1 *Proof.* First notice that since f is a constant function always equals to 1, $m_i = M_i = 1$ for all i, for any partition P. Therefore, for any partition P,

$$U(P, f, \alpha) = \sum_i M_i \Delta\alpha_i = \sum \Delta\alpha_i = \alpha(b) - \alpha(a),$$

as the sum telescopes. Similarly, for any partition P,

$$L(P, f, \alpha) = \sum_i m_i \Delta\alpha_i = \sum \Delta\alpha_i = \alpha(b) - \alpha(a).$$

Since the upper and lower Stieltjes sums both remain the same no matter the partition we choose, hence based on the definition of upper and lower Stieltjes integrals,

$$\overline{\int_a^b} f \, \mathrm{d}\alpha = \alpha(b) - \alpha(a) = \underline{\int_a^b} f \, \mathrm{d}\alpha.$$

And since the upper and lower Stieltjes integrals agree,

$$\int_a^b f \, \mathrm{d}\alpha = \alpha(b) - \alpha(a).$$

∎

2 *Proof.* First notice that given any partition P (assuming all points are distinct), $M_i = 1$ for any i, since in any non-trivial closed sub-intervals of $[a, b]$, there are points that are not in S. Therefore, for any partition P,

$$U(P, f, \alpha) = \sum_i M_i \Delta x_i = \sum_i \Delta x_i = b - a.$$

This means that,

$$\overline{\int_a^b} f \, \mathrm{d}x = b - a.$$

It remains to show the lower Riemann integral is also $b - a$, i.e., $\sup_P L(P, f) = b - a$.

First notice that $b - a$ is an upper bound of all the lower Riemann sums as $m_i \leq 1$ for all i, for any partition P. Now let $\varepsilon > 0$ be given. We show that there exists a partition P such that $b - a - \varepsilon < L(P, f)$, which establishes that the supremum of all the lower Riemann integrals is indeed $b - a$. Note that for any partition P,

$$L(P, f) = b - a - \sum_j \Delta x_j,$$

where the $\sum_j \Delta x_j$ term comes from the sub-intervals from P that contains points of S. We thus choose a partition P such that $\Delta x_i < \varepsilon / k$. Since $\sum_j \Delta x_j$ contains at most k summands, $\sum_j \Delta x_j < \varepsilon$ in this partition P, which gives $L(P, f) > b - a - \varepsilon$.

Since the lower and upper Riemann integral agree and equal to $b - a$, f is Riemann-integrable and $\int_a^b f \, \mathrm{d}x = b - a$.

■

3 *Proof.* Let α be just the identity function on $[a, b]$ and let f be a modified Dirichlet function:

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \cap \mathbb{Q}, \\ -1 & \text{otherwise.} \end{cases}$$

Note that given any partition P, all the non-trivial sub-intervals $[x_{i-1}, x_i]$ from P contain both irrational and rational points. Therefore, $M_i = 1$ and $m_i = -1$ for all i, for any partition P, which means that the upper Stieltjes sum (which is just the upper Riemann sum) is always $b - a$ and the lower Stieltjes sum (which is just the lower Riemann sum) is always $a - b$, no matter the partition we choose. This means that

$$b - a = \overline{\int_a^b} \mathrm{d}\alpha \neq \underline{\int_a^b} \mathrm{d}\alpha = a - b,$$

as $b \neq a$. Therefore, $f \notin \mathcal{R}_\alpha[a, b]$. However, notice that $|f|$ is the constant 1 function, which is always Stieltjes-integrable as it is continuous. ■

4(a) *Proof.* This function is Stieltjes-integrable and $\int_{-1}^{1} f \, \mathrm{d}\alpha = -1$. First note that given any partition P, $\Delta\alpha_i = 0$ if $[x_{i-1}, x_i] \subseteq [-1, 0)$ or $[x_{i-1}, x_i] \subseteq [0, 1]$. Therefore, P has the same effect as its sub-partition $Q = \{x_i, x_j\} \subseteq P$ where $[x_i, x_j]$ contains the origin ($x_i \neq 0$). Now take the refinement Q^* by adding the origin, i.e., $Q^* = [x_i, x_{i^*}, x_j]$ where $x_{i^*} = 0$. Hence, we see that

$$U(P) \geq U(Q^*) = M_{i^*}(\alpha(0) - \alpha(x_i)) = -1 \cdot 1 = -1.$$

Similarly,

$$L(P) \leq L(Q^*) = m_{i^*}(\alpha(0) - \alpha(x_i)) = -1 \cdot 1 = -1.$$

Since P is arbitrary, we see that $\sup_P L(P) = -1 = \inf_P U(P)$, i.e. the lower integral agrees with the upper integral. Hence, the integral value is -1.

