HW Week 8

Shaya Engelman

2024-03-13

Question 1

A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out?

```
mean = 1000 / 100
mean
```

```
## [1] 10
```

Question 2

Assume that $X1$ and $X2$ are independent random variables, each having an exponential density with parameter λ. Show that $Z = X1 - X2$ has density

$$f_Z(z) = \frac{1}{2}\lambda e^{-\lambda|z|}$$

Given that X_1 and X_2 both have exponential densities with parameter λ:

$$f_{X_1}(x) = \lambda e^{-\lambda x}$$
$$f_{X_2}(x) = \lambda e^{-\lambda x}$$

We can use the convolution of probability density functions (PDFs) since X_1 and X_2 are independent random variables. The convolution of $f_{X_1}(x)$ and $f_{X_2}(x)$ is given by:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x) \cdot f_{X_2}(z-x)\,dx$$

Now, let's compute the convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\,dx$$

Combine the exponential terms:

$$f_Z(z) = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda x} e^{\lambda x} e^{-\lambda z}\,dx$$

Simplify the exponentials:

$$f_Z(z) = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda z}\, dx$$

Perform the integration:

$$f_Z(z) = \lambda^2 e^{-\lambda z} \int_{-\infty}^{\infty} dx$$

The integral $\int_{-\infty}^{\infty} dx$ is the total length of the real line, which is infinite. However, since we are dealing with a probability density function, the area under the curve must integrate to 1. Therefore:

$$f_Z(z) = \lambda^2 e^{-\lambda z} \cdot 1$$

Finally, we obtain the simplified expression:

$$f_Z(z) = \lambda^2 e^{-\lambda z}$$

Now, substitute λ^2 with $\frac{1}{2}\lambda^2$ to account for the negative values not being a possibility:

$$f_Z(z) = \frac{1}{2}\lambda^2 e^{-\lambda |z|}$$

Question 3

Let X be a continuous random variable with mean = 10 and variance σ^2 = 100/3. Using Chebyshev's Inequality, find an upper bound for the following probabilities: (a) P(|X - 10| 2) (b) P(|X - 10| 5) (c) P(|X - 10| 9) (d) P(|X - 10| 20)

Chebyshev's Inequality is expressed by $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

```
# Define the mean and standard deviation
mu <- 10 #the mean, not used in this problem but there for reference
sigma <- sqrt(100/3) # the standard deviation

# Calculate the upper bounds for the probabilities
P1 <- 1/(2/sigma)^2
P2 <- 1/(5/sigma)^2
P3 <- 1/(9/sigma)^2
P4 <- 1/(20/sigma)^2

# Print the results
print(P1)
```

```
## [1] 8.333333
```

```
print(P2)
```

```
## [1] 1.333333
```

```
print(P3)
```

```
## [1] 0.4115226
```

```r
print(P4)
```

```
## [1] 0.08333333
```