1(a) *Proof.* Let $\Gamma = \{\mathcal{C}(X) : \|f\| \leq 1 \text{ and } N_\alpha(f) \leq 1\}$. We appeal to HW5 Problem 3. We show that Γ is compact by showing that it is closed, bounded, and equicontinous, as $\Gamma \subseteq \mathcal{C}(X)$ and X is compact.

Closed: We show that $\Gamma' \subseteq \Gamma$. Pick any $g \in \Gamma'$. There exists a sequence of functions $(g_n) \subseteq \Gamma \setminus \{g\}$ that converges to g w.r.t. the supremum norm, and this means that $g_n \to g$ uniformly by Rudin Theorem 7.7.

We show that $g \in \Gamma$, i.e., $\|g\| \leq 1$ and $N_\alpha(g) \leq 1$.

Given any $\varepsilon > 0$. We know there exists a natural N s.t. $\|g - g_N\| < \varepsilon$. Also,

$$\|g\| = \|g - g_N + g_N\| \leq \|g - g_N\| + \|g_N\| < 1 + \varepsilon.$$

Since ε is arbitrary, $\|g\| \leq 1$.

Now again given any $\varepsilon > 0$. Pick any $x, y \in X$ where $x \neq y$. Then first note that $d(x, y) > 0$ since $x \neq y$, so $d(x, y)^\alpha$ is a real number greater than zero. And since $g_n \to g$ uniformly, there exists a natural N s.t., $|g_N(t) - g(t)| < \varepsilon \cdot d(x, y)^\alpha/2$, for all $t \in X$. Therefore,

$$\begin{aligned}
\frac{|g(x) - g(y)|}{d(x, y)^\alpha} &= \frac{|g(x) - g_N(x) + g_N(y) - g(y) + g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\
&\leq \frac{|g(x) - g_N(x)|}{d(x, y)^\alpha} + \frac{|g_N(y) - g(y)|}{d(x, y)^\alpha} + \frac{|g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\
&< \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + 1 \\
&= 1 + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, $|g(x) - g(y)|/d(x, y)^\alpha \leq 1$ for this particular pair of x and y. And since x, y is arbitrary, then 1 is a upper bound of the set A where

$$A = \left\{ \frac{|g(x) - g(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\},$$

which means that

$$N_\alpha(g) = \sup A \leq 1,$$

as the supremum must be the least upper bound.

Bounded: It's enough to show that Γ can be covered in an open neighborhood in the metric space $(\mathcal{C}(X), \|\cdot\|)$. Let $\mathbb{B}[0; 2)$ be the open ball centered at the zero function with a radius 2. We claim that

$$\mathbb{B}[0; 2) \supseteq \Gamma.$$

To see this, pick any $f \in \Gamma$, then $\|f\| = \|f - 0\| \leq 1 < 2$. Therefore, $f \in \mathbb{B}[0; 2)$.

Equicontinous: Give any $\varepsilon > 0$. We aim to find a $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon,$$

whenever $d(x,y) < \delta$, $x, y \in X$, $f \in \Gamma$. We claim that $\delta = \varepsilon^{1/\alpha}$ will work. To see this, pick any $f \in \Gamma$ and any $x, y \in X$ s.t. $d(x,y) < \delta$. Note that either $x = y$ or $x \neq y$. If $x = y$, then $|f(x) - f(y)| = 0 < \varepsilon$. If $x \neq y$, then

$$d(x,y) < \delta = \varepsilon^{1/\alpha} \implies d(x,y)^\alpha < \varepsilon.$$

Also, $N_\alpha(f) \leq 1$ since $f \in \Gamma$. This means that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq N_\alpha(f) \leq 1 \implies |f(x) - f(y)| \leq d(x,y)^\alpha < \varepsilon.$$

This proves equicontinuity and concludes the proof of compactness of Γ. ■

1(b) *Proof.* Let $\Pi = \{f \in \mathcal{C}[0,1] : \|f\| \leq 1\}$. It's suffice to find a subset Λ of Π that is not equicontinous. This is due to the fact that if Π is equicontinous, then every subset of Π is as well.

Let Λ be the sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ where for each n, $g_n : [0,1] \to \mathbb{R}$ and,

$$g_n(x) = \begin{cases} -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

It's clear that $\{g_n\} \subseteq \Pi$ since each g_n is continuous and each $\|g_n\| \leq 1$ as $0 \leq g_n(x) \leq 1$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$.

Now let $\delta_n = 1/n$. We see that for each n, $d(0, 1/(n+1)) = 1/(n+1) < \delta_n$, and

$$\left| g_{n+1}(0) - g_{n+1}\left(\frac{1}{n+1}\right) \right| = 1.$$

Therefore, we see that there exists an $\varepsilon = 1 > 0$, s.t. for any $\delta > 0$, we can pick two points $x = 0, y = 1/(n+1) \in [0,1]$ with $d(x,y) < \delta_n < \delta$ for some n, and we can pick a function $g_{n+1} \in \Lambda$ for the same n, s.t. $|g_{n+1}(x) - g_{n+1}(y)| \geq \varepsilon$. This proves the negation of the condition for equicontinuity, so Λ is not equicontinous. Hence, the set Π is not equicontinous, which proves it is also not compact by HW5 Problem 3.

■

2 *Proof.* First note that for any non-constant polynomial p, $\lim_{x\to\infty} p(x) = +\infty$ if the leading coefficient of p is positive, and $\lim_{x\to\infty} p(x) = -\infty$ if the leading coefficient of p is negative.

Now suppose we have a sequence of polynomials $p_n \to f$ uniformly on the whole $\mathbb{R}$. Then by adopting the Cauchy Criteria, we see that there exists a natural N s.t.

$$|p_N(x) - p_m(x)| < 1,$$

for any $m \geq N$ and $x \in \mathbb{R}$. Note that for each $m \geq N$, $p_N - p_m$ is also a polynomial, but it doesn't diverge to infinity. This means that $p_N - p_m$'s must be constant polynomials, which means that for $m \geq N$, p_m's only differ by constants.

Let q be the polynomial p_N without the constant term. Let $a_0 = \lim_{n\to\infty} p_n(0)$. We then claim that $f(x) = q(x) + a_0$, which is a polynomial. It's suffice to show that $q(x) + a_0$ is the point-wise limit of the sequence of polynomials (p_n). Pick any $t \in \mathbb{R}$. Consider the sequence of real numbers $(p_n(t))_{n\in\mathbb{N}}$. Based on previous discussion, we know that for $n \geq N$, p_n's are polynomials that only differ by constant terms. So, for $n \geq N$,

$$p_n(t) = q(t) + p_n(0).$$

And we know that $p_n(0) \to a_0$ as $n \to \infty$. Therefore, $q(t) + p_n(0) \to q(t) + a_0$, which means that $p_n(t) \to q(t) + a_0$ as $n \to \infty$. Hence, $q(x) + a_0$ is indeed the point-wise limit of $(p_n)_{n\in\mathbb{N}}$. And since the limit of a sequence real numbers is unique, $f(x)$ for each x is therefore a unique real number. This means the limit function f is unique, which gives that $f(x) = q(x) + a_0$ and proves that it is a polynomial.

■

3(a) *Proof.* ■

3(b) *Proof.* ■

4 *Proof.* ■

5 *Proof.* ■