Fibonacci Sequence

Notes due to Steve Tanny

1   Background

- $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13$, and so on.

- Introduced by Leonardo Bonacci (c. 1175-1250) of Pisa. Fibonacci is a contraction of Filius Bonacci (son of Bonacci). In his book Liber Abaci in 1202 (note that we only have the second edition of the book from 1228 which mentions the existence of the first edition) he writes along the following lines: Assume every pair of rabbits begins breeding after it is one month old, and produces one pair of offspring after one month at the end of every month. If we begin with one pair of newly born rabbits, after one month there is still only one pair of rabbits. After two months there will be two pairs of rabbits since the initial pair has now begun to breed. After three months there will be three pairs: the two that existed after two months and the additional pair produced by the initial pair (note that the rabbits produced during the second month are still too young to breed). And so on. Assume that no rabbits die. How many rabbits will there be after n months? Fibonacci did not use sequence or recursion notation. If he had, he might have written something like the following: let F_n be the number of rabbits at the beginning of the nth month (starting from 1), then F_n is the number of pairs of rabbits alive at the beginning of the $(n-1)^{th}$ month plus the pairs of babies that are produced by the (sufficiently mature) rabbits, those that are alive at the beginning of the $(n-2)^{th}$ month. That is, we get the nth Fibonacci number: the sum of the two preceding numbers!

- DeMoivre was the first to write the Fibonacci numbers as a recursion: $F_n = F_{n-1} + F_{n-2}$. Together with the initial values $F_1 = F_2 = 1$, this defines the Fibonacci sequence as a recurrence relationship. DeMoivre, and independently Jacques Phillipe Marie Binet (1786-1856) and Daniel Bernoulli (1700-1782), solved this (second-order homogeneous, constant coefficient) recurrence relation to obtain the following formula :

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \tag{1}$$

- Edward Lucas (of Tower of Hanoi fame and much more) named the sequence after Fibonacci in 1876. Lucas worked with these numbers extensively in the last half of the 19th century. Originally he called them the series of Lamé.

Note: In some sources the sequence has initial conditions $F_0 = F_1 = 1$. In this case the sequence is advanced one term relative to the one we have been discussing. **Be aware of this difference**

when reading the literature.

Here are some interesting facts about Fibonacci numbers:

1. "...Fibonacci numbers appear very frequently in nature. The branch of botany that studies the arrangement of leaves around stems, the scales on cones, and so on, is called phyllotaxis. Usually, the leaves that appear on any given stem or branch point out in different directions. The second leaf is rotated from the first by a certain angle, the third leaf from the second by the same angle, and so on until some leaf points in the same direction as the first."

   - For example, if the angle is $144°$, then the fifth leaf is the first one pointing in the same direction as the first, since $5 \cdot 144° = 720°$. Let n be the number of leaves before returning to the same direction as the first leaf, and let m be the number of complete $360°$ turns which have been made before this leaf is encountered. Inexplicably both n and m always have Fibonacci numbers as values:

Plant	ANGLE	n	m
Elm	$180°$	2	1
Alder, Birch	$120°$	3	1
Rose	$144°$	5	2
Cabbage	$135°$	8	3

     Schips (1922), Batchelet (1971)
     From the table, we see that the leaves of a cabbage are spaced around its stem at $135°$ intervals. Since $8 \times 135 = 1080 = 3 \times 360$; $n = 8$ will be the number of leaves before turning to the same direction as the first leaf, and it will take $m = 3$ full turns to come to that second leaf that points in the same direction as the first.

   Source: Roberts, Applied Combinatorics, Prentice Hall, 1984, pp. 202-203

2. A male bee (drone) is produced asexually from a female (queen). Each female has two parents, a male and a female. Here is the family tree of a drone:
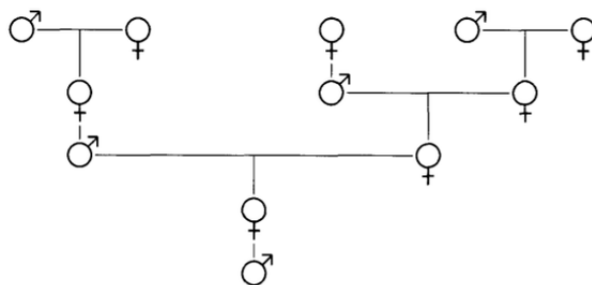
   Each Drone has:
   1 grandfather and 1 grandmother,
   1 great-grandfather and 2 great-grandmothers,

