

**1(a) Proof.** Let  $\Gamma = \{\mathcal{C}(X) : \|f\| \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ . We appeal to HW5 Problem 3. We show that  $\Gamma$  is compact by showing that it is closed, bounded, and equicontinuous, as  $\Gamma \subseteq \mathcal{C}(X)$  and  $X$  is compact.

**Closed:** We show that  $\Gamma' \subseteq \Gamma$ . Pick any  $g \in \Gamma'$ . There exists a sequence of functions  $(g_n) \subseteq \Gamma \setminus \{g\}$  that converges to  $g$  w.r.t. the supremum norm, and this means that  $g_n \rightarrow g$  uniformly by Rudin Theorem 7.7.

We show that  $g \in \Gamma$ , i.e.,  $\|g\| \leq 1$  and  $N_\alpha(g) \leq 1$ .

Given any  $\varepsilon > 0$ . We know there exists a natural  $N$  s.t.  $\|g - g_N\| < \varepsilon$ . Also,

$$\|g\| = \|g - g_N + g_N\| \leq \|g - g_N\| + \|g_N\| < 1 + \varepsilon.$$

Since  $\varepsilon$  is arbitrary,  $\|g\| \leq 1$ .

Now again given any  $\varepsilon > 0$ . Pick any  $x, y \in X$  where  $x \neq y$ . Then first note that  $d(x, y) > 0$  since  $x \neq y$ , so  $d(x, y)^\alpha$  is a real number greater than zero. And since  $g_n \rightarrow g$  uniformly, there exists a natural  $N$  s.t.,  $|g_N(t) - g(t)| < \varepsilon \cdot d(x, y)^\alpha / 2$ , for all  $t \in X$ . Therefore,

$$\begin{aligned} \frac{|g(x) - g(y)|}{d(x, y)^\alpha} &= \frac{|g(x) - g_N(x) + g_N(y) - g(y) + g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\ &\leq \frac{|g(x) - g_N(x)|}{d(x, y)^\alpha} + \frac{|g_N(y) - g(y)|}{d(x, y)^\alpha} + \frac{|g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\ &< \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + 1 \\ &= 1 + \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary,  $|g(x) - g(y)| / d(x, y)^\alpha \leq 1$  for this particular pair of  $x$  and  $y$ . And since  $x, y$  is arbitrary, then 1 is an upper bound of the set  $A$  where

$$A = \left\{ \frac{|g(x) - g(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\},$$

which means that

$$N_\alpha(g) = \sup A \leq 1,$$

as the supremum must be the least upper bound.

**Bounded:** It's enough to show that  $\Gamma$  can be covered in an open neighborhood in the metric space  $(\mathcal{C}(X), \|\cdot\|)$ . Let  $\mathbb{B}[0; 2)$  be the open ball centered at the zero function with a radius 2. We claim that

$$\mathbb{B}[0; 2) \supseteq \Gamma.$$

To see this, pick any  $f \in \Gamma$ , then  $\|f\| = \|f - 0\| \leq 1 < 2$ . Therefore,  $f \in \mathbb{B}[0; 2)$ .

**Equicontinuous:** Give any  $\varepsilon > 0$ . We aim to find a  $\delta > 0$  s.t.

$$|f(x) - f(y)| < \varepsilon,$$

whenever  $d(x, y) < \delta$ ,  $x, y \in X$ ,  $f \in \Gamma$ . We claim that  $\delta = \varepsilon^{1/\alpha}$  will work. To see this, pick any  $f \in \Gamma$  and any  $x, y \in X$  s.t.  $d(x, y) < \delta$ . Note that either  $x = y$  or  $x \neq y$ . If  $x = y$ , then  $|f(x) - f(y)| = 0 < \varepsilon$ . If  $x \neq y$ , then

$$d(x, y) < \delta = \varepsilon^{1/\alpha} \implies d(x, y)^\alpha < \varepsilon.$$

Also,  $N_\alpha(f) \leq 1$  since  $f \in \Gamma$ . This means that

$$\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leq N_\alpha(f) \leq 1 \implies |f(x) - f(y)| \leq d(x, y)^\alpha < \varepsilon.$$

This proves equicontinuity and concludes the proof of compactness of  $\Gamma$ . ■

**1(b)** *Proof.* Let  $\Pi = \{f \in \mathcal{C}[0, 1] : \|f\| \leq 1\}$ . It's suffice to find a subset  $\Lambda$  of  $\Pi$  that is not equicontinuous. This is due to the fact that if  $\Pi$  is equicontinuous, then every subset of  $\Pi$  is as well.

