Robotics

Estimation and Learning
with Dan Lee

Basic Intro to Probability

Penn Engineering

Why Learn About Probability?

- The real world has huge aspects of randomness and uncertainty.
- Still, we hope to make useful predictions and inferences.
- Randomness often follows reliable laws.
- The language of these laws is the language of probability (and statistics).

Contents

1. Definition of Probability

- Consider an (potentially abstract) experiment
  - A **sample space** Ω is the set of *all possible* outcomes of that experiment
  - An **elementary event** ω is a single outcome of the set.
- Example (Rolling two dice):
  - Sample space $\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,5), (6,6)\}$
  - Each member $\omega \in \Omega$ is an elementary event
  - Example of non-elementary event : Rolling doubles

    $B = \{(1,1), (2,2), (3,3), \ldots, (6,6)\}$

1. Definition of Probability

- Consider a finite* set Ω
- A **probability space** (Ω, P) is a sample space Ω, together with a function P, satisfying the following:
  i. $0 \leq P(\omega) \leq 1$ for all $\omega \in \Omega$
  ii. $\sum_{\omega \in \Omega} P(\omega) = 1$
  iii. For any event $A \subseteq \Omega$, $P(A) = \sum_{\omega \in \mathrm{A}} P(\omega)$
- The function P is called **probability measure**.

*To deal with countably infinite or uncountable space, we need third element called sigma algebra, but here we are simplifying.

1. Definition of Probability

- Basic Consequences

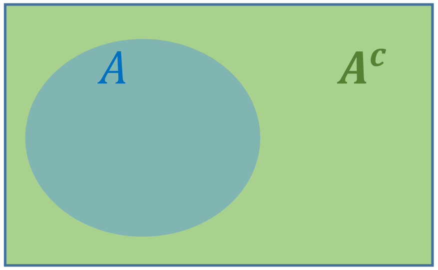
$$P(\emptyset) = 0$$

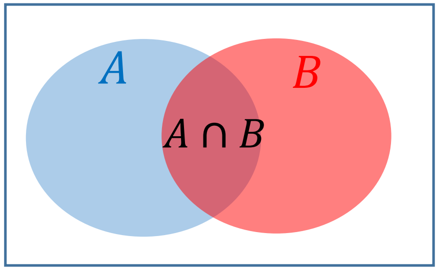
$$P(\Omega) = 1$$

$$P(A^C) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$





2. Independence

- Given a probability space, two events A and B are **independent** if and only if:

$$P(A \cap B) = P(A)P(B)$$

- Two events are dependent if they are not independent.

2. Independence

- Example (two coin flip): $\Omega = \{HH, HT, TH, TT\}$
  - Assume $\forall \omega \in \Omega$, $P(\omega) = \frac{1}{|\Omega|} = \frac{1}{4}$.
  - Define event A: "First flip is H." $A = \{HH, HT\}$
  - Define event B: "Second flip is H." $B = \{HH, TH\}$
  - Are A and B Independent?

    i) $P(A \cap B) = P(\{HH\}) = 1/4$

    ii) $P(A)P(B) = 1/2 * 1/2 = 1/4$ → Yes

2. Independence

- Example (two coin flip): $\Omega = \{HH, HT, TH, TT\}$
    - Assume $\forall \omega \in \Omega$, $P(\omega) = \frac{1}{|\Omega|} = \frac{1}{4}$.
    - Define event A: "First flip is H." $A = \{HH, HT\}$
    - Define event B: "Contains a T." $B = \{HT, TH, TT\}$
    - Are A and B Independent?

        i) $P(A \cap B) = P(\{HT\}) = 1/4$

        ii) $P(A)P(B) = 1/2 * 3/4 = 3/8$ → No

3. Conditional Probability

- Given some probability space (Ω, P), for any two events A and B, if $P(B) \neq 0$, then we define the **conditional probability** $P(A|B)$ that A occurs given that B occurs as,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- From this follows **Chain Rule**:

$$P(A \cap B) = P(A|B)\,P(B)$$

$$\begin{aligned} P(A \cap B \cap C) &= P(A|B \cap C)\,P(B \cap C) \\ &= P(A|B \cap C)\,P(B|C)P(C) \end{aligned}$$

and so on..

3. Conditional Probability

- Example (two coin flip): What is the probability that both are head, GIVEN at least one is head?
- Probability Problem:

$\Omega = \{HH, HT, TH, TT\}$
$B = \{HH, HT, TH\}$ → "At least one is head."
$A = \{HH\}$ → "Both are heads."
$P(A \cap B) = P(\{HH\}) = 1/4$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

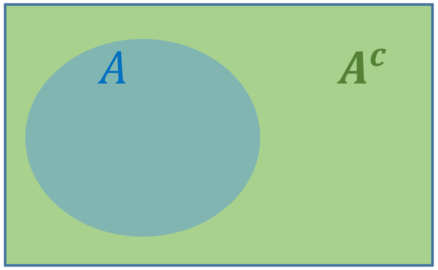
3. Conditional Probability

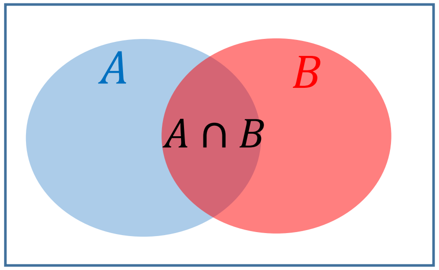
- Consequences

$$P(\emptyset|B) = 0$$

$$P(\mathrm{B}|\mathrm{B}) = 1$$

$$P(A|B) = 1 - P(A^C|B)$$





4. Bayes Rule

- From chain rule,
  $P(A \cap B) = P(A|B)\, P(B)$
  $P(B \cap A) = P(B|A)\, P(A)$
- Intersections are commutative, $A \cap B = B \cap A$.

$$
\begin{array}{ccccc}
P(A \cap B) & = & P(A|B) & P(B) \\
\| & & \| & \\
P(B \cap A) & = & P(B|A) & P(A)
\end{array}
$$

- As a result, we have **Bayes Rule**:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

4. Bayes Rule

- Each term is often called:

$$\underbrace{P(A|B)}_{\text{Posterior}} = \frac{\overbrace{P(B|A)}^{\text{Likelihood}}\overbrace{P(A)}^{\text{Prior}}}{\underbrace{P(B)}_{\text{Evidence}}}$$

5. Random Variables

- Given some probability space (Ω, P), a **random variable** $X: \Omega \to R$ is a *function* that maps the sample space to the reals.
- When we say $P(X = a)$, we actually mean the probability of the inverse image $X^{-1}(a)$. That is,

$$P(X = a) = P\left(X^{-1}(a)\right) = P(\{\omega \in \Omega | X(\omega) = a\})$$

- Example (single coin flip): $\Omega = \{Head, Tail\}$

$$X(Head) = 1, X(Tail) = 0$$

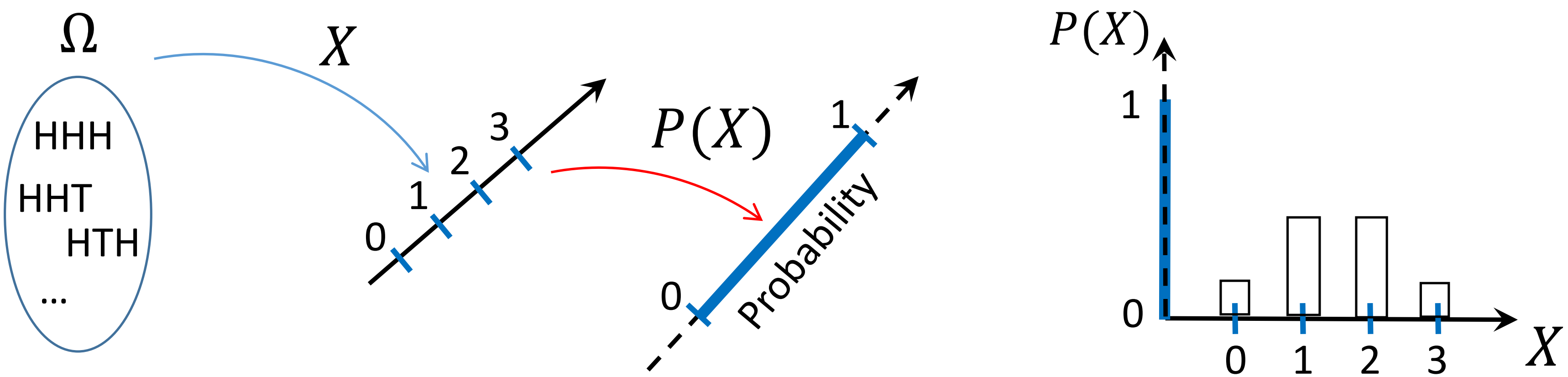
$$P(X = 1) = P\left(X^{-1}(1)\right) = P(Head)$$

5. Random Variables

- Example: 3 coin flips
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - Let us define $X(\omega)$ to be the number of Heads in a given flip. Then,

    $X(HHH) = 3, X(HHT) = X(HTH) = 2, \ldots, X(TTT) = 0$

    $P(X = 3) = 1/8, P(X = 2) = P(X = 1) = 3/8, P(X = 0) = 1/8$

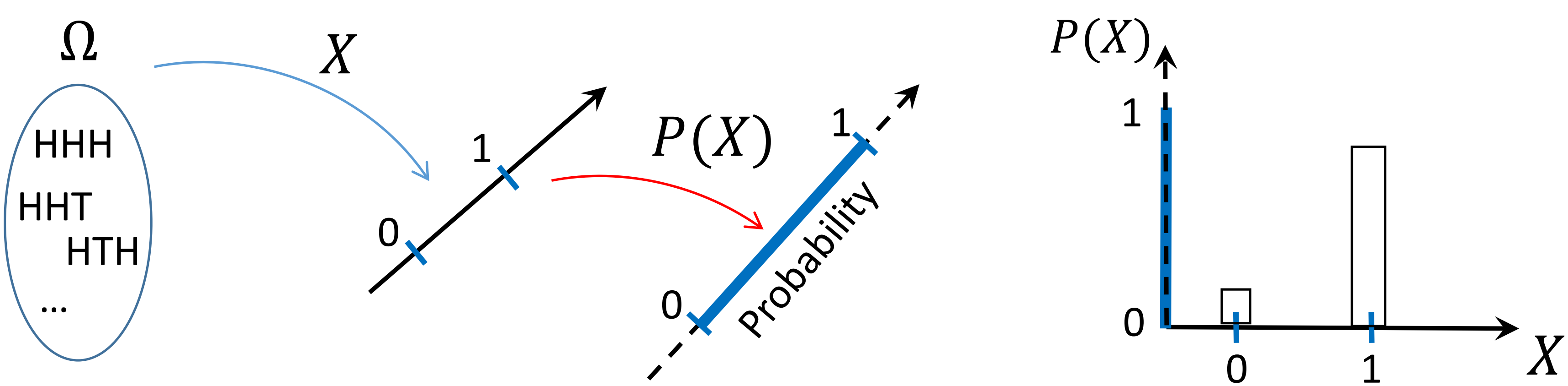


5. Random Variables

- Example: 3 coin flips
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - This time, let us define $X(\omega)$ to be 1 if H appears in a given flip, otherwise is 0. Then,

    $X(HHH) = X(HHT) = \cdots = 1\,, X(TTT) = 0$

    $P(X = 1) = 7/8, P(X = 0) = 1/8$

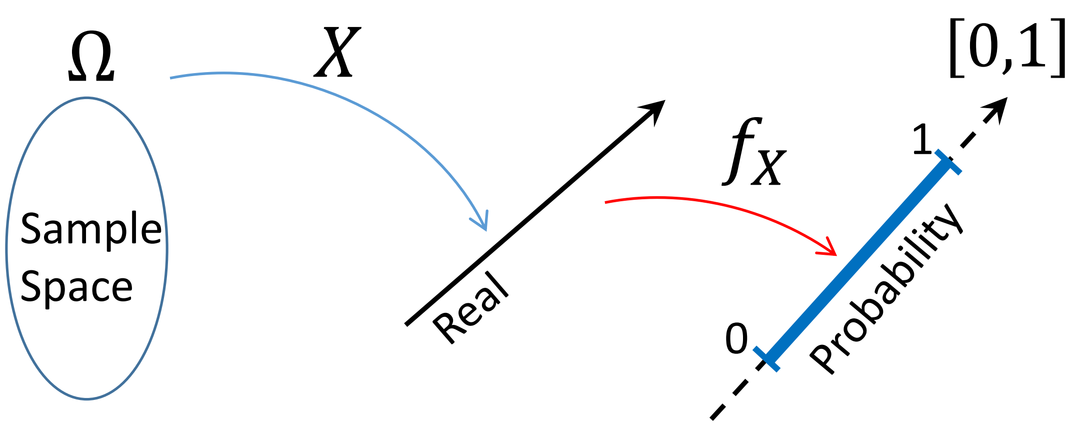


6. Density/Distribution Functions

- **Probability mass function** (pmf) of **discrete** RVs
- **Probability density function** (pdf) of **continuous** RVs

$f: R \rightarrow [0,1]$

$\forall a \in R, f_X(a) = P(X = a)$



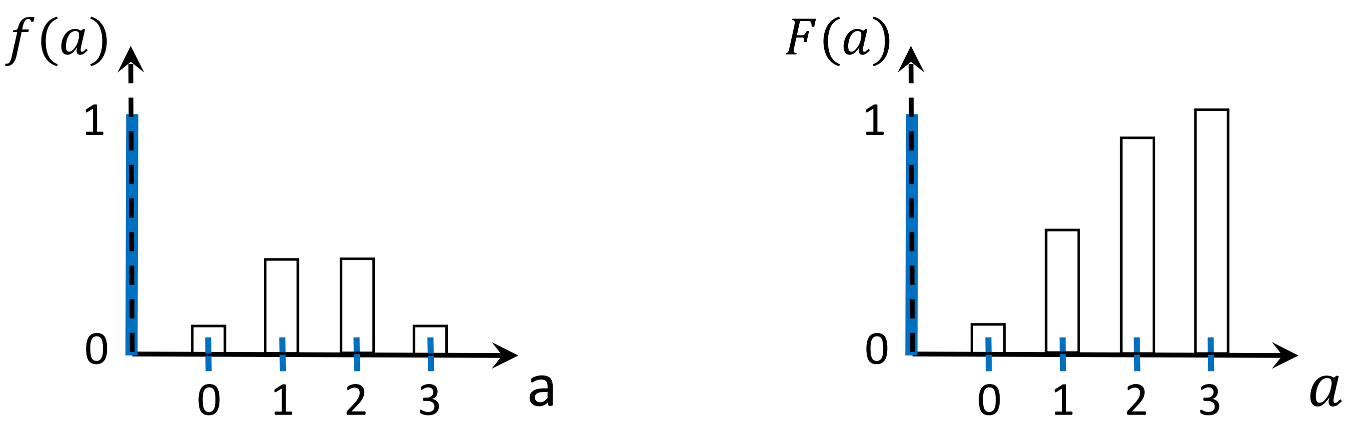
6. Density/Distribution Functions

- **Cumulative distribution functions** (cdf)

  $F: R \to [0,1]$

  $\forall a \in R, F_X(a) = P(X \leq a)$
- A cdf is a monotonic nondecreasing function, i.e.,

  $\forall x \leq y, F(x) \leq F(y)$



Acknowledgement

- Thanks to Daniel Moroz, Dan Lee's master student at the University of Pennsylvania, for allowing us to take parts of contents from his slides.

Robotics

Estimation and Learning
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Week 1.
Gaussian Model Learning

1.2.1 1D Gaussian Distribution

Gaussian Distribution

Why Gaussian?

- The two parameters (mean and variance) are easy to compute and interpret.

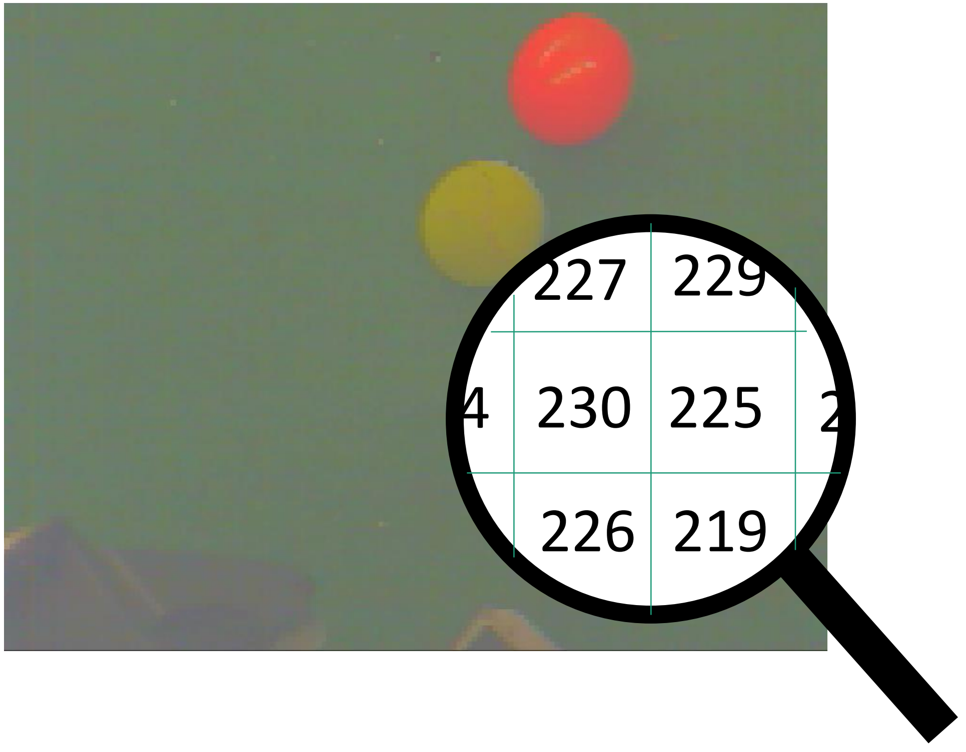
- Good mathematical properties:
  e.g., product of Gaussian distributions forms Gaussian.

