1(a) *Proof.* Following the convention in Rudin Theorem 6.19 (Change of Variable), we let $\varphi(t) = e^t$. Moreoever,

$$f(x) = \cos(x), \qquad \alpha(x) = \ln(x), \text{ and}$$
$$g(t) = f(\varphi(t)) = \cos(e^t), \qquad \beta(t) = \alpha(\varphi(t)) = \ln(e^t) = t.$$

Therefore, by Theorem 6.19,

$$\int_a^b \cos(e^t)\,\mathrm{d}t = \int_a^b g(t)\,\mathrm{d}\beta(t) = \int_{\varphi(a)}^{\varphi(b)} f(x)\,\mathrm{d}\alpha(x) = \int_{e^a}^{e^b} f(x)\,\mathrm{d}\alpha(x).$$

Also note that the change of integrand from $\beta(t)$ to $\alpha(x)$ makes sense since the domain of the natural log, ln, is $(0, +\infty)$ and we are assuming $[a, b] \subseteq (0, +\infty)$.

Now note that $\alpha'(x) = 1/x \in \mathcal{R}[a, b]$ where $[a, b] \subseteq (0, +\infty)$. Therefore, by Rudin Theorem 6.17,

$$\int_{e^a}^{e^b} f(x)\,\mathrm{d}\alpha = \int_{e^a}^{e^b} f(x)\alpha'(x)\,\mathrm{d}x = \int_{e^a}^{e^b} \frac{\cos(x)}{x}\,\mathrm{d}x.$$

Now we apply Rudin Theorem 6.22 (Integration by Parts), letting $g(x) = \cos(x)$, $F(x) = 1/x$. Hence, $G(x) = \sin(x)$, $f(x) = -1/x^2$. Therefore,

$$\int_{e^a}^{e^b} \frac{\cos(x)}{x}\,\mathrm{d}x = \left[\frac{\sin(x)}{x}\right]_{e^a}^{e^b} + \int_{e^a}^{e^b} \frac{\sin(x)}{x^2}\,\mathrm{d}x = e^{-b}\sin(e^b) - e^{-a}\sin(e^a) + \int_{e^a}^{e^b} \frac{\sin(x)}{x^2}\,\mathrm{d}x.$$

Let $r(a, b) = \int_{e^a}^{e^b} \frac{\sin(x)}{x^2}\,\mathrm{d}x$. Note that

$$\begin{aligned}
|r(a,b)| &= \left|\int_{e^a}^{e^b} \frac{\sin(x)}{x^2}\,\mathrm{d}x\right| \\
&\leq \int_{e^a}^{e^b} \left|\frac{\sin(x)}{x^2}\right|\,\mathrm{d}x && \text{(by Rudin Theorem 6.13(b))} \\
&= \int_{e^a}^{e^b} \frac{|\sin(x)|}{x^2}\,\mathrm{d}x && \text{(as } x > 0 \text{ on } [e^a, e^b]) \\
&\leq \int_{e^a}^{e^b} \frac{1}{x^2}\,\mathrm{d}x && \text{(as } |\sin(x)| \leq 1) \\
&= \left[-x^{-1}\right]_{e^a}^{e^b} \\
&= e^{-a} - e^{-b}.
\end{aligned}$$

Therefore, we have recovered the equation in question that,

$$\int_a^b \cos(e^t)\,\mathrm{d}t = e^{-b}\sin(e^b) - e^{-a}\sin(e^a) + r(a, b),$$

where $|r(a, b)| \leq e^{-a} - e^{-b}$. ∎

1(b) *Proof.* Let $\varepsilon > 0$ be given. Note that for any natural numbers n, m,

$$\begin{aligned}
|I_n - I_m| = \left|\int_m^n \cos(e^t)\,\mathrm{d}t\right| &= \left|e^{-n}\sin(e^n) - e^{-m}\sin(e^m) + r(m,n)\right| \\
&\le \left|e^{-n}\sin(e^n)\right| + \left|e^{-m}\sin(e^m)\right| + |r(m,n)| \\
&\le e^{-n}\cdot 1 + e^{-m}\cdot 1 + e^{-m} - e^{-n} \\
&= 2e^{-m}.
\end{aligned}$$

Therefore, choose a natural N such that $e^N > 2/\varepsilon$. Then for all $n, m \ge N$, $e^m \ge e^N > 2/\varepsilon \implies e^{-m} < \varepsilon/2 \implies 2e^{-m} < \varepsilon$. This means that $|I_n - I_m| < \varepsilon$. $(I_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. ■

1(c) *Proof.* Let $\varepsilon > 0$ be arbitrary. We need a natural N such that for any real $x > N$,

$$\left|\int_0^x \cos(e^t)\,\mathrm{d}t - L\right| < \varepsilon.$$

Since the sequence $\left(\int_0^n \cos(e^t)\,\mathrm{d}t\right)_{n\in\mathbb{N}} \to L$, there exists a natural N_1 such that for all $n > N_1$, $\left|\int_0^n \cos(e^t)\,\mathrm{d}t - L\right| < \varepsilon/2$. And we choose another N_2 where $e^{N_2} > 4/\varepsilon$ so that $n > N_2 \implies e^n > e^{N_2} > 4/\varepsilon \implies 2e^{-n} < \varepsilon/2$. We thus pick $N > \max\{N_1, N_2\}$.

Also note that the bound we have established in part(b) that $|I_n - I_m| \le 2e^{-m}$ works for any reals $n, m \in (0, +\infty)$ as well.

Now note that for any $x > N$, we have $x \in [\hat{n}_x, \hat{n}_x + 1]$ where $\hat{n}_x \in \mathbb{N}$ and $\hat{n}_x \ge N$. Thus,

$$\begin{aligned}
\left|\int_0^x \cos(e^t)\,\mathrm{d}t - L\right| &= \left|\int_0^{\hat{n}_x} \cos(e^t)\,\mathrm{d}t - L + \int_{\hat{n}_x}^x \cos(e^t)\,\mathrm{d}t\right| \\
&\le \left|\int_0^{\hat{n}_x} \cos(e^t)\,\mathrm{d}t - L\right| + \left|\int_{\hat{n}_x}^x \cos(e^t)\,\mathrm{d}t\right| \\
&= \left|\int_0^{\hat{n}_x} \cos(e^t)\,\mathrm{d}t - L\right| + |I_x - I_{\hat{n}_x}| \\
&\le \left|\int_0^{\hat{n}_x} \cos(e^t)\,\mathrm{d}t - L\right| + 2e^{-\hat{n}_x} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

■

2(a) *Proof.* ■

2(b) *Proof.* ■

2(c) *Proof.* ■

2(d) *Proof.* ■

2(e) *Proof.* ■