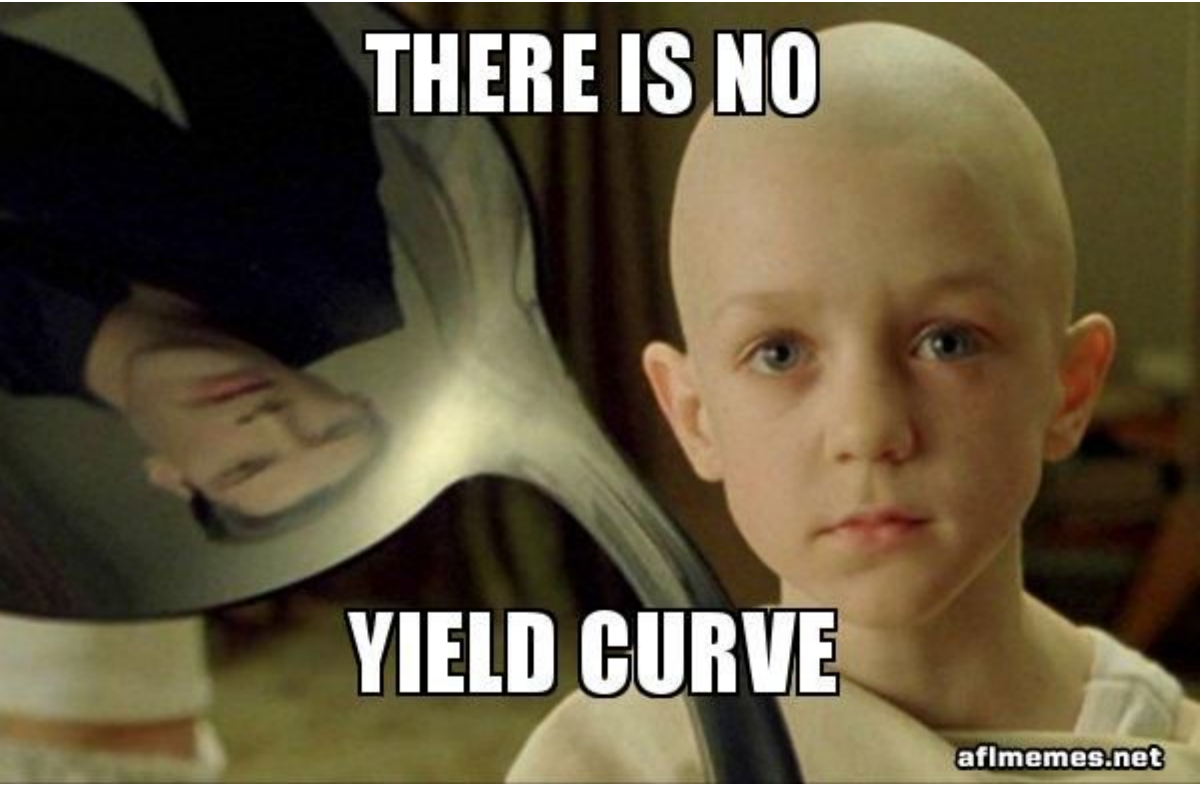


DSA302

Isaac Wong, Ker Zi Yi, Megan Chong, Shawn Lim, Tan Zen Wei

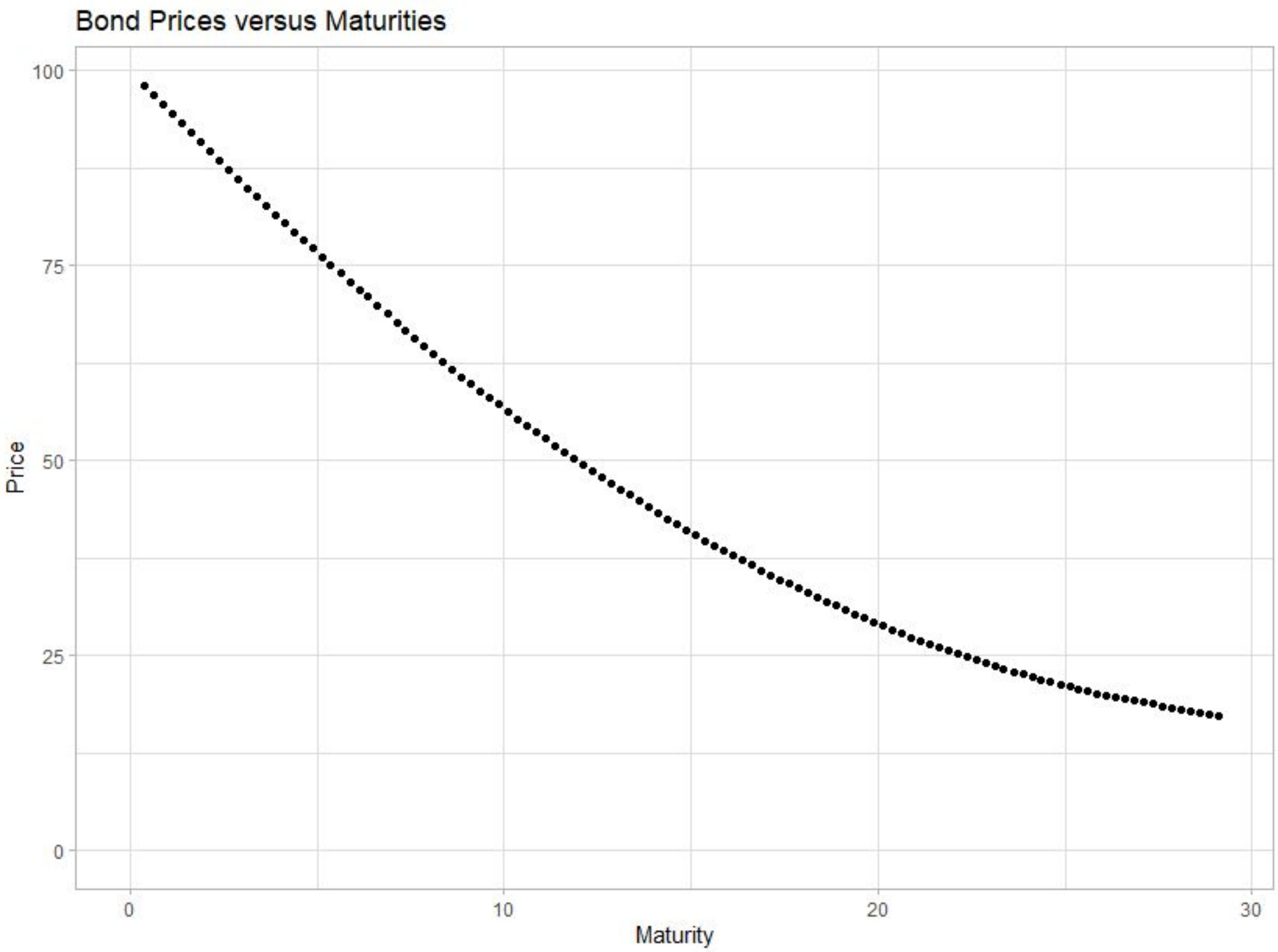
TABLE OF CONTENTS

PART 1.1

Plot the Bond prices versus their maturities

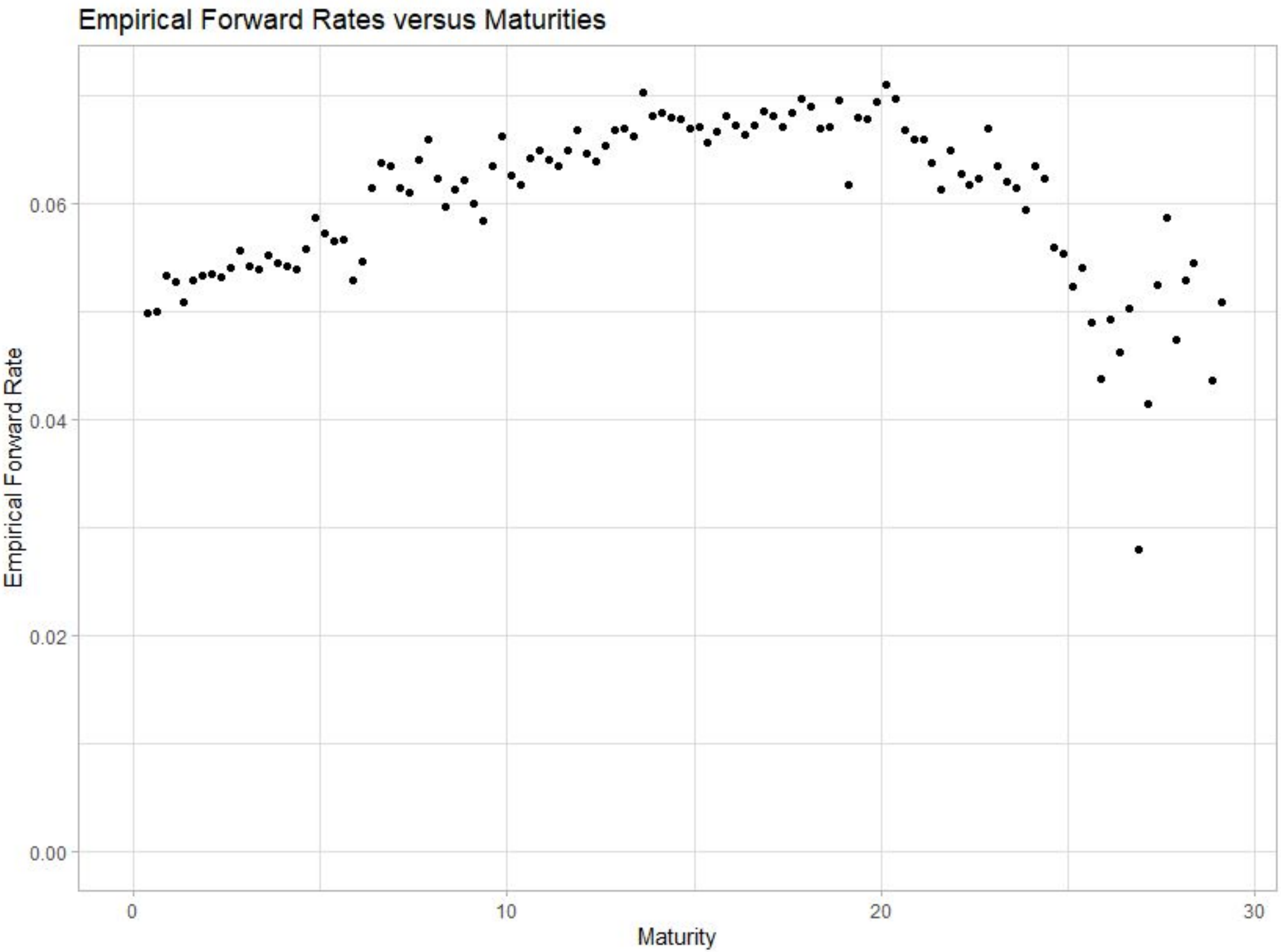
Equation 1

$$P(t) = 100\, e^{-r(t)t}$$



PART 1.2

Plot the empirical forward rates as computed in equation (3) versus maturities

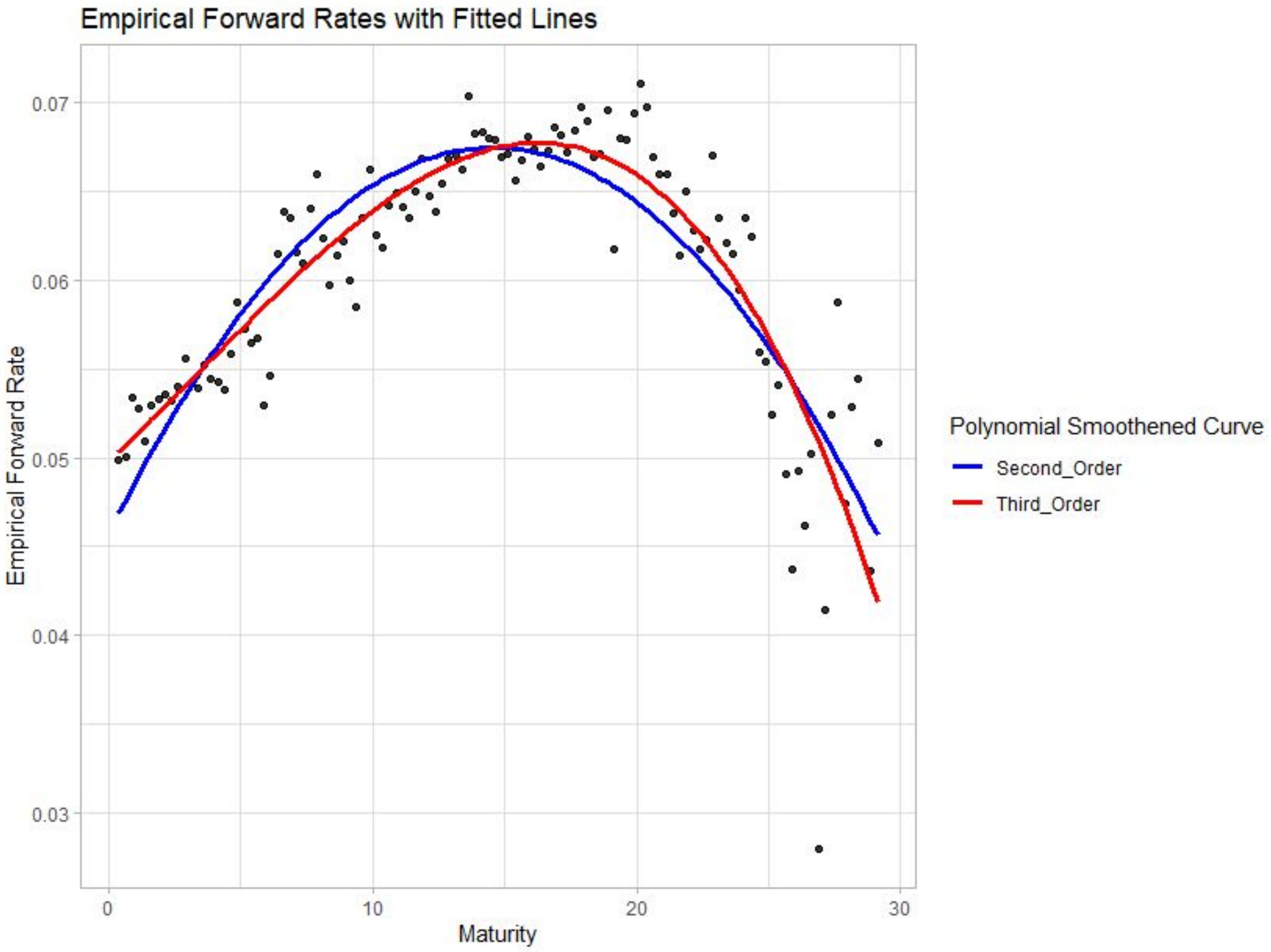


