1(a) *Proof.* Fix any x where $|x| < 1$. First if $x = 0$. Since $e^0 = 1$ by Rudin 8.27. And $e^{\log y} = y$ for all positive y by Rudin 8.36. Therefore, $\log(1) = 0$. And it's clear that

$$0 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{0^k}{k}.$$

Thus, the equation is true in this case.

Now suppose $x \in (0, 1)$. Then for any $t \in [1, 1 + x]$, $|1 - t| = t - 1 \leq x < 1$. Therefore, by the formula for geometric series,

$$\frac{1}{t} = \frac{1}{1 - (1 - t)} = \sum_{k=0}^{\infty} (1 - t)^k.$$

Therefore,

$$\int_1^{1+x} \frac{1}{t} \,\mathrm{d}t = \int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \,\mathrm{d}t.$$

Furthermore, note that the above infinite series has radius of convergence $(0, 2)$ as $|1 - t| < 1 \implies 0 < t < 2$. Thus that by Rudin Theorem 8.1, $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1, 1 + x]$, as $[1, 1 + x] \subsetneq (0, 2)$. Hence, by Rudin Theorem 7.16,

$$\int_1^{1+x} \sum_{k=0}^{\infty} (1 - t)^k \,\mathrm{d}t = \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \,\mathrm{d}t.$$

Combining the above results, and evaluating each integrals, we have

$$\begin{aligned}
\int_1^{1+x} \frac{1}{t} \,\mathrm{d}t &= \sum_{k=0}^{\infty} \int_1^{1+x} (1 - t)^k \,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} \left[\frac{-(1 - t)^{k+1}}{k + 1} \right]_1^{1+x} \\
&= \sum_{k=0}^{\infty} \frac{-(-x)^{k+1}}{k + 1} \\
&= \sum_{k=1}^{\infty} \frac{-(-x)^k}{k} \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}.
\end{aligned}$$

And since Rudin 8.39, $\log(1 + x) = \int_1^{1+x} \frac{1}{t} \,\mathrm{d}t$, the equation is thus true in this case.

Now suppose $x \in (-1, 0)$. Then again for any $t \in [1 + x, 1]$, $|1 - t| = 1 - t \leq -x < 1$. And the series $\sum_{k=0}^{\infty} (1 - t)^k$ converges uniformly on $[1 + x, 1]$, as $[1, 1 + x] \subsetneq (0, 2)$. Therefore, the above decomposition applies still. This completes the proof. ∎

1(b) *Proof.* We apply Rudin 8.2 (Abel's Theorem) here. First note that $\sum_{n=1}^{\infty}(-1)^{n-1}1/n$ converges by the alternating series test. Therefore, by Rudin 8.2 and by results from (a),

$$\lim_{x\to 1^-}\log(1+x)=\lim_{x\to 1^-}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

And since e^x is and strictly increasing and continuous, then $\log(y)$, being the inverse of e^x, is also strictly increasing continuous. Therefore,

$$\lim_{x\to 1^-}\log(1+x)=\log(2).$$

This gives that

$$\log(2)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

■

2 *Proof.* First note that since $\sum_{n=0}^{\infty} a_n = \infty$, then given any $c > 0$, we also have

$$c \cdot \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c \cdot a_n = \infty.$$

Let $M > 0$ be arbitrary. Choose $c = 1/2$. This means that there exists a natural K s.t. for $N > K$,

$$\sum_{n=0}^{N} c \cdot a_n > M.$$

Note that there exists $0 < \delta < 1$ such that $(1-\delta)^N = c$, since $1-\delta = c^{1/N}$ where $0 < c^{1/N} < 1$. We thus pick this δ. Then note that for any $1 - \delta < x < 1$, and any $n \leq N$

$$x^n > (1-\delta)^n = c^{n/N} \geq c,$$

which, with the fact that each $a_n \geq 0$, gives that,

$$\sum_{n=0}^{N} a_n x^n \geq \sum_{n=0}^{N} c \cdot a_n.$$

Note also that the partial sums of the series $\sum_{n=0}^{\infty} a_n$ are monotone increasing on $(0, 1)$, since $a_n \geq 0$ for all n. Therefore, for $0 < x < 1$,

$$\sum_{n=0}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n.$$

Therefore, combining the above results, we get that for any $x \in (1 - \delta, 1)$,

$$\sum_{n=0}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n \geq \sum_{n=0}^{N} c \cdot a_n > M.$$

This completes the proof that $\sum_{n=0}^{\infty} a_n x^n \to \infty$ as $x \to 1^-$.

