GeneratingSeaSurface

This function based on a Monte-Carlo based method for generating one-way propagating ocean wave with given DWS (directional wave spectrum) model. This function is coded following the patent of invention by Chen Ping, Zou Shihao 2018.

- Chen Ping, Zou Shihao. “A Monte-Carlo-based method to generate one-way propagating ocean waves.” China, PatentNo. 201811400287.1

0. Preliminary Knowledge

The DWS (directional wave spectrum) is defined as the Fourier transform of the autocorrelation of sea surface. i. e.

$$F(\overrightarrow{k}) = FT\left(\left\langle \xi(\overrightarrow{x})\xi^*(\overrightarrow{x} + \overrightarrow{\Delta x})\right\rangle\right)$$

x^* denotes the conjugate of x, $< \ldots >$ denotes the set average.

1. Monte-Carlo Method

At time t, a Monte-Carlo method establishes the sea surface at given position through the following equation [1]:

$$\xi(x_p, y_q, t) = \Sigma_m \Sigma_n a_{mn} \frac{b_{mn}}{2} exp\left[j\left(k_m x_p + k_n y_q - \omega_{mn} t + \phi_{mn}\right)\right]$$
$$k_m = \frac{2\pi}{L_x} \cdot m$$
$$k_n = \frac{2\pi}{L_y} \cdot n$$
$$x_p = \Delta x \cdot p$$
$$y_q = \Delta y \cdot q$$

$$m, p = -\frac{M}{2}, -\frac{M}{2} + 1, \ldots \frac{M}{2}. \text{ where } M = \frac{L_x}{\Delta x}$$

$$n, q = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots \frac{N}{2}. \text{ where } N = \frac{L_y}{\Delta y}$$

k_m, k_n denotes the ***m***th ***n***th discreted value of wavenumber along k_x, k_y axis. x_p, y_q denotes the ***p***th ***q***th discreted value of space position along x, y axis. ω_{mn} denotes the angular frequency which is related to wavenumber by dispersion equation $\omega = \sqrt{kg}, g = 9.8m/s^2$. ϕ_{mn} denotes the initial phase following uniform distribution and $\phi_{mn} = \phi_{-m-n}$. L_x, L_y denotes the simulated size of ocean area. b_{mn} denotes a random variable with none unit subject to **Rayleigh distribution** with

$\mu = 0, \sigma = 1$. a_{mn} denotes the normalized amplitudes as a function of wavenumber $\overrightarrow{k}$. Its expression is:

$$a_{mn} = \sqrt{2F(k_m, k_n)\Delta k_x \Delta k_y}$$
$$\Delta k_x = \frac{2\pi}{L_x}$$
$$\Delta k_y = \frac{2\pi}{L_y}$$

$\Delta k_x \Delta k_y$ denotes the resolution along k_x, k_y axis. $F(k_m, k_n)$ denotes the value of **non symmetric DWS** value at given position in wavenumber domain, its unit is m^4.

2. Correctness of this Method

How do we know if the generated sea surface is correct? The simplest way is to calculate the DWS of generated sea surface. If the DWS coincides with the given ground truth, then the generated sea surface is correct. The correctness of this method is insured theoretically. Now, let's work it through.

First, we derive the autocorrelation of a given sea surface generated by the above method, which would be like this:

$$\left\langle \xi(\overrightarrow{x})\xi(\overrightarrow{x} + \overrightarrow{\Delta x}) \right\rangle = \Sigma_{m_1,n_1,m_2,n_2} \frac{\langle b_{m_1 n_1} b_{m_2 n_2} \rangle}{4} \cdot X$$

Now, let's ignore what does X look like and pay attention to the first term at the right side. We know that b_{mn} is an indenpendent and identical variable. That gives us $\langle b_{m_1 n_1} b_{m_2 n_2} \rangle \neq 0$ only when $m_1 = m_2$ and $n_1 = n_2$. With this condition, we are able to simplify the expression of X, then we have:

$$\left\langle \xi(\overrightarrow{x})\xi(\overrightarrow{x} + \overrightarrow{\Delta x}) \right\rangle = \Sigma_{m,n} a_{mn}^2 \frac{\langle b_{mn}^2 \rangle}{4} exp\left[j\left(k_m \Delta x + k_n \Delta y\right)\right]$$

The second original moment of Rayleigh distribution is $2\sigma^2$. Substitute it into the above equation:

$$\left\langle \xi(\overrightarrow{x})\xi(\overrightarrow{x} + \overrightarrow{\Delta x}) \right\rangle = \Sigma_{m,n} \frac{a_{mn}^2}{2} exp\left[j\left(k_m \Delta x + k_n \Delta y\right)\right]$$