Empirical Forward Rates versus Maturities
Empirical Forward Rate
0.00
0.02
0.04
0.06
0
10
20
30
Maturity

PART 1.3

Smooth the empirical forward rates using second order and third order polynomials

Conclusion:
3rd order graph is better



PART 1.4

Estimate the empirical spot rates for t in (t1; tn) using equation (4)

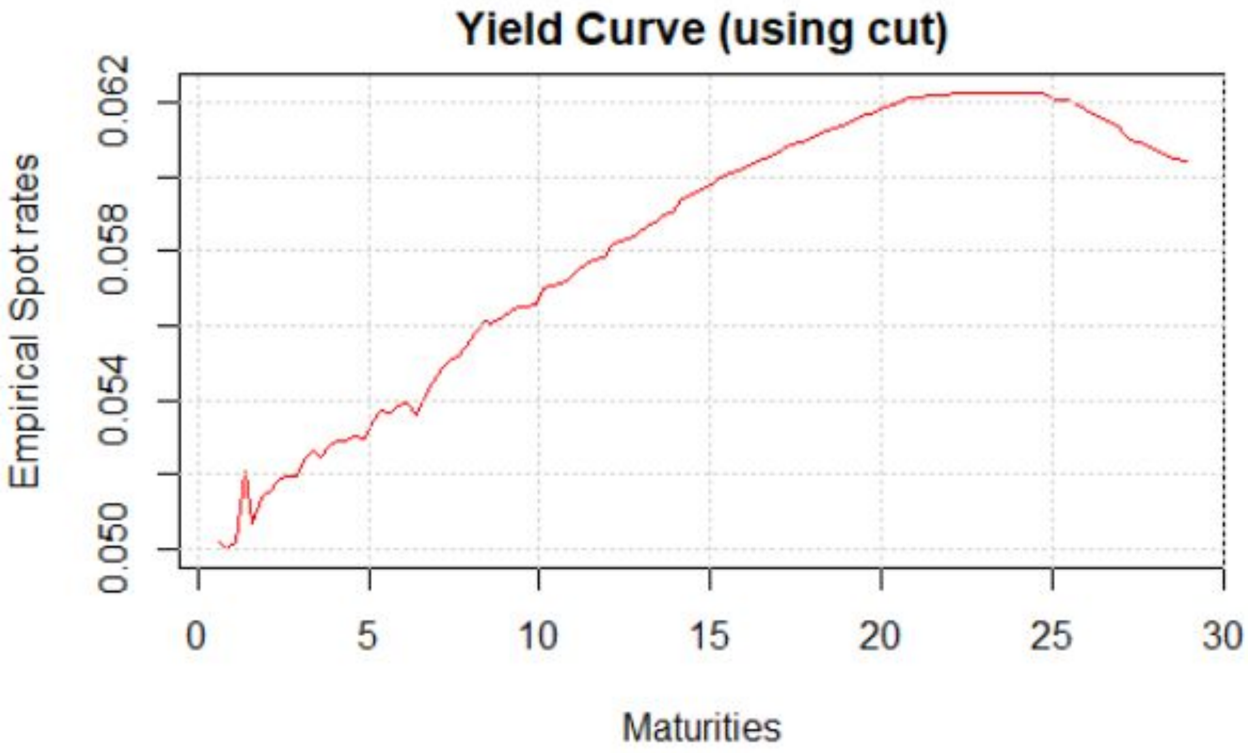
Using the Cut() Function to Derive Time Intervals

