

5004 Homework 2

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Question 1:

1. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation. If T satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ and } T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

then find

$$T \begin{bmatrix} 8 \\ 3 \\ 7 \end{bmatrix}$$

.

Answer

We know that

$$2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}$$

So in conclusion,

$$2T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3T \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 - 3 \\ 6 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Question 2:

2. Find the Jacobian matrix of the following vector-valued multi-variable functions.

(a) $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by $f(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$.

(b) $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \mathbf{x}\mathbf{x}^T \mathbf{a}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n$.

Answer

Assume that:

$$\begin{aligned} \mathbf{A}_i &= [A_{ij}]_{j=1}^n, \mathbf{A}_i \in \mathbb{R}^n \\ b_i &= \mathbf{b}_i, b_i \in \mathbb{R} \\ f_i(\mathbf{x}) &= \langle \mathbf{A}_i, \mathbf{x} \rangle - b_i \end{aligned}$$

We know that:

$$\begin{aligned}
 f(\mathbf{x}) &= \mathbf{Ax} - \mathbf{b} \\
 f(\mathbf{x}) &= \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix} \\
 Df(\mathbf{x}) &= \begin{bmatrix} \nabla f_1(\mathbf{x})^T \\ \vdots \\ \nabla f_n(\mathbf{x})^T \end{bmatrix} \\
 Df(\mathbf{x}) &= \begin{bmatrix} \nabla f_1(\mathbf{x})^T \\ \vdots \\ \nabla f_n(\mathbf{x})^T \end{bmatrix} \\
 Df(\mathbf{x}) &= \begin{bmatrix} \mathbf{A}_1^T \\ \vdots \\ \mathbf{A}_n^T \end{bmatrix} = \mathbf{A}
 \end{aligned}$$

We know that Jacobian matrix is the differentiation of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.
So In conclusion, the Jacobian matrix is \mathbf{A} .

(b) Assume that:

$$\begin{aligned}
 \mathbf{X} &= \mathbf{xx}^T = [x_i x_j]_{i=1, j=1}^{n, n}, \mathbf{X} \in \mathbb{R}^{n \times n} \\
 \mathbf{X}_i &= [x_i x_j]_{j=1}^n, \mathbf{X}_i \in \mathbb{R}^n \\
 f_i(\mathbf{x}) &= \langle \mathbf{X}_i, \mathbf{a} \rangle
 \end{aligned}$$

if $i = k$

$$\frac{\partial f_i}{\partial x_k} = \sum_{j=1}^n x_j a_j + x_k a_k$$

if $i \neq k$

$$\frac{\partial f_i}{\partial x_k} = x_i a_k$$

$$\begin{aligned}
 f(\mathbf{x}) &= \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix} \\
 Df(\mathbf{x}) &= \begin{bmatrix} \nabla f_1(\mathbf{x})^T \\ \vdots \\ \nabla f_n(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} x_1 a_1 & x_1 a_2 & \cdots & x_1 a_n \\ \vdots & \vdots & \cdots & \vdots \\ x_n a_1 & x_n a_2 & \cdots & x_n a_n \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n x_j a_j & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^n x_j a_j \end{bmatrix} = \mathbf{xa}^T + (\mathbf{x}^T \mathbf{a}) \mathbf{I}
 \end{aligned}$$

Question 3:

3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $g(x, y) = (x^2 y, x - y)$ and $h = f \circ g = f(g(x, y))$. Find $\frac{\partial h}{\partial x}|_{x=1, y=2}$ if $\frac{\partial f}{\partial x}|_{x=2, y=-1} = 3$ and $\frac{\partial f}{\partial y}|_{x=2, y=-1} = -2$. (Hint: use the chain rule)

Answer :

We know that:

$$\begin{aligned}g_1(x, y) &= x^2y \\g_2(x, y) &= x - y \\h(x, y) &= f(g_1(x, y), g_2(x, y)) \\\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial x} \frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g_2}{\partial x} \\\frac{\partial h}{\partial x} \Big|_{x=1, y=2} &= 3 \frac{\partial g_1}{\partial x} - 2 \frac{\partial g_2}{\partial x} = 3 * (2xy) \Big|_{x=1, y=2} - 2 * 1 = 10\end{aligned}$$

Question 4:

Let $f(t) = f_1(t) * f_2(t)$ be the convolution of two functions $f_1(t)$ and $f_2(t)$ on \mathbb{R} , i.e.,

$$f(t) = \int_{-\infty}^{+\infty} f_1(t-s)f_2(s)ds$$

Let a, a_1, a_2 be real number.

