1D Convolution

Two kinds of 1D convolutions are illustrated here: linear and circular convolutions.

Circular convolution

Consider the following two real vectors $\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4\times1}$ $\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}_{4\times1}$

The circular convolution of $\mathbf{a}$ and $\mathbf{b}$ results in a third vector $\boldsymbol{\omega}$.

The elements of $\boldsymbol{\omega}$ can be schematically computed as follows:

① Keep vector $\mathbf{a}^T$ in a fixed horizontal position position.

② Define a temporary vector by repeating the elements of $\mathbf{b}^T$ in a reversed order.

			a_0	a_1	a_2	a_3	
b_3	b_2	b_1	b_0	b_3	b_2	b_1	b_0

③ Place the temporary vector right below $\mathbf{a}^T$ so that b_0 is aligned with a_0.

④ Select the elements of the temporary vector that are aligned with $\mathbf{a}^T$. In our example, they are represented in red and the remaining elements in gray.

⑤ Compute the first element of $\boldsymbol{\omega}$ by multiplying and adding the aligned elements:

$$\omega_0 = b_0 \cdot a_0 + b_3 \cdot a_1 + b_2 \cdot a_2 + b_1 \cdot a_3$$

⑥ The succeeding elements of $\boldsymbol{\omega}$ are computed by sliding the temporary vector to right and repeating the steps above:

			a_0	a_1	a_2	a_3					
b_3	b_2	b_1	b_0	b_3	b_2	b_1	b_0				$\rightarrow \omega_0$
	b_3	b_2	b_1	b_0	b_3	b_2	b_1	b_0			$\rightarrow \omega_1$
		b_3	b_2	b_1	b_0	b_3	b_2	b_1	b_0		$\rightarrow \omega_2$
			b_3	b_2	b_1	b_0	b_3	b_2	b_1	b_0	$\rightarrow \omega_3$

unused elements

unused elements

The red elements shown above form a circulant matrix $\mathbf{C}$

This iterative scheme used to compute the elements of $\boldsymbol{\omega}$ is equivalent to the following matrix-vector product:

$$\boldsymbol{\omega} = \mathbf{C}\,\mathbf{a}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_3 & b_2 & b_1 \\ b_1 & b_0 & b_3 & b_2 \\ b_2 & b_1 & b_0 & b_3 \\ b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

This is a circulant matrix formed by the elements of $\mathbf{b}$

$$c_{ij} = b_{(i-j) \bmod N}$$

Linear convolution

Consider the following two real vectors

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4\times 1} \qquad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}_{3\times 1}$$

$$N_\omega = \underbrace{N_a}_{\text{number of elements of } \mathbf{a}} + \underbrace{N_b}_{\text{number of elements of } \mathbf{b}} - 1$$

N_ω: number of elements of $\boldsymbol{\omega}$

The linear convolution of $\mathbf{a}$ and $\mathbf{b}$ results in a third vector $\boldsymbol{\omega}$.

The elements of $\boldsymbol{\omega}$ can be schematically computed as follows:

1. Define a temporary vector by appending N_b zeros at the end of $\mathbf{a}^T$

$$a_0 \quad a_1 \quad a_2 \quad a_3 \quad \overbrace{0 \quad 0 \quad 0}^{N_b \text{ additional elements}}$$

$$\underbrace{0 \quad 0 \quad 0 \quad 0}_{N_a \text{ additional elements placed at the beginning}} \quad b_2 \quad b_1 \quad b_0 \quad \underbrace{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0}_{N_\omega = N_a + N_b - 1 \text{ additional elements placed at the end}}$$

2. Define a temporary vector by rearranging $\mathbf{b}^T$ in a reversed order and appending zeros to its beginnig and end.

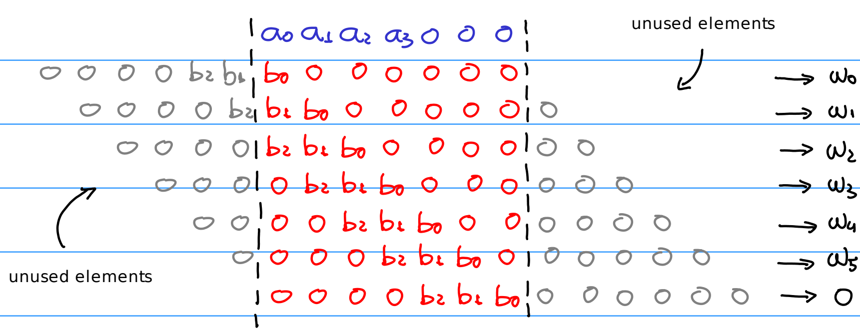
3. Align the temporary vectors so that b_0 is placed right below a_0

4. Select the elements of the temporary vectors that are vertically aligned. In our example, they are represented in red and the remaining elements in gray.

5. Compute the first element of $\boldsymbol{\omega}$ by multiplying and adding the aligned elements:

$$\omega_0 = b_0 \cdot a_0 + 0 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 + 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0$$

6. The succeeding elements of $\boldsymbol{\omega}$ are computed by sliding the temporary vector to right and repeating the steps above:



The red elements shown above form a Toeplitz matrix

This iterative scheme used to compute the elements of $\boldsymbol{\omega}$ is equivalent to the following matrix-vector product:

$$\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{a} \\ \mathbf{0}_{N_b} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{0}_{N_b} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} b_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_0 & 0 & 0 & 0 & 0 & 0 \\ b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & 0 & b_2 & b_1 & b_0 \end{bmatrix}$$

This is a Toeplitz matrix formed by the elements of $\mathbf{b}$