2D convolution

Circular convolution

$M = 4$
$N = 4$

$$\underset{M\times N}{A} = \begin{bmatrix} 00 & 01 & 02 & 03 \\ 10 & 11 & 12 & 13 \\ 20 & 21 & 22 & 23 \\ 30 & 31 & 32 & 33 \end{bmatrix} \qquad \underset{M\times N}{B} = \begin{bmatrix} 00 & 01 & 02 & 03 \\ 10 & 11 & 12 & 13 \\ 20 & 21 & 22 & 23 \\ 30 & 31 & 32 & 33 \end{bmatrix}$$

a_{31} $\qquad$ b_{32}

Note that the elements of matrices A and B are represented by using only their indices. Otherwise I would get crazy.

The circular convolution of A and B produces a third matrix W .

The elements of W can be computed by following the steps below:

① Put matrix A in a fixed position

$$\begin{bmatrix} 00 & 01 & 02 & 03 \\ 10 & 11 & 12 & 13 \\ 20 & 21 & 22 & 23 \\ 30 & 31 & 32 & 33 \end{bmatrix}$$

② Create a temporary matrix by flipping the rows and columns of matrix B

$$\begin{bmatrix} 33 & 32 & 31 & 30 \\ 23 & 22 & 21 & 20 \\ 13 & 12 & 11 & 10 \\ 03 & 02 & 01 & 00 \end{bmatrix}$$

③ Expand the temporary matrix by repeating the flipped matrix

④ Put the expanded matrix over matrix A

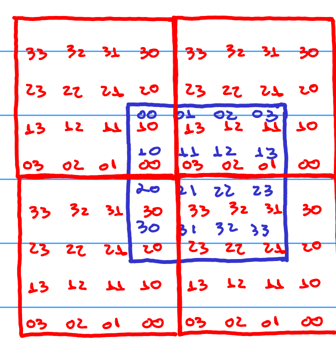
⑤ Compute the 00 element of matrix W by adding all the products of superposed elements

$$\begin{aligned} w_{00} = {} & a_{00}b_{00} + a_{01}b_{03} + a_{02}b_{02} + a_{03}b_{01} + \\ & + a_{10}b_{30} + a_{11}b_{33} + a_{12}b_{32} + a_{13}b_{31} + \\ & + a_{20}b_{20} + a_{21}b_{23} + a_{22}b_{22} + a_{23}b_{21} + \\ & + a_{30}b_{10} + a_{31}b_{13} + a_{32}b_{12} + a_{33}b_{11} \end{aligned}$$

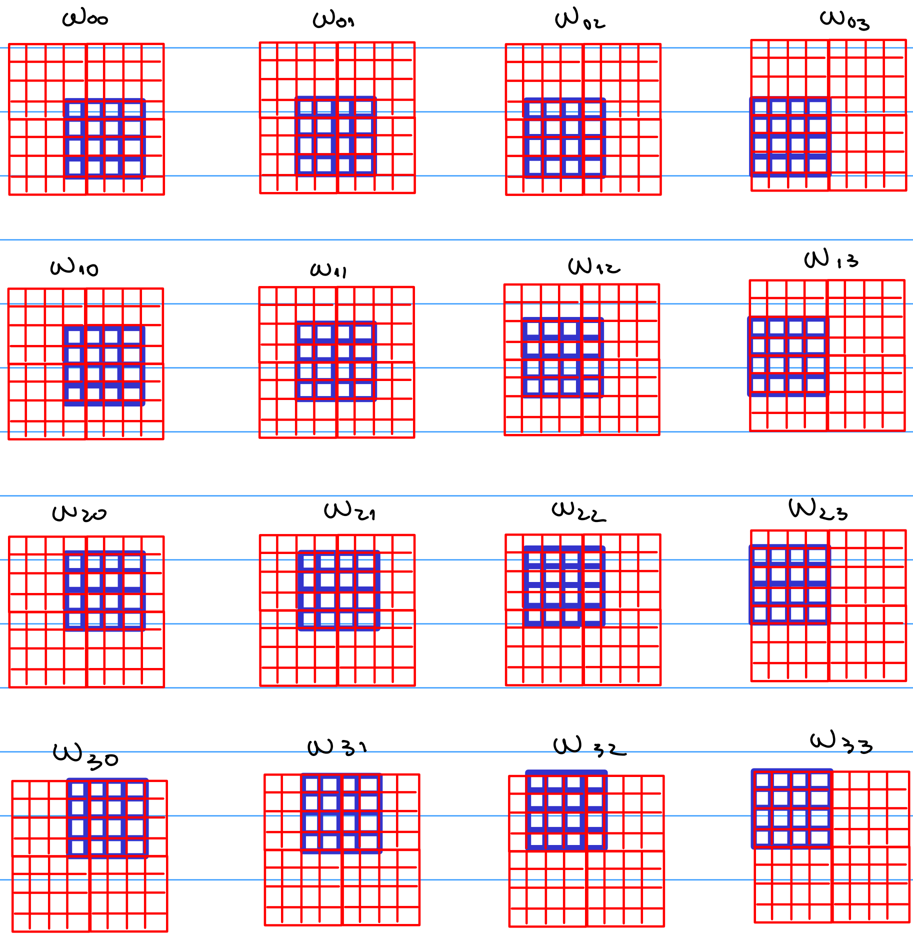
⑥ The remaining elements of W are computed by sliding the temporary matrix and adding the products of superposed elements