(i) Prove the following identity:

$$f_1(t-a) * f_2(t) = f_1(t) * f_2(t-a) = f(t-a).$$

(ii) Prove the following identity:

$$f_1(t-a_1) * f_2(t-a_2) = f(t-a_1-a_2).$$

Answer :

(i) Prove $f_1(t-a) * f_2(t) = f(t-a)$:

$$f(t-a) = \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds$$

$$\text{Assume } f'_1(t) = f_1(t-a)$$

$$f(t-a) = \int_{-\infty}^{+\infty} f'_1(t-s)f_2(s)ds = f'_1(t) * f_2(t) = f_1(t-a) * f_2(t)$$

Prove $f_1(t) * f_2(t-a) = f(t-a)$:

$$f(t-a) = \int_{-\infty}^{+\infty} f_1(t-s-a)f_2(s)ds$$

$$\text{Assume } f'_2(t) = f_2(t-a)$$

$$\begin{aligned}f(t-a) &= \int_{-\infty}^{+\infty} f_1(t-(s+a))f_2(s)d(s+a) = \int_{-\infty}^{+\infty} f_1(t-(s+a))f'_2(s+a)d(s+a) \\&= \int_{-\infty}^{+\infty} f_1(t-u)f'_2(u)du \\&= f_1(t) * f'_2(t) \\&= f_1(t) * f_2(t-a)\end{aligned}$$

(ii)

$$\begin{aligned} f_1(t - a_1) * f_2(t - a_2) &= \int_{-\infty}^{+\infty} f_1(t - a_1 - s) f_2(s - a_2) ds \\ &= \int_{-\infty}^{+\infty} f_1(t - a_1 - s) f_2(s - a_2) d(s - a_2) \\ &= \int_{-\infty}^{+\infty} f_1(t - a_1 - a_2 - (s - a_2)) f_2(s - a_2) d(s - a_2) \\ &= \int_{-\infty}^{+\infty} f_1(t - (a_1 + a_2) - u) f_2(u) du \\ &= f(t - a_1 - a_2). \end{aligned}$$

Question 5:

5. Let V_1 and V_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle_{V_1}$, and $\langle \cdot, \cdot \rangle_{V_2}$, respectively. Let $T \in \mathcal{L}(V_1, V_2)$, i.e., $T : V_1 \rightarrow V_2$ be a bounded linear operator.

(a) Let $S : V_2 \rightarrow V_1$ be an operator satisfying $\langle T\mathbf{x}, \mathbf{y} \rangle_{V_2} = \langle \mathbf{x}, S\mathbf{y} \rangle_{V_1}$ for any $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$. Prove that S is a bounded linear operator. (Consequently, S is the adjoint of T , i.e., $S = T^*$)

(b) Prove that $(T^*)^* = T$.

(c) Prove that $\|T\| = \|T^*\|$.

Answer

(a) We know

$$\|\mathbf{Ax}\|_{V_2} \leq \|\mathbf{A}\| \|\mathbf{x}\|_{V_1} \tag{1}$$

We know, by using Cauchy-Schwarz inequality and (1):

$$\langle T\mathbf{x}, \mathbf{y} \rangle_{V_2} \leq \|T\mathbf{x}\|_{V_2} \|\mathbf{y}\|_{V_2} \leq \|T\| \|\mathbf{x}\|_{V_1} \|\mathbf{y}\|_{V_2}$$

We choose \mathbf{x} as $S\mathbf{y}$

$$\begin{aligned} \langle TS\mathbf{y}, \mathbf{y} \rangle_{V_2} &= \langle S\mathbf{y}, S\mathbf{y} \rangle_{V_1} = \|S\mathbf{y}\|_{V_1}^2 \leq \|TS\mathbf{y}\|_{V_2} \|\mathbf{y}\|_{V_2} \leq \|T\| \|S\mathbf{y}\|_{V_1} \|\mathbf{y}\|_{V_2} \\ \|S\mathbf{y}\|_{V_1}^2 &\leq \|T\| \|S\mathbf{y}\|_{V_1} \|\mathbf{y}\|_{V_2} \\ \|S\mathbf{y}\|_{V_1} &\leq \|T\| \|\mathbf{y}\|_{V_2} \leq \infty \end{aligned}$$

Thus, S is a bounded linear operator.