Let  $\Lambda$  be the sequence of functions  $\{g_n\}_{n \in \mathbb{N}}$  where for each  $n$ ,  $g_n : [0, 1] \rightarrow \mathbb{R}$  and,

$$g_n(x) = \begin{cases} -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

It's clear that  $\{g_n\} \subseteq \Pi$  since each  $g_n$  is continuous and each  $\|g_n\| \leq 1$  as  $0 \leq g_n(x) \leq 1$  for all  $n \in \mathbb{N}$  and all  $x \in [0, 1]$ .

Now let  $\delta_n = 1/n$ . We see that for each  $n$ ,  $d(0, 1/(n+1)) = 1/(n+1) < \delta_n$ , and

$$\left| g_{n+1}(0) - g_{n+1}\left(\frac{1}{n+1}\right) \right| = 1.$$

Therefore, we see that there exists an  $\varepsilon = 1 > 0$ , s.t. for any  $\delta > 0$ , we can pick two points  $x = 0, y = 1/(n+1) \in [0, 1]$  with  $d(x, y) < \delta_n < \delta$  for some  $n$ , and we can pick a function  $g_{n+1} \in \Lambda$  for the same  $n$ , s.t.  $|g_{n+1}(x) - g_{n+1}(y)| \geq \varepsilon$ . This proves the negation of the condition for equicontinuity, so  $\Lambda$  is not equicontinuous. Hence, the set  $\Pi$  is not equicontinuous, which proves it is also not compact by HW5 Problem 3. ■

**2 Proof.** First note that for any non-constant polynomial  $p$ ,  $\lim_{x \rightarrow \infty} p(x) = +\infty$  if the leading coefficient of  $p$  is positive, and  $\lim_{x \rightarrow \infty} p(x) = -\infty$  if the leading coefficient of  $p$  is negative.

Now suppose we have a sequence of polynomials  $p_n \rightarrow f$  uniformly on the whole  $\mathbb{R}$ . Then by adopting the Cauchy Criteria, we see that there exists a natural  $N$  s.t.

$$|p_N(x) - p_m(x)| < 1,$$

for any  $m \geq N$  and  $x \in \mathbb{R}$ . Note that for each  $m \geq N$ ,  $p_N - p_m$  is also a polynomial, but it doesn't diverge to infinity. This means that  $p_N - p_m$ 's must be constant polynomials, which means that for  $m \geq N$ ,  $p_m$ 's only differ by constants.

Let  $q$  be the polynomial  $p_N$  without the constant term. Let  $a_0 = \lim_{n \rightarrow \infty} p_n(0)$ . We then claim that  $f(x) = q(x) + a_0$ , which is a polynomial. It's suffice to show that  $q(x) + a_0$  is the point-wise limit of the sequence of polynomials  $(p_n)$ . Pick any  $t \in \mathbb{R}$ . Consider the sequence of real numbers  $(p_n(t))_{n \in \mathbb{N}}$ . Based on previous discussion, we know that for  $n \geq N$ ,  $p_n$ 's are polynomials that only differ by constant terms. So, for  $n \geq N$ ,

$$p_n(t) = q(t) + p_n(0).$$

And we know that  $p_n(0) \rightarrow a_0$  as  $n \rightarrow \infty$ . Therefore,  $q(t) + p_n(0) \rightarrow q(t) + a_0$ , which means that  $p_n(t) \rightarrow q(t) + a_0$  as  $n \rightarrow \infty$ . Hence,  $q(x) + a_0$  is indeed the point-wise limit of  $(p_n)_{n \in \mathbb{N}}$ . And since the limit of a sequence real numbers is unique,  $f(x)$  for each  $x$  is therefore a unique real number. This means the limit function  $f$  is unique, which gives that  $f(x) = q(x) + a_0$  and proves that it is a polynomial. ■

**3(a)** *Proof.* Let  $G = \{e^{-nx} : n = 0, 1, 2, 3, \dots\}$  be a set of real-valued functions on  $[0, 1]$ . And let  $\mathcal{A}$  be an algebra of real-valued continuous functions generated by  $G$ , and it's clear that  $\mathcal{A}$  has the form,

$$\mathcal{A} = \{c_0 + c_1 e^{-x} + c_2 e^{-2x} + \dots + c_n e^{-nx} : n \in \mathbb{N}, c_0, c_1, \dots, c_k \in \mathbb{R}\}.$$