$\blacksquare$

3(a) *Proof.* Note that for any fixed i, the sequence of numbers $(a_{i,1}, a_{i,2}, \cdots)$ contains both $a_{i,i}$ and $a_{i,i+1}$, i.e, it contains both 1 and -1 with the rest of entries zero. This means that for fixed i, $\sum_{j=1}^{\infty} = a_{i,j} = 0$. Therefore,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = 0 + 0 + \cdots = 0.$$

However, for fixed j, the sequence $(a_{1,j}, a_{2,j}, \cdots)$ contains both non-zero terms $a_{j,j}$ and $a_{j-1,j}$ (i.e., 1 and -1) only if $j \geq 2$; for $j = 1$, the sequence only contains $a_{j,j}$ (i.e., 1) as its only non-zero term. This means that $\sum_{i=1}^{\infty} a_{i,1} = 1$, whereas $\sum_{i=1}^{\infty} a_{i,j} = 0$ for $j \geq 2$. Therefore,

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j} &= \sum_{j=1}^{1}\sum_{i=1}^{\infty} a_{i,j} + \sum_{j=2}^{\infty}\sum_{i=1}^{\infty} a_{i,j} \\ &= \sum_{i=1}^{\infty} a_{i,1} + \sum_{j=2}^{\infty} 0 \\ &= 1 + 0 \\ &= 1.\end{aligned}$$

■

3(b) *Proof.* Let $A = \left\{\sum_{i,j\in I} a_{ij} : \text{ finite } I \subseteq \mathbb{N}\times\mathbb{N}\right\}$. *Note that in the following we restrict our discussion to the case that* $\sup A < +\infty$*; otherwise it's clear that both double sums are infinite as the prompt suggested.* We show both that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sup A$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sup A$. It's suffice to show the former as the argument for the later is symmetrical.

Let $T = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. We first claim that if $T = +\infty$, then $\sup A = +\infty$ (the proof of the claim right below). The point is that since we've established that $\sup A < +\infty$, thus $T < +\infty$. Let's prove that claim. Suppose, for the sake of contradiction, $\sup A = M \in \mathbb{R}$. Since $T = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \infty$, there must exist a natural N s.t. $\sum_{i=1}^{N}\sum_{j=1}^{\infty} a_{ij} > 2M$. Note that since each $a_{ij} \geq 0$, then for any fixed i, either $\sum_{j=1}^{\infty} a_{ij}$ converges or diverges to $+\infty$ by monotone convergence theorem.

Now, if there exists $\sum_{j=1}^{\infty} a_{ij} = \infty$ for some $i = 1, 2, \cdots, N$, then it's clear that $\sup A = +\infty$, which is a contradiction. If, however, $\sum_{j=1}^{\infty} a_{ij} < +\infty$ for each $i = 1, 2, \cdots, N$, then as the tail of convergent series goes to zero, for each i, there exists N_i s.t. $\sum_{j=N_i+1}^{\infty} < M/N$ or $-\sum_{j=N_i+1}^{\infty} > -M/N$. This means that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N_i} a_{ij} &= \sum_{i=1}^{N}\left(\sum_{j=1}^{\infty} a_{ij} - \sum_{j=N_i+1}^{\infty} a_{ij}\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^{N}\sum_{j=N_i+1}^{\infty} a_{ij} && \text{(can split cuz all series converges)} \\ &> 2M - N\cdot M/N \\ &= M,\end{aligned}$$

which contradicts the fact that M is an upper bound for A.

