Graph Algorithms

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Main Topics

- Graph representation
- Graph traversal
  - Breadth-first search
  - Depth-first search

Graphs

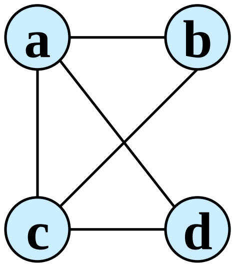
- *Graph* $G = (V, E)$
  - V = set of vertices
  - E = set of edges $\subseteq (V \times V)$
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - Dense: $|E| \approx |V|^2$.
  - Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

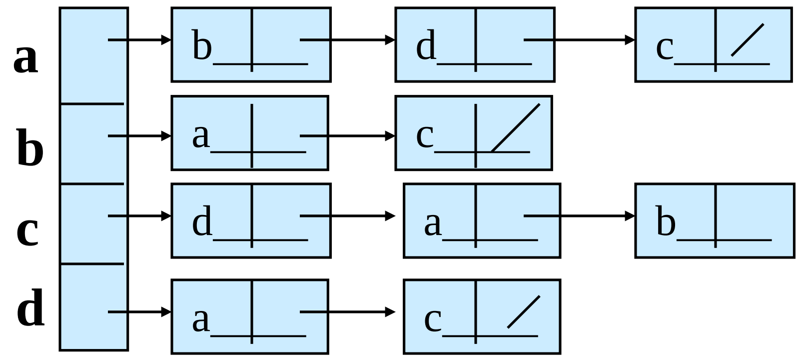
GRAPHS

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - Symmetric if G is undirected.
  - Not necessarily so if G is directed.
- Path
  - a **path** in a graph is a finite sequence of edges which connect a sequence of vertices.
    - Sequence of alternating vertices and edges.
    - Start from a vertex and ends to a vertex.
  - Simple path
    - All vertices and edges are distinct.
- Cycle
  - A path whose start and end vertex is same.

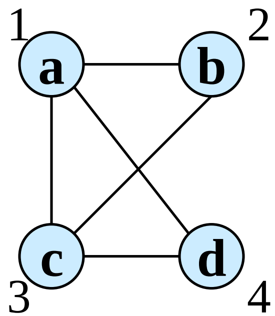
Representation of Graphs

- Two standard ways.
  - Adjacency Lists.





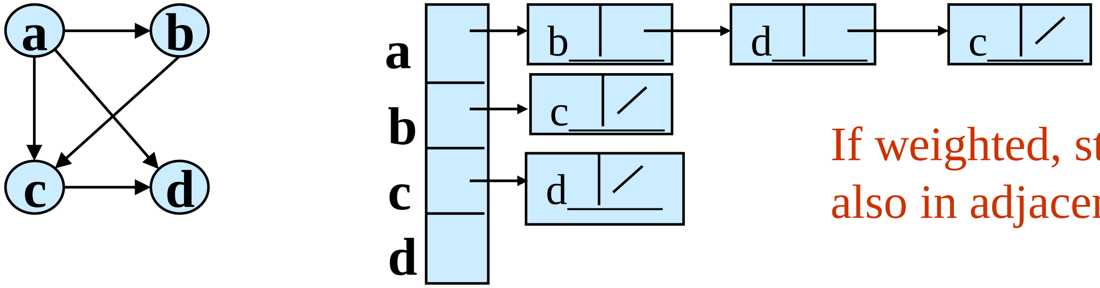
  - Adjacency Matrix.



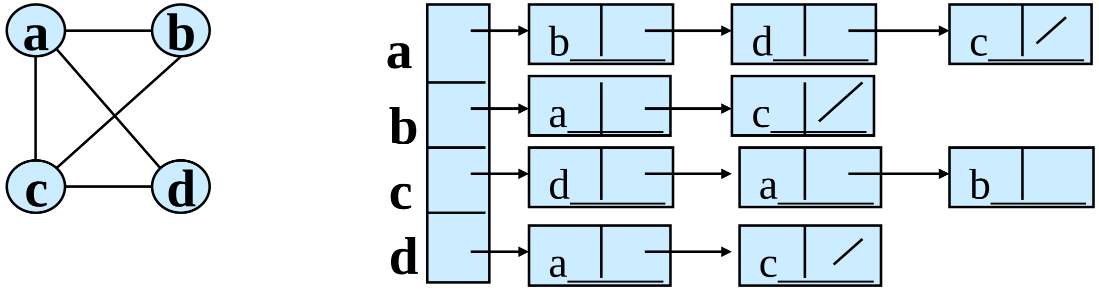
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

ADJACENCY LISTS

- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.



If weighted, store weights also in adjacency lists.



Storage Requirement

- For directed graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

No. of edges leaving *v*

  - Total storage: $\Theta(V+E)$
- For undirected graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

No. of edges incident on *v*. Edge (u,v) is incident on vertices *u* and *v*.

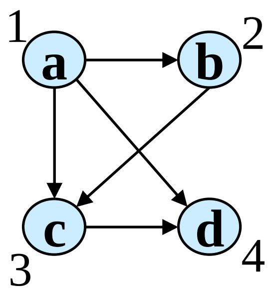
  - Total storage: $\Theta(V+E)$

Pros and Cons: adj list

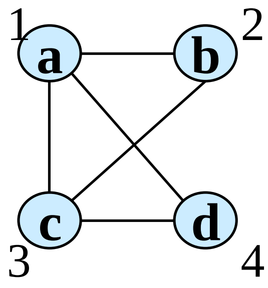
- Pros
  - Space-efficient, when a graph is sparse.
  - Can be modified to support many graph variants.
- Cons
  - Determining if an edge $(u,v) \in G$ is not efficient.
    - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.
    - $\Theta(V)$ in the worst case.

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by: $A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



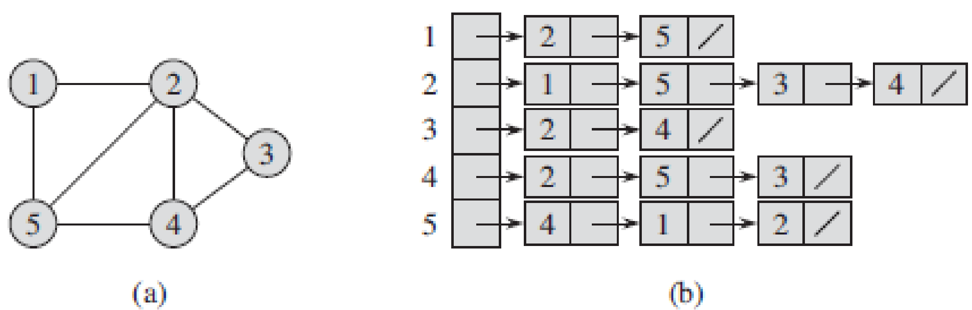
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(V^2)$.
  - Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

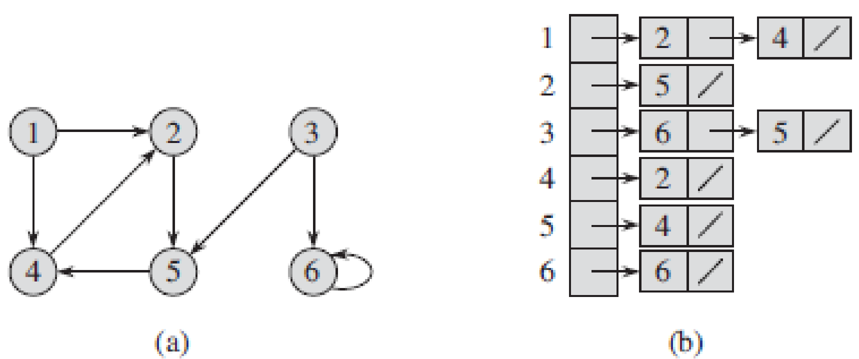
Graph Representation – List vs. Matrix



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

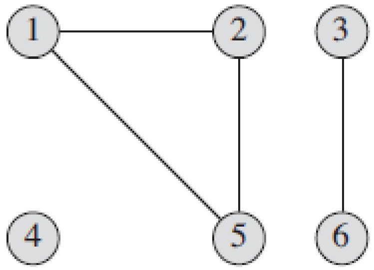
(c)

Graphs

- Reachable checking
  - If a path exists
- Cycle checking
  - a vertex is reachable from itself.

