1(a) *Proof.* We claim that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$.

First note that if $x = 0$, then $f_n = n \cdot 0/(1+n^2 \cdot 0) = 0$ for all n. Therefore, $\lim_{n\to\infty} f_n(0) = 0$. Now if $x \in \mathbb{R} \setminus \{0\}$. Then $nx \neq 0$ for any n, so

$$|f_n(x)| = \left|\frac{nx}{1+n^2x^2}\right| = \frac{|nx|}{1+|nx|^2} = \frac{1}{\frac{1}{|nx|} + |nx|}.$$

Note that $1/|nx| + |nx| \to \infty$ as $n \to \infty$. Therefore, $|f_n(x)| \to 0$ as $n \to \infty$. This proves that $\{f_n(x)\}_{n\in\mathbb{N}}$ converge pointwisely and the limit is 0. ∎

1(b) *Proof.* Before we prove the whole statement, we prove a few useful results that would be helpful.

We first claim that $\varphi(x) = |f_N(x)|$ is decreasing on $[1/N, +\infty)$ and increasing on $(-\infty, -1/N]$, for N fixed. Pick any $p, q \in \mathbb{R}$ such that $N < p < q$, which also means that $1/N > 1/p > 1/q$. We want to show that $\varphi(1/p) < \varphi(1/q)$. Note that

$$\varphi(1/p) = \frac{N/p}{1+(N/p)^2} = \frac{1}{\frac{1}{N/p} + N/p} = \frac{1}{\frac{p^2+N^2}{Np}} = \frac{1}{\frac{qp^2+qN^2}{Npq}},$$
$$\varphi(1/q) = \frac{N/q}{1+(N/q)^2} = \frac{1}{\frac{1}{N/q} + N/q} = \frac{1}{\frac{q^2+N^2}{Nq}} = \frac{1}{\frac{pq^2+pN^2}{Nqp}}.$$

Also note that

$$\frac{qp^2+qN^2}{Npq} - \frac{pq^2+pN^2}{Npq} = \frac{pq(p-q)+N^2(q-p)}{Npq} = \frac{(q-p)(pq-N^2)}{Npq} > 0,$$

as $p < q$, and $N^2 < pq$ from $N < p, q$. Therefore,

$$\frac{qp^2+qN^2}{Npq} > \frac{pq^2+pN^2}{Npq} \implies \frac{1}{\frac{qp^2+qN^2}{Npq}} < \frac{1}{\frac{pq^2+pN^2}{Npq}} \implies \varphi(1/p) < \varphi(1/q).$$

φ is decreasing on $[1/N, +\infty)$ as desired, and by a similar argument, φ is increasing on $(-\infty, -1/N]$.

* * *

We now first prove the backward direction. Suppose that $0 \notin A'$. Then there exists $\delta > 0$ such that $(-\delta, +\delta) \cap A = \varnothing$. Thus, we can show that $\{f_n(x)\}$ converges uniformly to 0 on A by showing that $\{f_n(x)\}$ uniformly converges to 0 on $\mathbb{R} \setminus (-\delta, +\delta)$, since $A \subseteq \mathbb{R} \setminus (-\delta, +\delta)$. Let $I = \mathbb{R} \setminus (-\delta, +\delta)$.

Given $\varepsilon > 0$. We want a natural N such that $n \geq N$ implies $|f_n(x)| < \varepsilon$ for all $x \in I$. Consider the points $x = \pm\delta$. First there must be a natural N_1 such that $\delta \in [1/N_1, +\infty)$. Therefore, by previous results $\varphi(x) = |f_{N_1}(x)|$ is decreasing on $[1/N_1, +\infty)$ and increasing on $(-\infty, -1/N_1]$. Since $-\delta < -1/N_1 < 1/N_1 < \delta$, $\varphi(x)$ is decreasing on $[\delta, +\infty)$ and increasing on $(-\infty, -\delta]$ as well, i.e., $\sup_{x\in I} \varphi(x) = \varphi(\delta) = \varphi(-\delta)$. And since $\{f_n(\delta)\} \to 0$ as $n \to \infty$, there must be another N_2 such that $n \geq N_2$ implies that $|f_n(\delta)| = |f_n(-\delta)| < \varepsilon$. We thus take $N > \max\{N_1, N_2\}$. Note that $N > N_1 \implies 1/N < 1/N_1 < \delta$. Hence for any $n \geq N$, $\sup_{x\in I} f_n(x) = |f_n(\delta)| = |f_n(-\delta)| < \varepsilon$. This proves that the function converges uniformly to zero.

* * *

We now proceed to the forward direction. We do proof by contraposition. Suppose that $0 \in A'$. Then there exists a sequence $(a_i)_{i\in\mathbb{N}} \subseteq A$ converges to 0 where $|a_{i+1}| \leq |a_i|$ for all i. To see this, note that there exists $x_1 \in (-1, 1) \cap A$; set $a_1 = x_1$, and for any $i \geq 2$, there exists $x_i \in (\max\{-1/i, -|x_{i-1}|\}, \min\{1/i, |x_{i-1}|\})$; set $a_i = x_i$.

We want to show that $\{f_n(x)\}$ does not converge uniformly to 0, i.e., there exists $\varepsilon > 0$, and there exist infinitely many N's such that for each N, $|f_N(x)| > \varepsilon$ for some x.

We pick $\varepsilon = 1/4$. Now consider the $(a_i)_{i\in\mathbb{N}}$ generated before. Note that for each a_i, there exists a natural N such that $|a_i| \in [1/N, 1/(N-1)]$. And note that $\varphi(x) = |f_N(x)|$ is decreasing on $[1/N, +\infty)$ and increasing on $(-\infty, -1/N]$, and note that fact that $\varphi(x)$ is even. Thus,

$$f_N\left(\frac{1}{N-1}\right) \leq f_N(a_i) \leq f_N\left(\frac{1}{N}\right).$$

Also note that for any natural $N \geq 2$,

$$\begin{aligned} f_N\left(\frac{1}{N-1}\right) &= \frac{\frac{N}{N-1}}{1 + \left(\frac{N}{N-1}\right)^2} = \frac{1}{\frac{N-1}{N} + \frac{N}{N-1}} = \frac{N(N-1)}{(N-1)^2 + N^2} = \frac{N^2 - N}{2(N^2 - N) + 1} = \frac{1}{2 + \frac{1}{N^2 - N}} \\ &> \varepsilon. \end{aligned}$$

To put everything together, since (a_i) is such a sequence that the absolute value is decreasing and converge to zero, then it generates an infinite sequence of $(N_i)_{i\in\mathbb{N}}$ where each $|a_i| \in [1/N_i, 1/(N_i - 1)]$, and there are infinitely many distinct elements in the sequence $(N_i)_{i\in\mathbb{N}}$. And for each N_i,

$$f_{N_i}(a_i) \geq f_{N_i}\left(\frac{1}{N-1}\right) > \varepsilon.$$

This completes the proof that $\{f_n(x)\}$ does not converge uniformly on A.

■

2(a) *Proof.* ■

2(b) *Proof.* ■

2(c) *Proof.* ■

3 *Proof.* ■

4(a) *Proof.* ∎

4(b) *Proof.* ∎

4(c) *Proof.* ∎