MSBD5007 Optimization and Matrix Computation
Homework 1

Due date: 28 February, Friday

1. Given two vectors $\boldsymbol{a} = [a_0, a_1, \dots, a_{N-1}]^T$ and $\boldsymbol{b} = [b_0, b_1, \dots, b_{N-1}]^T$, their *circular convolution* is defined by

$$(\boldsymbol{a} \circledast \boldsymbol{b})_k = \sum_{j=0}^{N-1} a_j \, b_{k-j}, \quad k = 0, 1, \dots, N-1,$$

where $\boldsymbol{b}$ is extended periodically, i.e., $b_{k-j} = b_{(k-j)+N}$ if $-N \le k-j \le -1$. Let $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ be vectors in $\mathbb{R}^N$. Prove that the circular convolution satisfies:

   (a) $\boldsymbol{f} \circledast \boldsymbol{g} = \boldsymbol{g} \circledast \boldsymbol{f}$.

   (b) $\boldsymbol{f} \circledast (\boldsymbol{g} \circledast \boldsymbol{h}) = (\boldsymbol{f} \circledast \boldsymbol{g}) \circledast \boldsymbol{h}$.

2. Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ -3 & -1 & 2 \end{bmatrix}$ be a 3×3 matrix.

   (a) Find the LU decomposition of the matrix $\boldsymbol{A}$. The final result will look like this:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

   (b) Use the result in (a) to solve the system:

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \\ -3x_1 - x_2 + 2x_3 &= 5 \end{aligned}$$

3. Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a tri-diagonal matrix (i.e., $a_{ij} = 0$ if $|i-j| > 1$). The pattern of nonzero entries is illustrated below:

$$\begin{bmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \ddots & \ddots & \ddots & \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}.$$

Develop an algorithm with complexity $O(n)$ to compute the LU decomposition of $\boldsymbol{A}$, assuming all the pivots are non-zero.

4. To accelerate matrix multiplications, the *CoppersmithWinograd* algorithm reduces the number of scalar multiplications by cleverly reformulating the inner product. Assume that n is even and define, for any vector $\boldsymbol{x} \in \mathbb{R}^n$,

$$f(\boldsymbol{x}) = \sum_{i=1}^{n/2} x_{2i-1}\, x_{2i}.$$

   (a) Prove that for all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, the inner product can be re-expressed as

$$\boldsymbol{x}^T \boldsymbol{y} = \sum_{i=1}^{n/2} \big((x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1}) \big) - f(\boldsymbol{x}) - f(\boldsymbol{y}).$$

   (b) Now consider the matrix product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$, where $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$. Devise an algorithm to compute $\boldsymbol{C}$ using only $\frac{n^3}{2} + O(n^2)$ scalar multiplications.

   Note: A standard matrix multiplication requires n^3 scalar multiplications. By combining this method with other techniques, one can obtain the CoppersmithWinograd algorithm, which has an asymptotic complexity of $O(n^{2.375})$.