- Split the time intervals into 116 equal parts
- Define values to be the upper limit of each interval
- Use these value in the formula for Spot Rates.

```
[1] "(0.341,0.618]" "(0.618,0.866]" "(0.866,1.11]"
[4] "(1.11,1.36]"   "(1.36,1.61]"   "(1.61,1.86]"
[7] "(1.86,2.11]"   "(2.11,2.35]"   "(2.35,2.6]"
```

$$r(t) = \frac{1}{t}\left[\sum_{i=1}^{j} f_{i-1}(t_i - t_{i-1}) + f_j(t - t_j)\right]$$

Plotting the Empirical Yield Curve



PART 1.5

Smooth the empirical spot rates using second order and third order polynomials

Smoothening using Higher Order Polynomials

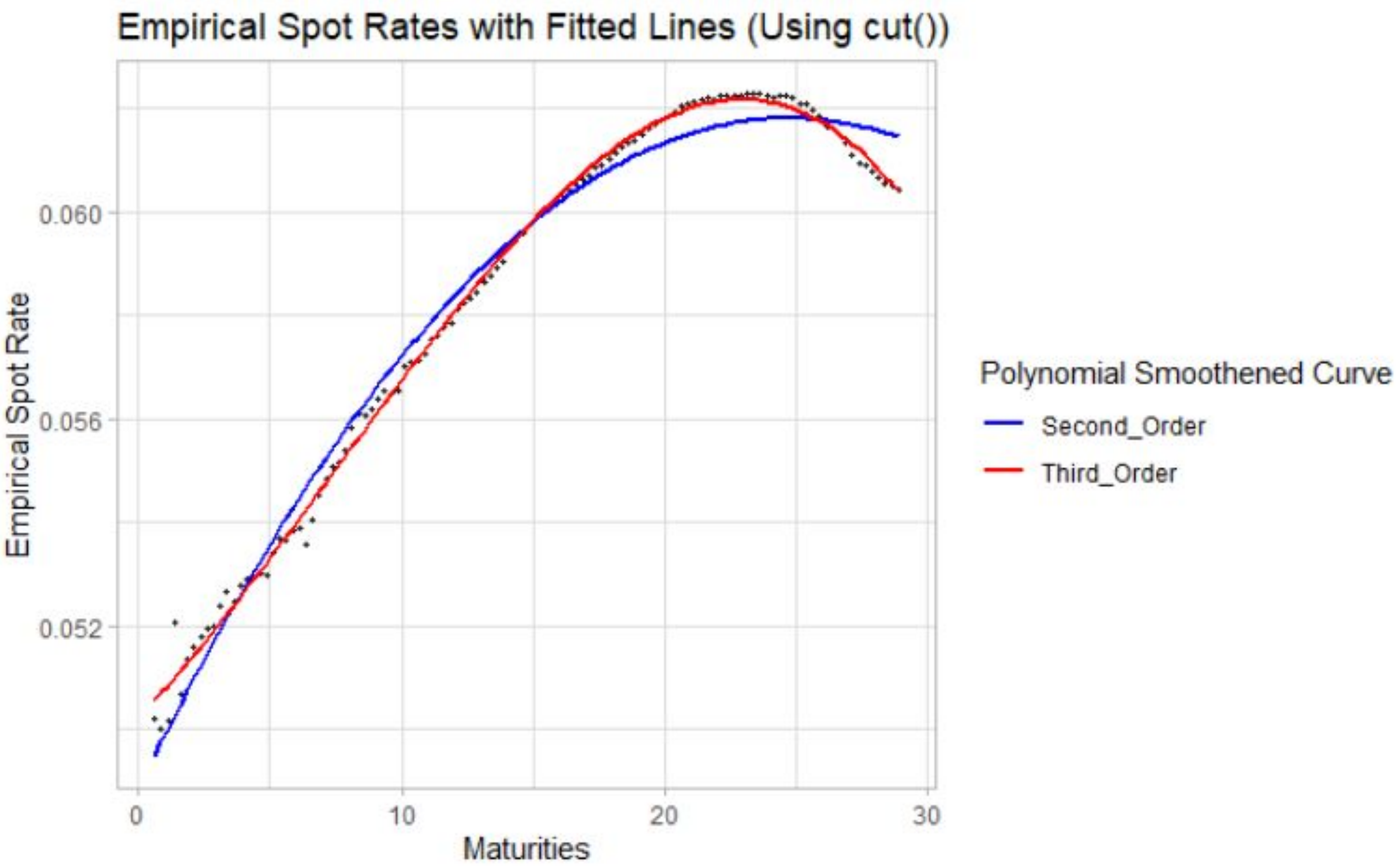
- To get a smoothened curved, we used the following equations:

$$Spot\ Rate = \beta_0\ Maturity + \beta_1\ Maturity^2$$

$$Spot\ Rate = \beta_0\ Maturity + \beta_1\ Maturity^2 + \beta_2\ Maturity^3$$

- The lm() function was then used to minimise the sum of squared errors

Resulting Smoothed Curves



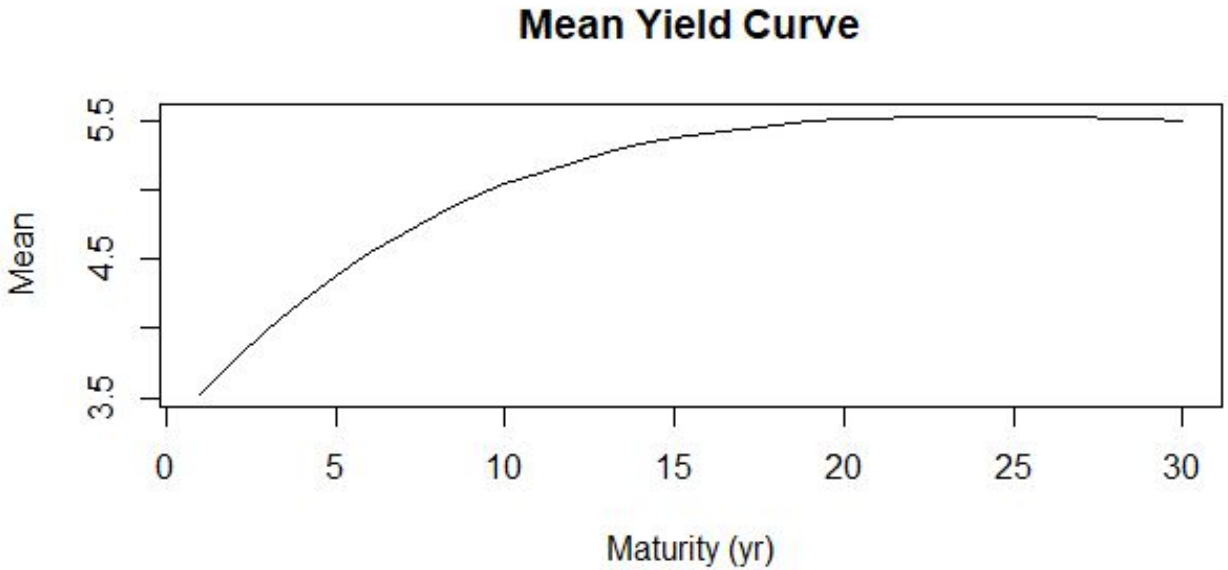
Our Comments

- Generally, the empirical yield curve is a standard upward sloping curve.
- However, the empirical spot rate seem to decrease after maturity => 25
- The 3rd order polynomial fits better when modelling both the empirical rates.

