Homework 1: Solving an ODE

Contents

Given an ordinary differential equation:

$$\frac{d}{dt}x(t) = 1 - x(t)$$

with initial condition $x(t=0) = 0.0$

Part 1: Julia’s ODE solver

Please **solve** the ODE using `OrdinaryDiffEq.jl` (optionally `ModelingToolkit.jl`) for $t \in [0, 5]$ and **plot** the time series. **Compare** it to the analytical solution *in one plot*.

Part 2: The forward Euler method

Please try a range of time steps (e.g. from 0.1 to 1.5) to solve the ODE using the forward Euler method **below** for $t \in [0.0, 5.0]$, plot the time series, and compare them to the analytical solution *in one figure*. What happened when dt gets larger? Describe your results and show them in the figure.

About the forward Euler method

We plot the trajectory as a straight line locally. In each step, the next state variables ($\vec{u}_{n+1}$) are accumulated by the time step (dt) multiplied the derivatives (slope) at the current state ($\vec{u}_n$):

$$\vec{u}_{n+1} = \vec{u}_n + dt \cdot f(\vec{u}_n, t_n)$$

The ODE model. Exponential decay in this example.

```
function model(u, p, t)
    return 1 .- u[1]
end
```

```
model (generic function with 1 method)
```

Forward Euler method

Inputs:

- `model`: the ODE model which returns the derivative(s) of the state variable(s)
- `u`: state variable(s)
- `p`: parameter(s)
- `t`: time in the model
- `dt`: time step

Outputs: state variable(s) of the next time step.

```
euler(model, u, p, t, dt) = u .+ dt .* model(u, p, t)
```

```
euler (generic function with 1 method)
```

A simple ODE solver

Inputs:

- `model`: the ODE model which returns the derivative(s) of the state variable(s)
- `u0` : initial conditions of the state variable(s)
- `tspan` : time span in the model, written as `(tstart, tend)`
- `p`: parameters
- `dt`: time step
- `method`: stepping method. Defaults to the Forward Euler method `euler`

```julia
function mysolve(model, u0, tspan, p; dt=0.1, method=euler)
    # Time points
    ts = tspan[begin]:dt:tspan[end]
    # States at those time points
    us = zeros(length(ts), length(u0))
    # Initial conditions
    us[1, :] .= u0
    # Iterations in a for loop
    for i in 1:length(ts)-1
        us[i+1, :] .= method(model, us[i, :], p, ts[i], dt)
    end
    # Return results
    return (t = ts, u = us)
end
```

```
mysolve (generic function with 1 method)
```

Time span, parameter(s), and initial condition(s)

```julia
tspan = (0.0, 5.0)
p = nothing
u0 = 0.0
```

```
0.0
```

Solve the problem

```julia
sol01 = mysolve(model, u0, tspan, p, dt=0.1, method=euler)
sol1 = mysolve(model, u0, tspan, p, dt=1.0, method=euler)
```

```
(t = 0.0:1.0:5.0, u = [0.0; 1.0; … ; 1.0; 1.0;;])
```

