MSBD 5007 HW2

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Question1

Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 4 \end{bmatrix}$. Use the QR factorization to solve the least square problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^4} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2.$$

(a)

(b)

Question2

Find the best rank-1 approximation with respect to the Frobenius norm to the matrix.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Question3

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two matrices of dimensions $m \times n$. Consider the optimization problem

$$\min_{Q \in \mathbb{R}^{m \times n}} \|\boldsymbol{X} - \boldsymbol{Q}\boldsymbol{Y}\|_F \text{ subject to } \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}_m.$$

, where $\|\dot{\|}_F$ denotes the Frobenius norm and I_m. is the $m \times n$ identity matrix. This problem aries when we want to align two datasets $\boldsymbol{X}$ and $\boldsymbol{Y}$ up to an orthogonal linear transformation. Prove that the solution is given by $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{V}^T$, where $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ is the singular value decomposition of $\boldsymbol{X}\boldsymbol{Y}^T$, i.e. $\boldsymbol{X}\boldsymbol{Y}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

Question4

Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix with $m \geq n$ and full column rank, i.e., $\text{rank}(\boldsymbol{A}) = n$. A *Given rotation* matrix $\boldsymbol{G}_{i,j,\theta} \in \mathbb{R}^{n \times n}$ (for indices $1 \leq i \leq j \leq n$) is defined by

$$\boldsymbol{G}_{i,j,\theta} = \begin{bmatrix} \boldsymbol{I}_{i-1} & & & & \\ & \cos(\theta) & & -\sin(\theta) & \\ & & \boldsymbol{I}_{j-i-1} & & \\ & \sin(\theta) & & \cos(\theta) & \\ & & & & \boldsymbol{I}_{n-j} \end{bmatrix}$$