HW Week9

Shaya Engelman

March 21, 2024

Q1

The price of one share of stock in the Pilsdorff Beer Company is given by Yn on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $= 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_{365} is: (a) 100. (b) 110. (c) 120.

```r
# Given parameters
mean <- 100 # givent he mu is 0, all the values of Yn will be 100
sd <- sqrt(1/4) # given the variance is 1/4
n <- 365 # number of days

# (a) Probability that Y_365 >= 100
prob_a <- pnorm(100, mean, sd * sqrt(n)) # using the CLT to calculate the probability as a normal distr
cat("Probability that Y_365 >= 100:", prob_a, "\n")
```

```
## Probability that Y_365 >= 100: 0.5
```

```r
# (b) Probability that Y_365 >= 110
prob_b <- pnorm(110, mean, sd * sqrt(n))
cat("Probability that Y_365 >= 110:", prob_b, "\n")
```

```
## Probability that Y_365 >= 110: 0.8524151
```

```r
# (c) Probability that Y_365 >= 120
prob_c <- pnorm(120, mean, sd * sqrt(n))
cat("Probability that Y_365 >= 120:", prob_c, "\n")
```

```
## Probability that Y_365 >= 120: 0.9818565
```

Q2

Calculate the expected value and variance of the binomial distribution using the moment generating function.

The moment generating function of the binomial distribution is given by:

$$M(t) = (1 - p + pe^t)^n$$

Expected value

The expected value $E[X]$ of a random variable X is given by the first derivative of the MGF evaluated at $t = 0$:

$$E[X] = M'_X(0)$$

Taking the derivative of $M_X(t)$ with respect to $t = 0$:

$$M'_X(t) = npe^0(pe^0 + 1 - p)^{n-1}$$

$$M'_X(0) = np$$

Therefore, the expected value of the binomial distribution is np.

variance

The variance $Var(X)$ of a random variable X is given by the second derivative of the MGF evaluated at $t = 0$ plus the square of the expected value:

$$Var(X) = M''_X(0) + [M'_X(0)]^2$$

Taking the second derivative of $M_X(t)$ with respect to $t = 0$:

$$M''_X(0) = npe^0(pe^0 + 1 - p)^{n-1} + np^2e^{2*0}(pe^0 + 1 - p)^{n-2}$$

$$M''_X(0) = np + np^2$$

$$M'_X(0) = np$$

Plugging these values into the variance formula:

$$Var(X) = np + np^2 + (np)^2$$

$$Var(X) = np + np^2 + n^2p^2$$

$$Var(X) = np(1 - p)$$

Therefore, the variance of the binomial distribution is $np(1 - p)$.

Q3

Calculate the expected value and variance of the exponential distribution using the moment generating function.

For the exponential distribution with rate parameter λ, the PDF is given by:

$$f(x|\lambda) = \lambda e^{-\lambda x}$$

The MGF of the exponential distribution is:

$$M_X(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

Expected value

The expected value $E[X]$ of a random variable X is given by the first derivative of the MGF evaluated at $t = 0$:

$$E[X] = -M'_X(0)$$

$$E[X] = -\left(-\frac{\lambda}{\lambda^2}\right) = \frac{1}{\lambda}$$

Therefore, the expected value of the exponential distribution is $\frac{1}{\lambda}$.

Variance

The variance $Var(X)$ of a random variable X is given by the second derivative of the MGF evaluated at $t = 0$, plus the square of the expected value:

$$Var(X) = M''_X(0) + (E[X])^2$$

The second derivative of the MGF is:

$$M''_X(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$M''_X(0) = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$$

Therefore, the variance of the exponential distribution is $\frac{2}{\lambda^2}$.