

MSBD5007 Optimization and Matrix Computation

Homework 5

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1. Find explicit formulas for the projection of $y \in \mathbb{R}^n$ onto the following non-empty, closed, and convex sets $S \subset \mathbb{R}^n$, respectively.

- (a) The unit ∞ -norm ball

$$S = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}.$$

- (b) The closed halfspace

$$S = \{x \in \mathbb{R}^n \mid a^T x \leq b\},$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$ are given.

2. Consider the optimization problem in $x = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$\begin{aligned} \min_{x \in \mathbb{R}^3} & \frac{1}{2}(x_1^2 + 4x_2^2 + 9x_3^2) - (4x_1 + 2x_2), \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 3, \\ & x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

- (a) Write down the KKT conditions (stationarity, feasibility, complementary slackness).
 - (b) Solve the KKT system to find the optimal solution x^* , the Lagrange multiplier λ^* for the equality constraint, and the multipliers $\mu^* = (\mu_1, \mu_2, \mu_3)$ for the inequality constraints.
3. We wish to compute the projection of $y \in \mathbb{R}^n$ onto the unit ℓ_1 ball, i.e. solve

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & \frac{1}{2} \|x - y\|_2^2 \\ \text{s.t.} \quad & \|x\|_1 \leq 1. \end{aligned}$$

- (a) Derive the Lagrange dual problem and show it can be written as

$$\max_{\lambda \geq 0} d(\lambda), \quad d(\lambda) = \sum_{i=1}^n h_\lambda(y_i) - \lambda,$$

where $h_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ is the so-called Huber's function (which is a smooth function consisting of a quadratic and two linear pieces) defined by

$$h_\lambda(t) = \begin{cases} \frac{1}{2}t^2, & |t| \leq \lambda, \\ \lambda|t| - \frac{1}{2}\lambda^2, & |t| \geq \lambda. \end{cases}$$

- (b) Prove that strong duality holds.
 - (c) Find the optimal dual multiplier λ^* .
 - (d) Give an expression of the projection in terms of λ^* .
4. Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set, and let $\|\cdot\|_2$ denote 2-norm. The projection of any point $y \in \mathbb{R}^n$ onto S is defined by

$$\mathcal{P}_S(y) = \arg \min_{x \in S} \|x - y\|_2.$$

Prove that the projection $\mathcal{P}_S : \mathbb{R}^n \rightarrow S$ is nonexpansive: for all $x, y \in \mathbb{R}^n$,

$$\|\mathcal{P}_S(x) - \mathcal{P}_S(y)\|_2 \leq \|x - y\|_2.$$

1 Answer

1.1 (1)

1.1.1 (a)

We know the projection:

$$\mathcal{P}_S(y) = \arg \min_{x \in S} \|\mathbf{x} - \mathbf{y}\|_2.$$

We consider

$$\min_{\mathbf{x} \in \mathbb{S}} \|\mathbf{x} - \mathbf{y}\|_2$$

This means:

$$\min_{\mathbf{x} \in \mathbb{S}} \sum_{i=1}^n (x_i - y_i)^2$$

We can solve for each component x_i independently:

$$\min_{x_i \in [-1, 1]} (x_i - y_i)^2$$

This is just the Euclidean projection of a scalar onto the interval $[-1, 1]$, which gives:

$$x_i = \min(\max(y_i, -1), 1)$$

and

$$\mathbf{x} = [x_i]_1^n$$

1.1.2 (b)

We know the projection:

$$\mathcal{P}_S(y) = \arg \min_{\mathbf{x} \in S} \|\mathbf{x} - \mathbf{y}\|_2$$

$$\mathcal{L}(\mathbf{x}, \lambda) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 + \lambda(\mathbf{a}^T \mathbf{x} - b), \quad \lambda \geq 0$$

We know the KKT conditions:

$$\begin{aligned} \nabla_x \mathcal{L} &= \mathbf{x} - \mathbf{y} + \lambda \mathbf{a} = 0 \\ \mathbf{x}^* &= \mathbf{y} - \lambda \mathbf{a} \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}^T \mathbf{x}^* &\leq b \\ \mathbf{a}^T (\mathbf{y} - \lambda \mathbf{a}) &\leq b \\ \mathbf{a}^T \mathbf{y} - \lambda \|\mathbf{a}\|^2 &\leq b \\ \lambda &\geq \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2} \end{aligned}$$

and we know:

$$\begin{aligned} \lambda(\mathbf{a}^T \mathbf{x} - b) &= 0 \\ \lambda(\mathbf{a}^T \mathbf{y} - \lambda \|\mathbf{a}\|^2 - b) &= 0 \end{aligned}$$

So we get $\lambda = \max(0, \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2})$

So we can get

$$\mathcal{P}_S(\mathbf{y}) = \mathbf{y} - \max(0, \frac{\mathbf{a}^T \mathbf{y} - b}{\|\mathbf{a}\|^2}) \mathbf{a}$$

