1(a) *Proof.* We fix $c \in (a,b)$. We need to show that LHS $\leq$ RHS and LHS $\geq$ RHS. Let partition P on $[a,b]$ be arbitrary. Then let P^* by its refinement by adding the point c. Then split P^* into P_1 and P_2, where P_1 is a partition for $[a,c]$ and P_2 is a partition for $[c,b]$. Then,

$$\begin{aligned}U(P,f|_{[a,b]}) &\geq U(P^*,f|_{[a,b]}) = U(P_1,f|_{[a,c]}) + U(P_2,f|_{[c,b]}) \geq \inf_P U(P,f|_{[a,c]}) + \inf_P U(P,f|_{[c,b]}) \\ &= \overline{\int_a^c} f \,\mathrm{d}\alpha + \overline{\int_c^b} f \,\mathrm{d}\alpha.\end{aligned}$$

This means that $\overline{\int_a^c} f \,\mathrm{d}\alpha + \overline{\int_c^b} f \,\mathrm{d}\alpha$ is a lower bound for $U(P,f|_{[a,b]})$ over all partitions of $[a,b]$, so it must be smaller than the greatest lower bound. Therefore,

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf_P U(P,f|_{[a,b]}) \geq \overline{\int_a^c} f \,\mathrm{d}\alpha + \overline{\int_c^b} f \,\mathrm{d}\alpha.$$

Hence, LHS $\geq$ RHS.

Let $\varepsilon > 0$ be given. By the property of infinum, there must be partition P_1 of $[a,c]$ and partition P_2 of $[c,b]$ such that

$$U(P_1,f|_{[a,c]}) < \overline{\int_a^c} f \,\mathrm{d}\alpha + \frac{\varepsilon}{2}, \text{ and } U(P_2,f|_{[c,b]}) < \overline{\int_c^b} f \,\mathrm{d}\alpha + \frac{\varepsilon}{2}.$$

Note that $P_1 \cup P_2$ is a partition for $[a,b]$. Therefore,

$$\overline{\int_a^b} f \,\mathrm{d}\alpha \leq U(P_1 \cup P_2, f|_{[a,b]}) = U(P_1,f|_{[a,c]}) + U(P_2,f|_{[c,b]}) < \overline{\int_a^c} f \,\mathrm{d}\alpha + \overline{\int_c^b} f \,\mathrm{d}\alpha + \varepsilon.$$

Since ε is arbitrary,

$$\overline{\int_a^b} f \,\mathrm{d}\alpha \leq \overline{\int_a^c} f \,\mathrm{d}\alpha + \overline{\int_c^b} f \,\mathrm{d}\alpha.$$

Hence, LHS $\leq$ RHS. ■

1(b) *Proof.* First note that over any sub-interval $[x_{i-1}, x_i] \subseteq [a,b]$,

$$\sup(f+g) \leq \sup f + \sup g. \tag{$*$}$$

To see this, let $\varepsilon > 0$ be given. Then there exists $x \in [x_{i-1}, x_i]$ s.t. $\sup(f+g) - \varepsilon < (f+g)(x)$. Therefore,

$$\sup(f+g) - \varepsilon < f(x) + g(x) \leq \sup f + \sup g.$$

And since ε is arbitrary, $\sup(f+g) \leq \sup f + \sup g$.

Now we are ready to prove the inequality in question. Again let $\varepsilon > 0$ be given. Then there exist partitions P_1, P_2 of $[a,b]$ s.t.

$$U(P_1,f) < \overline{\int_a^b} f \,\mathrm{d}\alpha + \frac{\varepsilon}{2}, \text{ and } U(P_2,g) < \overline{\int_a^b} g \,\mathrm{d}\alpha + \frac{\varepsilon}{2}.$$

Let $P^* = P_1 \cup U_2$. By $(*)$, $U(P^*, f+g) \leq U(P^*, f) + U(P^*, g)$. Also, by property of refinement, $U(P^*, f) \leq U(P_1, f)$ and $U(P^*, g) \leq U(P_2, g)$. Hence,

$$\overline{\int_a^b} f \,\mathrm{d}\alpha \leq U(P^*, f+g) \leq U(P^*, f) + U(P^*, g) < \overline{\int_a^b} f \,\mathrm{d}\alpha + \overline{\int_a^b} g \,\mathrm{d}\alpha + \varepsilon.$$

And since ε is arbitrary, LHS $\leq$ RHS as desired.

