

1D Convolution

Two kinds of 1D convolutions are illustrated here: linear and circular convolutions.

Circular convolution

Consider the following two real vectors

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4 \times 1}$$

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}_{4 \times 1}$$

The circular convolution of \mathbf{a} and \mathbf{b} results in a third vector \mathbf{w} .

The elements of \mathbf{w} can be schematically computed as follows:

① Keep vector \mathbf{a}^T in a fixed horizontal position.

$$a_0 \quad a_1 \quad a_2 \quad a_3$$

②

Define a temporary vector by repeating the elements of \mathbf{b}^T in a reversed order.

$$b_3 \quad b_2 \quad b_1 \quad b_0 \quad b_3 \quad b_2 \quad b_1 \quad b_0$$

③ Place the temporary vector right below \mathbf{a}^T so that b_0 is aligned with a_0 .

④

Select the elements of the temporary vector that are aligned with \mathbf{a}^T . In our example, they are represented in red and the remaining elements in gray.

⑤ Compute the first element of \mathbf{w} by multiplying and adding the aligned elements:

$$w_0 = b_0 \cdot a_0 + b_3 \cdot a_1 + b_2 \cdot a_2 + b_1 \cdot a_3$$

⑥ The succeeding elements of \mathbf{w} are computed by sliding the temporary vector to right and repeating the steps above:

				a_0	a_1	a_2	a_3		
b_3	b_2	b_1		b_0	b_3	b_2	b_1	b_0	unused elements
	b_3	b_2		b_1	b_0	b_3	b_2	b_1	b_0
		b_3		b_2	b_1	b_0	b_3	b_2	b_1
			b_3	b_2	b_1	b_0	b_3	b_2	b_1

unused elements

$\rightarrow w_0$

$\rightarrow w_1$

$\rightarrow w_2$

$\rightarrow w_3$

This iterative scheme used to compute the elements of \mathbf{w} is equivalent to the following matrix-vector product:

$$\mathbf{w} = \mathbf{C} \mathbf{a}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_3 & b_2 & b_1 \\ b_1 & b_0 & b_3 & b_2 \\ b_2 & b_1 & b_0 & b_3 \\ b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

This is a circulant matrix formed by the elements of \mathbf{b}

$$c_{ij} = b_{(i-j) \bmod N}$$

The red elements shown above form a circulant matrix \mathbf{C}

Linear convolution

Consider the following two real vectors

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4 \times 1}$$

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}_{3 \times 1}$$

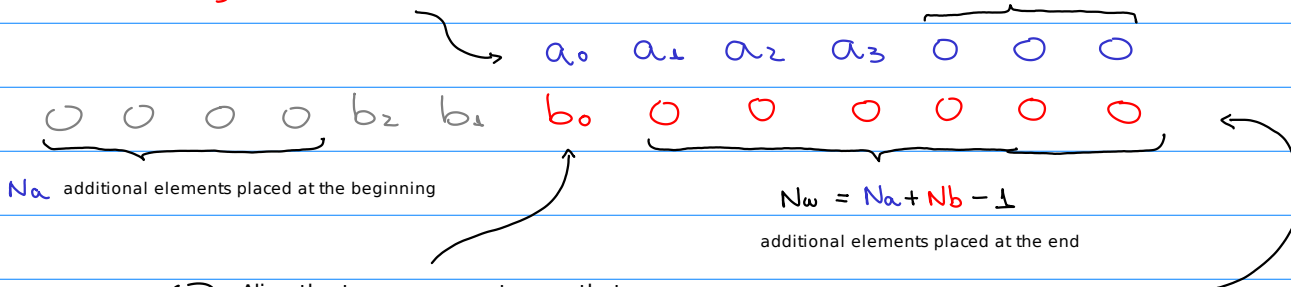
number of elements of \mathbf{a}
 number of elements of \mathbf{b}
 $N_w = N_a + N_b - 1$
 number of elements of \mathbf{b}

The linear convolution of \mathbf{a} and \mathbf{b} results in a third vector \mathbf{w} .

The elements of \mathbf{w} can be schematically computed as follows:

- ① Define a temporary vector by appending N_b zeros at the end of \mathbf{a}^T

N_b additional elements



- ③ Align the temporary vectors so that b_0 is placed right below a_0

- ② Define a temporary vector by rearranging \mathbf{b}^T in a reversed order and appending zeros to its beginning and end.

- ④ Select the elements of the temporary vectors that are vertically aligned. In our example, they are represented in red and the remaining elements in gray.

- ⑤ Compute the first element of \mathbf{w} by multiplying and adding the aligned elements:

$$w_0 = b_0 \cdot a_0 + 0 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0$$

- ⑥ The succeeding elements of \mathbf{w} are computed by sliding the temporary vector to right and repeating the steps above:

