

Gaussian elimination

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}_{N \times N}$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}_{N \times 1}$$

$$N = 4$$

$$K=0 \quad C^{(0)} = C \quad \rightarrow \quad C = \left[A \mid y \right]_{N \times (N+1)}$$

$$K=1 \quad C^{(1)} = \left(I - t^{(1)} \cdot u^{(0)T} \right) C^{(0)}$$

$$K=2 \quad C^{(2)} = \left(I - t^{(2)} \cdot u^{(1)T} \right) C^{(1)}$$

$$K=3 \quad C^{(3)} = \left(I - t^{(3)} \cdot u^{(2)T} \right) C^{(2)}$$

$$\underline{K=1}$$

this is the pivot for iteration $k = 1$

$$C^{(0)} = \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix}$$

$$t^{(1)} = \begin{bmatrix} t_0^{(1)} \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{a_{10}^{(0)}}{a_{00}^{(0)}} \\ \frac{a_{20}^{(0)}}{a_{00}^{(0)}} \\ \frac{a_{30}^{(0)}}{a_{00}^{(0)}} \end{bmatrix} \quad u^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$t^{(k)}[k:] = \frac{C^{(k-1)}[k:, k-1]}{C^{(k-1)}[k-1, k-1]} \rightarrow \begin{bmatrix} a_{10}^{(0)} \\ a_{20}^{(0)} \\ a_{30}^{(0)} \end{bmatrix}$$

$$C^{(1)} = (I - t^{(1)} \cdot u^{(0)T}) C^{(0)} = C^{(0)} - t^{(1)} \cdot (u^{(0)T} C^{(0)}) =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix} - \begin{bmatrix} 0 \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ t_1^{(1)} a_{00}^{(0)} & t_1^{(1)} a_{01}^{(0)} & t_1^{(1)} a_{02}^{(0)} & t_1^{(1)} a_{03}^{(0)} & t_1^{(1)} y_0^{(0)} \\ t_2^{(1)} a_{00}^{(0)} & t_2^{(1)} a_{01}^{(0)} & t_2^{(1)} a_{02}^{(0)} & t_2^{(1)} a_{03}^{(0)} & t_2^{(1)} y_0^{(0)} \\ t_3^{(1)} a_{00}^{(0)} & t_3^{(1)} a_{01}^{(0)} & t_3^{(1)} a_{02}^{(0)} & t_3^{(1)} a_{03}^{(0)} & t_3^{(1)} y_0^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} - t_1^{(1)} a_{00}^{(0)} & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\ a_{20}^{(0)} - t_2^{(1)} a_{00}^{(0)} & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\ a_{30}^{(0)} - t_3^{(1)} a_{00}^{(0)} & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ 0 & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\ 0 & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\ 0 & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)} \end{bmatrix}$$

$$\begin{cases} a_{10}^{(0)} - t_1^{(1)} a_{00}^{(0)} = a_{10}^{(0)} - \left(\frac{a_{10}^{(0)}}{a_{00}^{(0)}} \right) a_{00}^{(0)} = a_{10}^{(0)} - a_{10}^{(0)} = 0 \\ a_{20}^{(0)} - t_2^{(1)} a_{00}^{(0)} = a_{20}^{(0)} - \left(\frac{a_{20}^{(0)}}{a_{00}^{(0)}} \right) a_{00}^{(0)} = a_{20}^{(0)} - a_{20}^{(0)} = 0 \\ a_{30}^{(0)} - t_3^{(1)} a_{00}^{(0)} = a_{30}^{(0)} - \left(\frac{a_{30}^{(0)}}{a_{00}^{(0)}} \right) a_{00}^{(0)} = a_{30}^{(0)} - a_{30}^{(0)} = 0 \end{cases}$$

$$C^{(1)} = \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ 0 & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\ 0 & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\ 0 & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)} \end{bmatrix}$$

The first line is equal to that in the original matrix C

$$= \begin{bmatrix} a_{00}^{(1)} & a_{01}^{(1)} & a_{02}^{(1)} & a_{03}^{(1)} & y_0^{(1)} \\ 0 & a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & y_1^{(1)} \\ 0 & a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & y_2^{(1)} \\ 0 & a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & y_3^{(1)} \end{bmatrix}$$

replace by

$$C^{(1)}[1:, 0] = \begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix}$$

All the math to compute matrix C(1) lead to these zeros in the first column. Besides, the "space" occupied by these zeros can be better used by storing the non-null elements of the Gauss vector (the Gauss multipliers)

Note that the elements in black can be written as follows:

$$C^{(1)}[1:, 1:] = C^{(1)}[1:, 0] \cdot C^{(0)}[0, 1:]$$

$$\begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \cdot \begin{bmatrix} a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \end{bmatrix}$$

Actually, we overwrite matrix C so that

$$C[k:, k:] = C[k:, k-1] \cdot C[k-1, k:] \text{ for any } k$$

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}_{N \times N}$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}_{N \times 1}$$

$$N = 4$$

$$K=0$$

$$C^{(0)} = C$$

$$K=1$$

$$C^{(1)} = \left(I - \underbrace{t^{(1)} \cdot u^{(0)T}}_{M^{(1)}} \right) C^{(0)}$$

$$C = \begin{bmatrix} A & y \end{bmatrix}_{N \times (N+1)}$$

$$C^{(0)} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$$t^{(1)} = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

$$u^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M^{(1)} = \underbrace{\begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}}_{t^{(1)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{u^{(0)T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \end{bmatrix}$$

$$C^{(1)} = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$$= \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$K=1$

$$C = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$$C_{K_i, 0}^{K_i, 0} = \frac{C_{[1:, 0]}^{K_i, 0}}{C_{[0, 0]}^{K_i, 0}} = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

$$C = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$

$$C_{[1:, 1:]}^{K_i, K_i} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} - \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} - \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$$