EMA

Let's assume the parameter μ is set to 0.1.

Initial Setup

1. At time $t = 0$, Agent A has no prior information about Agent B's strategy. Let's initialize the moving average $\bar{y}(0)$ to be equal for all actions:
   $\bar{y}(0) = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$
   This represents an equal probability for Rock, Paper, and Scissors.

Observations and Updates

2. At time $t = 1$, Agent B plays Rock. The unit vector representation for Rock is:
   $\mathbf{u}_{\text{Rock}} = [1, 0, 0]$

Agent A updates its estimate using the EMA formula:
$\bar{y}(1) = (1 - \mu)\bar{y}(0) + \mu\mathbf{u}_{\text{Rock}}$
$\bar{y}(1) = (1 - 0.1)\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] + 0.1[1, 0, 0]$
$\bar{y}(1) = [0.3, 0.3, 0.3] + [0.1, 0, 0]$
$\bar{y}(1) = [0.4, 0.3, 0.3]$

3. At time $t = 2$, Agent B plays Scissors. The unit vector representation for Scissors is:
   $\mathbf{u}_{\text{Scissors}} = [0, 0, 1]$

Agent A updates its estimate again:
$\bar{y}(2) = (1 - \mu)\bar{y}(1) + \mu\mathbf{u}_{\text{Scissors}}$
$\bar{y}(2) = (1 - 0.1)[0.4, 0.3, 0.3] + 0.1[0, 0, 1]$
$\bar{y}(2) = [0.36, 0.27, 0.27] + [0, 0, 0.1]$
$\bar{y}(2) = [0.36, 0.27, 0.37]$

4. At time $t = 3$, Agent B plays Paper. The unit vector representation for Paper is:
   $\mathbf{u}_{\text{Paper}} = [0, 1, 0]$

Agent A updates its estimate:
$\bar{y}(3) = (1 - \mu)\bar{y}(2) + \mu \mathbf{u}_{\text{Paper}}$
$\bar{y}(3) = (1 - 0.1)[0.36, 0.27, 0.37] + 0.1[0, 1, 0]$
$\bar{y}(3) = [0.324, 0.243, 0.333] + [0, 0.1, 0]$
$\bar{y}(3) = [0.324, 0.343, 0.333]$

Interpretation

After these updates, Agent A's estimate of Agent B's strategy is:
$\bar{y}(3) = [0.324, 0.343, 0.333]$

This means Agent A estimates that Agent B is choosing Rock approximately 32.4% of the time, Paper 34.3% of the time, and Scissors 33.3% of the time. As Agent A continues to observe more actions from Agent B, it will keep updating this estimate, giving more weight to recent actions due to the nature of the EMA method.

Bayesian Estimation

Consider a scenario where the opponent (Agent B) can choose one of three actions: Rock, Paper, or Scissors. We'll discretize the strategy space into a grid with mixed strategies. For simplicity, assume we have a limited grid of possible strategies:

- $y_1 = [0.8, 0.1, 0.1]$: Plays Rock 80% of the time, Paper 10% of the time, and Scissors 10% of the time.
- $y_2 = [0.1, 0.8, 0.1]$: Plays Paper 80% of the time, Rock 10% of the time, and Scissors 10% of the time.
- $y_3 = [0.1, 0.1, 0.8]$: Plays Scissors 80% of the time, Rock 10% of the time, and Paper 10% of the time.

The prior probabilities for these strategies are equal:
$P(y_1) = P(y_2) = P(y_3) = \frac{1}{3}$

Observations and Updates

We observe the following actions from Agent B over three time steps: Rock, Scissors, Rock. We'll use Bayesian inference to update our belief about Agent B's strategy.

Initial Setup

1. **Prior probabilities:**

$$P(y_1) = P(y_2) = P(y_3) = \frac{1}{3}$$

2. **No observations yet:**

$$P(H|y_i) = 1 \quad \text{for all} \quad y_i$$

Step-by-Step Process

First Observation: Rock

1. **Likelihoods:**

$$P(\text{Rock}|y_1) = 0.8$$

$$P(\text{Rock}|y_2) = 0.1$$

$$P(\text{Rock}|y_3) = 0.1$$

2. **Posterior probabilities using Bayes' rule:**

$$P(y_1|\text{Rock}) = \frac{P(\text{Rock}|y_1)P(y_1)}{P(\text{Rock}|y_1)P(y_1) + P(\text{Rock}|y_2)P(y_2) + P(\text{Rock}|y_3)P(y_3)}$$

$$P(y_1|\text{Rock}) = \frac{0.8 \cdot \frac{1}{3}}{0.8 \cdot \frac{1}{3} + 0.1 \cdot \frac{1}{3} + 0.1 \cdot \frac{1}{3}} = \frac{0.8}{0.8 + 0.1 + 0.1} = \frac{0.8}{1} = 0.8$$