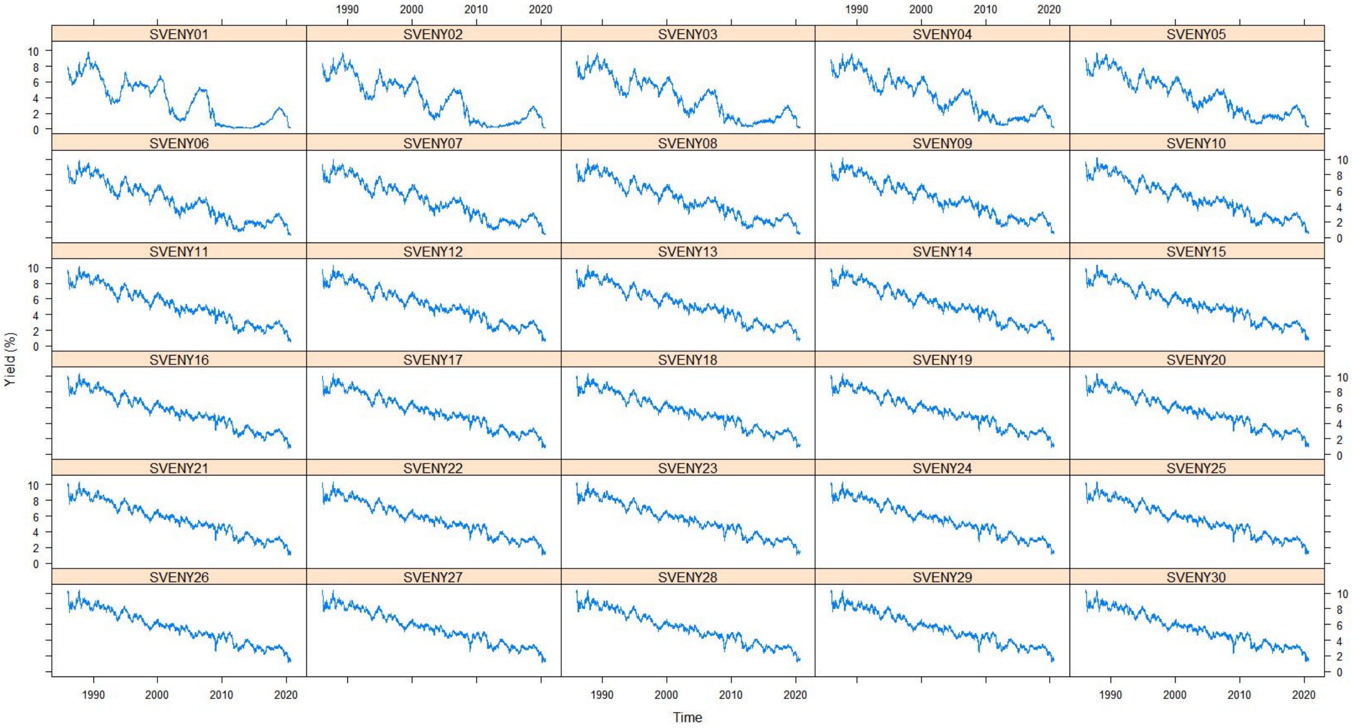
PART 2.1

DATA PRESENTATION

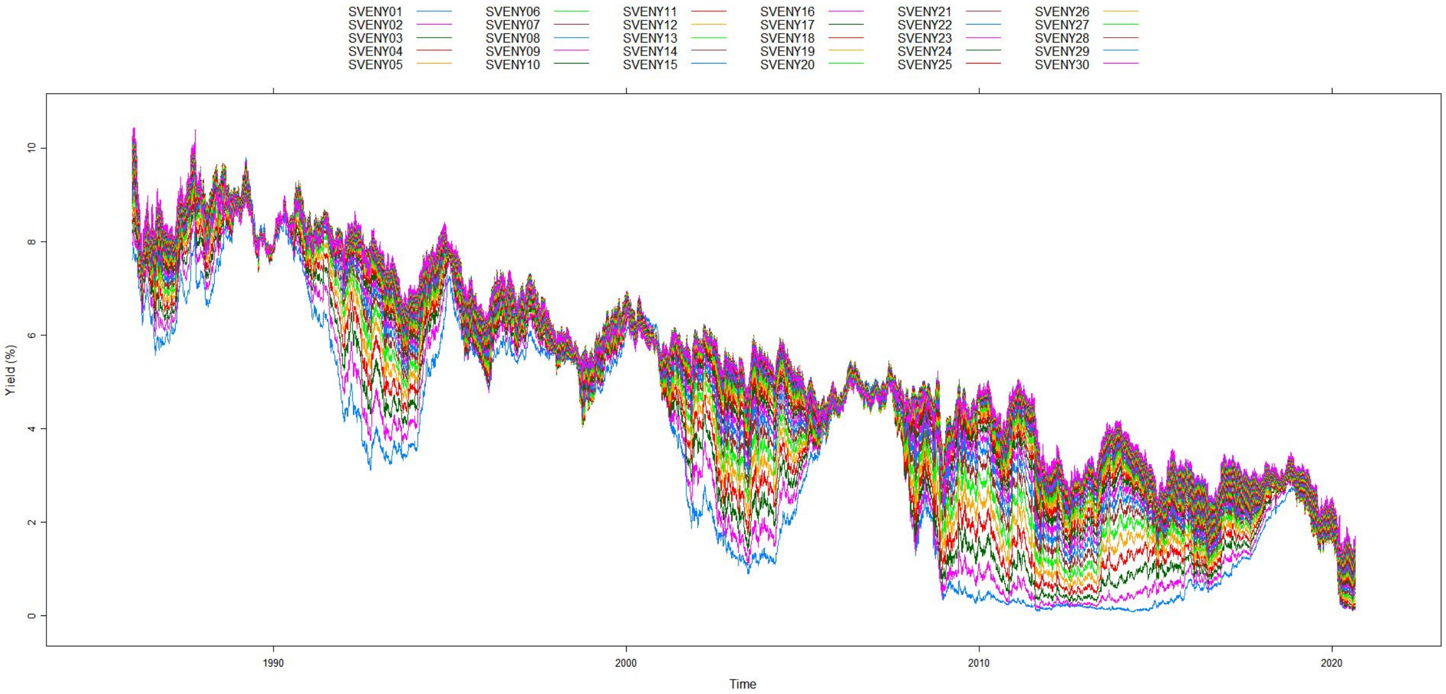
```
    SVENY01           SVENY05           SVENY10           SVENY20           SVENY30      
 Min.   :0.0828   Min.   :0.2218   Min.   : 0.5202   Min.   : 0.9581   Min.   : 1.250  
 1st Qu.:0.8794   1st Qu.:2.0398   1st Qu.: 2.9330   1st Qu.: 3.6734   1st Qu.: 3.727  
 Median :3.5109   Median :4.4592   Median : 4.8532   Median : 5.4626   Median : 5.399  
 Mean   :3.5201   Mean   :4.3786   Mean   : 5.0441   Mean   : 5.5147   Mean   : 5.507  
 3rd Qu.:5.6410   3rd Qu.:6.3240   3rd Qu.: 6.7711   3rd Qu.: 7.2055   3rd Qu.: 7.297  
 Max.   :9.8020   Max.   :9.7455   Max.   :10.1805   Max.   :10.3236   Max.   :10.429  
```



