Gaussian elimination

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}_{N\times N} \qquad \boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}_{N\times 1} \qquad N = 4$$

$$\mathbf{C} = \left[\, \mathbf{A} \,\vdots\, \boldsymbol{y} \,\right]_{N\times(N+1)}$$

$$k=0 \quad \mathbf{C}^{(0)} = \mathbf{C}$$

$$k=1 \quad \mathbf{C}^{(1)} = \left(\mathbf{I} - \boldsymbol{t}^{(1)} \cdot \boldsymbol{u}^{(0)^T}\right) \mathbf{C}^{(0)}$$

$$k=2 \quad \mathbf{C}^{(2)} = \left(\mathbf{I} - \boldsymbol{t}^{(2)} \cdot \boldsymbol{u}^{(1)^T}\right) \mathbf{C}^{(1)}$$

$$k=3 \quad \mathbf{C}^{(3)} = \left(\mathbf{I} - \boldsymbol{t}^{(3)} \cdot \boldsymbol{u}^{(2)^T}\right) \mathbf{C}^{(2)}$$

$\underline{\underline{k=1}}$

this is the pivot for iteration k = 1

$$\mathbf{C}^{(0)} = \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix}$$

$$\boldsymbol{t}^{(1)} = \begin{bmatrix} t_0^{(1)} \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\frac{a_{10}^{(0)}}{a_{00}^{(0)}}\right) \\ \left(\frac{a_{20}^{(0)}}{a_{00}^{(0)}}\right) \\ \left(\frac{a_{30}^{(0)}}{a_{00}^{(0)}}\right) \end{bmatrix} \qquad \boldsymbol{u}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{t}^{(k)}[k:] = \frac{\mathbf{C}^{(k-1)}[k:, k-1]}{\underbrace{\mathbf{C}^{(k-1)}[k-1, k-1]}_{a_{00}^{(0)}}} \qquad \mathbf{C}^{(k-1)}[k:, k-1] \rightarrow \begin{bmatrix} a_{10}^{(0)} \\ a_{20}^{(0)} \\ a_{30}^{(0)} \end{bmatrix}$$

$$\mathbf{C}^{(1)} = \left(\mathbf{I} - \mathbf{t}^{(1)} \cdot \mathbf{u}^{(0)T}\right) \mathbf{C}^{(0)} = \mathbf{C}^{(0)} - \mathbf{t}^{(1)} \cdot \left(\mathbf{u}^{(0)T}\, \mathbf{C}^{(0)}\right) =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix} - \begin{bmatrix} 0 \\ t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} & a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & y_1^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & a_{23}^{(0)} & y_2^{(0)} \\ a_{30}^{(0)} & a_{31}^{(0)} & a_{32}^{(0)} & a_{33}^{(0)} & y_3^{(0)} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ t_1^{(1)} a_{00}^{(0)} & t_1^{(1)} a_{01}^{(0)} & t_1^{(1)} a_{02}^{(0)} & t_1^{(1)} a_{03}^{(0)} & t_1^{(1)} y_0^{(0)} \\ t_2^{(1)} a_{00}^{(0)} & t_2^{(1)} a_{01}^{(0)} & t_2^{(1)} a_{02}^{(0)} & t_2^{(1)} a_{03}^{(0)} & t_2^{(1)} y_0^{(0)} \\ t_3^{(1)} a_{00}^{(0)} & t_3^{(1)} a_{01}^{(0)} & t_3^{(1)} a_{02}^{(0)} & t_3^{(1)} a_{03}^{(0)} & t_3^{(1)} y_0^{(0)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ a_{10}^{(0)} - t_1^{(1)} a_{00}^{(0)} & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\ a_{20}^{(0)} - t_2^{(1)} a_{00}^{(0)} & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\ a_{30}^{(0)} - t_3^{(1)} a_{00}^{(0)} & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\ 0 & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\ 0 & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\ 0 & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)} \end{bmatrix}$$

$$\begin{cases} a_{10}^{(0)} - t_1^{(1)} a_{00}^{(0)} = a_{10}^{(0)} - \left(\dfrac{a_{10}^{(0)}}{a_{00}^{(0)}}\right) a_{00}^{(0)} = a_{10}^{(0)} - a_{10}^{(0)} = 0 \\ a_{20}^{(0)} - t_2^{(1)} a_{00}^{(0)} = a_{20}^{(0)} - \left(\dfrac{a_{20}^{(0)}}{a_{00}^{(0)}}\right) a_{00}^{(0)} = a_{20}^{(0)} - a_{20}^{(0)} = 0 \\ a_{30}^{(0)} - t_3^{(1)} a_{00}^{(0)} = a_{30}^{(0)} - \left(\dfrac{a_{30}^{(0)}}{a_{00}^{(0)}}\right) a_{00}^{(0)} = a_{30}^{(0)} - a_{30}^{(0)} = 0 \end{cases}$$

$$
C^{(1)} = \begin{bmatrix}
a_{00}^{(0)} & a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \\
0 & a_{11}^{(0)} - t_1^{(1)} a_{01}^{(0)} & a_{12}^{(0)} - t_1^{(1)} a_{02}^{(0)} & a_{13}^{(0)} - t_1^{(1)} a_{03}^{(0)} & y_1^{(0)} - t_1^{(1)} y_0^{(0)} \\
0 & a_{21}^{(0)} - t_2^{(1)} a_{01}^{(0)} & a_{22}^{(0)} - t_2^{(1)} a_{02}^{(0)} & a_{23}^{(0)} - t_2^{(1)} a_{03}^{(0)} & y_2^{(0)} - t_2^{(1)} y_0^{(0)} \\
0 & a_{31}^{(0)} - t_3^{(1)} a_{01}^{(0)} & a_{32}^{(0)} - t_3^{(1)} a_{02}^{(0)} & a_{33}^{(0)} - t_3^{(1)} a_{03}^{(0)} & y_3^{(0)} - t_3^{(1)} y_0^{(0)}
\end{bmatrix}
$$

The first line is equal to that in the original matrix C

$$
= \begin{bmatrix}
a_{00}^{(1)} & a_{01}^{(1)} & a_{02}^{(1)} & a_{03}^{(1)} & y_0^{(1)} \\
0 & a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & y_1^{(1)} \\
0 & a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & y_2^{(1)} \\
0 & a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & y_3^{(1)}
\end{bmatrix}
$$

All the math to compute matrix C(1) lead to these zeros in the first column. Besides, the "space" occupied by these zeros can be better used by storing the non-null elements of the Gauss vector (the Gauss multipliers)

replace by

$$
C^{(1)}[1:, 0] = \begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix}
$$

Note that the elements in black can be written as follows:

$$
C^{(1)}[1:, 1:] \mathrel{-}= C^{(1)}[1:, 0] \cdot C^{(0)}[0, 1:]
$$

$$
\begin{bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_3^{(1)} \end{bmatrix} \cdot \begin{bmatrix} a_{01}^{(0)} & a_{02}^{(0)} & a_{03}^{(0)} & y_0^{(0)} \end{bmatrix}
$$

Actually, we overwrite matrix C so that

$$
C[k:, k:] \mathrel{-}= C[k:, k-1] \cdot C[k-1, k:] \quad \text{for any } k
$$