1(a) *Proof.* Let $\Gamma = \{\mathcal{C}(X) : \|f\| \leq 1 \text{ and } N_\alpha(f) \leq 1\}$. We appeal to HW5 Problem 3. We show that Γ is compact by showing that it is closed, bounded, and equicontinous, as $\Gamma \subseteq \mathcal{C}(X)$ and X is compact.

Closed: We show that $\Gamma' \subseteq \Gamma$. Pick any $g \in \Gamma'$. There exists a sequence of functions $(g_n) \subseteq \Gamma \setminus \{g\}$ that converges to g w.r.t. the supremum norm, and this means that $g_n \to g$ uniformly by Rudin Theorem 7.7.

We show that $g \in \Gamma$, i.e., $\|g\| \leq 1$ and $N_\alpha(g) \leq 1$.

Given any $\varepsilon > 0$. We know there exists a natural N s.t. $\|g - g_N\| < \varepsilon$. Also,

$$\|g\| = \|g - g_N + g_N\| \leq \|g - g_N\| + \|g_N\| < 1 + \varepsilon.$$

Since ε is arbitrary, $\|g\| \leq 1$.

Now again given any $\varepsilon > 0$. Pick any $x, y \in X$ where $x \neq y$. Then first note that $d(x, y) > 0$ since $x \neq y$, so $d(x, y)^\alpha$ is a real number greater than zero. And since $g_n \to g$ uniformly, there exists a natural N s.t., $|g_N(t) - g(t)| < \varepsilon \cdot d(x, y)^\alpha/2$, for all $t \in X$. Therefore,

$$\begin{aligned}
\frac{|g(x) - g(y)|}{d(x, y)^\alpha} &= \frac{|g(x) - g_N(x) + g_N(y) - g(y) + g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\
&\leq \frac{|g(x) - g_N(x)|}{d(x, y)^\alpha} + \frac{|g_N(y) - g(y)|}{d(x, y)^\alpha} + \frac{|g_N(x) - g_N(y)|}{d(x, y)^\alpha} \\
&< \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + \frac{\varepsilon \cdot d(x, y)^\alpha}{2d(x, y)^\alpha} + 1 \\
&= 1 + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, $|g(x) - g(y)| / d(x, y)^\alpha \leq 1$ for this particular pair of x and y. And since x, y is arbitrary, then 1 is a upper bound of the set A where

$$A = \left\{ \frac{|g(x) - g(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\},$$

which means that

$$N_\alpha(g) = \sup A \leq 1,$$

as the supremum must be the least upper bound.

Bounded: It's enough to show that Γ can be covered in an open neighborhood in the metric space $(\mathcal{C}(X), \|\cdot\|)$. Let $\mathbb{B}[0; 2)$ be the open ball centered at the zero function with a radius 2. We claim that

$$\mathbb{B}[0; 2) \supseteq \Gamma.$$

To see this, pick any $f \in \Gamma$, then $\|f\| = \|f - 0\| \leq 1 < 2$. Therefore, $f \in \mathbb{B}[0; 2)$.

Equicontinous: Give any $\varepsilon > 0$. We aim to find a $\delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon,$$

whenever $d(x,y) < \delta$, $x, y \in X$, $f \in \Gamma$. We claim that $\delta = \varepsilon^{1/\alpha}$ will work. To see this, pick any $f \in \Gamma$ and any $x, y \in X$ s.t. $d(x,y) < \delta$. Note that either $x = y$ or $x \neq y$. If $x = y$, then $|f(x) - f(y)| = 0 < \varepsilon$. If $x \neq y$, then

$$d(x,y) < \delta = \varepsilon^{1/\alpha} \implies d(x,y)^\alpha < \varepsilon.$$

Also, $N_\alpha(f) \leq 1$ since $f \in \Gamma$. This means that

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \leq N_\alpha(f) \leq 1 \implies |f(x) - f(y)| \leq d(x,y)^\alpha < \varepsilon.$$

This proves equicontinuity and concludes the proof of compactness of Γ. ■

1(b) *Proof.* Let $\Pi = \{f \in \mathcal{C}[0,1] : \|f\| \leq 1\}$. It's suffice to find a subset Λ of Π that is not equicontinous. This is due to the fact that if Π is equicontinous, then every subset of Π is as well.