■

4(b) *Proof.* We claim that $f \in \mathcal{R}_\alpha[-1, 1]$ and $\int_{-1}^{1} \mathrm{d}\alpha = 1$. First note that given any partition P, $\Delta\alpha_i = 0$ if $[x_{i-1}, x_i] \subseteq [-1, 0]$ or $[x_{i-1}, x_i] \subseteq (0, 1]$. Therefore, P has the same effect as its sub-partition $Q = \{x_i, x_j\} \subseteq P$ where $[x_i, x_j]$ contains the origin ($x_j \neq 0$). Now take the refinement Q^* by adding the origin, i.e., $Q^* = \{x_i, x_{j^*}, x_j\}$ where $x_{j^*} = 0$. Hence,

$$U(P) \geq U(Q^*) = M_j(\alpha(x_j) - \alpha(0)) = 1 \cdot 1 = 1.$$

Similarly,

$$L(P) \leq L(Q^*) = m_j(\alpha(x_j) - \alpha(0)) = 1 \cdot 1 = 1.$$

Since P is arbitrary, we see that $\sup_P L(P) = 1 = \inf_P U(P)$, i.e., the lower integral agrees with the upper integral. Hence the integral is 1. ■

4(c) *Proof.* This function is not Stieltjes-integrable. It is suffice to show that $U(P) - L(P) \geq 2$ for all partition P, then by Theorem 6.6, $f \notin \mathcal{R}_\alpha[-1, 1]$.

Let partition P on $[-1, 1]$ be arbitrary. Let P^* be the refinement of P by adding the point 0 in the partition. Then by Theorem 4.4, $U(P) - L(P) \geq U(P^*) - L(P^*)$. Suppose within P, $x_I = 0$ for some I (assume all the partition points are distinct). Then, $\Delta\alpha_i = 0$ for any $i \neq I$, and $M_I = 1, m_I = -1, \Delta\alpha_I = 1$. Therefore,

$$U(P) - L(P) \geq U(P^*) - L(P^*) = (M_I - m_I)\Delta\alpha_I = 2 \cdot 1 = 2.$$

■

4(d) *Proof.* Note that f is continuous on $[-1, 1]$; hence by Rudin Theorem 6.8, $f \in \mathcal{R}_\alpha[-1, 1]$.

We claim that $\int_{-1}^{1} f \, \mathrm{d}\alpha = 1$. We need to show that $\sup_P L(P) = 1 = \inf_P U(P)$. Now take any partition P. We take the refinement P^* that includes the origin. Then $L(P) \leq L(P^*)$ and $U(P^*) \leq U(P)$. Also note that since α is a piece-wise function comprised of constant functions, $\Delta\alpha_i = 0$ if $[x_{i-1}, x_i] \subseteq (-1, 0]$ or $[x_{i-1}, x_i] \subseteq (0, 1)$. Therefore, P^* has the same effect as $Q = \{-1, x_1, 0, x_3, x_4, 1\}$ where $x_1 \in (-1, 0), x_3, x_4 \in (0, 1)$. We can further assume that $x_1 = -1 + \varepsilon, x_3 = \varepsilon, x_4 = 1 - \varepsilon$ for some ε some enough. Therefore,

$$\begin{aligned} U(Q) &= \sum_{i=1}^{5} M_i \Delta\alpha_i = M_1 \cdot 1 + M_2 \cdot 0 + M_3 \cdot 2 + M_4 \cdot 0 + M_5 \cdot 2 \\ &= M_1 + 2M_3 + 2M_5 \\ &= (-1 + \varepsilon) + 2\varepsilon + 2 \\ &= 1 + 3\varepsilon. \end{aligned}$$