2

2 great-great-grandfathers and 3 great-great-grandmothers,

$\vdots$

F_{n+1} great n^{th} grandfathers, F_{n+2} great n^{th}-grandmothers

Source: Graham, Knuth, Patashnik (1994), p.29

3. The golden section is determined by the point M on a line segment AB such that the ratio of the segment MB to AM is exactly the same as that of the segment AM to the line AB. Equivalently, $\frac{AB}{AM} = \frac{AM}{MB} = \tau$. This ratio is called the golden mean (section). The Fibonacci sequence relates to the golden mean (the divine proportion). Define the growth rate $G(n)$ of the sequence F_n as $G(n) = \frac{F_{n+1}}{F_n}$. Then we get the following values for $G(n)$:

n	$G(n)$
1	$\dfrac{F(1)}{F(0)} = \dfrac{1}{1} = 1$
2	$\dfrac{F(2)}{F(1)} = \dfrac{2}{1} = 2$
3	$\dfrac{F(3)}{F(2)} = \dfrac{3}{2} = 1.5$
4	$\dfrac{F(4)}{F(3)} = \dfrac{5}{3} \approx 1.667$
5	$\dfrac{F(5)}{F(4)} = \dfrac{8}{5} = 1.6$
6	$\dfrac{F(6)}{F(5)} = \dfrac{13}{8} = 1.625$
7	$\dfrac{F(7)}{F(6)} = \dfrac{21}{13} \approx 1.615$
8	$\dfrac{F(8)}{F(7)} = \dfrac{34}{21} \approx 1.619$

$G(n)$ tends quickly to its limit (sometimes denoted by ϕ or τ or α), the golden mean, $\frac{1+\sqrt{5}}{2} = 1.618034\ldots$. This was first proved by Scottish mathematician Robert Simson in 1753. Notice that the successive ratios of Fibonacci numbers alternately overshoot and undershoot the golden mean. It is easy to show that the quantity $\frac{1-\sqrt{5}}{2} = -\frac{1}{\tau}$ and is approximately $-0.61803\ldots$. It shares many of the same properties as τ and is often called $\hat{\tau}$ - 'tau-hat'.

4. The most aesthetic rectangle is one with sides in the ratio of $\tau : 1$ (a golden rectangle). See the book De Divina Proportione (On Divine Proportion) by the artist Luca Pacioli (1445-1517), where the significance of a golden rectangle and its prevalence in the masterpieces of Leonardo da Vinci was documented. (Source: Ron Knott, Fibonacci Numbers and the Golden Section in Art, Architecture and Music, University of Surrey Department of Mathematics and Statistics and Department of Computing, 2004.)

5. τ is very nearly the number of kilometers in a mile (the exact number is 1.609344 km/mile). So it follows that a distance of F_{n+1} kilometers is very close to F_n miles!

6. According to European scholars, extensive empirical research indicates that on average, the ratio of ones height to the height of ones navel (from the ground, presumably) is approximately 1.618! Consider the following diagram: (missing figure)
   In fact each of the lengths of the numbered lines in the right side of the picture above is in τ proportion to the preceding coloured line. (i.e. the line4: line3 ratio is τ and the line3: line2 ratio is τ.)

2   Selected Fibonacci Sequence Properties

$$\text{Define:} \quad F_0 = 0, \quad F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \; ; \; n \geq 2 \tag{2}$$

We have already shown that F_{n+2} counts the number of subsets of $[n]$ with no two elements consecutive, and that $F_{n+2} = \sum_k \binom{n-k+1}{k}$. Further, we have shown that the number of circular k-choices from $[n]$ with no two elements consecutive is $\binom{n-k}{k} + \binom{n-k-1}{k-1}$, so that the number of subsets of $[n]$ with no two elements consecutive and where we count n and 1 as consecutive is

$$\sum_k \binom{n-k}{k} + \sum_k \binom{n-k-1}{k-1}$$

But the second sum in the above is

$$\sum_k \binom{n-k-1}{k-1} = \sum_k \binom{n-2-(k-1)}{k-1} = F_{n-1}$$

Thus, this latter count is just $F_{n+1} + F_{n-1}$.

Let $L_n = F_{n+1} + F_{n-1}$. Then

$$L_{n+1} = F_{n+2} + F_n = (F_{n+1} + F_n) + (F_{n+1} + F_{n-2}) = L_n + L_{n-1}$$

So, if we set $L_0 = 2$, and since $L_1 = F_2 + F_0 = 1$, we get the Lucas numbers.

$$\text{Lucas:} \quad L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \; ; \; n \geq 2 \tag{3}$$

Note: $L_2 = 3 = F_3 + F_1$, so the recursion for L_n holds for $n = 2$. This supports the choice of $L_0 = 2$. We will provide another justification for this choice later. Some define the initial terms of the Lucas sequence slightly differently: $L_0 = 1, L_1 = 3$ are common choices. This leads to essentially the same sequence. The only difference from the one we have defined above is that the initial term equal to 2 is dropped.