1.2 (2)

1.2.1 (a)

We can write the Lagrangian Function:

$$\mathcal{L}(x, \lambda, \mu) = \frac{1}{2}(x_1^2 + 4x_2^2 + 9x_3^2) - (4x_1 + 2x_2) - \lambda(x_1 + x_2 + x_3 - 3) - \sum_{i=1}^3 \mu_i x_i$$

$$\begin{aligned}
\nabla_x \mathcal{L} &= 0 \\
x_1 - 4 - \lambda - \mu_1 &= 0 \\
4x_2 - 2 - \lambda - \mu_2 &= 0 \\
9x_3 - \lambda - \mu_3 &= 0
\end{aligned}$$

$$\begin{aligned}
x_1 + x_2 + x_3 &= 3 \\
x_i &\geq 0
\end{aligned}$$

$$\begin{aligned}
\mu_i &\geq 0 \\
\mu_i x_i &= 0
\end{aligned}$$

1.2.2 (b)

We need to do analysis of the solution first.

$$\begin{aligned}
x_1 > 0, x_2 > 0, x_3 > 0 &\quad \text{invalid} \\
x_1 = 0, x_2 > 0, x_3 > 0 &\quad \text{invalid} \\
x_1 > 0, x_2 = 0, x_3 > 0 &\quad \text{invalid} \\
x_1 > 0, x_2 > 0, x_3 = 0 &\quad \text{valid}
\end{aligned}$$

if $x_3 = 0$, we know $\mu_1 = 0, \mu_2 = 0$. Therefore, we get:

$$\begin{aligned}
x_1 &= \lambda + 4 \\
x_2 &= \frac{\lambda + 2}{4} \\
\mu_3 &= -\lambda
\end{aligned}$$

We know $x_1 + x_2 = 3$.

$$\begin{aligned}
\lambda + 4 + \frac{\lambda + 2}{4} &= 3 \\
\lambda &= -\frac{6}{5}
\end{aligned}$$

s.t.

$$\begin{aligned}x_1 &= \frac{14}{5} \\x_2 &= \frac{1}{5} \\x_1 &= 0 \\\mu_3 &= \frac{6}{5}\end{aligned}$$

Therefore, we can conclude:

$$\begin{aligned}x_1 &= \frac{14}{5} \\x_2 &= \frac{1}{5} \\x_1 &= 0 \\\mu_1 &= 0 \\\mu_2 &= 0 \\\mu_3 &= \frac{6}{5} \\\lambda &= -\frac{6}{5}\end{aligned}$$

1.3 (3)

1.3.1 (a)

We know the Lagrange formula:

$$\mathcal{L}(\mathbf{x}, \lambda) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 + \lambda(\|\mathbf{x}\|_1 - 1)$$

And the primal problem can be written as:

$$\min_{\mathbf{x}} \max_{\lambda \geq 0} \mathcal{L}(\mathbf{x}, \lambda)$$

And the dual problem can be written as:

$$\begin{aligned}\max_{\lambda \geq 0} \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda) \\ \max_{\lambda \geq 0} \min_{\mathbf{x}} \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 + \lambda(\|\mathbf{x}\|_1 - 1)\end{aligned}$$

The problem is separable in \mathbf{x} , so we can reformulate the minimization problem as:

$$\min_{x_i} \frac{1}{2} (x_i - y_i)^2 + \lambda |x_i|$$

Apparently, this is a soft-thresholding operator which have been solved in hw4. We can derive:

$$x_i^* = \begin{cases} y_i - \lambda & \text{if } y_i > \lambda \\ y_i + \lambda & \text{if } y_i < -\lambda \\ 0 & \text{if } |y_i| \leq \lambda \end{cases}$$

If $|y_i| \leq \lambda$:

$$\min_{x_i} \frac{1}{2}(x_i - y_i)^2 + \lambda|x_i| = \frac{1}{2}y_i^2$$

If $y_i > \lambda$:

$$\begin{aligned} \min_{x_i} \frac{1}{2}(x_i - y_i)^2 + \lambda|x_i| \\ &= \frac{1}{2}\lambda^2 + \lambda y_i - \lambda^2 \\ &= \lambda y_i - \frac{1}{2}\lambda^2 \end{aligned}$$

If $y_i < -\lambda$:

$$\begin{aligned} \min_{x_i} \frac{1}{2}(x_i - y_i)^2 + \lambda|x_i| \\ &= \frac{1}{2}\lambda^2 - \lambda y_i - \lambda^2 \\ &= -\lambda y_i - \frac{1}{2}\lambda^2 \end{aligned}$$

In conclusion,

$$h_\lambda(y_i) = \begin{cases} \frac{1}{2}y_i^2 & \text{if } |y_i| \leq \lambda \\ \lambda|y_i| - \frac{1}{2}\lambda^2 & \text{if } |y_i| > \lambda \end{cases}$$

And the dual problem can be written as:

$$\max_{\lambda \geq 0} d(\lambda), \quad d(\lambda) = \sum_{i=1}^n h_\lambda(y_i) - \lambda$$

1.3.2 (b)

The objective $\frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2$ and the constraint $\|\mathbf{x}\|_1 \leq 1$ is convex.

