1(a) *Proof.* Fix any x where $|x| < 1$. First if $x = 0$. Since $e^0 = 1$ by Rudin 8.27. And $e^{\log y} = y$ for all positive y by Rudin 8.36. Therefore, $\log(1) = 0$. And it's clear that

$$0 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{0^k}{k}.$$

Thus, the equation is true in this case.

Now suppose $x \in (0, 1)$. Then for any $t \in [1, 1 + x]$, $|1 - t| = t - 1 \leq x < 1$. Therefore, by the formula for geometric series,

$$\frac{1}{t} = \frac{1}{1 - (1 - t)} = \sum_{k=0}^{\infty} (1 - t)^k.$$

Therefore,

$$\int_1^{1+x} \frac{1}{t} \, \mathrm{d}t = \int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t.$$

Furthermore, note that the above infinite series has radius of convergence $(0, 2)$ as $|1 - t| < 1 \implies 0 < t < 2$. Thus that by Rudin Theorem 8.1, $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1, 1 + x]$, as $[1, 1 + x] \subsetneq (0, 2)$. Hence, by Rudin Theorem 7.16,

$$\int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \, \mathrm{d}t = \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t.$$

Combining the above results, and evaluating each integrals, we have

$$\begin{aligned}
\int_1^{1+x} \frac{1}{t} \, \mathrm{d}t &= \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \left[\frac{-(1 - t)^{k+1}}{k + 1} \right]_1^{1+x} \\
&= \sum_{k=0}^{\infty} \frac{-(-x)^{k+1}}{k + 1} \\
&= \sum_{k=1}^{\infty} \frac{-(-x)^k}{k} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.
\end{aligned}$$

And since Rudin 8.39, $\log(1 + x) = \int_1^{1+x} \frac{1}{t} \, \mathrm{d}t$, the equation is thus true in this case.

Now suppose $x \in (-1, 0)$. Then again for any $t \in [1 + x, 1]$, $|1 - t| = 1 - t \leq -x < 1$. And the series $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1 + x, 1]$, as $[1, 1 + x] \subsetneq (0, 2)$. Therefore, the above decomposition applies still. This completes the proof. ∎

1(b) *Proof.* We apply Rudin 8.2 (Abel's Theorem) here. First note that $\sum_{n=1}^{\infty}(-1)^{n-1}1/n$ converges by the alternating series test. Therefore, by Rudin 8.2 and by results from (a),

$$\lim_{x\to 1^-}\log(1+x) = \lim_{x\to 1^-}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

And since e^x is and strictly increasing and continuous, then $\log(y)$, being the inverse of e^x, is also strictly increasing continuous. Therefore,

$$\lim_{x\to 1^-}\log(1+x) = \log(2).$$

This gives that

$$\log(2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

■

2 *Proof.* First note that since $\sum_{n=0}^{\infty} a_n = \infty$, then given any $c > 0$, we also have

$$c \cdot \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c \cdot a_n = \infty.$$

Let $M > 0$ be arbitrary. Choose $c = 1/2$. This means that there exists a natural K s.t. for $N > K$,

$$\sum_{n=0}^{N} c \cdot a_n > M.$$

Note that there exists $0 < \delta < 1$ such that $(1-\delta)^N = c$, since $1-\delta = c^{1/N}$ where $0 < c^{1/N} < 1$. We thus pick this δ. Then note that for any $1 - \delta < x < 1$, and any $n \leq N$

$$x^n > (1-\delta)^n = c^{n/N} \geq c,$$

which, with the fact that each $a_n \geq 0$, gives that,

$$\sum_{n=0}^{N} a_n x^n \geq \sum_{n=0}^{N} c \cdot a_n.$$

Note also that the partial sums of the series $\sum_{n=0}^{\infty} a_n$ are monotone increasing on $(0, 1)$, since $a_n \geq 0$ for all n. Therefore, for $0 < x < 1$,

$$\sum_{n=0}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n.$$

Therefore, combining the above results, we get that for any $x \in (1-\delta, 1)$,

$$\sum_{n=0}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n \geq \sum_{n=0}^{N} c \cdot a_n > M.$$

This completes the proof that $\sum_{n=0}^{\infty} a_n x^n \to \infty$ as $x \to 1^-$.

