For fixed $\mathbf{r}_1$, fixed centres $\mathbf{c}_3, \mathbf{c}_4$, fixed constant $\alpha_3, \alpha_4 > 0$, and with variable $r > 0$, define

$$f(r) = \int_{S^2} e^{-\alpha_3|\mathbf{r}_1+r\mathbf{w}-\mathbf{c}_3|-\alpha_4|\mathbf{r}_1+r\mathbf{w}-\mathbf{c}_4|}\,\mathrm{d}\mathbf{w}.$$

Note that

$$|r\mathbf{w}| - |\mathbf{r}_1 - \mathbf{c}_3| \le |\mathbf{r}_1 - \mathbf{c}_3 + r\mathbf{w}| \le |r\mathbf{w}| + |\mathbf{r}_1 - \mathbf{c}_3|, \text{ and } |r\mathbf{w}| = r.$$

Therefore,

$$r - |\mathbf{r}_1 - \mathbf{c}_3| \le |\mathbf{r}_1 - \mathbf{c}_3 + r\mathbf{w}| \le r + |\mathbf{r}_1 - \mathbf{c}_3|.$$

Since $\alpha_3 > 0$,

$$\begin{aligned} &-\alpha_3 r + \alpha_3|\mathbf{r}_1 - \mathbf{c}_3|) \ge -\alpha_3|\mathbf{r}_1 - \mathbf{c}_3 + r\mathbf{w}| \ge -\alpha_3 r - \alpha_3|\mathbf{r}_1 - \mathbf{c}_3|, \text{ and similarly} \\ &-\alpha_4 r + \alpha_4|\mathbf{r}_1 - \mathbf{c}_4|) \ge -\alpha_4|\mathbf{r}_1 - \mathbf{c}_4 + r\mathbf{w}| \ge -\alpha_4 r - \alpha_4|\mathbf{r}_1 - \mathbf{c}_4|. \end{aligned}$$

Therefore,

$$-r(\alpha_3+\alpha_4) + \alpha_3|\mathbf{r}_1 - \mathbf{c}_3| + \alpha_4|\mathbf{r}_1 - \mathbf{c}_4| \ge \text{exponent} \ge -r(\alpha_3+\alpha_4) - \alpha_3|\mathbf{r}_1 - \mathbf{c}_3| - \alpha_4|\mathbf{r}_1 - \mathbf{c}_4|.$$

And as e^x is monotone increasing,

$$e^{-r(\alpha_3+\alpha_4)+\alpha_3|\mathbf{r}_1-\mathbf{c}_3|+\alpha_4|\mathbf{r}_1-\mathbf{c}_4|} \ge \text{ integrand } \ge e^{-r(\alpha_3+\alpha_4)-\alpha_3|\mathbf{r}_1-\mathbf{c}_3|-\alpha_4|\mathbf{r}_1-\mathbf{c}_4|}.$$

Take the integral on all three,

$$\int_{S^2} e^{-r(\alpha_3+\alpha_4)+\alpha_3|\mathbf{r}_1-\mathbf{c}_3|+\alpha_4|\mathbf{r}_1-\mathbf{c}_4|}\,\mathrm{d}\mathbf{w} \ge f(r) \ge \int_{S^2} e^{-r(\alpha_3+\alpha_4)-\alpha_3|\mathbf{r}_1-\mathbf{c}_3|-\alpha_4|\mathbf{r}_1-\mathbf{c}_4|}\,\mathrm{d}\mathbf{w}.$$

Notice that the integral on the LHS and RHS doesn't involve $\mathbf{w}$, so we can pull it out, and note that $\int_{S^2}\mathrm{d}\mathbf{w} = 4\pi$, which is the surface area of the unit sphere. Therefore,

$$\boxed{4\pi\mu \cdot e^{-r(\alpha_3+\alpha_4)} \ge f(r) \ge 4\pi\frac{1}{\mu} \cdot e^{-r(\alpha_3+\alpha_4)},}$$

where $\mu = \alpha_3|\mathbf{r}_1 - \mathbf{c}_3| + \alpha_4|\mathbf{r}_1 - \mathbf{c}_4|$.

Now for fixed centres $\mathbf{c}_3, \mathbf{c}_4$ and fixed constants $\alpha_3, \alpha_3 > 0$, define

$$g(\mathbf{r}_1) = \int_0^\infty r \int_{S^2} e^{-\alpha_3|\mathbf{r}_1 + r\mathbf{w} - \mathbf{c}_3| - \alpha_4|\mathbf{r}_1 + r\mathbf{w} - \mathbf{c}_4|}\,\mathrm{d}\mathbf{w}\,\mathrm{d}r.$$

Note that

$$g(\mathbf{r}_1) = \int_0^\infty r f_{\mathbf{r}_1}(r)\,\mathrm{d}r.$$

Based on the previous results,

$$4\pi\mu_{\mathbf{r}_1} \cdot \int_0^\infty re^{-(\alpha_3+\alpha_4)r}\,\mathrm{d}r \geq g(\mathbf{r}_1) \geq 4\pi\frac{1}{\mu_{\mathbf{r}_1}} \cdot \int_0^\infty re^{-(\alpha_3+\alpha_4)r}\,\mathrm{d}r,$$

where $\mu_{\mathbf{r}_1} = \alpha_3 |\mathbf{r}_1 - \mathbf{c}_3| + \alpha_4 |\mathbf{r}_1 - \mathbf{c}_4|$. And using integration by parts twice,

$$\int_0^\infty re^{-(\alpha_3+\alpha_4)r}\,\mathrm{d}r = \left[-\frac{r}{\alpha_3+\alpha_4}e^{-(\alpha_3+\alpha_4)r}\right]_0^\infty - \left[\frac{1}{(\alpha_3+\alpha_4)^2}e^{-(\alpha_3+\alpha_4)r}\right]_0^\infty = \frac{1}{(\alpha_3+\alpha_4)^2}.$$

Therefore, putting everything together, we have

$$\boxed{\kappa\left(\alpha_3 |\mathbf{r}_1 - \mathbf{c}_3| + \alpha_4 |\mathbf{r}_1 - \mathbf{c}_4|\right) \geq g(\mathbf{r}_1) \geq \frac{\kappa}{\left(\alpha_3 |\mathbf{r}_1 - \mathbf{c}_3| + \alpha_4 |\mathbf{r}_1 - \mathbf{c}_4|\right)},}$$

where $\kappa = 4\pi/(\alpha_3 + \alpha_4)^2$.