MSBD5007 Optimization and Matrix Computation
Homework 5

Due date: 4 May, Sunday

1. Find explicit formulas for the projection of $\boldsymbol{y} \in \mathbb{R}^n$ onto the following non-empty, closed, and convex sets $S \subset \mathbb{R}^n$, respectively.
   (a) The unit ∞-norm ball
   $$S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_\infty \leq 1\}.$$
   (b) The closed halfspace
   $$S = \left\{\boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \boldsymbol{a}^T \boldsymbol{x} \leq b\right\},$$
   where $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, and $b \in \mathbb{R}$ are given.

2. Consider the optimization problem in $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$:
   $$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^3} \quad & \tfrac{1}{2}\left(x_1^2 + 4x_2^2 + 9x_3^2\right) \;-\; \left(4x_1 + 2x_2\right), \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 3, \\ & x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$
   (a) Write down the KKT conditions (stationarity, feasibility, complementary slackness).
   (b) Solve the KKT system to find the optimal solution $\boldsymbol{x}^*$, the Lagrange multiplier λ^* for the equality constraint, and the multipliers $\boldsymbol{\mu}^* = (\mu_1, \mu_2, \mu_3)$ for the inequality constraints.

3. We wish to compute the projection of $\boldsymbol{y} \in \mathbb{R}^n$ onto the unit $\ell_1$ball, i.e. solve
   $$\min_{\boldsymbol{x} \in \mathbb{R}^n} \frac{1}{2}\left\|\boldsymbol{x} - \boldsymbol{y}\right\|_2^2 \quad \text{s.t.} \quad \left\|\boldsymbol{x}\right\|_1 \leq 1.$$
   (a) Derive the Lagrange dual problem and show it can be written as
   $$\max_{\lambda \geq 0} \; d(\lambda), \qquad d(\lambda) \;=\; \sum_{i=1}^{n} h_\lambda(y_i) \;-\; \lambda,$$
   where $h_\lambda : \mathbb{R} \to \mathbb{R}$ is the so-called Huber's function (which is a smooth function consisting of a quadratic and two linear pieces) defined by
   $$h_\lambda(t) = \begin{cases} \tfrac{1}{2}\, t^2, & |t| \leq \lambda, \\ \lambda\,|t| \;-\; \tfrac{1}{2}\,\lambda^2, & |t| \geq \lambda. \end{cases}$$
   (b) Prove that strong duality holds.
   (c) Find the optimal dual multiplier λ^*.
   (d) Give an expression of the projection in terms of λ^*.

4. Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set, and let $\|\cdot\|_2$ denote 2-norm. The projection of any point $\boldsymbol{y} \in \mathbb{R}^n$ onto S is defined by

$$\mathcal{P}_S(\boldsymbol{y}) \;=\; \arg\min_{\boldsymbol{x} \in S} \left\|\boldsymbol{x} - \boldsymbol{y}\right\|_2.$$

Prove that the projection $\mathcal{P}_S : \mathbb{R}^n \to S$ is *nonexpansive*: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$\left\|\mathcal{P}_S(\boldsymbol{x}) \;-\; \mathcal{P}_S(\boldsymbol{y})\right\|_2 \;\leq\; \left\|\boldsymbol{x} - \boldsymbol{y}\right\|_2.$$