Gaussian elimination with partial pivoting

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}_{N \times N} \qquad \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}_{N \times 1} \qquad N = 4$$

$$\mathbf{C} = \left[\mathbf{A} \,\vdots\, \mathbf{y} \right]_{N \times (N+1)}$$

$$K=0 \qquad \mathbf{C}^{(0)} = \mathbf{C}$$

$$K=1 \qquad \mathbf{C}^{(1)} = \left(\mathbf{I} - \mathbf{t}^{(1)} \cdot \mathbf{u}^{(0)^T} \right) \mathbf{P}^{(1)} \mathbf{C}^{(0)}$$

$$K=2 \qquad \mathbf{C}^{(2)} = \left(\mathbf{I} - \mathbf{t}^{(2)} \cdot \mathbf{u}^{(1)^T} \right) \mathbf{P}^{(2)} \mathbf{C}^{(1)}$$

$$K=3 \qquad \mathbf{C}^{(3)} = \left(\mathbf{I} - \mathbf{t}^{(3)} \cdot \mathbf{u}^{(2)^T} \right) \mathbf{P}^{(3)} \mathbf{C}^{(2)}$$

$K = 1$

The pivot is chosen as that element having the greater absolute value in column 0

$$\mathbf{C}^{(0)} = \begin{bmatrix} a^{(0)}_{00} & a^{(0)}_{01} & a^{(0)}_{02} & a^{(0)}_{03} & y^{(0)}_0 \\ a^{(0)}_{10} & a^{(0)}_{11} & a^{(0)}_{12} & a^{(0)}_{13} & y^{(0)}_1 \\ a^{(0)}_{20} & a^{(0)}_{21} & a^{(0)}_{22} & a^{(0)}_{23} & y^{(0)}_2 \\ a^{(0)}_{30} & a^{(0)}_{31} & a^{(0)}_{32} & a^{(0)}_{33} & y^{(0)}_3 \end{bmatrix}$$

Lets consider that, in the example shown at left, the element with indices 20 is that with the greatest absolute value in column 0. In this case, we should interchange rows 0 and 2. To do this, we premultiply matrix C(0) by the following permutation matrix:

$$\mathbf{P}^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The partial pivoting consists in reordering the rows of the matrix above so that the pivot (in this case, the element with indices 00) has the greatest absolute value

$$\tilde{\mathbf{C}}^{(0)} = \mathbf{P}^{(1)} \mathbf{C}^{(0)}$$

$$\tilde{\mathbf{C}}^{(0)} = \begin{bmatrix} \tilde{a}^{(0)}_{00} & \tilde{a}^{(0)}_{01} & \tilde{a}^{(0)}_{02} & \tilde{a}^{(0)}_{03} & \tilde{y}^{(0)}_0 \\ \tilde{a}^{(0)}_{10} & \tilde{a}^{(0)}_{11} & \tilde{a}^{(0)}_{12} & \tilde{a}^{(0)}_{13} & \tilde{y}^{(0)}_1 \\ \tilde{a}^{(0)}_{20} & \tilde{a}^{(0)}_{21} & \tilde{a}^{(0)}_{22} & \tilde{a}^{(0)}_{23} & \tilde{y}^{(0)}_2 \\ \tilde{a}^{(0)}_{30} & \tilde{a}^{(0)}_{31} & \tilde{a}^{(0)}_{32} & \tilde{a}^{(0)}_{33} & \tilde{y}^{(0)}_3 \end{bmatrix}$$

$$\tilde{a}^{(0)}_{00} = a^{(0)}_{20}$$

element 00 in matrix $\tilde{\mathbf{C}}^{(0)}$

element 20 in matrix $\mathbf{C}^{(0)}$

this is the pivot for iteration k = 1

$$\tilde{C}^{(0)} = \begin{bmatrix} \tilde{a}_{00}^{(0)} & \tilde{a}_{01}^{(0)} & \tilde{a}_{02}^{(0)} & \tilde{a}_{03}^{(0)} & \tilde{y}_0^{(0)} \\ \tilde{a}_{10}^{(0)} & \tilde{a}_{11}^{(0)} & \tilde{a}_{12}^{(0)} & \tilde{a}_{13}^{(0)} & \tilde{y}_1^{(0)} \\ \tilde{a}_{20}^{(0)} & \tilde{a}_{21}^{(0)} & \tilde{a}_{22}^{(0)} & \tilde{a}_{23}^{(0)} & \tilde{y}_2^{(0)} \\ \tilde{a}_{30}^{(0)} & \tilde{a}_{31}^{(0)} & \tilde{a}_{32}^{(0)} & \tilde{a}_{33}^{(0)} & \tilde{y}_3^{(0)} \end{bmatrix}$$

$$\mathbf{t}^{(1)} = \begin{bmatrix} t_0^{(1)} \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\frac{\tilde{a}_{10}^{(0)}}{\tilde{a}_{00}^{(0)}}\right) \\ \left(\frac{\tilde{a}_{20}^{(0)}}{\tilde{a}_{00}^{(0)}}\right) \\ \left(\frac{\tilde{a}_{30}^{(0)}}{\tilde{a}_{00}^{(0)}}\right) \end{bmatrix} \qquad \mathbf{u}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{t}^{(k)}[k:] = \frac{\tilde{C}[k:, k-1]}{\underbrace{\tilde{C}[k-1, k-1]}_{\tilde{a}_{00}^{(0)}}} \rightarrow \begin{bmatrix} \tilde{a}_{10}^{(0)} \\ \tilde{a}_{20}^{(0)} \\ \tilde{a}_{30}^{(0)} \end{bmatrix}$$

