1(a) *Proof.* First note that for any fixed $a > 0$ and $x \neq 0$.

$$\frac{a^x - 1}{x} = \frac{e^{x \log a} - e^{x \log 1}}{x} = \left[\frac{e^{xt}}{x}\right]_{t=\log 1}^{t=\log a} = \int_{\log 1}^{\log a} e^{xt}\, \mathrm{d}t.$$

Now consider any real sequence $(x_n) \subseteq \mathbb{R} \setminus \{0\}$ s.t. $x_n \to 0$. Define $f : I \to \mathbb{R}$, where I is the closed interval between $\log a$ and $\log 1 (= 0)$, as

$$f_n(t) = e^{x_n t}.$$

We claim that $f_n(t)$ converges uniformly to $f(t) = 1$ on I. To see this, first note that for each n, f_n is monotone on I. Therefore,

$$\begin{aligned} M_n &= \sup_{t \in I} |f_n(t) - f(t)| && \text{(by the def. of } M_n) \\ &= \sup_{t \in I} |f_n(t) - f_n(0)| && \text{(since } f_n(0) = f(t), \forall n, t) \\ &= \sup_{t \in I} |f_n(\log a) - f_n(0)| && \text{(since } f_n \text{ monotone on } I) \\ &= \sup_{t \in I} |f_n(\log a) - 1| \,. && \text{(since } f_n(0) = 1) \end{aligned}$$

Therefore, $M_n \to 0$ since $f_n(\log a) \to 1$ as $n \to \infty$. This concludes the proof that $f_n(t)$ converges uniformly to 1. Hence by Rudin Theorem 7.16,

$$\lim_{n \to \infty} \int_{\log 1}^{\log a} f_n(t)\, \mathrm{d}t = \int_{\log 1}^{\log a} f(t)\, \mathrm{d}t, \text{ i.e., } \lim_{n \to \infty} \int_{\log 1}^{\log a} e^{x_n t}\, \mathrm{d}t = \int_{\log 1}^{\log a} 1\, \mathrm{d}t = \log a.$$

Since (x_n) is an arbitrary sequence in $\mathbb{R} \setminus \{0\}$ that converges to 0, by Rudin Theorem 4.2,

$$\lim_{x \to 0} \int_{\log 1}^{\log a} e^{xt} dt = \log a.$$

Combining with the previous results, we can thus conclude that,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$

■

1(b) *Proof.* First note that by HW7(a), if $x \neq 0$ is small enough, say $|x| < 1$,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This means that for a fixed $x \neq 0$ small enough,

$$\begin{aligned} \frac{1}{x^2} \left[\log(1 + x) - x\right] &= \frac{1}{x^2} \left[-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right] \\ &= -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots && \text{(since } \textstyle\sum_{k=2}^{\infty} (-1)^{k-1} \frac{x^k}{k} \text{ converges)} \\ &= -\frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k+2}. \end{aligned}$$

Note that since

$$\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k+2}} = 1,$$

the series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ has raidus of convergence of 1. This means that on $(-1,1)$, the function $f(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ is continuous by Rudin Theorem 8.1. Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{\log(1+x)-x}{x^2} &= \lim_{x\to 0} -\frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since they agree on } |x|<1) \\
&= -\frac{1}{2} + \lim_{x\to 0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since the series converges)} \\
&= -\frac{1}{2} + f(0) && \text{(since } f \text{ is continous at } 0) \\
&= -\frac{1}{2}. && \text{(since } f(0)=0)
\end{aligned}$$

■

1(c) *Proof.* First note that with the results in Rudin Theorem 3.31, it's clear that for any sequence $(a_n) \subseteq \mathbb{R}$ where $a_n \to \infty$,

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

We thus fix $x \in \mathbb{R}$. If $x = 0$, then

$$\lim_{n\to\infty}\left(1+\frac{0}{n}\right)^n = \lim_{n\to\infty} 1 = 1 = e^0.$$

