2D convolution

Circular convolution

$M_a = 4$

$N_a = 4$

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$M_a \times N_a$

a_{31}

$$B = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$M_b = 4$

$N_b = 4$

$M_b \times N_b$

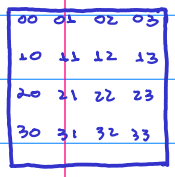
b_{32}

Note that the elements of matrices A and B are represented by using only their indices. Otherwise I would get crazy.

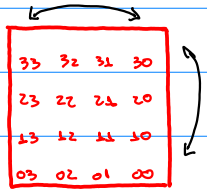
The circular convolution of A and B produces a third matrix W.

The elements of W can be computed by following the steps below:

1. Put matrix A in a fixed position

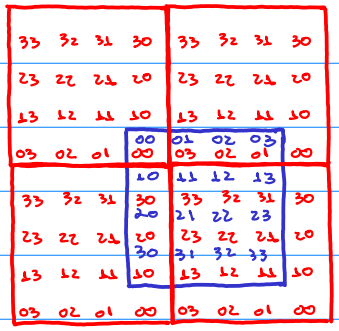


2. Create a temporary matrix by flipping the rows and columns of matrix B



3. Expand the temporary matrix by repeating the flipped matrix

4. Put the expanded matrix over matrix A



5. Compute the 00 element of matrix W by adding all the products of superposed elements

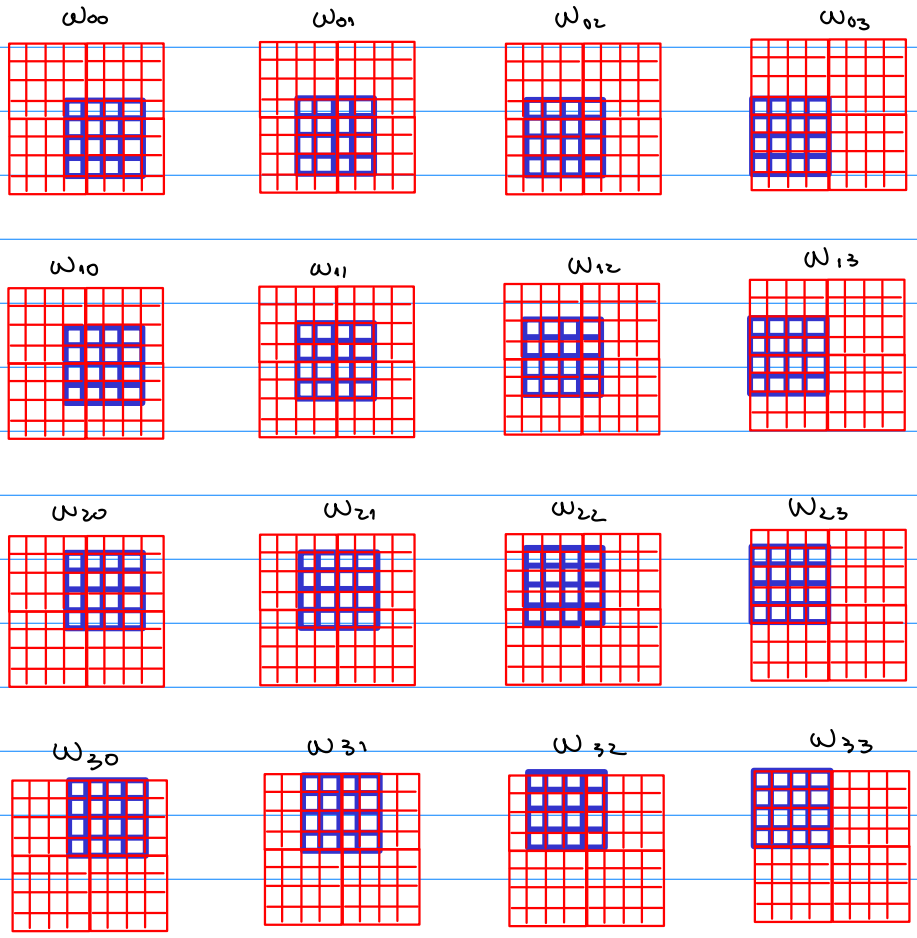
$$\begin{aligned} w_{00} = {} & a_{00}b_{00} + a_{01}b_{03} + a_{02}b_{02} + a_{03}b_{01} + \\ & + a_{10}b_{30} + a_{11}b_{33} + a_{12}b_{32} + a_{13}b_{31} + \\ & + a_{20}b_{20} + a_{21}b_{23} + a_{22}b_{22} + a_{23}b_{21} + \\ & + a_{30}b_{10} + a_{31}b_{13} + a_{32}b_{12} + a_{33}b_{11} \end{aligned}$$