w_{01}

w_{10}



By repeating step 6, we obtain all elements of W:



Note that:

1) All elements are conveniently represented by using only their indices in order to make things as clean as possible;

2) All plus "+" symbols are conveniently omitted in order to make things as clean as possible;

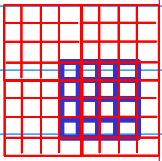
3) The elements of A are arranged by column.

$$C_k = \underbrace{\begin{bmatrix} b_{0k} & b_{3k} & b_{2k} & b_{1k} \\ b_{1k} & b_{0k} & b_{3k} & b_{2k} \\ b_{2k} & b_{1k} & b_{0k} & b_{3k} \\ b_{3k} & b_{2k} & b_{1k} & b_{0k} \end{bmatrix}}_{M\times M}$$

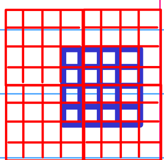
$$\omega_k = \underbrace{\begin{bmatrix} \omega_{0k} \\ \omega_{1k} \\ \omega_{2k} \\ \omega_{3k} \end{bmatrix}}_{M\times 1} \qquad a_k = \underbrace{\begin{bmatrix} a_{0k} \\ a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix}}_{M\times 1}$$

$$\underset{N\times M}{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$$

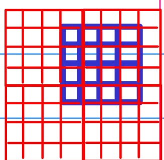
$$\underset{N\times M}{W} = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$$



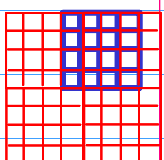
$$00 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 10 & 20 & 30 & 01 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 00 & 30 & 20 & 10 & 03 & 33 & 23 & 13 & 02 & 32 & 22 & 12 & 01 & 31 & 21 & 11 \end{array}$$



$$10 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 10 & 20 & 30 & 01 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 10 & 00 & 30 & 20 & 13 & 03 & 33 & 23 & 12 & 02 & 32 & 22 & 11 & 01 & 31 & 21 \end{array}$$

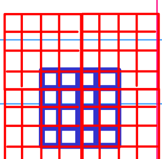


$$20 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 10 & 20 & 30 & 01 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 20 & 10 & 00 & 30 & 23 & 13 & 03 & 33 & 22 & 12 & 02 & 32 & 21 & 11 & 01 & 31 \end{array}$$

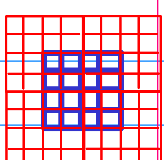


$$30 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 10 & 20 & 30 & 01 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 30 & 20 & 10 & 00 & 33 & 23 & 13 & 03 & 32 & 22 & 12 & 02 & 31 & 21 & 11 & 01 \end{array}$$

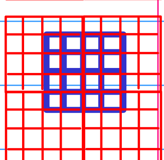
$$\omega_0 = C_0 a_0 + C_3 a_1 + C_2 a_2 + C_1 a_3$$



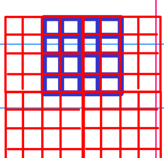
$$01 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 01 & 02 & 03 & 10 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 01 & 31 & 21 & 11 & 00 & 30 & 20 & 10 & 03 & 33 & 23 & 13 & 02 & 32 & 22 & 12 \end{array}$$



$$11 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 01 & 02 & 03 & 10 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 11 & 01 & 31 & 21 & 10 & 00 & 30 & 20 & 13 & 03 & 33 & 23 & 12 & 02 & 32 & 22 \end{array}$$

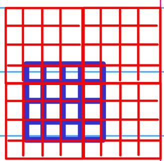


$$21 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 01 & 02 & 03 & 10 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 21 & 11 & 01 & 31 & 20 & 10 & 00 & 30 & 23 & 13 & 03 & 33 & 22 & 12 & 02 & 32 \end{array}$$

