Jacobson 2.1.1 *Proof.* Pick any continuous functions $f, g, h \in C$.

$(C, +, 0)$ is an abelian group.

Closure: Clearly that $f + g$ is a continuous function as well.

Associativity: For all $x \in \mathbb{R}$, $[f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)]$ by the additive associativity of $\mathbb{R}$. Thus, $(f + g) + h = f + (g + h)$.

Identity: Since the integer 0 is the identity in $\mathbb{R}$, $f(x) + 0 = f(x) = 0 + f(x)$ for all x, i.e., the zero function 0 is the identity here, and note that it is continuous.

Inverse: Note that for all $x \in \mathbb{R}$, $f(x) + (-1)f(x) = x - x = 0$. Thus, the additive inverse of function f is $(-1)f$ or simply $-f$, and it is continuous.

Commutative: The abelianess of C follows from that of $\mathbb{R}$. Note that for all $x \in \mathbb{R}$, $f(x) + g(x) = g(x) + f(x)$. Therefore, $f + g = g + f$.

* * *

$(C, \circ, \mathrm{id}_{\mathbb{R}})$ is a monoid.

Closure: Clearly that $f \circ g$ is a continuous function as well.

Associativity: For all $x \in \mathbb{R}$, $((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x)$. Thus, $f \circ (g \circ h) = (f \circ g) \circ h$.

Identity: For all $x \in \mathbb{R}$, $(f \circ \mathrm{id}_{\mathbb{R}})(x) = f(\mathrm{id}_{\mathbb{R}}(x)) = f(x) = \mathrm{id}_{\mathbb{R}}(f(x)) = (\mathrm{id}_{\mathbb{R}} \circ f)(x)$. Thus, $f \circ \mathrm{id}_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}} \circ f$.

* * *

$(C, +, \circ)$ is not a ring as it violates the distributive law. Let $f(x) = |x|, g(x) = 2, h(x) = -2$ for all $x \in \mathbb{R}$. Then for all x,

$$(f \circ (g + h))(x) = 0 \neq 4 = (f \circ g + f \circ h)(x).$$

■

Jacobson 2.1.4 *Proof.* Pick any $a+b\sqrt{-3}, c+d\sqrt{-3} \in I$.

First, I is a subgroup of the additive group of $\mathbb{C}$. Note that$(a+b\sqrt{-3})-(c+d\sqrt{-3}) = (a-c)+(b-d)\sqrt{-3}$. Then,

Case 1: If all $a, b, c, d \in \mathbb{Z}$, then $a-c, b-d \in \mathbb{Z}$.

Case 2: If all a, b, c, d are halfs of odd integers, i.e., $a = a'+1/2, b = b'+1/2, c = c'+1/2, d = d'+1/2$, for some $a', b', c', d' \in \mathbb{Z}$. So, $a-c = a'-c' \in \mathbb{Z}, b-d = b'-d' \in \mathbb{Z}$.

Case 3: If only $a, b \in \mathbb{Z}$ but c, d are halfs of odd integers, i.e., $c = c'+1/2, d = d'+1/2$. Then $a-c = (a-c')-1/2, b-d = (b-d')-1/2$ where $a-c', b-d' \in \mathbb{Z}$. So, $a-c, b-d$ are halfs of odd integers.

Case 4: If a, b are halfs of odd integers but $c, d \in \mathbb{Z}$. Then similar as in case 3, both $a-c, b-d$ are halfs of odd integers.

Therefore, in all four cases, $(a-c)+(b-d)\sqrt{-3} \in I$. Based on the subgroup criteria, I is an additive subgroup of $\mathbb{C}$.

Next, I is a submonoid of the multiplicative monoid of $\mathbb{C}$. First note that since $1, 0 \in \mathbb{Z}$, then $1 = 1+0\sqrt{-3} \in I$. It remains to check that I is closed under multiplication. Note that $(a+b\sqrt{-3})(c+d\sqrt{-3}) = (ac-3bd)+(ad+bc)\sqrt{-3}$. Then,

Case 1: If all $a, b, c, d \in \mathbb{Z}$, then $ac-3bd, ad+bc \in \mathbb{Z}$ as well.

Case 2: If all a, b, c, d are halfs of odd integers as before. Then,

$$\begin{aligned} ac-3bd &= (a'+1/2)(c'+1/2)-3(b'+1/2)(d'+1/2) \\ &= a'c'+a'/2+c'/2+1/4-3(b'd'+b'/2+d'/2+1/4) \\ &= a'c'-3b'd'+1/2(a'+c'-3b'-3d')-1/2, \end{aligned}$$

which is either an integer or a half of an odd integer. Note that $ac-3bd \in \mathbb{Z}$ iff $a'+c'-3b'-3d'$ is odd, and $ac-3bd$ is a half of an odd integer iff $a'+c'-3b'-3d'$ is even.

