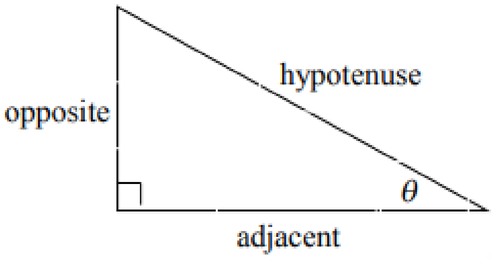
# Trigonometric Formulas

## Right-Triangle Definition:

For this definition we assume that

$0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.




$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

## Reciprocal Identities:

$$\sin x = \frac{1}{\csc x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \frac{1}{\sec x} \qquad \sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{1}{\cot x} \qquad \cot x = \frac{1}{\tan x}$$

## Ratio Identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sin x = \cos x \tan x \qquad \cos x = \sin x \cot x$$

## Tangent and Cotangent Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

## Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

## Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

## Half Angle Formulas:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

## Sum and Difference Formulas/Identities:

$$\sin(u+v) = \sin u\cos v + \cos u\sin v$$
$$\sin(u-v) = \sin u\cos v - \cos u\sin v$$

$$\cos(u+v) = \cos u\cos v - \sin u\sin v$$
$$\cos(u-v) = \cos u\cos v + \sin u\sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

## Double-angle formulas:

Substitute $\alpha = \beta$ in the previous sum formulas, then we find the double-angle formulas: :

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad (7)$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Two useful forms of (7) are derived by replacing $\cos^2\alpha$ by $1 - \sin^2\alpha$, resp. $\sin^2\alpha$ by $1 - \cos^2\alpha$ :

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$

And so:

$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

## Inverse Trig Functions:

**Definition**

$y = \sin^{-1} x$ is equivalent to $x = \sin y$

$y = \cos^{-1} x$ is equivalent to $x = \cos y$

$y = \tan^{-1} x$ is equivalent to $x = \tan y$

**Domain and Range**

| Function | Domain | Range |
|---|---|---|
| $y = \sin^{-1} x$ | $-1 \le x \le 1$ | $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ |
| $y = \cos^{-1} x$ | $-1 \le x \le 1$ | $0 \le y \le \pi$ |
| $y = \tan^{-1} x$ | $-\infty < x < \infty$ | $-\frac{\pi}{2} < y < \frac{\pi}{2}$ |

**Inverse Properties**

$\cos\left(\cos^{-1}(x)\right) = x \qquad \cos^{-1}\left(\cos(\theta)\right) = \theta$

$\sin\left(\sin^{-1}(x)\right) = x \qquad \sin^{-1}\left(\sin(\theta)\right) = \theta$

$\tan\left(\tan^{-1}(x)\right) = x \qquad \tan^{-1}\left(\tan(\theta)\right) = \theta$

**Alternate Notation**

$\sin^{-1} x = \arcsin x$

$\cos^{-1} x = \arccos x$

$\tan^{-1} x = \arctan x$

## Sum to Product Formulas:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

## Degrees to Radians Formulas:

If $x$ is an angle in degrees and $t$ is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

## Cofunction Formulas:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

# Law of Sines, Cosines and Tangents:

**Law of Sines**

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

**Law of Cosines**

$$a^2=b^2+c^2-2bc\cos\alpha$$

$$b^2=a^2+c^2-2ac\cos\beta$$

$$c^2=a^2+b^2-2ab\cos\gamma$$

**Mollweide's Formula**

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}$$

**Law of Tangents**

$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

## How to Find Reference Angles:

Step 1: Determine which quadrant the angle is in

Step 2: Use the appropriate formula

| | | | | |
|---|---|---|---|---|
| Quad I | = | is the angle itself | | |
| Quad II | = | $180 - \theta$ | or | $\pi - \theta$ |
| Quad III | = | $\theta - 180$ | or | $\theta - \pi$ |
| Quad IV | = | $360 - \theta$ | or | $2\pi - \theta$ |

## Odd-Even Identities:

| | |
|---|---|
| Sin $(-x)$ = $-\sin x$ | Csc $(-x)$ = $-\csc x$ |
| Cos $(-x)$ = $\cos x$ | Sec $(-x)$ = $\sec x$ |
| Tan $(-x)$ = $-\tan x$ | Cot $(-x)$ = $-\cot x$ |

# Facts and Properties

## Period:

The period of a function is the number, $T$, such that $f(\theta+T)=f(\theta)$. So, if $\omega$ is a fixed number and $\theta$ is any angle we have the following periods.

$$\sin(\omega\theta) \rightarrow T=\frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \rightarrow T=\frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \rightarrow T=\frac{\pi}{\omega}$$

$$\csc(\omega\theta) \rightarrow T=\frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \rightarrow T=\frac{2\pi}{\omega}$$

$$\cot(\omega\theta) \rightarrow T=\frac{\pi}{\omega}$$

## Domain:

The domain is all the values of $\theta$ that can be plugged into the function.