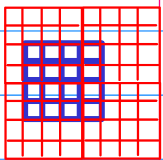


$$31 = \begin{array}{cccc|cccc|cccc|cccc} 00 & 01 & 02 & 03 & 10 & 11 & 21 & 31 & 02 & 12 & 22 & 32 & 03 & 13 & 23 & 33 \\ 31 & 21 & 11 & 01 & 30 & 20 & 10 & 00 & 33 & 23 & 13 & 03 & 32 & 22 & 12 & 02 \end{array}$$

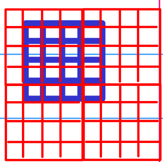
$$\omega_1 = C_1 a_0 + C_0 a_1 + C_3 a_2 + C_2 a_3$$



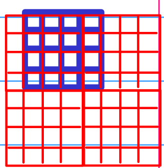
02 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
02 32 22 12   01 31 21 11   00 30 20 10   03 33 23 13



12 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
12 02 32 22   11 01 31 21   10 00 30 20   13 03 33 23

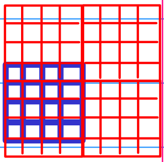


22 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
22 12 02 32   21 11 01 31   20 10 00 30   23 13 03 33

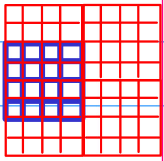


32 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
32 22 12 02   31 21 11 01   30 20 10 00   33 23 13 03

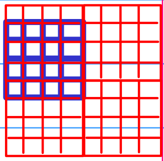
$$\omega_2 = C_2 a_0 + C_1 a_1 + C_0 a_2 + C_3 a_3$$



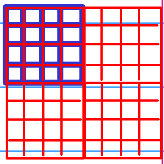
03 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
03 33 23 13   02 32 22 12   01 31 21 11   00 30 20 10



13 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
13 03 33 23   12 02 32 22   11 01 31 21   10 00 30 20



23 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
23 13 03 33   22 12 02 32   21 11 01 31   20 10 00 30



33 = 00 01 02 03   10 11 21 31   02 12 22 32   03 13 23 33
33 23 13 03   32 22 12 02   31 21 11 01   30 20 10 00

$$\omega_3 = C_3 a_0 + C_2 a_1 + C_1 a_2 + C_0 a_3$$

This is a BCCB matrix

$$\underset{QP \times 1}{\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}} = \underset{QP \times QP}{\begin{bmatrix} C_0 & C_3 & C_2 & C_1 \\ C_1 & C_0 & C_3 & C_2 \\ C_2 & C_1 & C_0 & C_3 \\ C_3 & C_2 & C_1 & C_0 \end{bmatrix}} \underset{QP \times 1}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}$$

The BCCB matrix is formed by a grid of Q x Q blocks, where each block is a P x P circulant matrix formed by a given column of the input matrix B, Q = N (number of columns of the input matrices), and P = M (number of rows of the input matrices).

$$Q = N$$
$$P = M$$

Linear convolution

$M_a = 4$
$N_a = 5$

$$\underset{M_a \times N_a}{A} = \begin{bmatrix} 00 & 01 & 02 & 03 & 04 \\ 10 & 11 & 12 & 13 & 14 \\ 20 & 21 & 22 & 23 & 24 \\ 30 & 31 & 32 & 33 & 34 \end{bmatrix} \qquad \underset{M_b \times N_b}{B} = \begin{bmatrix} 00 & 01 & 02 \\ 10 & 11 & 12 \end{bmatrix}$$

$M_b = 2$
$N_b = 3$

Note that the elements of matrices A and B are represented by using only their indices. Otherwise I would get crazy.

$$\overset{5}{M_w} = \overset{4}{M_a} + \overset{2}{M_b} - 1$$

$$\underset{7}{N_w} = \underset{5}{N_a} + \underset{3}{N_b} - 1$$

The circular convolution of A and B produces a third matrix W.

