~~Problem~~ Problem 1

Algorithms
HW1
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1. $\log_2 n \in O(\log_{10} n)$

Definition $O(g(n)) = \{h(n) \mid \exists c, n_0, \ s.t, \ h(n) \le c g(n), \forall n \ge n_0\}$

$\because \log_2 n = \frac{\log_{10} n}{\log_{10} 2} = 0.3 \log_{10} n \Rightarrow$ pick $c = 0.4$, $n_0 = 1$

$[\, 0.3 \log_{10} n \le 0.4 \log_{10} n, \ \forall n \ge 1 \,]$

$\Rightarrow [\, \log_2 n \le 0.4 \log_{10} n, \ \forall n \ge 1 \,]$ #

$\therefore \log_2 n \in O(\log_{10} n)$ is proved

2. $f(n) \in \Omega(n^3)$, then $f(n) \in \omega(n^2)$

Definition $\Omega(g(n)) = \{h(n) \mid \exists c_1, n_1, \ s.t. \ c g(n) \le h(n), \ \forall n \ge n_1\}$

$\omega(g(n)) = \{h(n) \mid \forall c_2 > 0, \exists n_2 > 0, \ s.t \ c g(n) \le h(n), \ \forall n \ge n_2\}$

take $f(n) = 3n^3$ pick $c_1 = 2$, $n_0 = 1$

$[\, 2n^3 \le 3n^3, \ \forall n \ge 1 \,]$

$\therefore 3n^3 \in \Omega(n^3)$

$\because \forall c_2 > 0$, pick $n_2 = c_2$ $[\, c_2 n^2 \le 3n^3, \ \forall n \ge \underset{c_2}{\overset{\parallel}{n_2}} \,]$

$\therefore 3n^3 \in \omega(n^2)$

$\therefore f(n) \in \Omega(n^3)$ then $f(n) \in \omega(n^2)$ is proved.

Problem 2.

$f_1 = [(2n)!,\ n^n,\ n!]$

$f_2 = (\log_2 n)!$

$f_3 = 2^{2n}$

$f_4 = 2^{3\log_2 n}$

$f_5 = n^3$

$f_6 = [\log_2 n,\ \log_e n]$

$f_7 = n^{0.5}$

$f_8 = n^{0.01}$

$(\log_2 n)! \in \theta((\log_2 n)^{\log_2 n})$

$2^{2n} \in \theta((\log_2 4)^{n\log_2 4})$

$2^{3\log_2 n} \in \theta((\log_2 4)^{3\log_2 n})$

$[f(n),\ g(n)] = [(2n)!,\ n^n]$

$= [(2n)!,\ n!]$

$= [n^n,\ n!]$

$= [\log_2 n,\ \log_e n]$

Problem 3.

1. $T(n) = 9T(\frac{n}{3}) + n^3,\quad T(1) = 5$

$= 9\left[9T(\frac{n}{9}) + \frac{n^3}{3^3}\right] + n^3 = 9^2 T(\frac{n}{9}) + \frac{n^3}{3} + n^3$

$= 9^2\left[9T(\frac{n}{27}) + \frac{n^3}{3^{3\times2}}\right] + \frac{n^3}{3} + n^3 = 9^3 T(\frac{n}{27}) + \frac{n^3}{9} + \frac{n^3}{3} + n^3$

$\vdots$

$= 9^k T(\frac{n}{3^k}) + \frac{n^3}{3^{k-1}} + \frac{n^3}{3^{k-2}} + \cdots + \frac{n^3}{9} + \frac{n^3}{3} + n^3$

$n = 3^k,\ k \in \mathbb{N}$

$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$

$= n^2 T(1) + n^3 \times \left(\frac{1}{1-\frac{1}{3}}\right)$

$1 + \frac{1}{3} + (\frac{1}{3})^2 + \cdots (\frac{1}{3})^{k-1}$

$= n^2 T(1) + n^3 \times \frac{3}{2}$

$= \frac{3}{2} n^3 + 5n^2$

$\therefore T(n) \in \theta(\frac{3}{2} n^3 + 5n^2)$

2. $T(n) = 9T(\frac{n}{3}) + n^2 + 20n\log n + 3,\ T(1) = 5$

Master theorem :

