

# MSBD 5007 HW1

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## Question1

Given two vectors  $\mathbf{a} = [a_0, a_1, \dots, a_{N-1}]$  and  $\mathbf{b} = [b_0, b_1, \dots, b_{N-1}]^T$ , their *circular convolution* is defined by

$$(\mathbf{a} \circledast \mathbf{b})_k = \sum_{j=0}^{N-1} a_j b_{k-j}, k = 0, 1, \dots, N-1,$$

where  $\mathbf{b}$  is extended periodically, i.e.,  $b_{k-j} = b_{(k-j)+N}$  if  $-N \leq k-j \leq -1$ . Let  $\mathbf{f}, \mathbf{g}$ , and  $\mathbf{h}$  be vectors in  $\mathbb{R}^N$ . Prove that the circular convolution satisfies:

(a)  $\mathbf{f} \circledast \mathbf{g} = \mathbf{g} \circledast \mathbf{f}$ .

(b)  $\mathbf{f} \circledast (\mathbf{g} \circledast \mathbf{h}) = (\mathbf{f} \circledast \mathbf{g}) \circledast \mathbf{h}$ .

(a)

$$\begin{aligned} (\mathbf{f} \circledast \mathbf{g})_k &= \sum_{j=0}^{N-1} f_j g_{k-j} \\ &= \sum_{j=k+1}^{N-1} f_j g_{k-j} + \sum_{j=0}^k f_j g_{k-j} \\ &= \sum_{j=k+1}^{N-1} f_j g_{N+k-j} + \sum_{j=0}^k f_j g_{k-j} \\ &= \sum_{i=k+1}^{N-1} f_{N+k-i} g_i + \sum_{j=0}^k f_j g_{k-j} \\ &= \sum_{i=k+1}^{N-1} g_i f_{k-i} + \sum_{i=0}^k f_{k-i} g_i \\ &= \sum_{i=k+1}^{N-1} g_i f_{k-i} + \sum_{i=0}^k g_i f_{k-i} \\ &= \sum_{i=0}^{N-1} g_i f_{k-i} \\ &= (\mathbf{g} \circledast \mathbf{f})_k \end{aligned}$$

Therefore, we can conclude that:  $\mathbf{f} \circledast \mathbf{g} = \mathbf{g} \circledast \mathbf{f}$ .

(b)

$$\begin{aligned}
(\mathbf{f} \circledast (\mathbf{g} \circledast \mathbf{h}))_k &= (\mathbf{f} \circledast (\mathbf{h} \circledast \mathbf{g}))_k \\
&= \sum_{j=0}^{N-1} f_j (\mathbf{h} \circledast \mathbf{g})_{k-j} \\
&= \sum_{j=0}^{N-1} f_j \sum_{i=0}^{N-1} h_i g_{k-j-i} \\
&= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} f_j h_i g_{k-j-i} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f_j h_i g_{k-j-i} \\
&= \sum_{i=0}^{N-1} h_i \sum_{j=0}^{N-1} f_j g_{k-j-i} \\
&= \sum_{i=0}^{N-1} h_i (\mathbf{f} \circledast \mathbf{g})_{k-j} \\
&= (\mathbf{h} \circledast (\mathbf{f} \circledast \mathbf{g}))_k \\
&= ((\mathbf{f} \circledast \mathbf{g}) \circledast \mathbf{h})_k
\end{aligned}$$

Therefore, we can conclude that:  $\mathbf{f} \circledast (\mathbf{g} \circledast \mathbf{h}) = (\mathbf{f} \circledast \mathbf{g}) \circledast \mathbf{h}$ .

## Question2

Let  $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ -3 & -1 & 2 \end{bmatrix}$  be a  $3 \times 3$  matrix.

