(unique) prime factorization. This means that n is evenly divided by some prime number q. But notice that q cannot be any of the primes between 2 and p, because these will all have a remainder of 1. Therefore, n must be a prime factor greater than p. This is a contradiction.

PHIL 322—Modal Logic
Homework 1

September 15, 2022

entered at the top of the file. 1. Call a binary relation R on a set S *asymmetric* if for no $a, b \in S$ both $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$. Specify a relation R and a set S such that R is a relation on S and R is reflexive and asymmetric.

2. Prove that the transitive closure, R^+ of a binary relation R is actually transitive. You can do this however you like. Here is one suggestion:

a. Prove that, where X is a set and $R \subseteq X^2$ (i.e., it is a binary relation on X), then $\langle a, b\rangle \in R^n$ (as defined on page 191 of your text) iff there is a path of length n between a and b in the graph of R.

b. Use this result to then argue that if $\langle a, b\rangle \in R^+$ and $langleb, c\rangle \in R^+$, then $\langle a, c\rangle \in R^+$.

3. Let $R \subseteq X^2$ be a binary relation on X that is antisymmetric, transitive and total/connex (notice that totality implies reflexivity). Show that

$$R' := \{\langle x, y\rangle \in X^2 : \langle y, x\rangle \notin R\}$$

is irreflexive, transitive, and connected/semi-connex. (A relation on X is emphconnected, or *semi-connex*, if, for all $x, y \in X$, if $x \neq y$ then either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$. A relation is *connex*, or *total*, when, for all $x, y \in X$, either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$.)

4. Call a relation R on a set S *universal* when, for all $x, y \in S$, $\ langlex, y\rangle \in R$. Let R be an equivalence relation and define, for each $x \in S$, the set

$$[x] = \{y \in S : \langle x, y\rangle \in R\}$$

Call this the *equivalence class* of x (with respect to R). Prove that all of the following are true:

a. $x \in [x]$, for all $x \in S$;

b. R is universal on each equivalence class $[x]$;

c. Every element of S is in one and only one equivalence class.

5. Prove that the set $\{\neg, \wedge, \vee\}$ is truth-functionally complete. Hint: Consider the following truth table for some arbitrary binary connective $\circ$:

φ	ψ	$\varphi \circ \psi$
t	t	v_1
t	f	v_2
f	t	v_3
f	f	v_4

where v_i is the truth value for that row. There are at least two options now. One is a brute force argument that is effective, but not so exciting. See if you can come up with a different one. Start by thinking about what you could do if all the values are f. What would express such a function? Then think about what would happen if one of the values were t. How might you express that? Then go from there...