To see  $\mathcal{A}$  is an algebra, pick any  $\alpha \in \mathbb{R}$ ,  $a = c_0 + c_1 e^{-x} + \dots + c_n e^{-nx}$ ,  $b = d_0 + d_1 e^{-x} + \dots + d_m e^{-mx} \in \mathcal{A}$ . Note that we can assume  $n = m$  because if not, say  $n > m$ , then we can add on zero terms, i.e.,  $d_i e^{-ix}$  where  $d_i = 0$ , to the end of  $b$ . Therefore,

$$a + b = (c_0 + d_0) + (c_1 + d_1)e^{-x} + (c_2 + d_2)e^{-2x} + \dots + (c_n + d_n)e^{-nx} \in \mathcal{A}.$$

And,

$$a \cdot b = c_0 d_0 + (c_0 d_1 + c_1 d_0)e^{-x} + \dots + \left( \sum_{i+j=n+m} c_i d_j \right) e^{-(n+m)x} \in \mathcal{A}.$$

Lastly,

$$\alpha b = \alpha d_0 + \alpha d_1 e^{-x} + \dots + \alpha d_m e^{-mx} \in \mathcal{A}.$$

So,  $\mathcal{A}$  is closed under addition, multiplication, and scalar multiplication, which makes  $\mathcal{A}$  an algebra.

Also note that on  $[0, 1]$ ,  $e^{-nx}$  is strictly monotone decreasing for  $n \geq 1$ , which makes  $\mathcal{A}$  separate points. And on  $[0, 1]$ ,  $e^{-nx}$  is strictly positive for  $n \geq 0$ , which makes  $\mathcal{A}$  vanishes at no points. Therefore, by Stone-Weistrass Theorem, the uniform closure of  $\mathcal{A}$  is  $\mathcal{C}([0, 1])$ .

Now pick any continuous real-valued function  $f$  on  $[0, 1]$ . Based on our discussion before, there exists a sequence of functions  $(\varphi_n) \subseteq \mathcal{A}$  s.t.  $\varphi_n \rightarrow f$  uniformly. Note that since each  $\varphi_n$  is of the form

$$c_0 e^{-0x} + c_1 e^{-x} + \dots + c_k e^{-kx},$$

by the linearity of Stieltjes integrals,

$$\begin{aligned} \int_0^1 \varphi_n d\alpha &= c_0 \int_0^1 e^{-0x} d\alpha + c_1 \int_0^1 e^{-x} d\alpha + \dots + c_k \int_0^1 e^{-kx} d\alpha, \text{ and} \\ \int_0^1 \varphi_n d\beta &= c_0 \int_0^1 e^{-0x} d\beta + c_1 \int_0^1 e^{-x} d\beta + \dots + c_k \int_0^1 e^{-kx} d\beta. \end{aligned}$$

And since  $\int_0^1 e^{-mx} d\alpha = \int_0^1 e^{-mx} d\beta$  for each  $m = 0, 1, 2, \dots$ ,  $\int_0^1 \varphi_n d\alpha = \int_0^1 \varphi_n d\beta$  for each  $n \in \mathbb{N}$ . Therefore,  $(\int_0^1 \varphi_n d\alpha)_{n \in \mathbb{N}}$  and  $(\int_0^1 \varphi_n d\beta)_{n \in \mathbb{N}}$  are the same sequence of real numbers, and hence their limits are the same, if they exist.

Also, since  $\varphi_n \rightarrow f$  uniformly on  $[0, 1]$ , by Rudin Theorem 7.16,

$$\begin{aligned} \int_0^1 f d\alpha &= \lim_{n \rightarrow \infty} \int_0^1 \varphi_n d\alpha, \text{ and similarly,} \\ \int_0^1 f d\beta &= \lim_{n \rightarrow \infty} \int_0^1 \varphi_n d\beta. \end{aligned}$$

And since the two limits are the same,

$$\int_0^1 f d\alpha = \int_0^1 f d\beta,$$

as desired. ■

**3(b) Proof.** The claim is true and we give a direct proof. First note that  $\alpha(x) = \beta(x)$  when  $x = 0$  since they are both zero when  $x = 0$ .