Note that the Gauss vector is computed with the permuted matrix

$$C^{(1)} = \left(\mathbf{I} - \mathbf{t}^{(1)} \cdot \mathbf{u}^{(0)T}\right) \tilde{C}^{(0)} = \tilde{C}^{(0)} - \mathbf{t}^{(1)} \cdot \left(\mathbf{u}^{(0)T} \tilde{C}^{(0)}\right) =$$

$$= \begin{bmatrix} \tilde{a}_{00}^{(0)} & \tilde{a}_{01}^{(0)} & \tilde{a}_{02}^{(0)} & \tilde{a}_{03}^{(0)} & \tilde{y}_0^{(0)} \\ \tilde{a}_{10}^{(0)} & \tilde{a}_{11}^{(0)} & \tilde{a}_{12}^{(0)} & \tilde{a}_{13}^{(0)} & \tilde{y}_1^{(0)} \\ \tilde{a}_{20}^{(0)} & \tilde{a}_{21}^{(0)} & \tilde{a}_{22}^{(0)} & \tilde{a}_{23}^{(0)} & \tilde{y}_2^{(0)} \\ \tilde{a}_{30}^{(0)} & \tilde{a}_{31}^{(0)} & \tilde{a}_{32}^{(0)} & \tilde{a}_{33}^{(0)} & \tilde{y}_3^{(0)} \end{bmatrix} - \begin{bmatrix} 0 \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \begin{bmatrix} \tilde{a}_{00}^{(0)} & \tilde{a}_{01}^{(0)} & \tilde{a}_{02}^{(0)} & \tilde{a}_{03}^{(0)} & \tilde{y}_0^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} \tilde{a}_{00}^{(0)} & \tilde{a}_{01}^{(0)} & \tilde{a}_{02}^{(0)} & \tilde{a}_{03}^{(0)} & \tilde{y}_0^{(0)} \\ 0 & \tilde{a}_{11}^{(0)} - t_1^{(1)}\tilde{a}_{01}^{(0)} & \tilde{a}_{12}^{(0)} - t_1^{(1)}\tilde{a}_{02}^{(0)} & \tilde{a}_{13}^{(0)} - t_1^{(1)}\tilde{a}_{03}^{(0)} & \tilde{y}_1^{(0)} - t_1^{(1)}\tilde{y}_0^{(0)} \\ 0 & \tilde{a}_{21}^{(0)} - t_2^{(1)}\tilde{a}_{01}^{(0)} & \tilde{a}_{22}^{(0)} - t_2^{(1)}\tilde{a}_{02}^{(0)} & \tilde{a}_{23}^{(0)} - t_2^{(1)}\tilde{a}_{03}^{(0)} & \tilde{y}_2^{(0)} - t_2^{(1)}\tilde{y}_0^{(0)} \\ 0 & \tilde{a}_{31}^{(0)} - t_3^{(1)}\tilde{a}_{01}^{(0)} & \tilde{a}_{32}^{(0)} - t_3^{(1)}\tilde{a}_{02}^{(0)} & \tilde{a}_{33}^{(0)} - t_3^{(1)}\tilde{a}_{03}^{(0)} & \tilde{y}_3^{(0)} - t_3^{(1)}\tilde{y}_0^{(0)} \end{bmatrix}$$

The first line is equal to that in the original matrix C

$$C^{(1)} = \begin{bmatrix} a_{00}^{(1)} & a_{01}^{(1)} & a_{02}^{(1)} & a_{03}^{(1)} & y_0^{(1)} \\ 0 & a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & y_1^{(1)} \\ 0 & a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & y_2^{(1)} \\ 0 & a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & y_3^{(1)} \end{bmatrix}$$

All the math to compute matrix C(1) lead to these zeros in the first column. Besides, the "space" occupied by these zeros can be better used by storing the non-null elements of the Gauss vector (the Gauss multipliers)

replace by

$$C^{(1)}[1:, 0] = \begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix}$$

Note that the elements in black can be written as follows:

$$C^{(1)}[1:, 1:] \mathrel{-}= C^{(1)}[1:, 0] \cdot \tilde{C}^{(0)}[0, 1:]$$

First line of the permuted matrix

$$\begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \tilde{a}_{01}^{(0)} & \tilde{a}_{02}^{(0)} & \tilde{a}_{03}^{(0)} & \tilde{y}_0^{(0)} \end{bmatrix}$$

Actually, we overwrite matrix C so that

$$C[k:, k:] \mathrel{-}= C[k:, k-1] \cdot C[k-1, k:]$$ for any k