Connected Component

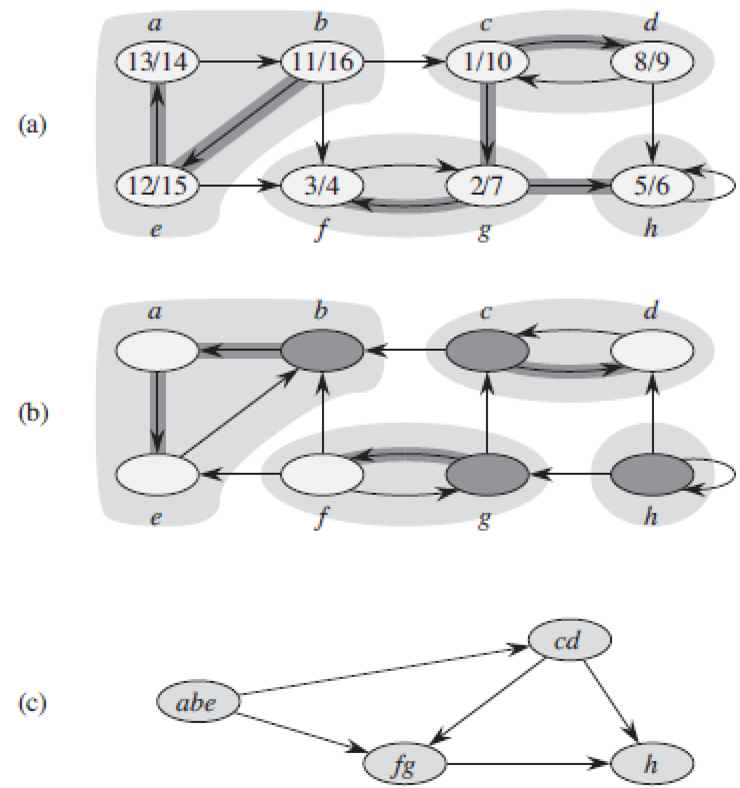
- An undirected graph is ***connected if there is a path between every pair of v***ertices.
- A **connected component** of an undirected graph is a maximal set of nodes such that each pair of nodes is connected by a path.



(b)

Strongly Connected Components

- A graph is **strongly connected** if every vertex is reachable from every other vertex.
- The ***strongly connected components of a directed graph*** *is the subgraph where every vertex is reachable from every other vertices.*



Connected Components

- Connected (Strongly or not) components form a **partition** of the set of graph vertices, meaning
  - connected components are non-empty,
  - they are pair-wise **disjoints**, and
  - the union of connected components forms the set of all vertices.

- Algorithm to find the strong connectivity of a graph takes linear time ($\Theta(V+E)$).

Connected Components

- Connected components count
  - Count of disjoint connected components
- Size of connected component
  - number of vertices in a component

- Figure
  - Fig 1: has 3 connected components with 3 different sizes
  - Fig 2: has 4 connected components with 3 different sizes

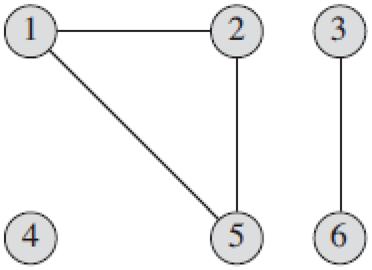


Fig 1

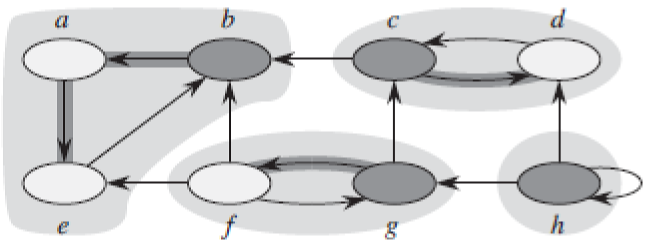


Fig 2

TREE

- **Tree**: Connected Acyclic undirected graph
- **Forest**: If an undirected graph is acyclic but possibly disconnected, it is a ***forest***
- **Spanning Tree**: is a tree which includes all of the vertices of the graph.

Appendix B Sets, Etc.

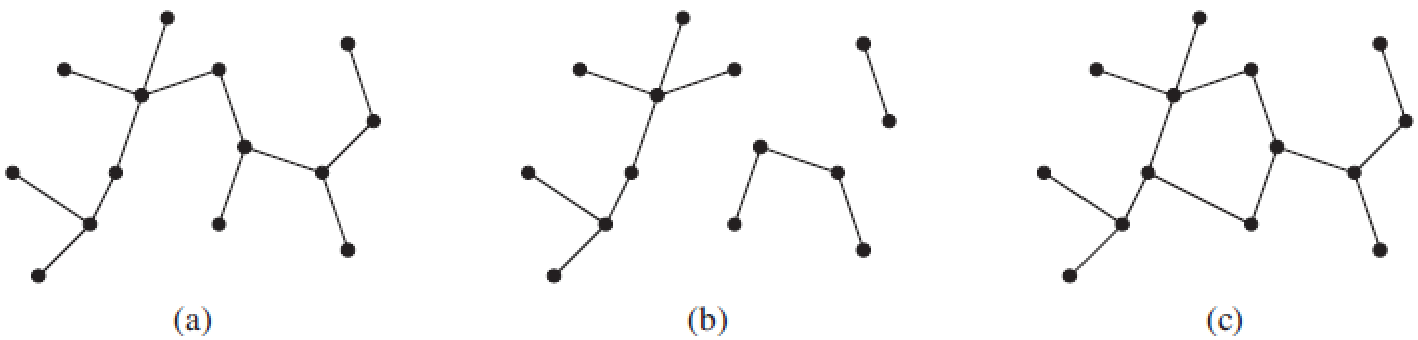


Figure B.4 (a) A free tree. (b) A forest. (c) A graph that contains a cycle and is therefore neither a tree nor a forest.

Properties of Free Trees

- Let G ={V;E} be an undirected graph. The following statements are equivalent.
  - Any two vertices in G are connected by a unique simple path.
  - G is connected, acyclic and $|E|=|V| - 1$.
  - G is connected, but if any edge is removed from E, the resulting graph is disconnected.
  - G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

