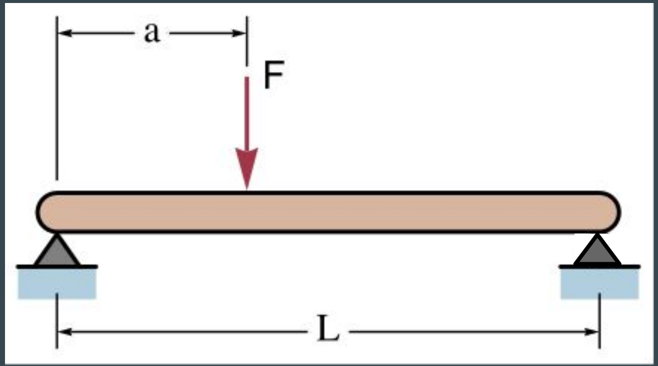
Minimizing Energy to Describe the Shape of an Elastic Beam



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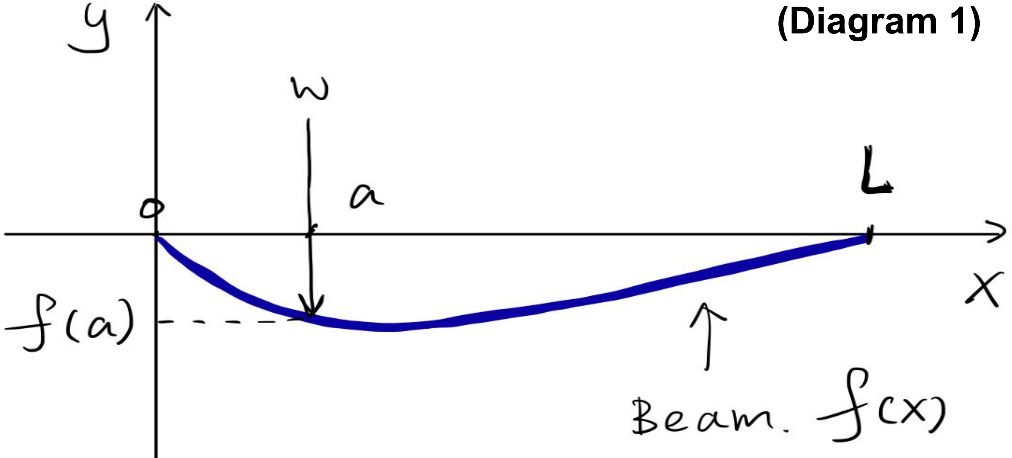
Problem Statement

- Set up/Assumptions (*D. 1*)
- Goal: Find the beam function f(x)
- Find f(x) by minimizing energy
  - Energy functional $\mathcal{E}[f]$ (*Eq. 1*)
  - Euler-Lagrange equation (*Eq. 2*)
  - ODE (*Eq. 3*)
  - Boundary/Initial conditions (*Eq. 4*)
  - Unique solution to the IVP (*Eq. 5*)

(Diagram 1)



$$\mathcal{E}[f] = \frac{EI}{2}\int_0^L (f''(x))^2 dx - Wf(a)$$

(1)
$$= \frac{EI}{2}\int_0^L (f''(x))^2 dx - W\int_0^L f(x)\delta(x-a)dx$$

(2)
$$\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f''}\right) = 0$$

where, $\mathcal{L}(x, f, f', f'') = \frac{EI}{2}(f''(x))^2 + Wf(x)\delta(x-a)$

(3)
$$f^{(4)}(x) = -\frac{W}{EI}\delta(x-a)$$

(4)
$$f(0) = f(L) = 0$$
$$f''(0) = f''(L) = 0$$

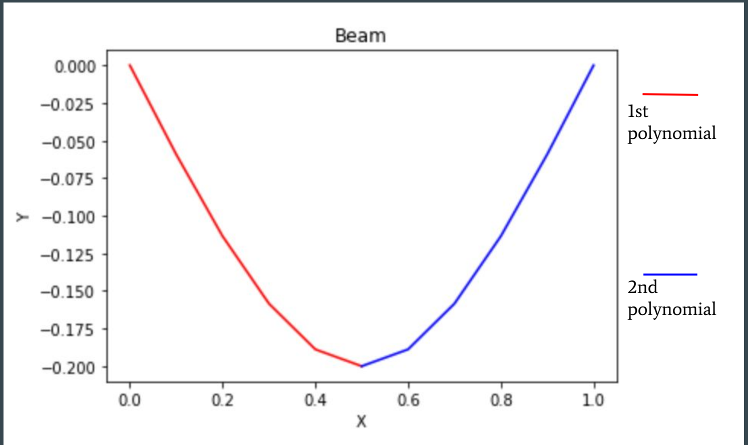
(5)
$$\begin{cases} f(x) = -\frac{Wbx}{6EIL}(L^2 - x^2 - b^2), & \text{if } x < a \\ f(x) = -\frac{Wav}{6EIL}(L^2 - v^2 - a^2), & \text{if } x \geq a \end{cases}$$