$\sin\theta$ , $\theta$ can be any angle

$\cos\theta$ , $\theta$ can be any angle

$\tan\theta$ , $\theta \neq \left(n+\frac{1}{2}\right)\pi,\quad n=0,\pm1,\pm2,\ldots$

$\csc\theta$ , $\theta \neq n\pi,\quad n=0,\pm1,\pm2,\ldots$

$\sec\theta$ , $\theta \neq \left(n+\frac{1}{2}\right)\pi,\quad n=0,\pm1,\pm2,\ldots$

$\cot\theta$ , $\theta \neq n\pi,\quad n=0,\pm1,\pm2,\ldots$

## Range:

The range is all possible values to get out of the function.

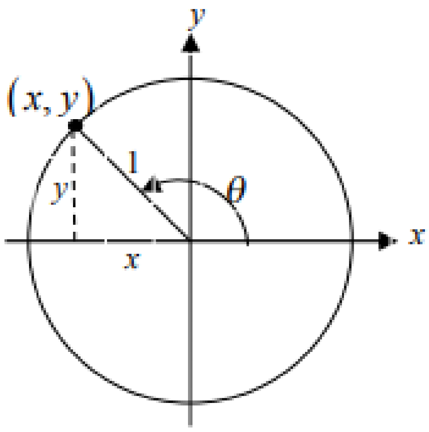
$-1 \leq \sin\theta \leq 1$ $\qquad$ $\csc\theta \geq 1$ and $\csc\theta \leq -1$

$-1 \leq \cos\theta \leq 1$ $\qquad$ $\sec\theta \geq 1$ and $\sec\theta \leq -1$

$-\infty < \tan\theta < \infty$ $\qquad$ $-\infty < \cot\theta < \infty$

# Unit circle definition:

For this definition $\theta$ is any angle.

