1(a) *Proof.* Fix any x where $|x| < 1$. First if $x = 0$. Since $e^0 = 1$ by Rudin 8.27. And $e^{\log y} = y$ for all positive y by Rudin 8.36. Therefore, $\log(1) = 0$. And it's clear that

$$0 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{0^k}{k}.$$

Thus, the equation is true in this case.

Now suppose $x \in (0, 1)$. Then for any $t \in [1, 1 + x]$, $|1 - t| = t - 1 \leq x < 1$. Therefore, by the formula for geometric series,

$$\frac{1}{t} = \frac{1}{1 - (1 - t)} = \sum_{k=0}^{\infty} (1 - t)^k.$$

Therefore,

$$\int_1^{1+x} \frac{1}{t} \, \mathrm{d}t = \int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t.$$

Furthermore, note that the above infinite series has radius of convergence $(0, 2)$ as $|1 - t| < 1 \implies 0 < t < 2$. Thus that by Rudin Theorem 8.1, $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1, 1 + x]$, as $[1, 1 + x] \subsetneq (0, 2)$. Hence, by Rudin Theorem 7.16,

$$\int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t = \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t.$$

Combining the above results, and evaluating each integrals, we have

$$\begin{aligned} \int_1^{1+x} \frac{1}{t} \, \mathrm{d}t &= \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t \\ &= \sum_{k=0}^{\infty} \left[\frac{-(1 - t)^{k+1}}{k + 1} \right]_1^{1+x} \\ &= \sum_{k=0}^{\infty} \frac{-(-x)^{k+1}}{k + 1} \\ &= \sum_{k=1}^{\infty} \frac{-(-x)^k}{k} \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}. \end{aligned}$$

And since Rudin 8.39, $\log(1 + x) = \int_1^{1+x} \frac{1}{t} \, \mathrm{d}t$, the equation is thus true in this case.

Now suppose $x \in (-1, 0)$. Then again for any $t \in [1 + x, 1]$, $|1 - t| = 1 - t \leq -x < 1$. And the series $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1 + x, 1]$, as $[1, 1 + x] \subsetneq (0, 2)$. Therefore, the above decomposition applies still. This completes the proof. ■

1(b) *Proof.* ■

2 *Proof.* ■

3(a) *Proof.* ■

3(b) *Proof.* ■

3(c) *Proof.* ■

4 *Proof.* ■

5(a) *Proof.* ■

5(b) *Proof.* ■

5(c) *Proof.* ■