- Central limit theorem:
  Expectation of the mean of any random variables converges to Gaussian.

Gaussian Distribution : Example

Ball color distribution

Color Image

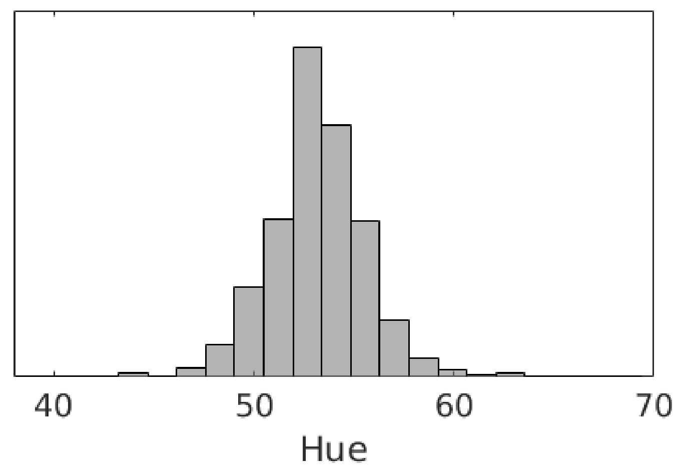


“Yellow”?
“Red”?

Gaussian Distribution : Example

Ball color distribution

Color Image



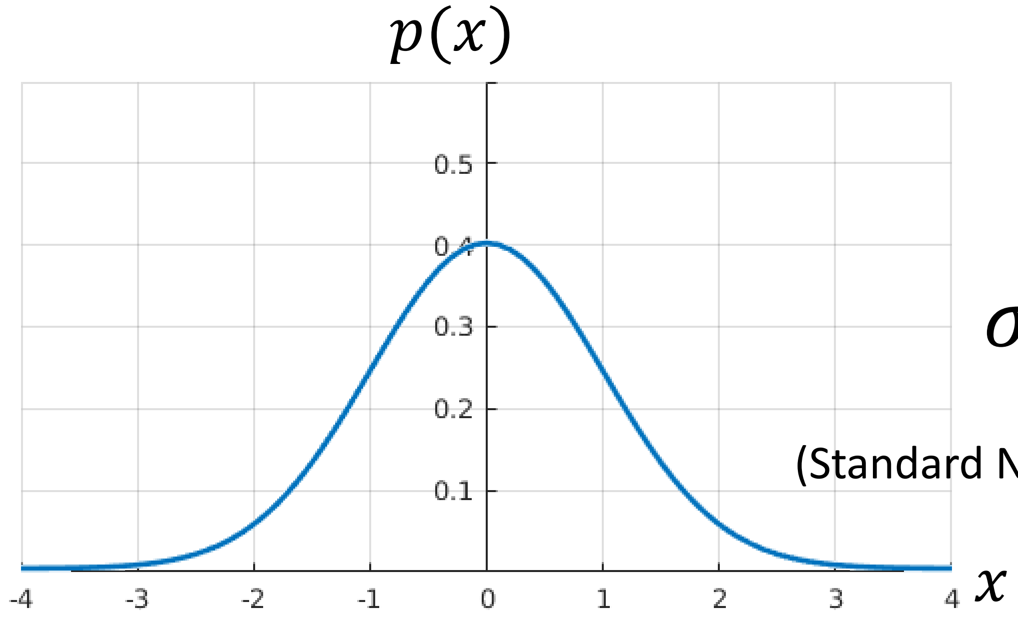
Gaussian Distribution (1D)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

x	Variable
μ	Mean
σ^2	Variance
σ	Standard deviation

Gaussian Distribution (1D)

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



$$\mu = 0$$

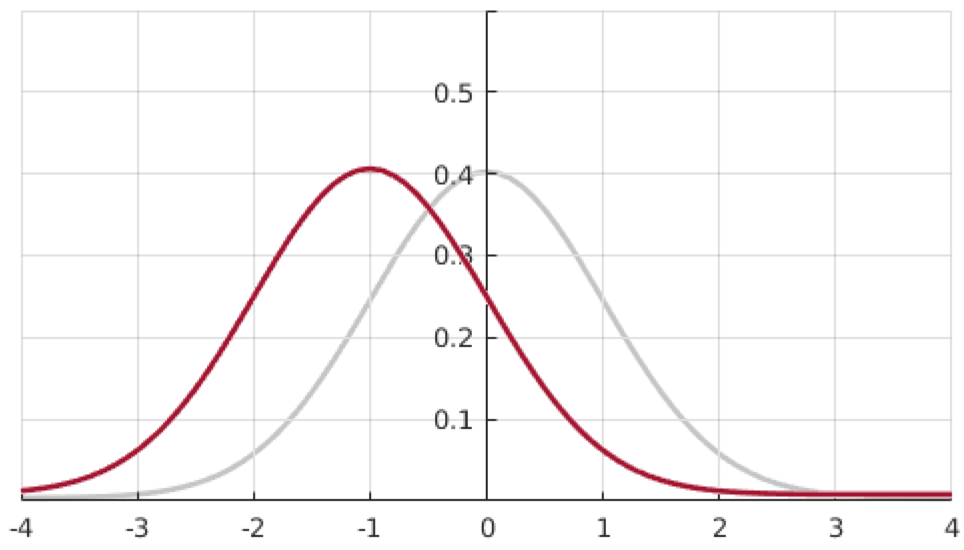
$$\sigma^2 = 1$$

(Standard Normal Distribution)

Gaussian Distribution (1D)

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}}$$

$p(x)$



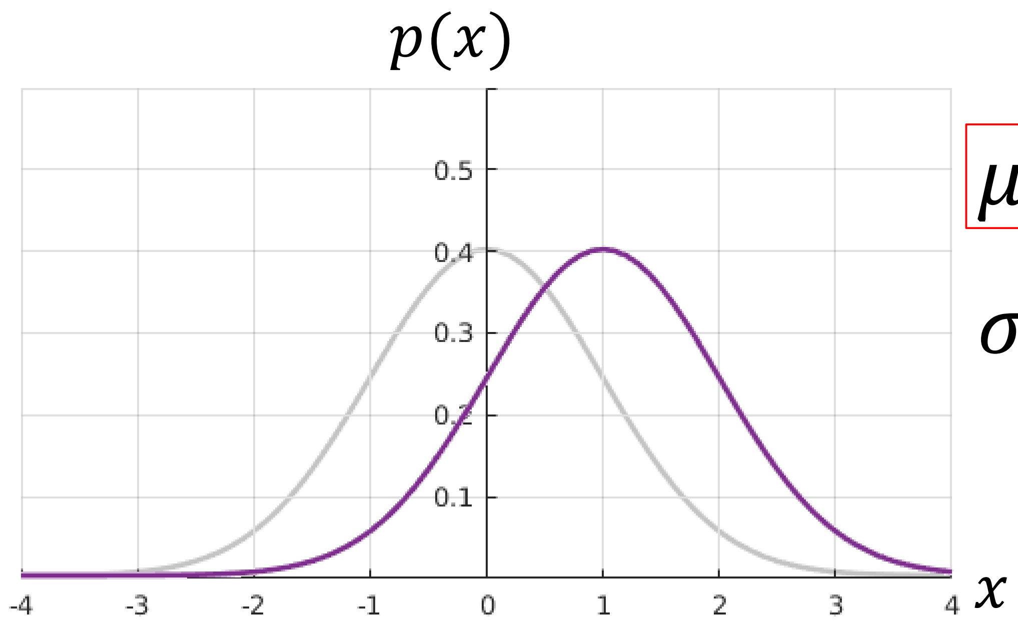
x

$\boxed{\mu = -1}$

$\sigma^2 = 1$

Gaussian Distribution (1D)

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

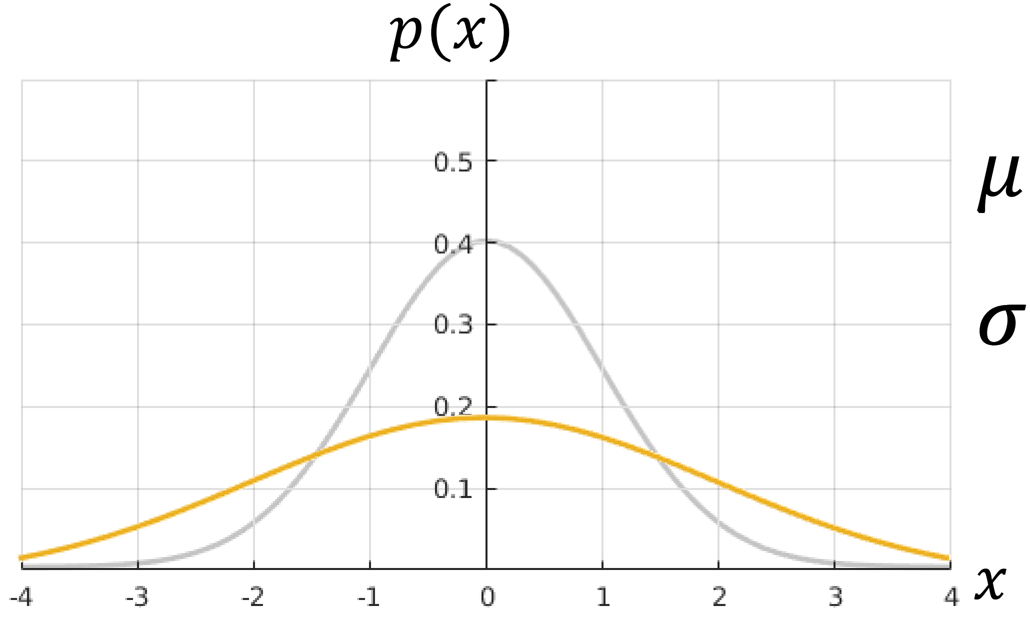


$\boxed{\mu = 1}$

$\sigma^2 = 1$

Gaussian Distribution (1D)

$$p(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}$$

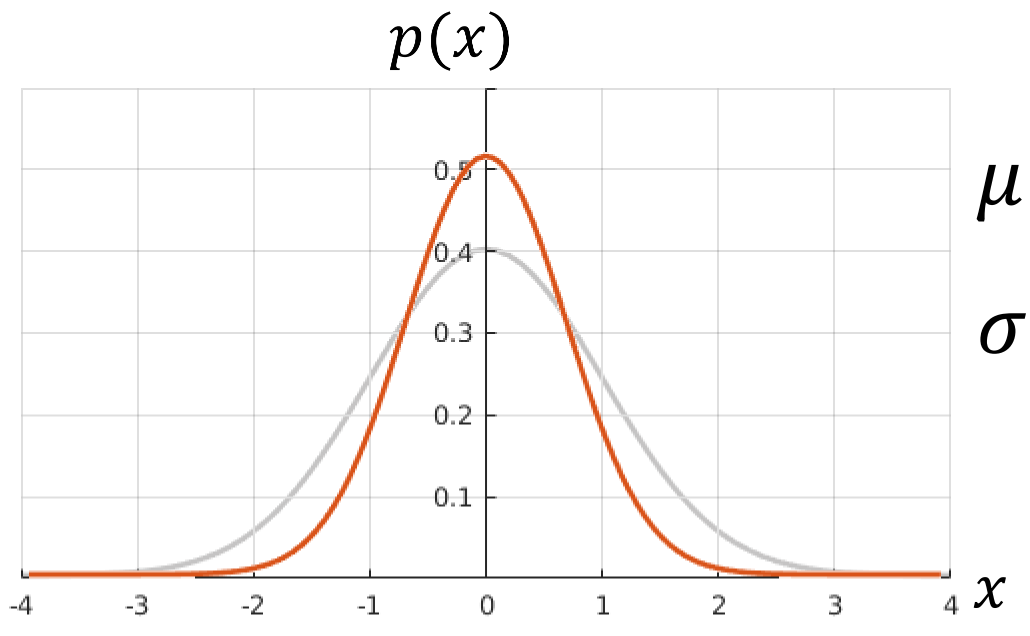


$\mu = 0$

$\sigma^2 = 2$

Gaussian Distribution (1D)

$$p(x) = \frac{1}{\sqrt{1.4\pi}} e^{-\frac{x^2}{1.4}}$$

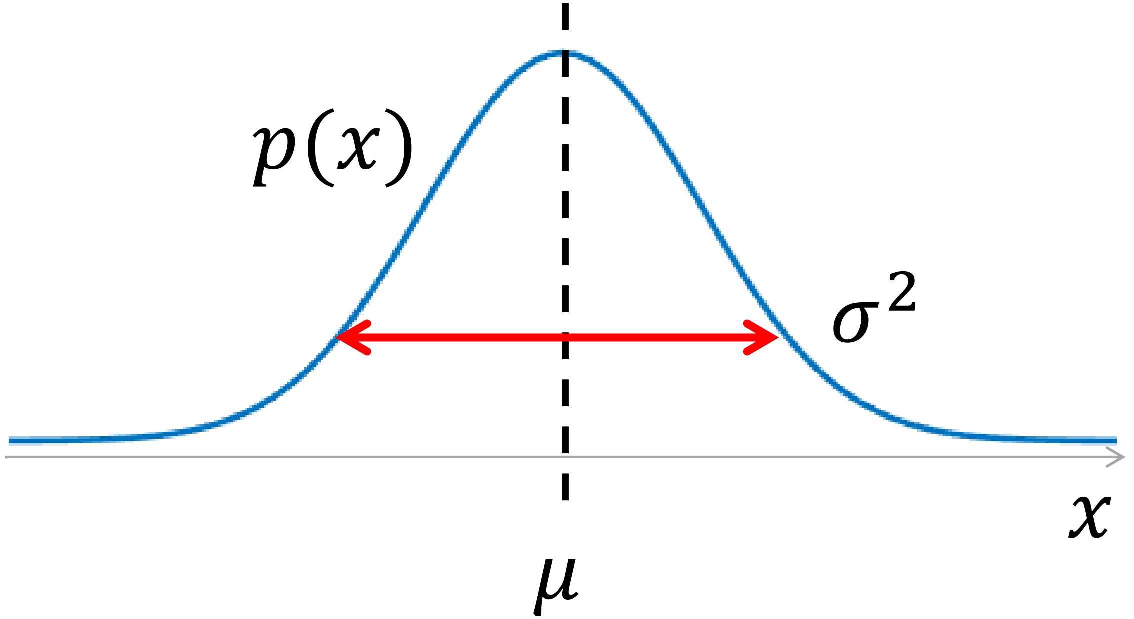


$\mu = 0$

$\sigma^2 = 0.7$

Gaussian Distribution (1D)

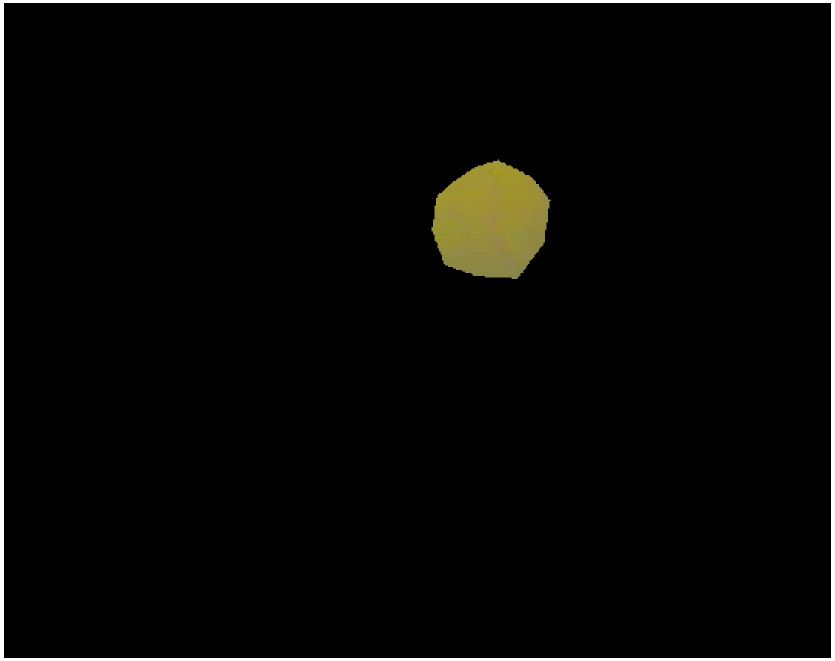
$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

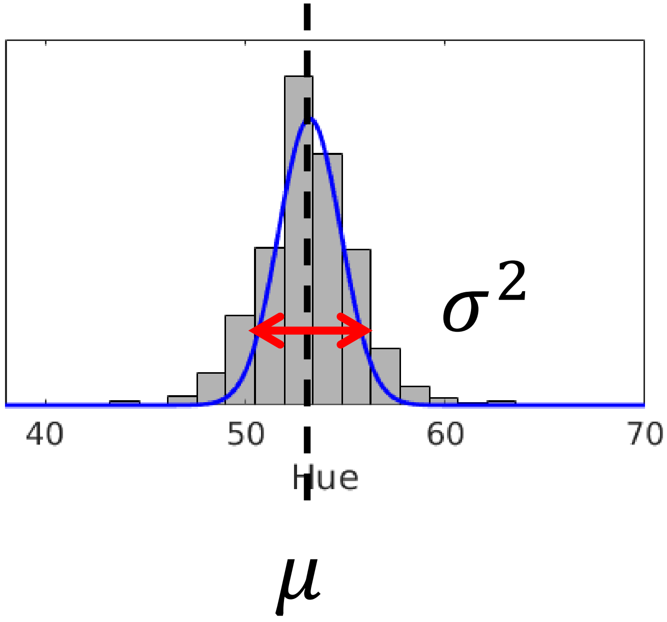


Gaussian Distribution : Example

Ball color distribution

Color Image





Acknowledgement

- Thanks to Rei Suzuki, Dan Lee’s master student at the University of Pennsylvania, for helping us create the lectures for WEEK 1.