Case ① : $f(n) \in O(n^{\log_b a} n^{-\varepsilon}),\ \varepsilon > 0 \Rightarrow T(n) \in \Theta(n^{\log_b a})$

Case ② : $f(n) \in \Theta(n^{\log_b a} \log^k n),\ k \geq 0 \Rightarrow T(n) \in \Theta(f(n)\log n)$

Case ③ : $\begin{cases} f(n) \in \Omega(n^{\log_b a} n^{\varepsilon}) \\ af(\frac{n}{b}) \leq cf(n),\ c < 1 \end{cases},\ \varepsilon > 0,\ \Rightarrow T(n) \in \Theta(f(n))$

$a = 9,\ b = 3,\ f(n) = n^2 + 20n\log n + 3$

$\in$ Case ②, $f(n) \in \Theta(n^2 \log^k n),\ k \geq 0$

$\therefore T(n) \in \Theta(n^2 \log n)$

Problem 4.

1. $f_1(n), f_2(n), \cdots, f_i(n), \cdots, f_i(n) = O(n)$

$g(k) = \sum_{j=1}^{k} f_j(j)$

$\because \{\exists\ c_i\ n_i,\ s.t.\ f_i(n) \leq c_i n,\ \forall n \geq n_i\}$ for $i = 1$ to n

IF $g(n) = O(n^2)$

$\exists\ c_g, n_g,\ s.t.\ g(n) \leq c_g n^2,\ \forall n \geq n_g$

$$\Rightarrow \underbrace{\underset{\leq c_1}{f_1(1)} + \underset{\leq 2c_2}{f_2(2)} + \cdots + \underset{\leq c_n n}{f_n(n)}}_{c_M(\frac{n(1+n)}{2})} \leq c_g n_g^2$$

$\boxed{\text{take } c_M = \max\{c_1, c_2 \cdots, c_n\}}$

$$c_M \times \frac{n^2+n}{2} \leq c_g n^2$$

pick $c_n = 1$ $c_g = 4$, $n_1 = n_2 = \cdots = n_g = 5$

$$1 \times \frac{n^2+n}{2} \le 4n^2, \forall n \ge 5$$

$\therefore g(n) = O(n^2)$ #

2. $f(n) = O(n) \Rightarrow \{\exists c_1, n_1, \text{ s.t } f(n) \le c_1 n, \forall n \ge n_1\}$

Let $g(k) = \sum_{j=1}^{k} f(j)$

IF $g(n) = O(n^2)$

$\{\exists c_2, n_2, \text{ s.t. } g(n) \le c_2 n^2, \forall n \ge n_2\}$

$$\Rightarrow \underbrace{\underbrace{f(1)}_{\Downarrow\, c_1} + \underbrace{f(2)}_{\Downarrow\, 2c_1} + \cdots + \underbrace{f(n)}_{\Downarrow\, c_1 n}}_{} \le c_2 n^2, \forall n \ge n_2$$

$$c_1\left(\frac{n(1+n)}{2}\right) \le c_2 n^2$$

$$c_1\left(\frac{n^2+n}{2}\right) \le c_2 n^2$$

pick $c_1 = 1$, $c_2 = 4$, $n_1 = n_2 = 1$

$$1 \times \frac{n^2+n}{2} \le 4n^2, \forall n \ge 1$$

$\therefore g(n) = O(n^2)$

Problem 5.

1. $T(n) = T(n-2) + 2T(\lceil \frac{n}{2} \rceil) + n$, for all $n \geq 3$. $T(1) = T(2) = 1$

Guess $T(k) \leq c2^k$ $\forall k$ for some const c

Verify ① base case

$$\underset{1}{T(1)} \leq c \times 2$$

$\Rightarrow$ need $c \geq \frac{1}{2}$

$$T(2) \leq c \times 2^2$$

$\Rightarrow$ need $c \geq \frac{1}{4}$

Assume $T(k) \leq c2^k$, $\forall k < n$

$$T(n) = T(n-2) + 2T(\lceil \frac{n}{2} \rceil) + n$$

$$\leq c2^{n-2} + 2c2^{\frac{n}{2}} + n$$

$$\leq c2^n$$

$\Rightarrow \frac{c}{4}2^n + 2c2^{\frac{n}{2}} + n \leq c2^n$, true when $c \geq \frac{1}{2}$