Let Λ be the sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ where for each n, $g_n : [0,1] \to \mathbb{R}$ and,

$$g_n(x) = \begin{cases} -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

It's clear that $\{g_n\} \subseteq \Pi$ since each g_n is continuous and each $\|g_n\| \leq 1$ as $0 \leq g_n(x) \leq 1$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$.

Now let $\delta_n = 1/n$. We see that for each n, $d(0, 1/(n+1)) = 1/(n+1) < \delta_n$, and

$$\left| g_{n+1}(0) - g_{n+1}\left(\frac{1}{n+1}\right) \right| = 1.$$

Therefore, we see that there exists an $\varepsilon = 1 > 0$, s.t. for any $\delta > 0$, we can pick two points $x = 0, y = 1/(n+1) \in [0,1]$ with $d(x,y) < \delta_n < \delta$ for some n, and we can pick a function $g_{n+1} \in \Lambda$ for the same n, s.t. $|g_{n+1}(x) - g_{n+1}(y)| \geq \varepsilon$. This proves the negation of the condition for equicontinuity, so Λ is not equicontinous. Hence, the set Π is not equicontinous, which proves it is also not compact by HW5 Problem 3.

■

2 *Proof.* First note that for any non-constant polynomial p, $\lim_{x\to\infty} p(x) = +\infty$ if the leading coefficient of p is positive, and $\lim_{x\to\infty} p(x) = -\infty$ if the leading coefficient of p is negative.

Now suppose we have a sequence of polynomials $p_n \to f$ uniformly on the whole $\mathbb{R}$. Then by adopting the Cauchy Criteria, we see that there exists a natural N s.t.

$$|p_N(x) - p_m(x)| < 1,$$

for any $m \geq N$ and $x \in \mathbb{R}$. Note that for each $m \geq N$, $p_N - p_m$ is also a polynomial, but it doesn't diverge to infinity. This means that $p_N - p_m$'s must be constant polynomials, which means that for $m \geq N$, p_m's only differ by constants.

Let q be the polynomial p_N without the constant term. Let $a_0 = \lim_{n\to\infty} p_n(0)$. We then claim that $f(x) = q(x) + a_0$, which is a polynomial. It's suffice to show that $q(x) + a_0$ is the point-wise limit of the sequence of polynomials (p_n). Pick any $t \in \mathbb{R}$. Consider the sequence of real numbers $(p_n(t))_{n\in\mathbb{N}}$. Based on previous discussion, we know that for $n \geq N$, p_n's are polynomials that only differ by constant terms. So, for $n \geq N$,

$$p_n(t) = q(t) + p_n(0).$$

And we know that $p_n(0) \to a_0$ as $n \to \infty$. Therefore, $q(t) + p_n(0) \to q(t) + a_0$, which means that $p_n(t) \to q(t) + a_0$ as $n \to \infty$. Hence, $q(x) + a_0$ is indeed the point-wise limit of $(p_n)_{n\in\mathbb{N}}$. And since the limit of a sequence real numbers is unique, $f(x)$ for each x is therefore a unique real number. This means the limit function f is unique, which gives that $f(x) = q(x) + a_0$ and proves that it is a polynomial.

■

3(a) *Proof.* Let $G = \{e^{-nx} : n = 0, 1, 2, 3, \cdots\}$ be a set of real-valued functions on $[0, 1]$. And let $\mathcal{A}$ be an algebra of real-valued continuous functions generated by G, and it's clear that $\mathcal{A}$ has the form,

$$\mathcal{A} = \left\{c_0 + c_1e^{-x} + c_2e^{-2x} + \cdots + c_ne^{-nx} : n \in \mathbb{N}, c_0, c_1, \cdots, c_k \in \mathbb{R}\right\}.$$

To see $\mathcal{A}$ is an algebra, pick any $\alpha \in \mathbb{R}$, $a = c_0 + c_1e^{-x} + \cdots + c_ne^{-nx}$, $b = d_0 + d_1e^{-x} + \cdots + d_me^{-mx} \in \mathcal{A}$. Note that we can assume $n = m$ because if not, say $n > m$, then we can add on zero terms, i.e., d_ie^{-ix} where $d_i = 0$, to the end of b. Therefore,

$$a + b = (c_0 + d_0) + (c_1 + d_1)e^{-x} + (c_2 + d_2)e^{-2x} + \cdots + (c_n + d_n)e^{-nx} \in \mathcal{A}.$$

And,

$$a \cdot b = c_0d_0 + (c_0d_1 + c_1d_0)e^{-x} + \cdots + \left(\sum_{i+j=n+m} c_id_j\right) e^{-(n+m)x} \in \mathcal{A}.$$

Lastly,

$$\alpha b = \alpha d_0 + \alpha d_1e^{-x} + \cdots + \alpha d_me^{-mx} \in \mathcal{A}.$$

So, $\mathcal{A}$ is closed under addition, multiplication, and scalar multiplication, which makes $\mathcal{A}$ an algebra.