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Estimation and Learning
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Week 1.
Gaussian Model Learning

1.2.2 Maximum Likelihood Estimate

Penn Engineering

Maximum Likelihood Estimate of Gaussian Model Parameters

- Objective

  Estimate the mean and the variance given observed data

Likelihood: $p(\{x_i\}|\mu, \sigma)$

Observed data Unknown parameters

Maximum Likelihood Estimate of Gaussian Model Parameters

- Objective

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} p(\{x_i\}|\mu, \sigma)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- Objective

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} p(\{x_i\}|\mu, \sigma)$$

Assuming independence of observations,

$$p(\{x_i\}|\mu, \sigma) = \prod_{i=1}^{N} p(x_i|\mu, \sigma)$$

$$\boxed{\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu, \sigma)}$$

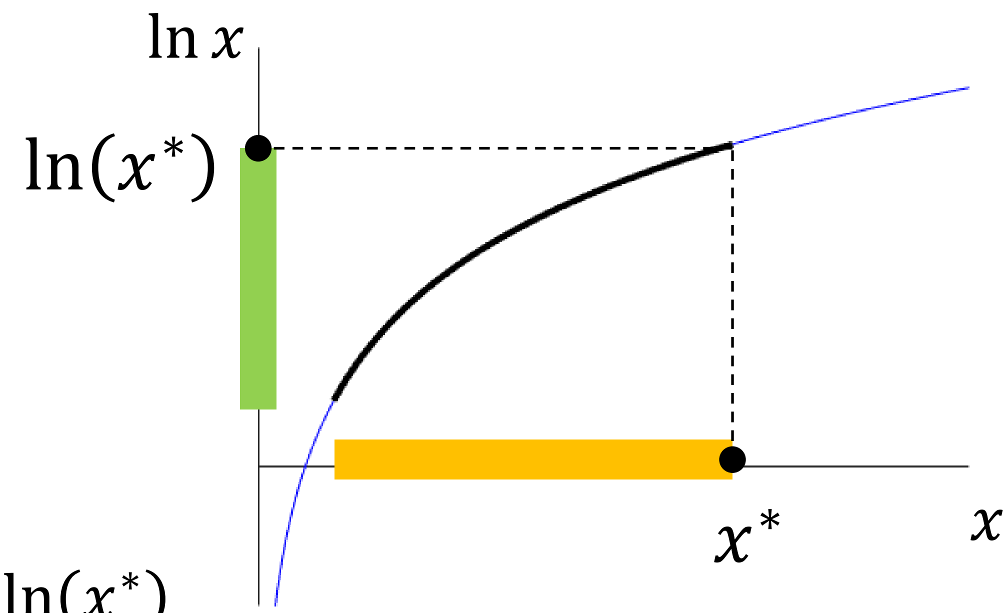
Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i | \mu, \sigma)$$

(1) Take the log!

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i | \mu, \sigma)$$



NOTE 1:

$$x \leq x^* \leftrightarrow \ln(x) \leq \ln(x^*)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu,\sigma)$$

(1) $$\arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu,\sigma) = \arg\max_{\mu,\sigma} \ln\left\{\prod_{i=1}^{N} p(x_i|\mu,\sigma)\right\}$$

NOTE 1:

$$x \leq x^* \leftrightarrow \ln(x) \leq \ln(x^*)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i | \mu, \sigma)$$

(1) $$\arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i | \mu, \sigma) = \arg\max_{\mu,\sigma} \ln \left\{ \prod_{i=1}^{N} p(x_i | \mu, \sigma) \right\}$$

$$= \arg\max_{\mu,\sigma} \boxed{\sum_{i=1}^{N} \ln p(x_i | \mu, \sigma)}$$

NOTE 2:

$$\log(x_1 \times x_2 \times \cdots \times x_k) = \log(x_1) + \log(x_2) + \cdots + \log(x_k)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln \underline{p(x_i|\mu, \sigma)}$$

(2) Gaussian!

$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x_i - \mu)^2}{2\sigma^2}\right\}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln p(x_i|\mu,\sigma)$$

$$\ln \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}$$

$$= \left\{-\frac{(x_i-\mu)^2}{2\sigma^2} - \ln\sigma - \ln\sqrt{2\pi}\right\}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} - \ln \sigma \right\}$$

$$\updownarrow$$

$$\boxed{\hat{\mu}, \hat{\sigma} = \arg\min_{\mu,\sigma} \sum_{i=1}^{N} \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln \sigma \right\}}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\min_{\mu,\sigma} \underbrace{\sum_{i=1}^{N} \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln \sigma \right\}}_{J(\mu,\sigma)}$$

- At optimum,

$$\boxed{\frac{\partial J}{\partial \mu} = 0 \longrightarrow \hat{\mu}} \qquad \boxed{\frac{\partial J(\hat{\mu}, \sigma)}{\partial \sigma} = 0 \rightarrow \hat{\sigma}}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- The MLE Solution:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \hat{\mu})^2$$

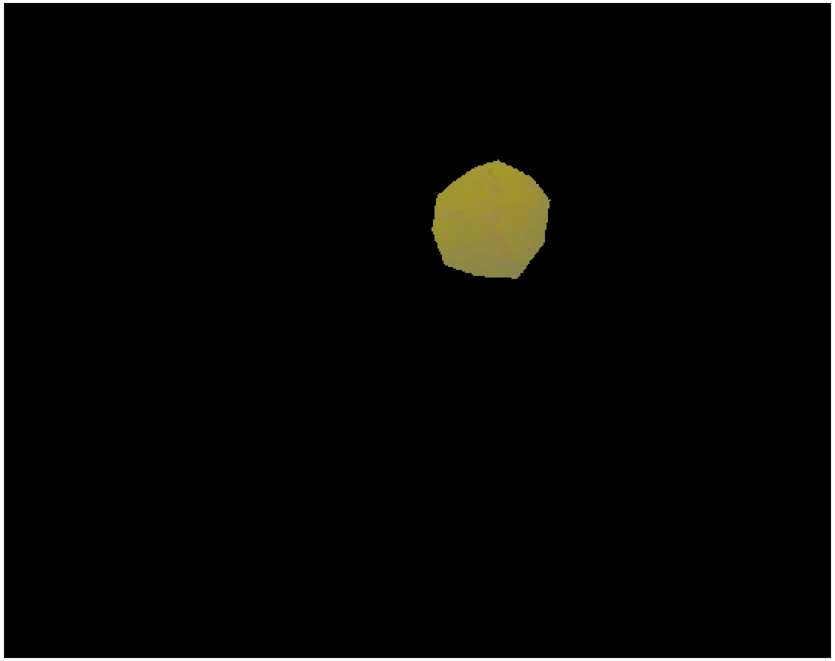
MLE Estimate: Example

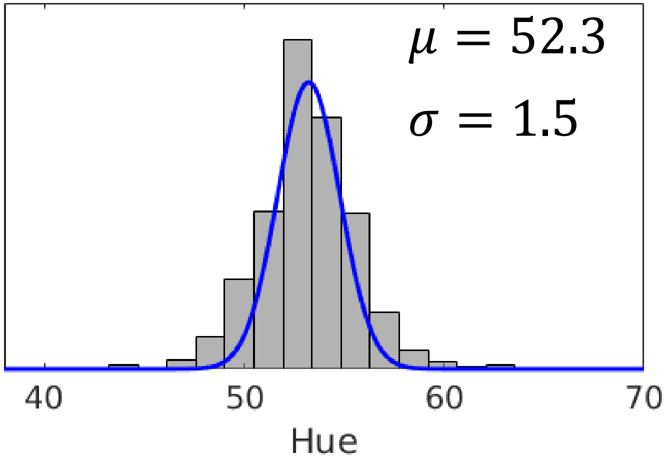
Ball color distribution

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$$

Segmented Ball Image





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Supplementary Notes on
MLE for Univariate Gaussian

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Maximum Likelihood Estimate of Gaussian Model Parameters

- Objective

  Estimate the mean and the variance given observed data

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} p(\{x_i\}|\mu, \sigma)$$

- Assuming independence of observations,

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu, \sigma)$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- To obtain $\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu,\sigma)$

(1) $\arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i|\mu,\sigma) = \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln p(x_i|\mu,\sigma)$

(2) $p(x_i|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}$ (Gaussian)

Maximum Likelihood Estimate of Gaussian Model Parameters

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \prod_{i=1}^{N} p(x_i | \mu, \sigma)$$

$$= \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln p(x_i | \mu, \sigma) \qquad \text{from (1)}$$

$$= \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln \left\{ \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\} \right\} \quad \text{from (2)}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

(continued)

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln\left\{\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}\ \right\}$$

$$= \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \left\{-\frac{(x_i-\mu)^2}{2\sigma^2} - \ \ln\sqrt{2\pi}\sigma\right\}$$

$$= \arg\min_{\mu,\sigma} \sum_{i=1}^{N} \left\{\frac{(x_i-\mu)^2}{2\sigma^2} + \ln\sigma + const\right\}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

Let $$J(\mu, \sigma) = \sum_{i=1}^{N} \left\{ \frac{(x_i - \mu)^2}{2\sigma^2} + \ln\sigma \right\}$$

Then $$\hat{\mu}, \hat{\sigma} = \arg\min_{\mu,\sigma} J(\mu, \sigma)$$

① $$\frac{\partial J}{\partial \mu} = 0 \longrightarrow \hat{\mu}$$

② $$\frac{\partial J(\hat{\mu}, \sigma)}{\partial \sigma} = 0 \longrightarrow \hat{\sigma}$$

Maximum Likelihood Estimate of Gaussian Model Parameters

① $$\boxed{\frac{\partial J}{\partial \mu} = 0 \longrightarrow \hat{\mu}}$$

$$\frac{\partial J}{\partial \mu} = \frac{\partial}{\partial \mu}\sum_{i=1}^{N}\left\{\frac{(x_i-\mu)^2}{2\sigma^2} + \ln\sigma\right\}$$

$$= \sum_{i=1}^{N}\left\{\frac{\partial}{\partial \mu}\frac{(x_i-\mu)^2}{2\sigma^2}\right\}$$

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N}x_i \longleftarrow = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu) = 0$$

Maximum Likelihood Estimate of Gaussian Model Parameters

② $$\boxed{\frac{\partial J(\hat{\mu},\sigma)}{\partial \sigma} = 0 \longrightarrow \hat{\sigma}}$$

$$\frac{\partial J}{\partial \sigma} = \frac{\partial}{\partial \sigma}\sum_{i=1}^{N}\left\{\frac{(x_i-\hat{\mu})^2}{2\sigma^2} + \ln\sigma\right\}$$

$$= \left(\frac{\partial}{\partial \sigma}\frac{1}{2\sigma^2}\right)\left(\sum_{i=1}^{N}(x_i-\hat{\mu})^2\right) - \frac{N}{\sigma}$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\hat{\mu})^2 \longleftarrow = \frac{1}{\sigma}\left(N - \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\hat{\mu})^2\right) = 0$$

Maximum Likelihood Estimate of Gaussian Model Parameters

- In summary, we have

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} (x_i - \hat{\mu})^2$$

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Week 1. Gaussian Model Learning

1.3.1 Multivariate Gaussian Distribution

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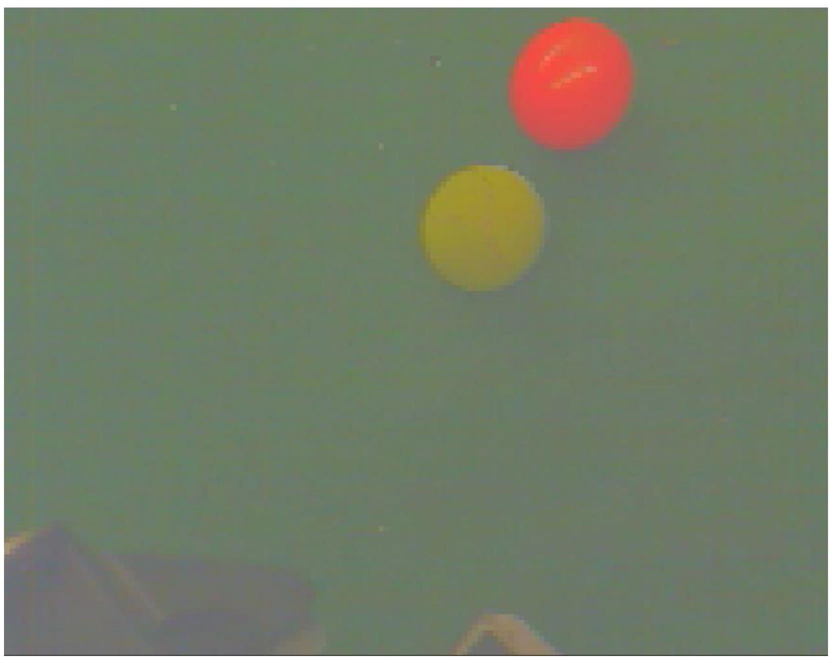
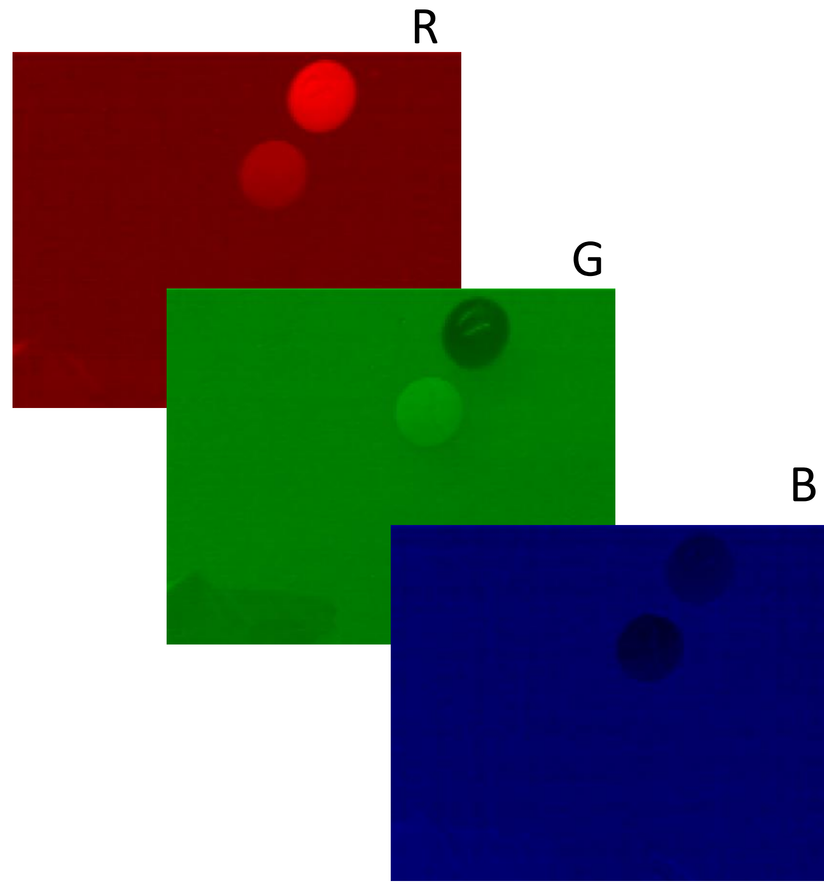
Multivariate Gaussian : Example