If $x \neq 0$, then

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \left[\left(1+\frac{1}{n/x}\right)^{n/x}\right]^x = e^x,$$

since as $n\to\infty$, $n/x \to \infty$. ■

2(a) *Proof.* ■

2(b) *Proof.* ■

3(a) *Proof.* Let's first compute each c_m. By Rudin (62),

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,\mathrm{d}x.$$

If $m = 0$, then with the periodicity of the f,

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0.$$

If $m \neq 0$, then with integration by parts,

$$\begin{aligned}
c_m &= \frac{1}{2\pi}\int_{-\pi}^{0} \frac{-\pi - x}{2}e^{-imx}\,\mathrm{d}x + \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\pi - x}{2}e^{-imx}\,\mathrm{d}x \\
&= \frac{-i\pi m + e^{i\pi m} - 1}{4\pi m^2} + \frac{-i\pi m - e^{-i\pi m} + 1}{4\pi m^2}.
\end{aligned}$$

Therefore, when m is even,

$$c_m = \frac{-i\pi m + 1 - 1}{4\pi m^2} + \frac{-i\pi m - 1 + 1}{4\pi m^2} = \frac{-i}{2m},$$

and similarly, when m is odd,

$$c_m = \frac{-i\pi m + (-1) - 1}{4\pi m^2} + \frac{-i\pi m - (-1) + 1}{4\pi m^2} = \frac{-i}{2m}.$$

* * *

Now note that for $m \neq 0$,

$$\begin{aligned}
c_m e^{imx} &= -\frac{1}{2m}e^{i(mx+\pi/2)} \\
&= -\frac{1}{2m}\left(\cos(mx+\pi/2) + i\sin(mx+\pi/2)\right) \\
&= \frac{1}{2m}\left(\sin(mx) - i\sin(mx+\pi/2)\right) \\
&= \frac{1}{2m}\left(\sin(mx) - i\cos(mx)\right).
\end{aligned}$$

And similarly,

$$\begin{aligned}
c_{-m}e^{-imx} &= \frac{1}{2(-m)}\left(\sin(-mx) - i\sin(-mx+\pi/2)\right) \\
&= \frac{1}{2m}\left(\sin(mx) + i\sin(-mx+\pi/2)\right) \\
&= \frac{1}{2m}\left(\sin(mx) + i\sin(-(mx-\pi/2))\right) \\
&= \frac{1}{2m}\left(\sin(mx) - i\sin(mx-\pi/2)\right) \\
&= \frac{1}{2m}\left(\sin(mx) + i\cos(mx)\right).
\end{aligned}$$

This means that,

$$c_m e^{imx} + c_{-m} e^{-imx} = \frac{1}{m}\sin(mx).$$

Therefore,

$$\sum_{m=-\infty}^{\infty} c_m e^{imx} = \sum_{n=0}^{\infty} \frac{1}{n}\sin(nx).$$

Also since $f(x)$ is differentiable on $[-\pi, 0)$ and $(0, \pi]$, $f(x)$ is thus Lipschitz continuous on there as well. Therefore, by Rudin Theorem 8.14, on those intervals, $f(x)$'s Fourier series converge to $f(x)$, which means that on those intervals,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n}\sin(nx).$$

■

3(b) *Proof.* We plug in $x = \pi/2$ into above equation as $\pi/2$ is within in the said intervals. Notice that on one side that

$$f\left(\frac{\pi}{2}\right) = \frac{\pi - \pi/2}{2} = \frac{\pi}{4}.$$

On another

$$f\left(\frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right).$$

And when n is even, $\sin(n\pi/2) = 0$, when $n \equiv 1 \pmod 4$, $\sin(n\pi/2) = 1$, and when $n \equiv 3 \pmod 4$, $\sin(n\pi/2) = -1$. This concludes the proof that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

■

3(c) *Proof.* ■

4(a) *Proof.* First notice that for any $x \in (0, \pi/2)$,$(\pi - x, \pi + x) \subseteq (\pi/2, 3\pi/2)$. This means that $\pi/2 < s < 3\pi/2$, so $\pi/4 < s/2 < 3\pi/4$. Therefore, $\sin(s/2) \neq 0$ on any $(\pi - x, \pi + x)$. This means that the integral of $D_N(s)$ over $(\pi - x, \pi + x)$ is well-defined.

