1(a) *Proof.* First note that for any fixed $a > 0$ and $x \neq 0$.

$$\frac{a^x - 1}{x} = \frac{e^{x\log a} - e^{x\log 1}}{x} = \left[\frac{e^{xt}}{x}\right]_{t=\log 1}^{t=\log a} = \int_{\log 1}^{\log a} e^{xt}\,\mathrm{d}t.$$

Now consider any real sequence $(x_n) \subseteq \mathbb{R}\setminus\{0\}$ s.t. $x_n \to 0$. Define $f : I \to \mathbb{R}$, where I is the closed interval between $\log a$ and $\log 1 (= 0)$, as

$$f_n(t) = e^{x_n t}.$$

We claim that $f_n(t)$ converges uniformly to $f(t) = 1$ on I. To see this, first note that for each n, f_n is monotone on I. Therefore,

$$\begin{aligned}
M_n &= \sup_{t\in I}|f_n(t) - f(t)| && \text{(by the def. of } M_n) \\
&= \sup_{t\in I}|f_n(t) - f_n(0)| && (\text{since } f_n(0) = f(t), \forall n, t) \\
&= \sup_{t\in I}|f_n(\log a) - f_n(0)| && (\text{since } f_n \text{ monotone on } I) \\
&= \sup_{t\in I}|f_n(\log a) - 1|. && (\text{since } f_n(0) = 1)
\end{aligned}$$

Therefore, $M_n \to 0$ since $f_n(\log a) \to 1$ as $n \to \infty$. This concludes the proof that $f_n(t)$ converges uniformly to 1. Hence by Rudin Theorem 7.16,

$$\lim_{n\to\infty}\int_{\log 1}^{\log a} f_n(t)\,\mathrm{d}t = \int_{\log 1}^{\log a} f(t)\,\mathrm{d}t, \text{ i.e., } \lim_{n\to\infty}\int_{\log 1}^{\log a} e^{x_n t}\,\mathrm{d}t = \int_{\log 1}^{\log a} 1\,\mathrm{d}t = \log a.$$

Since (x_n) is an arbitrary sequence in $\mathbb{R}\setminus\{0\}$ that converges to 0, by Rudin Theorem 4.2,

$$\lim_{x\to 0}\int_{\log 1}^{\log a} e^{xt}dt = \log a.$$

Combining with the previous results, we can thus conclude that,

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \log a$$

■

1(b) *Proof.* First note that by HW7(a), if $x \neq 0$ is small enough, say $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This means that for a fixed $x \neq 0$ small enough,

$$\begin{aligned}
\frac{1}{x^2}\left[\log(1+x) - x\right] &= \frac{1}{x^2}\left[-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right] \\
&= -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots && \left(\text{since } \textstyle\sum_{k=2}^{\infty}(-1)^{k-1}\frac{x^k}{k} \text{ converges}\right) \\
&= -\frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}.
\end{aligned}$$

Note that since

$$\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k+2}} = 1,$$

the series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ has raidus of convergence of 1. This means that on $(-1,1)$, the function $f(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2}$ is continuous by Rudin Theorem 8.1. Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{\log(1+x)-x}{x^2} &= \lim_{x\to 0} -\frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since they agree on } |x|<1\text{)}\\
&= -\frac{1}{2} + \lim_{x\to 0}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k+2} && \text{(since the series converges)}\\
&= -\frac{1}{2} + f(0) && \text{(since } f \text{ is continous at 0)}\\
&= -\frac{1}{2}. && \text{(since } f(0)=0\text{)}
\end{aligned}$$

■

1(c) *Proof.* First note that with the results in Rudin Theorem 3.31, it's clear that for any sequence $(a_n) \subseteq \mathbb{R}$ where $a_n \to \infty$,

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

We thus fix $x \in \mathbb{R}$. If $x = 0$, then

$$\lim_{n\to\infty}\left(1+\frac{0}{n}\right)^{n} = \lim_{n\to\infty} 1 = 1 = e^0.$$

If $x \neq 0$, then

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = \left[\left(1+\frac{1}{n/x}\right)^{n/x}\right]^x = e^x,$$

since as $n \to \infty$, $n/x \to \infty$. ■

2(a) *Proof.* ■

2(b) *Proof.* ■

3(a) *Proof.* Let's first compute each c_m. By Rudin (62),

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,\mathrm{d}x.$$

If $m = 0$, then with the periodicity of the f,

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0.$$

If $m \neq 0$, then with integration by parts,

$$\begin{aligned} c_m &= \frac{1}{2\pi}\int_{-\pi}^{0} \frac{-\pi - x}{2}e^{-imx}\,\mathrm{d}x + \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\pi - x}{2}e^{-imx}\,\mathrm{d}x \\ &= \frac{-i\pi m + e^{i\pi m} - 1}{4\pi m^2} + \frac{-i\pi m - e^{-i\pi m} + 1}{4\pi m^2}. \end{aligned}$$

