1(a) *Proof.* We claim that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$.

First note that if $x = 0$, then $f_n = n \cdot 0/(1+n^2 \cdot 0) = 0$ for all n. Therefore, $\lim_{n\to\infty} f_n(0) = 0$. Now if $x \in \mathbb{R} \setminus \{0\}$. Then $nx \neq 0$ for any n, so

$$|f_n(x)| = \left|\frac{nx}{1+n^2x^2}\right| = \frac{|nx|}{1+|nx|^2} = \frac{1}{\frac{1}{|nx|} + |nx|}.$$

Note that $1/|nx| + |nx| \to \infty$ as $n \to \infty$. Therefore, $|f_n(x)| \to 0$ as $n \to \infty$. This proves that $\{f_n(x)\}_{n\in\mathbb{N}}$ converge pointwisely and the limit is 0. ■

1(b) *Proof.* Before we prove the whole statement, we prove a few useful results that would be helpful.

We first claim that $\varphi(x) = |f_N(x)|$ is decreasing on $[1/N, +\infty)$ and increasing on $(-\infty, -1/N]$, for N fixed. Pick any $p, q \in \mathbb{R}$ such that $N < p < q$, which also means that $1/N > 1/p > 1/q$. We want to show that $\varphi(1/p) < \varphi(1/q)$. Note that

$$\varphi(1/p) = \frac{N/p}{1+(N/p)^2} = \frac{1}{\frac{1}{N/p} + N/p} = \frac{1}{\frac{p^2+N^2}{Np}} = \frac{1}{\frac{qp^2+qN^2}{Npq}},$$
$$\varphi(1/q) = \frac{N/q}{1+(N/q)^2} = \frac{1}{\frac{1}{N/q} + N/q} = \frac{1}{\frac{q^2+N^2}{Nq}} = \frac{1}{\frac{pq^2+pN^2}{Nqp}}.$$

Also note that

$$\frac{qp^2+qN^2}{Npq} - \frac{pq^2+pN^2}{Npq} = \frac{pq(p-q)+N^2(q-p)}{Npq} = \frac{(q-p)(pq-N^2)}{Npq} > 0,$$

as $p < q$, and $N^2 < pq$ from $N < p, q$. Therefore,

$$\frac{qp^2+qN^2}{Npq} > \frac{pq^2+pN^2}{Npq} \implies \frac{1}{\frac{qp^2+qN^2}{Npq}} < \frac{1}{\frac{pq^2+pN^2}{Npq}} \implies \varphi(1/p) < \varphi(1/q).$$

φ is decreasing on $[1/N, +\infty)$ as desired, and by a similar argument, φ is increasing on $(-\infty, -1/N]$.

Also it's clear that $\varphi(x)$ is an even function.

* * *

We now first prove the backward direction. Suppose that $0 \notin A'$. Then there exists $\delta > 0$ such that $(-\delta, +\delta) \cap A = \varnothing$. Thus, we can show that $\{f_n(x)\}$ converges uniformly to 0 on A by showing that $\{f_n(x)\}$ uniformly converges to 0 on $\mathbb{R} \setminus (-\delta, +\delta)$, since $A \subseteq \mathbb{R} \setminus (-\delta, +\delta)$. Let $I = \mathbb{R} \setminus (-\delta, +\delta)$.

Given $\varepsilon > 0$. We want a natural N such that $n \geq N$ implies $|f_n(x)| < \varepsilon$ for all $x \in I$. Consider the points $x = \pm\delta$. First there must be a natural N_1 such that $\delta \in [1/N_1, +\infty)$. Therefore, by previous results $\varphi(x) = |f_{N_1}(x)|$ is decreasing on $[1/N_1, +\infty)$ and increasing on

$(-\infty, -1/N_1]$. Since $-\delta < -1/N_1 < 1/N_1 < \delta$, $\varphi(x)$ is decreasing on $[\delta, +\infty)$ and increasing on $(-\infty, -\delta]$ as well, i.e., $\sup_{x\in I}\varphi(x) = \varphi(\delta) = \varphi(-\delta)$. And since $\{f_n(\delta)\} \to 0$ as $n \to \infty$, there must be another N_2 such that $n \geq N_2$ implies that $|f_n(\delta)| = |f_n(-\delta)| < \varepsilon$. We thus take $N > \max\{N_1, N_2\}$. Note that $N > N_1 \implies 1/N < 1/N_1 < \delta$. Hence for any $n \geq N$, $\sup_{x\in I} f_n(x) = |f_n(\delta)| = |f_n(-\delta)| < \varepsilon$. This proves that the function converges uniformly to zero.

