \boldsymbol{x}, S\boldsymbol{y} \rangle_{V_1}$ for any $\boldsymbol{x} \in V_1$ and $\boldsymbol{y} \in V_2$. Prove that S is a bounded linear operator. (Consequently, S is the adjoint of T, i.e., $S = T^*$)
(b) Prove that $(T^*)^* = T$.
(c) Prove that $\|T\| = \|T^*\|$.

Answer

(a)We know

$$\|\boldsymbol{A}\boldsymbol{x}\|_{V_2} \leq \|\boldsymbol{A}\| \|\boldsymbol{x}\|_{V_1} \tag{1}$$

We know, by using Cauchy–Schwarz inequality and (1):

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} \leq \|\boldsymbol{T}\boldsymbol{x}\|_{V_2} \|\boldsymbol{y}\|_{V_2} \leq \|\boldsymbol{T}\| \|\boldsymbol{x}\|_{V_1} \|\boldsymbol{y}\|_{V_2}$$

We choose $\boldsymbol{x}$ as $\boldsymbol{S}\boldsymbol{y}$

$$\begin{aligned} &\langle \boldsymbol{T}\boldsymbol{S}\boldsymbol{y}, \boldsymbol{y} \rangle_{V_2} = \langle \boldsymbol{S}\boldsymbol{y}, \boldsymbol{S}\boldsymbol{y} \rangle_{V1} = \|\boldsymbol{S}\boldsymbol{y}\|_{V_1}^2 \leq \|\boldsymbol{T}\boldsymbol{S}\boldsymbol{y}\|_{V_2} \|\boldsymbol{y}\|_{V_2} \leq \|\boldsymbol{T}\| \|\boldsymbol{S}\boldsymbol{y}\|_{V_1} \|\boldsymbol{y}\|_{V_2} \\ &\quad \|\boldsymbol{S}\boldsymbol{y}\|_{V_1}^2 \leq \|\boldsymbol{T}\| \|\boldsymbol{S}\boldsymbol{y}\|_{V_1} \|\boldsymbol{y}\|_{V_2} \\ &\quad \|\boldsymbol{S}\boldsymbol{y}\|_{V_1} \leq \|\boldsymbol{T}\| \|\boldsymbol{y}\|_{V_2} \leq \infty \end{aligned}$$

Thus, S is a bounded linear operator.

(b) We know:

$$\langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} = \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y} \rangle_{V_1}$$

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y} \rangle_{V_1} &= \langle (\boldsymbol{T}^*)^*\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} \\ \langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} &= \langle (\boldsymbol{T}^*)^*\boldsymbol{x}, \boldsymbol{y} \rangle_{V_2} \end{aligned}$$

Therefore

$$\boldsymbol{T} = (\boldsymbol{T}^*)^*$$

(c) Proof of

$$\|\boldsymbol{T}\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle$$

By using Cauchy Schwarz inequality

$$\langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle \leq \|\boldsymbol{x}\| \|\boldsymbol{T}\boldsymbol{y}\|$$

Assume $\|\boldsymbol{x}\| = 1$, s.t.

$$\langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle \leq \|\boldsymbol{T}\boldsymbol{y}\|$$
$$\|\boldsymbol{T}\boldsymbol{y}\| = \sup_{\|x\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle$$

Therefore

$$\|\boldsymbol{T}\| = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}\boldsymbol{y} \rangle$$

Then

$$\begin{aligned}\|\boldsymbol{T}^*\| &= \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{T}^*\boldsymbol{y}\| = \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{x}, \boldsymbol{T}^*\boldsymbol{y} \rangle \\ &= \sup_{\|\boldsymbol{y}\|=1} \sup_{\|\boldsymbol{x}\|=1} \langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \sup_{\|\boldsymbol{y}\|=1} \langle \boldsymbol{T}\boldsymbol{x}, \boldsymbol{y} \rangle = \sup_{\|\boldsymbol{x}\|=1} \|\boldsymbol{T}\boldsymbol{x}\| = \|\boldsymbol{T}\|\end{aligned}$$

−2

In conclusion, $\|\boldsymbol{T}^*\| = \|\boldsymbol{T}\|$

Question 6:

Consider the vector space ℓ_∞ equipped with the norm $\|\cdot\|_\infty$. Define the operator $T : \ell_\infty \to \ell_\infty$ by $T(\{x_n\}_{n\in N}) = \{y_n\}_{n\in N}$ where $y_n = x_{n+1}$.
(a) Prove that T is a linear operator.
(b) Prove that T is a bounded operator.
(c) Prove that $\|T\| = 1$.

Answer

(a) Additivity: Proof:
For any $\{x_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \in \ell_\infty$

$$T(\{x_n\}_{n\in\mathbb{N}} + \{z_n\}_{n\in\mathbb{N}}) = T(\{x_n + z_n\}_{n\in\mathbb{N}}) =$$
$$\{(x_n + z_n)_{n+1}\}_{n\in\mathbb{N}} = \{x_{n+1} + z_{n+1}\}_{n\in\mathbb{N}} = \{x_{n+1}\}_{n\in\mathbb{N}} + \{z_{n+1}\}_{n\in\mathbb{N}} = T(\{x_n\}_{n\in\mathbb{N}}) + T(\{z_n\}_{n\in\mathbb{N}})$$

Homogeneity Proof:

$$T(\alpha\{x_n\}_{n\in\mathbb{N}}) = T(\{\alpha x_n\}_{n\in\mathbb{N}}) = \{\alpha x_{n+1}\}_{n\in\mathbb{N}} = \alpha\{x_{n+1}\}_{n\in\mathbb{N}} = \alpha T(\{x_n\}_{n\in\mathbb{N}})$$

(b) To prove T is bounded operator, we can prove, for a constance C, that :

$$\begin{aligned}
&\|T(\{x_n\}_{n\in\mathbb{N}})\|_\infty \leq C\|\{x_n\}_{n\in\mathbb{N}}\|_\infty, \forall\{x_n\}_{n\in\mathbb{N}} \in \ell_\infty \\
&\|T(\{x_n\}_{n\in\mathbb{N}})\|_\infty = \sup_{n\in\mathbb{N}} |y_n| = \sup_{n\in\mathbb{N}} |x_{n+1}| \leq \sup_{n\in\mathbb{N}} |x_n| = \|\{x_n\}_{n\in\mathbb{N}}\|_\infty \\
&\|T(\{x_n\})\|_\infty \leq \|\{x_n\}\|_\infty
\end{aligned}$$

In conclusion, T is a bounded operator.

(c) From (b), we know that $\|T(\{x_n\})\|_\infty \leq \|\{x_n\}\|_\infty$, s.t. $\|T\| \leq 1$.
Consider the sequence $\{x_n\}_{n\in\mathbb{N}} = \{1, 1, \cdots\} = \{x_{n+1}\}_{n\in\mathbb{N}}$ By definition, we know:

$$\begin{aligned}
&\|T\| \geq \frac{\|T(\{x_n\})\|_\infty}{\|\{x_n\}\|_\infty} = 1 \\
&1 \leq \|T\| \leq 1
\end{aligned}$$

Thus, we conclude that $\|T\| = 1$