2.1 Challenges

Establish the following properties of the Fibonacci numbers. Be careful to check out the indices carefully; some might be slightly different from the usual notation as some authors use $F_0 = 1$. Try to find interesting combinatorial proofs where possible.

Challenge 0: Show each the following:

Lucas-Fibonacci connection : 1) $L_{n+1} = F_{n+2} + F_n$ 2) $F_{2n} = F_n \cdot L_n$

Binet's formulas: 1) $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ 2) $L_n = \alpha^n + \beta^n$, where

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}$$

Asymptotic behavior: $\dfrac{F_{n+1}}{F_n} \to \alpha \quad (n \to \infty)$

Pythagorean triples: If w, x, y, and z are 4 consecutive Fibonacci numbers, then $(wz, 2xy, yz - wx)$ is a Pythagorean triple.

Matrix form:
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$$

Challenge 1: Prove that any two consecutive Fibonacci numbers are relatively prime.

Challenge 2: Prove that F_{n-1} is even if and only if n is a multiple of 3. (watch initial conditions here)

Challenge 3: Prove $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$ (running sum identity)

Challenge 4: Prove $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ (sum of squares identity)

Challenge 5: Prove $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$ (convolution property)

Challenge 6: Prove $\displaystyle\sum_{k=0}^{n-1} \binom{n}{k+1} F_k = F_{2n-1}$

Challenge 7: Prove $F_2 + F_5 + \ + F_{3n-1} = \dfrac{F_{3n+1} - 1}{2}$

Challenge 8: Prove $nF_0 + (n-1)F_1 + \cdots + 2F_{n-2} + F_{n-1} = F_{n+3} - (n+3)$

Challenge 9: Prove $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (Cassinis formula,1680)

Challenge 10: Prove $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

Challenge 11: Prove $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$

Challenge 12: Prove $F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n} = -F_{2n-1} + 1$

Challenge 13: Prove $F_{n-1}^2 + F_n^2 = F_{2n-1}$

Challenge 14: Prove $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$

Challenge 15: Prove $F_n^3 + F_{n+1}^3 - F_{n-1}^3 = F_{3n}$

Challenge 16: Call a set of integers fat if each of its elements is at least as large as its cardinality. For example, $\{6, 10, 11, 20, 33, 34\}$ is fat, but $\{2, 100, 200\}$ is not. Let $f(n)$ be the number of fat subsets of $[n]$. Count the null set as fat. Notice that $f(3) = 5$ (the subsets are $\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}$). Prove that $f(n) = F_{n+2}$.

Challenge 17: Prove $2^{n+1} F_{n+1} = \sum_{k=0}^{n} 2^k L_k$

$[L_0 = 2,\ L_1 = 1 \ldots L_n = L_{n-1} + L_{n-2}\ (n \geq 2)]$

Challenge 18: Prove $\sum_{k \geq 0} 5^k \binom{n}{2k+1} = 2^{n-1} F_n$

Challenge 19: Prove $\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$

Challenge 20: Prove that F_{kn} is a multiple of F_n for all integers k and n.

Challenge 21: Prove that if $n > 2$ and if F_m is a multiple of F_n, then m is a multiple of n (partial converse of above).

Challenge 22: Let $gcd(\alpha, \beta)$ be the greatest common divisor of α and β. Prove that $gcd(F_m, F_n) = F_{gcd(m,n)}$.

2.1.1 Negative Indices

We can define the Fibonacci numbers for negative indices in a natural way by just running the recursion backwards, so to speak:

$F_{-1} = F_1 - F_0 = 1 - 0 = 1$
$F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$
$F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$

Its easy to show (do it!) that $F_{-n} = (-1)^n F_n$. The sequence $\{|F_n|\}$ so defined for all integers n is palindromic (that is, it reads the same right-to-left as left-to-right) about the 0 index. It now follows that Cassinis identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ holds for all integers n. Similarly, the convolution property $F_{n+k} = F_k F_{n+1} + F_{k-1}F_n$ holds for all integers n and k. This definition of the Fibonacci sequence for negative indices is also consistent with our choice of initial values for the Lucas sequence.

Challenge 23: Let the initial conditions be $F_0 = a$ and $F_1 = b$ (for original Fibonacci, we have $a = 0$ and $b = 1$). For what choices of a and b do we get sequences that are palindromic about some index (or between two indices) in absolute value?
Note: We may get the same sequence (only shifted) for different values of a and b. For example, if $a = -8$ and $b = 5$ we get the usual Fibonacci sequence, so this doesnt count as a new palindromic sequence in absolute value.