* * *

We now proceed to the forward direction. We do proof by contraposition. Suppose that $0 \in A'$. Then there exists a sequence $(a_i)_{i\in\mathbb{N}} \subseteq A$ converges to 0 where $|a_{i+1}| \leq |a_i|$ for all i. To see this, note that there exists $x_1 \in (-1, 1) \cap A$; set $a_1 = x_1$, and for any $i \geq 2$, there exists $x_i \in (\max\{-1/i, -|x_{i-1}|\}, \min\{1/i, |x_{i-1}|\})$; set $a_i = x_i$.

We want to show that $\{f_n(x)\}$ does not converge uniformly to 0, i.e., there exists $\varepsilon > 0$, and there exist infinitely many N's such that for each N, $|f_N(x)| > \varepsilon$ for some x.

We pick $\varepsilon = 1/4$. Now consider the $(a_i)_{i\in\mathbb{N}}$ generated before. Note that for each a_i, there exists a natural N such that $|a_i| \in [1/N, 1/(N-1)]$. And note that $\varphi(x) = |f_N(x)|$ is decreasing on $[1/N, +\infty)$ and increasing on $(-\infty, -1/N]$, and note that fact that $\varphi(x)$ is even. Thus,

$$f_N\left(\frac{1}{N-1}\right) \leq f_N(a_i) \leq f_N\left(\frac{1}{N}\right).$$

Also note that for any natural $N \geq 2$,

$$f_N\left(\frac{1}{N-1}\right) = \frac{\frac{N}{N-1}}{1 + \left(\frac{N}{N-1}\right)^2} = \frac{1}{\frac{N-1}{N} + \frac{N}{N-1}} = \frac{N(N-1)}{(N-1)^2 + N^2} = \frac{N^2 - N}{2(N^2 - N) + 1} = \frac{1}{2 + \frac{1}{N^2 - N}} > \varepsilon.$$

Putting everything together, we know that since (a_i) is such a sequence that its absolute value is decreasing and it converges to zero, then it generates an infinite sequence of $(N_i)_{i\in\mathbb{N}}$ where each $|a_i| \in [1/N_i, 1/(N_i - 1)]$, and there are infinitely many distinct elements in the sequence $(N_i)_{i\in\mathbb{N}}$. And for each N_i,

$$f_{N_i}(a_i) \geq f_{N_i}\left(\frac{1}{N-1}\right) > \varepsilon.$$

This completes the proof that $\{f_n(x)\}$ does not converge uniformly on A.

■

2(a) *Proof.* First note that if $x = 0$ or $x = 1$, then for any $n \in \mathbb{N}$ and any $x \in [0, 1]$, $f_n(x) = 0$. If however, $x \in (0, 1)$, then $0 < 1 - x^4 < 1$, i.e., $1 - x^4 = 1/(1 + p)$ for some $p > 0$. Therefore, by Rudin Theorem 3.20(d), for any fixed $x \in [0, 1]$

$$\lim_{n\to\infty} n^c(1 - x^4)^n = \lim_{n\to\infty} \frac{n^c}{(1 + p)^n} = 0,$$

so,

$$\lim_{n\to\infty} n^c x^3(1 - x^4)^n = \lim_{n\to\infty} x^3 \cdot \lim_{n\to\infty} \frac{n^c}{(1 + p)^n} = x^3 \cdot 0 = 0.$$

The limit function $f(x)$ thus exists and $f(x) = 0$ for all $x \in [0, 1]$. ■

2(b) *Proof.* We claim that the convergence is not uniform for $c \geq 3/4$ but it is uniform for $c < 3/4$.