(b) We know:

$$\langle T\mathbf{x}, \mathbf{y} \rangle_{V_2} = \langle \mathbf{x}, T^*\mathbf{y} \rangle_{V_1}$$

$$\begin{aligned} \langle \mathbf{x}, T^*\mathbf{y} \rangle_{V_1} &= \langle (T^*)^*\mathbf{x}, \mathbf{y} \rangle_{V_2} \\ \langle T\mathbf{x}, \mathbf{y} \rangle_{V_2} &= \langle (T^*)^*\mathbf{x}, \mathbf{y} \rangle_{V_2} \\ T &= (T^*)^* \end{aligned}$$

(c) Proof of

$$\|T\| = \sup_{\|y\|=1} \|Ty\| = \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, Ty \rangle$$

By using Cauchy Schwarz inequality

$$\begin{aligned} \langle x, Ty \rangle &\leq \|x\| \|Ty\| \\ \|x\| &= 1 \\ \langle x, Ty \rangle &\leq \|Ty\| \\ \|Ty\| &= \sup_{\|x\|=1} \langle x, Ty \rangle \\ \|T\| &= \sup_{\|y\|=1} \|Ty\| = \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, Ty \rangle \end{aligned}$$

Then

$$\begin{aligned} \|T^*\| &= \sup_{\|y\|=1} \|T^*y\| = \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, T^*y \rangle \\ &= \sup_{\|y\|=1} \sup_{\|x\|=1} \langle Tx, y \rangle = \sup_{\|x\|=1} \sup_{\|y\|=1} \langle Tx, y \rangle = \sup_{\|x\|=1} \|Tx\| = \|T\| \end{aligned}$$

In conclusion, $\|T^*\| = \|T\|$

Question 6:

Consider the vector space ℓ_∞ equipped with the norm $\|\cdot\|_\infty$. Define the operator $T : \ell_\infty \rightarrow \ell_\infty$ by $T(\{x_n\}_{n \in \mathbb{N}}) = \{y_n\}_{n \in \mathbb{N}}$ where $y_n = x_{n+1}$.

- (a) Prove that T is a linear operator.
- (b) Prove that T is a bounded operator.
- (c) Prove that $\|T\| = 1$.

Answer

(a) Additivity: Proof:

For any $\{x_n\}_{n \in \mathbb{N}}, \{z_n\}_{n \in \mathbb{N}} \in \ell_\infty$

$$\begin{aligned} T(\{x_n\}_{n \in \mathbb{N}} + \{z_n\}_{n \in \mathbb{N}}) &= T(\{x_n + z_n\}_{n \in \mathbb{N}}) = \\ \{(x_n + z_n)_{n+1}\}_{n \in \mathbb{N}} &= \{x_{n+1} + z_{n+1}\}_{n \in \mathbb{N}} = \{x_{n+1}\}_{n \in \mathbb{N}} + \{z_{n+1}\}_{n \in \mathbb{N}} = T(\{x_n\}_{n \in \mathbb{N}}) + T(\{z_n\}_{n \in \mathbb{N}}) \end{aligned}$$

Homogeneity Proof:

$$T(\alpha\{x_n\}_{n \in \mathbb{N}}) = T(\{\alpha x_n\}_{n \in \mathbb{N}}) = \{\alpha x_{n+1}\}_{n \in \mathbb{N}} = \alpha\{x_{n+1}\}_{n \in \mathbb{N}} = \alpha T(\{x_n\}_{n \in \mathbb{N}})$$

(b) To prove T is bounded operator, we can prove, for a constance C , that :

$$\begin{aligned} \|T(\{x_n\}_{n \in \mathbb{N}})\|_\infty &\leq C\|\{x_n\}_{n \in \mathbb{N}}\|_\infty, \forall \{x_n\}_{n \in \mathbb{N}} \in \ell_\infty \\ \|T(\{x_n\}_{n \in \mathbb{N}})\|_\infty &= \sup_{n \in \mathbb{N}} |y_n| = \sup_{n \in \mathbb{N}} |x_{n+1}| \leq \sup_{n \in \mathbb{N}} |x_n| = \|\{x_n\}_{n \in \mathbb{N}}\|_\infty \\ \|T(\{x_n\})\|_\infty &\leq \|\{x_n\}\|_\infty \end{aligned}$$

In conclusion, T is a bounded operator.

(c) From (b), we know that $\|T(\{x_n\})\|_\infty \leq \|\{x_n\}\|_\infty$, s.t. $\|T\| \leq 1$.

Consider the sequence $\{x_n\}_{n \in \mathbb{N}} = \{1, 1, \dots\} = \{x_{n+1}\}_{n \in \mathbb{N}}$ By definition, we know:

$$\begin{aligned}\|T\| &\geq \frac{\|T(\{x_n\})\|_\infty}{\|\{x_n\}\|_\infty} = 1 \\ 1 &\leq \|T\| \leq 1\end{aligned}$$

Thus, we conclude that $\|T\| = 1$