2.1.2 Zeckendorf Representations

Every positive integer has a unique representation of the form

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \text{ where } k_1 \gg k_2 \gg \ldots \gg k_r \gg 0 \tag{4}$$

($a \gg b \iff a \geq b + 2$ so none of the k_i's are consecutive).
This is called the *Zeckendorf* representation of n.

For example,

$$
\begin{aligned}
10^6 &= 832040 + 121393 + 46368 + 144 + 55 \\
&= F_{30} + F_{26} + F_{24} + F_{12} + F_{10}
\end{aligned}
$$

This naturally leads to a Fibonacci number system:

$$n = (b_m b_{m-1} \cdots b_2)_F \iff n = \sum_{k=2}^{m} b_k F_k \tag{5}$$

where all the b_i are 0 or 1, $b_m = 1$, and there are never two adjacent 1s.

Example: $5 = (1000)_F$ $\qquad$ $19 = (101001)_F$

Note: Confirm that the Fibonacci representation has more bits than the binary representation, in part because it doesnt allow consecutive 1s and also because the Fibonacci numbers dont grow as fast as powers of 2 (recall that the ratio of consecutive Fibonacci numbers is the golden mean, which is about 1.61).

3 A Combinatorial Interpretation of Fibonacci Numbers [1]

- Count the number of sequences fn of 1s and 2s that sum to n. For example, $f_4 = 5$ since $4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 2 + 2$.

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[1]See Benjamin and Quinn references for more on this topic.

- Clearly, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Set $f_0 = 1$ and $f_1 = 1$. Notice that $f_n = F_{n+1}$, the Fibonacci number.

- Visually, we can think of the 1s as 1×1 squares and the 2s as 1×2 dominoes. Then f_n (equivalently, F_{n+1}) counts the number of ways to tile a $1 \times n$ board (equivalently, tile an n-board) with squares and dominoes.

- We can use this combinatorial interpretation of F_{n+1} to prove Fibonacci sequence identities. By conditioning upon some specific property we show that both sides of the identity count the same thing in two ways (remember H. Weyls saying).

3.1 Combinatorial proofs of some Fibonacci sequence identities

Example 1: For $n \geq 0$, $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$ (Chall. 3: Running Sum identity)

Proof. The R.H.S counts the number of ways to tile an $(n+2)$-board with at least one domino (that is, it excludes the possibility of using only squares). The L.H.S counts these by conditioning on the location of the last domino in the tiling. Suppose it covers cells $k+1$ and $k+2$ for some $k \geq 0$. There are f_k ways to tile the first k cells; cover the next two cells with a domino. Extend each of these to an $(n+2)$-tiling of the board with at least one domino by completing the tiling with all squares. Use the sum rule to complete the proof. □

Example 2: For $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$ (Challenge 10).

Proof. The R.H.S counts the number of tilings f_{2n+1} of a $(2n+1)$-board. Note that since the $(2n+1)$-board has an odd number of cells, any tiling must contain an odd number of squares (since the dominoes cover an even number of cells). Here is how to see that the L.H.S counts the same thing: condition on the location of the last square in the tiling. This square must occupy an odd-numbered cell, since it is preceded by an even number of squares (possibly 0) and some number of dominoes (which cover an even number of squares: note the parity principle at work here!). When the last square covers cell $(2k+1)$, then there are f_{2k} tilings of the initial $2k$-board. Each tiling of the initial $(2k+1)$-board is extended to a tiling of the entire board by adding dominoes. Apply the sum rule to complete the proof. □

Example 3: For $m, n \geq 0$, $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$ (Chall. 5: Convolution Property)

Proof. The L.H.S counts the number of tilings of an $(m+n)$-board. The R.H.S counts this by first tiling an m-board and then an n-board. But to do this we have to know that the tiling can be broken up this way. Any tiling of an n-board is breakable at cell k if the tiling can be decomposed into two tilings, one covering cells 1 through k, the other covering cells $k+1$ through n. A tiling is unbreakable at cell k if and only if a domino occupies cells k and $k+1$. Any n-tiling is always breakable at cell n (its followed by an empty board and so an empty tiling).