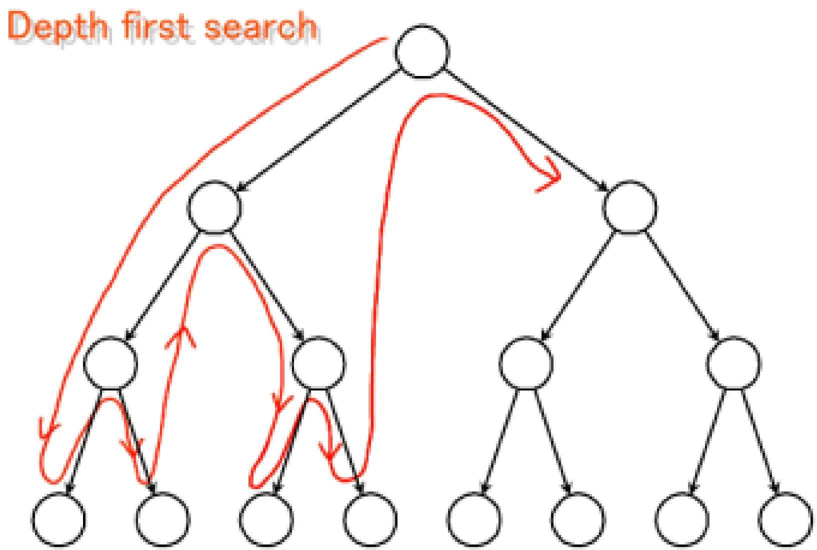
Graph-searching Algorithms

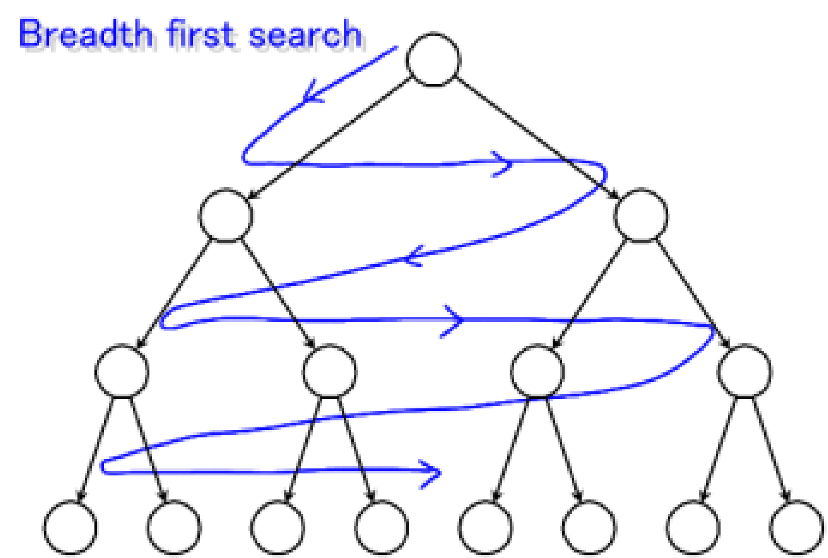
- Given: a graph G = (V, E), directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree
  - Note: might also build a *forest if graph is not connected*
- Used to discover the structure of a graph.

Graph-searching Algorithms

- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).
    - "Search as deep as possible first."

Graph Traversals





- Both take time: O(V+E)