Now that $T \in \mathbb{R}$, we show that

1. T is an upper bound for the set A,
2. $T - \varepsilon$ is not an upper bound for A, for any $\varepsilon > 0$.

The first point is rather clear. If we pick any $a = \sum_{i,j \in I} a_{ij} \in A$. Note that a only sums over finitely many a_{ij}'s. And since each $a_{ij} \geq 0$, T, being the sum of all a_{ij}'s, is necessarily no smaller than a. This gives that $T \geq a$. So T is indeed an upper bound for A as a is arbitrary.

To show the second point, let $\varepsilon > 0$ be given. Then there must exists a natural N s.t.

$$\sum_{i=1}^{N} \sum_{j=1}^{\infty} a_{ij} > T - \varepsilon/2.$$

Similarly, for each i, there exists a natural N_i s.t.

$$\sum_{j=N_i+1}^{\infty} a_{ij} < \varepsilon/(2N) \text{ or } -\sum_{j=N_i+1}^{\infty} a_{ij} > -\varepsilon/(2N).$$

This means that

$$\sum_{i=1}^{N} \sum_{j=1}^{N_i} a_{ij} = \sum_{i=1}^{N} \left(\sum_{j=1}^{\infty} a_{ij} - \sum_{j=N_i+1}^{N_i} a_{ij} \right) = \sum_{i=1}^{N} \sum_{j=1}^{\infty} a_{ij} - \sum_{i=1}^{N} \sum_{j=N_i+1}^{N_i} a_{ij} > T - \frac{\varepsilon}{2} - \frac{N\varepsilon}{2N} = T - \varepsilon,$$

which means that $T - \varepsilon$ is not an upper bound. This completes the proof. ■

3(c) *Proof.* First note that each $a_{ij}^{+}, a_{ij}^{-} \geq 0$. Therefore, by using the simple comparison test against $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$, we see that both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{+}$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{-}$ must converge. And therefore with the results in $3(b)$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{+} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}^{+} \text{ and } \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^{-} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}^{-}.$$

This also means that for every fixed i, $\sum_{j=1}^{\infty} a_{ij}^{+}$ converges; for each fixed j, $\sum_{i=1}^{\infty} a_{ij}^{+}$ converges, and similarly for a_{ij}^{-}'s.

Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(a_{ij}^{+} - a_{ij}^{-}) \\
&= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{+} - \sum_{j=1}^{\infty} a_{ij}^{-}\right) && \text{(as each } \textstyle\sum_{j=1}^{\infty} a_{ij}^{+}, \sum_{j=1}^{\infty} a_{ij}^{-} \text{ converges)} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^{+} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^{-} && \text{(as each double sum converges)} \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}^{+} - \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}^{-} && \text{(by results above)} \\
&= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{+} - \sum_{i=1}^{\infty} a_{ij}^{-}\right) && \text{(as each } \textstyle\sum_{i=1}^{\infty} a_{ij}^{+}, \sum_{i=1}^{\infty} a_{ij}^{-} \text{ converges)} \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(a_{ij}^{+} - a_{ij}^{-}) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.
\end{aligned}$$

■

4 *Proof.* We first show that $f \in \mathcal{R}[0, R]$. By Rudin Theorem 8.1, we know that f is continuous on $[0, R)$ since f is a power series with radius of convergence R. Also note that f is bounded on the whole $[0, R]$ since f is bounded on $[0, R)$ and $f(R) \in \mathbb{R}$. This means that $f(x)$ on $[0, R]$ is a bounded function with possibly $x = R$ being its only discontinuity. Therefore, by Rudin Theorem 6.10, $f \in \mathcal{R}[0, R]$, i.e., the expression $\int_0^R f(x)\,\mathrm{d}x$ is well-defined.

We next show that the series $\sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$ actually converges. First note that for each n,

$$a_n \frac{R^{n+1}}{n+1} = a_n R^n \cdot \frac{R}{n+1}.$$

Let $\alpha_n = a_n R^n$ and $\beta_n = R/(n+1)$. Since $\sum_{n=0}^{\infty} \alpha_n$ converges, the partial sums A_n of $\sum_{n=0}^{\infty} \alpha_n$ form a bounded sequence. Also $\beta_0 \geq \beta_1 \geq \beta_n \geq \cdots$, with $\lim_{n\to\infty} \beta_n = 0$. Therefore, by Rudin Theorem 3.42, $\sum_{n=0}^{\infty} \alpha_n \beta_n$ converges, i.e., $\sum_{n=0}^{\infty} a_n R^{n+1}/(n+1)$ converges.