For $c \geq 3/4$, let $x_n = n^{-1/4}$, then

$$f_n(x_n) = n^{c-\frac{3}{4}} \left(1 - \frac{1}{n}\right)^n.$$

Note that $(1 - 1/n)^n \to e^{-1}$ as $n \to \infty$. Therefore, when $c = 3/4$, $f_n(x_n) \to e^{-1}$ as $n \to \infty$. And when $c > 3/4$, $n^{c-3/4}$ diverges, so $f_n(x_n)$ diverges as $n \to \infty$. In either case, $f_n(x_n)$ is at least $e^{-1}/2$ for infinitely many n's. This concludes that $\{f_n(x)\}$ cannot be converging to zero uniformly in this case.

For $c < 3/4$. First note that $1 + t \leq e^t$ for all $x \in \mathbb{R}$. We also set $y = nx^4$. Thus,

$$|f_n(x)| = \left|n^c x^3(1 - x^4)^n\right| \leq \left|n^c x^3 e^{-nx^4}\right| = \left|n^{c-\frac{3}{4}}(nx^4)^{\frac{3}{4}} e^{-nx^4}\right| = \left|n^{c-\frac{3}{4}} y^{\frac{3}{4}} e^{-y}\right|.$$

Note that $y^{3/4}e^{-y} \to 0$ as exponential grows faster than polynomial. Therefore the function $\varphi(y) = y^{3/4}e^{-y}$ is bounded. Also note that $n^{c-3/4} \to 0$ as $n \to \infty$ since $c - 3/4 < 0$. Hence, $\left|n^{c-3/4}y^{3/4}e^{-y}\right| \to 0$ as $n \to \infty$. This means that $\sup_{x\in[0,1]} |f_n(x)| \to 0$, which shows that $\{f_n(x)\}$ converges uniformly to zero in this case. ■

2(c) *Proof.* First note that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0.$$

We then use change of variable and let $g(x) = 1 - x^4$, so $g'(x) = -4x^3$; note that the derivative is continuous. Thus,

$$\begin{aligned}
\int_0^1 x^3(1 - x^4)^n\,\mathrm{d}x &= \frac{-1}{4}\int_0^1 (1 - x^4)^n(-4x^3)\,\mathrm{d}x = \frac{-1}{4}\int_0^1 (g(x))^n g'(x)\,\mathrm{d}x \\
&= \frac{-1}{4}\int_{g(0)}^{g(1)} u^n\,\mathrm{d}u = \frac{-1}{4}\int_1^0 u^n\,\mathrm{d}u \\
&= \frac{-1}{4}\left[\frac{u^{n+1}}{n+1}\right]_{u=1}^{u=0} \\
&= \frac{1}{4} \cdot \frac{1}{n+1}.
\end{aligned}$$

Therefore,

$$\int_0^1 f_n(x)\,\mathrm{d}x = n^c \int_0^1 x^3(1-x^4)^n\,\mathrm{d}x = \frac{n^c}{4(n+1)}.$$

And it's clear that the above integral converges to 0 iff $c < 1$. ∎

3 *Proof.* Given $\varepsilon > 0$. Since g is continuous at 0, there exists $\delta > 0$ s.t. $|g(x) - g(0)| < \varepsilon$ whenever $|x| < \delta$. And by (iii), there exists a natural N s.t. $f_n(x) < \varepsilon$ for all $n \geq N$ and for all $x \in [-1, -\delta] \cup [+\delta, +1]$. Also let $M = \sup_{x\in[-1,1]} g(x)$. Therefore, for all $n \geq N$,