SPOT RATES OVER TIME



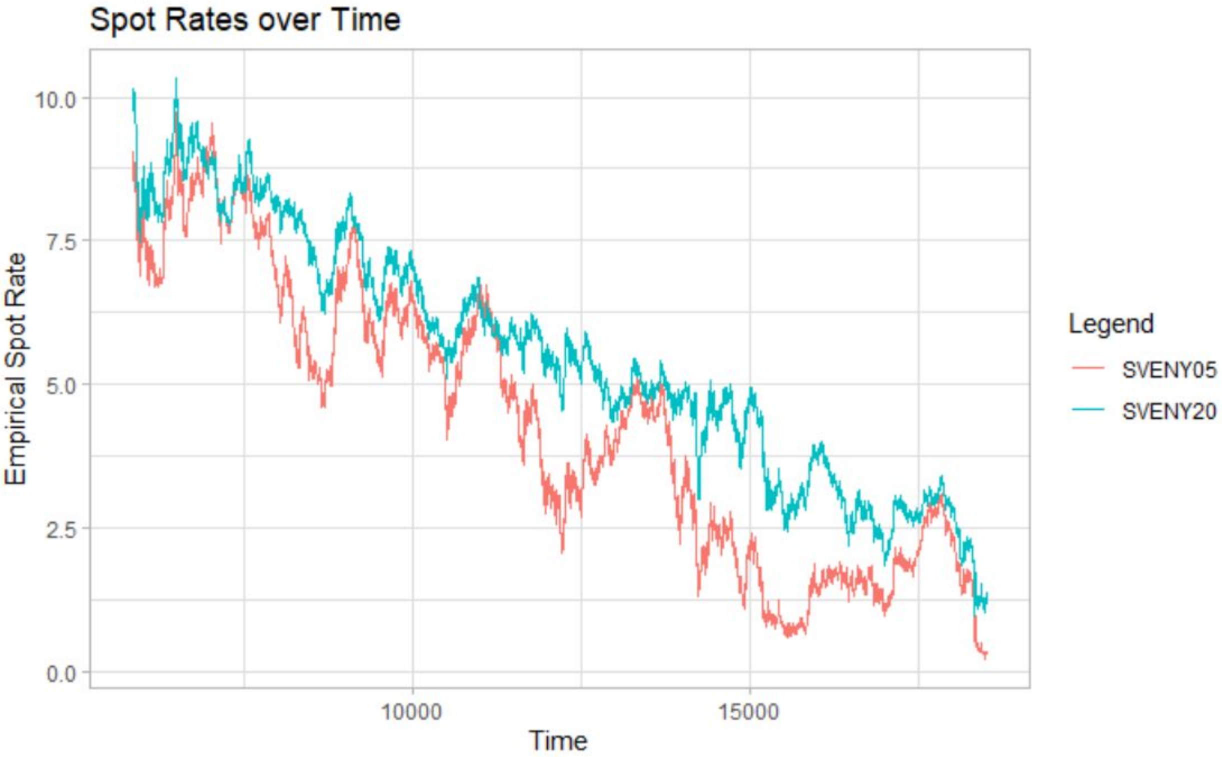
SPOT RATES OVER TIME

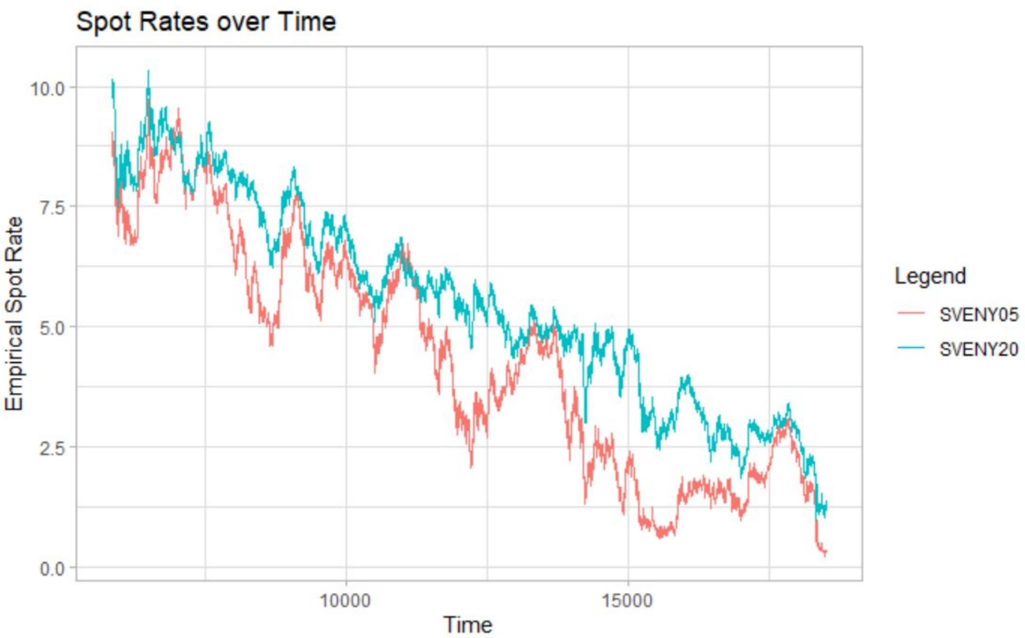


PART 2.2

COMPARISON OF SPOT RATES

COMPARING SPOT RATES



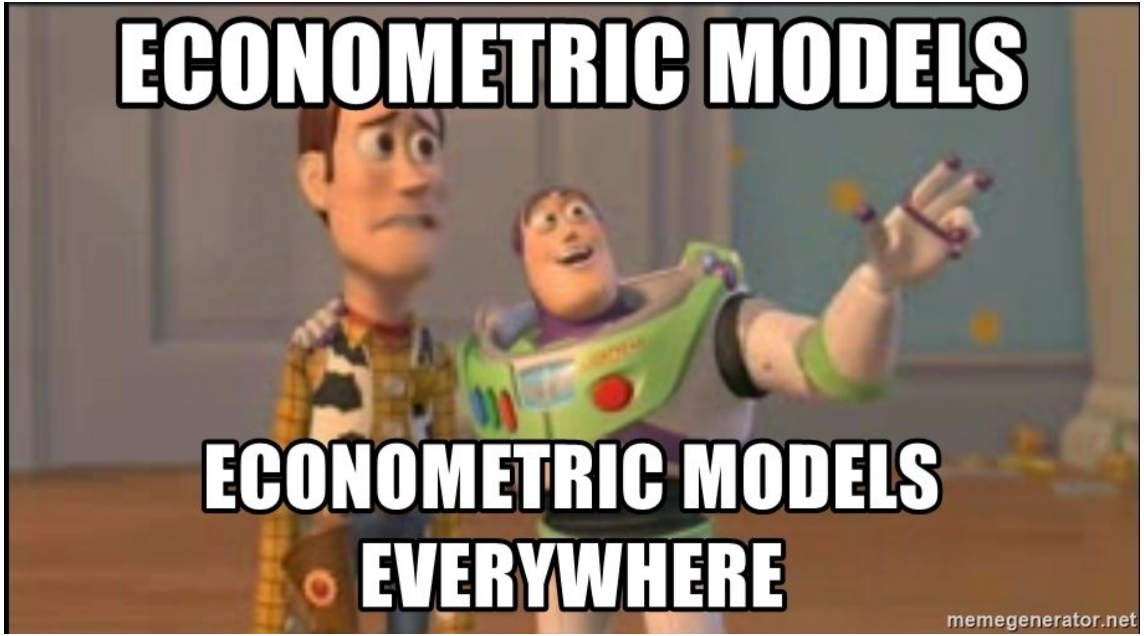


COMPARING SPOT RATES

- Spot rates generally decreased over time
- Level of fluctuation higher for bonds with shorter maturity periods
- However there are some periods where 5 year spot rate exceeds 30 year spot rate, indicating bleak economic outlook

PART 2.3

UNDERSTANDING THE MODEL



2.3 Understanding the Model

*Original equation as seen in the project pdf (NSS model)

$$r(t) = \theta_0 + \theta_1\left[\frac{1-e^{-\theta_3 t}}{\theta_3 t}\right] + \frac{\theta_2}{\theta_3}\left[\frac{1-e^{-\theta_3 t}}{\theta_3 t} - e^{-\theta_3 t}\right] + \frac{\theta_4}{\theta_5}\left[\frac{1-e^{-\theta_5 t}}{\theta_5 t} - e^{-\theta_5 t}\right]$$

*Re-writing $\boldsymbol{\theta}$ in terms of β

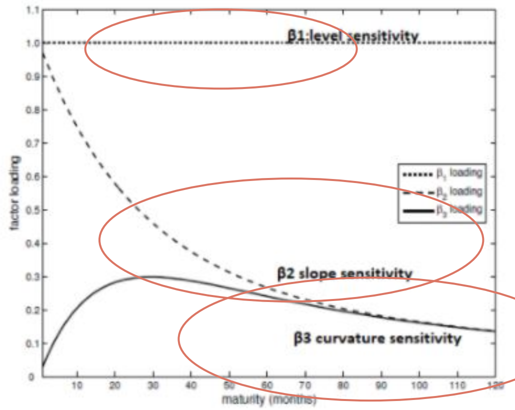
NS model

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{3,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}} - e^{-\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{4,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{2,t}}}}{\frac{\tau}{\lambda_{2,t}}} - e^{-\frac{\tau}{\lambda_{2,t}}}\right]$$

$\boldsymbol{\theta}_0=\beta_{1,t}$; $\boldsymbol{\theta}_1=\beta_{2,t}$; $\boldsymbol{\theta}_2/\boldsymbol{\theta}_3=\beta_{3,t}$; $\boldsymbol{\theta}_3=-\tau/\lambda_{1,t}$; $\boldsymbol{\theta}_4/\boldsymbol{\theta}_5=\beta_{4,t}$; $\boldsymbol{\theta}_5=-\tau/\lambda_{2,t}$

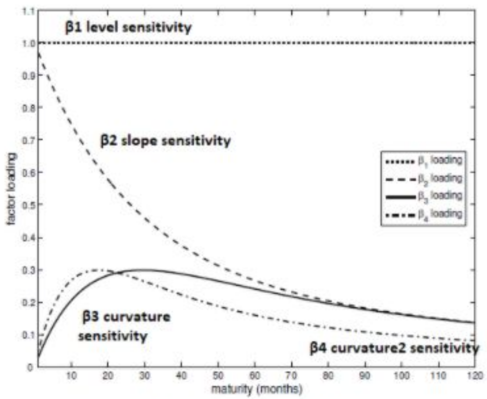
**The NS model does not have the second medium term component($\boldsymbol{\theta}_4/\boldsymbol{\theta}_5$) and the 2nd decay rate ($\boldsymbol{\theta}_5$)

Nelson Siegel (NS) model



[a] three-factor model

Nelson Siegel Svensson (NSS) model



[c] Svensson model

Table 1: Understanding the models

The equations above are rewritten from the project paper so we can interpret them more easily.

