Algorithms Homework #4

Due May 26, 2022 before class.

Collaboration policy: You can discuss the problem with other students, but you must obtain and write the final solution by yourself. Please specify all of your collaborators (name and student id) for each question. If you solve some problems by yourself, please also specify ”no collaborators”.

Problem 1 (15%)

n different houses are located along a road. The goal of this problem to build some fire stations to protect these houses. Each fire station must be built near one of the houses. Building a fire station near house i costs c_i and can cover houses $a_i, a_i+1, a_i+2, \ldots, b_i$. (You may assume that $a_i \leq i \leq b_i$.) Design an $O(n \log n)$-time algorithm to find the set of fire stations which covers all houses with minimum cost. Briefly justify the correctness and analyze the running time.

Problem 2 (15%)

Given n different currencies $c_1, c_2, \ldots, c_n$. 1 dollar in currency c_i can buy $w_{ij} > 0$ dollars in currency c_j (assuming no transaction fees). Given all exchange rates w_{ij}, we want to find out the best way to buy currency c_n starting with some money in currency c_1.

1. (5%) If the exchange rates are arbitrary and can be unrealistic, does the optimal exchange rate always exist? Why?
2. (10%) Given that the optimal exchange rate exists, design a polynomial time algorithm to find the optimal exchange rate. Briefly justify the correctness of your algorithm and analyze the running time.

Problem 3 (15%)

In this problem, there are n visitors $v_1, v_2, \ldots, v_n$ from other countries and m hotels $h_1, h_2, \ldots, h_m$ that are eligible for quarantine visitors. Each visitor has exactly 3 hotels that he/she is willing to stay in. Given the capacity (maximum number of visitors allowed) of every hotel and the preferences of the visitors, design a polynomial-time algorithm that assigns visitors to hotels or reports that no such assignment is possible. (Each visitor must be assigned to one of the 3 hotels that he/she is willing to stay in.) Briefly justify the correctness and analyze the running time.

Problem 4 (15%)

Given an undirected graph G in which all edge costs are distinct. Let C be any simple cycle in G. Prove the following statements or give counterexamples.

1. (8%) The most expensive edge in C does not belong to the minimum spanning tree.
2. (7%) The cheapest edge in C must belong to the minimum spanning tree.

Problem 5 (15%)

Given a flow network with source s and sink t. We add an extra restriction that the maximum amount of flow that can go through vertex v_i is at most d_i (i.e. the total incoming flow to v_i is at most d_i). Given all edge capacities c_e and all vertex capacities d_i, design a polynomial time algorithm to find the maximum flow from s to t. Briefly justify the correctness and analyze the running time.

For problems 6 and 7, you must either

1. design a polynomial-time algorithm, briefly justify the correctness,

   or

2. prove that the problem is NP-complete. You must describe the reduction in detail and briefly explain the correctness of your reduction.

You may use the fact that SAT, 3-SAT, Vertex Cover, Set Cover, Independent Set, Hamiltonian Path, and Hamiltonian Cycle problems are all NP-complete. Also, you do not need to prove that problems 5, 6 are in NP.

Problem 6 (15%)

Given an undirected graph G. Each edge has a (possibly negative) edge cost. The problem is to determine whether the graph G has a simple cycle of total cost 0.

Problem 7 (15%)

Given a directed acyclic graph G. The problem is to determine whether the graph G has a Hamiltonian path.