Ball color in multi-channels

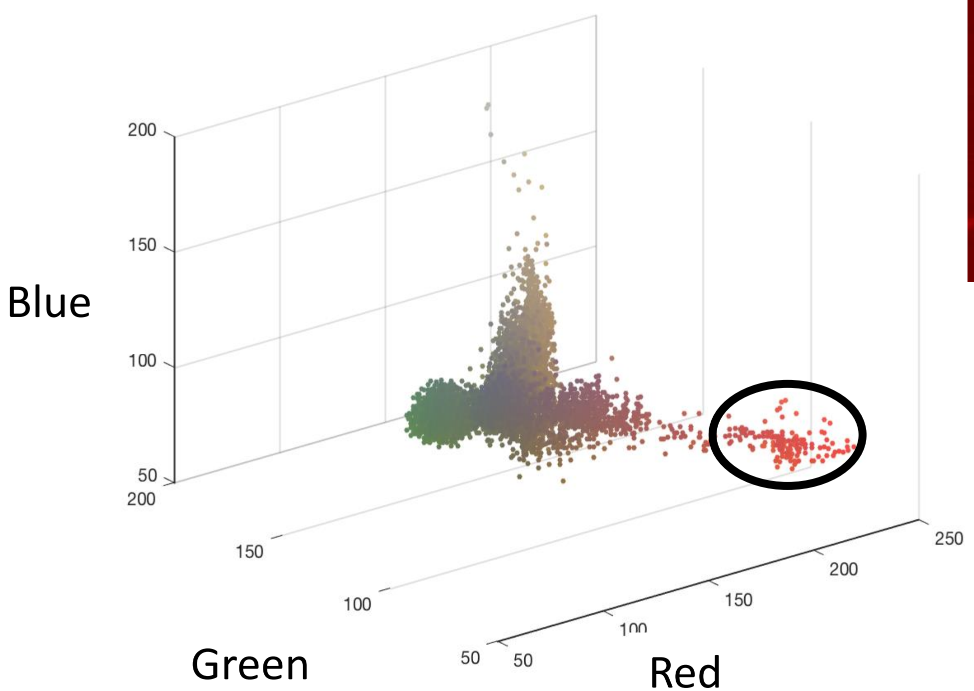
RGB Image

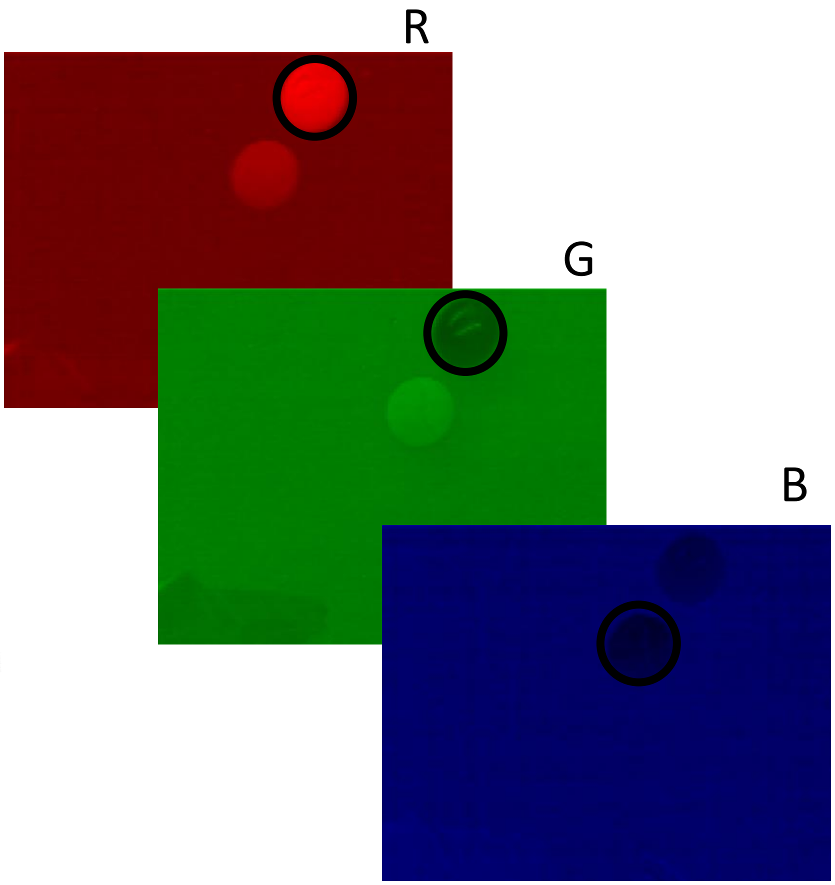




Multivariate Gaussian : Example

Ball color in multi-channels





Multivariate Gaussian

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\boldsymbol{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

D Number of Dimensions

$\mathbf{x}$ Variable

$\boldsymbol{\mu}$ Mean *vector*

Σ Covariance *matrix*

(Dimension = 1)

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}$$

Multivariate Gaussian

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Σ Covariance *matrix*

* Diagonal terms: variance
* Off-diagonal terms: correlation

(Dimension = 2)

$$\boxed{\Sigma = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2}^2 \\ \sigma_{x_2x_1}^2 & \sigma_{x_2}^2 \end{bmatrix}} \quad (\sigma_{x_1x_2}^2 = \sigma_{x_2x_1}^2)$$

Multivariate Gaussian

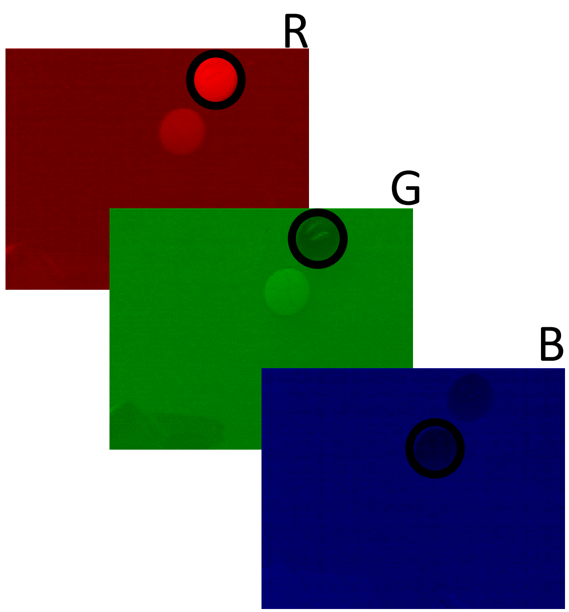
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Determinant of Σ

Multivariate Gaussian

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Ball color in multi-channels



$$D = 3$$

$$\mathbf{x} = [x_R \quad x_G \quad x_B]$$

$$\boldsymbol{\mu} = [\mu_R \quad \mu_G \quad \mu_B]$$

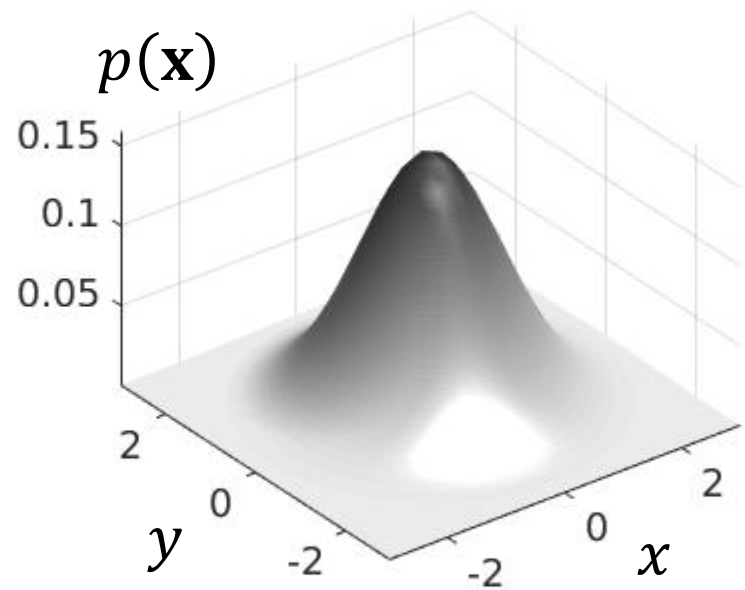
$$\Sigma = \begin{bmatrix} \sigma_{x_R}^2 & \sigma_{x_R x_G}^2 & \sigma_{x_R x_B}^2 \\ \sigma_{x_R x_G}^2 & \sigma_{x_G}^2 & \sigma_{x_G x_B}^2 \\ \sigma_{x_R x_B}^2 & \sigma_{x_G x_B}^2 & \sigma_{x_B}^2 \end{bmatrix}$$

Multivariate Gaussian: 2D

$$p(\mathbf{x}) = \frac{1}{2\pi} \exp\left\{-\frac{x^2 + y^2}{2}\right\}$$

- 2D Zero-mean Spherical Case

$$D = 2$$
$$\mathbf{x} = [x \quad y]^T$$
$$\boldsymbol{\mu} = [0 \quad 0]^T$$
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

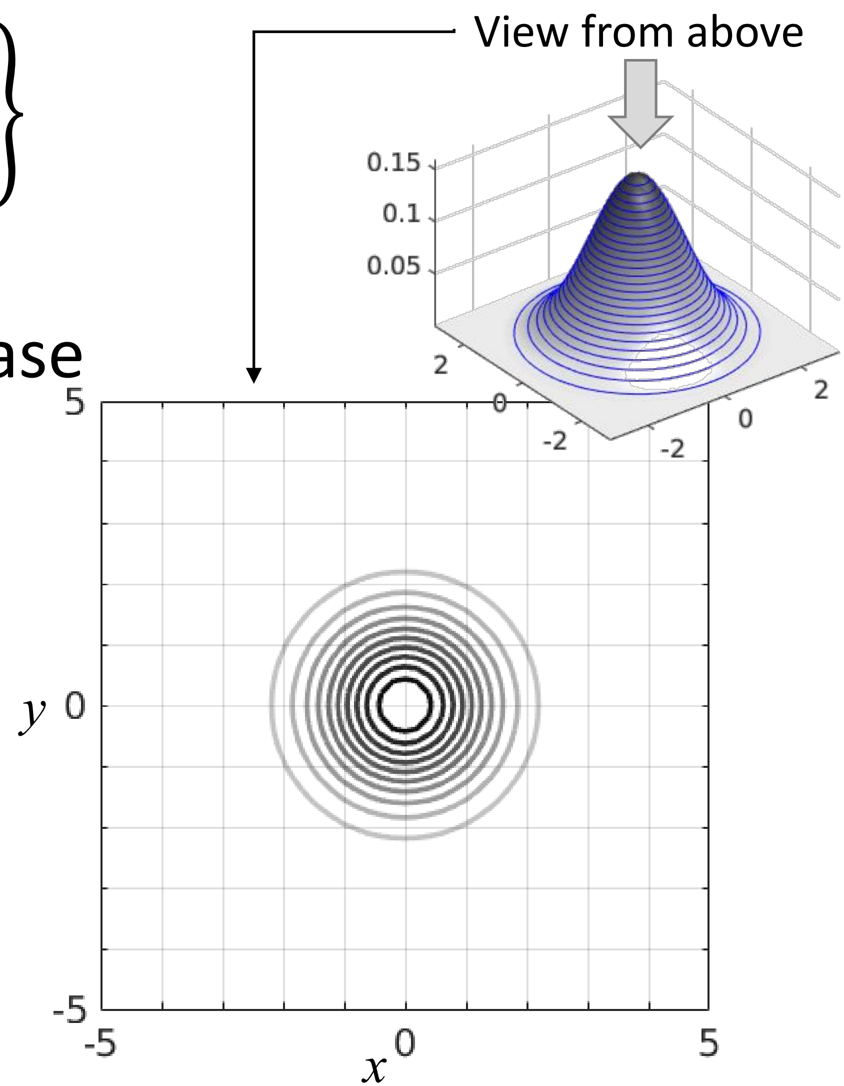


Multivariate Gaussian: 2D

$$p(\mathbf{x}) = \frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}$$

- 2D Zero-mean Spherical Case

$$D = 2$$
$$\mathbf{x} = [x \quad y]^T$$
$$\boldsymbol{\mu} = [0 \quad 0]^T$$
$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Multivariate Gaussian: 2D

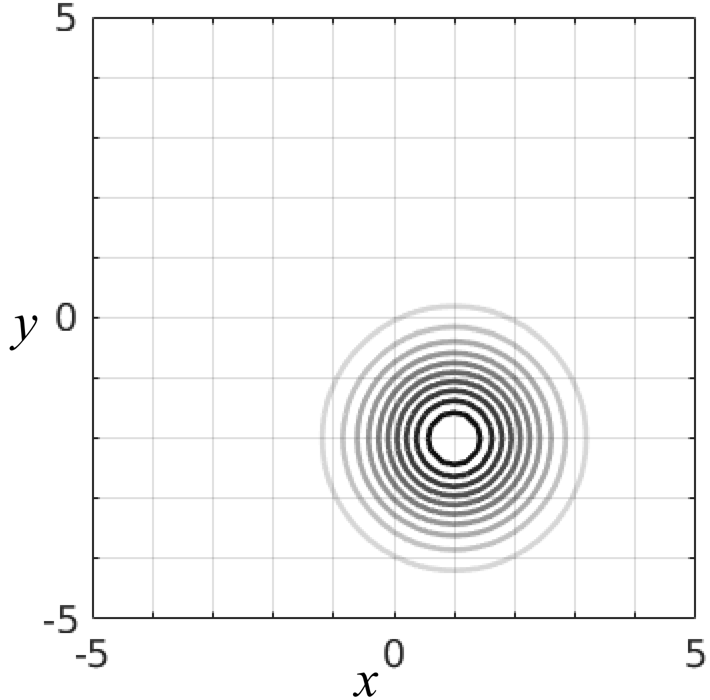
$$p(\mathbf{x}) = \frac{1}{2\pi} \exp\left\{-\frac{(x-\mu_x)^2+(y-\mu_y)^2}{2}\right\}$$

$$D = 2$$

$$\mathbf{x} = [x \quad y]^T$$

$$\boxed{\boldsymbol{\mu} = [\mu_x \quad \mu_y]^T}$$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Multivariate Gaussian: 2D

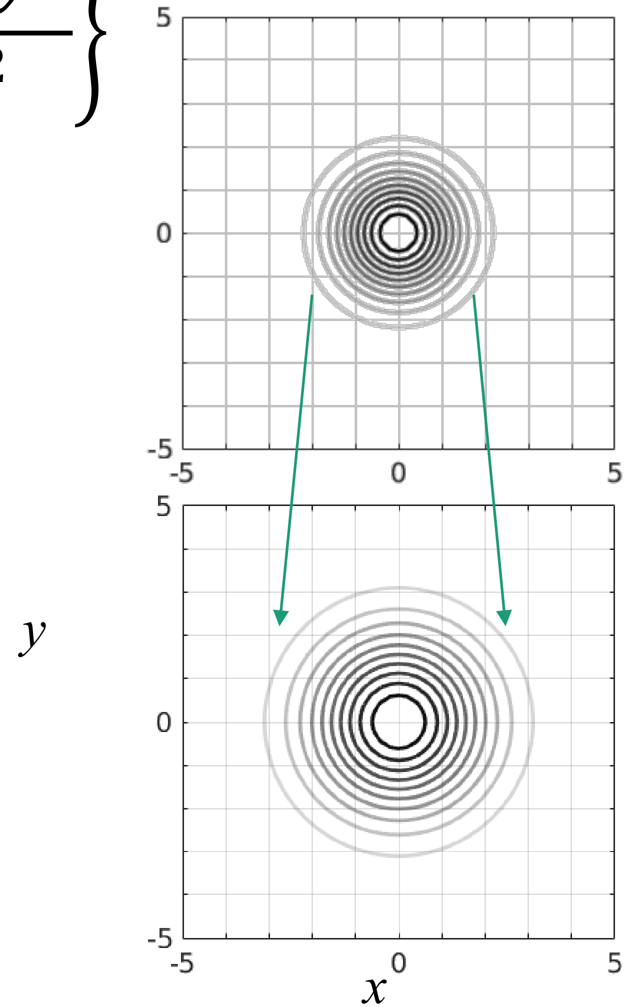
$$p(\mathbf{x}) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

$$D = 2$$

$$\mathbf{x} = [x \quad y]^T$$

$$\boldsymbol{\mu} = [0 \quad 0]^T$$

$$\boxed{\Sigma = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}}$$



Larger σ^2

Multivariate Gaussian: 2D

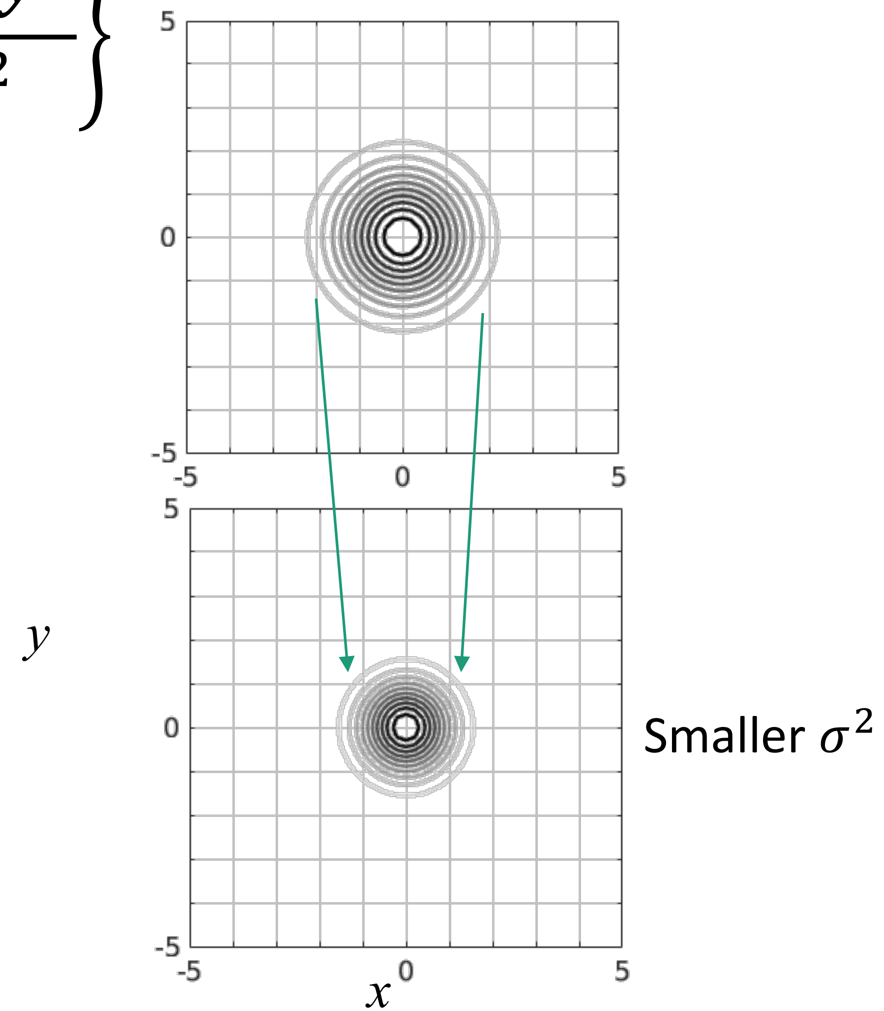
$$p(\mathbf{x}) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