Also, since  $e^{-0x} = 1$  for any  $x \in [0, 1]$ ,

$$\begin{aligned} \int_0^1 e^{-0x} d\alpha &= \int_0^1 d\alpha = \alpha(1) - \alpha(0) = \alpha(1), \text{ and similarly,} \\ \int_0^1 e^{-0x} d\beta &= \beta(1). \end{aligned}$$

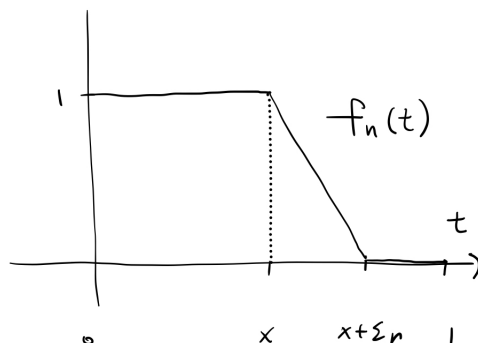
Since  $\int_0^1 e^{-0x} d\alpha = \int_0^1 e^{-0x} d\beta$ ,  $\alpha(x) = \beta(x)$  when  $x = 1$ .

Now for any  $x \in (0, 1)$ . We first construct a sequence of functions  $(f_n(t))$  on  $[0, 1]$  that approaches point-wise to the heavy-side step function on  $[0, 1]$ ,  $H_x(t)$  where,

$$H_x(t) = \begin{cases} 1, & t \in [0, x] \\ 0, & t \in (x, 1]. \end{cases}$$

First let  $\varepsilon_n = (1 - x)/2^n$ . It's clear that  $\varepsilon_n \rightarrow 0$ , each  $\varepsilon_n$  is nonzero, and  $x + \varepsilon_n \in (x, 1)$ . Now for each  $n$ , define

$$f_n(t) = \begin{cases} 1, & t \in [0, x] \\ -\frac{1}{\varepsilon_n}t + \frac{x+\varepsilon_n}{\varepsilon_n}, & t \in (x, x + \varepsilon_n), \\ 0, & t \in [x + \varepsilon_n, 1]. \end{cases}$$



Therefore, for each  $n$ ,

$$\int_0^1 f_n(t) d\alpha(t) = \int_0^x f_n(t) d\alpha(t) + \int_x^{x+\varepsilon_n} f_n(t) d\alpha(t) + \int_{x+\varepsilon_n}^1 f_n(t) d\alpha(t).$$

Note first that  $\int_0^x f_n(t) d\alpha(t) = \alpha(x)$  and  $\int_{x+\varepsilon_n}^1 f_n(t) d\alpha(t) = 0$ . Also,

$$\int_x^{x+\varepsilon_n} f_n(t) d\alpha(t) \leq \int_x^{x+\varepsilon_n} 1 d\alpha(t) = \alpha(x + \varepsilon_n) - \alpha(x).$$

Therefore, combining the above equalities/inequalities,

$$\int_0^1 f_n(t) d\alpha(t) \leq \alpha(x) + \alpha(x + \varepsilon_n) - \alpha(x) + 0 = \alpha(x + \varepsilon_n).$$

Also, since  $f_n(t) \geq 0$  on  $[x, x + \varepsilon_n]$ , therefore  $\int_x^{x+\varepsilon_n} f_n(t) d\alpha(t) \geq 0$ . This gives that

$$\alpha(x) = \int_0^x f_n(t) d\alpha(t) \leq \int_0^1 f_n(t) d\alpha(t).$$

Combining the above results, we have

$$\alpha(x) \leq \int_0^1 f_n(t) d\alpha(t) \leq \alpha(x + \varepsilon_n).$$

Since  $\alpha$  is continuous, as  $\varepsilon_n \rightarrow 0$ ,  $x + \varepsilon_n \rightarrow x$ , so  $\alpha(x + \varepsilon_n) \rightarrow \alpha(x)$ . Therefore,  $\int_0^1 f_n(t) d\alpha(t) \rightarrow \alpha(x)$  by Squeeze Theorem. And by a similar reasoning,  $\int_0^1 f_n(t) d\beta(t) \rightarrow \beta(x)$ .

Note that since each  $f_n$  is continuous, by the results of (a),  $(\int_0^1 f_n(t) d\alpha)$  and  $(\int_0^1 f_n(t) d\beta)$  are essentially the same sequence of real numbers. Since the limit of a sequence of real numbers is unique,  $\alpha(x) = \beta(x)$ . This concludes the proof that  $\alpha(x) = \beta(x)$  on the whole  $[0, 1]$ . ■

*4 Proof.*



**5** *Proof.*