(a) Find the LU decomposition of the matrix  $\mathbf{A}$ . The final result will look like this:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 0 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

(b) Use the result in (a) to solve the system:

$$\begin{aligned}
2x_1 - x_2 + 3x_3 &= 3 \\
x_1 + 2x_2 + x_3 &= 4 \\
-3x_1 - x_2 + 2x_3 &= 5
\end{aligned}$$

(a)

$$\begin{aligned}
&\begin{bmatrix} 2 & -1 & 3 \\ 1/2 & 5/2 & -1/2 \\ -3/2 & -5/2 & 13/2 \end{bmatrix}_{k=1} \\
&\begin{bmatrix} 2 & -1 & 3 \\ 1/2 & 5/2 & -1/2 \\ -3/2 & -1 & 6 \end{bmatrix}_{k=2}
\end{aligned}$$

From this, we can get:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5/2 & -1/2 \\ 0 & 0 & 6 \end{bmatrix}$$

(b)

From (a), we know:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5/2 & -1/2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Assume  $\mathbf{LUx} = \mathbf{b}$  as  $\mathbf{Ly} = \mathbf{b}$ , s.t.: From (a), we know:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

From these, we can solve:

$$\begin{aligned} y_1 &= 3 \\ y_2 &= 5/2 \\ y_3 &= 12 \end{aligned}$$

s.t.

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5/2 & -1/2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5/2 \\ 12 \end{bmatrix}$$

From these, we can solve:

$$\begin{aligned} x_1 &= -4/5 \\ x_2 &= 7/5 \\ x_3 &= 2 \end{aligned}$$

### Question3

Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a tri-diagonal matrix(i.e.  $a_{ij} = 0$  if  $|i - j| > 1$ ). The pattern of nonzero entries is illustrated below:

$$\begin{bmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \ddots & \ddots & \ddots & \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

Develop an algorithm with complexity  $O(n)$  to compute the LU decomposition of  $\mathbf{A}$ , assuming all the pivots are non-zero.

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**Algorithm 1** Update Matrix Elements

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1: for k = 1 to n do
2:    $a_{k,k+1} \leftarrow \frac{a_{k,k+1}}{a_{k,k}}$ 
3:    $a_{k+1,k+1} \leftarrow a_{k+1,k+1} - a_{k,k+1} \cdot a_{k+1,k}$ 
4: end for
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## Question4

To accelerate matrix multiplications, the *Coppersmith Winograd* algorithm reduces the number of scalar multiplications by cleverly reformulating the inner product. Assume that  $n$  is even and define, for any vector  $x \in \mathbb{R}^n$ ,

$$f(\mathbf{x}) = \sum_{i=1}^{n/2} x_{2i-1}x_{2i}$$

(a) Prove that for all vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , the inner product can be re-expressed as

$$\mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n/2} ((x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1})) - f(\mathbf{x}) - f(\mathbf{y}).$$

(b) Now consider the matrix product  $\mathbf{C} = \mathbf{A}\mathbf{B}$ , where  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ . Devise an algorithm to compute  $\mathbf{C}$  using  $\frac{n^3}{2} + O(n^2)$  scalar multiplications.

*Note:* A standard matrix multiplication requires  $n^3$  scalar multiplications. By combining this method with other techniques, one can obtain the Coppersmith Winograd algorithm, which has an asymptotic complexity of  $O(n^{2.375})$ .

(a)

$$\begin{aligned} & \sum_{i=1}^{n/2} ((x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1})) \\ &= \sum_{i=1}^{n/2} x_{2i-1}x_{2i} + y_{2i}x_{2i} + x_{2i-1}y_{2i-1} + y_{2i}y_{2i-1} \\ &= \sum_{i=1}^{n/2} x_{2i-1}x_{2i} + \sum_{i=1}^{n/2} y_{2i}x_{2i} + \sum_{i=1}^{n/2} x_{2i-1}y_{2i-1} + \sum_{i=1}^{n/2} y_{2i}y_{2i-1} \\ &= \mathbf{x}^T \mathbf{y} + f(\mathbf{x}) + f(\mathbf{y}) \end{aligned}$$

So we can conclude that:

$$\mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n/2} ((x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1})) - f(\mathbf{x}) - f(\mathbf{y}).$$

(b)