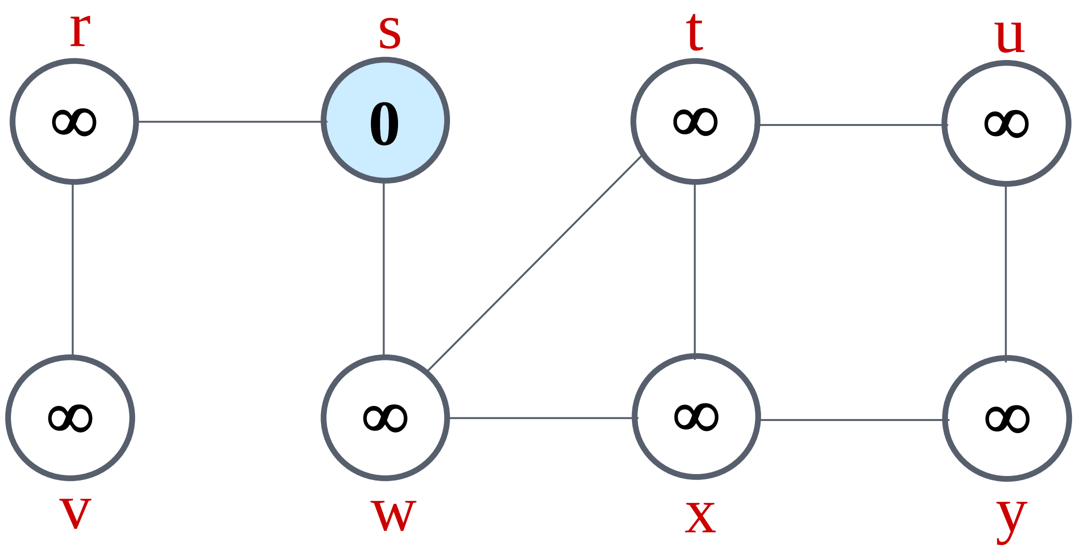
BFS(G,s)

```
for each vertex u in V[G] – {s}
    do color[u] ← white
       d[u] ← ∝
       π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
    do u ← dequeue(Q)
        for each v in Adj[u]
            do if color[v] = white
                then color[v] ← gray
                     d[v] ← d[u] + 1
                     π[v] ← u
                     enqueue(Q,v)
        color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

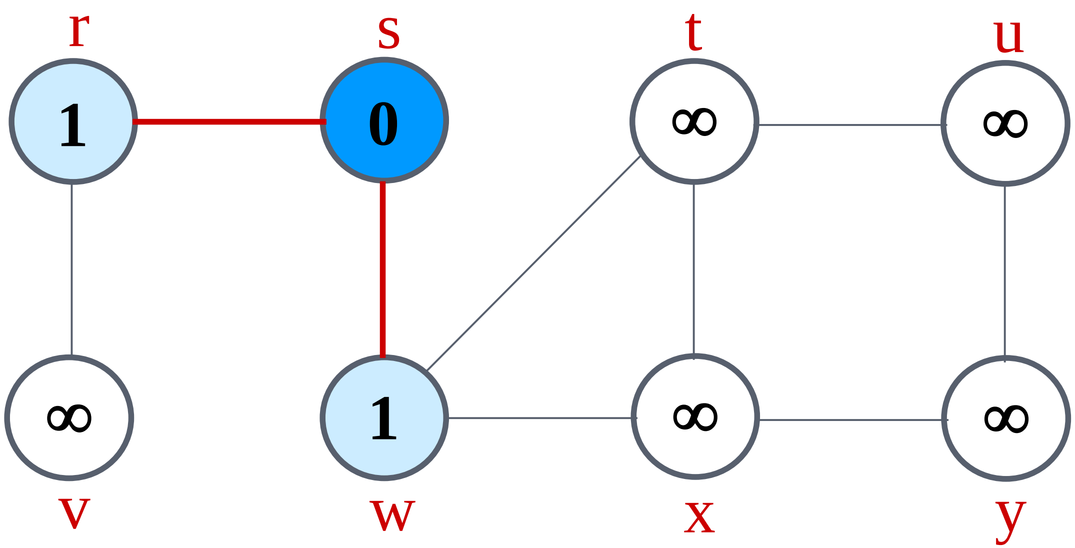
Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

EXAMPLE (BFS)



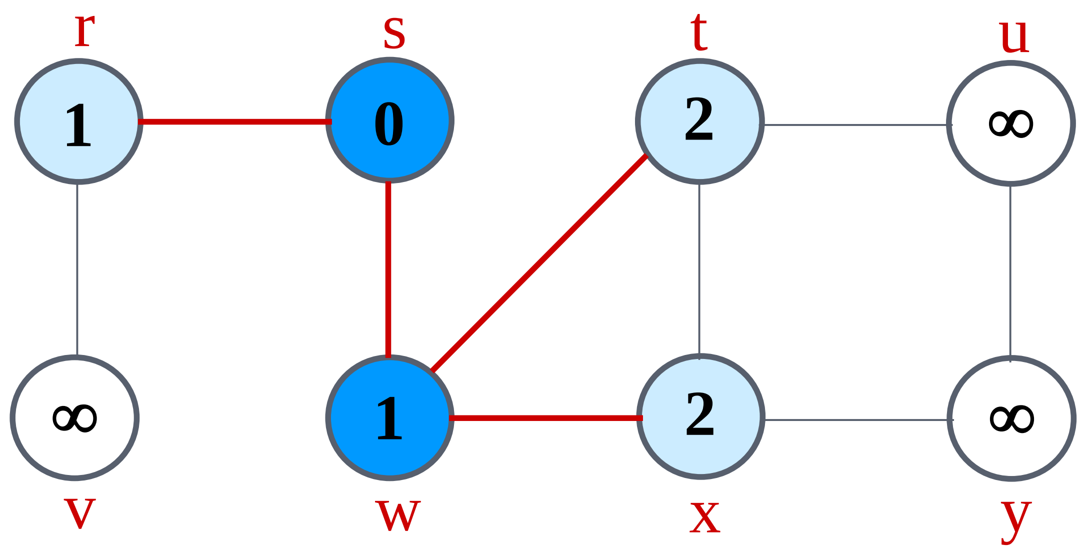
Q:	s
	0

Example (BFS)



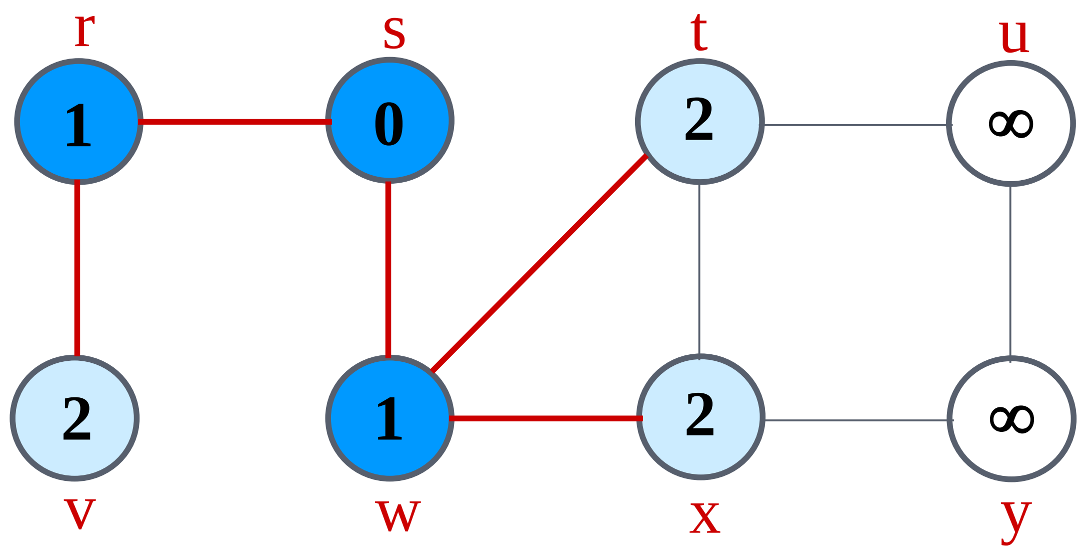
Q:	w	r
	1	1

Example (BFS)



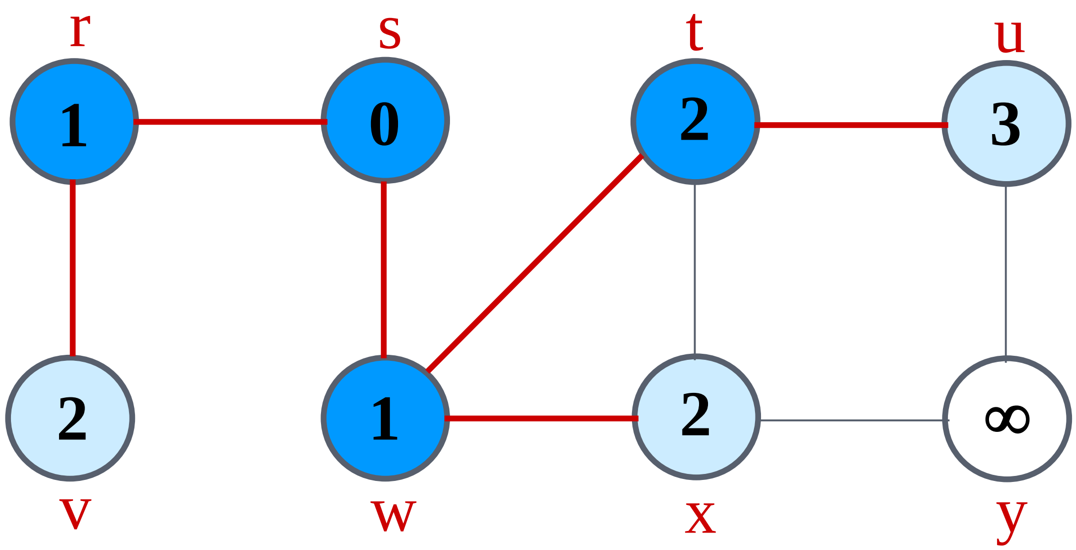
Q:	r	t	x
	1	2	2

Example (BFS)



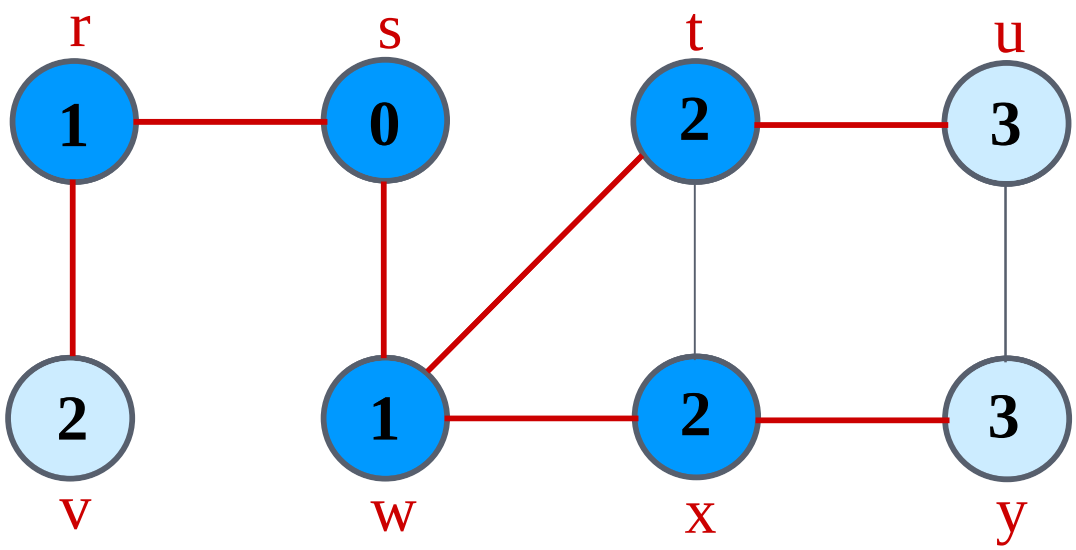
Q: t x v
2 2 2

Example (BFS)



Q: x v u
2 2 3

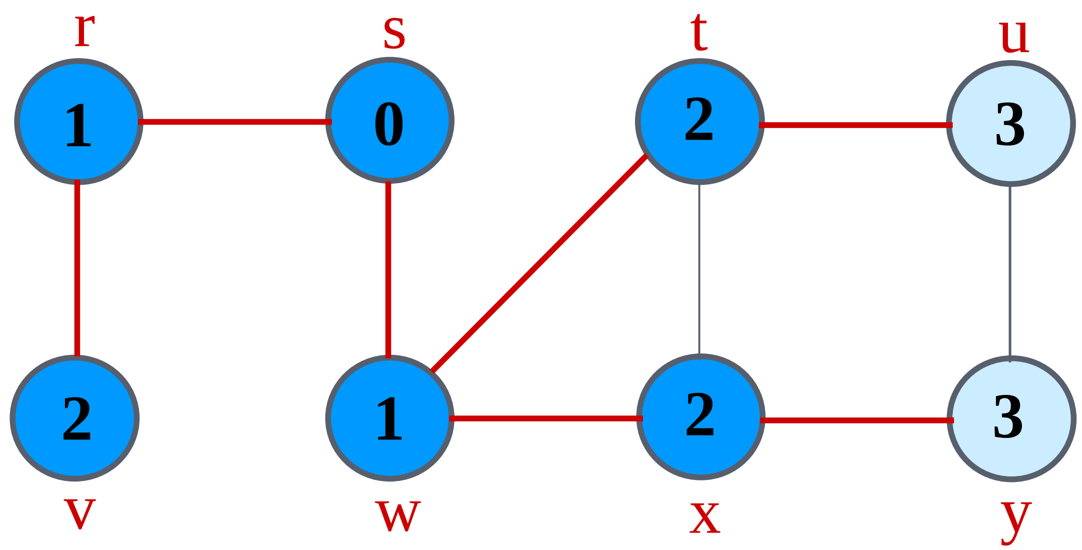