And we know for Slater's condition, we can easily find $\mathbf{x} = 0, \mathbf{x} \in \mathbb{R}^n$, s.t. $\|\mathbf{x}\|_1 < 1$.

Hence, strong duality holds.

1.3.3 (c)

$$d(\lambda) = \sum_{i=1}^n h_{\lambda}(y_i) - \lambda,$$

$$d'(\lambda) = \sum_{i=1}^n h'_{\lambda}(y_i) - 1$$

$$h_{\lambda}(y_i) = \begin{cases} \frac{1}{2}y_i^2 & \text{if } |y_i| \leq \lambda \\ \lambda|y_i| - \frac{1}{2}\lambda^2 & \text{if } |y_i| > \lambda \end{cases}$$

$$h'_{\lambda}(y_i) = \begin{cases} 0 & \text{if } |y_i| \leq \lambda \\ |y_i| - \lambda & \text{if } |y_i| > \lambda \end{cases}$$

$$d'(\lambda) = \sum_{i=1}^n h'_{\lambda}(y_i) - 1$$

$$d'(\lambda) = \sum_{i=1}^n \max\{|y_i| - \lambda, 0\} - 1$$

Hence

$$\lambda^* = \frac{\sum_{i=n-k+1}^n |y_i| - 1}{k}$$

where $|y_i|$ is rearrange by descending order, and k is the number of $|y_i|$ values greater than λ .

1.3.4 (d)

$$x_i^* = \begin{cases} y_i - \lambda^* & \text{if } y_i > \lambda \\ y_i + \lambda^* & \text{if } y_i < -\lambda \\ 0 & \text{if } |y_i| \leq \lambda \end{cases}$$

$$\lambda^* = \frac{\sum_{i=n-k+1}^n |y_i| - 1}{k}$$

where $|y_i|$ is rearrange by descending order, and k is the number of $|y_i|$ values greater than λ .

1.4 (4)

$$\|\mathcal{P}_S(\mathbf{x}) - \mathcal{P}_S(\mathbf{y})\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2$$

Due to convexity, we know:

$$\begin{aligned} \langle \mathbf{y} - \mathcal{P}_S \mathbf{y}, \mathbf{z} - \mathcal{P}_S \mathbf{y} \rangle &\leq 0, \forall \mathbf{z} \in S \\ \langle \mathbf{x} - \mathcal{P}_S \mathbf{x}, \mathbf{z} - \mathcal{P}_S \mathbf{x} \rangle &\leq 0, \forall \mathbf{z} \in S \end{aligned}$$

Substitute \mathbf{z} as $\mathcal{P}_S \mathbf{x}$ and $\mathcal{P}_S \mathbf{y}$ independently, and add the inequality up.

$$\begin{aligned} \langle \mathbf{y} - \mathcal{P}_S \mathbf{y}, \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle &\leq 0, \forall \mathbf{z} \in S \\ \langle \mathbf{x} - \mathcal{P}_S \mathbf{x}, \mathcal{P}_S \mathbf{y} - \mathcal{P}_S \mathbf{x} \rangle &\leq 0, \forall \mathbf{z} \in S \\ \langle \mathbf{y} - \mathcal{P}_S \mathbf{y}, \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle + \langle \mathbf{x} - \mathcal{P}_S \mathbf{x}, \mathcal{P}_S \mathbf{y} - \mathcal{P}_S \mathbf{x} \rangle &\leq 0 \\ \langle \mathbf{y} - \mathcal{P}_S \mathbf{y} - \mathbf{x} + \mathcal{P}_S \mathbf{x}, \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle &\leq 0 \\ \langle \mathbf{y} - \mathbf{x} - (\mathcal{P}_S \mathbf{y} - \mathcal{P}_S \mathbf{x}), \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle &\leq 0 \\ \langle \mathbf{x} - \mathbf{y}, \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle &\geq \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2^2 \end{aligned}$$

By CS-inequality,

$$\langle \mathbf{x} - \mathbf{y}, \mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y} \rangle \leq \|\mathbf{x} - \mathbf{y}\|_2 \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2$$

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|_2 \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2 &\geq \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2^2 \\ \|\mathbf{x} - \mathbf{y}\|_2 &\geq \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2 \\ \|\mathbf{x} - \mathbf{y}\|_2 &\geq \|\mathcal{P}_S \mathbf{x} - \mathcal{P}_S \mathbf{y}\|_2 \end{aligned}$$