$$
\begin{aligned}
&\left|\int_{-1}^{1} f_n(x)g(x) - g(0)\,\mathrm{d}x\right| \\
=&\left|\int_{-1}^{1} f_n(x)[g(x) - g(0)]\,\mathrm{d}x\right| \\
\leq&\left|\int_{-1}^{-\delta} f_n(x)[g(x) - g(0)]\,\mathrm{d}x\right| + \left|\int_{-\delta}^{+\delta} f_n(x)[g(x) - g(0)]\,\mathrm{d}x\right| + \left|\int_{+\delta}^{+1} f_n(x)[g(x) - g(0)]\,\mathrm{d}x\right| \\
\leq&\int_{-1}^{-\delta} f_n(x)\,|g(x) - g(0)|\,\mathrm{d}x + \int_{-\delta}^{+\delta} f_n(x)\,|g(x) - g(0)|\,\mathrm{d}x + \int_{+\delta}^{+1} f_n(x)\,|g(x) - g(0)|\,\mathrm{d}x \\
\leq&\int_{-1}^{-\delta} \varepsilon\,|g(x) - g(0)|\,\mathrm{d}x + \int_{-\delta}^{+\delta} f_n(x)\varepsilon\,\mathrm{d}x + \int_{+\delta}^{+1} \varepsilon\,|g(x) - g(0)|\,\mathrm{d}x \\
&\hfill \text{(by above assumptions)} \\
\leq&\varepsilon\int_{-1}^{-\delta} |g(x) - g(0)|\,\mathrm{d}x + \varepsilon\int_{-\delta}^{+\delta} f_n(x)\,\mathrm{d}x + \varepsilon\int_{+\delta}^{+1} |g(x) - g(0)|\,\mathrm{d}x \\
\leq&\varepsilon 2M(1-\delta) + \varepsilon\int_{-1}^{1} f_n(x) + \varepsilon 2M(1-\delta) && \text{(since } f_n(x) \geq 0\text{)} \\
\leq&\varepsilon 4M(1-\delta) + \varepsilon \\
\leq&\varepsilon 4M + \varepsilon \\
\leq&\varepsilon(4M + 1).
\end{aligned}
$$

This concludes the proof.

■

4(a) *Proof.* First note that for all $x \in \mathbb{R}$, $0 \le \varphi(x) \le 1$. Also note that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series hence converge and equals 1. Now fix $t \in \mathbb{R}$. Note that $f_1(t)$ is a series of non-negative terms that's bounded above, thus

$$0 \le f_1(t) = \sum_{n=1}^{\infty} \frac{\varphi(3^{2n-2}t)}{2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

So, $f_1(t)$ converges, which means that f_1 is well-defined. With a similar reasoning, f_2 is well-defined as well. And $0 \le f_i(t) \le 1$ for all t.

* * *

We show that $f_1(t)$ is continuous by showing that it is the limit of a sequence of uniform convergent continuous functions; with a similar reasoning, $f_2(t)$ is continuous as well. Let

$$S_I(t) = \sum_{i=1}^{I} \frac{\varphi(3^{2i-2}t)}{2^i}.$$

Note that each $S_I(t)$ is a continuous function on $\mathbb{R}$, since it is a finite sum of compositions of continuous functions.

Now let $\varepsilon > 0$ be given. Note that $\sum_{i=1}^{\infty} 1/2^i$ converges, so there exists a natural N s.t. $I, J \ge N$ implies

$$\left| \sum_{i=I}^{J} \frac{1}{2^i} \right| < \varepsilon.$$

And note that for any $I, J \ge N$,

$$|S_I(t) - S_J(t)| = \left| \sum_{i=I}^{J} \frac{\varphi(3^{2i-2})}{2^i} \right| \le \left| \sum_{i=I}^{J} \frac{1}{2^i} \right| < \varepsilon.$$

Therefore, by the Cauchy Criterion, $\{S_I\}_{I\in\mathbb{N}}$ converges uniformly. Since $f_1(t)$ is its limiting function, we conclude that $f_1(t)$ must be continuous as well, by Rudin Theorem 7.12.

■

4(b) *Proof.* sorry I didn't have the time to finish this question... ■

4(c) *Proof.* We just need to follow the results established before. Note that

$$\begin{aligned}
f(c) &= (f_1(c), f_2(c)) \\
&= \left(\sum_{n=1}^{\infty} \frac{\varphi(3^{2n-2}c)}{2^n}, \sum_{n=1}^{\infty} \frac{\varphi(3^{2n-1}c)}{2^n}\right) \\
&= \left(\sum_{n=1}^{\infty} \frac{c_{2n-1}}{2^n}, \sum_{n=1}^{\infty} \frac{c_{2n}}{2^n}\right) && \text{(by 4(b))} \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n}{2^n}, \sum_{n=1}^{\infty} \frac{b_n}{2^n}\right) && \text{(by definition of } c_i\text{)} \\
&= (a, b).
\end{aligned}$$

■