$$D = 2$$

$$\mathbf{x} = [x \quad y]^T$$

$$\boldsymbol{\mu} = [0 \quad 0]^T$$

$$\boxed{\Sigma = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}}$$



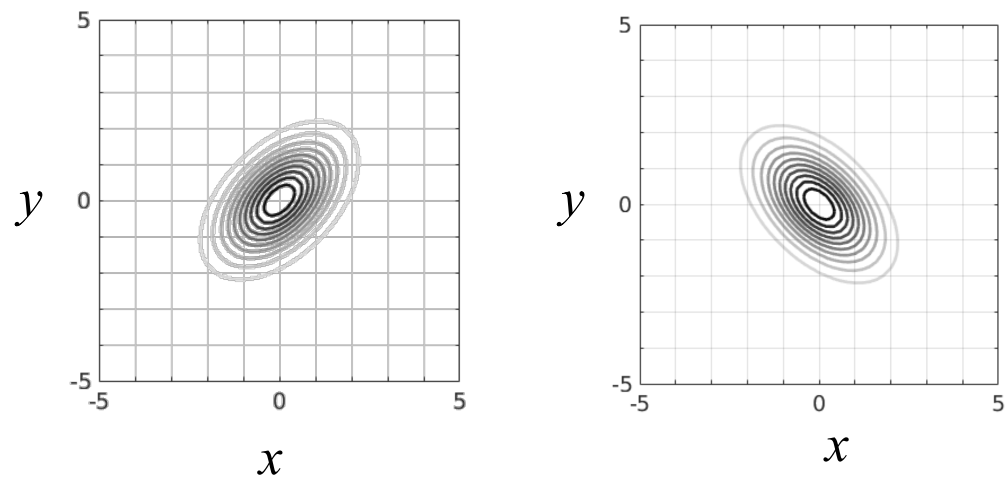
Multivariate Gaussian: 2D

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\boldsymbol{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

- 2D General Case

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix}$$

$\sigma_{xy}^2 > 0$ $\sigma_{xy}^2 < 0$



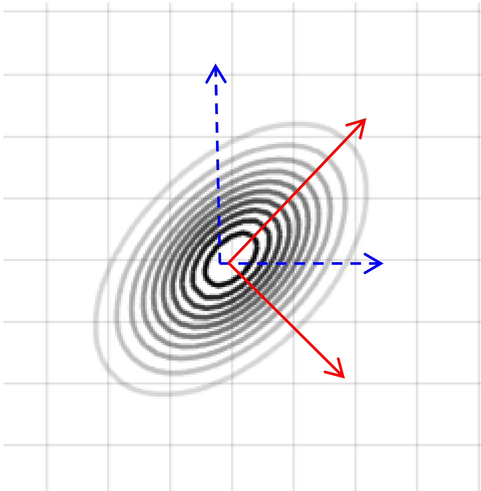
Multivariate Gaussian: 2D

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\boldsymbol{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

- Properties of Covariance Matrix Σ

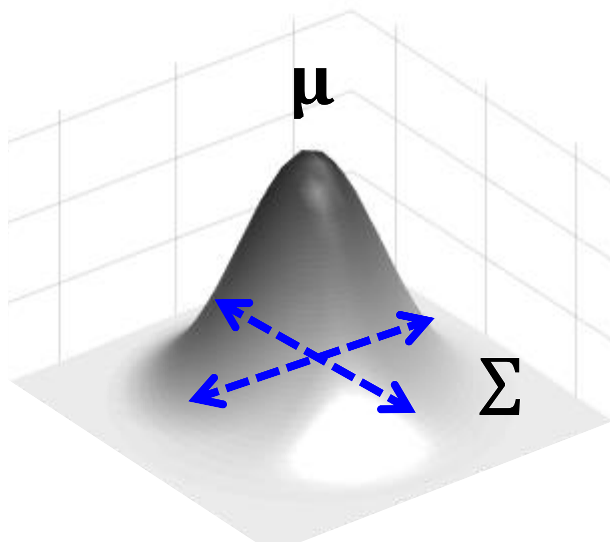
1) Σ is Symmetric and Positive Definite.

2) Diagonalization: Σ can be decomposed in the form of UDU^T.
(D is a Diagonal matrix.)



Multivariate Gaussian: 2D

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$



Robotics

Estimation and Learning
with Dan Lee

Week 1.
Gaussian Model Learning

1.3.2 MLE of Multivariate Gaussian

Penn Engineering

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- Objective

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\boldsymbol{\Sigma}} p(\{\mathbf{x}_i\}|\boldsymbol{\mu}, \Sigma)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- Objective

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\boldsymbol{\Sigma}} p(\{\mathbf{x}_i\}|\boldsymbol{\mu}, \Sigma)$$

Assuming independence of observations,

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\boldsymbol{\Sigma}} \prod_{i=1}^{N} p(\mathbf{x}_i|\boldsymbol{\mu}, \Sigma)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

(1) Take the log!

$$\arg\max \, likelihood \leftrightarrow \arg\max \ln(likelihood)$$

$$\log(x_1 \times x_2 \times \cdots \times x_k) = \log(x_1) + \log(x_1) + \cdots + \log(x_k)$$

$$\boxed{\arg\max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma) \Rightarrow \arg\max_{\boldsymbol{\mu}, \Sigma} \sum_{i=1}^{N} \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)}$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \underline{\ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)}$$

(2) Gaussian!

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

$$\boxed{\ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)} \longrightarrow \left\{-\frac{1}{2}(\mathbf{x}_i-\boldsymbol{\mu})^{T}\Sigma^{-1}(\mathbf{x}_i-\boldsymbol{\mu}) - \frac{1}{2}\ln|\Sigma| + c\right\}$$

$c = -D/2 \ln(2\pi)$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \left\{ -\frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu})^{\boldsymbol{T}} \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) - \frac{1}{2} \ln|\Sigma| + c \right\}$$

$$\boxed{\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\min_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \left\{ \frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu})^{\boldsymbol{T}} \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln|\Sigma| \right\}}$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\underbrace{\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\min_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^{\boldsymbol{T}} \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln|\Sigma| \right\}}_{J(\boldsymbol{\mu}, \Sigma)}$$

- At optimum,

① $\dfrac{\partial J}{\partial \boldsymbol{\mu}} = \mathbf{0} \longrightarrow \hat{\boldsymbol{\mu}}$ ② $\dfrac{\partial J(\hat{\boldsymbol{\mu}}, \Sigma)}{\partial \Sigma} = 0 \longrightarrow \hat{\Sigma}$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- In summary, we have

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$$

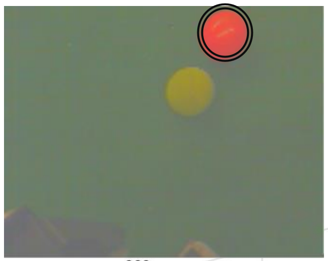
$$\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^\intercal$$

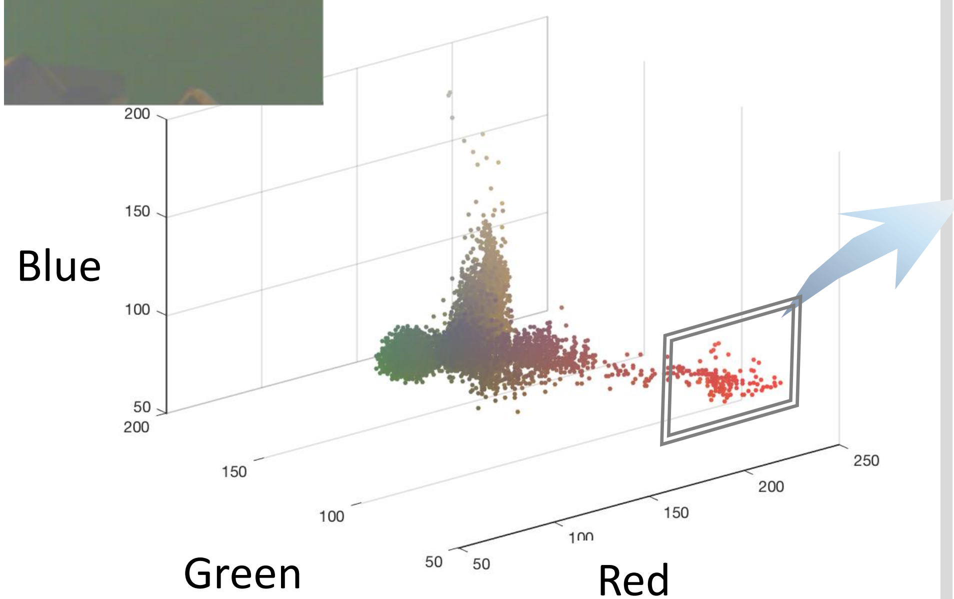
Multi-dimension Distribution: Example

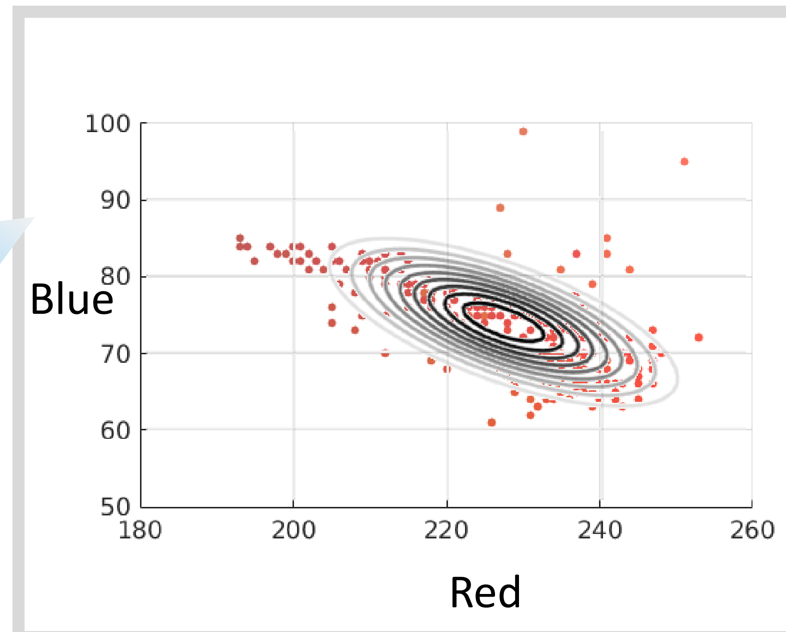
- Ball color in multi-channels

$$\hat{\boldsymbol{\mu}} = [227 \quad 74]$$

$$\hat{\Sigma} = \begin{bmatrix} 150 & -48 \\ -48 & 46 \end{bmatrix}$$







Robotics

Estimation and Learning
with Dan Lee

Supplementary Notes on
MLE for Multivariate Gaussian

Penn Engineering

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- Objective

  Estimate the mean and the variance given observed data

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \Sigma} p(\{\mathbf{x}_i\} | \boldsymbol{\mu}, \Sigma)$$

- Assuming independence of observations,

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- To obtain $\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$

(1) $\arg\max_{\boldsymbol{\mu},\Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma) = \arg\max_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$

(2) $p(\mathbf{x}) = \dfrac{1}{(2\pi)^{D/2} |\Sigma|^{1/2}} \exp\left\{ -\dfrac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\}$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

$$= \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

from (1)

$$= \arg\max_{\mu,\sigma} \sum_{i=1}^{N} \left\{ -\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^{\boldsymbol{T}} \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) - \frac{1}{2} \ln|\Sigma| + c \right\}$$

from (2)

$$= \arg\min_{\mu,\sigma} \sum_{i=1}^{N} \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^{\boldsymbol{T}} \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln|\Sigma| \right\}$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

Let $J(\boldsymbol{\mu}, \Sigma) = \sum_{i=1}^{N} \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} \ln|\Sigma| \right\}$

Then $\widehat{\boldsymbol{\mu}}, \widehat{\Sigma} = \arg\min_{\boldsymbol{\mu}, \Sigma} J(\boldsymbol{\mu}, \Sigma)$

① $\frac{\partial J}{\partial \boldsymbol{\mu}} = \mathbf{0} \longrightarrow \widehat{\boldsymbol{\mu}}$ ② $\frac{\partial J(\widehat{\boldsymbol{\mu}}, \Sigma)}{\partial \Sigma} = 0 \longrightarrow \widehat{\Sigma}$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

① $$\boxed{\frac{\partial J}{\partial \boldsymbol{\mu}} = \mathbf{0} \longrightarrow \hat{\boldsymbol{\mu}}}$$

$$\frac{\partial J}{\partial \boldsymbol{\mu}} = \frac{\partial}{\partial \boldsymbol{\mu}} \sum_{i=1}^{N} \left\{ \frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) + \frac{1}{2} |\Sigma| \right\}$$

$$= \frac{\partial}{\partial \boldsymbol{\mu}} \sum_{i=1}^{N} \left\{ \frac{1}{2} \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu} - \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{x}_i \right\}$$

$$= \Sigma^{-1} \sum_{i=1}^{N} \{ \boldsymbol{\mu} - \mathbf{x}_i \} = \mathbf{0} \longrightarrow \hat{\boldsymbol{\mu}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

② $$\boxed{\frac{\partial J(\hat{\boldsymbol{\mu}}, \Sigma)}{\partial \Sigma} = 0 \longrightarrow \hat{\Sigma}}$$

$$\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^{\top}$$

$$\frac{\partial J}{\partial \Sigma} = \frac{\partial}{\partial \Sigma}\sum_{i=1}^{N}\left\{\frac{1}{2}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^{T}\Sigma^{-1}(\mathbf{x}_i - \hat{\boldsymbol{\mu}}) + \frac{1}{2}\ln|\Sigma|\right\}$$

$$= \frac{1}{2}\sum_{i=1}^{N}\{-\Sigma^{-1}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^{T}\Sigma^{-1} + \Sigma^{-1}\}$$

$$= \frac{1}{2}\Sigma^{-1}\left[-\left\{\sum_{i=1}^{N}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^{T}\right\}\Sigma^{-1} + N\cdot \mathrm{I}\right] = 0$$

Maximum Likelihood Estimate of Multivariate Gaussian Parameters

- In summary, we have

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i$$

$$\hat{\Sigma} = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^{\intercal}$$

