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## ECE368: Probabilistic Reasoning
## Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(z_i|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{N-1},\hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. **(1 pt)**

$\alpha(\cdot)$: forward message, $\beta(\cdot)$: backward message, both normalized

$$P(z_i|(\hat{x}_0,\hat{y}_0),\cdots,(\hat{x}_{N-1},\hat{y}_{N-1})) = \frac{\alpha(z_i)\beta(z_i)}{P((\hat{x}_0,\hat{y}_0),\cdots,(\hat{x}_{N-1},\hat{y}_{N-1}))}$$

$$= \frac{\alpha(z_i)\beta(z_i)}{\sum_{z_i}(\alpha(z_i)\beta(z_i))}$$

Initialization:

$$\alpha(z_0) = P(z_0)P((\hat{x}_0,\hat{y}_0)|z_0)$$

$$\beta(z_{N-1}) = 1$$

Recursion Relations:

$$\alpha(z_i) = P((\hat{x}_i,\hat{y}_i)|z_i)\cdot\sum_{z_{i-1}}\alpha(z_{i-1})P(z_i|z_{i-1})$$

$$\beta(z_i) = \sum_{z_{i+1}}\beta(z_{i+1})P((\hat{x}_{i+1},\hat{y}_{i+1})|z_{i+1})P(z_{i+1}|z_i)$$

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(z_{99}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. **(2 pt)**

$$P(z_{99}|(\hat{x}_0,\hat{y}_0),\cdots,(\hat{x}_{99},\hat{y}_{99})) = \begin{cases} 0.81026 & z_{99}=(11,0,\text{'stay'}) \\ 0.17961 & z_{99}=(11,0,\text{'right'}) \\ 0.01013 & z_{99}=(10,1,\text{'down'}) \\ 0 & \text{otherwise} \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(z_{30}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. **(1 pt)**

$$P(z_{30}|(\hat{x}_0,\hat{y}_0),\cdots,(\hat{x}_{99},\hat{y}_{99})) = \begin{cases} 0.91304 & z_{30}=(6,7,\text{'right'}) \\ 0.04348 & z_{30}=(5,7,\text{'right'}) \\ 0.04348 & z_{30}=(5,7,\text{'stay'}) \\ 0 & \text{otherwise} \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $z_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 pt)

$$z_i^* = \underset{z_i}{\arg\max}\ W_i(z_i)$$

$$\text{Initialization: } W_0(z_0) = \ln P(z_0) + \ln P((\hat{x}_0, \hat{y}_0) \mid z_0)$$

$$\text{Recursion: } W_i(z_i) = \ln P((\hat{x}_i, \hat{y}_i) \mid z_i) + \underset{z_{i-1}}{\max}\{\ln P(z_i \mid z_{i-1}) + W_{i-1}(z_{i-1})\}$$

$$\forall i = 1, \ldots, N-1$$

$$z_{i-1}^* = \phi_i(z_i) = \underset{z_{i-1}}{\arg\max}(W_i(z_i^*)) \quad \text{back tract to find } z^*\text{'s}$$

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 pt)

$$z_{90} = (11, 5, \text{'down'}) \qquad z_{96} = (4, 7, \text{'left'})$$
$$z_{91} = (11, 6, \text{'down'}) \qquad z_{97} = (8, 7, \text{'left'})$$
$$z_{92} = (11, 7, \text{'down'}) \qquad z_{98} = (7, 7, \text{'left'})$$
$$z_{93} = (11, 7, \text{'stay'}) \qquad z_{99} = (6, 7, \text{'left'})$$
$$z_{94} = (11, 7, \text{'stay'})$$
$$z_{95} = (10, 7, \text{'left'})$$

4. Compute and compare the error probabilities of $\{\bar{z}_i\}$ and $\{\check{z}_i\}$ using the data in test_missing.txt. The error probability of $\{\bar{z}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{z}_i\}$ is $\boxed{0.02}$. (1 pt)

5. Is sequence $\{\check{z}_i\}$ a valid sequence? If not, please find a small segment $\check{z}_i, \check{z}_{i+1}$ that violates the transition model for some time step $i$. You answer should specify the value of $i$ as well as the corresponding states $\check{z}_i, \check{z}_{i+1}$. (1 pt)

$$z_{64} = (3, 7, \text{'stay'}) \qquad \text{if the rover stays for}$$
$$z_{65} = (2, 7, \text{'stay'}) \qquad \text{state 65, it should be}$$
$$\underset{\smile}{\quad} \text{this violates} \qquad \text{at the same position}$$
$$\text{as state 64, which is}$$
$$(3, 7)$$