1(a) *Proof.* First note that for any fixed $a > 0$ and $x \neq 0$.

$$\frac{a^x - 1}{x} = \frac{e^{x \log a} - e^{x \log 1}}{x} = \left[\frac{e^{xt}}{x}\right]_{t=\log 1}^{t=\log a} = \int_{\log 1}^{\log a} e^{xt}\,\mathrm{d}t.$$

Now consider any real sequence $(x_n) \subseteq \mathbb{R} \setminus \{0\}$ s.t. $x_n \to 0$. Define $f : I \to \mathbb{R}$, where I is the closed interval between $\log a$ and $\log 1 (= 0)$, as

$$f_n(t) = e^{x_n t}.$$

We claim that $f_n(t)$ converges uniformly to $f(t) = 1$ on I. To see this, first note that for each n, f_n is monotone on I. Therefore,

$$\begin{aligned}
M_n &= \sup_{t \in I} |f_n(t) - f(t)| && \text{(by the def. of } M_n) \\
&= \sup_{t \in I} |f_n(t) - f_n(0)| && \text{(since } f_n(0) = f(t), \forall n, t) \\
&= \sup_{t \in I} |f_n(\log a) - f_n(0)| && \text{(since } f_n \text{ monotone on } I) \\
&= \sup_{t \in I} |f_n(\log a) - 1|. && \text{(since } f_n(0) = 1)
\end{aligned}$$

Therefore, $M_n \to 0$ since $f_n(\log a) \to 1$ as $n \to \infty$. This concludes the proof that $f_n(t)$ converges uniformly to 1. Hence by Rudin Theorem 7.16,

$$\lim_{n\to\infty} \int_{\log 1}^{\log a} f_n(t)\,\mathrm{d}t = \int_{\log 1}^{\log a} f(t)\,\mathrm{d}t, \text{ i.e., } \lim_{n\to\infty} \int_{\log 1}^{\log a} e^{x_n t}\,\mathrm{d}t = \int_{\log 1}^{\log a} 1\,\mathrm{d}t = \log a.$$

Since (x_n) is an arbitrary sequence in $\mathbb{R} \setminus \{0\}$ that converges to 0, by Rudin Theorem 4.2,

$$\lim_{x\to 0} \int_{\log 1}^{\log a} e^{xt} dt = \log a.$$

Combining with the previous results, we can thus conclude that,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

■

1(b) *Proof.* First note that by HW7(a), if $x \neq 0$ is small enough, say $|x| < 1$,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This means that for a fixed $x \neq 0$ small enough,

$$\begin{aligned}
\frac{1}{x^2}[\log(1 + x) - x] &= \frac{1}{x^2}\left[-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right] \\
&= -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots && \text{(since } \textstyle\sum_{k=2}^{\infty}(-1)^{k-1}\frac{x^k}{k} \text{ converges)} \\
&= -\frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}.
\end{aligned}$$

Note that since

$$\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k+2}} = 1,$$

the series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ has raidus of convergence of 1. This means that on $(-1,1)$, the function $f(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ is continuous by Rudin Theorem 8.1. Therefore,

$$\begin{aligned}
\lim_{x\to 0} \frac{\log(1+x)-x}{x^2} &= \lim_{x\to 0} -\frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since they agree on } |x|<1\text{)} \\
&= -\frac{1}{2} + \lim_{x\to 0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since the series converges)} \\
&= -\frac{1}{2} + f(0) && \text{(since } f \text{ is continous at 0)} \\
&= -\frac{1}{2}. && \text{(since } f(0)=0\text{)}
\end{aligned}$$

■

1(c) *Proof.* First note that with the results in Rudin Theorem 3.31, it's clear that for any sequence $(a_n) \subseteq \mathbb{R}$ where $a_n \to \infty$,

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

We thus fix $x \in \mathbb{R}$. If $x = 0$, then

$$\lim_{n\to\infty}\left(1+\frac{0}{n}\right)^{n} = \lim_{n\to\infty} 1 = 1 = e^0.$$

If $x \neq 0$, then

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = \left[\left(1+\frac{1}{n/x}\right)^{n/x}\right]^{x} = e^x,$$

since as $n\to\infty$, $n/x\to\infty$. ■

2(a) *Proof.* ■

2(b) *Proof.* ■

3(a) *Proof.* ∎

3(b) *Proof.* ∎

3(c) *Proof.* ∎

4(a) *Proof.* ∎

4(b) *Proof.* ∎

4(c) *Proof.* ∎