If the $(m+n)$-tiling is breakable at cell m, then it consists of an m-tiling followed by an n-tiling and by the product rule there are $f_m f_n$ such tilings. If the $(m+n)$-tiling is not breakable at cell m, then there is a domino covering cells m and $m+1$. So the $(m+n)$-tiling consists of an $(m-1)$-tiling, a domino and an $(n-1)$-tiling. Again, by the product rule there are $f_{m-1} f_{n-1}$ such tilings. Now apply the sum rule to complete the proof. □

Example 4: For $n \geq 0$, $F_n = \sum_k \binom{n-k}{k}$

Proof. The L.H.S counts the number of tilings of an n-board. The R.H.S counts these by conditioning on the number of dominoes in the tiling, which must be $\leq \frac{n}{2}$. A tiling with k dominoes necessarily contains $n-2k$ squares so uses a total of $n-k$ tiles. Choose the k tiles that are dominoes in $\binom{n-k}{k}$ ways and apply the sum rule to complete the proof. $\square$

Example 5: For $n \geq 0$, $f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}$

Proof. The L.H.S counts the number of ways to tile a $(2n-1)$-board. The R.H.S counts these by conditioning on the number of squares appearing among the first n tiles. Any $(2n-1)$-tiling must have at least n tiles, and by the parity principle at least 1 of these is a square. Suppose the first n tiles consist of k squares and $n-k$ dominoes. Then these n tiles can be arranged in $\binom{n}{k}$ ways and cover cells 1 through $2n-k$. The remaining board had length $k-1$ and can be tiled in f_{k-1} ways. Apply the product and sum rules to complete the proof. $\square$

Example 6: For $n \geq 1$, $3f_n = f_{n+2} + f_{n-2}$

Proof. We find a 3:1 correspondence: fn counts the number of tilings of an n-board, $f_{n+2} + f_{n-2}$ counts the number of tilings of an $(n+2)$-board or an $(n-2)$-board. For each n-tiling, we create the following three tilings:

a) append a domino, giving an $(n+2)$-tiling ending in a domino

b) append 2 squares, giving an $(n+2)$-tiling ending in two squares

c) There are two cases for the last tiling. If the n-tiling ends with a square, insert a domino before this last square to get an $(n+2)$-tiling that ends in a single square preceded by a domino. Together with (a) and (b) we now have accounted for all possible $(n+2)$-tilings.

d) If the n-tiling ends with a domino, remove it. This creates all possible $(n-2)$-tilings.

1. Using this method, we see that every tiling of length $(n+2)$ is created exactly once from a unique n-tiling.

   Proof. Given any $(n+2)$-tiling, we may find the n-tiling that made it by examining the ending. If it ends in a domino, remove it (a 1:1 correspondence with an n-tiling). If it ends in two squares, remove them (a 1:1 correspondence with an n-tiling). If it ends in a domino followed by a square, remove the domino to get a 1:1 correspondence with those n-tilings that end in a square. Unlike the earlier two possibilities it omits some of the n-tilings, namely, those that end in a domino. But we now fix this by adding them in. $\square$

2. Using this method, every tiling of length $(n-2)$ is created exactly once from a unique n-tiling.

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Proof. Given any $(n-2)$-tiling, append a domino. This gives a 1:1 correspondence between $(n-2)$-tilings and the n-tilings that end in a domino, precisely the n-tilings we were missing from the set in the final correspondence described in 1. □

3. It follows that $f_{n+2} + f_{n-2}$ counts all of the n-tilings precisely 3 times.

To summarize:

f_{n+2} counts: all $(n+2)$-tilings that end in a domino (f_n) + all $(n+2)$-tilings that end in 2 squares (f_n) + all $(n+2)$-tilings that end in a square preceded by a domino (n-tilings that end in a square), and f_{n-2} counts: all $(n-2)$ tilings (which is equivalent to the n-tilings that end in a domino). □

Alternate proof:

Proof. We show that for $n \geq 1$, $3f_n - f_{n-2} = f_{n+2}$.

f_{n+2} counts the number of ways to tile an $(n+2)$-board. We make the following correspondence:

- If the $(n+2)$-tiling starts with a domino, remove the domino to get an n-tiling.

- If the $(n+2)$-tiling starts and ends with a square, remove the two squares to get an n-tiling.

- If the $(n+2)$-tiling ends with a domino, remove the domino to get an n-tiling.

Thus, we get three n-tilings from the $(n+2)$ tiling BUT the correspondence isnt 1:1. We double counted precisely the $(n+2)$-tilings that start and end with a domino. These correspond exactly to all the possible $(n-2)$-tilings (take any $n-2$-tiling and put a domino at each end a 1:1 correspondence). So we need to subtract f_{n-2} from the L.H.S. □

4   More Direct Counting Techniques

4.1   Tail Swapping

Can we use direct counting techniques to prove more complicated identities, such as Cassinis identity: for $n \geq 0$, $f_n^2 = f_{n+1}f_{n-1} + (-1)^n$?

f_n^2 suggests that we need 2 n-boards to tile. But how do we count $(-1)^n$? We need a new technique: **tail swapping**.

Here is an example of tail swapping with two 11-tilings: We offset them by 1 tile,
Faults are at 1,2,5,6 and 9. The tail is the piece in each tiling after the last fault.
Now the tail swap ...