$\blacksquare$

1(c) *Proof.* Let $[a, b] = [0, 1]$ Consider $\alpha(x) = x$ on $[0, 1]$. And let

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 1 & \text{otherwise,} \end{cases} \quad \text{and } g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ -1 & \text{otherwise.} \end{cases}$$

Then $f + g = 0$ on $[a, b]$. Since for any sub-intervals $[x_{i-1}, x_i]$ of $[0, 1]$ contain both rational and irrational points, $M_i = 1$ for both f and g for all i. Therefore,

$$0 = \overline{\int_0^1} (f+g) \,\mathrm{d}x < \overline{\int_0^1} f \,\mathrm{d}x + \overline{\int_0^1} g \,\mathrm{d}x = 1 + 1 = 2.$$

$\blacksquare$

2 *Proof.* For the sake of contradiction, suppose that there is indeed a bounded $\rho \in \mathcal{R}[-1,1]$ s.t. $\int_{-1}^{1} f(x)\rho(x)\,\mathrm{d}x = f(0)$ for any $f \in C[-1,1]$. Consider the following sequence of continuous functions on $[-1,1]$,

$$f_n = \begin{cases} n^2x + n & \text{if } x \in [-\frac{1}{n}, 0], \\ -n^2x + n & \text{if } x \in [0, \frac{1}{n}], \\ 0 & \text{otherwise.} \end{cases}$$

Note that each f_n forms a thin triangle of area of exactly 1 on $[-1,1]$ where $f_n(0) = n$.

Now let $M \in \mathbb{R}$ be such that $\sup_{x\in[-1,1]} |\rho(x)| \leq M$. Such M exists as ρ is bounded on $[-1,1]$. Then, for all $n \in \mathbb{N}$,

$$n = f_n(0) = \int_{-1}^{1} f(x)\rho(x)\,\mathrm{d}x \leq \int_{-1}^{1} f(x)M\,\mathrm{d}x = M\int_{-1}^{1} f(x)\,\mathrm{d}x = M.$$

This is a contradiction since $\mathbb{N}$ is not bounded above. ■

3 *Proof.* ■

4(a) *Proof.* Let $\varepsilon > 0$ be arbitrary. We show that there exists a natural N, such that for all $n > N$, $|U(P_n, f) - L(P_n, f) - 0| < \varepsilon$, or simply $U(P_n, f) - L(P_n, f) < \varepsilon$ since $U(P_n, f) \geq L(P_n, f)$ always true.

Since $f \in \mathcal{R}[a, b]$, then there exists a partition $Q = \{q_0, q_1, \cdots, q_k\}$ of $[a, b]$ where $U(Q, f) - L(Q, f) < \varepsilon/2$. Let l_{min} be the minimum length of the closed intervals $[q_{j-1}, q_j]$ where $j = 1, \cdots, k$.

We also know that f has to be bounded. Therefore, let $M \geq |f(x)|$ for all $x \in [a, b]$.

Let $P_n = \{x_0, x_1, \cdots, x_n\}$. We choose N so large that $d_N = \frac{b-a}{N} < l_{min}$. Then by generalized pigeonhole principle, each interval $[x_{i-1}, x_i]$ from P_n contains at most one point from Q. And note that for all $n > N$, such property still holds. And make N so large that $d_N < \frac{\varepsilon/2}{2M(k-1)}$.

Consider the partition $Q \cup P_n$, which is a refinement of both Q and P_n. Therefore, we know

$$U(Q \cup P_n, f) - L(Q \cup P_n, f) \leq U(P_n, f) - L(P_n, f),$$

and

$$U(Q \cup P_n, f) - L(Q \cup P_n, f) \leq U(Q, f) - L(Q, f) < \varepsilon/2.$$

Now look more closely, we first see that there are $k - 1$ points from Q that are not the end points. And therefore, there are exactly $k - 1$ intervals in the form of $[x_{i-1}, x_i]$ from P_n $(n > N)$ that contains exactly one point of Q, where $i \in \{2, 3, \cdots, n - 1\}$.