Similarly,

$$P(y_2|\text{Rock}) = \frac{0.1 \cdot \frac{1}{3}}{1} = 0.1$$

$$P(y_3|\text{Rock}) = \frac{0.1 \cdot \frac{1}{3}}{1} = 0.1$$

After observing Rock, the updated probabilities are:
$P(y_1|\text{Rock}) = 0.8, \quad P(y_2|\text{Rock}) = 0.1, \quad P(y_3|\text{Rock}) = 0.1$

Second Observation: Scissors

1. **Likelihoods:**

$$P(\text{Scissors}|y_1) = 0.1$$

$$P(\text{Scissors}|y_2) = 0.1$$

$$P(\text{Scissors}|y_3) = 0.8$$

2. **Posterior probabilities:**

$$P(y_1|\text{Rock, Scissors}) = \frac{P(\text{Scissors}|y_1)P(y_1|\text{Rock})}{P(\text{Scissors}|y_1)P(y_1|\text{Rock}) + P(\text{Scissors}|y_2)P(y_2|\text{Rock}) + P}$$

$$P(y_1|\text{Rock, Scissors}) = \frac{0.1 \cdot 0.8}{0.1 \cdot 0.8 + 0.1 \cdot 0.1 + 0.8 \cdot 0.1} = \frac{0.08}{0.08 + 0.01 + 0.08} = \frac{0.08}{0.17}$$

$$\frac{P(\text{Scissors}|y_1)P(y_1|\text{Rock})}{P(\text{Scissors}|y_2)P(y_2|\text{Rock}) + P(\text{Scissors}|y_3)P(y_3|\text{Rock})}$$

$$= \frac{0.08}{0.08 + 0.01 + 0.08} = \frac{0.08}{0.17} \approx 0.47$$

After observing Scissors, the updated probabilities are:

$$P(y_1|\text{Rock, Scissors}) = 0.47, \quad P(y_2|\text{Rock, Scissors}) = 0.06, \quad P(y_3|\text{Rock, Scissors}) = 0.47$$

Bayesian Formula

Bayes' theorem can be written as:
$P(y|H) = \frac{P(H|y)P(y)}{P(H)}$
Where:

- $P(y|H)$ is the posterior probability of the hypothesis y given the history H of observations.
- $P(H|y)$ is the likelihood of the history H given the hypothesis y.
- $P(y)$ is the prior probability of the hypothesis y.
- $P(H)$ is the marginal likelihood of the history H.

Applying to Our Problem

In the context of our problem, we want to update our belief about the strategy y_1 after observing the actions Rock and Scissors. The updated (posterior) probability of y_1 given the observations (Rock and Scissors) is calculated using Bayes' theorem:

$P(y_1|\text{Rock, Scissors}) = \frac{P(\text{Scissors}|\text{Rock}, y_1)P(y_1|\text{Rock})}{P(\text{Scissors}|\text{Rock})}$

Where:

- $P(y_1|\text{Rock, Scissors})$ is the posterior probability of strategy y_1 after observing Rock and Scissors.
- $P(\text{Scissors}|\text{Rock}, y_1)$ is the likelihood of observing Scissors given that the opponent followed strategy y_1 after Rock.
- $P(y_1|\text{Rock})$ is the prior probability of strategy y_1 after observing Rock.
- $P(\text{Scissors}|\text{Rock})$ is the marginal likelihood of observing Scissors after Rock, summed over all possible strategies.

Breaking Down the Numerator

The numerator in the Bayes' theorem is:
$P(\text{Scissors}|\text{Rock}, y_1)P(y_1|\text{Rock})$

Let's break down each component:

1. **Likelihood** $P(\text{Scissors}|\text{Rock}, y_1)$:
   - This term represents the probability of observing Scissors given that the opponent is following strategy y_1 after having observed Rock.
   - For y_1 (which plays Rock 80% of the time, Paper 10% of the time, and Scissors 10% of the time), this likelihood is simply $P(\text{Scissors}|y_1) = 0.1$.
2. **Prior** $P(y_1|\text{Rock})$:
   - This is the updated probability of the strategy y_1 after observing Rock.
   - From the previous update, we calculated $P(y_1|\text{Rock}) = 0.8$.

Forming the Numerator

Combining these two components:
$P(\text{Scissors}|\text{Rock}, y_1)P(y_1|\text{Rock}) = 0.1 \times 0.8 = 0.08$

This product forms the numerator of the Bayesian update formula because it combines the likelihood of the new observation given the strategy and the prior probability of the strategy after the previous observation.