More formally, we say that a pair of offset n-tilings has a fault at cell i ($2 \leq i \leq n$) if both tilings are breakable at cell i. We say there is a fault at cell 1 (11) if the first (second) tiling is breakable at cell 1 (11). This is equivalent to saying that the pair has a fault at cell i if neither tiling has a domino covering cells i and $i+1$. Again, the tails of a tiling pair are the tiles after the last fault. Swapping tails means placing the lower tail on the upper tiling and vice versa. In the new pair of tilings the upper tiling has length $(n+1)$ and the lower has length $(n-1)$ and this pair of tiles has the same faults as the initial pair.

Example 7: For $n \geq 0$, $f_n^2 = f_{n+1}f_{n-1} + (-1)^n$ (Cassini's identity)

Proof. **Set 1**: two n-boards, top and bottom. The number of tilings is f_n^2.
Set 2: an $(n+1)$-board on top, an $(n-1)$-board on bottom. The number of tilings is $f_{n+1}f_{n-1}$.
Correspondance:

a) Suppose n is odd. If $n = 1$, we have nothing to prove. Take $n \geq 3$. Each n-board has at least one square. Note that if there is a square in cell i of either board, then there is a fault at cell i or at cell $i-1$. Swap the tails. We get a pair of tilings with the same faults, one $(n+1)$-tiling and one $(n-1)$-tiling. So we have a 1:1 correspondence between all pairs of n-tilings and all tiling pairs of sizes $(n+1)$ and $(n-1)$ that have faults. However, we still need to account for tilings that are fault free. Those are the tilings that consist entirely of dominoes (recall, n is odd, so $n+1$ and $n-1$ are even). For example,
This pair could not be obtained from our tail-swapping procedure, so it is not counted in our 1:1 correspondence. Thus, for n odd, $f_n^2 = f_{n+1}f_{n-1} - 1$.

b) Suppose n is even. Once again tail swapping produces a 1:1 correspondence between the tiling pairs with faults. And as before the only fault free tiling pair consists of an all-dominoes pair, but this time a pair of n-tilings. (Shown below for $n = 10$.)
Thus, our correspondence doesnt count one of the pairs counted by f_n^2. So for n even, $f_n^2 = f_{n+1}f_{n-1} + 1$.

$\square$

4.2   Binary Sequences and n-Tilings

Example 8: Show that for $n \geq 2$, $f_n + f_{n-1} + \sum_{k=o}^{n-2} f_k 2^{n-k-2} = 2^n$.

Proof. To encode an n-tiling as a binary sequence of length n, translate each square as a 1 and each domino as 01. Equivalently, the ith term of the binary sequence is 1 if and only if the tiling is breakable at cell i. Note that the binary sequence resulting from this tiling encoding has no consecutive 0s and always ends in 1. There are $f_n = F_{n+1}$ such sequences.

Conversely, given an n-binary sequence with no consecutive 0s that ends with a 1, we get a unique n-tiling. If an n-binary sequence with no consecutive 0s ends in a 0, it represents a unique $(n-1)$-tiling (ignore the last 0). Note that the R.H.S counts all binary sequences of length n. We now associate a unique tiling with each binary n-sequence. If a sequence has no consecutive 0s, it must end either in 11, 01, or 10. We associate it with a unique $(n-1)$-tiling if it ends in a 0, or a

unique n-tiling if it ends in a 1. This gives us the first two terms on the L.H.S above. Otherwise, the binary sequence contains at least one pair of 0s. We condition on the location of the first pair of zeros.

Let the first pair of zeroes occur in cells $k + 1$ and $k + 2$ for some $k, 0 \leq k \leq n - 2$. For this sequence, associate a k-tiling with the first k terms of the binary sequence (if $k > 0$, then the kth digit must be 1, since otherwise the first pair of consecutive 0s would occur in cells k and $k + 1$). There are $n - k - 2$ binary digits in the n-sequence following this first pair of consecutive 0s, so $2n - k - 2$ binary sequences each give the same k-tiling. Hence, by the product rule, there are $f_k 2^{n-2-k}$ such binary sequences where the first pair of consecutive 0s occurs in cells $k + 1, k + 2$. Now apply the sum rule. $\qquad\square$

4.3 Connection between Lucas and Fibonacci numbers

We now demonstrate an identity linking the Lucas numbers and the Fibonacci numbers that leads to a useful understanding for generalizations of these sequences. Recall the recursion for the Lucas numbers L_n:

$$L_0 = 2, L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \ n \geq 2.$$

It is easy to show that L_n counts the number of ways to tile a circular n-board with curved 1×1 squares and 1×2 dominoes. Cells are labeled 1 to n and such a tiling is called an n-bracelet. An n-bracelet is out of phase if the same domino covers cells n and 1. Otherwise the bracelet is in phase. There are fn in phase n-bracelets and f_{n-2} out of phase n-bracelets. (Think of L_0 as representing two empty bracelets, one in phase and the other out of phase.) Thus, for $n \geq 2, L_n = f_n + f_{n-2} = F_{n+1} + F_{n-1}$.