Similarly,

$$\begin{aligned} ad+bc &= a'd'+a'/2+d'/2+1/4+b'c'+b'/2+c'/2+1/4 \\ &= a'd'+b'c'+1/2(a'+b'+c'+d')+1/2 \end{aligned}$$

which is either an integer or a half of an odd integer. Note that $ad+bc \in \mathbb{Z}$ iff $a'+b'+c'+d'$ is odd, and $ad+bc$ is a half of an odd integer iff $a'+b'+c'+d'$ is even.

Now, if $(ad+bc) \in \mathbb{Z}$, then $a'+b'+c'+d'$ is odd. This means that either one or three of a', b', c', d' are odd. This gives that $a'+c'-3b'-3d'$ is also odd, as multiplying by -3 does not change the parity of an integer, which in turn gives that $(ac-3bd) \in \mathbb{Z}$. And by a similar argument, $(ac-3bd) \in \mathbb{Z} \implies (ad+bc) \in \mathbb{Z}$.

Case 3: If $a, b \in \mathbb{Z}$ but c, d are halfs of odd integers. Then,

$$\begin{aligned} ac - 3bd &= a(c' + 1/2) - 3b(d' + 1/2) \\ &= ac' + a/2 - 3bd' - 3b/2 \\ &= (ac' - 3bd') - (a - 3b)/2, \end{aligned}$$

which means that $ac - 3bd$ is either an integer or a half of an integer. Note that $ac - 3bd \in \mathbb{Z}$ iff $a - 3b$ is even, and $ac - 3bd$ a half of an odd integer iff $a - 3b$ is odd.

Similarly,

$$\begin{aligned} ad + bc &= a(d' + 1/2) + b(c' + 1/2) \\ &= (ad' + bc') + (a + b)/2, \end{aligned}$$

which means that $ad + bc$ is either an integer or a half on odd integer. Note that $ad + bc \in \mathbb{Z}$ iff $a + b$ is even, and $ad + bc$ is half of an odd integer iff $a + b$ is odd.

Now, if $ac - 3bd \in \mathbb{Z}$, then $a - 3b$ is even. So a, b have the same parity, which means that $a + b$ is even as well. This gives that $ad + bc$ is an integer as well. And by a similar reasoning on parity, $ad + bc \in \mathbb{Z} \implies ac - 3bd \in \mathbb{Z}$.

Case 4: If $c, d \in \mathbb{Z}$ but a, b are halfs of odd integers. Then similar to case 3, it reaches the same conclusion.

Therefore, in all four cases, $ac - 3bd$ and $ad + bc$ are either both integers of both halfs of odd integers. This means that $(ac - 3bd) + (ad + bc)\sqrt{-3} \in I$, i.e., I is closed under multiplication. I is a subring of $\mathbb{C}$. ■

Jacobson 2.2.1 *Proof.* Let a finite domain R be given. Pick any nonzero element $a \in R$. We aim to show that there exists $a^{-1} \in R$ such that $a^{-1}a = 1 = aa^{-1}$. It is suffice to show that a has both a right inverse a_R^{-1} and a left inverse a_L^{-1}; if so, we have,

$$a_R^{-1} = (a_L^{-1}a)a_R^{-1} = a_L^{-1}(aa_R^{-1}) = a_L^{-1},$$

i.e., the left inverse equals to the right inverse, which means the inverse a^{-1} exists.

First note that since R is a domain, then the left cancellation law holds. To see this, assume $ax = ay$ for some $x, y \in R$, then $ax - ay = 0$, which gives $a(x - y) = 0$. Since $a \neq 0$ and we are in a domain, a is thus not a zero-divisor, which means that $x - y = 0 \implies x = y$.

Now since R is finite, we can enumerate all elements of R as

$$r_1, r_2, \cdots, r_k,$$

for some positive integer k. And we claim that the following is also an enumeration of all elements of R,

$$ar_1, ar_2, \cdots, ar_k,$$

as it contains k distinct elements of R. Note that they are distinct because if $ar_i = ar_j$, then by left cancellation law established above, $r_i = r_j$. Therefore, there exists $1 \leq s \leq k$ such that $ar_s = 1$, i.e., $a_R^{-1} = r_s$. And by a similar argument as above, a must also have a left inverse a_L^{-1}. This proves that a^{-1} exists, which concludes the proof. ■

Jacobson 2.2.4 *Proof.* We show that $1-ba$ has both a left inverse and a right inverse, and they are equal.

