Algorithm Analysis / Big Theta exercise

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SUMMARY :

The purpose of this document is to demonstrate a method of proof that seeks to show $f(n) = \Theta(g(n))$.

EXPLANATION :

LINE1: Show that $f(n) = \Theta(g(n))$ where $f(n) = \sqrt{n + 10}$ and $g(n) = \sqrt{n}$.

(1) Essentially, this statement is saying that $f(n)$ grows no faster or slower than $g(n)$.

LINE2: If $f(n)$ grows no faster than $g(n)$ then $f(n) = O(g(n))$.
If $f(n)$ grows no slower than $g(n)$ then $f(n) = \Omega(g(n))$.

We can formally say that $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

LINE3: **(2)** Definition of $f(n) = O(g(n))$

Let $g(n)$ be some function, c some constant, assume $f(n)\ and\ g(n)$ are asymptotically non-negative.
$O(g(n)) = \{ f(n) \mid \exists c > 0,\ \exists n_0 > 0,\ \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n) \}$

LINE4: **(3)** Definition of $f(n) = \Omega(g(n))$

Let $g(n)$ be some function, c some constant, assume $f(n)\ and\ g(n)$ are asymptotically non-negative.
$\Omega(g(n)) = \{ f(n) \mid \exists x > 0,\ \exists n_0 > 0,\ \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n) \}$

LINE5: We can express **(1)** as an inequality using **(2)** and **(3)**.

$0 \leq c_1\sqrt{n} \leq \sqrt{n + 10} \leq c_2\sqrt{n}$

LINE6: By finding values for c_1, c_2 and n_0 then we assert the existence of a single case proving the statement.

LINE7: $0 \leq c_1 \cdot \sqrt{n} \leq \sqrt{n + 10} \leq c_2 \cdot \sqrt{n}$ // Initial expression

LINE8: $0 \leq c_1^2 \cdot n \leq n + 10 \leq c_2^2 \cdot n$ // We squared all sides of the inequality to simplify operations.

LINE9: $c_1^2 \cdot n \leq n + 10$ AND $n + 10 \leq c_2^2 \cdot n$ // Split the inequality.

LINE10: $-10 \leq n - (c_1^{\,2} \cdot n)$ AND $10 \leq c_2^{\,2} \cdot n - n$ // Isolate n on both sides.

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LINE11: $-10 \leq (1 - c_1^{\,2})n$ AND $10 \leq (c_2^{\,2} - 1)n$ // factor n on both sides.

LINE12: $-10 \leq (1 - 1)n$ AND $10 \leq (2 - 1)n$ // Chose a c that'll allow us to select an n that'll make these inequalities true.
$c_1 = 1,\ c_2 = \sqrt{2}$

LINE13: $-10 \leq 0$ AND $10 \leq n$ // Simplify

LINE14: By setting $c_1 = 1,\ c_2 = \sqrt{2},\ n_0 = 10$ we've satisfied the requirements and illustrated that $f(n) = \Theta(g(n))$.

$Q.E.D.$