The Fourier transform FT is defined as:

$$FT\left(g(x)\right) = \frac{1}{2\pi} \int g(x) e^{-jkx} dx$$

Therefore, the FT of the autocorrelation of sea surface is:

$$
\begin{aligned}
FT(< \dots >) &= \frac{1}{4\pi^2}\iint \Sigma_{m,n}\frac{a_{mn}^2}{2}exp\left[j\left(k_m\Delta x + k_n\Delta y\right)\right]exp\left[-j\left(k_{m'}\Delta x + k_{n'}\Delta y\right)\right]dxdy \\
&= \frac{1}{4\pi^2}\cdot F(k_m, k_n)\Delta k_x\Delta k_y\cdot L_xL_y \\
&= F(k_m, k_n)
\end{aligned}
$$

See, the generated spectrum is exactly equal to the ground truth. That insured the correctness of this Monte-Carlo method.

3. Construction Unsymetric DWS

It should be noticed that to generate a one-way propagating ocean wave, we shoule first construct a **non symmetric DWS model**. However, all the empirical wave models are symmetric, including PM, DV, EL, JSON. That means $F(\overrightarrow{k}) = F(\overrightarrow{-k})$. In this way, the opposite propagating waves split the energy averagely which causes an odd wave that would not transmit over time. See Fig.1 Below.

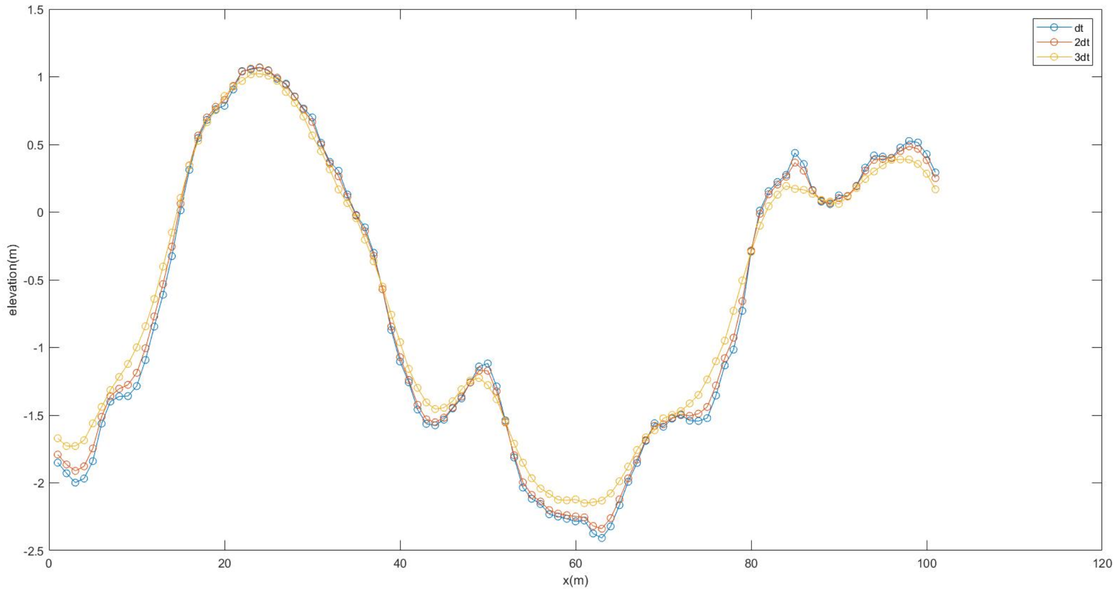

Fig.1 Generated ocean wave using symmetric DWS (not propagating)

To insure conservation of energy meanwhile eliminate the one-way propagation of ocean waves, the fake direction area should be set to 0, and allocate all the energy to direction toward which the wave is supposed to propagate. Mathematically, 2 steps are taken:

$$
\begin{aligned}
&\text{For all } \Phi \text{ that satisfies } |\Phi - \Phi_0| \geq 90^o, F(K,\Phi) = 0. \\
&\text{For all } \Phi \text{ that satisfies } |\Phi - \Phi_0| < 90^o, F(K,\Phi) = 2F(K,\Phi).
\end{aligned}
$$

Φ_0 denotes the peak wave pdrection for one single wave model. Cause you have to take the above 2 steps for each wave model if a mixed sea condition is to simulate. With the processed DWS, the generated wave is one-way propagating. See Fig.2