Robotics

Estimation and Learning
with Dan Lee

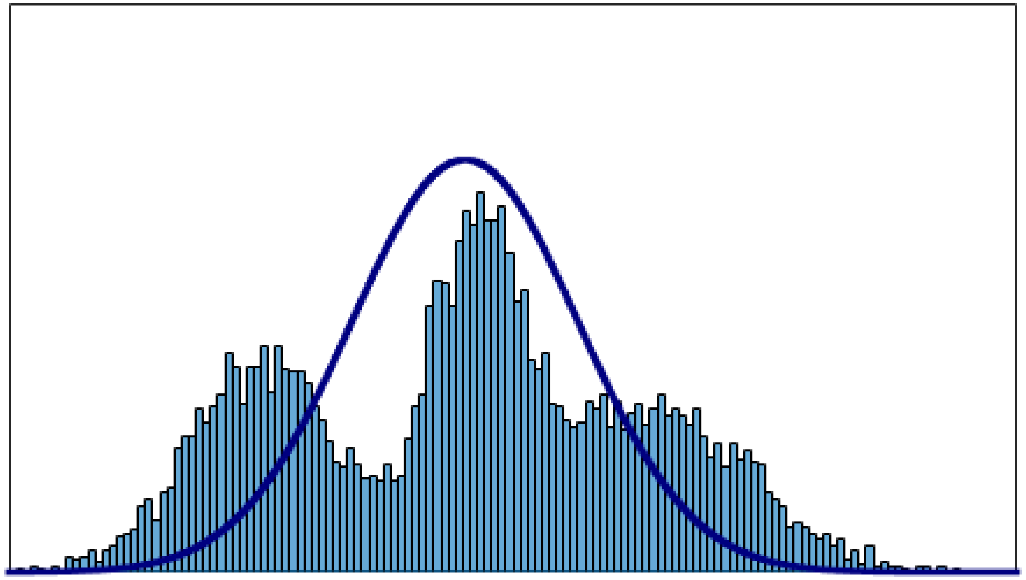
Week 1.
Gaussian Model Learning

1.4.1 Gaussian Mixture Model

Penn Engineering

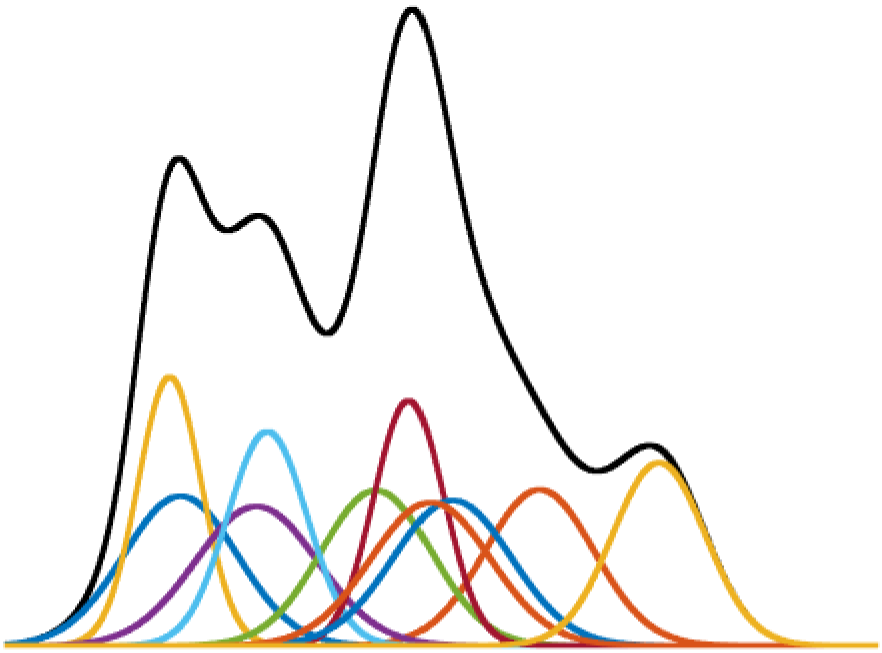
Limitations of Single Gaussian

- Single Mode
- Symmetric



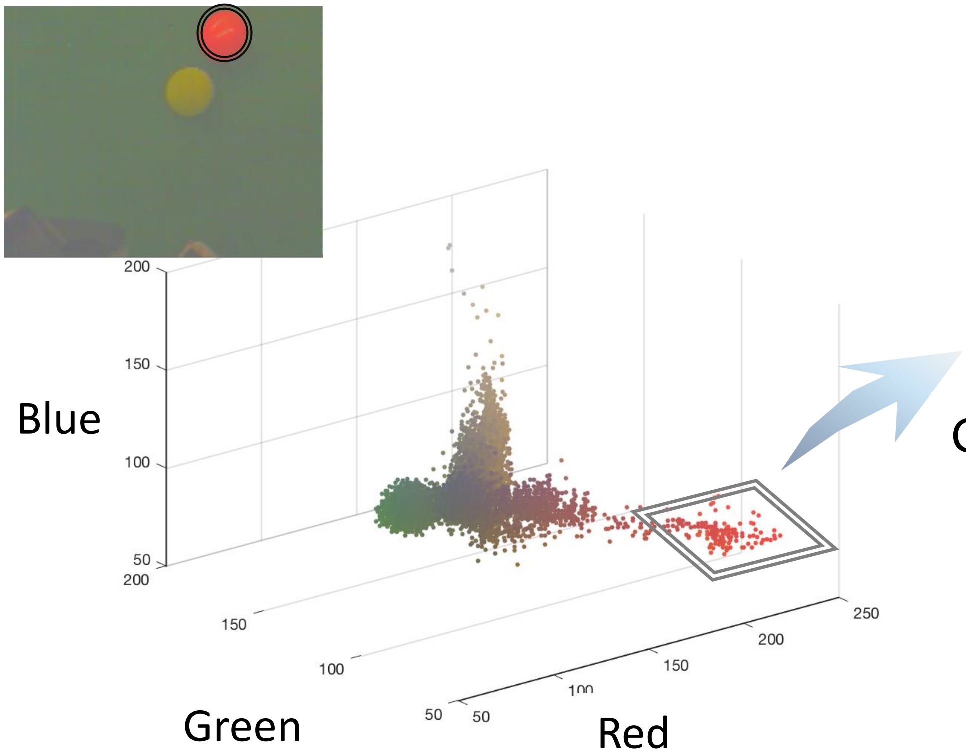
Gaussian Mixture Model

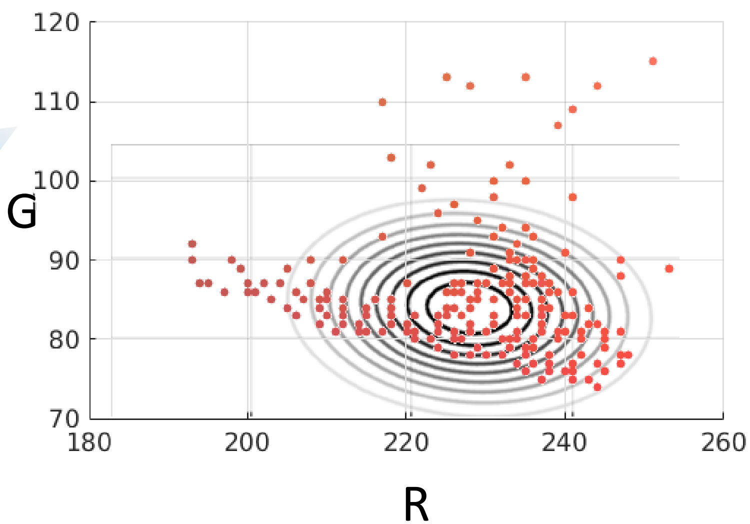
- Mixture (=**Sum**) of Gaussians



Multi-dimension Distribution: Example

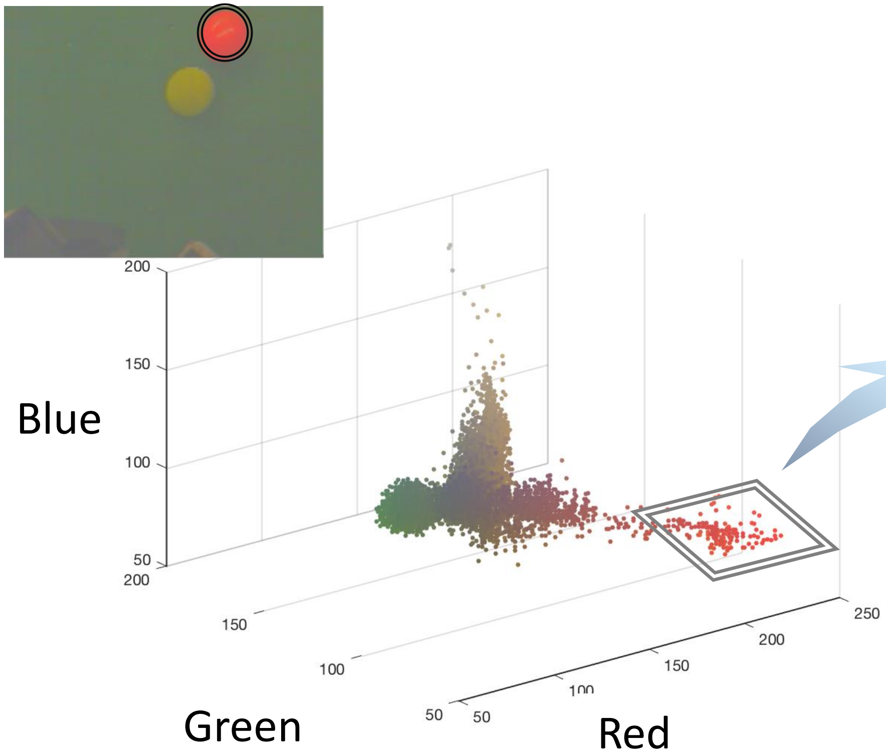
- Ball color in multi-channels

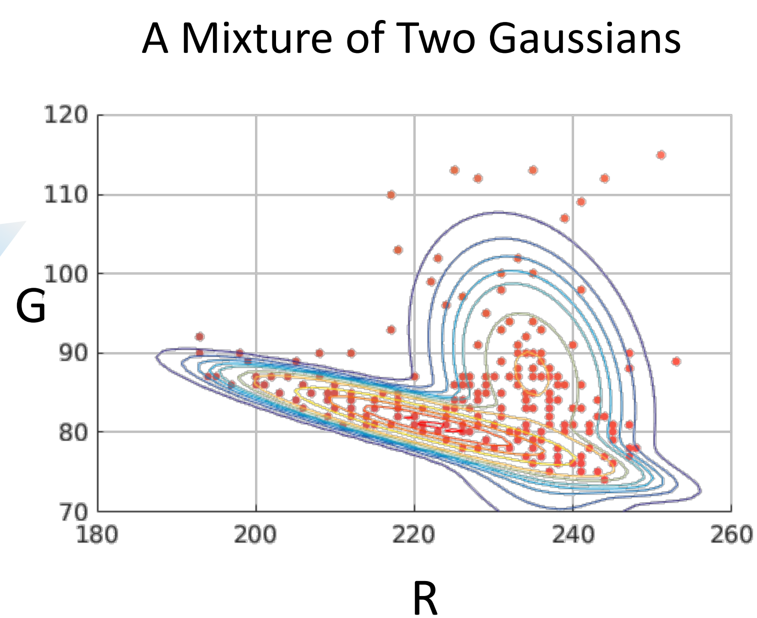




Multi-dimension Distribution: Example

- Ball color in multi-channels





Gaussian Mixture Model

- Mixture of Gaussians

$$p(\mathbf{x}) = \sum_{k=1}^{K} w_k g_k(\mathbf{x}|\boldsymbol{\mu}_k, \Sigma_k)$$

g_k: Gaussian with $\boldsymbol{\mu}_k$ and Σ_k

w_k: mixing coefficient (weight, a prior) $w_k > 0, \sum_{k=1}^{K} w_k = 1$

Using GMM

Flexibility

Parameters ↑

- No analytic solution
- Overfitting

$$\boldsymbol{\mu} = \{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K\}$$

$$\Sigma = \{\Sigma_1, \Sigma_2, \dots, \Sigma_K\}$$

$$\boldsymbol{w} = \{w_1, w_2, \dots, w_K\}$$

K: Number of Components

Using GMM

Flexibility

\# Parameters ↑

- No analytic solution
- Overfitting

$$\boldsymbol{\mu} = \{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K\}$$

$$\Sigma = \{\Sigma_1, \Sigma_2, \dots, \Sigma_K\}$$

$$\boldsymbol{w} = w = 1/K$$

K: given number of Components

Robotics

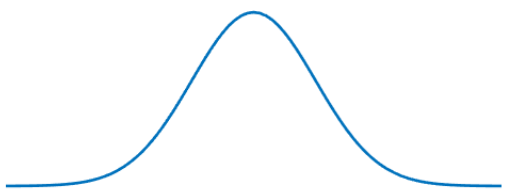
Estimation and Learning
with Dan Lee

Week 1.
Gaussian Model Learning

1.4.2 GMM Parameter Estimation via EM

Penn Engineering

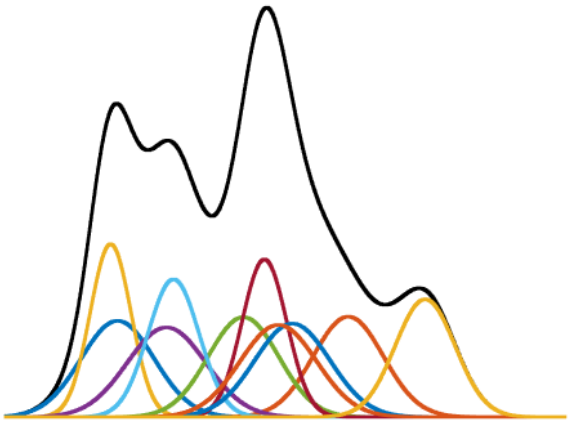
- Single Gaussian



$\boldsymbol{\mu}$

Σ

- Mixture of Gaussians



$$\boxed{\begin{aligned}\boldsymbol{\mu} &= \{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K\} \\ \Sigma &= \{\Sigma_1, \Sigma_2, \dots, \Sigma_K\}\end{aligned}}$$

$w = 1/K$

K: Given

Learning GMM Parameters

Likelihood: $p(\{\mathbf{x}_i\}|\boldsymbol{\mu}, \Sigma)$

Observed data　　Unknown parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \Sigma} p(\{\mathbf{x}_i\}|\boldsymbol{\mu}, \Sigma)$$

$$\boldsymbol{\mu} = \{\boldsymbol{\mu}_k\}$$

$$\Sigma = \{\Sigma_k\} \quad k = 1, 2, \ldots, K$$

Learning GMM Parameters

- Objective

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} p(\{\mathbf{x}_i\} | \boldsymbol{\mu}, \Sigma)$$

Assuming independence of observations,

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

Learning GMM Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \arg\max_{\boldsymbol{\mu},\Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)$$

(1) Take the log!

$$\arg\max \, likelihood \leftrightarrow \arg\max \ln(likelihood)$$

$$\log(x_1 \times x_2 \times \cdots \times x_k) = \log(x_1) + \log(x_1) + \cdots + \log(x_k)$$

$$\boxed{\arg\max_{\boldsymbol{\mu},\Sigma} \prod_{i=1}^{N} p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma) \Rightarrow \arg\max_{\boldsymbol{\mu},\Sigma} \sum_{i=1}^{N} \ln p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)}$$

Learning GMM Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \underset{\boldsymbol{\mu},\Sigma}{\arg\max} \sum_{i=1}^{N} \ln \underline{p(\mathbf{x}_i | \boldsymbol{\mu}, \Sigma)}$$

(2) Gaussian Mixture Model!

$$p(\mathbf{x}) = \sum_{k=1}^{K} w_k g_k(\mathbf{x} | \boldsymbol{\mu}_k, \Sigma_k)$$

g_k: Gaussian with $\boldsymbol{\mu}_k$ and Σ_k

$$\boxed{\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \underset{\boldsymbol{\mu},\Sigma}{\arg\max} \sum_{i=1}^{N} \ln \left\{ \frac{1}{K} \sum_{k=1}^{K} g_k(\mathbf{x}_i | \boldsymbol{\mu}_k, \Sigma_k) \right\}}$$

Learning GMM Parameters

$$\hat{\boldsymbol{\mu}}, \hat{\Sigma} = \underset{\boldsymbol{\mu}, \Sigma}{\arg\max} \sum_{i=1}^{N} \ln \left\{ \frac{1}{K} \sum_{k=1}^{K} g_k(\mathbf{x}_i | \boldsymbol{\mu}_k, \Sigma_k) \right\}$$

where

$$g_k(\mathbf{x}) = \frac{1}{(2\pi)^{D/2} |\Sigma_k|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_k)^{\boldsymbol{T}} {\Sigma_k}^{-1} (\mathbf{x} - \boldsymbol{\mu}_k) \right\}$$

→ No closed form solution exist.