Example (BFS)



Q: v u y

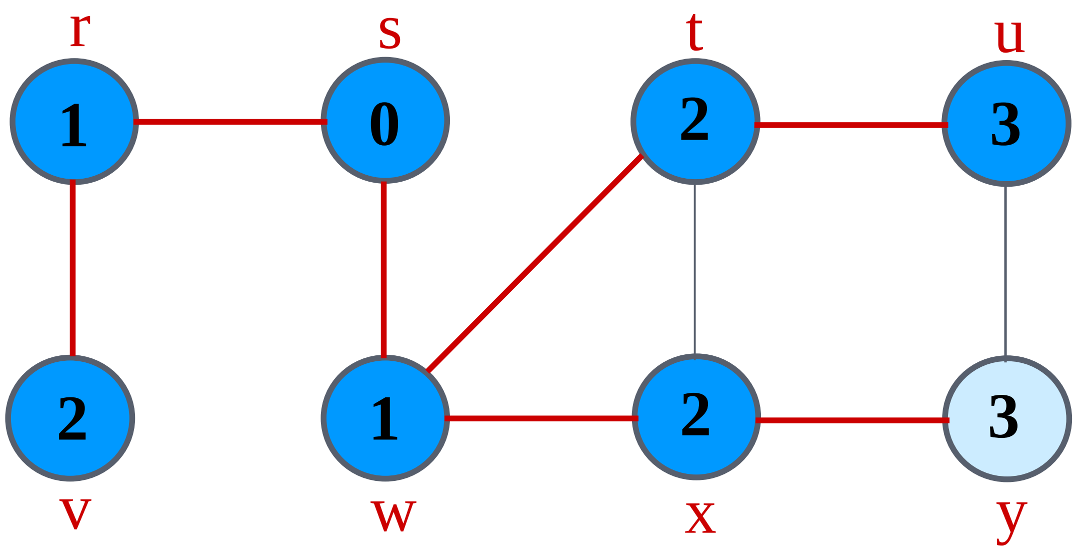
2 3 3

Example (BFS)



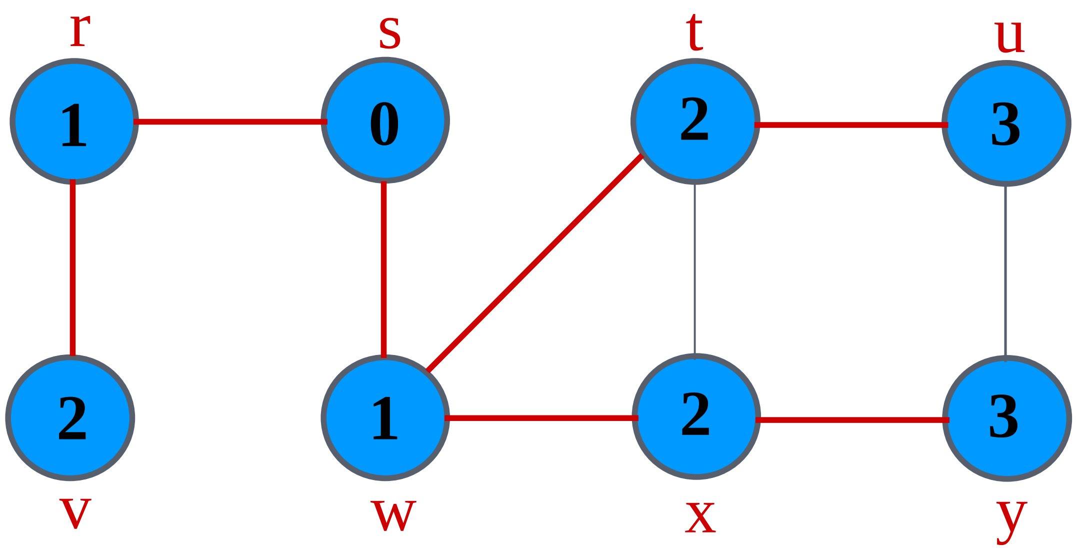
Q:	u	y
	3	3

Example (BFS)



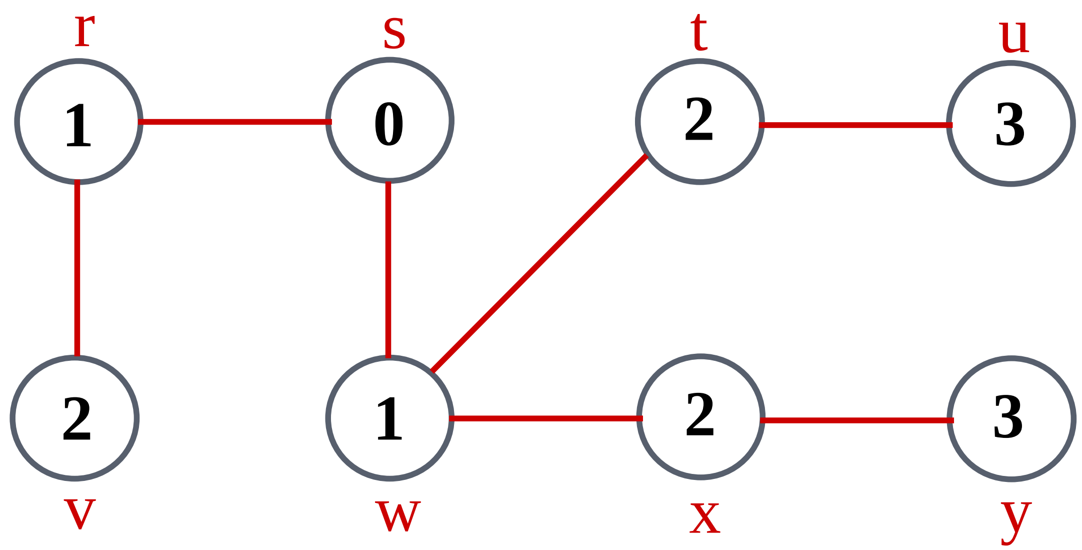
Q: y
3

Example (BFS)



Q: $\varnothing$

Example (BFS)



BF Tree

Application of BFS

- Finding Shortest Path
  - network (Computer, transportation[highway, flight])
  - GPS finding direction with shortest distance
  - Broadcasting of network node
  - Social network find people within k distance
  - **Crawlers in Search Engines**

DEPTH-FIRST SEARCH

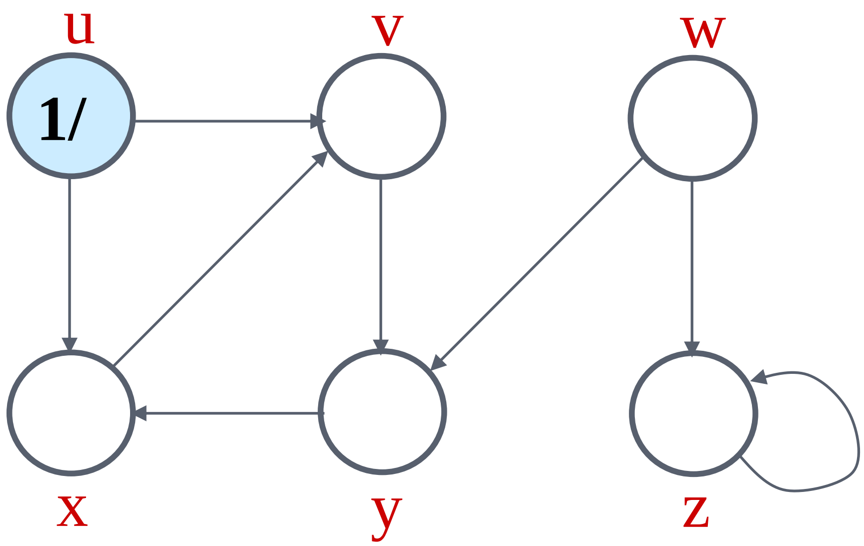
PSEUDO-CODE

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1                  // white vertex u has just been discovered
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]            // explore edge (u, v)
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK                  // blacken u; it is finished
time = time + 1
u.f = time
```