Let $D_n = [U(P_n, f) - L(P_n, f)] - [U(Q \cup P_n, f) - L(Q \cup P_n, f)]$. And suppose $[x_{i-1}, x_i]$ contains exactly one q_j where $j = 2, \cdots, k - 2$, define

$$\Lambda_j = \left(\sup_{x \in [x_{i-1}, q_j]} f(x) - \inf_{x \in [x_{i-1}, q_j]} f(x) \right) |x_{i-1} - q_j| + \left(\sup_{x \in [q_j, x_i]} f(x) + \inf_{x \in [q_j, x_i]} f(x) \right) |q_j - x_i| .$$

Notice that $\Lambda_j \geq 0$ always.

Therefore, we have

$$\begin{aligned}
D_N &= [U(P_N, f) - L(P_N, f)] - [U(Q \cup P_N, f) - L(Q \cup P_N, f)] \\
&= \sum_{[x_{i-1}, x_i] \cap \{q_1, \cdots, q_j, \cdots, q_{k-1}\} \neq \varnothing} (M_i - m_i) \Delta x_i - \Lambda_j \\
&\leq \sum_{[x_{i-1}, x_i] \cap \{q_1, \cdots, q_j, \cdots, q_{k-1}\} \neq \varnothing} (M_i - m_i) \Delta x_i \\
&= d_N \cdot \left(\sum_{[x_{i-1}, x_i] \cap \{q_1, \cdots, q_j, \cdots, q_{k-1}\} \neq \varnothing} (M_i - m_i) \right) \\
&\leq d_N \cdot \left(\sum_{[x_{i-1}, x_i] \cap \{q_1, \cdots, q_j, \cdots, q_{k-1}\} \neq \varnothing} 2M \right) \\
&= d_N \cdot 2M(k - 1) \\
&< \varepsilon/2.
\end{aligned}$$

And note that as for all $n > N$, we will still have $D_n \leq d_n \cdot 2M(k-1) < \varepsilon/2$.

Therefore, we see that for all $n > N$,

$$U(P_n, f) - L(P_n, f) = U(Q \cup P_n, f) - L(Q \cup P_n, f) + D_n < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

■

4(b) *Proof.* We give a counter-example. Let $[a, b] = [0, 1]$. Let ϕ be the Golden Ratio (≈ 0.618), a irrational number. Define

$$f(x) = \begin{cases} 0, & \text{if } x < \phi \\ 1, & \text{if } x \geq \phi, \end{cases}$$

and

$$\alpha(x) = \begin{cases} 0, & \text{if } x \leq \phi \\ 1, & \text{if } x > \phi. \end{cases}$$

We can see that $f \in \mathcal{R}(\alpha)$ on $[0, 1]$. Note that if we choose partition $P = [0, \phi, 1]$, then

$$\begin{aligned} U(P, f) - L(P, f) &= (M_1 - m_1)(\alpha(\phi) - \alpha(0)) + (M_2 - m_2)(\alpha(1) - \alpha(\phi)) \\ &= (M_2 - m_2)(\alpha(1) - \alpha(\phi)) && (\text{as } \alpha(1) = \alpha(\phi)) \\ &= 0. && (\text{as } M_2 = m_2) \end{aligned}$$

This means that for all $\varepsilon > 0$, there exists a partition P such that $U(P, f) - L(P, f) < \varepsilon$. Thus, by Rudin Theorem 6.6, we have $f \in \mathcal{R}(\alpha)$ on $[0, 1]$.

Now note that for all $n \in \mathbb{N}$, all points in P_n remain to be rational. Therefore, no matter how large n is, $\phi \neq p$ for all $p \in P_n$. And there is exactly one interval $[x_{i-1}, x_i]$ from P_n that contains ϕ.

Therefore, for any $n \in \mathbb{N}$,

$$\begin{aligned} U(P_n, f, \alpha) - L(P_n, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\ &= \sum_{\phi \in [x_{i-1}, x_i]} (M_i - m_i)\Delta\alpha_i \\ &= (1 - 0) \cdot (1 - 0) \\ &= 1 \end{aligned}$$

Therefore, the limit cannot go to zero. ■