Expectation-Maximization (EM)

1) Special case : EM for GMM Parameter Estimation
2) General EM Algorithm

EM for GMM

Initial $\boldsymbol{\mu}$ and Σ

Latent variable z

EM for GMM

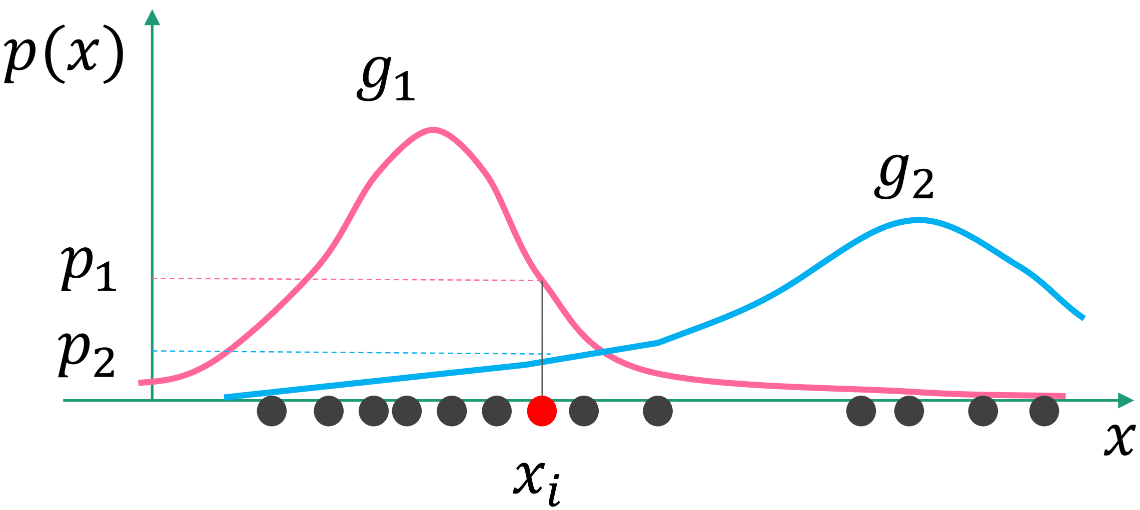
- Latent Variable

$$z_k^i = \frac{g_k(\mathbf{x}_i|\boldsymbol{\mu}_k, \Sigma_k)}{\sum_{k=1}^{K} g_k(\mathbf{x}_i|\boldsymbol{\mu}_k, \Sigma_k)}$$

EM for GMM

- Latent Variable Example

$$z_k^i = \frac{g_k(\mathbf{x}_i | \boldsymbol{\mu}_k, \Sigma_k)}{g_1(\mathbf{x}_i | \boldsymbol{\mu}_1, \Sigma_1) + g_2(\mathbf{x}_i | \boldsymbol{\mu}_2, \Sigma_2)}$$



$$z_1^i = \frac{p_1}{p_1 + p_2}$$

$$z_2^i = \frac{p_2}{p_1 + p_2}$$

EM for GMM

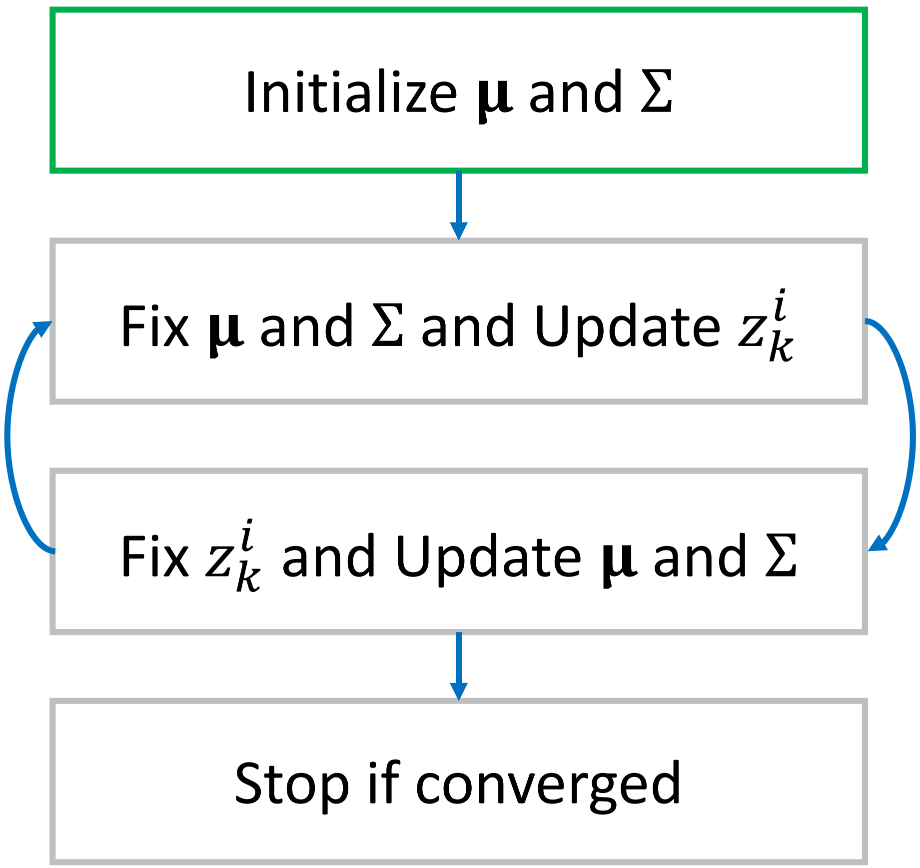
- Mean and Covariance Matrix

$$\widehat{\boldsymbol{\mu}}_k = \frac{1}{z_k} \sum_{i=1}^{N} z_k^i \mathbf{x}_i$$

$$\widehat{\Sigma}_k = \frac{1}{z_k} \sum_{i=1}^{N} z_k^i (\mathbf{x}_i - \widehat{\boldsymbol{\mu}}_k)(\mathbf{x}_i - \widehat{\boldsymbol{\mu}}_k)^{\top}$$

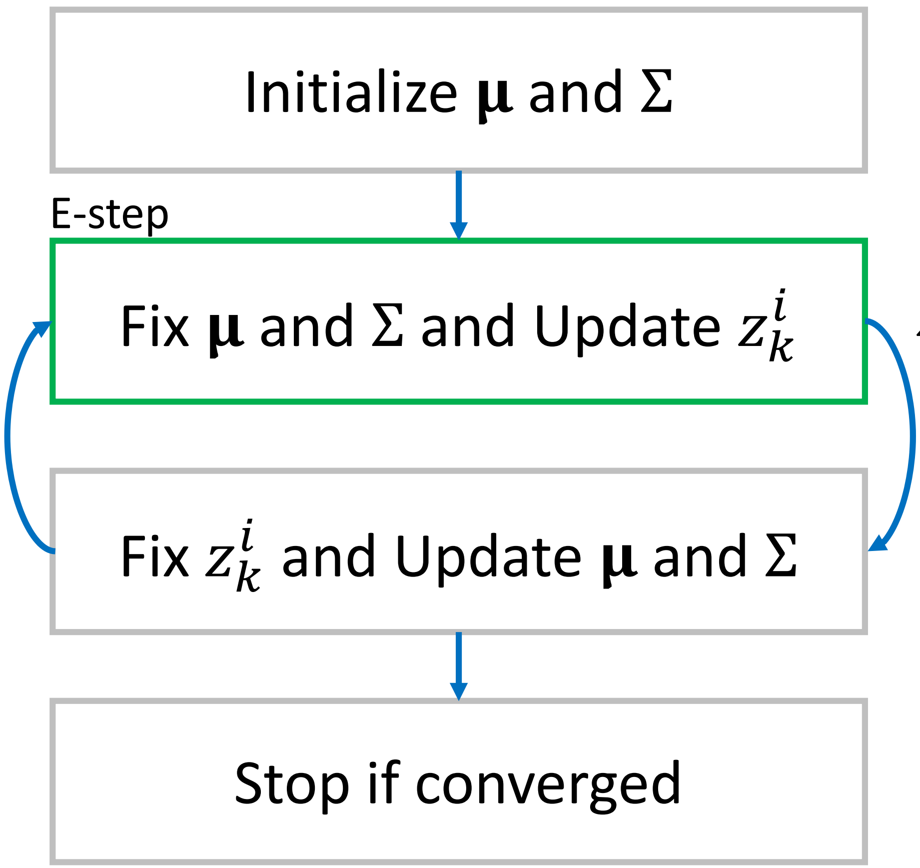
$$z_k = \sum\nolimits_{i=1}^{N} z_k^i$$

EM for GMM



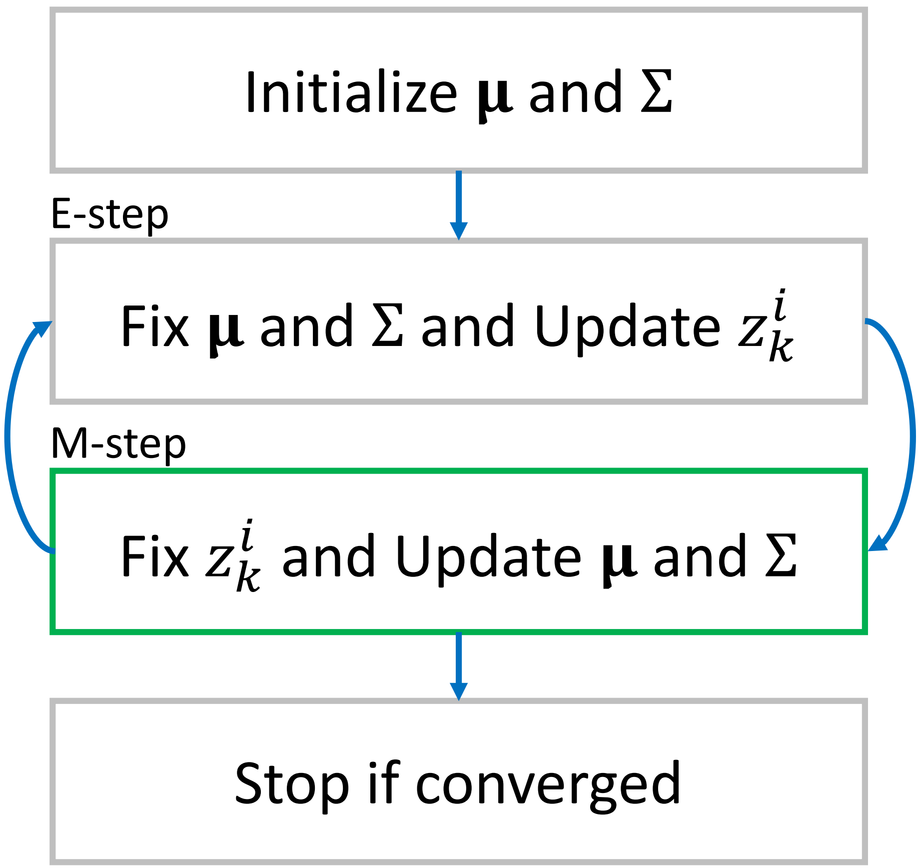
Initialize $\boldsymbol{\mu}$ and Σ
Fix $\boldsymbol{\mu}$ and Σ and Update z_k^i
Fix z_k^i and Update $\boldsymbol{\mu}$ and Σ
Stop if converged

EM for GMM



$$z_k^i = \frac{g_k(\mathbf{x}_i|\boldsymbol{\mu}_k, \Sigma_k)}{\sum_{k=1}^{K} g_k(\mathbf{x}_i|\boldsymbol{\mu}_k, \Sigma_k)}$$

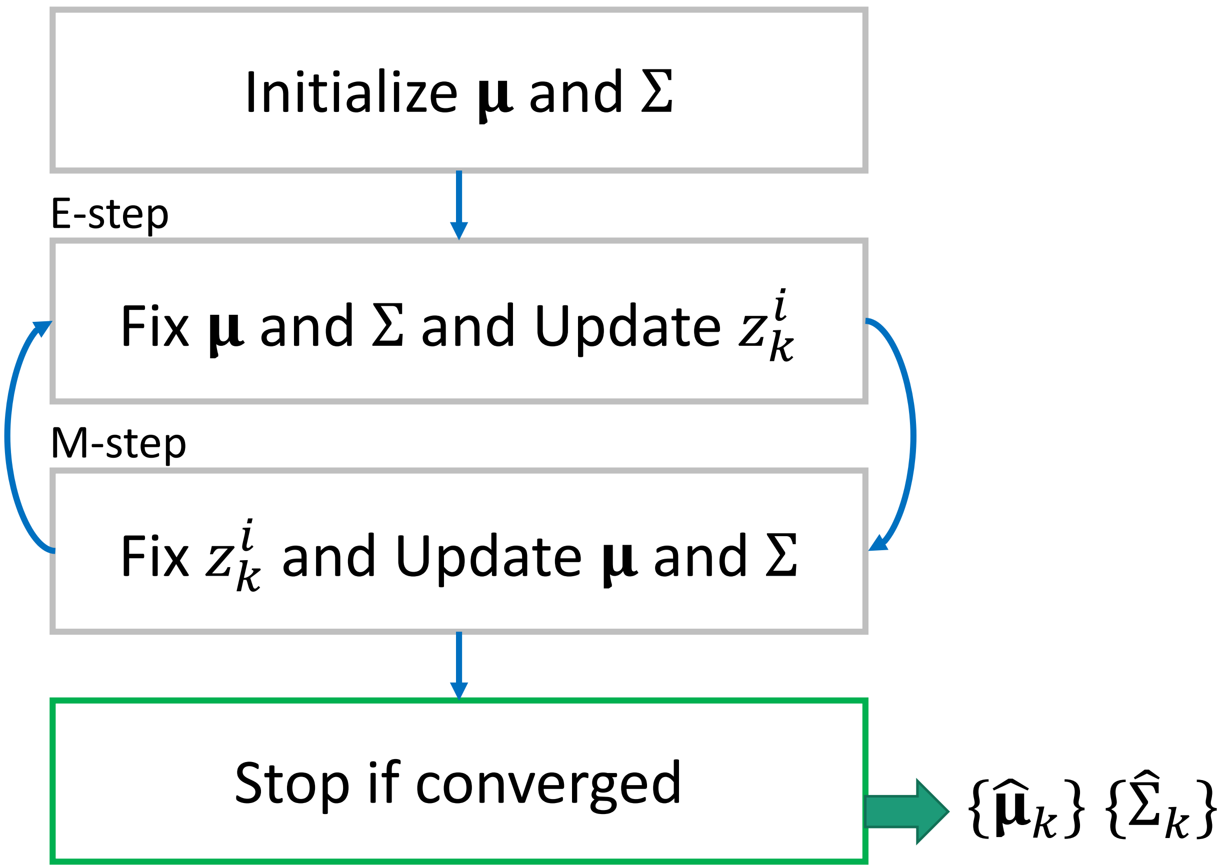
EM for GMM



$$\hat{\boldsymbol{\mu}}_k = \frac{1}{z_k}\sum_{i=1}^{N} z_k^i \mathbf{x}_i$$

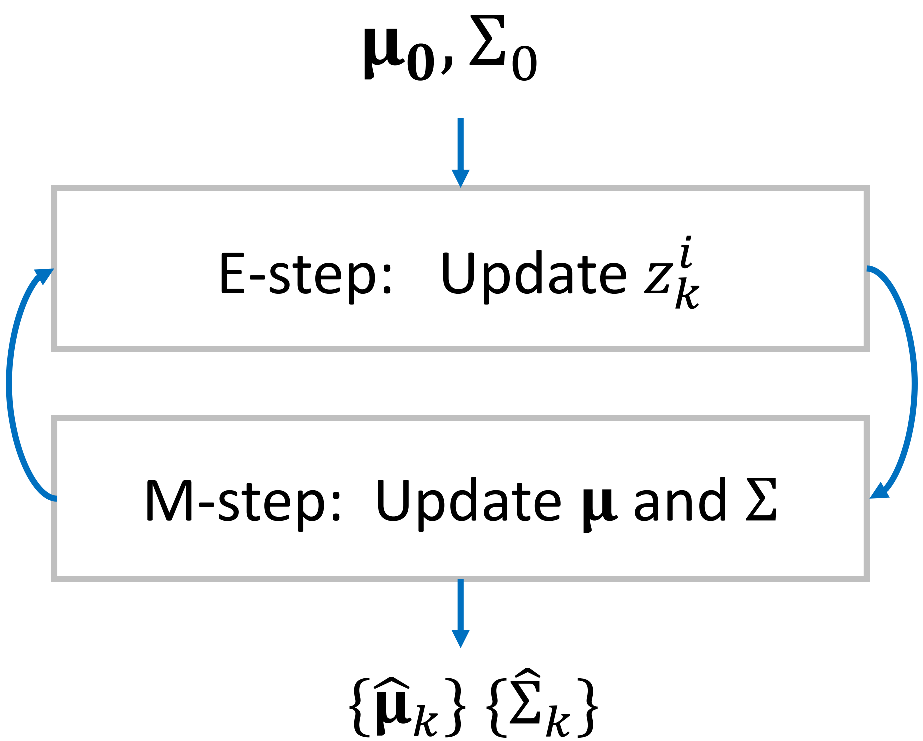
$$\hat{\Sigma}_k = \frac{1}{z_k}\sum_{i=1}^{N} z_k^i (\mathbf{x}_i - \hat{\boldsymbol{\mu}}_k)(\mathbf{x}_i - \hat{\boldsymbol{\mu}}_k)^{\top}$$

EM for GMM



Initialize $\boldsymbol{\mu}$ and Σ
E-step
Fix $\boldsymbol{\mu}$ and Σ and Update z_k^i
M-step
Fix z_k^i and Update $\boldsymbol{\mu}$ and Σ
Stop if converged
$\{\hat{\boldsymbol{\mu}}_k\}$ $\{\hat{\Sigma}_k\}$

EM for GMM


$\boldsymbol{\mu}_{\mathbf{0}}, \Sigma_0$
E-step: Update z_k^i
M-step: Update $\boldsymbol{\mu}$ and Σ
$\{\widehat{\boldsymbol{\mu}}_k\}$ $\{\widehat{\Sigma}_k\}$

Week 1.
Gaussian Model Learning

1.4.3. EM Algorithm [Advanced]

Robotics

Estimation and Learning
with Dan Lee

Week 1.
Gaussian Model Learning

1.4.3 EM Algorithm [Advanced]

Penn Engineering

Expectation-Maximization (EM)

- EM as lower-bound maximization

$$\arg\max_{\theta} \sum_i \ln p(x_i|\theta)$$

θ : All parameters

(1) Jensen's inequality

(2) Latent variable and marginal probability

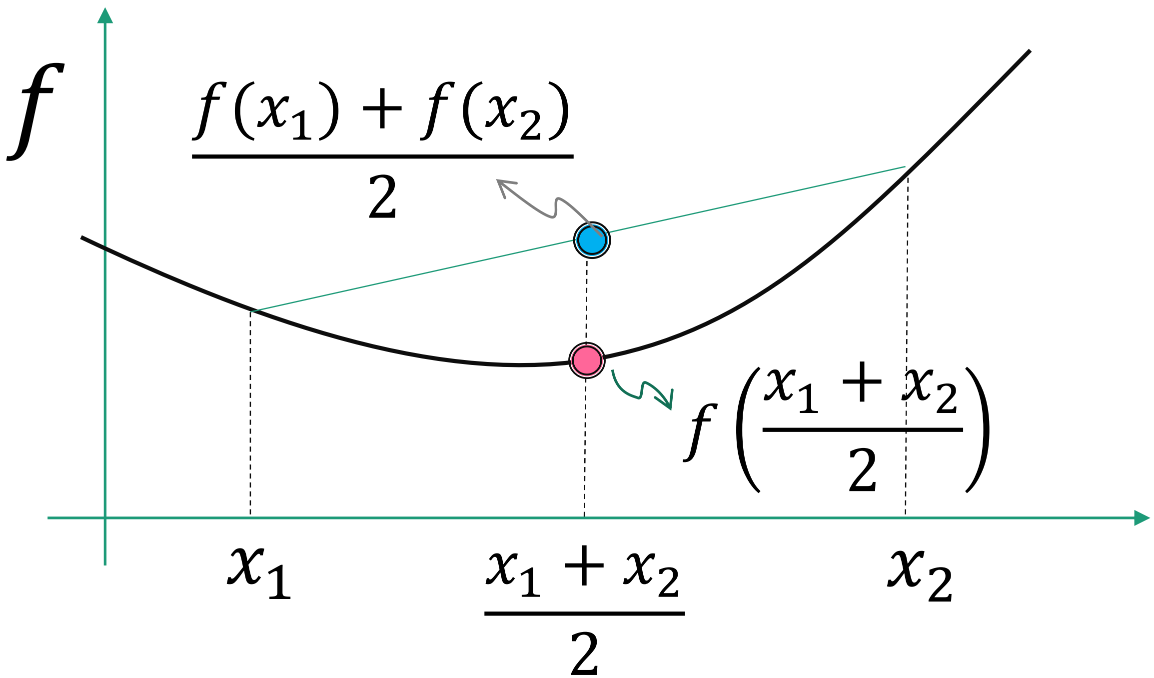
(3) Procedure : E-step and M-step.