Uses a global timestamp ***time***.

EXAMPLE (DFS)



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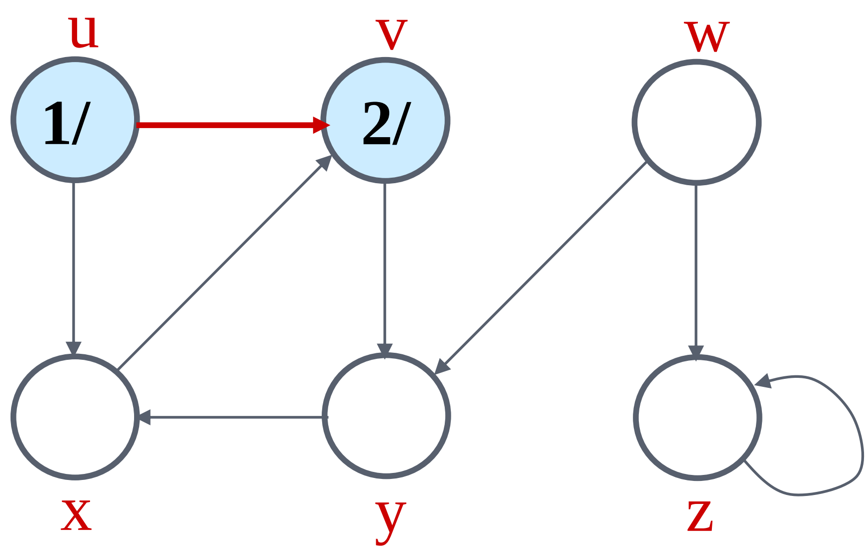

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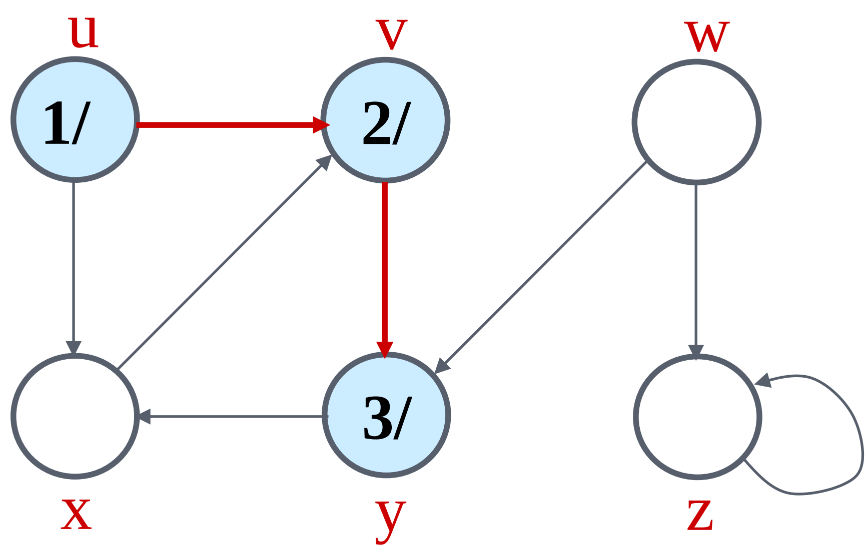

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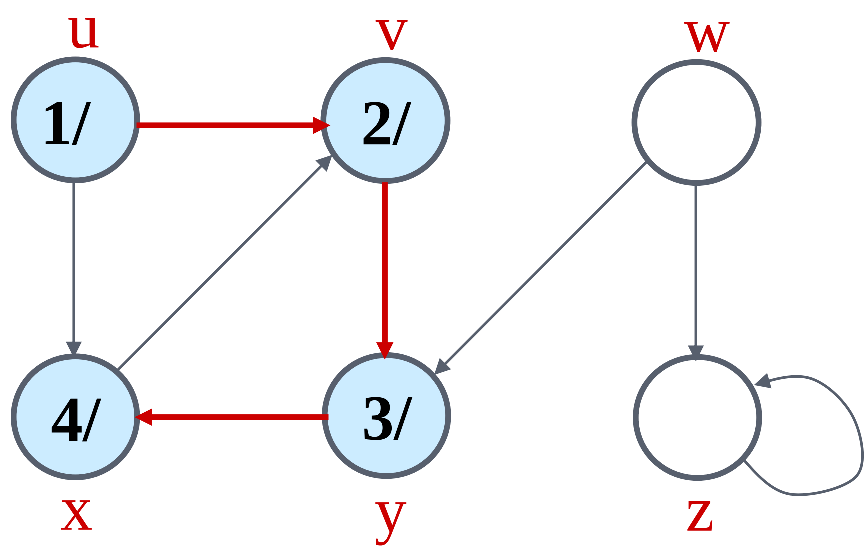

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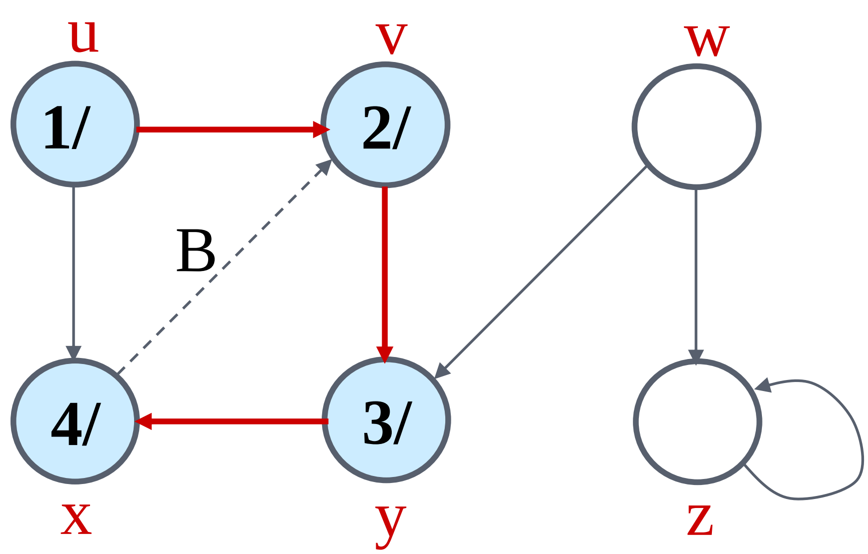

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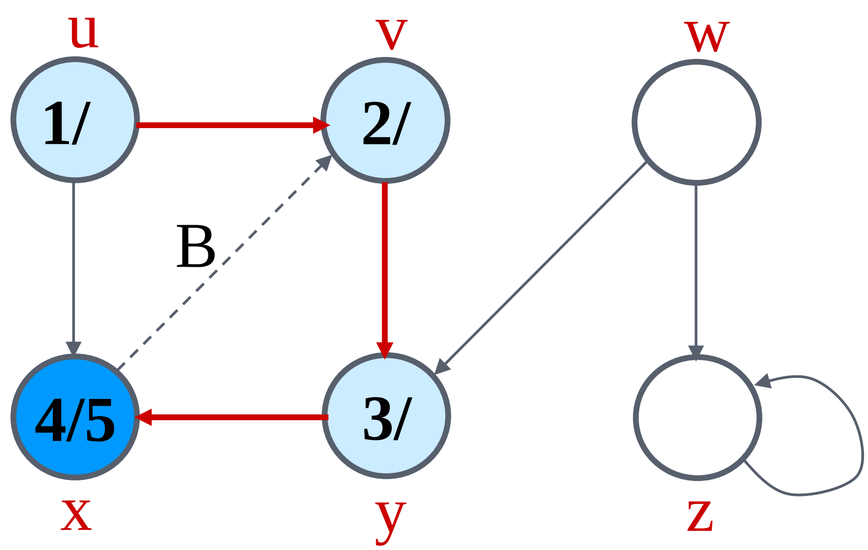