$\therefore T(n) \in O(2^n)$

Guess $T(k) \geq cn^2$, $\forall k$ for some const c

Verify ① base case

$\underset{1}{T(1)} \geq c \times 1^2 \Rightarrow$ need $c \leq 1$

$T(2) \geq 4c \Rightarrow$ need $c \leq \frac{1}{4}$

Assume $T(k) \geq ck^2$, $\forall k < n$

$$T(n) = T(n-2) + 2T(\lceil \frac{n}{2} \rceil) + n$$

$$\geq c(n-2)^2 + 2c(\frac{n}{2})^2 + n \geq cn^2$$

$$cn^2 - 4cn + 4c + \frac{cn^2}{2} + n \geq cn^2$$

$$\frac{3}{2}cn^2 + (1-4c)n + 4c \geq cn^2, \text{ true when } c \leq \frac{1}{4}$$

$$\therefore T(n) \in \Omega(n^2)$$

2. $T(n) = 3T(\frac{n}{3}) + \frac{n}{2\log_3 n}, \quad T(1) = 1$

$$= 3 \times \left[3 \times T(\tfrac{n}{9}) + \frac{\frac{n}{3}}{2\log_3 \frac{n}{3}}\right] + \frac{n}{2\log_3 n}$$

$$= 3^2 T(\tfrac{n}{9}) + \frac{n}{2(\log_3 n - 1)} + \frac{n}{2\log_3 n}$$

$$= 3^2 \left[3 \times T(\tfrac{n}{3^3}) + \frac{\frac{n}{9}}{2(\log_3 n - 2)}\right] + \frac{n}{2(\log_3 n - 1)} + \frac{n}{2\log_3 n}$$

$$= 3^3 T(\tfrac{n}{3^3}) + \frac{n}{2(\log_3 n - 2)} + \frac{n}{2(\log_3 n - 1)} + \frac{n}{2\log_3 n}$$

$$= \cdots \quad n = 3^k, \ k \in \mathbb{N}$$

$$= 3^k T(\tfrac{n}{3^k}) + \frac{n}{2(\log_3 n - (k-1))} + \frac{n}{2(\log_3 n - (k-2))} + \cdots$$

$$= nT(1) + \frac{n}{2 \times 1} + \frac{n}{2 \times 2} + \cdots + \frac{n}{2 \times \log_3 n}$$

$$= n + \frac{n}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\log_3 n}\right)$$

$$= n + \frac{n}{2}\ln(\log_3 n) \quad \#$$

$$\therefore T(n) \in \Theta\left(n + \frac{n}{2}\ln(\log_3 n)\right)$$

Problem 6

1. $T(n) = 2T(\frac{n}{2}) + f(n)$ and $f(n) = \Theta(n^2)$

by Master Theorem

$a=2,\ b=2,\ f(n) = \Theta(n^2)$

$n^{\log_2 2} = n$

$f(n) = \Theta(n^2) = \Omega(n \times n^{\varepsilon})$, where $\varepsilon = 1$

case 3

$2f(\frac{n}{2}) \le c f(n)$

$2 \times \frac{n^2}{4} - cn^2 \le 0$

$n^2(\frac{1}{2} - c) \le 0$ true for $\frac{1}{2} \le c < 1$

$\therefore T(n) = \Theta(f(n))$ for all $n = 2^k$ is proved !