Therefore, when m is even,

$$c_m = \frac{-i\pi m + 1 - 1}{4\pi m^2} + \frac{-i\pi m - 1 + 1}{4\pi m^2} = \frac{-i}{2m},$$

and similarly, when m is odd,

$$c_m = \frac{-i\pi m + (-1) - 1}{4\pi m^2} + \frac{-i\pi m - (-1) + 1}{4\pi m^2} = \frac{-i}{2m}.$$

* * *

Now note that for $m \neq 0$,

$$\begin{aligned} c_m e^{imx} &= -\frac{1}{2m}e^{i(mx+\pi/2)} \\ &= -\frac{1}{2m}\left(\cos(mx + \pi/2) + i\sin(mx + \pi/2)\right) \\ &= \frac{1}{2m}\left(\sin(mx) - i\sin(mx + \pi/2)\right) \\ &= \frac{1}{2m}\left(\sin(mx) - i\cos(mx)\right). \end{aligned}$$

And similarly,

$$\begin{aligned} c_{-m} e^{-imx} &= \frac{1}{2(-m)}\left(\sin(-mx) - i\sin(-mx + \pi/2)\right) \\ &= \frac{1}{2m}\left(\sin(mx) + i\sin(-mx + \pi/2)\right) \\ &= \frac{1}{2m}\left(\sin(mx) + i\sin(-(mx - \pi/2))\right) \\ &= \frac{1}{2m}\left(\sin(mx) - i\sin(mx - \pi/2)\right) \\ &= \frac{1}{2m}\left(\sin(mx) + i\cos(mx)\right). \end{aligned}$$

This means that,

$$c_m e^{imx} + c_{-m} e^{-imx} = \frac{1}{m} \sin(mx).$$

Therefore,

$$\sum_{m=-\infty}^{\infty} c_m e^{imx} = \sum_{n=0}^{\infty} \frac{1}{n} \sin(nx).$$

Also since $f(x)$ is differentiable on $[-\pi, 0)$ and $(0, \pi]$, $f(x)$ is thus Lipschitz continuous on there as well. Therefore, by Rudin Theorem 8.14, on those intervals, $f(x)$'s Fourier series converge to $f(x)$, which means that on those intervals,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n} \sin(nx).$$

■

3(b) *Proof.* ■

3(c) *Proof.* ■

4(a) *Proof.* First notice that for any $x \in (0, \pi/2)$,$(\pi - x, \pi + x) \subseteq (\pi/2, 3\pi/2)$. This means that $\pi/2 < s < 3\pi/2$, so $\pi/4 < s/2 < 3\pi/4$. Therefore, $\sin(s/2) \neq 0$ on any $(\pi - x, \pi + x)$. This means that the integral of $D_N(s)$ over $(\pi - x, \pi + x)$ is well-defined.

Now we fix $x \in (0, \pi/2)$. Note that

$$\begin{aligned}
\int_{\pi-x}^{\pi+x} D_N(s) &= \int_{\pi-x}^{\pi+x} \frac{\sin(Ns + s/2)}{s/2}\,\mathrm{d}s \\
&= \int_{\pi-x}^{\pi+x} \frac{\sin(Ns)\cos(s/2) + \cos(Ns)\sin(s/2)}{\sin(s/2)}\,\mathrm{d}s \\
&= \int_{\pi-x}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \int_{\pi-x}^{\pi+x} \cos(Ns)\,\mathrm{d}s.
\end{aligned}$$

Now, define $g : [-\pi, \pi] \to \mathbb{R}$ as

$$g(s) = \begin{cases} \cot(s/2) & \text{if } s \in (\pi - x, \pi + x), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(s) \in \mathcal{R}[-\pi, \pi]$, since $\cot(s/2)$ is continous on $\pi - x, \pi + x$, which means that $g(s)$ has finitely many discontinuities on $[-\pi, \pi]$. Therefore, what we've learned in class,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \sin(Ns)g(s)\,\mathrm{d}s = 0.$$

Noice also that for each N,

$$\int_{-\pi}^{\pi} \sin(Ns)g(s)\,\mathrm{d}s = \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s.$$

Thus,

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s = 0.$$

And with a reasoning along with the peridocity of sin and cot, we also get that

$$\lim_{N\to\infty} \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s = 0.$$

Therefore,

$$\begin{aligned}
&\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s \\
=&\lim_{N\to\infty} \left(\int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2)\,\mathrm{d}s \right) \\
=&\lim_{N\to\infty} \int_{\pi-x}^{\pi} \sin(Ns)\cot(s/2)\,\mathrm{d}s + \lim_{n\to\infty} \int_{\pi}^{\pi+x} \sin(Ns)\cot(s/2) \\
=&0 + 0 \\
=&0.
\end{aligned}$$

With a similar reasoning, we can get that

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} \cos(Ns)\,\mathrm{d}s = 0.$$

Therefore,

$$\lim_{N\to\infty} \int_{\pi-x}^{\pi+x} D_n(s)\,\mathrm{d}s, \text{ which gives that } \lim_{N\to\infty} r_N(x) = 0.$$

■

4(b) *Proof.* ■

4(c) *Proof.* ■