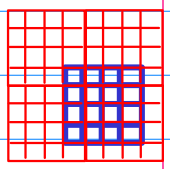
6. The remaining elements of W are computed by sliding the temporary matrix and adding the products of superposed elements

w_{01}

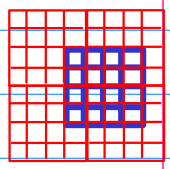
w_{10}

By repeating step 6, we obtain all elements of W:

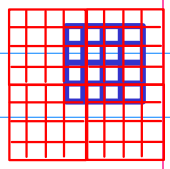




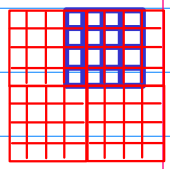
$\omega_0 = \begin{matrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{00} & a_{30} & a_{20} & a_{10} \end{matrix} \quad \begin{matrix} a_{01} & a_{11} & a_{21} & a_{31} \\ a_{03} & a_{33} & a_{23} & a_{13} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{02} & a_{32} & a_{22} & a_{12} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{01} & a_{31} & a_{21} & a_{11} \end{matrix}$



$\omega_1 = \begin{matrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{10} & a_{00} & a_{30} & a_{20} \end{matrix} \quad \begin{matrix} a_{01} & a_{11} & a_{21} & a_{31} \\ a_{13} & a_{03} & a_{33} & a_{23} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{12} & a_{02} & a_{32} & a_{22} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{11} & a_{01} & a_{31} & a_{21} \end{matrix}$



$\omega_2 = \begin{matrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{20} & a_{10} & a_{00} & a_{30} \end{matrix} \quad \begin{matrix} a_{01} & a_{11} & a_{21} & a_{31} \\ a_{23} & a_{13} & a_{03} & a_{33} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{22} & a_{12} & a_{02} & a_{32} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{21} & a_{11} & a_{01} & a_{31} \end{matrix}$



$\omega_3 = \begin{matrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{30} & a_{20} & a_{10} & a_{00} \end{matrix} \quad \begin{matrix} a_{01} & a_{11} & a_{21} & a_{31} \\ a_{33} & a_{23} & a_{13} & a_{03} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{32} & a_{22} & a_{12} & a_{02} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{31} & a_{21} & a_{11} & a_{01} \end{matrix}$

$$\omega_0 = C_0 a_0 + C_3 a_1 + C_2 a_2 + C_1 a_3$$

Note that:

1) All elements are conveniently represented by using only their indices in order to make things as clean as possible;

2) All plus "+" symbols are conveniently omitted in order to make things as clean as possible;

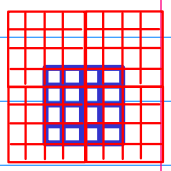
3) The elements of A are arranged by column.

$$C_k = \begin{bmatrix} b_{0k} & b_{3k} & b_{2k} & b_{1k} \\ b_{1k} & b_{0k} & b_{3k} & b_{2k} \\ b_{2k} & b_{1k} & b_{0k} & b_{3k} \\ b_{3k} & b_{2k} & b_{1k} & b_{0k} \end{bmatrix}$$

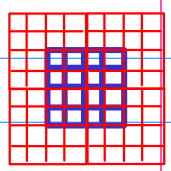
$$\omega_k = \begin{bmatrix} \omega_{0k} \\ \omega_{1k} \\ \omega_{2k} \\ \omega_{3k} \end{bmatrix} \qquad a_k = \begin{bmatrix} a_{0k} \\ a_{1k} \\ a_{2k} \\ a_{3k} \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}$$

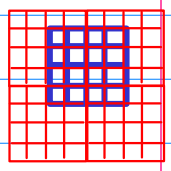
$$W = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$$



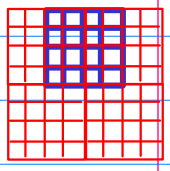
$\omega_1 = \begin{matrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{31} & a_{21} & a_{11} \end{matrix} \quad \begin{matrix} a_{10} & a_{11} & a_{21} & a_{31} \\ a_{00} & a_{30} & a_{20} & a_{10} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{33} & a_{23} & a_{13} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{02} & a_{32} & a_{22} & a_{12} \end{matrix}$



