Problem Set 1 - Graphical Adventures: Navigating the Nodes!

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Warm-up Problems

Warm-Up 1

Suppose G is a simple, connected, non-complete graph. Prove that there exists three nodes u, v and w where uv, $vw \in$ but $uw \notin E(G)$.

Solution:

Prove by contradiction!

Warm-Up 2

Suppose that in a graph G, there are exactly 2 odd vertices. Prove that there must be a path between these 2 odd nodes.

Solution:

Prove by contradiction again :)

Warm-Up 3 (a theorem to be exact)

Suppose G is a directed graph with a set of nodes v_i. Prove that

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v)$$

where $d^+(v)$ denotes the out-degree of node v and $d^-(v)$ denotes the in-degree of node v.

Problems

Problem 1: Node Necessity Conundrum

In a graph G, if we remove the node v, then G will not be connected. Prove that there exists 2 nodes in G like a and b where all the paths from a to b pass through v.

Problem 2: Minimum Length of a Path

Prove that in a graph G, there exists a path of length at least δ.

Problem 3: Really Simple

Prove that in every graph G, the following equation holds:

$$\delta \leq \frac{2m}{n} \leq \Delta$$

Problem 4: Triangle Inequality

Prove that for any 3 nodes x, y, and z in a graph G, the following inequality holds:

$$d(x, y) + d(x, z) \geq d(x, z)$$

Problem 5: How to Stay Connected?

Prove that if G is self-complementary, then it is connected.

Problem 6: Edge-Vertex Relationship in Simple Connected Graphs

Suppose in a simple and connected graph G, we have $diam(G) = 2$ and $\Delta = n - 2$. Prove that

$$m \geq 2n - 4$$

where n is the number of vertices and m is the number of edges in G.

Problem 7: Bonus Question

Note: Your solution for the bonus question will be reviewed only after you have solved all the previous questions, and I have approved your solution. Moreover, keep in mind that you must communicate your solution in person; otherwise, no points will be awarded!

Suppose $V = \{A_1, A_2, A_3, \ldots, A_n\}$ is a subset of n points in space, such that the distance between any two points is at least 1. Prove that at most $3n$ pairs of these points can have distances exactly equal to 1.