Now we fix $x \in (0, \pi/2)$. Note that

$$\begin{aligned}\int_{\pi-x}^{\pi+x} D_N(s) &= \int_{\pi-x}^{\pi+x} \frac{\sin(Ns + s/2)}{s/2}\,\mathrm{d}s \\ &= \int_{\pi-x}^{\pi+x} \frac{\sin(Ns)\cos(s/2) + \cos(Ns)\sin(s/2)}{\sin(s/2)}\,\mathrm{d}s \\ &= \int_{\pi-x}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \int_{\pi-x}^{\pi+x} \cos(Ns)\,\mathrm{d}s.\end{aligned}$$

Now, define $g : [-\pi, \pi] \to \mathbb{R}$ as

$$g(s) = \begin{cases} \cot(s/2) & \text{if } s \in (\pi - x, \pi + x), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(s) \in \mathcal{R}[-\pi, \pi]$, since $\cot(s/2)$ is continous on $\pi - x, \pi + x$, which means that $g(s)$ has finitely many discontinuities on $[-\pi, \pi]$. And since the sequence of Fourier cofficients c_N goes to zero as $N \to \infty$,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \sin(Ns) g(s)\,\mathrm{d}s = 0.$$

Noice also that for each N,

$$\int_{-\pi}^{\pi} \sin(Ns) g(s)\,\mathrm{d}s = \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s.$$

Thus,

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s = 0.$$

And with a reasoning along with the peridocity of sin and cot, we also get that

$$\lim_{N\to\infty} \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s = 0.$$

Therefore,

$$\begin{aligned}&\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s \\ =& \lim_{N\to\infty} \left(\int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s \right) \\ =& \lim_{N\to\infty} \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \lim_{n\to\infty} \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2) \\ =& 0 + 0 \\ =& 0.\end{aligned}$$

With a similar reasoning, we can get that

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} \cos(Ns)\,\mathrm{d}s = 0.$$

Therefore,

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} D_n(s)\,\mathrm{d}s, \text{ which gives that } \lim_{N\to\infty} r_N(x) = 0.$$

■

4(b) *Proof.* It's suffice to show that

$$\frac{\pi}{2}s_N(x_N) \to \int_0^x \frac{\sin(t)}{t}\,\mathrm{d}t.$$

Note that

$$\frac{\pi}{2}s_N(x_N) = \int_0^{x_N} \frac{\sin(N+1/2)t}{2\sin(t/2)}\,\mathrm{d}t + \frac{\pi}{2}r_N(x_N).$$

Let $u = (N+1/2)t$. Therefore,

$$\int_0^{x_N} \frac{\sin(N+1/2)t}{2\sin(t/2)}\,\mathrm{d}t = \frac{1}{2N+1}\int_0^{\pi} \frac{\sin u}{\sin\left(\frac{u}{2N+1}\right)}\,\mathrm{d}u.$$

Note that as $N\to\infty$, $\sin\left(\frac{u}{2N+1}\right) \to \frac{u}{2N+1}$. Therefore, above RHS approaches $\int_0^{\pi} \frac{\sin t}{t}\,\mathrm{d}t$.

It remains to show that $r_N(x_N) \to 0$ as $N\to\infty$. First note that there exists $\delta > 0$ such that $\sin(s/2) > 1/2$ for all $s/2 \in (\pi/2 - \delta/2, \pi/2 + \delta/2)$ or for all $s \in (\pi-\delta, \pi+\delta)$. This means that $1/|\sin(s/2)| < 2$ for all $s \in (\pi-\delta,\pi+\delta)$. And note that on $(\pi-\delta,\pi+\delta)$, for all N,

$$|D_N(s)| = \left|\frac{\sin(N+1/2)s}{\sin(s/2)}\right| \le \frac{1}{|\sin(s/2)|} < 2.$$

Given $\varepsilon > 0$. Since $x_N \to 0$, then there exists M such that for all $N > M$, $|x_N| < \varepsilon$. This also means that there exists L such that for all $N > L$, $(\pi - x_N, \pi + x_N) \subseteq (\pi-\delta,\pi+\delta)$. Therefore, for all $N > \max\{M, L\}$,

$$|r_N(x_N)| \le \frac{1}{2\pi}\int_{\pi-\varepsilon}^{\pi+\varepsilon} |D_N(s)|\,\mathrm{d}s \le \frac{1}{2\pi}\int_{\pi-\varepsilon}^{\pi+\varepsilon} 2\,\mathrm{d}s = \frac{2\varepsilon}{\pi}.$$

This concludes the proof that $r_N(x_N) \to 0$. ■