Question 1:

Determine whether each of the following scalar-valued functions of n-vectors is linear. If it is a linear function, give its inner product representation, ie., an n-vector $\boldsymbol{a}$ for which $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$ for all $\boldsymbol{x}$. If it is not linear, give specific $\boldsymbol{x}, \boldsymbol{y}$, α and β such that

$$f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) \neq \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}).$$

(a) The spread of values of the vector, defined as $f(\boldsymbol{x}) = max_k x_k - min_k x_k$.
(b) The difference of the last element and the first, $f(\boldsymbol{x}) = x_n - x_1$.

Answer :

(a) Take $\boldsymbol{x} = (1, 2, 3)$ and $\alpha = 1, \beta = 1$ for example:

$$\begin{aligned} f(\boldsymbol{x}) &= 3 - 1 = 2 \\ f(-\boldsymbol{x}) &= -1 + 3 = 2 \\ f(\boldsymbol{0}) &= 0 - 0 = 0 \\ f(\boldsymbol{x} + (-\boldsymbol{x})) &= f(\boldsymbol{0}) = 0 \\ f(\boldsymbol{x}) + f(-\boldsymbol{x}) &= 2 + 2 = 4 \\ f(\boldsymbol{x} + (-\boldsymbol{x})) &\neq f(\boldsymbol{x}) + f(-\boldsymbol{x}) \end{aligned}$$