Therefore, $\inf_P U(P) = 1$ as ε can be arbitrarily small. Similarly,

$$\begin{aligned} L(Q) &= \sum_{i=1}^{5} m_i \Delta\alpha_i = m_1 \cdot 1 + m_2 \cdot 0 + m_3 \cdot 2 + m_4 \cdot 0 + m_5 \cdot 2 \\ &= m_1 + 2m_3 + 2m_5 \\ &= -1 + 2 \cdot 0 + 2 \cdot (1 - \varepsilon) \\ &= 1 - 2\varepsilon. \end{aligned}$$

Therefore, $\sup_P L(P) = 1$. Hence, the integral is 1. ■

5 *Proof.* We show that $f \in \mathcal{R}[0,1]$ by showing that for all $\varepsilon > 0$, there exists a partition P_ε such that $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$.

We first claim that given any partition $P = \{x_0, x_1, \cdots, x_n\}$, $L(P, f) = 0$ always. To see this, note that given any closed interval $[a, b] \subseteq [0, 1]$ where $a < b$, there are always irrational numbers within $[a, b]$ as irrational numbers are dense in $\mathbb{R}$. Therefore, we have $\inf_{x \in [a,b]} f(x) = 0$. This means that

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$$

It remains to show that given any $\varepsilon > 0$, there exists a partition P_ε, such that $U(P_\varepsilon, f) < \varepsilon$.

Let $\varepsilon > 0$ be given. Then by Archimedean Property, we know there exists $q \in \mathbb{N}$, such that $\frac{1}{q} < \frac{\varepsilon}{2}$. We note that there are only finitely many of x's within $[0, 1]$ where $f(x) \geq \frac{1}{q}$; let $\mathcal{F} \subseteq [0, 1]$ be such set of x's and let $|\mathcal{F}| = N$ (we choose q big enough so that $N \geq 1$). We also note that the range of f on $[0, 1]$ is bounded above by 1.

Given the above information we create the partition P_ε in the following way. We list the elements of $\mathcal{F}$ for the smallest to the largest, i.e., $\mathcal{F} = \{z_1, z_2, \cdots, z_N\}$. For each i from 1 to N, let $[a_i, b_i]$ be the closed interval contains z_i where $0 < b_i - a_i < \frac{\varepsilon}{2N}$. Then shrink each $[a_i, b_i]$ if necessary to make sure all $[a_i, b_i]$ intervals are pairwise disjoint. Add all final a_i's and b_i's to P_ε.

Now list $P_\varepsilon = \{x_0, x_1, \cdots, x_K\}$ where K is some natural number. Now we know $U(P_\varepsilon, f) = \sum_{j=1}^{K} M_j \Delta x_j$. Thus for each j from 1 to K, either $[x_{j-1}, x_j] \cap \mathcal{F} \neq \varnothing$ or not by above construction. And if $[x_{j-1}, x_j] \cap \mathcal{F} \neq \varnothing$, then $M_i < 1$; note that we have precisely N number of such intervals. If $[x_{j-1}, x_j] \cap \mathcal{F} = \varnothing$, then $M_i < \frac{1}{q}$; note that we have precisely $K - N$ number of such intervals. Hence,

$$\begin{aligned}
U(P_\varepsilon, f) &= \sum_{j=1}^{K} M_i \Delta x_j \\
&= \sum_{j | [x_{j-1}, x_j] \cap \mathcal{F} \neq \varnothing} M_j \Delta x_j + \sum_{j | [x_{j-1}, x_j] \cap \mathcal{F} = \varnothing} M_j \Delta x_j \\
&< \sum_{j | [x_{j-1}, x_j] \cap \mathcal{F} \neq \varnothing} 1 \cdot \frac{\varepsilon}{2N} + \sum_{j | [x_{j-1}, x_j] \cap \mathcal{F} = \varnothing} \frac{1}{q} \cdot \Delta x_j \\
&\leq \sum_{j | [x_{j-1}, x_j] \cap \mathcal{F} \neq \varnothing} \frac{\varepsilon}{2N} + \sum_{j=1}^{K} \frac{1}{q} \cdot \Delta x_j \\
&= N \cdot \frac{\varepsilon}{2N} + \frac{1}{q}(x_K - x_0) \\
&= \frac{\varepsilon}{2} + \frac{1}{q} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

And since the lower partition always equal to zero, the lower integral must also be zero. This means that the integral is zero as well. ■