Components	Description	Sensitivity
Long-term component	This is the component on θ_0 because it is constant at 1 and remains the same for every maturity.	$\theta_{0,1}$ (Level factor sensitivity)
Short-term component	This is the component $[(1-\exp(-\theta_3 t))/\theta_3 t)]$ on θ_1 because it starts at 1 but then decays to zero at an exponential rate. The rate of this decay is determined by $\theta_3 t$, where a smaller value means a faster rate of decay	θ_1 (slope factor sensitivity
Medium-term component	This is the component$[(1-\exp(-\theta_3 t))/(\theta_3 t))) -\exp(-\theta_3 t_t)]$ on θ_2/θ_3, which starts at 0, increases for medium maturities and then decays to zero again thereby creating a hump-shape. The $\theta_3 t$ component determines at which rate the medium term component reaches its maximum	θ_2/θ_3(curvature factor sensitivity)
Rate of decay	The rate of decay is $\theta_3 t=\tau/\lambda_t$	NA

Table 2: NS model

NSS model

2.3 Understanding the Model

*Original equation as seen in the project pdf (NSS model)

$$r(t) = \theta_0 + \theta_1\left[\frac{1-e^{-\theta_3 t}}{\theta_3 t}\right] + \frac{\theta_2}{\theta_3}\left[\frac{1-e^{-\theta_3 t}}{\theta_3 t} - e^{-\theta_3 t}\right] + \frac{\theta_4}{\theta_5}\left[\frac{1-e^{-\theta_5 t}}{\theta_5 t} - e^{-\theta_5 t}\right]$$

θ in terms of β

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{3,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}} - e^{-\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{4,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{2,t}}}}{\frac{\tau}{\lambda_{2,t}}} - e^{-\frac{\tau}{\lambda_{2,t}}}\right]$$

$\boldsymbol{\theta}_0=\beta_{1,t}$; $\boldsymbol{\theta}_1=\beta_{2,t}$; $\boldsymbol{\theta}_2/\boldsymbol{\theta}_3=\beta_{3,t}$; $\boldsymbol{\theta}_3=-\boldsymbol{\tau}/\lambda_{1,t}$; $\boldsymbol{\theta}_4/\boldsymbol{\theta}_5=\beta_{4,t}$; $\boldsymbol{\theta}_5=-\boldsymbol{\tau}/\lambda_{2,t}$

**The NS model does not have the second medium term component($\boldsymbol{\theta}_4/\boldsymbol{\theta}_5$) and the 2nd decay rate ($\boldsymbol{\theta}_5$)

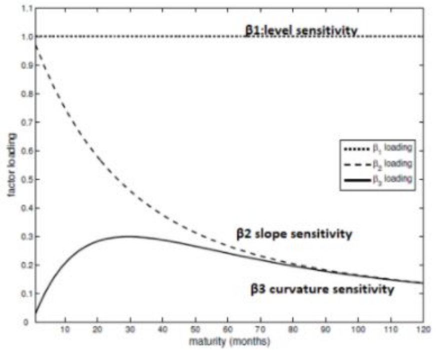
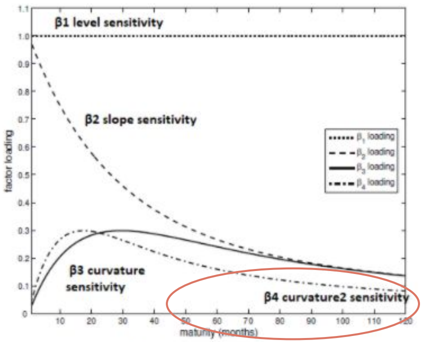
Nelson Siegel (NS) model	Nelson Siegel Svensson (NSS) model
 [a] three-factor model	 [c] Svensson model

Table 1: Understanding the models

The equations above are rewritten from the project paper so we can interpret them more easily.

Components	Description	Sensitivity
2nd medium-term component	This is the component[(1-exp(-$\boldsymbol{\theta}_3$t))/ ($\boldsymbol{\theta}_3$t)) -exp(-$\boldsymbol{\theta}_3$t)] on $\boldsymbol{\theta}_4/\boldsymbol{\theta}_5$, which starts at 0, increases for medium maturities and then decays to zero again thereby creating a hump-shape. The $\boldsymbol{\theta}_5$t component determines at which rate the medium term component reaches its maximum	$\boldsymbol{\theta}_4/\boldsymbol{\theta}_{5t}$ (2nd curvature factor sensitivity)
2nd rate of decay	The second rate of decay is $\boldsymbol{\theta}_5$t=$\boldsymbol{\tau}/\lambda_{2,t}$	NA

- As the NSS model has an additional medium term component, it is able to fit term structure shapes that <u>have more than one local maximum or minimum along the maturity spectrum</u> more easily.

Strategy for θ values

2.3.1 Importance of **θ** (sensitivity values)

Parameter **θ**	Low **θ**	High **θ**
θ_1 (slope factor) is used for the identification of spread strategies.	Low value (negative value) → steep upward sloping yield curve→ curve get flatter over longer maturity	High value(positive value) → inverted sloping yield curve with a negative slope → curve get steeper over longer maturity
θ_2/θ_3 & **θ_4/θ_5**(curvature factor) is used for the identification of interest rate term structure curvatures strategies	Low value (absolute value) →increase in yield curve 's curvature is expected→indicates steepening is expected	High value (absolute) value→reduction in yield curve 's curvature is expected→indicates flattening is expected

Table 4: Importance of ϑ

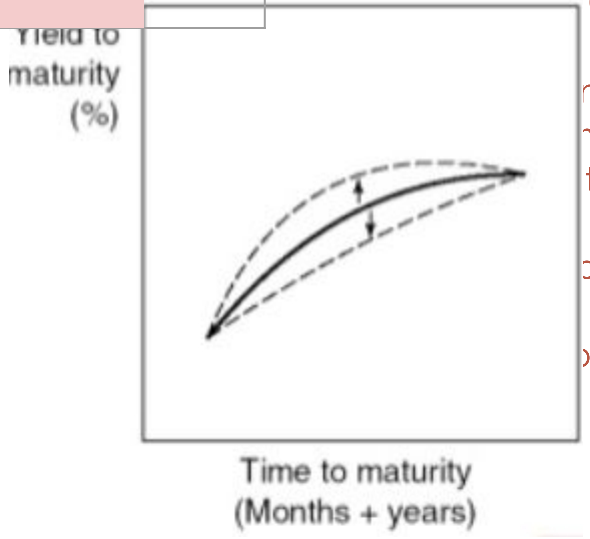
- According to the diagram above, where **θ**1=**β**2 and **θ**2/**θ**3=**β**3.
- A short or long position can be determined for short or long maturities.
- Short position = sell stock (anticipate value to fall in short run)
- Long position = buy stock (anticipate value to fall in short run)

SHORT TERM

β2	Short maturity	Long maturity
Low	Short	Long
High	BUY	SELL

MIDDLE TERM

β3	Middle maturity
Low	Short
High	BUY



Strategy for θ values

- According to the diagram above, θ1=β2 and θ2/θ3=β3.

t or long position can be
nined for short or long
ties.

oosition = selling high now, in
of buying low later
oosition = buying low now, in
of selling high later

2.3 Understanding the Model

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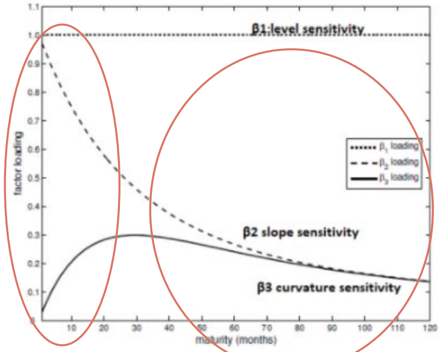
*Re-writing $\boldsymbol{\theta}$ in terms of β

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{3,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{1,t}}}}{\frac{\tau}{\lambda_{1,t}}} - e^{-\frac{\tau}{\lambda_{1,t}}}\right] + \beta_{4,t}\left[\frac{1-e^{-\frac{\tau}{\lambda_{2,t}}}}{\frac{\tau}{\lambda_{2,t}}} - e^{-\frac{\tau}{\lambda_{2,t}}}\right]$$

$\boldsymbol{\theta}_0=\beta_{1,t}$; $\boldsymbol{\theta}_1=\beta_{2,t}$; $\boldsymbol{\theta}_2/\boldsymbol{\theta}_3=\beta_{3,t}$; $\boldsymbol{\theta}_3=-\tau/\lambda_{1,t}$; $\boldsymbol{\theta}_4/\boldsymbol{\theta}_5=\beta_{4,t}$; $\boldsymbol{\theta}_5=-\tau/\lambda_{2,t}$

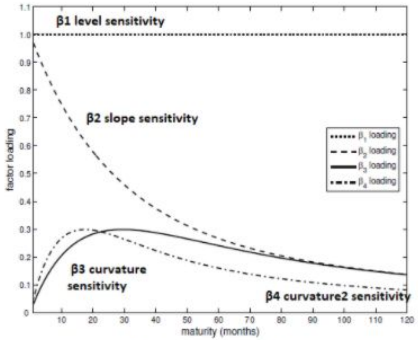
**The NS model does not have the second medium term component($\boldsymbol{\theta}_4/\boldsymbol{\theta}_5$) and the 2nd decay rate ($\boldsymbol{\theta}_5$)

Nelson Siegel (NS) model



[a] three-factor model

Nelson Siegel Svensson (NSS) model



[c] Svensson model

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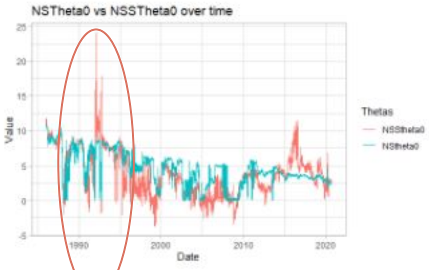
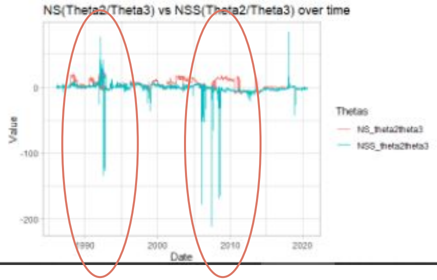
Components	Description	Sensitivity
Long-term component	This is the component on $\boldsymbol{\theta}_0$ because it is constant at 1 and remains the same for every maturity.	$\boldsymbol{\theta}_0$ (Level factor sensitivity)
Short-term component	This is the component $[(1-\exp(-\boldsymbol{\theta}_3 t))/\ \boldsymbol{\theta}_3 t)]$ on $\boldsymbol{\theta}_1$ because it starts at 1 but then decays to zero at an exponential rate. The rate of this decay is determined by $\boldsymbol{\theta}_3$t, where a smaller value means a faster rate of decay	$\boldsymbol{\theta}_1$ (slope factor sensitivity)
Medium-term component	This is the component$[(1-\exp(-\boldsymbol{\theta}_3 t))/\ (\boldsymbol{\theta}_3 t)) -\exp(-\boldsymbol{\theta}_3 t_t)]$ on $\boldsymbol{\theta}_2/\boldsymbol{\theta}_3$, which starts at 0, increases for medium maturities and then decays to zero again thereby creating a hump-shape. The $\boldsymbol{\theta}_3$t component determines at which rate the medium term component reaches its maximum	$\boldsymbol{\theta}_2/\boldsymbol{\theta}_3$(curvature factor sensitivity)
Rate of decay	The rate of decay is $\boldsymbol{\theta}_3 t=\tau/\lambda_t$	NA

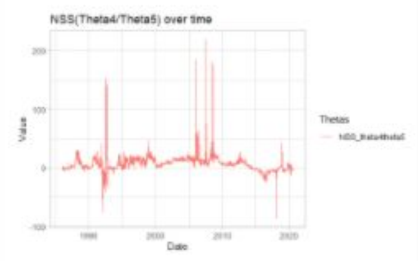
Table 2: NS model

Table of results

Graphs	NS model ($\boldsymbol{\theta}$)	NSS model($\boldsymbol{\theta}$)
	$\boldsymbol{\theta}_0$ decreases over time	$\boldsymbol{\theta}_0$ decreases over time, but more volatile here
	$\boldsymbol{\theta}_1$ is volatile	$\boldsymbol{\theta}_1$ is more volatile here
	$\boldsymbol{\theta}_2/\boldsymbol{\theta}_3$ is volatile	$\boldsymbol{\theta}_2/\boldsymbol{\theta}_3$ is more volatile

θ Results

- The level for both models are generally constant. However, the NSS model is able to capture more of the level's volatility over time
- Value alternates between positive and negative. Tells you that the yield curve changes from upwards sloping to downward sloping
- Tells you yield curve shifts between getting flatter and steeper (signals). NSS was able to capture the curvature's volatility during GFC 2008 & Black wednesday 1992

	NA	θ_4/θ_5 is only volatile at some points	The curvature for the second medium component is generally constant but volatile during crises such as GFC 2008 and Black Wednesday (1992)

Conclusion

- In conclusion, as NSS adds a fourth factor to raise the flexibility of in-sample fit, it allows it to better view the volatility of the yield curve. This is evidenced that in 1992 (Black Wednesday), the volatility was better captured by the NSS model.
- Additionally, looking at the curvature factors (θ2/θ3=B3 & θ4/θ5=B4) and the slope factors(θ1=β2), the aforementioned strategies can be adopted when trading.

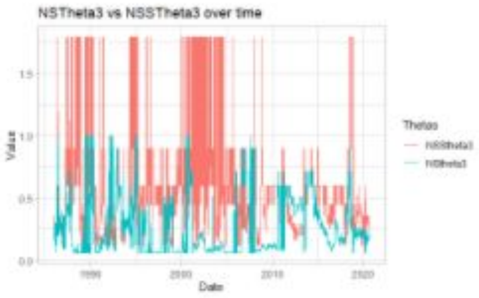
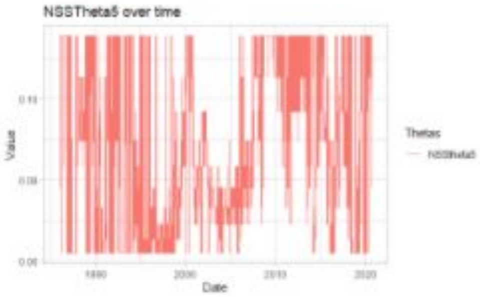
	$\boldsymbol{\theta}_3$ is volatile	$\boldsymbol{\theta}_3$ is more volatile volatile here	The rate of decay $\boldsymbol{\theta}_3$ is much larger in the nss model than the ns model
	NA	$\boldsymbol{\theta}_5$ is very volatile	Rate of decay $\boldsymbol{\theta}_5$fluctuates at a low value between 0.06 and 0.12

Table 5: Table of results

PART 2.4

DATA FITTING &
MODEL SELECTION

MODEL FITTING

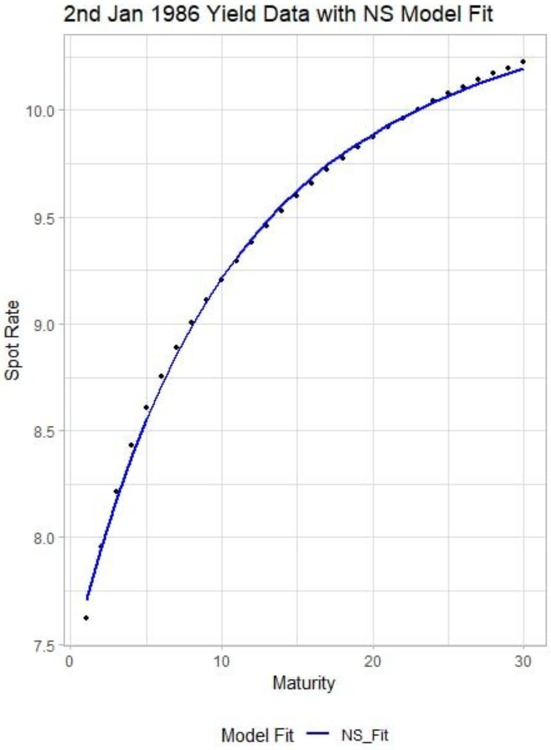
- Minimize the sum of squared errors non linearly for both NS and NSS
- We took advantage of the “YieldCurve” package but rewrote the source code to output SSR, AIC and BIC for Model Selection