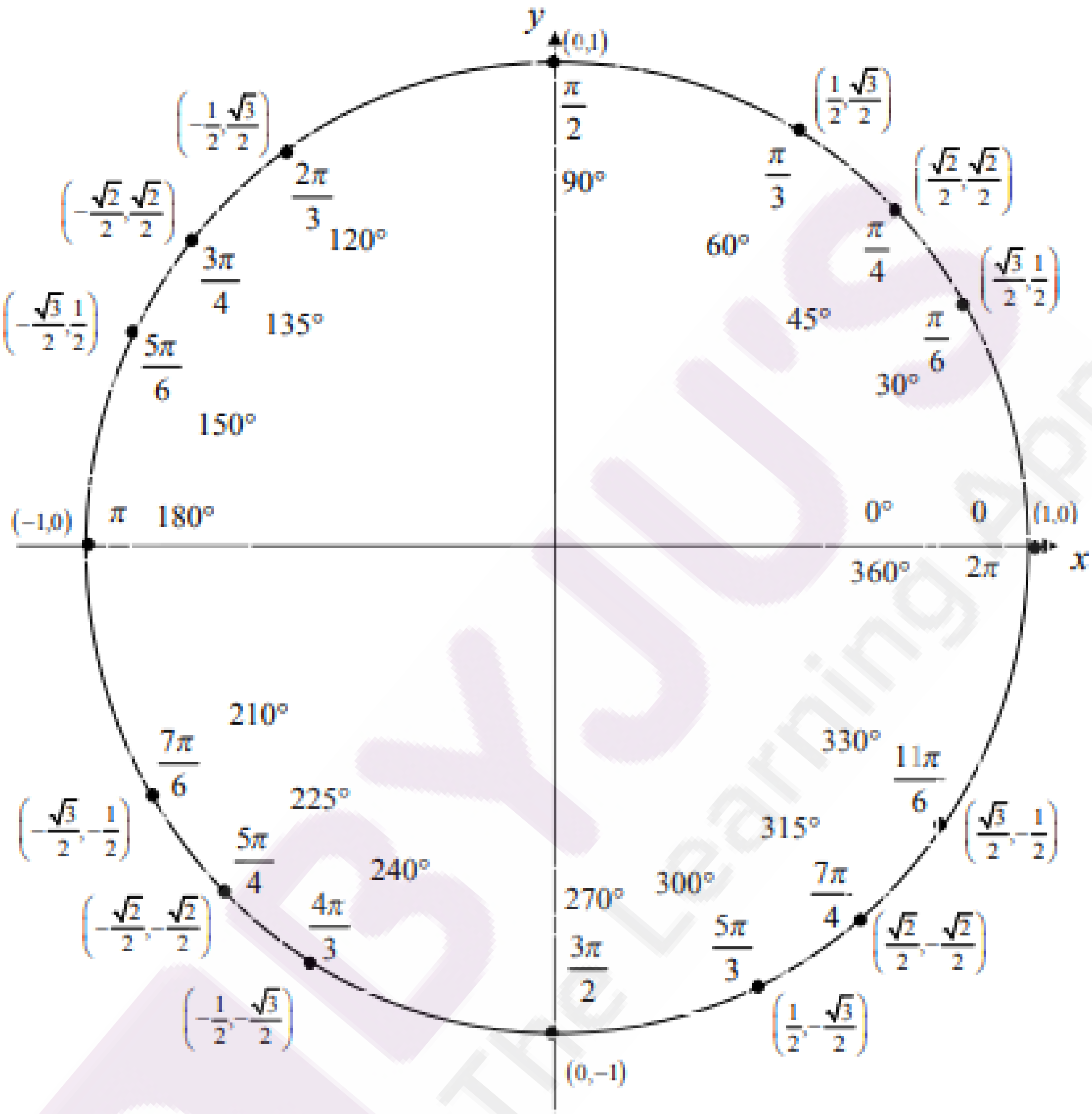



$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

# Unit Circle:



y
x
(0,1)
(1,0)
(−1,0)
(0,−1)
$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° $2\pi$
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
$\pi$ 180°
210° $\frac{7\pi}{6}$
225° $\frac{5\pi}{4}$
240° $\frac{4\pi}{3}$
270° $\frac{3\pi}{2}$
300° $\frac{5\pi}{3}$
315° $\frac{7\pi}{4}$
330° $\frac{11\pi}{6}$

# Radian and Degree Measures of Angles:

$$1\ \text{rad} = \frac{180^\circ}{\pi} \approx 57^\circ 17' 45''$$

$$1^\circ = \frac{\pi}{180}\ \text{rad} \approx 0.017453\ \text{rad}$$

$$1' = \frac{\pi}{180 \cdot 60}\ \text{rad} \approx 0.000291\ \text{rad}$$

$$1'' = \frac{\pi}{180 \cdot 3600}\ \text{rad} \approx 0.000005\ \text{rad}$$

| Angle (degrees) | 0 | 30 | 45 | 60 | 90 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|
| Angle (radians) | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

# Signs of Trigonometric Functions:

| Quadrant | Sin $\alpha$ | Cos $\alpha$ | Tan $\alpha$ | Cot $\alpha$ | Sec $\alpha$ | Cosec $\alpha$ |
|---|---|---|---|---|---|---|
| I | + | + | + | + | + | + |
| II | + | | | | | + |
| III | | | + | + | | |
| IV | | + | | | + | |

# Trigonometric Functions of Common Angles:

| $\alpha°$ | $\alpha$ rad | $\sin\alpha$ | $\cos\alpha$ | $\tan\alpha$ | $\cot\alpha$ | $\sec\alpha$ | $\operatorname{cosec}\alpha$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | $\infty$ | 1 | $\infty$ |
| 30 | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ | $\sqrt{3}$ | $\frac{2}{\sqrt{3}}$ | 2 |
| 45 | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ |
| 60 | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{\sqrt{3}}$ | 2 | $\frac{2}{\sqrt{3}}$ |
| 90 | $\frac{\pi}{2}$ | 1 | 0 | $\infty$ | 0 | $\infty$ | 1 |
| 120 | $\frac{2\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{\sqrt{3}}$ | $-2$ | $\frac{2}{\sqrt{3}}$ |
| 180 | $\pi$ | 0 | $-1$ | 0 | $\infty$ | $-1$ | $\infty$ |
| 270 | $\frac{3\pi}{2}$ | $-1$ | 0 | $\infty$ | 0 | $\infty$ | $-1$ |
| 360 | $2\pi$ | 0 | 1 | 0 | $\infty$ | 1 | $\infty$ |

| $\alpha°$ | $\alpha$ rad | $\sin\alpha$ | $\cos\alpha$ | $\tan\alpha$ | $\cot\alpha$ |
|---|---|---|---|---|---|
| 15 | $\frac{\pi}{12}$ | $\frac{\sqrt{6}-\sqrt{2}}{4}$ | $\frac{\sqrt{6}+\sqrt{2}}{4}$ | $2-\sqrt{3}$ | $2+\sqrt{3}$ |
| 18 | $\frac{\pi}{10}$ | $\frac{\sqrt{5}-1}{4}$ | $\frac{\sqrt{10+2\sqrt{5}}}{4}$ | $\sqrt{\frac{5-2\sqrt{5}}{5}}$ | $\sqrt{5+2\sqrt{5}}$ |
| 36 | $\frac{\pi}{5}$ | $\frac{\sqrt{10-2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}+1}{4}$ | $\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$ | $\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$ |
| 54 | $\frac{3\pi}{10}$ | $\frac{\sqrt{5}+1}{4}$ | $\frac{\sqrt{10-2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$ | $\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$ |
| 72 | $\frac{2\pi}{5}$ | $\frac{\sqrt{10+2\sqrt{5}}}{4}$ | $\frac{\sqrt{5}-1}{4}$ | $\sqrt{5+2\sqrt{5}}$ | $\sqrt{\frac{5-2\sqrt{5}}{5}}$ |
| 75 | $\frac{5\pi}{12}$ | $\frac{\sqrt{6}+\sqrt{2}}{4}$ | $\frac{\sqrt{6}-\sqrt{2}}{4}$ | $2+\sqrt{3}$ | $2-\sqrt{3}$ |

## Most Important Formulas:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sec^2\alpha - \tan^2\alpha = 1$$

$$\csc^2\alpha - \cot^2\alpha = 1$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$\tan\alpha \cdot \cot\alpha = 1$$

$$\sec\alpha = \frac{1}{\cos\alpha}$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha}$$