Also note that on $[0, 1]$, e^{-nx} is strictly monotone decreasing for $n \geq 1$, which makes $\mathcal{A}$ separate points. And on $[0, 1]$, e^{-nx} is strictly positive for $n \geq 0$, which makes $\mathcal{A}$ vanishes at no points. Therefore, by Stone-Weistrass Theorem, the uniform closure of $\mathcal{A}$ is $\mathcal{C}([0, 1])$.

Now pick any continuous real-valued function f on $[0, 1]$. Based on our discussion before, there exists a sequence of functions $(\varphi_n) \subseteq \mathcal{A}$ s.t. $\varphi_n \to f$ uniformly. Note that since each φ_n is of the form

$$c_0e^{-0x} + c_1e^{-x} + \cdots + c_ke^{-kx},$$

by the linearity of Stieltjes integrals,

$$\int_0^1 \varphi_n \,\mathrm{d}\alpha = c_0 \int_0^1 e^{-0x} \,\mathrm{d}\alpha + c_1 \int_0^1 e^{-x} \,\mathrm{d}\alpha + \cdots + c_k \int_0^1 e^{-kx} \,\mathrm{d}\alpha, \text{ and}$$
$$\int_0^1 \varphi_n \,\mathrm{d}\beta = c_0 \int_0^1 e^{-0x} \,\mathrm{d}\beta + c_1 \int_0^1 e^{-x} \,\mathrm{d}\beta + \cdots + c_k \int_0^1 e^{-kx} \,\mathrm{d}\beta.$$

And since $\int_0^1 e^{-mx} \,\mathrm{d}\alpha = \int_0^1 e^{-mx} \,\mathrm{d}\beta$ for each $m = 0, 1, 2, \cdots$, $\int_0^1 \varphi_n \,\mathrm{d}\alpha = \int_0^1 \varphi_n \,\mathrm{d}\beta$ for each $n \in \mathbb{N}$. Therefore, $(\int_0^1 \varphi_n \,\mathrm{d}\alpha)_{n\in\mathbb{N}}$ and $(\int_0^1 \varphi_n \,\mathrm{d}\beta)_{n\in\mathbb{N}}$ are the same sequence of real numbers, and hence their limits are the same, if they exist.

Also, since $\varphi_n \to f$ uniformly on $[0, 1]$, by Rudin Theorem 7.16,

$$\int_0^1 f \,\mathrm{d}\alpha = \lim_{n\to\infty} \int_0^1 \varphi_n \,\mathrm{d}\alpha, \text{ and similarly,}$$
$$\int_0^1 f \,\mathrm{d}\beta = \lim_{n\to\infty} \int_0^1 \varphi_n \,\mathrm{d}\beta.$$

And since the two limits are the same,

$$\int_0^1 f \, \mathrm{d}\alpha = \int_0^1 f \, \mathrm{d}\beta,$$

as desired. ■

3(b) *Proof.* The claim is true and we give a direct proof. First note that $\alpha(x) = \beta(x)$ when $x = 0$ since they are both zero when $x = 0$.

Also, since $e^{-0x} = 1$ for any $x \in [0, 1]$,

$$\begin{aligned} &\int_0^1 e^{-0x} \, \mathrm{d}\alpha = \int_0^1 \mathrm{d}\alpha = \alpha(1) - \alpha(0) = \alpha(1), \text{ and similarly,} \\ &\int_0^1 e^{-0x} \, \mathrm{d}\beta = \beta(1). \end{aligned}$$

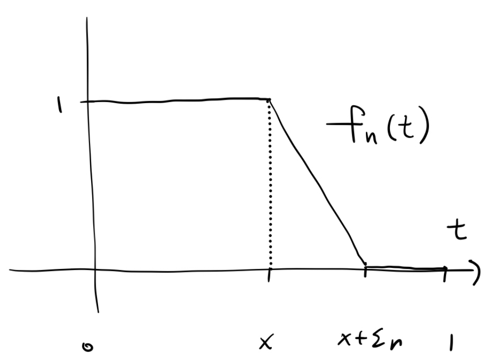
Since $\int_0^1 e^{-0x} \, \mathrm{d}\alpha = \int_0^1 e^{-0x} \, \mathrm{d}\beta$, $\alpha(x) = \beta(x)$ when $x = 1$.