Now note that by Rudin Theorem 8.1, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R - \xi]$ given any $R > \xi > 0$. Thus, by Rudin Theorem 7.16,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^{R-\xi} a_n x^n\,\mathrm{d}x.$$

And by integrating each term,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1}.$$

For the rest of this proof, we show that as $\xi \to 0^+$,

$$\int_0^{R-\xi} f(x)\,\mathrm{d}x \to \int_0^{R} f(x)\,\mathrm{d}x, \text{ and } \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} \to \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

And since the functional limit is unique, this would complete the proof that $\int_0^R f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$.

Let's show the former first. Let $\varepsilon > 0$ be given. Note first that since $f(x)$ is bounded on $[0, R]$, then there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [0, R]$. Let $\delta = \varepsilon/M$. Then for any $0 < \xi < \delta$,

$$\begin{aligned}
\left| \int_0^{R-\xi} f(x)\,\mathrm{d}x - \int_0^{R} f(x)\,\mathrm{d}x \right| &= \left| \int_{R-\xi}^{R} f(x)\,\mathrm{d}x \right| \\
&\leq \int_{R-\xi}^{R} |f(x)|\,\mathrm{d}x && \text{(Rudin 6.13(b))} \\
&\leq \int_{R-\xi}^{R} M\,\mathrm{d}x \\
&= M\xi \\
&< M \cdot \frac{\varepsilon}{M} \\
&< \varepsilon.
\end{aligned}$$

This completes the proof of the former functional limit.

Let's move on to the later. Notice that if we do a change of variable with $t = (R - \xi)/\xi$,

$$\sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} \left(\frac{R-\xi}{R}\right)^{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1},$$

where $0 < t < 1$. Note that as $\sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}$ converges by above, by Rudin Theorem 8.2,

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

Now we do the change of variable again,

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} t^{n+1} = \lim_{\xi \to 0^+} \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1} \left(\frac{R-\xi}{R}\right)^{n+1}.$$

Therefore, by combining the above results,

$$\lim_{\xi \to 0^+} \sum_{n=0}^{\infty} a_n \frac{(R-\xi)^{n+1}}{n+1} = \sum_{n=0}^{\infty} a_n \frac{R^{n+1}}{n+1}.$$

This completes the latter functional limit, which completes the whole proof. ■

5(a) *Proof.* Consider

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n.$$

Using ratio test, we get,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

which gives that the radius of convergence is 1.

At $x = 1$, $f(1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \log(2)$ as proven in 1(b). So the power series $f(x)$ converges at $x = 1$. However,

$$\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1},$$

which diverges.

■

5(b) *Proof.* Consider

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n.$$

Using ratio test, we get,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = 1^2 = 1,$$

which gives that the radius of convergence is 1.

Now let $f_n(x) = x^n/n^2$ defined on $[-1, 1]$. Let $M_n = 1/n^2$. Then it's clear that for each n, and each x on $[-1, 1]$

$$|f_n(x)| = \frac{|x|^n}{n^2} \leq \frac{1}{n^2} = M_n.$$

And $\sum_{n=0}^{\infty} M_n$ converges by the p-test. Therefore, by Rudin Theorem 7.10 (M-test), $f(x) = \sum_{n=0}^{\infty} f_n$ converges uniformly on $[-1, 1]$. ■

5(c) *Proof.* Consider the geometric series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n,$$

which we know already has the radius of convergence 1.

Also note that on $(-1, 1)$, $f(x) = \frac{1}{1-x}$, which is an unbounded function (diverges to $+\infty$). However, for each partial sum of $f(x)$, $S_m(x) = \sum_{n=1}^{m} x^n$ is a bounded function on $(-1, 1)$. Therefore, for any $m \in \mathbb{N}$, there will be an $x \in (-1, 1)$ s.t. the difference between $f(x)$ and S_m would be at least 1. This means that the power series doesn't converge uniformly to $f(x)$ on $(-1, 1)$.

$\blacksquare$