一、Define

1. $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix}$ weight

2. $\tilde{r} = \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \vdots \\ \tilde{r}_N \end{pmatrix}$ return

3. $V = \begin{pmatrix} V_{11} & \cdots & V_{1N} \\ V_{21} & \cdots & V_{2N} \\ & \ddots & \end{pmatrix}$ $V_{ij} = Cov(\tilde{r}_i, \tilde{r}_j)$

4. $E(U(\theta)) = (1 - \sum_{k=1}^{N} \theta_k) r_f + \sum_{k=1}^{N} \theta_k E(\tilde{r}_k) - \frac{1}{2t} \sum_{k=1}^{N} \sum_{l=1}^{N} V_{kl} \theta_k \theta_l$

$t \hookrightarrow -\frac{U''}{U'}$

$\mu = E(\tilde{r}) - r_f$

$= r_f + \theta' \mu - \frac{1}{2t} \theta' V \theta$

二、Optimal

Max $E(U(\theta))$

F.O.C

$\frac{\partial \pi}{\partial \theta} = 0, \quad \mu - \frac{1}{2t} 2V\theta = 0$

$\Rightarrow \theta^* = t \cdot V^{-1} \cdot \mu$

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HW: Find θ^*

① μ pool 自訂.

international is better.

② Freq 可自訂.

(IMF所有大國 Data)

If the correlation ρ drops to 0.8, like what happened after the announcement of the Russian default, the volatility more than doubles:

$$\sigma' = \sqrt{\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 - 2\rho'\theta_1\theta_2\sigma_1\sigma_2} \simeq 65.8\%.$$

APPENDIX 1: PORTFOLIO CHOICE WITH SEVERAL RISKY ASSETS

Consider a situation where n different risky assets are available to the investor. These assets are indexed by $k = 1,\ldots,n$. We denote by μ the n-vector of excess expected returns (i.e., $\mu_k = \mathcal{E}(\tilde{r}_k) - r_F$ for all k) and V the $n \times n$ matrix of variance–covariance of the returns (i.e. $v_{kl} = \text{cov}(\tilde{r}_k, \tilde{r}_l)$ for all $k, l = 1,\ldots,n$). We assume that V is non-singular, which means that its inverse V^{-1} is well-defined.[9]

A portfolio is represented by a n-vector $\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$, where θ_k represents the fraction of the investor's wealth that is invested in risky asset k. By difference, $1 - \sum_{k=1}^{n} \theta_k$ is the fraction invested in the riskless asset. The optimal portfolio for an investor who uses the mean–variance criterion defined by formula (9.1) is the n-vector θ^* that maximizes:

$$U(\theta_1,\ldots,\theta_n) = \left(1 - \sum_{k=1}^{n} \theta_k\right) r_F + \sum_{k=1}^{n} \theta_k \mathcal{E}(\tilde{r}_k) - \frac{1}{2t}\,\text{var}\left(\sum_{k=1}^{n} \theta_k \tilde{r}_k\right)$$

$$= r_F + \sum_{k=1}^{n} \theta_k \mu_k - \frac{1}{2t} \sum_{k=1}^{n} \sum_{l=1}^{n} v_{kl}\theta_k\theta_l.$$

Using vector notations, this is also equal to:

$$U(\theta) = r_F + \theta \cdot \mu - \frac{1}{2t}\theta^t(V\theta),$$

where $\theta^t = (\theta_1,\ldots,\theta_n)$ is the transposed (row vector) of column vector θ.

The optimal portfolio θ^* is characterized by:

$$\frac{\partial U}{\partial \theta_k}(\theta^*) = \mu_k - \frac{1}{t}\sum_{l=1}^{n} v_{km}\theta_m^* = 0 \text{ for all } k,$$

or in vector notation:

$$\mu = \frac{1}{t} V\theta^*.$$

Because V is non-singular, the optimal portfolio is uniquely defined by:

$$\theta^* = tV^{-1}\mu. \tag{9.13}$$

Notice that (9.13) is just the multidimensional extension of (9.4).

As an illustration, consider the numerical example presented in the LTCM case study above. Data concerning the United States for the period on 1990 to 1997, for a horizon of 1 year:

- Average return on Treasury bills (riskless asset) $r = 5.36\%$
- Average return on Corporate bonds $\mathcal{E}(\tilde{r}_1) = 7.28\%$
- Average return on Treasury bonds $\mathcal{E}(\tilde{r}_2) = 5.75\%$
- Volatilities: $\sigma_1 = 6.58\%$ (T-bonds), $\sigma_2 = 5.47\%$ (corporates) with a correlation between corporate return and T-bonds returns of $\rho = 96.54\%$.

To calibrate the model, one uses a benchmark investor who holds simultaneously some stocks and some T-bills over a 1-year horizon. The risk tolerance index t of this investor is determined by using his indifference between investing in stocks (with 1 year expected return 10% and a yearly volatility of $\sigma = 20\%$) and in T-Bills (with 1 year return 2%). The mean–variance criterion gives $0.1 - \frac{1}{2t}[0.2]^2$ for stocks and 0.02 for T-bills. Equality implies that $2t = \frac{0.04}{0.08}$ or $t = 0.25$.

Makes its E(U) is indifferent.

$\boxed{0.02 = 0.1 - \frac{1}{2t}[0.2]^2 \Rightarrow t = \frac{1}{4}}$

We can now compute the optimal portfolio of corporate bonds, treasury bonds, and the risky asset for an equity value of \$1, and the risk tolerance index determined above.

Excess returns are $\mu_1 = (7.28 - 5.36)\% = 1.92\%$ for corporate bonds, and $\mu_2 = (5.75 - 5.36)\% = 0.39\%$ for T-Bonds. The variance–covariance matrix is

$$V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = 10^{-4}\begin{pmatrix} (5.47)^2 & 0.9654(5.47)(6.58) \\ 0.9654(5.47)(6.58) & (6.58)^2 \end{pmatrix}.$$

Its inverse is:

$$\Sigma^{-1} = 10^4\begin{pmatrix} 0.49 & -0.39 \\ -0.39 & 0.34 \end{pmatrix}$$

The optimal portfolio for \$1 equity is

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = t\Sigma^{-1}\mu = (0.25)10^4\begin{pmatrix} 0.49 & -0.39 \\ -0.39 & 0.34 \end{pmatrix}\begin{pmatrix} 1.92\% \\ 0.39\% \end{pmatrix} \simeq \begin{pmatrix} 20 \\ -16 \end{pmatrix}.$$

The optimal investment for \$1 of the investor's own money (what can be called equity) is thus \$20 of corporate bonds, with short positions of \$16 in T-~~bills~~ bond and \$3 in T-~~bonds~~ bill. Leverage is thus huge:

$$\lambda = \frac{\text{Debt}}{\text{Equity}} = 19.$$

APPENDIX 2: DERIVING THE CAPM FORMULA

The CAPM, elaborated by Markovitz, Lintner, and Sharpe, is a very convenient tool for evaluating the risk premia of risky assets. It is obtained by determining the expected returns of risky assets in an equilibrium situation. Consider a group of I mean–variance investors (indexed by $i = 1,\ldots,I$) characterized by risk tolerance indices t^i and wealths W^i, $i = 1,\cdots,I$. The portfolios optimally chosen by these investors are thus given by formula (9.13):

$$\theta^i = t^i V^{-1}\mu \quad \text{for all } i,$$