Problem Set 1 - Graphical Adventures: Navigating the Nodes!

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October 2024

Problems

Problem 1: Exactly 2 Odd Vertices

Suppose that in a graph G, there are exactly 2 odd vertices. Prove that there must be a path between these 2 odd nodes.

Problem 2: Minimum Length of a Path

Prove that in a graph G, there exists a path of length at least δ.

Problem 3: Really Simple

Prove that in every graph G, the following equation holds:

$$\delta \leq \frac{2m}{n} \leq \Delta$$

Problem 4: Triangle Inequality

Prove that for any 3 nodes x, y, and z in a graph G, the following inequality holds:

$$d(x, y) + d(x, z) \geq d(x, z)$$

Problem 5: How to Stay Connected?

Prove that if G is self-complementary, then it is connected.

Problem 6: Edge-Vertex Relationship in Simple Connected Graphs

Suppose in a simple and connected graph G, we have $diam(G) = 2$ and $\Delta = n - 2$. Prove that

$$m \geq 2n - 4$$

where n is the number of vertices and m is the number of edges in G.

Problem 7: Bonus Question

Note: Your solution for the bonus question will be reviewed only after you have solved all the previous questions, and I have approved your solution. Moreover, keep in mind that you must communicate your solution in person; otherwise, no points will be awarded!

Suppose $V = A_1, A_2, A_3, \ldots, A_n$ is a subset of n points in space, such that the distance between any two points is at least 1. Prove that at most $3n$ pairs of these points can have distances exactly equal to 1.