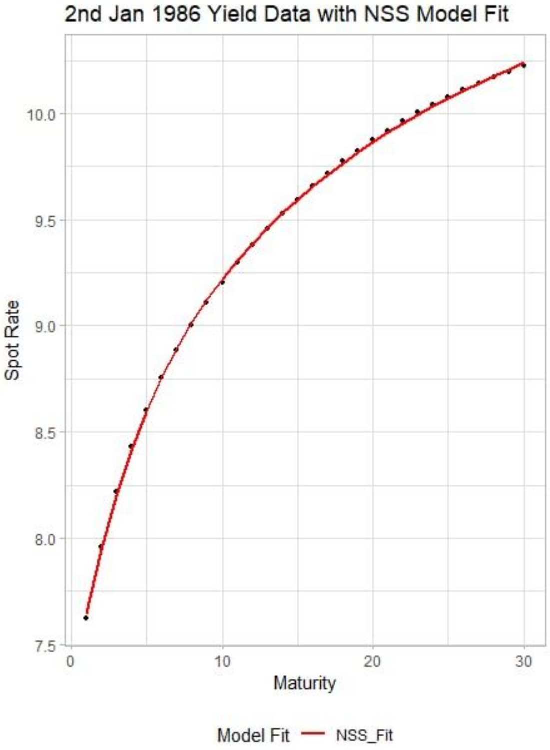
MODEL FITTING - Sample re-written code

```r
NSFit <- function (rate, maturity)
{
  rate <- try.xts(rate, error = as.matrix)
  if (ncol(rate) == 1)
    rate <- matrix(as.vector(rate), 1, nrow(rate))
  pillars.number <- length(maturity)
  lambdaValues <- seq(maturity[1], maturity[pillars.number],
                      by = 0.5)
  FinalResults <- matrix(0, nrow(rate), 7)
  colnames(FinalResults) <- c("beta_0", "beta_1",
                              "beta_2", "lambda","SSR","AIC", "BIC")
  j <- 1
  while (j <= nrow(rate)) {
    InterResults <- matrix(0, length(lambdaValues), 7)
    colnames(InterResults) <- c("beta0", "beta1",
                                "beta2", "lambda", "SSR","AIC", "BIC")
    for (i in 1:length(lambdaValues)) {
      lambdaTemp <- optimize(.FactorBeta2, interval = c(0.001,
                             1), maturity = lambdaValues[i], maximum = TRUE)$maximum
      InterEstimation <- NSestimator(as.numeric(rate[j,
                             ]), maturity, lambdaTemp)
      BetaCoef <- InterEstimation$Par
      AIC <- InterEstimation$AIC
      BIC <- InterEstimation$BIC
```

MODEL SELECTION

- Visually examine the fit of both models





MODEL SELECTION

- Compute the BIC, MSE and MSFE (more on MSFE later) for both models.

	NSS Model	NS Model
% of time chosen by BIC	97%	3%
MSE	0.0000644	0.00229
MSFE	0.00352	0.00987

PART 2.5

PREDICTION

PREDICTION

- Research (Swanson & Xiong, May 2018) has shown that NS and NSS models are the best performers for mean square forecast error (MSFE).
- We took inspiration from their prediction protocol to get our test MSFE as well as predict the spot rates for September 2020

MSFE DERIVATION

1) **Split Data and fit NS / NSS model to Training Data**
   Test set = Aug 2020

2) **Fit best ARIMA model to each parameter based on BIC**

$$\beta_t' = c + \varphi_1 \beta_{t-1}' + \cdots + \varphi_p \beta_{t-p}' + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

3) **Then, predict the parameters for Aug 2020**

4) **Predict the spot rates for Aug 2020 for both NS and NSS**

$$\widehat{y}_{t+h}(\tau) = \widehat{\beta}^f_{1,t+h} + \widehat{\beta}^f_{2,t+h} \cdot \left[\frac{1-\exp(-\lambda_t \tau)}{\lambda_t \tau}\right] + \widehat{\beta}^f_{3,t+h} \cdot \left[\frac{1-\exp(-\lambda_t \tau)}{\lambda_t \tau} - \exp(-\lambda_t \tau)\right],$$

$$\begin{aligned}\widehat{y}_{t+h}(\tau) = \widehat{\beta}^f_{1,t+h} + \widehat{\beta}^f_{2,t+h} \cdot \left[\frac{1-\exp(-\lambda_{1,t} \tau)}{\lambda_{1,t} \tau}\right] + \widehat{\beta}^f_{3,t+h} \cdot \left[\frac{1-\exp(-\lambda_{1,t} \tau)}{\lambda_{1,t} \tau} - \exp(-\lambda_{1,t} \tau)\right] \\ + \widehat{\beta}^f_{4,t+h} \cdot \left[\frac{1-\exp(-\lambda_{2,t} \tau)}{\lambda_{2,t} \tau} - \exp(-\lambda_{2,t} \tau)\right]\end{aligned}$$

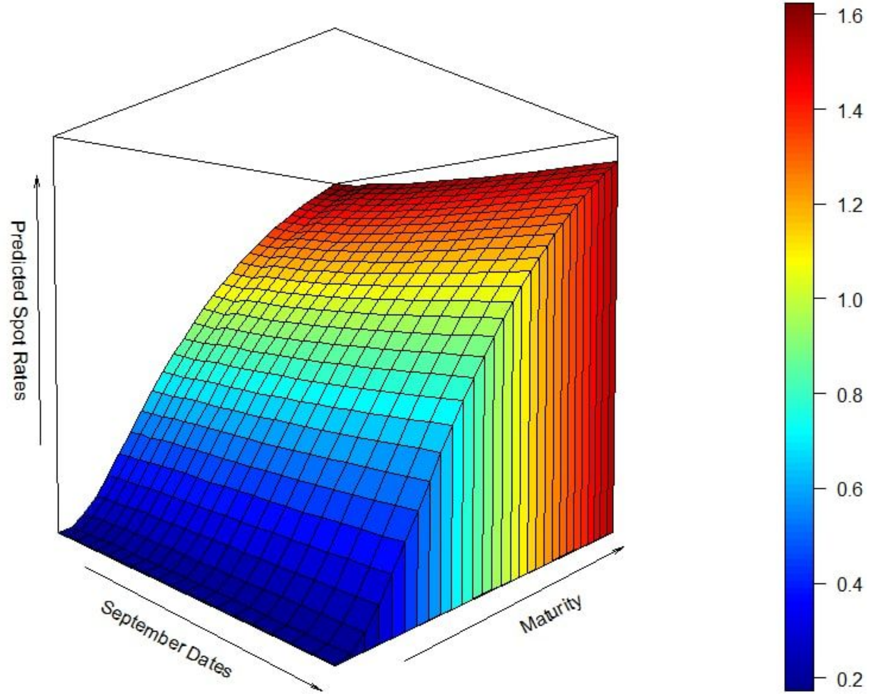
5) **Get MSFE:**

$$\mathrm{MSFE}_h(\tau) = \sum_{t=1}^{P} (\hat{y}_{t+h}(\tau) - y_{t+h}(\tau))^2$$

NSS: 0.00352, NS = 0.00987

PREDICTED YIELD CURVES FOR SEP 2020

- Standard upward sloping Yield Curve for all maturities
- However, spot rates are still much lower than pre COVID-19 levels.
- Light at the end of the tunnel amidst COVID-19.



PROJECT REVIEW / FOLLOW UPS

- Given the volatility of the parameters of the NSS models, we can consider an ARIMA-GARCH model instead or intervention models to capture the historical crises
- Can consider using Vector Autoregressions (ie other bonds as predictors) instead of ARIMA
- Can the models be used for real time data? Swanson and Xiong (2018) proposing using real time data and diffusion indexes in the DNS model

PROJECT REVIEW / FOLLOW UPS

The Diffusion Index Formula is

$$\text{Diffusion Index (DI)} = (\text{Advances} - \text{Declines}) + \text{PDIV}$$

where:

$\text{Advances} = \text{Number of stocks moving higher}$

$\text{Declines} = \text{Number of stocks moving lower}$

$\text{PDIV} = \text{Previous DI value}$

PROJECT REVIEW / FOLLOW UPS

- Diffusion Index and Modifications to DNS model by Swanson and Xiong (2018)

$$\begin{pmatrix} F_{y,t+h} \\ x_t \end{pmatrix} = \begin{pmatrix} c_y \\ c_x \end{pmatrix} + \begin{bmatrix} \Gamma_y & \Gamma_x \\ 0 & \Gamma_{xx} \end{bmatrix} \begin{pmatrix} F_{y,t} \\ F_{x,t} \end{pmatrix} + \begin{pmatrix} e_{y,t+h} \\ e_{x,t} \end{pmatrix}$$

Diffusion Index

THANK YOU

Do you have any questions?

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