We show that: $2^{n+1} f_n = \displaystyle\sum_{k=0}^{n} 2^k L_k.$

We need a meaning for each side of this identity. Suppose each cell of the $1 \times n$ board can be coloured white or black, and suppose the squares and dominoes are transparent. Then there are $2^n f_n$ coloured tilings of the $1 \times n$ board. Equivalently: the $1 \times n$ (ordinary, uncoloured) board is tiled with white squares, black squares, and four different colours of dominoes (white-white, white-black, black-white, and black-black). For convenience, think of four different colours of dominoes (say red, yellow, green, and blue). Call this a coloured tiling of a $1 \times n$ board. Similarly, there are $2^n L_n$ coloured tilings of an n-bracelet.

Let T be such a coloured tiling of an n-board. We create two coloured tilings of a k-bracelet for some $k \leq n$. If T is not the all white square tiling, then let k denote the last cell covered by a non-white tile (either a black square or a coloured domino) (so only white squares follow this tile). Define:

B_1: Remove cells $k + 1$ through n. B_1 is the in phase k-bracelet ending in a non-white tile produced by joining cells k and 1 together.

Since the last tile in this bracelet is non-white, and the bracelet so formed is in phase, this correspondence omits those bracelets that end in a white square or an out of phase domino. So define:

B_2: If cell k is covered by a black square, then B_2 is the in phase k-bracelet obtained by replacing the kth cell of B_1 with a white square. If cell k is covered by a coloured domino, then B_2 is the out of phase k-bracelet obtained by rotating the tiles of B_1 clockwise one cell.

Every coloured k-bracelet, $1 \leq k \leq n$, is obtained exactly once this way. The process is easily reversed by examining the last tile of the coloured bracelet. The case where our coloured tiling T consists of all white squares is identified with the two empty bracelets for L_0. Thus, we have a 2 to 1 mapping, so

$$2 \cdot 2^n f_n = \sum_{k=0}^{n} 2^k L_k$$

The initial factor of 2 accounts for the 2:1 correspondence. The term $2^n f_n$ counts all such coloured n-boards, while the sum on the R.H.S counts all such coloured n-bracelets.

As it turns out, there is a much easier way to derive this identity.

Let $a_n = 2^{n+1} f_n, n \geq 0$, and $b_n = 2^n L_n$. Then $a_n = \sum_{k=0}^{n} b_k \Leftrightarrow b_n = a_n - a_{n-1}$. But,

$$
\begin{aligned}
b_n = a_n - a_{n-1} &\Longleftrightarrow & 2^n L_n = 2^{n+1} f_n - 2^n f_{n-1} \\
&\Longleftrightarrow & L_n = 2f_n - f_{n-1} \\
&\Longleftrightarrow & L_n = f_n + (f_n - f_{n-1}) \\
&\Longleftrightarrow & L_n = f_n + f_{n-2}
\end{aligned}
$$

Since we know this to be true we are done. So the moral of the story is: don't get too carried away with any one single technique or approach, even a beautiful direct counting argument. It can sometimes lead to too much work.

4.4 Generalized Fibonacci and Lucas Sequences by Coloured Tilings

The generalized Fibonacci (and Lucas) sequences satisfy analogues of many of the properties and identities for the ordinary Fibonacci and Lucas sequences. The approach by colouring provides an intuitive combinatorial understanding of this generalization.

Define $F_0 = 0, F_1 = 1$, and for $n \geq 2, F_n = aF_{n-1} + bF_{n-2}, \ a, b \in \mathbb{N}$

Also $L_0 = 2, L_1 = a$, and for $n \geq 2, L_n = aL_{n-1} + bL_{n-2}, \ a, b \in \mathbb{N}$

Note: If $a = b = 1$, we get the ordinary Fibonacci and Lucas sequences.

