Jacobson 2.1.1 *Proof.* Pick any functions $f, g, h \in C$.

$(C, +, 0)$ is an abelian group.

Closure: Clearly that $f + g$ is a function as well.

Associativity: For all $x \in \mathbb{R}$, $[f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)]$ by the additive associativity of $\mathbb{R}$. Thus, $(f + g) + h = f + (g + h)$.

Identity: Since the integer 0 is the identity in $\mathbb{R}$, $f(x) + 0 = f(x) = 0 + f(x)$ for all x, i.e., the zero function 0 is the identity here.

Inverse: Note that for all $x \in \mathbb{R}$, $f(x) + (-1)f(x) = x - x = 0$. Thus, the additive inverse of function f is $(-1)f$ or simply $-f$.

Commutative: The abelianess of C follows from that of $\mathbb{R}$. Note that for all $x \in \mathbb{R}$, $f(x) + g(x) = g(x) + f(x)$. Therefore, $f + g = g + f$.

* * *

$(C, \circ, \mathrm{id}_{\mathbb{R}})$ is a monoid.

Closure: Clearly that $f \circ g$ is a function as well.

Associativity: For all $x \in \mathbb{R}$, $((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x)$. Thus, $f \circ (g \circ h) = (f \circ g) \circ h$.

Identity: For all $x \in \mathbb{R}$, $(f \circ \mathrm{id}_{\mathbb{R}})(x) = f(\mathrm{id}_{\mathbb{R}}(x)) = f(x) = \mathrm{id}_{\mathbb{R}}(f(x)) = (\mathrm{id}_{\mathbb{R}} \circ f)(x)$. Thus, $f \circ \mathrm{id}_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}} \circ f$.

* * *

$(C, +, \circ)$ is not a ring as it violates the distributive law. Let $f(x) = |x|, g(x) = 2, h(x) = -2$ for all $x \in \mathbb{R}$. Then for all x,

$$(f \circ (g + h))(x) = 0 \neq 4 = (f \circ g + f \circ h)(x).$$

∎

Jacobson 2.1.4 *Proof.* Pick any $a+b\sqrt{-3}, c+d\sqrt{-3} \in I$.

First, I is a subgroup of the additive group of $\mathbb{C}$. Note that$(a+b\sqrt{-3})-(c+d\sqrt{-3}) = (a-c)+(b-d)\sqrt{-3}$. Then,

Case 1: If all $a,b,c,d \in \mathbb{Z}$, then $a-c, b-d \in \mathbb{Z}$.

Case 2: If all a,b,c,d are halfs of odd integers, i.e., $a = a'+1/2, b = b'+1/2, c = c'+1/2, d = d'+1/2$, for some $a', b', c', d' \in \mathbb{Z}$. So, $a-c = a'-c' \in \mathbb{Z}, b-d = b'-d' \in \mathbb{Z}$.

Case 3: If only $a,b \in \mathbb{Z}$ but c,d are halfs of odd integers, i.e., $c = c'+1/2, d = d'+1/2$. Then $a-c = (a-c')-1/2, b-d = (b-d')-1/2$ where $a-c', b-d' \in \mathbb{Z}$. So, $a-c, b-d$ are halfs of odd integers.

Case 4: If a,b are halfs of odd integers but $c,d \in \mathbb{Z}$. Then similar as in case 3, both $a-c, b-d$ are halfs of odd integers.

Therefore, in all four cases, $(a-c)+(b-d)\sqrt{-3} \in I$. Based on the subgroup criteria, I is an additive subgroup of $\mathbb{C}$.

Next, I is a submonoid of the multiplicative monoid of $\mathbb{C}$. First note that since $1, 0 \in \mathbb{Z}$, then $1 = 1+0\sqrt{-3} \in I$. It remains to check that I is closed under multiplication. Note that $(a+b\sqrt{-3})(c+d\sqrt{-3}) = (ac-3bd)+(ad+bc)\sqrt{-3}$. Then,

Case 1: If all $a,b,c,d \in \mathbb{Z}$, then $ac-3bd, ad+bc \in \mathbb{Z}$ as well.

Case 2: If all a,b,c,d are halfs of odd integers as before. Then, $ac-3bd = (a'+1/2)(c'+1/2)-3(b'+1/2)(d'+1/2) = a'c'+a'/2+c'/2+1/4-3(b'd'+b'/2+d'/2+1/4) = a'c'-3b'd'+1/2(a'+c'-3b'-3d')-1/2$, which is an integer if $a'+c'-3b'-3d'$ is odd, and is a half of an odd integer of $a'+c'-3b'-3d'$ is even. Similarly, $ad+bc = a'd'+a'/2+d'/2+1/4+b'c'+b'/2+c'/2+1/4 = a'd'+b'c'+1/2(a'+b'+c'+d')+1/2$, which is either an integer of a half of an odd integer.

Case 3: If $a,b \in \mathbb{Z}$ but c,d are halfs of odd integers. Then consider $2(ac-3bd) = 2a(c'+1/2)-6b(d'+1/2) = 2ac'+1-6bd'-3 \in \mathbb{Z}$, which means that $ac-3bd$ is either itself an integer of a half of an odd integer. Similarly, consider $2(ad+bc) = 2a(d'+1/2)+2b(c'+1/2) = 2ad'+a+2bc'+b \in \mathbb{Z}$, which means that $ad+bc$ is either itself an integer of a half on odd integer.

Case 4: If $c,d \in \mathbb{Z}$ but a,b are halfs of odd integers. Then similar to case 3, it reaches the same conclusion.

Therefore, in all four cases, $(ac-3bd)+(ad+bc)\sqrt{-3} \in I$, i.e., I is closed under multiplication. I is a subring of $\mathbb{C}$. ∎

Jacobson 2.2.1 *Proof.* Let a finite domain R be given. Pick any nonzero element $a \in R$. We aim to show that there exists $a^{-1} \in R$ such that $a^{-1}a = 1 = aa^{-1}$. It is suffice to show that a has both a right inverse a_R^{-1} and a left inverse a_L^{-1}; if so, we have,

$$a_R^{-1} = (a_L^{-1}a)a_R^{-1} = a_L^{-1}(aa_R^{-1}) = a_L^{-1},$$

i.e., the left inverse equals to the right inverse, which means the inverse a^{-1} exists.

First note that since R is a domain, then the left cancellation law holds. To see this, assume $ax = ay$ for some $x, y \in R$, then $ax - ay = 0$, which gives $a(x - y) = 0$. Since $a \neq 0$ and we are in a domain, a is thus not a zero-divisor, which means that $x - y = 0 \implies x = y$.

Now since R is finite, we can enumerate all elements of R as

$$r_1, r_2, \cdots, r_k,$$

for some positive integer k. And we claim that the following is also an enumeration of all elements of R,

$$ar_1, ar_2, \cdots, ar_k,$$

as it contains k distinct elements of R. Note that they are distinct because if $ar_i = ar_j$, then by left cancellation law established above, $r_i = r_j$. Therefore, there exists $1 \leq s \leq k$ such that $ar_s = 1$, i.e., $a_R^{-1} = r_s$. And by a similar argument as above, a must also have a left inverse a_L^{-1}. This proves that a^{-1} exists, which concludes the proof. ∎

Jacobson 2.2.4 *Proof.* ■

Jacobson 2.2.6 *Proof.* ■

Jacobson 2.2.7 *Proof.*

■