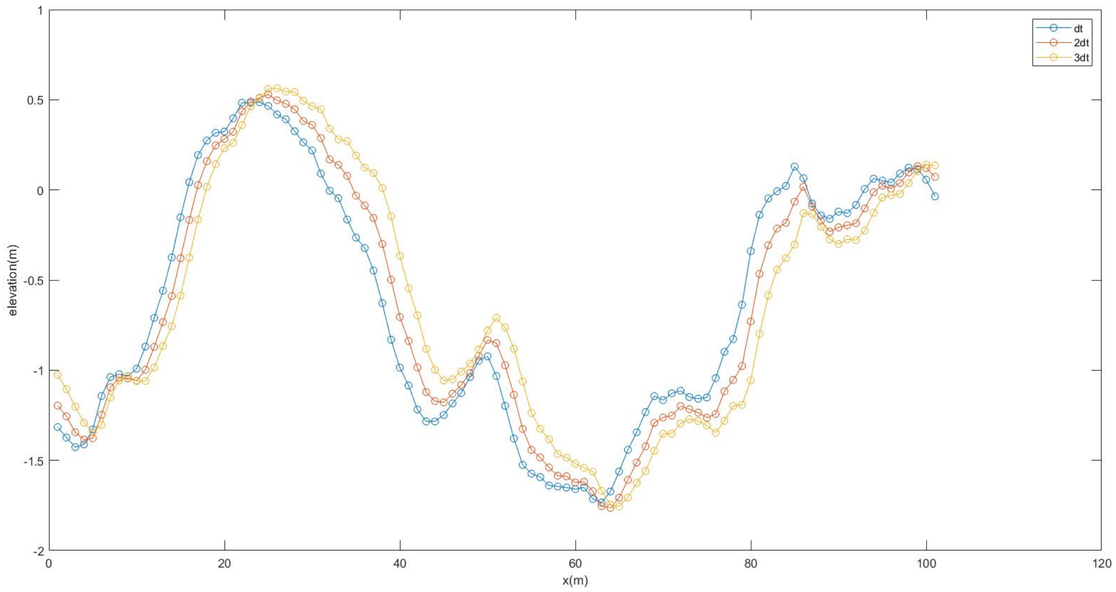

Fig.2 Generated ocean wave using Unsymmetric DWS (propagating)

4. Relationship between the IFT and MATLAB's IFFT2 function

According to the definition of DWS, the sea surface is actually an inversion Fourier transform of a certain function related to DWS:

$$\xi(\overrightarrow{x},t)=\int F(\overrightarrow{k})e^{-\omega(\overrightarrow{k})t+\phi(\overrightarrow{k})}e^{j\overrightarrow{k}\overrightarrow{x}}d\overrightarrow{k}$$

Thus we can speed up the sea surface generation simply by applying MATLAB IFFT2 function. However, you should notice that IFFT2 function does not do the IFT as we wish. Refering to the guide book of MATLAB, you can see that the IFFT of an array of length M is done by the following form:

$$X(p)=\frac{1}{M}\Sigma_m Y(m)e^{i\frac{2\pi(p-1)(m-1)}{M}}$$

This definition is totally different from what it is for continuous function. Thus, a correction msut be done after the IFFT is used. Now, let's derive the correction factor. We first discrete the continous definition of inverse Fourier transform below:

$$\xi(x_p)=DIFT(F(k_m))=\Sigma_m F(k_m)e^{i\cdot k_m x_p}\Delta k$$

For future convenience, we define some new discrete variables here:

$$k_m=-\frac{\pi}{\Delta x}+\Delta k\cdot m$$
$$\Delta k=\frac{2\pi}{L_x}$$
$$x_p=-\frac{L_x}{2}+\Delta x\cdot p$$

$$m, p = 0, 1, 2, 3...M, M = \frac{L_x}{\Delta x}$$

Now, substituting the above variables into the DIFT (discrete inverse Fourier transform), and assuming that M is a multiple of 4, we have:

$$\begin{aligned}\xi(x_p) = DIFT(F(k_m)) &= \Sigma_m F(k_m) e^{i\cdot\left(M\frac{\pi}{2} - p\pi - m\pi + \frac{pm}{M}2\pi\right)} \frac{2\pi}{L_x} \\ &= \frac{2\pi}{\Delta x} e^{-p\pi} \left\{ \frac{1}{M} \Sigma_m F(k_m) e^{-m\pi} e^{i\cdot\left(\frac{pm}{M}2\pi\right)} \right\}\end{aligned}$$

Comparing the above equation to the IFFT in MATLAB, the correction devided into 3 steps: First, we times a sign matrix to the original function F, then applying IFFT in MATLAB, after that an sign matrix is multiplied again with one more factor $\frac{2\pi}{\Delta x}$ to correct the output of IFFT. That is why there are 2 sign matrix assigned successively in the source code of function `GeneratingSeaSurface`. the sign matrix looks like this:

$$\left\{ \begin{matrix} 1 & -1 & 1 & -1 & ... \\ -1 & 1 & -1 & 1 & ... \\ ... \end{matrix} \right\}$$