The elements of W can be computed by following the steps below:

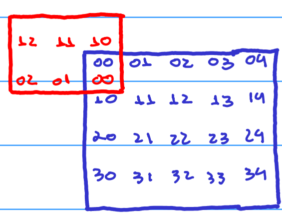
(1) Put matrix A in a fixed position

$$\begin{bmatrix} 00 & 01 & 02 & 03 & 04 \\ 10 & 11 & 12 & 13 & 14 \\ 20 & 21 & 22 & 23 & 24 \\ 30 & 31 & 32 & 33 & 34 \end{bmatrix}$$

(2) Create a temporary matrix by flipping the rows and columns of matrix B

$$\begin{bmatrix} 12 & 11 & 10 \\ 02 & 01 & 00 \end{bmatrix}$$

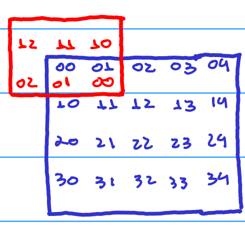
(3) Put the expanded matrix over matrix A



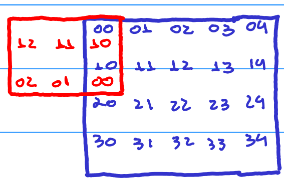
(4) Compute the 00 element of matrix W by adding the product of superposed elements

$$w_{00} = a_{00} b_{00}$$

(5) The remaining elements of W are computed by sliding the temporary matrix and adding the products of superposed elements

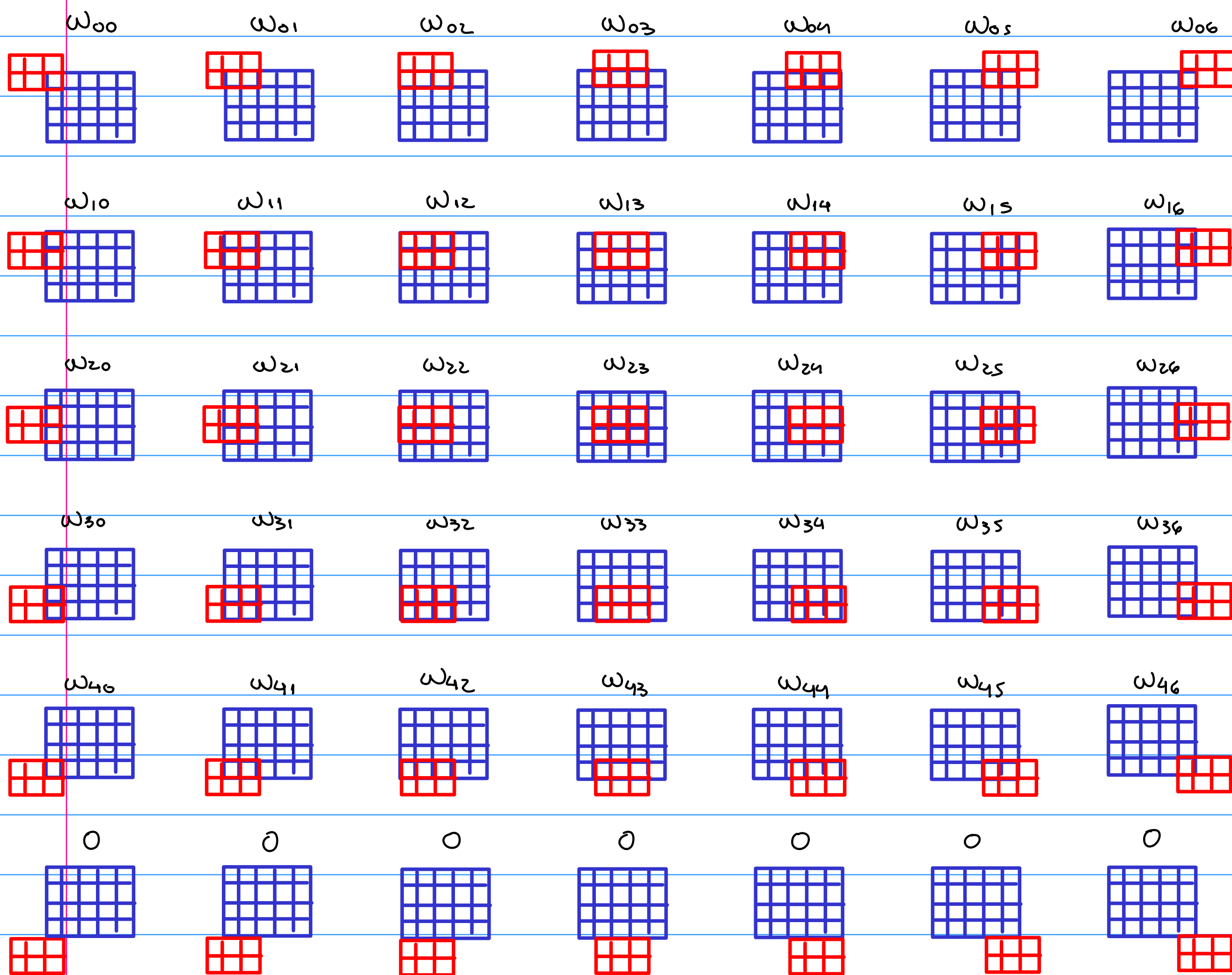


$$w_{01} = a_{00} b_{01} + a_{01} b_{00}$$



$$w_{10} = a_{00} b_{10} + a_{10} b_{00}$$

By repeating step 5, we obtain all elements of W:



$$\begin{bmatrix}\boldsymbol{\omega}_0\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_1\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_2\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_3\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_4\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_5\\0\end{bmatrix}\quad\begin{bmatrix}\boldsymbol{\omega}_6\\0\end{bmatrix}$$

$$\boldsymbol{\omega}_k = \begin{bmatrix} w_{0k}\\ w_{1k}\\ w_{2k}\\ w_{3k}\\ w_{4k}\end{bmatrix} \qquad \underset{M_w \times N_w}{\mathbf{W}} = \begin{bmatrix}\boldsymbol{\omega}_0 & \cdots & \boldsymbol{\omega}_6\end{bmatrix}$$

$$m_w = M_a + M_b - 1$$
$$5 = 4 + 2 - 1$$

$(M_w+1) \times 1$

$$\begin{bmatrix} \omega_0 \\ 0 \end{bmatrix} = T_0 \begin{bmatrix} a_0 \\ 0_{M_b} \end{bmatrix}$$

$(M_w+1) \times 1$

$$\begin{bmatrix} \omega_1 \\ 0 \end{bmatrix} = T_1 \begin{bmatrix} a_0 \\ 0_{M_b} \end{bmatrix} + T_0 \begin{bmatrix} a_1 \\ 0_{M_b} \end{bmatrix}$$

$$\begin{bmatrix} \omega_2 \\ 0 \end{bmatrix} = T_2 \begin{bmatrix} a_0 \\ 0_{M_b} \end{bmatrix} + T_1 \begin{bmatrix} a_1 \\ 0_{M_b} \end{bmatrix} + T_0 \begin{bmatrix} a_2 \\ 0_{M_b} \end{bmatrix}$$

$$\underset{\substack{M_a \times 1 \\ 4 \times 1}}{a_k} = \begin{bmatrix} a_{0k} \\ a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix} \qquad \underset{\substack{M_a \times N_a \\ 4 \times 5}}{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

$$T_k = \begin{bmatrix} b_{0k} & 0 & 0 & 0 & 0 & 0 \\ b_{1k} & b_{0k} & 0 & 0 & 0 & 0 \\ 0 & b_{1k} & b_{0k} & 0 & 0 & 0 \\ 0 & 0 & b_{1k} & b_{0k} & 0 & 0 \\ 0 & 0 & 0 & b_{1k} & b_{0k} & 0 \\ 0 & 0 & 0 & 0 & b_{1k} & b_{0k} \end{bmatrix}$$

$(M_w+1) \times (M_w+1)$

6×6

$$b_k = \underset{\substack{M_b \times 1 \\ 2 \times 1}}{\begin{bmatrix} b_{0k} \\ b_{1k} \end{bmatrix}} \qquad \underset{\substack{M_b \times N_b \\ 2 \times 3}}{B} = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}$$

This is a BTTB matrix

$$\underset{QP \times 1}{\begin{bmatrix} \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_3 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_4 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_5 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \omega_6 \\ 0 \end{pmatrix} \end{bmatrix}} = \underset{QP \times QP}{\begin{bmatrix} T_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_1 & T_0 & 0 & 0 & 0 & 0 & 0 \\ T_2 & T_1 & T_0 & 0 & 0 & 0 & 0 \\ 0 & T_2 & T_1 & T_0 & 0 & 0 & 0 \\ 0 & 0 & T_2 & T_1 & T_0 & 0 & 0 \\ 0 & 0 & 0 & T_2 & T_1 & T_0 & 0 \\ 0 & 0 & 0 & 0 & T_2 & T_1 & T_0 \end{bmatrix}} \underset{QP \times 1}{\begin{bmatrix} \begin{pmatrix} a_0 \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} a_1 \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} a_2 \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} a_3 \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} a_4 \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} 0_{M_a} \\ 0_{M_b} \end{pmatrix} \\ \begin{pmatrix} 0_{M_a} \\ 0_{M_b} \end{pmatrix} \end{bmatrix}}$$

$$Q = N_a + N_b$$
$$P = M_b + M_b$$

The BTTB matrix is formed by a grid of Q x Q blocks, where each block is a P x P Toeplitz matrix formed by a given column of the input matrix B.