Then we have the following properties:

- $F_n = F_m F_{n-m+1} + b F_{m-1} F_{n-m}$

- $(a + b - 1) \sum_{k=0}^{n} F_k = F_{n+1} + b F_n - 1$

- $a \cdot \left(\sum_{k=0}^{n-1} b^{n-1-k} F_{k+1}^2 \right) = F_n F_{n+1}$

- $F_{n-1} F_{n+1} = F_n^2 + (-1)^n b^n$ (generalized Cassinis identity)

- $\gcd(F_m, F_n) = F_{\gcd(m,n)}$

- $L_n = a F_n + 2 b F_{n-1}$

- $L_n = F_{n+1} + b F_{n-1}$

Benjamin and Quinn show that the interpretation of Fibonacci numbers as tilings of a $1 \times n$ board can be generalized to cover these identities as follows: suppose the 1×1 square can be coloured in a ways, the 1×2 dominoes in b ways. Then $f_1 = a$, $f_2 = a^2 + b$ (product and sum rules) and $f_3 = a^3 + 2ab$ (a 1×3 board is covered by 3 squares or 1 square and 1 domino, the latter with 2 possible orders). Again, $f_0 = 1$. Then for $n \geq 2$, $f_n = a f_{n-1} + b f_{n-2}$ (a board of length n ends in a square in a different ways, corresponding to colours and leaves a $1 \times (n-1)$ board to tile, or it ends in a domino in b ways, leaving a $1 \times (n-2)$ board to tile). Call these coloured tilings. Set $f_{-1} = 0$. Then $f_n = F_{n+1}$, the generalized Fibonacci numbers.

Example 9: Show that $f_n = f_m f_{n-m} + b f_{m-1} f_{n-m-1}$, where f_n counts the numbers of coloured tilings of a $1 \times n$ board.

Proof. For any $m \in ?\mathbb{N}$, the mth cell of the $1 \times n$ board is covered either by a 1×1 square or the second square of a 1×2 domino (in which case we say the tiling is breakable at the mth cell), or it is covered by the first cell of a 1×2 domino (in which case we say the tiling is unbreakable at the mth cell, but the tiling is breakable at the $(m + 1)^{th}$ cell: in fact, the tiling consists of a tiling of a $1 \times (m - 1)$ board, a 1×2 domino, and a tiling of a $1 \times (n - m - 1)$ board). Since there are b colours for the domino, there are $b f_{m-1} f_{n-m-1}$ tilings that are unbreakable at m, and $f_m f_{n-m}$ breakable ones. The result follows by the sum rule. $\square$

Example 10: Show that for $n \geq 1$, $f_n - 1 = (a - 1) f_{n-1} + (a + b - 1) \sum_{k=0}^{n-2} f_k$.

Proof. The L.H.S counts all coloured tilings of a $1 \times n$ board excluding the tiling consisting of only 1×1 squares, all of which have the same colour (say, white).

The R.H.S counts the coloured tilings according to the last tile that is not a white square. Suppose the last tile that is not a white square begins on cell k. Then if $k = n$, the remaining $1 \times (n - 1)$ board can be tiled in f_{n-1} ways and there are $(a1)$ choices for the colour of the 1×1 square on the nth cell of the $1 \times n$ board. This explains the first term on the R.H.S.

If $1 \le k \le (n-1)$, the tile covering cell k can be either a 1×1 non-white square or a 1×2 domino covering cells k and $(k+1)$. By the sum rule there are $(a - 1 + b)$ ways to choose this tile, while the previous cells in the $1 \times (k-1)$ board can be tiled in f_{k-1} ways. This yields the remaining term on the R.H.S. Apply the sum rule to conclude the argument. □

Note the similarity in the preceding argument and the typical combinatorial proof of the identity $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$, where the L.H.S counts $(k+1)$-subsets of $[n+1]$ with $(n+1)$ in, then $(n+1)$ not in but n in, then $(n+1)$ and n not in, but $(n-1)$ in, and so on. This can be expressed as the number of $(k+1)$-subsets with largest element i, where $i = k+1, k+2, \ldots, n+1$.

The preceding identity can be generalized further, using the interpretation of coloured tilings. The number of such tilings of a $1 \times n$ board that consist of only 1×1 squares all of colours $1, 2, \ldots, c$ $(c < a)$ is c^n, so $f_n - c^n$ counts all tilings omitting such kind.

Challenge 28: By a similar argument to the preceding identity, show

$$f_n - c^n = (a - c)f_{n-1} + ((a - c)c + b) \sum_{k=0}^{n-2} f_k c^{n-2-k}$$

Challenge 29: Find an analogous argument to show:

A) $f_{2n+1} = a \sum_{k=0}^{n} f_{2k} + (b - 1) \sum_{k=1}^{n} f_{2k-1}$

B) $f_{2n} - 1 = a \sum_{k=1}^{n} f_{2k-1} + (b - 1) \sum_{k=0}^{n-1} f_{2k}$

Hint: Partition according to the last tile that is not a black (say) domino.

Challenge 30: Show that for $n \ge 0$, $f_n f_{n+1} = a \sum_{k=0}^{n} f_k^2 b^{n-k}$

Challenge 31: Show that for $n \ge 1$, $f_n^2 = f_{n+1} f^{n-1} + (-1)^n b^n$