Now for any $x \in (0, 1)$. We first construct a sequence of functions $(f_n(t))$ on $[0, 1]$ that approaches point-wise to the heavy-side step function on $[0, 1]$, $H_x(t)$ where,

$$H_x(t) = \begin{cases} 1, & t \in [0, x] \\ 0, & t \in (x, 1]. \end{cases}$$

First let $\varepsilon_n = (1 - x)/2^n$. It's clear that $\varepsilon_n \to 0$, each ε_n is nonzero, and $x + \varepsilon_n \in (x, 1)$. Now for each n, define

$$f_n(t) = \begin{cases} 1, & t \in [0, x] \\ -\frac{1}{\varepsilon_n} t + \frac{x + \varepsilon_n}{\varepsilon_n}, & t \in (x, x + \varepsilon_n), \\ 0, & t \in [x + \varepsilon_n, 1]. \end{cases}$$



Therefore, for each n,

$$\int_0^1 f_n(t) \, \mathrm{d}\alpha(t) = \int_0^x f_n(t) \, \mathrm{d}\alpha(t) + \int_x^{x+\varepsilon_n} f_n(t) \, \mathrm{d}\alpha(t) + \int_{x+\varepsilon_n}^1 f_n(t) \, \mathrm{d}\alpha(t).$$

Note first that $\int_0^x f_n(t)\,\mathrm{d}\alpha(t) = \alpha(x)$ and $\int_{x+\varepsilon_n}^1 f_n(t)\,\mathrm{d}\alpha(t) = 0$. Also,

$$\int_x^{x+\varepsilon_n} f_n(t)\,\mathrm{d}\alpha(t) \le \int_x^{x+\varepsilon_n} 1\,\mathrm{d}\alpha(t) = \alpha(x+\varepsilon_n) - \alpha(x).$$

Therefore, combining the above equalities/inequalities,

$$\int_0^1 f_n(t)\,\mathrm{d}\alpha(t) \le \alpha(x) + \alpha(x+\varepsilon_n) - \alpha(x) + 0 = \alpha(x+\varepsilon_n).$$

Also, since $f_n(t) \ge 0$ on $[x, x+\varepsilon_n]$, therefore $\int_x^{x+\varepsilon_n} f_n(t)\,\mathrm{d}\alpha(t) \ge 0$. This gives that

$$\alpha(x) = \int_0^x f_n(t)\,\mathrm{d}\alpha(t) \le \int_0^1 f_n(t)\,\mathrm{d}\alpha(t).$$

Combining the above results, we have

$$\alpha(x) \le \int_0^1 f_n(t)\,\mathrm{d}\alpha(t) \le \alpha(x+\varepsilon_n).$$

Since α is continuous, as $\varepsilon_n \to 0$, $x+\varepsilon_n \to x$, so $\alpha(x+\varepsilon_n) \to \alpha(x)$. Therefore, $\int_0^1 f_n(t)\,\mathrm{d}\alpha(t) \to \alpha(x)$ by Squeeze Theorem. And by a similar reasoning, $\int_0^1 f_n(t)\,\mathrm{d}\beta(t) \to \beta(x)$.

Note that since each f_n is continuous, by the results of (a), $(\int_0^1 f_n(t)\,\mathrm{d}\alpha)$ and $(\int_0^1 f_n(t)\,\mathrm{d}\beta)$ are essentially the same sequence of real numbers. Since the limit of a sequence of real numbers is unique, $\alpha(x) = \beta(x)$. This concludes the proof that $\alpha(x) = \beta(x)$ on the whole $[0, 1]$. ■

4 *Proof.* Note that the algebra $\mathcal{A}$ in question either vanishes at no points in K or it vanishes at some points in K. If it's the former, then by Stone-Weistrass Theorem, $\overline{\mathcal{A}}$ consists of all continuous real-valued functions on K; this is case (i). If, however, $\mathcal{A}$ does vanish at some point in K, say $p \in K$. Let $\mathcal{F}$ by the set of all continuous functions on K that vanish at p. For case (ii), it remains to show that $\overline{\mathcal{A}} = \mathcal{F}$.