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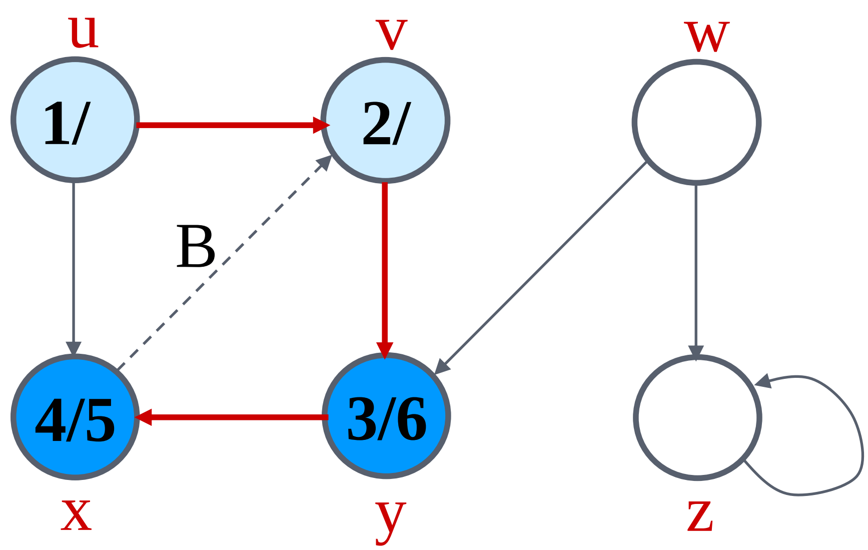

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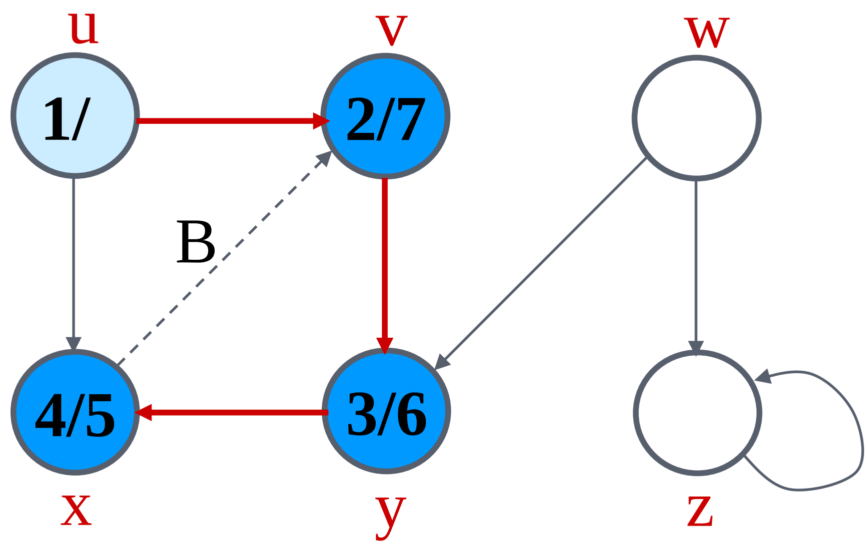

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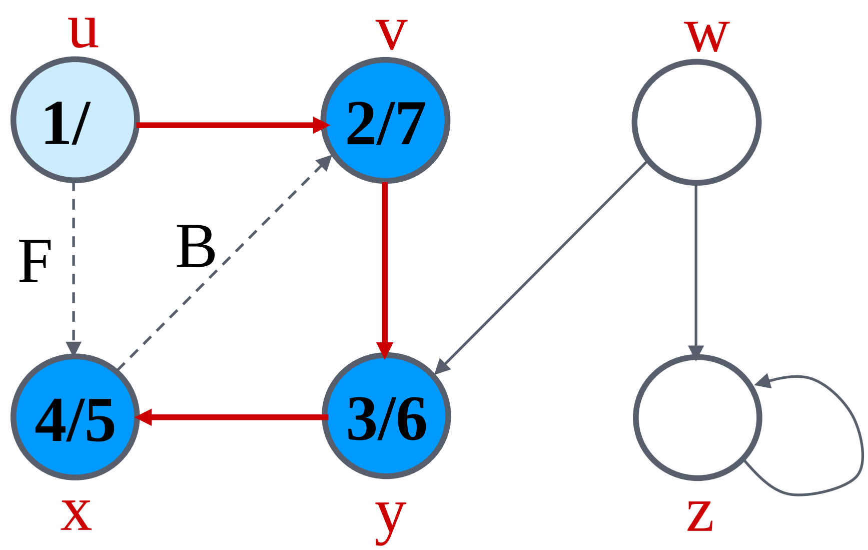

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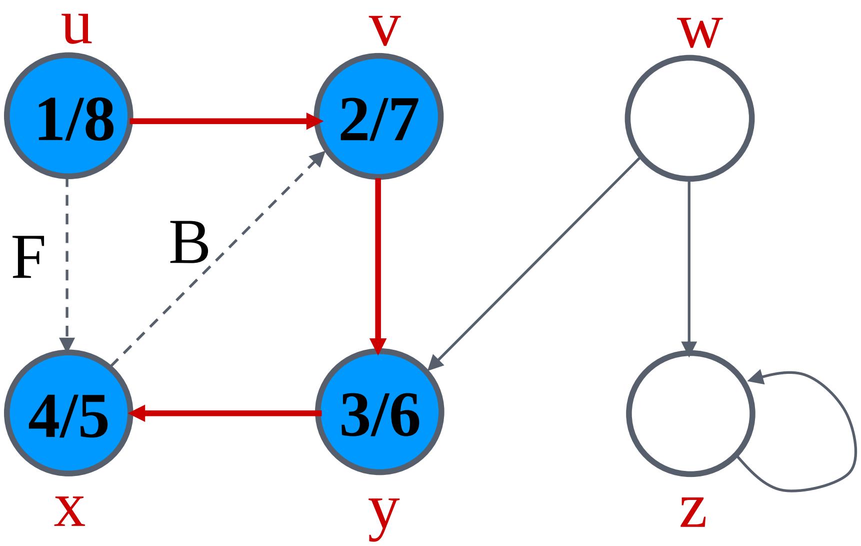

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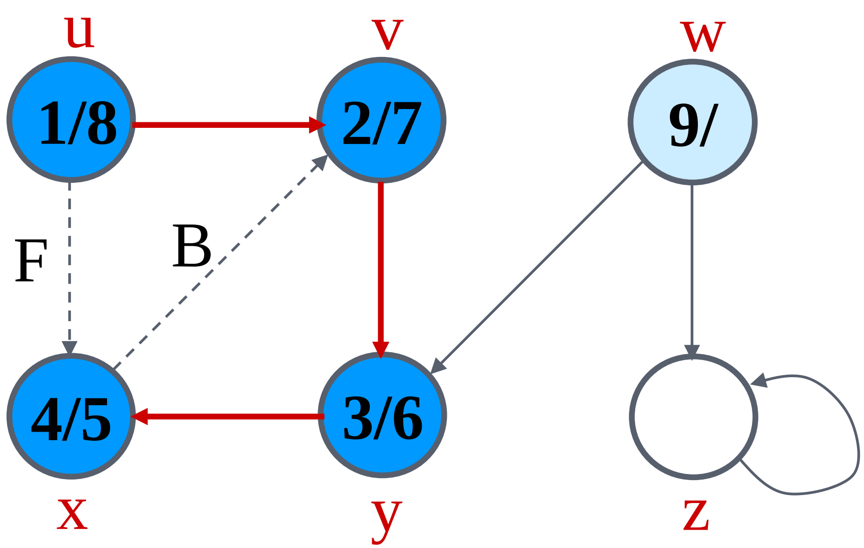

