MSBD 5007 HW1

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Question1

Given two vectors $\boldsymbol{a} = [a_0, a_1, \cdots, a_{N-1}]$ and $\boldsymbol{b} = [b_0, b_1, \cdots, b_{N-1}]^T$, their *circular convolution* is defined by

$$(\boldsymbol{a} \circledast \boldsymbol{b})_k = \sum_{j=0}^{N-1} a_j b_{k-j}, k = 0, 1, \cdots, N-1,$$

where $\boldsymbol{b}$ is extended periodically, i.e., $b_{k-j} = b_{(k-j)+N}$ if $-N \leq k-j \leq -1$. Let $\boldsymbol{f}, \boldsymbol{g}$, and $\boldsymbol{h}$ be vectors in $\mathbb{R}^N$. Prove that the circular convolution satisfies:

(a) $\boldsymbol{f} \circledast \boldsymbol{g} = \boldsymbol{g} \circledast \boldsymbol{f}$.

(b) $\boldsymbol{f} \circledast (\boldsymbol{g} \circledast \boldsymbol{h}) = (\boldsymbol{f} \circledast \boldsymbol{g}) \circledast \boldsymbol{h}$.

(a)

$$\begin{aligned}
(\boldsymbol{f} \circledast \boldsymbol{g})_k & \\
&= \sum_{j=0}^{N-1} f_j g_{k-j} \\
&= \sum_{j=k+1}^{N-1} f_j g_{k-j} + \sum_{j=0}^{k} f_j g_{k-j} \\
&= \sum_{j=k+1}^{N-1} f_j g_{N+k-j} + \sum_{j=0}^{k} f_j g_{k-j} \\
&= \sum_{i=k+1}^{N-1} f_{N+k-i} g_i + \sum_{j=0}^{k} f_j g_{k-j} \\
&= \sum_{i=k+1}^{N-1} g_i f_{k-i} + \sum_{i=0}^{k} f_{k-i} g_i \\
&= \sum_{i=k+1}^{N-1} g_i f_{k-i} + \sum_{i=0}^{k} g_i f_{k-i} \\
&= \sum_{i=0}^{N-1} g_i f_{k-i} \\
&= (\boldsymbol{g} \circledast \boldsymbol{f})_k
\end{aligned}$$

Therefore, we can conclude that: $\boldsymbol{f} \circledast \boldsymbol{g} = \boldsymbol{g} \circledast \boldsymbol{f}$.

(b)

$$
\begin{aligned}
(\boldsymbol{f} \circledast (\boldsymbol{g} \circledast \boldsymbol{h}))_k &= (\boldsymbol{f} \circledast (\boldsymbol{h} \circledast \boldsymbol{g}))_k \\
&= \sum_{j=0}^{N-1} f_j (\boldsymbol{h} \circledast \boldsymbol{g})_{k-j} \\
&= \sum_{j=0}^{N-1} f_j \sum_{i=0}^{N-1} h_i g_{k-j-i} \\
&= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} f_j h_i g_{k-j-i} \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f_j h_i g_{k-j-i} \\
&= \sum_{i=0}^{N-1} h_i \sum_{j=0}^{N-1} f_j g_{k-j-i} \\
&= \sum_{i=0}^{N-1} h_i (\boldsymbol{f} \circledast \boldsymbol{g})_{k-j} \\
&= (\boldsymbol{h} \circledast (\boldsymbol{f} \circledast \boldsymbol{g}))_k \\
&= ((\boldsymbol{f} \circledast \boldsymbol{g}) \circledast \boldsymbol{h})_k
\end{aligned}
$$

Therefore, we can conclude that: $\boldsymbol{f} \circledast (\boldsymbol{g} \circledast \boldsymbol{h}) = (\boldsymbol{f} \circledast \boldsymbol{g}) \circledast \boldsymbol{h}$.