$\omega_{11} = \begin{matrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{11} & a_{01} & a_{31} & a_{21} \end{matrix} \quad \begin{matrix} a_{10} & a_{11} & a_{21} & a_{31} \\ a_{10} & a_{00} & a_{30} & a_{20} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{13} & a_{03} & a_{33} & a_{23} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{12} & a_{02} & a_{32} & a_{22} \end{matrix}$

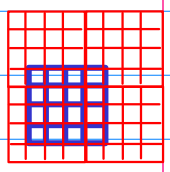


$\omega_{21} = \begin{matrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{21} & a_{11} & a_{01} & a_{31} \end{matrix} \quad \begin{matrix} a_{10} & a_{11} & a_{21} & a_{31} \\ a_{20} & a_{10} & a_{00} & a_{30} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{23} & a_{13} & a_{03} & a_{33} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{22} & a_{12} & a_{02} & a_{32} \end{matrix}$



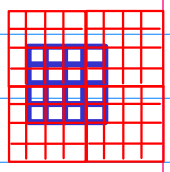
$\omega_{31} = \begin{matrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{31} & a_{21} & a_{11} & a_{01} \end{matrix} \quad \begin{matrix} a_{10} & a_{11} & a_{21} & a_{31} \\ a_{30} & a_{20} & a_{10} & a_{00} \end{matrix} \quad \begin{matrix} a_{02} & a_{12} & a_{22} & a_{32} \\ a_{33} & a_{23} & a_{13} & a_{03} \end{matrix} \quad \begin{matrix} a_{03} & a_{13} & a_{23} & a_{33} \\ a_{32} & a_{22} & a_{12} & a_{02} \end{matrix}$

$$\omega_1 = C_1 a_0 + C_0 a_1 + C_3 a_2 + C_2 a_3$$



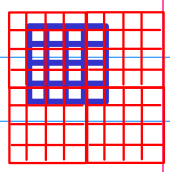
o_2 =

o_0 o_1 o_2 o_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
0_2 3_2 2_2 1_2	0_1 3_1 2_1 1_1	0_0 3_0 2_0 1_0	0_3 3_3 2_3 1_3



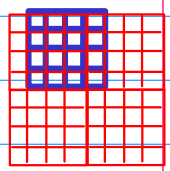
1_2 =

0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
1_2 0_2 3_2 2_2	1_1 0_1 3_1 2_1	1_0 0_0 3_0 2_0	1_3 0_3 3_3 2_3



2_2 =

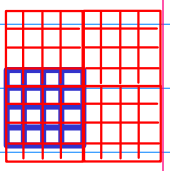
0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
2_2 1_2 0_2 3_2	2_1 1_1 0_1 3_1	2_0 1_0 0_0 3_0	2_3 1_3 0_3 3_3



3_2 =

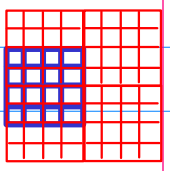
0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
3_2 2_2 1_2 0_2	3_1 2_1 1_1 0_1	3_0 2_0 1_0 0_0	3_3 2_3 1_3 0_3

$$\omega_2 = C_2 a_0 + C_1 a_1 + C_0 a_2 + C_3 a_3$$



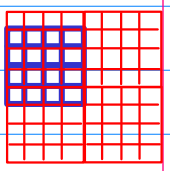
o_3 =

0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
0_3 3_3 2_3 1_3	0_2 3_2 2_2 1_2	0_1 3_1 2_1 1_1	0_0 3_0 2_0 1_0



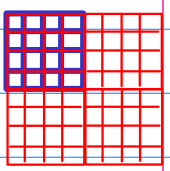
1_3 =

0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
1_3 0_3 3_3 2_3	1_2 0_2 3_2 2_2	1_1 0_1 3_1 2_1	1_0 0_0 3_0 2_0



2_3 =

0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
2_3 1_3 0_3 3_3	2_2 1_2 0_2 3_2	2_1 1_1 0_1 3_1	2_0 1_0 0_0 3_0



3_3 =

0_0 0_1 0_2 0_3	1_0 1_1 2_1 3_1	0_2 1_2 2_2 3_2	0_3 1_3 2_3 3_3
3_3 2_3 1_3 0_3	3_2 2_2 1_2 0_2	3_1 2_1 1_1 0_1	3_0 2_0 1_0 0_0

$$\omega_3 = C_3 a_0 + C_2 a_1 + C_1 a_2 + C_0 a_3$$

$$\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$