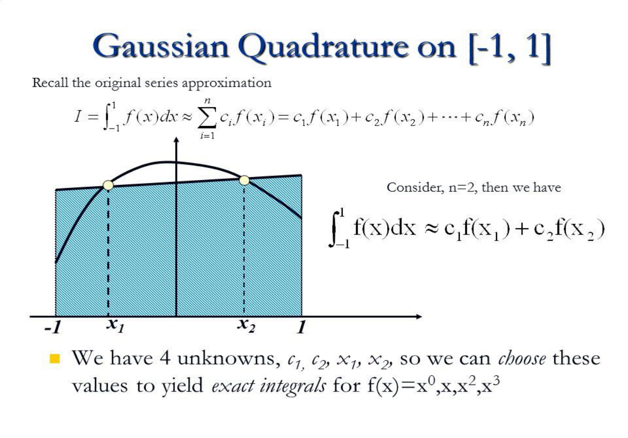
for $b = L - a$, and $v = L - x$.

Methods Used

$$\mathcal{E}[f] = \frac{EI}{2}\int_0^L (f''(x))^2 dx - W f(a)$$

- Natural Cubic Spline that interpolates between 3 points and returns two sets of coefficients for a piecewise polynomial that describes the shape of the beam
- Used definition of a derivative to calculate the 1st and 2nd derivatives of a function at a point
- 2-point Gaussian Quadrature to integrate function to reduce unnecessary errors
- Code to find minimum value of the function on an interval





Results

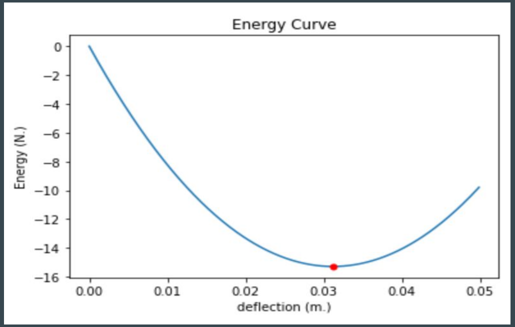
- Inputs: Beam Properties(Length, Elasticity, etc), Load Position, Load Mass
- Outputs: Polynomial of optimized beam, the amount of deflection at the load position, the energy of the optimized beam, and a graph of Energy as a function of the deflection at the load position
- We can compare our resulting deflections below the load position with the analytic results given by (1.), and we compare our piecewise polynomial with the function derived from the energy equation. (2.)
- The convexity of E(deflection) is unsurprising, and this convexity allowed us to tailor our optimization function to this particular function, decreasing the runtime of the algorithm.

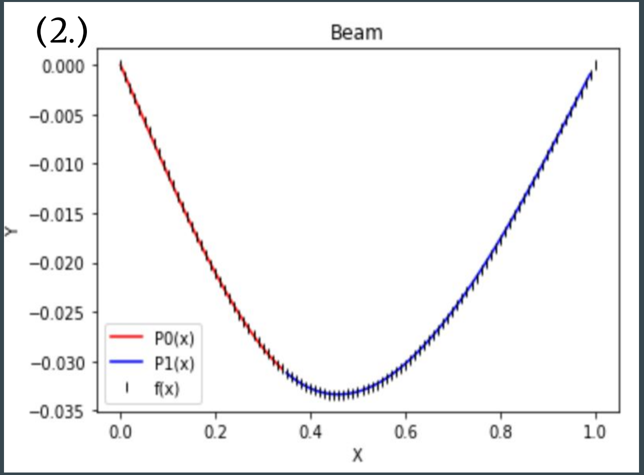
(1.)

$$f(x) = -\frac{Wbx}{6EIL}(L^2 - x^2 - b^2), x < a,$$

$$f(x) = -\frac{Wav}{6EIL}(L^2 - v^2 - a^2), x \geq a,$$

for $b = L - a$, and $v = L - x$.





Thank You