Pick any $f \in \overline{\mathcal{A}}$. This means that there exists a sequence of functions $(f_n) \subseteq \mathcal{A}$ that converges uniformly to f. This means that f is the limit function of f_n, i.e.,

$$f(p) = \lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} 0 = 0.$$

Therefore, $f \in \mathcal{F}$. This shows that $\overline{\mathcal{A}} \subseteq \mathcal{F}$.

Now pick any $f \in \mathcal{F}$.

First note that $\mathcal{A} + \mathbb{R}$ is an algebra of continuous real-valued functions that both separate points and vanishes at no points. To see $\mathcal{A} + \mathbb{R}$ separate points, note that $\mathcal{A}$ separate points and $\mathcal{A} = \mathcal{A} + 0 \subseteq \mathcal{A} + \mathbb{R}$. To see $\mathcal{A} + \mathbb{R}$ vanishes at no points, pick any $x \in K$. Then either $\mathcal{A}$ vanishes at x or not. If $\mathcal{A}$ doesn't not, then $\mathcal{A} + \mathbb{R}$ being a superset of clearly also doesn't not vanish at x; if it does, then there exists $\alpha \in \mathcal{A}$ s.t. $\alpha(x) = 0$, so $\alpha(x) + 1 \neq 0$ where $\alpha(x) + 1 \in \mathcal{A} + \mathbb{R}$, which means that $\mathcal{A} + \mathbb{R}$ doesn't not vanish at x, and hence it vanishes at no points in K at x is an arbitrary point in K.

Lastly, to see $\mathcal{A}$ is an algebra, pick any $\varphi + a, \psi + b \in \mathcal{A} + \mathbb{R}$ where $a, b \in \mathbb{R}$ and $c \in \mathbb{R}$. Note that

$$\varphi + a + \psi + b = (\varphi + \psi) + (a + b) \in \mathcal{A} + \mathbb{R},$$

as $\varphi + \psi \in \mathcal{A}$, $a + b \in \mathbb{R}$, since both $\mathcal{A}$ and $\mathbb{R}$ are closed under addition. Also,

$$(\varphi + a)(\psi + b) = (\varphi\psi + a\psi + b\varphi) + ab \in \mathcal{A} + \mathbb{R},$$

as $\varphi\psi + a\psi + b\varphi \in \mathcal{A}$ and $ab \in \mathbb{R}$. It's clear that $ab \in \mathbb{R}$ as $\mathbb{R}$ is closed under multiplication. To see $\varphi\psi + a\psi + b\varphi \in \mathcal{A}$, note that $\varphi\psi \in \mathcal{A}$ since $\mathcal{A}$ is closed under multiplication, $a\psi, b\varphi \in \mathcal{A}$ since $\mathcal{A}$ is also closed under scalar multiplication, so the sum of the three is in $\mathcal{A}$ since $\mathcal{A}$ is closed under addition. Also, it clearly that

$$c(\varphi + a) = c\varphi + ca \in \mathcal{A} + \mathbb{R},$$

as both $\mathcal{A}$ and $\mathbb{R}$ are closed under scalar multiplication.

Therefore, by Stone-Weistrass Theorem, $\overline{\mathcal{A} + \mathbb{R}} = \mathcal{C}(K)$. This means that there exists a sequence of functions $(f_n + c_n) \subseteq \mathcal{A} + \mathbb{R}$, where $(f_n) \subseteq \mathcal{A}$ and $(c_n) \subseteq \mathbb{R}$, that approaches uniformly to the function f in $\mathcal{F}$ we have picked before.

It remains to show that $f_n \to f$ uniformly on K, which gives $\mathcal{F} \subseteq \overline{\mathcal{A}}$. Given any $\varepsilon > 0$. Since $f_n + c_n \to f$ uniformly on K, there exists a natural N such that for $n > N$,

$$|(f_n(x) + c_n) - f(x)| < \frac{\varepsilon}{2},$$

for any $x \in K$. In particular,

$$|(f_n(p) + c_n) - f(p)| < \frac{\varepsilon}{2}, \text{ or simply } |c_n| < \frac{\varepsilon}{2},$$

since f and all f_n vanish at the point p. Now note that for any $n > N$

$$|f_n(x) - f(x)| = |(f_n(x) + c_n - f(x)) - c_n| \leq |f_n(x) + c_n - f(x)| + |c_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for any $x \in K$. This concludes the step that $f_n \to f$ uniformly on K, which concludes the entire proof. ■

5 *Proof.* ■