Analytical solution

```julia
analytical(t) = 1 - exp(-t)
```

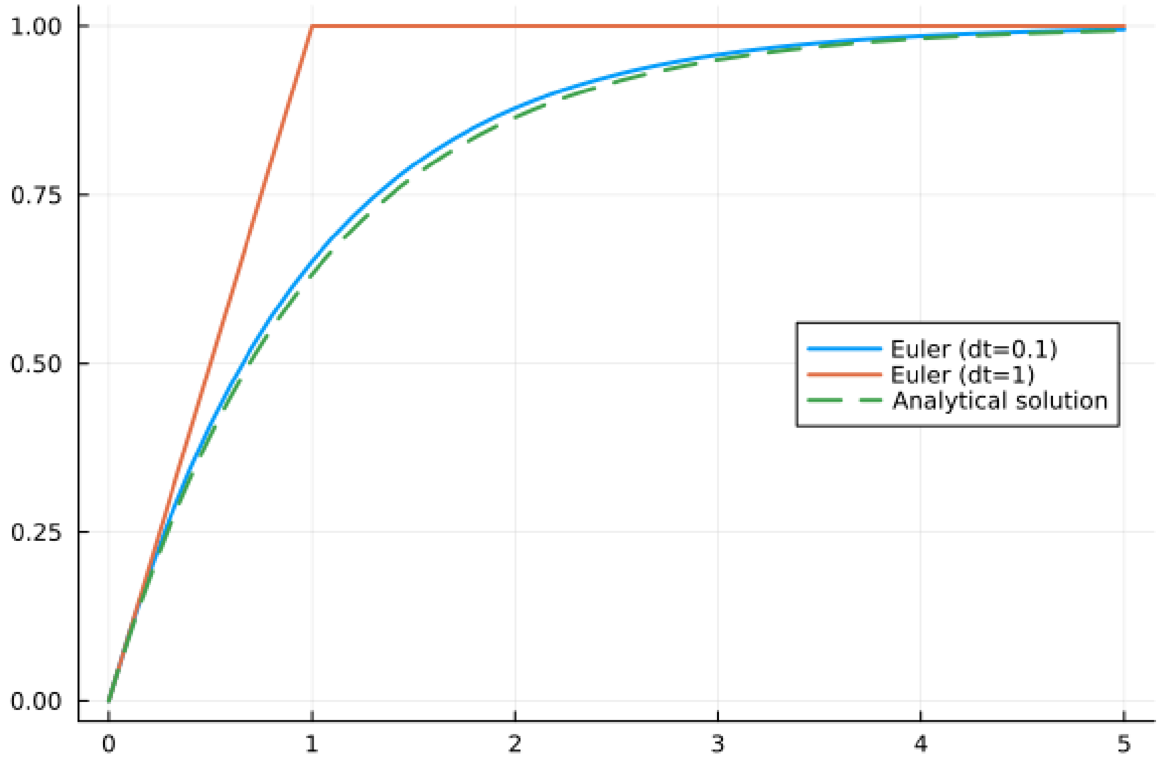
```
analytical (generic function with 1 method)
```

Visualization

```julia
```

```
using Plots
Plots.default(linewidth=2)

plot(sol01.t, sol01.u, label="Euler (dt=0.1)")
plot!(sol1.t, sol1.u, label = "Euler (dt=1)")
plot!(analytical, tspan[begin], tspan[end], label = "Analytical solution", linestyl
```



Part 3: The RK4 method

Please **try** some time steps to **solve** the ODE using the (home-grown) fourth order Runge-Kutta (RK4) method for $t \in [0.0, 5.0]$, **plot** the time series, and **compare** them to the analytical solution and the solution by the Forward Eular method (of the same dt). Which method is more accurate? Please show your results in the figure.

About the RK4 method

We use 4 additional intermediate steps to eliminate some of the nonlinear (higher order) errors in the integration. In each iteration, the next state $\vec{u}_{n+1}$ is:

$$
\begin{aligned}
k_1 &= dt \cdot f(\vec{u}_n, t_n) \\
k_2 &= dt \cdot f(\vec{u}_n + 0.5k_1, t_n + 0.5dt) \\
k_3 &= dt \cdot f(\vec{u}_n + 0.5k_2, t_n + 0.5dt) \\
k_4 &= dt \cdot f(\vec{u}_n + k_3, t_n + dt) \\
\vec{u}_{n+1} &= \vec{u}_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Hint: you can replace the Euler method with the RK4 one to reuse the `mysolve()` function.

```julia
# Forward Euler stepper
euler(model, u, p, t, dt) = u .+ dt .* model(u, p, t)
# Your RK4 stepper
function rk4(model, u, p, t, dt)
    ### calculate k1, k2, k3, and k4 here ###
    next = u .+ (k1 .+ 2k2 .+ 2k3 .+ k4) ./ 6
    return next
end

sol = mysolve(model, u0, tspan, p, dt=1.0, method=rk4)
```

This notebook was generated using [Literate.jl](https://github.com/fredrikekre/Literate.jl).