# Algorithms Midterm

Apr 07, 2022, 09:10 - 12:10

Please answer the following questions on the answer sheets provided. Be sure to write your name and student ID on all answer sheets you use. This is a closed book exam. No books, notes, or calculators may be used during the exam with the exception of one double-sided, hand-written A4 note. Read all questions first. You may not request for clarification after 9:40am.

You may assume that all pairwise comparisons and arithmetic operations (ex: additions and multiplications) take $\Theta(1)$ time. If you want to apply any result or theorem that has been taught in class (including homeworks), you may do so but you must state the result or theorem clearly before using it.

**Problem 1** Let all functions be positive functions. Prove or disprove the following three statements. You may only use the definitions of asymptotic notations. Any other property of asymptotic notations must be proved before using. Answering true/false without any explanation will not receive any credit.

1. (4%) $(\frac{1}{5}n)^2 \notin O(n)$.
2. (8%) If $f(n) \in O(h(n))$ and $g(n) \in O(h(n))$, then $\max\{f(n), g(n)\} \in O(h(n))$.
3. (8%) If $f(n) \notin \Omega(h(n))$ and $g(n) \notin \Omega(h(n))$, then $\max\{f(n), g(n)\} \notin \Omega(h(n))$.

**Problem 2** Solve the following recurrences. You only need to obtain the asymptotic solution (in $\Theta()$ notation). If you use the master theorem, you must specify all parameters and briefly verify all conditions.

1. (5%) $T(n) = T(\frac{n}{3}) + \log_3 n$, $T(1) = 1$.
2. (10%) $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lfloor \frac{n}{3} \rfloor) + T(\lfloor \frac{2n}{3} \rfloor) + n^2$, for all $n \geq 2$. $T(0) = T(1) = 1$.

**Problem 3** Given $2n + 1$ real numbers $a_1, a_2, \ldots, a_{2n+1}$ (unsorted). Every number appears exactly two times except one number (which only appears once). Only pairwise comparisons between these numbers are allowed.

1. (15%) Design an $O(n)$-time algorithm which finds the number that only appears once. Briefly justify the correctness and analyze the running time.
2. (15%) Formally prove that any algorithm which finds all $n$ pairs of identical numbers requires $\Omega(n \log n)$ time.

400+360 180 160 10 30 30 20

760 340 40 40 40 40 40 40

n=100 10 9 8 76

$a_i d_i \le \frac{n}{3} \le \frac{2}{3} a_i d_i$

$\frac{P_{greedy}}{3}$ $P_{opt}$

$n > a_i > 0$ $a_i > \frac{n}{3}$

Problems 4 and 5 are on the GreedyOne restaurant problem taught in class. The restaurant has $n$ seats. There are $m$ groups of people $G_1, G_2, \ldots, G_m$. Group $G_i$ has $a_i$ people and will provide a profit of $d_i$ dollars per person. The goal is to accept some of these groups to maximize the total profit subject to the constraint that the total number of people is at most $n$. For simplicity, assume that $d_1 \ge d_2 \ge \cdots \ge d_m$ and $a_j < n$ for every $j$.

$P_g \le \frac{2}{3} P_{opt}$

$P_{greedy} \ge \frac{2}{3} P_{opt}$

**Problem 4** Consider the greedy algorithm which selects groups $G_1, G_2, \ldots, G_k$ until group $G_{k+1}$ does not fit into the restaurant. (In the lectures, we have shown an example in which the profit of the algorithm is much worse than the optimal solution.) Denote the profit of the algorithm by $P_{greedy}$ and the profit of the optimal solution by $P_{opt}$.

1. (10%) Given an extra assumption that $a_i \le \frac{n}{3}$. Prove that $P_{opt} \le \frac{3}{2} P_{greedy}$ or provide a counterexample.
2. (5%) Given an extra assumption that $a_i \le \frac{n}{4}$. Prove that $P_{opt} \le \frac{5}{4} P_{greedy}$ or provide a counterexample.

496 125 75 200 $\times \frac{5}{4}$ 100 25 25 25 25 99 1 40 40 40 5 1 1 1 1 $\frac{2}{3} P_{opt} \le P_{greedy}$ $a_i > \frac{n}{3}$

**Problem 5**

Gi+2

1. (10%) Suppose we add an extra requirement that groups $G_i$ and $G_{i+1}$ cannot both be accepted, for every $i = 1, 2, \ldots, m-1$. Design an algorithm which finds the maximum profit and the corresponding choices of groups. The time complexity of your algorithm must be polynomial in $mn$. Briefly justify the correctness and analyze the running time.
2. (10%) Suppose we add an extra requirement that at least $k$ groups must be accepted, for a given constant $k$. Design an algorithm which finds the maximum profit. The time complexity of your algorithm must be polynomial in $mnk$. Briefly justify the correctness and analyze the running time. Notice that you do not need to find the optimal set of groups in this problem. Finding the maximum profit suffices.

$P_{opt} > \frac{4}{3} d_i$

340 $680 \times \frac{3}{2}$

99 $33 \times 5$ $680 \times \frac{5}{4}$

165+1 166 +33+33 199 $116 \times 3$ $232 \times \frac{3}{2}$ 348

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Administrative issues:

1. The exam score distribution will be announced by email before 4/21(Thu). Exam solutions (video) will be announced on COOL.

2. You may check exam scores on 4/21(Thu) afternoon from 14:00-16:00 in my office (MD 718).

3. If you think you are not performing well in the exam, you may redo this exam as a homework (all homework rules apply) and submit *online* before 04/14(Thu) midnight. This extra work will NOT affect your letter grade unless your total score for the whole semester is one of the following 3 cases:

a) an undergrad student with total score 55-60

b) a grad student with total score 65-70

c) a PhD student doing qualify exam subsitution with score 75-80 (please email me)

In the above three cases, you will be raised to the lowest passing grade if you perform reasonably well in this extra homework.