2. If $T(n) = 2T(\frac{n}{2}) + f(n)$, and $f(n) = \Omega(n^2)$

by master theorem

$a=2,\ b=2,\ f(n) = \Omega(n^2) \Rightarrow$ take $f(n) = \Theta(n \log^n n)$

$\therefore$ case 2

$T(n) \in \Theta(n \log^n n \log n)$

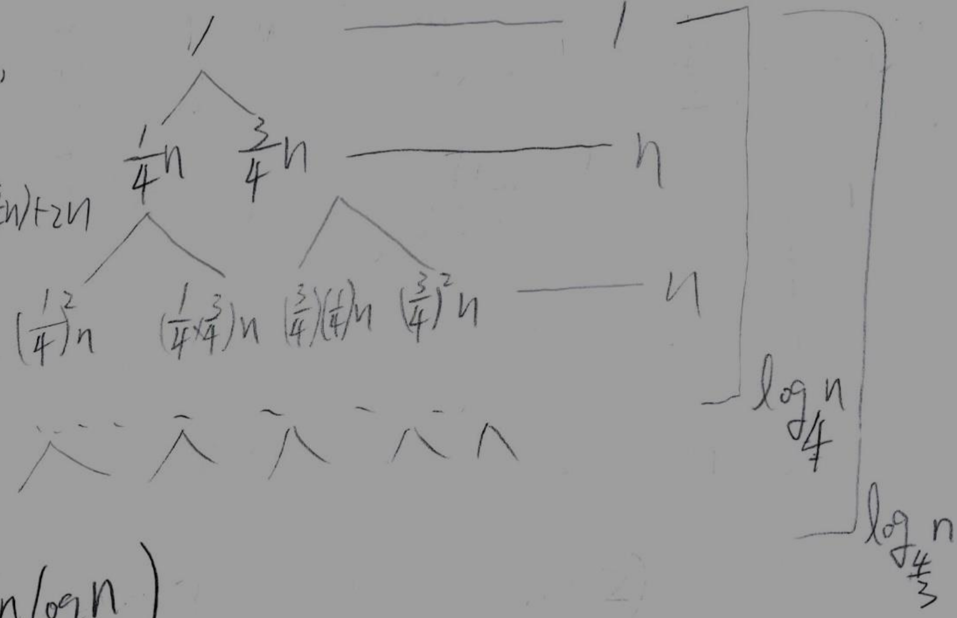
$\therefore T(n) = O(f(n)) = O(n \log^n n)$ is disproved !

Problem 7

1. ① if $n < 4$, solve directly — $\Theta(1)$

② else $n \geq 4$,

$\therefore T(n) = T(\frac{n}{4}) + T(\frac{3}{4}n) + 2n$



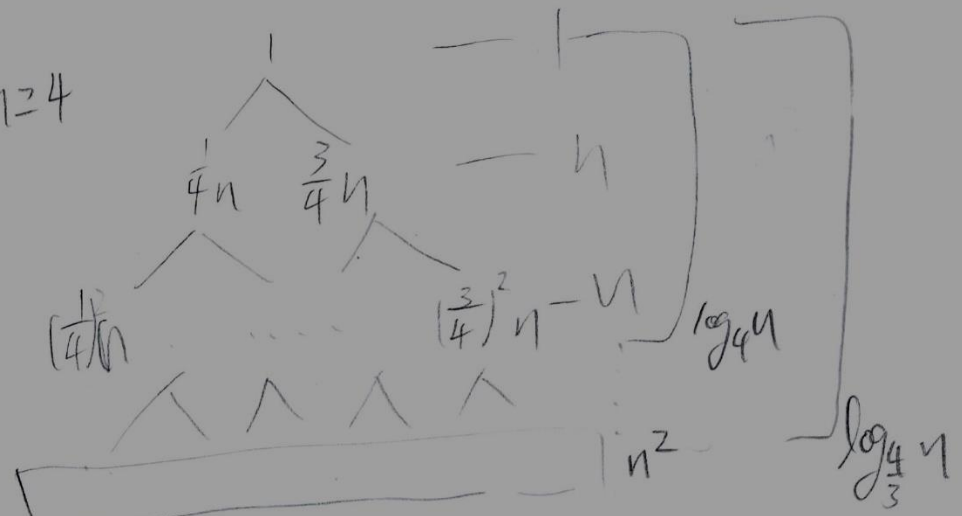
$T(n) = \Theta(n \log n)$

n^2 $T(1)$

n^2

2. ① if $n < 4$, solve directly — $\Theta(1)$

② else $n \geq 4$



$T(n) = \Theta(n^2 + n\log n) = \Theta(n^2)$