1(a) *Proof.* Fix any x where $|x| < 1$. First if $x = 0$. Since $e^0 = 1$ by Rudin 8.27. And $e^{\log y} = y$ for all positive y by Rudin 8.36. Therefore, $\log(1) = 0$. And it's clear that

$$0 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{0^k}{k}.$$

Thus, the equation is true in this case.

Now suppose $x \in (0, 1)$. Then for any $t \in [1, 1 + x]$, $|1 - t| = t - 1 \leq x < 1$. Therefore, by the formula for geometric series,

$$\frac{1}{t} = \frac{1}{1 - (1 - t)} = \sum_{k=0}^{\infty} (1 - t)^k.$$

Therefore,

$$\int_1^{1+x} \frac{1}{t} \, \mathrm{d}t = \int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t.$$

Furthermore, note that the above infinite series has radius of convergence $(0, 2)$ as $|1 - t| < 1 \implies 0 < t < 2$. Thus that by Rudin Theorem 8.1, $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1, 1 + x]$, as $[1, 1 + x] \subsetneq (0, 2)$. Hence, by Rudin Theorem 7.16,

$$\int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t = \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t.$$

Combining the above results, and evaluating each integrals, we have

$$\begin{aligned}
\int_1^{1+x} \frac{1}{t} \, \mathrm{d}t &= \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \left[\frac{-(1 - t)^{k+1}}{k + 1} \right]_1^{1+x} \\
&= \sum_{k=0}^{\infty} \frac{-(-x)^{k+1}}{k + 1} \\
&= \sum_{k=1}^{\infty} \frac{-(-x)^k}{k} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.
\end{aligned}$$

And since Rudin 8.39, $\log(1 + x) = \int_1^{1+x} \frac{1}{t} \, \mathrm{d}t$, the equation is thus true in this case.

Now suppose $x \in (-1, 0)$. Then again for any $t \in [1 + x, 1]$, $|1 - t| = 1 - t \leq -x < 1$. And the series $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1 + x, 1]$, as $[1, 1 + x] \subsetneq (0, 2)$. Therefore, the above decomposition applies still. This completes the proof. ■

1(b) *Proof.* We apply Rudin 8.2 (Abel's Theorem) here. First note that $\sum_{n=1}^{\infty}(-1)^{n-1}1/n$ converges by the alternating series test. Therefore, by Rudin 8.2 and by results from (a),

$$\lim_{x\to 1^-}\log(1+x)=\lim_{x\to 1^-}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

And since e^x is and strictly increasing and continuous, then $\log(y)$, being the inverse of e^x, is also strictly increasing continuous. Therefore,

$$\lim_{x\to 1^-}\log(1+x)=\log(2).$$

This gives that

$$\log(2)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

■

2 *Proof.* ■

3(a) *Proof.* ■

3(b) *Proof.* ■

3(c) *Proof.* ■

4 *Proof.* ■

5(a) *Proof.* ■

5(b) *Proof.* ■

5(c) *Proof.* ■