1(a) *Proof.* First note that for any fixed $a > 0$ and $x \neq 0$.

$$\frac{a^x - 1}{x} = \frac{e^{x \log a} - e^{x \log 1}}{x} = \left[\frac{e^{xt}}{x}\right]_{t=\log 1}^{t=\log a} = \int_{\log 1}^{\log a} e^{xt}\,\mathrm{d}t.$$

Now consider any real sequence $(x_n) \subseteq \mathbb{R} \setminus \{0\}$ s.t. $x_n \to 0$. Define $f : I \to \mathbb{R}$, where I is the closed interval between $\log a$ and $\log 1 (= 0)$, as

$$f_n(t) = e^{x_n t}.$$

We claim that $f_n(t)$ converges uniformly to $f(t) = 1$ on I. To see this, first note that for each n, f_n is monotone on I. Therefore,

$$\begin{aligned} M_n &= \sup_{t \in I} |f_n(t) - f(t)| && \text{(by the def. of } M_n) \\ &= \sup_{t \in I} |f_n(t) - f_n(0)| && (\text{since } f_n(0) = f(t), \forall n, t) \\ &= \sup_{t \in I} |f_n(\log a) - f_n(0)| && (\text{since } f_n \text{ monotone on } I) \\ &= \sup_{t \in I} |f_n(\log a) - 1|\,. && (\text{since } f_n(0) = 1) \end{aligned}$$

Therefore, $M_n \to 0$ since $f_n(\log a) \to 1$ as $n \to \infty$. This concludes the proof that $f_n(t)$ converges uniformly to 1. Hence by Rudin Theorem 7.16,

$$\lim_{n\to\infty} \int_{\log 1}^{\log a} f_n(t)\,\mathrm{d}t = \int_{\log 1}^{\log a} f(t)\,\mathrm{d}t, \text{ i.e., } \lim_{n\to\infty} \int_{\log 1}^{\log a} e^{x_n t}\,\mathrm{d}t = \int_{\log 1}^{\log a} 1\,\mathrm{d}t = \log a.$$

Since (x_n) is an arbitrary sequence in $\mathbb{R} \setminus \{0\}$ that converges to 0, by Rudin Theorem 4.2,

$$\lim_{x\to 0} \int_{\log 1}^{\log a} e^{xt} dt = \log a.$$

Combining with the previous results, we can thus conclude that,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

∎

1(b) *Proof.* ∎

1(c) *Proof.* ∎

2(a) *Proof.* ■

2(b) *Proof.* ■

3(a) *Proof.* ∎

3(b) *Proof.* ∎

3(c) *Proof.* ∎

4(a) *Proof.* ■

4(b) *Proof.* ■

4(c) *Proof.* ■