∎

3(a) *Proof.* ■

3(b) *Proof.* ■

3(c) *Proof.* ■

4 *Proof.* We first show that $f \in \mathcal{R}[0,R]$. By Rudin Theorem 8.1, we know that f is continuous on $[0,R)$ since f is a power series with radius of convergence R. Also note that f is bounded on the whole $[0,R]$ since f is bounded on $[0,R)$ and $f(R) \in \mathbb{R}$. This means that $f(x)$ on $[0,R]$ is a bounded function with possibly $x = R$ being its only discontinuity. Therefore, by Rudin Theorem 6.10, $f \in \mathcal{R}[0,R]$, i.e., the expression $\int_0^R f(x)\,\mathrm{d}x$ is well-defined.

We next show that the series $\sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$ actually converges. First note that for each n,

$$a_n \frac{R^{n+1}}{n+1} = a_n R^n \cdot \frac{R}{n+1}.$$

Let $\alpha_n = a_n R^n$ and $\beta_n = R/(n+1)$. Since $\sum_{n=0}^{\infty} \alpha_n$ converges, the partial sums A_n of $\sum_{n=0}^{\infty} \alpha_n$ form a bounded sequence. Also $\beta_0 \geq \beta_1 \geq \beta_n \geq \cdots$, with $\lim_{n\to\infty} \beta_n = 0$. Therefore, by Rudin Theorem 3.42, $\sum_{n=0}^{\infty} \alpha_n \beta_n$ converges, i.e., $\sum_{n=0}^{\infty} a_n R^{n+1}/(n+1)$ converges.

Now note that by Rudin Theorem 8.1, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R-\xi]$ given any $R > \xi > 0$. Thus, by Rudin Theorem 7.16,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^{R-\xi} a_n x^n\,\mathrm{d}x.$$

And by integrating each term,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1}.$$

For the rest of this proof, we show that as $\xi \to 0^+$,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x \to \int_0^R f(x)\,\mathrm{d}x, \text{ and } \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} \to \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

And since the functional limit is unique, this would complete the proof that $\int_0^R f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$.

Let's show the former first. Let $\varepsilon > 0$ be given. Note first that since $f(x)$ is bounded on $[0,R]$, then there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [0,R]$. Let $\delta = \varepsilon/M$. Then for any $0 < \xi < \delta$,

$$\begin{aligned}
\left| \int_0^{R-\xi} f(x)\,\mathrm{d}x - \int_0^R f(x)\,\mathrm{d}x \right| &= \left| \int_{R-\xi}^R f(x)\,\mathrm{d}x \right| \\
&\leq \int_{R-\xi}^R |f(x)|\,\mathrm{d}x && \text{(Rudin 6.13(b))} \\
&\leq \int_{R-\xi}^R M\,\mathrm{d}x \\
&= M\xi \\
&< M \cdot \frac{\varepsilon}{M} \\
&< \varepsilon.
\end{aligned}$$

This completes the proof of the former functional limit.

Let's move on to the later. Notice that if we do a change of variable with $t = (R - \xi)/\xi$,

$$\sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} \left(\frac{R-\xi}{R}\right)^{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1},$$

where $0 < t < 1$. Note that as $\sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$ converges by above, by Rudin Theorem 8.2,

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

Also note that by the change of variable,

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1} = \lim_{\xi \to 0^+} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} \left(\frac{R-\xi}{R}\right)^{n+1}.$$

Therefore, by combining the above results,

$$\lim_{\xi \to 0^+} \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

This completes the latter functional limit, which completes the whole proof. ■

5(a) *Proof.* ■

5(b) *Proof.* ■

5(c) *Proof.* ■