Expectation-Maximization (EM)

(1) Jensen's inequality

f : **convex** function

$$\boxed{f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}}$$



Expectation-Maximization (EM)

(1) Jensen's inequality

f : **convex** function

$$f\left(\sum a_i x_i\right) \leq \sum a_i f(x_i)$$

$$\sum a_i = 1$$

$$a_i \geq 0$$

Expectation-Maximization (EM)

(1) Jensen's inequality

f : **concave** function

$$f\left(\sum a_i x_i\right) \geq \sum a_i f(x_i)$$

$$\sum a_i = 1$$

$$a_i \geq 0$$

Expectation-Maximization (EM)

(1) Jensen's inequality

$\ln$ is *concave*

$$\ln\left(\sum a_i p_i\right) \geq \sum a_i \ln p_i$$

$$\sum a_i = 1$$

$$a_i \geq 0$$

Expectation-Maximization (EM)

(2) latent variable z

$$p(X|\theta) = \sum_{Z} p(X, Z|\theta)$$

(From definition of marginal probability)

Expectation-Maximization (EM)

(2) latent variable

$$\ln p(X|\theta) = \ln \sum_{Z} p(X, Z|\theta)$$

Log-likelihood

$$= \ln \sum_{Z} q(Z) \frac{p(X, Z|\theta)}{q(Z)} \geq \sum_{Z} q(Z) \ln \frac{p(X, Z|\theta)}{q(Z)}$$

Lower bound

Note: $q(Z)$ is a valid probability distribution over Z.

Expectation-Maximization (EM)

(2) latent variable

$$\ln p(X|\theta) = \ln \sum_{Z} p(X, Z|\theta) \;\geq \sum_{Z} q(Z) \ln \frac{p(X, Z|\theta)}{q(Z)}$$

Log-likelihood

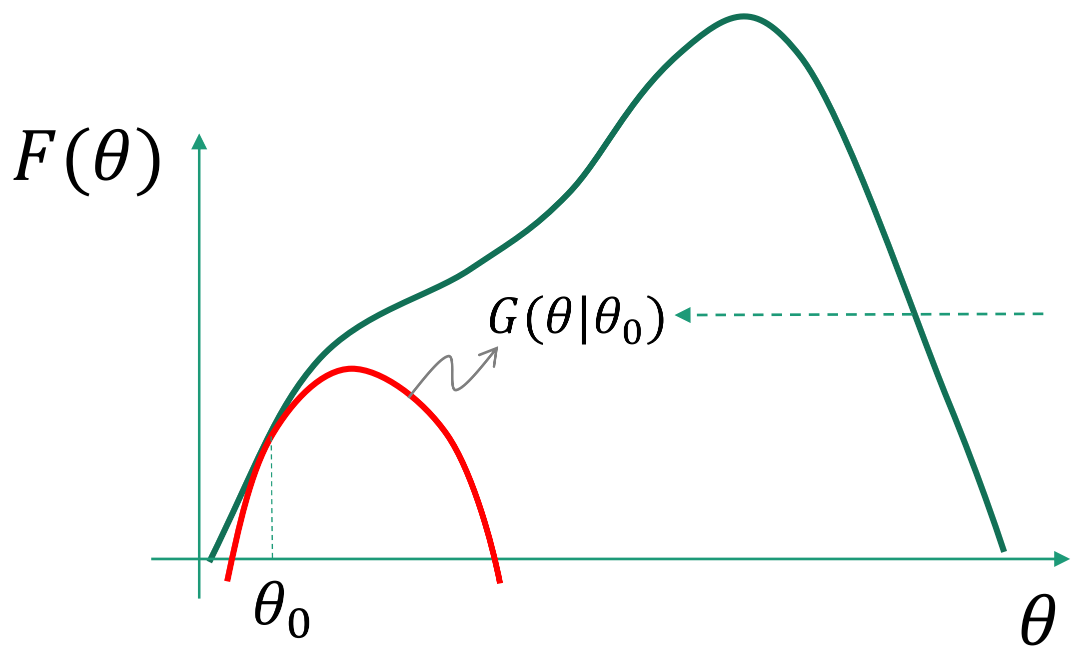
Lower bound

Find q!

Expectation-Maximization (EM)

(3a) Find a lower bound G with an initial guess

$$\underbrace{\ln p(X|\theta)}_{F} \geq \underbrace{\sum_{Z} q(Z) \ln \frac{p(X,Z|\theta)}{q(Z)}}_{G}$$

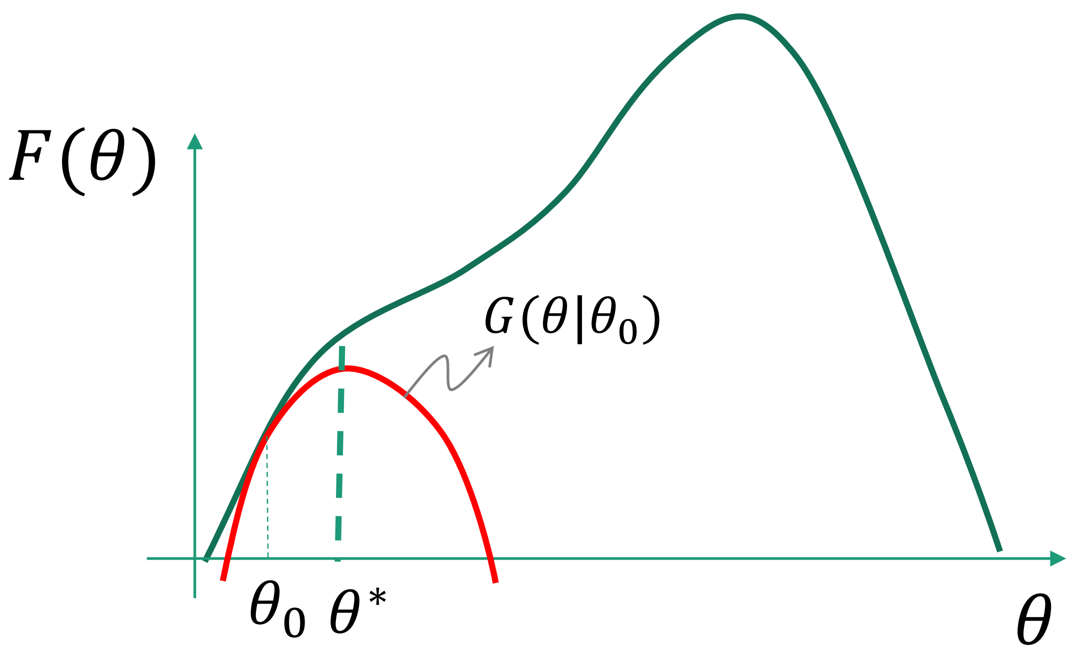


This Implies that $q(Z)$ is found based on θ_0.

Expectation-Maximization (EM)

(3b) Find $\theta^* = \arg\max_{\theta} G(\theta|\theta_0)$

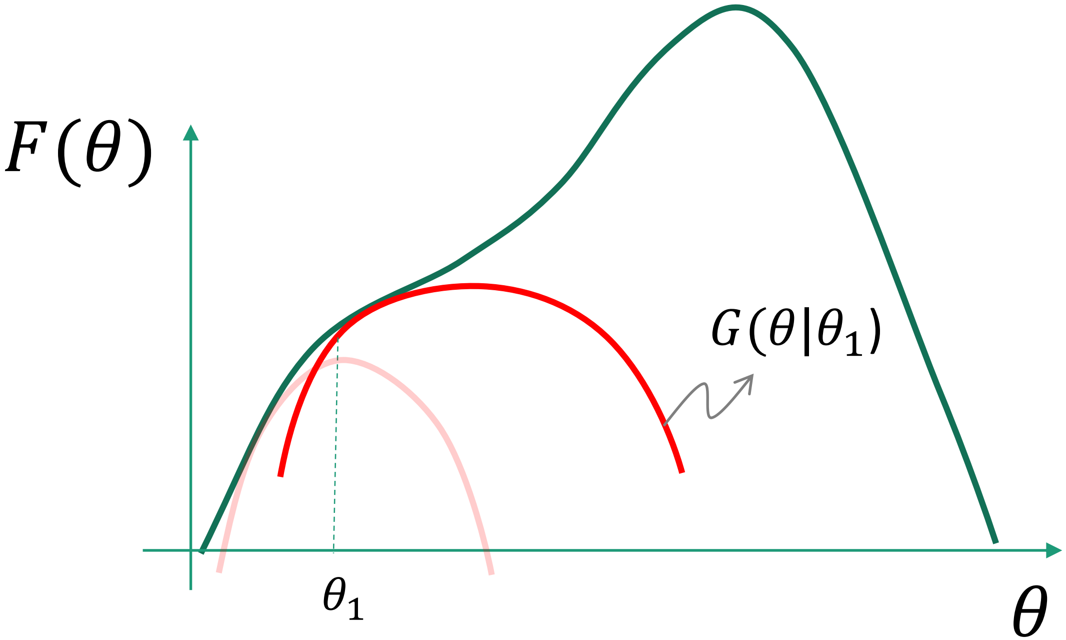
$$\underbrace{\ln p(X|\theta)}_{F} \geq \underbrace{\sum_{Z} q(Z) \ln \frac{p(X,Z|\theta)}{q(Z)}}_{G}$$



Expectation-Maximization (EM)

(3c) Find a new lower bound G with $\theta_1 \leftarrow \theta^*$

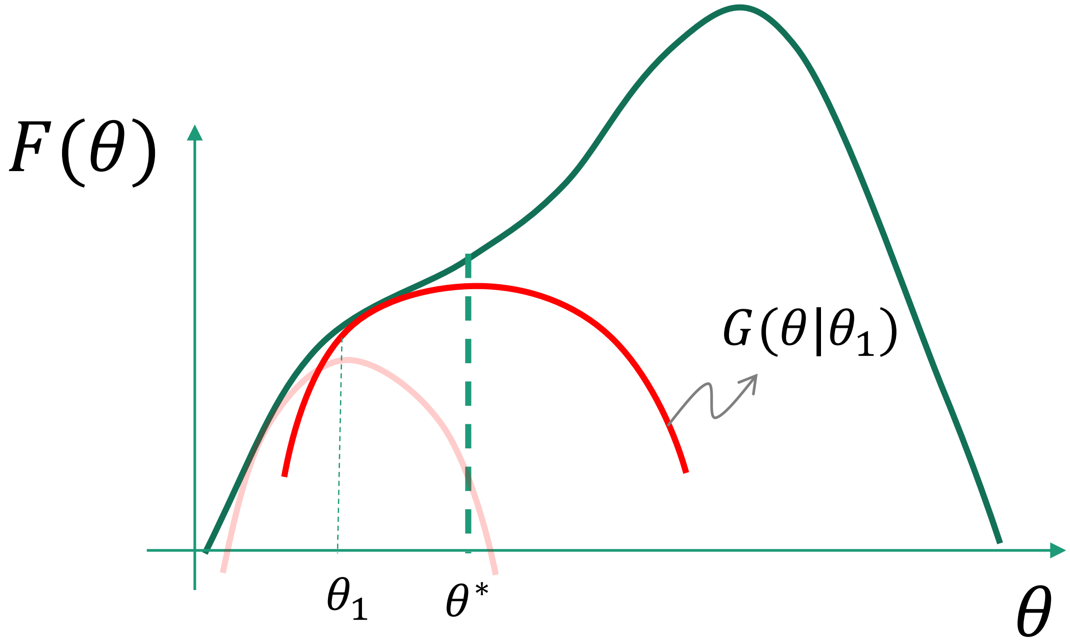
$$\underbrace{\ln p(X|\theta)}_{F} \geq \underbrace{\sum_{Z} q(Z) \ln \frac{p(X,Z|\theta)}{q(Z)}}_{G}$$



Expectation-Maximization (EM)

(3d) Find $\theta^* = \arg\max_{\theta} G(\theta|\theta_1)$

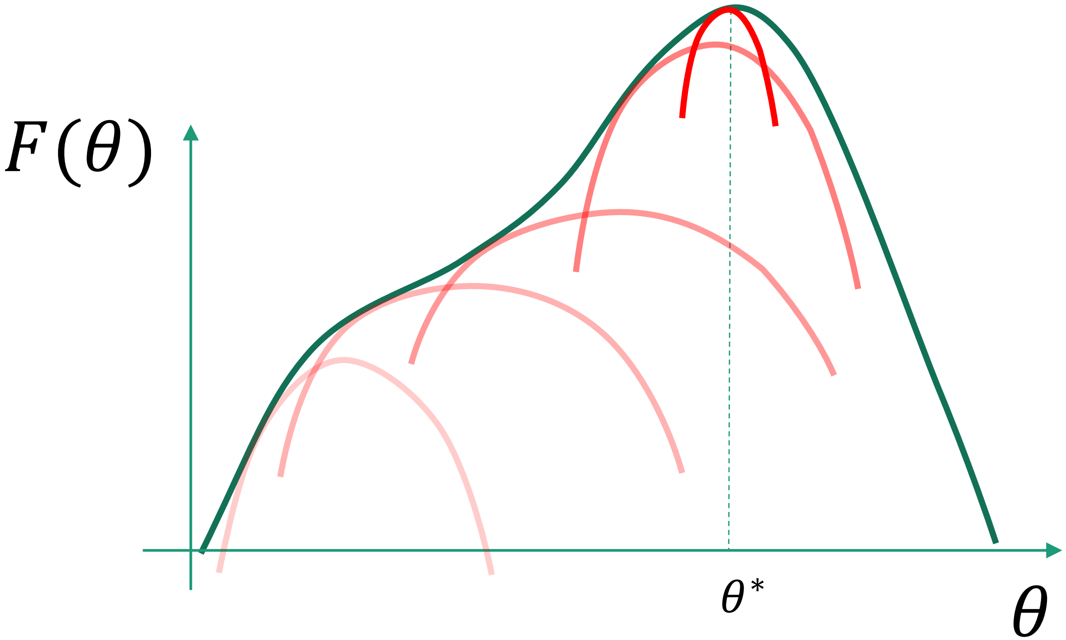
$$\underbrace{\ln p(X|\theta)}_{F} \geq \underbrace{\sum_{Z} q(Z) \ln \frac{p(X,Z|\theta)}{q(Z)}}_{G}$$



Expectation-Maximization (EM)

(4e) Repeat (until converged)

$$\underbrace{\ln p(X|\theta)}_{F} \geq \underbrace{\sum_{Z} q(Z) \ln \frac{p(X,Z|\theta)}{q(Z)}}_{G}$$



Expectation-Maximization (EM)

Initialize $\hat{\theta}$

E-step

Find a lower bound $G(\theta|\hat{\theta})$

This Implies that $q(Z)$ is found based on $\hat{\theta}$.

$$\hat{\theta}_{new} = \arg\max_{\theta} G(\theta|\hat{\theta})$$

Stop if converged

Expectation-Maximization (EM)

Initialize $\hat{\theta}$

Find a lower bound $G(\theta|\hat{\theta})$

M-step

$$\hat{\theta}_{new} = \arg\max_{\theta} G(\theta|\hat{\theta})$$

Stop if converged