We name the inverse of $1-ab$ as c. Then, we see that

$$\begin{aligned}(1-ab)c=1 &\implies c-abc=1 \implies bc-babc=b \implies bca-babca=ba\\ &\implies (1-ba)bca=ba \implies (1-ba)bca-ba=0\\ &\implies (1-ba)bca+(1-ba)=1\\ &\implies (1-ba)(bca+1)=1.\end{aligned}$$

Therefore, $(1-ba)_R^{-1}$ exists. Similarly,

$$\begin{aligned}c(1-ab)=1 &\implies c-cab=1 \implies ca-caba=a \implies bca-bcaba=ba\\ &\implies bca(1-ba)=ba \implies bca(1-ba)-ba=0\\ &\implies bca(1-ba)+(1-ba)=1\\ &\implies (bca+1)(1-ba)=1.\end{aligned}$$

Therefore, $(1-ba)_L^{-1}$ exists. Since $(1-ba)_L^{-1}=bca+1=(1-ba)_R^{-1}$, this concludes the proof. ■

Jacobson 2.2.6 *Proof.* We show equivalence by proving (1) $\implies$ (3), (3) $\implies$ (2), and (2) $\implies$ (1).

(1) $\implies$ (3) : Suppose u has two distinct right inverses, x_1, x_2. Then $ux_1 = 1 = ux_2$, which gives that $ux_1 - ux_2 = 0$. Then by distributive law, $u(x_1 - x_2) = 0$. Since $x_1 \neq x_2$, $x_1 - x_2 \neq 0$, which means that u is a left zero-divisor.

(3) $\implies$ (2) : Suppose u is a left zero-divisor, then there exists $x \neq 0$ such that $ux = 0$. Now for the sake of contradiction, suppose that u is a unit, then u^{-1} exists. Therefore,

$$x = 1 \cdot x = (u^{-1}u)x = u^{-1}(ux) = u^{-1} \cdot 0 = 0.$$

This contradictions $x \neq 0$. Hence, u is not a unit.

(2) $\implies$ (1) : We do proof by contraposition. Suppose that u has a unique right inverse, x. Then $ux = 1$, and $uxu = u$, or $uxu - u = 0$. Therefore, $uxu - u + ux = 1$, i.e., $u(xu - 1 + x) = 1$. Since u has a unique right inverse, then $xu - 1 + x = x$, which gives that $xu = 1$. So, x is also a left inverse u, which proves that u is a unit.

■

Jacobson 2.2.7 *Proof.* Suppose that element u in a ring R has one right inverse x. We then claim that the $\{x_n\}_{n\in\mathbb{N}}$ is an infinite collection of distinct right inverses of u, where,

$$x_n = x + (1 - xu)u^n.$$

We first see that each x_n is a right inverse of u. Note that

$$\begin{aligned} ux_n &= u(x + (1 - xu)u^n) = ux + u(1 - xu)u^n = ux + (u - (ux)u)u^n \\ &= ux + (u - u)u^n = ux + 0 \cdot u^n \\ &= ux \\ &= 1. \end{aligned}$$

We next see that $x_n \neq x_m$ if $n \neq m$. Suppose the contrary that $n \neq m$ but $x_n = x_m$ (WLOG assume $n > m$). First notice that since u has a right inverse, then u is not a right zero-divisor, meaning that u obeys the right cancellation law. To see this, suppose $au = bu$ for some $a, b \in R$. Then $(a - b)u = 0$. Note then that

$$0 = 0 \cdot x = [(a - b)u]x = (a - b)[ux] = a - b.$$

Therefore, $a = b$, i.e., we can cancel u on the right.

Going back the proof, if $x_n = x_m$, then

$$\begin{aligned} x + (1 - xu)u^n = x + (1 - xu)u^m &\implies (1 - xu)u^n = (1 - xu)u^m \\ &\implies (1 - xu)u^{n-m} = 1 - xu \quad \text{(cancel } u \text{ on the right)} \\ &\implies (1 - xu)u^{n-m} + xu = 1 \\ &\implies ((1 - xu)u^{n-m-1} + x)u = 1, \end{aligned}$$

i.e., u has a left inverse, which means that u is a unit. However, we have established in the previous exercise that u is not a unit since u has more than one right inverse. Therefore, $x_n \neq x_m$, i.e., all x_n's are distinct.

* * *

Consider the following counter-example. Let R be the set of continuous functions on $[0, +\infty)$. Then as in Ex 2.1.1, we see that $(R, \circ, \mathrm{id})$ is a monoid. Consider the function $f \in R$ where $f(x) = x^2$ for all $x \in [0, +\infty)$. Suppose g is a right inverse of f, then we must have

$$\begin{aligned} (f \circ g)(x) = x &\implies f(g(x)) = x \\ &\implies g(x)^2 = x \\ &\implies g(x) = \pm\sqrt{x}. \end{aligned}$$

Since g has to be continuous, then either $g(x) = +\sqrt{x}$ or $g(x) = -\sqrt{x}$, and only those two options. So f has exactly two right inverses, not infinitely many. ■