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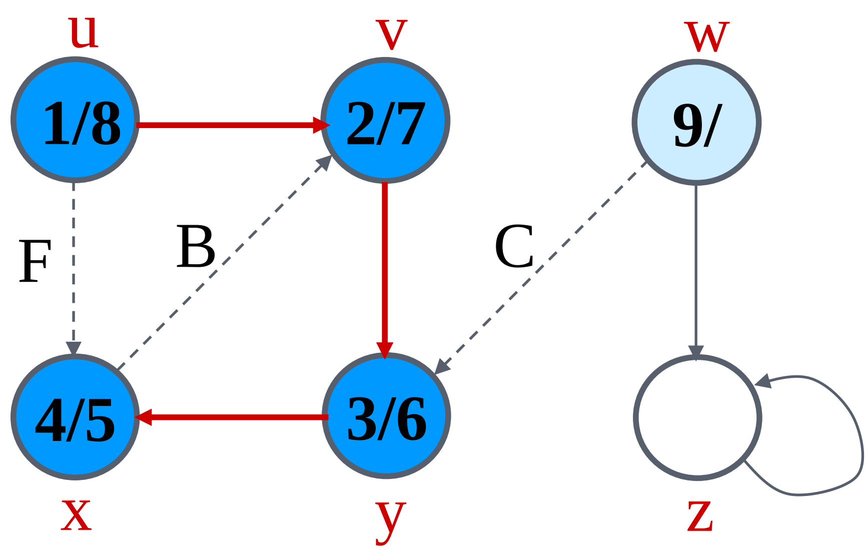

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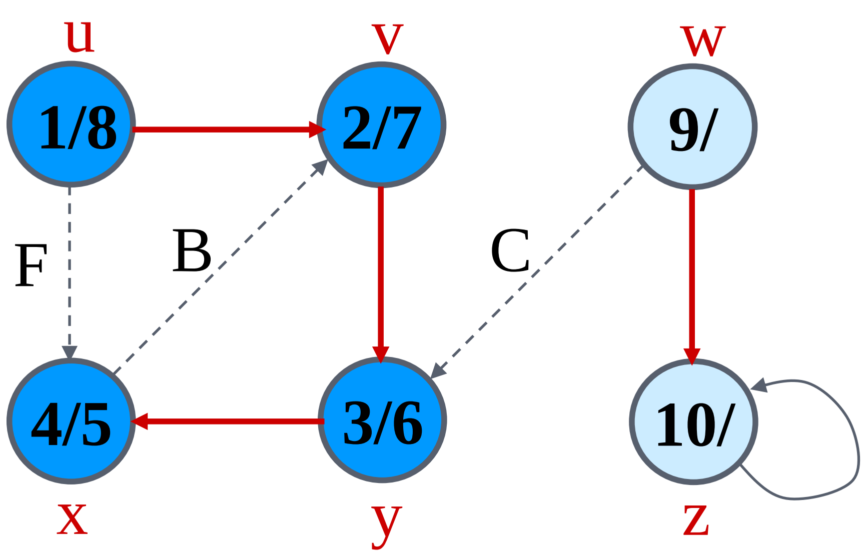

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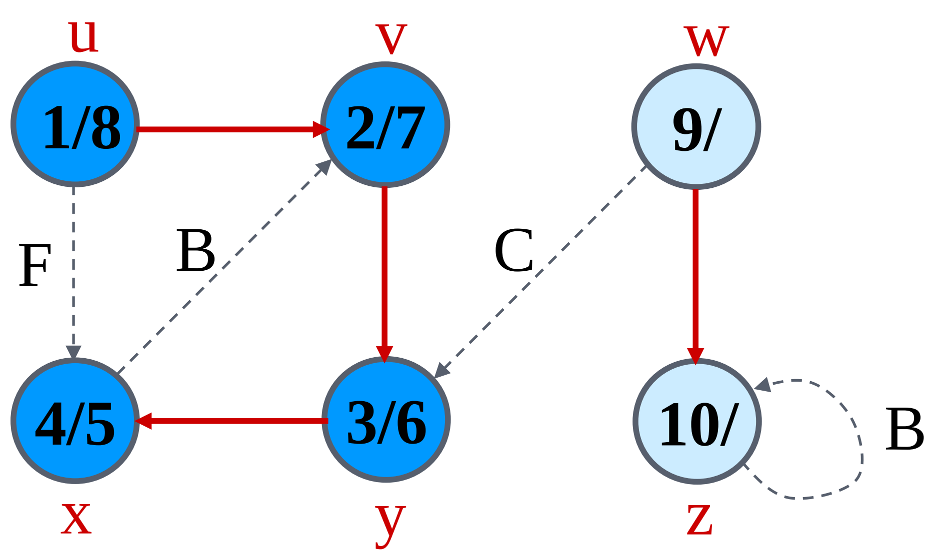

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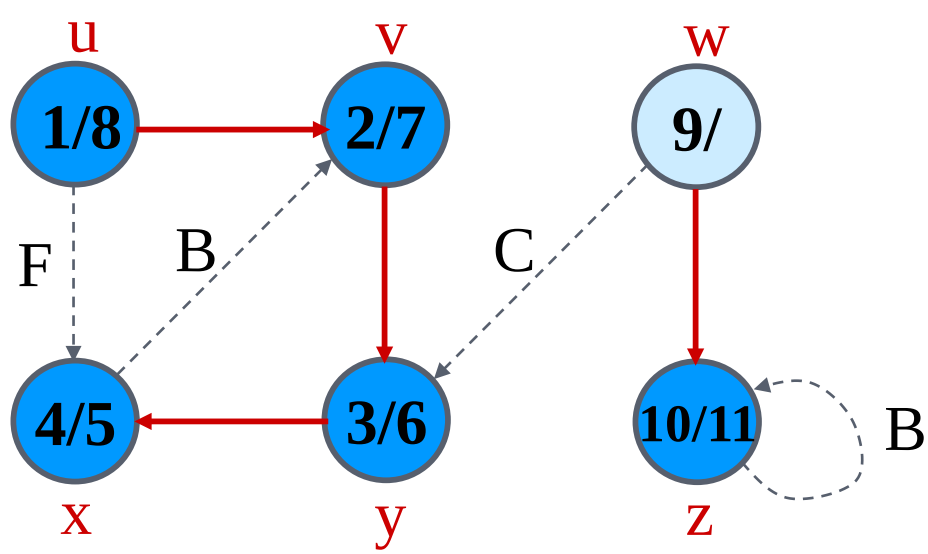

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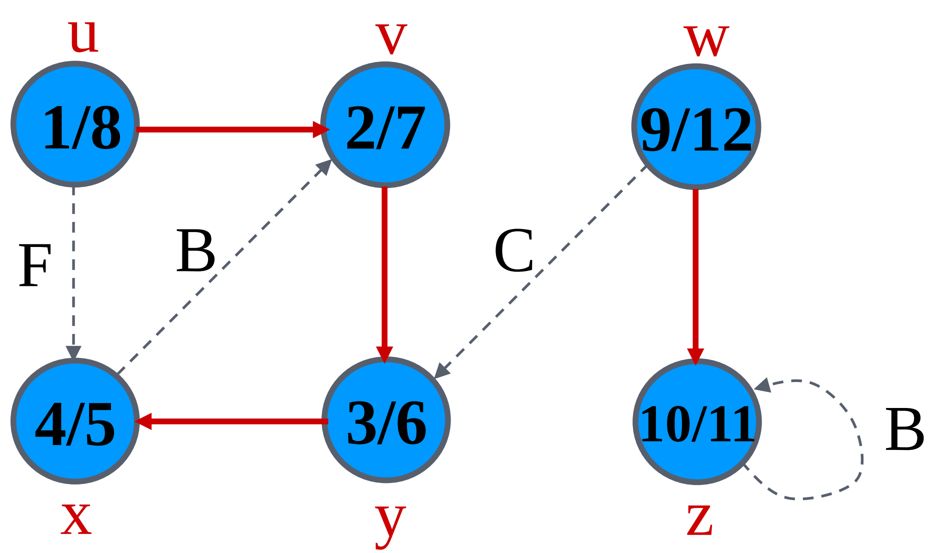

Application of DFS

- Finding path between 2 vertices.
- Finding connected components
- Topological sorting (Dependency resolution)
  - mainly used for scheduling a sequence of jobs or tasks based on their dependencies.

Reference

- Chapter 22 (22.1, 22.2, 22.3, 22.4) (Cormen)