Computational Exercise #3

Type of Activity: The least squares solution in the matrix form

In the computational exercise #1, the estimated parameters ($\hat{a}$ and $\hat{b}$) were computed by using estimators for parameters a and b given by sum of terms (see equations 5a and 5b in the exercise #1). Here, you will repeat the exercise #1 by using a matrix notation. In this case, the forward model can be expressed, in matrix notation, as

$$\bar{\mathbf{y}} = \bar{\bar{\mathbf{A}}}\,\bar{\mathbf{p}} \quad , \tag{3.1}$$

where $\bar{\boldsymbol{y}}$ is an N-dimensional vector of the exact data (see item 1.2 in the exercise #1) whose ith element is:

$$y_i = p_1 x_i + p_2,$$

where p_1 and p_2 are the coefficients describing the first-order polynomial, where p_1 is the slope. In equation 3.1, $\bar{\mathbf{p}}$ is a M-dimensional vector of parameters (M=2) given by

$$\bar{\mathbf{p}} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \tag{3.2}$$

and $\bar{\bar{\mathbf{A}}}$ is the N × M sensitivity matrix given by

$$\bar{\bar{\mathbf{A}}} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}. \tag{3.3}$$

Here, you will repeat the exercise #1 by using a matrix notation. Hence, you need to find the least squares solution for the model parameters (p_1 and p_2), i.e.:

$$\hat{\bar{\mathbf{p}}} \;=\; \left(\bar{\bar{\mathbf{A}}}^{\boldsymbol{T}}\bar{\bar{\mathbf{A}}}\right)^{-\mathbf{1}}\bar{\bar{\mathbf{A}}}^{\boldsymbol{T}}\;\bar{\boldsymbol{y}}^{\boldsymbol{o}} \quad , \tag{3.4}$$

where $\bar{\boldsymbol{y}}^{\boldsymbol{o}}$ is an N-dimensional vector of the noise-corrupted data (observed data set). See the item 1.3 in the exercise #1. Here, the predicted data is an N-dimensional vector given by:

$$\bar{\boldsymbol{y}}^{\boldsymbol{p}} = \bar{\bar{\mathbf{A}}}\,\hat{\bar{\mathbf{p}}} \tag{3.5}$$

and the residual vector is an N-dimensional vector of differences between observed data and corresponding model predictions given by

$$\bar{\boldsymbol{\varepsilon}} = \;\bar{\boldsymbol{y}}^{\boldsymbol{o}} - \bar{\boldsymbol{y}}^{\boldsymbol{p}}. \tag{3.6}$$

To compute the variance of the parameters, let us recall that the M × M covariance matrix of the parameters for a least squares solution is given by

$$\mathbf{Cov}(\hat{\bar{\mathbf{p}}}) = \; \sigma^2\left(\bar{\bar{\mathbf{A}}}^{\boldsymbol{T}}\bar{\bar{\mathbf{A}}}\right)^{-\mathbf{1}}, \tag{3.7}$$

where σ^2 is a constant and known variance of the errors.