

**1(a) Proof.** Let  $\Gamma = \{\mathcal{C}(X) : \|f\| \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ . We appeal to HW5 Problem 3. We show that  $\Gamma$  is compact by showing that it is closed, bounded, and equicontinuous, as  $\Gamma \subseteq \mathcal{C}(X)$  and  $X$  is compact.

**Closed:** We show that  $\Gamma' \subseteq \Gamma$ . Pick any  $g \in \Gamma'$ . There exists a sequence of functions  $(g_n) \subseteq \Gamma \setminus \{g\}$  that converges to  $g$  w.r.t. the supremum norm, and this means that  $g_n \rightarrow g$  uniformly by Rudin Theorem 7.7.

We show that  $g \in \Gamma$ , i.e.,  $\|g\| \leq 1$  and  $N_\alpha(g) \leq 1$ .

Given any  $\varepsilon > 0$ . We know there exists a natural  $N$  s.t.  $\|g - g_N\| < \varepsilon$ . Also,

$$\|g\| = \|g - g_N + g_N\| \leq \|g - g_N\| + \|g_N\| < 1 + \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $\|g\| \leq 1$ .

Now again given any  $\varepsilon > 0$ . Pick any  $x, y \in X$  where  $x \neq y$ . Then first note that  $d(x, y) > 0$  since  $x \neq y$ , so  $d(x, y)^\alpha$  is a real number greater than zero. And since  $g_n \rightarrow g$  uniformly, there exists a natural  $N$  s.t.,  $|g_N(t) - g(t)| < \varepsilon \cdot d(x, y)^\alpha / 2$ , for all  $t \in X$ . Therefore,

$$\begin{aligned} \frac{|g(x) - g(y)|}{d(x, y)^\alpha} &= \frac{|g(x) - g_N(x) + g_N(y) - g(y) + g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\ &\leq \frac{|g(x) - g_N(x)|}{d(x, y)^\alpha} + \frac{|g_N(y) - g(y)|}{d(x, y)^\alpha} + \frac{|g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\ &< \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + 1 \\ &= 1 + \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary,  $|g(x) - g(y)| / d(x, y)^\alpha \leq 1$  for this particular pair of  $x$  and  $y$ . And since  $x, y$  is arbitrary, then 1 is an upper bound of the set  $A$  where

$$A = \left\{ \frac{|g(x) - g(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\},$$

which means that

$$N_\alpha(g) = \sup A \leq 1,$$

as the supremum must be the least upper bound.

**Bounded:** It's enough to show that  $\Gamma$  can be covered in an open neighborhood in the metric space  $(\mathcal{C}(X), \|\cdot\|)$ . Let  $\mathbb{B}[0; 2)$  be the open ball centered at the zero function with a radius 2. We claim that

$$\mathbb{B}[0; 2) \supseteq \Gamma.$$

To see this, pick any  $f \in \Gamma$ , then  $\|f\| = \|f - 0\| \leq 1 < 2$ . Therefore,  $f \in \mathbb{B}[0; 2)$ .

**Equicontinuous:** Give any  $\varepsilon > 0$ . We aim to find a  $\delta > 0$  s.t.

$$|f(x) - f(y)| < \varepsilon,$$

whenever  $d(x, y) < \delta$ ,  $x, y \in X$ ,  $f \in \Gamma$ . We claim that  $\delta = \varepsilon^{1/\alpha}$  will work. To see this, pick any  $f \in \Gamma$  and any  $x, y \in X$  s.t.  $d(x, y) < \delta$ . Note that either  $x = y$  or  $x \neq y$ . If  $x = y$ , then  $|f(x) - f(y)| = 0 < \varepsilon$ . If  $x \neq y$ , then

$$d(x, y) < \delta = \varepsilon^{1/\alpha} \implies d(x, y)^\alpha < \varepsilon.$$

Also,  $N_\alpha(f) \leq 1$  since  $f \in \Gamma$ . This means that

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq N_\alpha(f) \leq 1 \implies |f(x) - f(y)| \leq d(x, y)^\alpha < \varepsilon.$$

This proves equicontinuity and concludes the proof of compactness of  $\Gamma$ . ■

**1(b)** *Proof.* Let  $\Pi = \{f \in \mathcal{C}[0, 1] : \|f\| \leq 1\}$ . It's suffice to find a subset  $\Lambda$  of  $\Pi$  that is not equicontinuous. This is due to the fact that if  $\Pi$  is equicontinuous, then every subset of  $\Pi$  is as well.

Let  $\Lambda$  be the sequence of functions  $\{g_n\}_{n \in \mathbb{N}}$  where for each  $n$ ,  $g_n : [0, 1] \rightarrow \mathbb{R}$  and,

$$g_n(x) = \begin{cases} -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

It's clear that  $\{g_n\} \subseteq \Pi$  since each  $g_n$  is continuous and each  $\|g_n\| \leq 1$  as  $0 \leq g_n(x) \leq 1$  for all  $n \in \mathbb{N}$  and all  $x \in [0, 1]$ .

Now let  $\delta_n = 1/n$ . We see that for each  $n$ ,  $d(0, 1/(n+1)) = 1/(n+1) < \delta_n$ , and

$$\left| g_{n+1}(0) - g_{n+1}\left(\frac{1}{n+1}\right) \right| = 1.$$

Therefore, we see that there exists an  $\varepsilon = 1 > 0$ , s.t. for any  $\delta > 0$ , we can pick two points  $x = 0, y = 1/(n+1) \in [0, 1]$  with  $d(x, y) < \delta_n < \delta$  for some  $n$ , and we can pick a function  $g_{n+1} \in \Lambda$  for the same  $n$ , s.t.  $|g_{n+1}(x) - g_{n+1}(y)| \geq \varepsilon$ . This proves the negation of the condition for equicontinuity, so  $\Lambda$  is not equicontinuous. Hence, the set  $\Pi$  is not equicontinuous, which proves it is also not compact by HW5 Problem 3. ■

*2 Proof.*



**3(a)** *Proof.*



**3(b)** *Proof.*



*4 Proof.*



**5** *Proof.*

